FACULTY OF SCIENCE
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PhD Thesis

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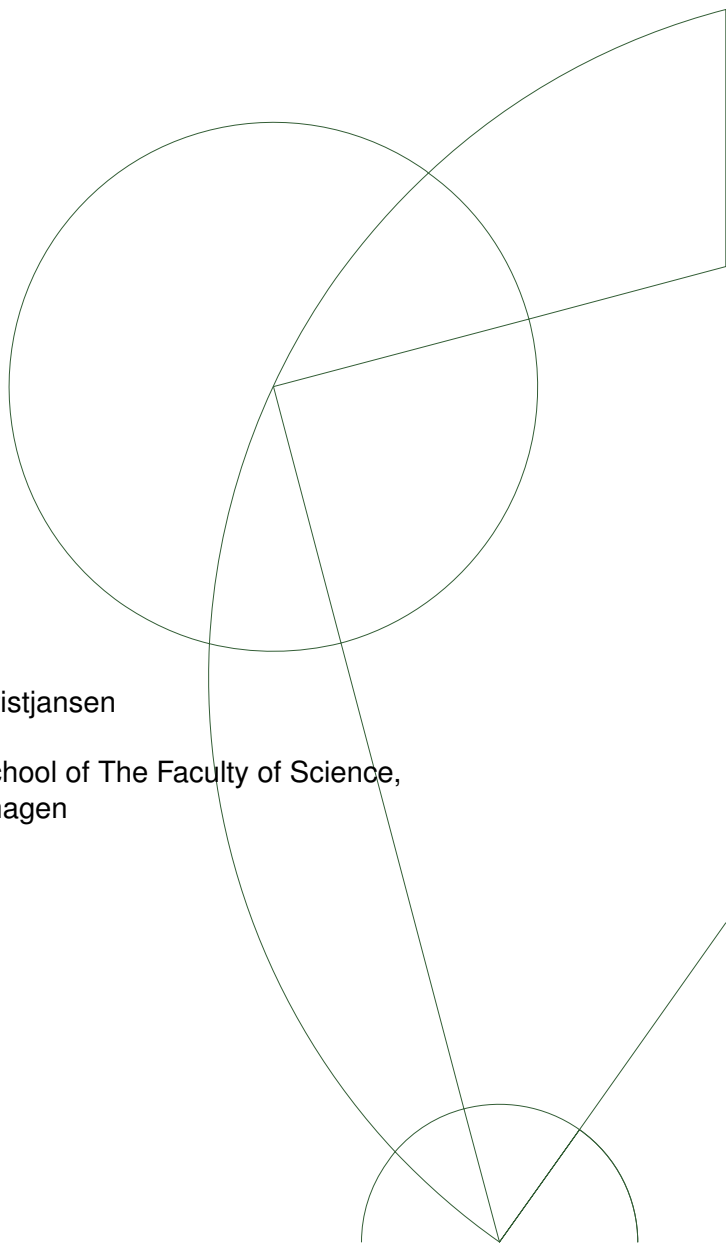
Boundaries of Integrability in AdS/dCFT

One- and two-point functions in probe brane field theories



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Summary

The topic of this thesis is the computation of correlation functions in defect conformal field theories (dCFTs) that holographically dual to certain probe brane configurations. The defect field theories discussed are all domain walls of $\mathcal{N} = 4$ super Yang-Mills theory that interface between a $U(N-k)$ gauge group and a $U(N)$ gauge group. dCFTs may have non-trivial one-point functions and for the $SO(3) \times SO(3)$ symmetric probe D7 defect we compute one-point functions at tree-level using integrability of the $\mathcal{N} = 4$ spectrum. The one-point functions are computed for $SU(2)$-sector operators with a small $M = 0, 2, 4, 6$ number of excitations and a general form for large operators is conjectured.

The explicit expressions for the one-point functions shows that the matrix product state for the $SO(3) \times SO(3)$ symmetric probe D7 defect is not an integrable spin chain state. In a related setup, the probe D5 defect, we present a new solution of the boundary Yang-Baxter equation that reduces to the $SO(6)$-sector matrix product state for zero rapidity.

Last, we consider the computation of two-point functions in the probe D5 defect for simple operators including the BMN vacuum of different lengths. In dCFTs the two-point functions can be expanded in conformal blocks providing a relation between the one-, two- and three-point functions.

Resumé

Emnet for denne afhandling er udregninger af korrelations funktioner i defekte konforme feltteorier, der er holografiske duale teorier for bestemte konfigurationer af braner. De defekte konforme feltteorier vi skal betragte er 'domain wall' løsninger af $\mathcal{N} = 4$ super Yang-Mills, der interpolerer mellem gauge grupperne $U(N-k)$ og $U(N)$. Defekte konforme feltteorier kan have ikke-trivielle et-punkt funktioner, og specielt udregner vi et-punkts funktionerne til ledende orden for den $SO(3) \times SO(3)$ symmetriske D7 defekt ved at anvende, at $\mathcal{N} = 4$ spektret er integrabelt. Et-punkts funktionerne er udregnet for $SU(2)$ operatorer med for lave eksitationer $M = 0, 2, 4, 6$, og et generalt udryk for store operatorer foreslåes.

Et-punkts funktioner viser at matrixprodukttilstanden svarerende til den $SO(3) \times SO(3)$ symmetriske D7 defekt, ikke udgør en integrabel spinkædetilstand. Et relateret setup er D5 defekten, her bestemmer vi en ny løsning til reflektionsligningen, der har den egenskab, at den reducerer til $SO(6)$ matrixprodukttilstanden for ingen rapiditet.

Tilsidst betragter vi udregningerne af to-punkts funktioner i D5 defekten for simple operatorer heriblandt BMN vacua af forskellig længde. I defekte konforme feltteorier kan to-punkts funktionen ekspanderes i konforme blokke. Dette giver relationer mellem et-, to- og tre-punkts funktionerne.

Preface

This thesis presents the research that I have done during my PhD studies at the Niels Bohr Institute, University of Copenhagen, Denmark from February 2016 to January 2019. The text is based on the research done under supervision of Charlotte Kristjansen and in collaboration with Marius de Leeuw, Asger C. Ipsen, Charlotte Kristjansen, & Matthias Wilhelm.

The published work discussed in the thesis is

- M. de Leeuw, A. C. Ipsen, C. Kristjansen, K. E. Vardinghus, and M. Wilhelm, *"Two-point functions in AdS/dCFT and the boundary conformal bootstrap equations," JHEP* **08** (2017) 020.

While yet-to-be published results are: The tree-level one-point function in the $SO(3) \times SO(3)$ symmetric probe D7-brane field theory, and the solution of the boundary Yang-Baxter equation that corresponds to the $SO(6)$-sector matrix product state.

I would like to thank my advisor Charlotte Kristjansen without whom, I would not have been led on to this exciting journey and her group, past and present, for providing a stimulating environment for research. In particular, I would like to thank my collaborators: Isak Buhl-Mortensen, Marius de Leeuw, Asger C. Ipsen and Matthias Wilhelm for the interesting discussions and insights shared. A thank goes to Matthias Staudacher and his group at Humboldt-Universität zu Berlin, and likewise to Pedro Vieira and his group at Perimeter Institute for Theoretical Physics, for their hospitality during my vists. I should also like to thank my former and current office mates: Isak Buhl-Mortensen, Carsten Fritzner, Lorenzo Menculini, Adam Mielke, Matthias Volk, Haopeng Yan. As well as the rest of the high-energy theory group.

Finally I would live to thank Linea and my family for their support and encouragement during these three years.

Kasper E. Vardinghus, January 2019

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1 Introduction

> *"So in writing this, I have to separate the subjects for clarity, but sometimes the right separation is not clear."*
>
> — J. G. Polchinski, Memories of a Theoretical Physicist

Quantum field theory is the framework in which modern theories of physics are formulated. The gauge theories – more precisely – Yang-Mills type theories, describes the interactions of matter through the exchange of spin 1 bosonic fields excitations. Yang-Mills theories are generalisations of electromagnetism obtained by enlarging the possible local symmetries of the theory to include non-Abelian continuous symmetries. A Yang-Mills theory is then classified by a Lie group, henceforth the *gauge group* and the possible matter in various representations of the local symmetry. The Standard Model of particle physics is precisely such a Yang-Mills theory with the gauge group $U(1)\times SU(2)\times SU(3)$ and various fundamental matter fields, i.e. the leptons and quarks.

The maximally supersymmetric Yang-Mills theory in four dimensions, $\mathcal{N} = 4$ super Yang-Mills (SYM)[1], distinguishes itself amongst the four-dimensional field theories for its particular simplicity. As it is maximally supersymmetric all the fields of $\mathcal{N} = 4$ SYM are related by supersymmetry to the vector field, i.e. it is the unique theory of a four-dimensional $\mathcal{N} = 4$ vector multiplet [1]. As a consequence of the large amount of supersymmetry $\mathcal{N} = 4$ SYM is also quantum conformal invariant [2, 3]. The conformal invariance enhances to the exact superconformal symmetry of $\mathcal{N} = 4$ SYM, for which the global symmetry is described by the supergroup $PSU(2, 2|4)$. Yang-Mills theories simplifies for large gauge groups, e.g. $SU(N)$ for $N \to \infty$, in the limit where the 't Hooft coupling, $\lambda = g^2_{\mathrm{YM}}N$, is held fixed. The perturbation series then arranges

[1] $\mathcal{N}$ denotes the number of supersymmetries.

itself as an expansion in $\frac{1}{N}$ with the leading contribution being the planar diagrams [4]. Planar $\mathcal{N} = 4$ is amongst the simplest and most symmetric four-dimensional field theories, and hence provide a neat toy model in the space of Yang-Mills theories [5].

In string theory, the effective field world-volume theories of D-branes are Yang-Mills theories. In particular, the world-volume theory of a stack of N D3-branes is $U(N)$ $\mathcal{N} = 4$ SYM. This enables us to think geometrically about, conjecture various properties of, and relations amongst the gauge theories (see e.g. [6]). A remarkable example of this is provided by the AdS/CFT correspondence[2] [7–9] (see also the review [10]). The correspondence conjectures the equivalence between type IIB superstring theory on an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background and $U(N)$ $\mathcal{N} = 4$ SYM under the identification of parameters

$$\frac{\lambda}{N} \sim g_{\mathrm{s}}\,, \quad \frac{L^4}{l_{\mathrm{s}}^4} \sim \lambda\,.$$

Here N is the rank of the gauge group / the five-form flux, g_{YM} is the gauge theory coupling, g_{s} is the string coupling, and L/l_{s} is the curvature of the background $\mathrm{AdS}_5 \times \mathrm{S}^5$ in string length l_{s}. The conjecture passes many non-trivial checks and is widely believed to be true. In the planar limit $N \to \infty$, the string coupling goes to zero. This equates classical strings, i.e. only tree-level diagrams, with planar $\mathcal{N} = 4$ SYM. The 't Hooft coupling λ then interpolates between planar $\mathcal{N} = 4$ SYM at weak coupling $\lambda \ll 0$ and supergravity at strong coupling $\lambda \gg 0$. The AdS/CFT correspondence provides 'explanations' for many phenomena: It gives a precise large N dual string theory [4], it provides a holographic description of gravity [11,12], and it connects $\mathcal{N} = 4$ SYM to the integrable 2-dimensional world-sheet theory of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring [13].

The AdS/CFT correspondence provides a weak/strong duality between gravity and gauge theory. Two ways – understand gravity from field theory or field theory from gravity.

1.1 AdS/CFT Integrability

In the planar limit, $\mathcal{N} = 4$ SYM exhibits integrability for many observables including anomalous dimensions and amplitudes [14]. In particular, the dilatation

[2] Many conjectured equivalences between a conformal gauge theory and some quantum gravity theory on AdS exists. In this thesis we will only consider $\mathrm{AdS}_5/\mathrm{CFT}_4$.

operator

$$D(g) = \sum_{n=0}^{\infty} g^{2n} D^{(n)} = D^{(0)} + \Gamma(g)$$

defines, order by order in $g^2 := \frac{\lambda}{16\pi^2}$, an integrable local spin Hamiltonian of range $2n$ that describes the mixing of single trace operators [15–18]. A closed sector is The SU(2)-subsector, consisting of two complex scalars. The anomalous dilatation operator at one-loop in the SU(2) sector is described by the Heisenberg $\mathrm{XXX}_{1/2}$. The Heisenberg spin chain was first solved by H. Bethe [19] using, what is now known as, the coordinate Bethe ansatz. The Bethe ansatz determines the energies and eigenstates from solutions to polynomial equations. To go beyond the asymptotic Bethe ansatz for large spin chains one needs to account of wrapping corrections, i.e. the planar diagrams where contractions encircles the operator [20]. The quantum spectral curve [21–23] is believed to provide a unified description of the scaling dimensions for the AdS/CFT integrable system that interpolates between weak and strong coupling.

When available the Bethe eigenstates gives the conformal operators. Based on integrability it has been possible to device techniques to compute higher point correlation functions, specifically three- and four-point functions at leading order [24, 25] and asymptotically [26, 27]. A recent approach is based on the quantum spectral curve [28]. It seem possible to systematically compute $\frac{1}{N}$ corrections by integrability techniques [29] or conformal bootstrap [30].

The list of other observables where integrability of $\mathcal{N} = 4$ SYM have been useful includes amplitudes [31], the Hagedorn temperature [32], Wilson loops [33]. The unifying algebraic structure is the Yangian algebra $Y[\mathfrak{psu}(2,2|4)]$ [34]. The equations of motion of planar $\mathcal{N} = 4$ have been shown to exhibit Yangian invariance [35].

1.2 Holography of defect CFTs

In the AdS/CFT setup, conformal field theories involving defects can be engineered from D-brane intersections. Consider a stack of N D3-branes intersected by a stack of N_p Dp-branes that share three of the four directions with the D3-branes. The near-horizon limit of N D3-branes is $\mathrm{AdS}_5 \times \mathrm{S}^5$. In the *probe brane limit* where $N_p \ll N_3$ the near-horizon geometry is not modified by the bisecting branes. Therefore, the Dp occupy an AdS_4 submanifold times a compact submanifold of the five-sphere [36, 37].

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	•	•	•	•						
D5	•	•	•		•	•	•			
D7	•	•	•		•	•	•	•	•	

Table 1.1 – The embedding of flat Dp-branes giving rise to defect conformal field theories in the decoupling limit near the D3 branes.

The dual field theory is a defect conformal theory where the open $3-p$ strings provides fundamental matter confined to co-dimension one defect in $\mathcal{N}=4$ SYM. These configurations also allows a number n of D3-branes to end on the Dp-branes. This provides a holographic dual of domain walls in $\mathcal{N}=4$ SYM separating a $U(N)$ gauge group for $x_\perp > 0$ and $U(N-n)$ for $x_\perp < 0$ [37, 38].

Defect conformal field theories are relevant for critical phenomena with boundaries or co-dimensional one sheets of matter, e.g. graphene. The probe-brane configurations have been extensively studied as top-down holographic models for such phenomena providing access to strong coupling results [39–41].

1.3 Outline of the thesis

This thesis is about field theory computations for the defect conformal theories dual, through AdS/dCFT correspondence, to the probe brane configurations mentioned in the introduction. Chapter 2 gives a brief review of the defect conformal field theories that will be discussed in the later chapters. The chapter also contains a discussion of the boundary fields for the probe D5-brane defect. In chapter 3 I review how to compute one-point functions by mapping the problem, using the integrability of $\mathcal{N}=4$ SYM to an overlap between a matrix product state and a Bethe eigenstate. The chapter then details the computation of one-point functions in D3$\perp$D7$_{k_1,k_2}$ for the $SU(2)$-sector explicitly demonstrating that the D3$\perp$D7$_{k_1,k_2}$ matrix product state is not integrable. Chapter 4 discusses the integrability of the one-point functions. In particular, I review how to find open integrable spin chains and how the SU(2) one-point functions in D3$\perp$D5$_2$ was proven using the six-vertex model. I then discuss the recent definition of an integrable spin chain boundary state and describe a new solution of the boundary Yang-Baxter equation for an open SO(6) integrable spin chain. Chapter 5 recounts the computation of two-point

functions of BPS operators at leading order and the possibility of mining the expressions for conformal data. The final chapter 6 contains some concluding remarks.

2 The defect theories

In this chapter, we shall review the various defect conformal field theories obtained as probe D-brane intersections and describes fuzzy funnel solutions of $\mathcal{N} = 4$ SYM. We shall define three different domain walls: D3⊥D5$_k$, D3⊥D7$_{k_1,k_2}$, D3⊥D7$_n$. Our primary motivation to study these field theories is to compute one-point correlation functions using integrability techniques. The one-point functions in these backgrounds are non-trivial already at tree-level and therefore provide a wonderful playground for integrability techniques. In chapter 5 we shall discuss the the leading contribution to the two-point functions in D3⊥D5$_k$ and for this we need the spectrum and propagators. I will therefore, briefly review the perturbative theory around the fuzzy funnel solutions. Last, I will recount the intricacies of determining the of the boundary fields of bulk operators in D3⊥D5$_k$ which was published in [42].

2.1 Domain wall solutions of $\mathcal{N} = 4$ SYM

In this section, we shall review the different defect theories based on $\mathcal{N} = 4$ SYM that are dual to the probe brane configurations discussed above. The action of $\mathcal{N} = 4$ SYM is [1, 43]

$$S_{\mathcal{N}=4} = \frac{2}{g_{\text{YM}}^2}\int \mathrm{d}^4x\ \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\phi_i D^\mu\phi_i + \frac{\mathrm{i}}{2}\bar{\Psi}\Gamma^\mu D_\mu\Psi \right.$$
$$\left. +\frac{1}{2}\bar{\Psi}\tilde{\Gamma}^i[\phi_i,\Psi] + \frac{1}{4}[\phi_i,\phi_j][\phi_i,\phi_j]\right). \quad (2.1)$$

Here ϕ_i are the 6 real scalars in the **6** representation of the $SO(6)$ R-symmetry group, Ψ is a ten-dimensional Majorana-Weyl fermion and $\{\Gamma^\mu, \tilde{\Gamma}^i\}$ are the corresponding ten-dimensional gamma matrices. The indices ranges as $\mu, \nu = 0, 1, 2, 3$ and $i, j = 1, \ldots, 6$. All the fields are in the adjoint representation of the gauge group hence, $D_\mu\phi_i = \partial_\mu\phi_i - \mathrm{i}[A_\mu, \phi_i]$.

The solutions of the scalar equations of motion provides backgrounds around which we may perturbatively quantize. The equations of motion for the scalars of $\mathcal{N} = 4$ SYM (i.e. setting the gauge field and fermions to zero) is [38]

$$\nabla^2 \phi_i - \sum_{j=1}^{6} [\phi_j, [\phi_j, \phi_i]] = 0 \,.$$

Where $\nabla^2 = \sum_i \partial_i^2$ is the spatial Laplacian. This should further be supplemented by the condition that the colour current vanishes

$$\sum_{i=1}^{6} [\phi_i, \nabla \phi_i] = 0$$

If we restrict to co-dimension one, flat defects, and have to preserve translational and rotational invariance along the directions parallel with the defect, the scalar VEVs can only depend on the distance to the defect $x_\perp$. Furthermore, there is no length scale if we demand conformal invariance. Therefore, the form of scalar solutions that preserve defect conformal invariance is

$$\phi_i(x_\perp) = \frac{T_i}{x_\perp} \,,$$

where T_i for $i = 1, \ldots, 6$ are $N \times N$ matrices that satisfy

$$T_i = \frac{1}{2} \sum_j [T_j, [T_j, T_i]] \,.$$

In the following we will discuss three different non-trivial vacua for $\mathcal{N}$=4 SYM.

Before discussing solutions let us quickly remark on the possibility of preserving a subset of the supersymmetries. Boundary conditions that preserve a subset of the supersymmetries of $\mathcal{N} = 4$ SYM are possible. The supersymmetry algebra implies schematically that $\{Q, Q\} \sim P$ where Q is the supersymmetry and P the translation generator. Therefore, it is impossible to preserve the full supersymmetry of $\mathcal{N} = 4$ SYM when breaking translational invariance. Boundary conditions that preserve the maximal possible subset, i.e. $\frac{1}{2}$-BPS boundary conditions, have been classified [44, 45]. In particular, the R-symmetry must be broken down to $SO(3) \times SO(3)$ and the full symmetry group $PSU(2,2|4)$ is reduced to a subgroup $OSp(4|4)$ [44]. Under supersymmetry the fermions transform into a combination of scalars and field strength. If the solutions with $\Psi = 0$ is to preserve supersymmetry this puts restrictions on the scalars by

demanding that the supersymmetry variation vanishes. Therefore, the scalars must satisfy the equation

$$\frac{d\phi_i}{dz} = \frac{\mathrm{i}}{2}\varepsilon_{ijk}[\phi_j, \phi_k]\,, \tag{2.2}$$

known as the Nahm equation [44].

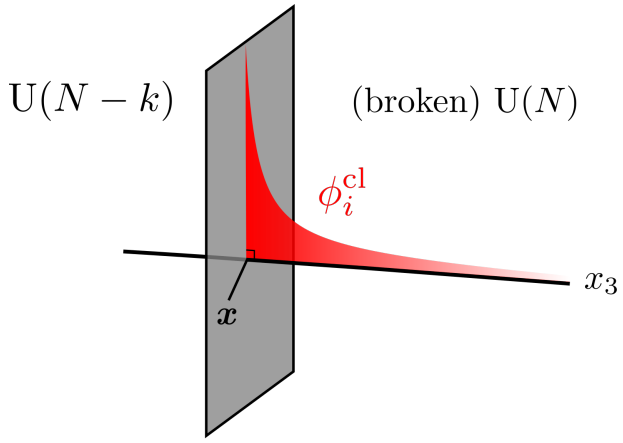

Figure 2.1 – The $\mathcal{N} = 4$ SYM domain wall separating $U(N)$ and $U(N-k)$ vacua.

D3⊥D5$_k$

The probe D5-brane intersection (see table 1.1) is supersymmetric. Let k be the difference in the number of D3-brane between $x_\perp > 0$ and $x_\perp < 0$. In the following we shall refer to the field theory that is dual to this probe brane configuration as the D3⊥D5$_k$ field theory. For $k = 0$, the dual field theory was fixed using the supersymmetry [46] and shown to be conformal invariant [47]. For k D3-branes dissolved into the D5-brane the matrix structure of the ϕ_i $i = 1, 2, 3$ defines a fuzzy funnel [48]. The solutions satisfying the Nahm equation (2.2) takes the form of a Nahm pole

$$\phi_i^{\mathrm{cl}}(x_\perp) = -\frac{1}{x_\perp} t_i^{(k)} \oplus {}_{(N-k)\times(N-k)}\,,$$

where $i = 1, 2, 3$ and $t_i^{(k)}$ is the k-dimensional irreducible representation of $\mathfrak{su}(2)$.

D3⊥D7$_{k_1,k_2}$ & D3⊥D7$_n$

The probe D7-brane has possible embeddings in the near-horizon geometry of the D3-branes: $AdS_4\times S^2\times S^2$ and $AdS_4\times S^4$ [38]. Note that the D7-brane configurations are not supersymmetric, and therefore the scalar VEVs in the dual field theory do not satisfy the Nahm equation.

The solutions corresponding to the $AdS_4\times S^2\times S^2$ is the product of fuzzy funnels providing a $\mathcal{N}=4$ SYM domain wall between $U(N)$ for $x_\perp>0$ and $U(N-k_1k_2)$ for $x_\perp<0$ gauge groups [38]

$$\phi_i^{\rm cl}(x_\perp) = -\frac{1}{x_\perp}(t_i^{k_1}\otimes 1_{k_2\times k_2})\oplus_{(N-k_1k_2)\times(N-k_1k_2)}, \quad i=1,2,3$$
$$\phi_i^{\rm cl}(x_\perp) = -\frac{1}{x_\perp}(1_{k_1\times k_1}\otimes t_i^{k_2})\oplus_{(N-k_1k_2)\times(N-k_1k_2)}, \quad i=4,5,6\,.$$

Here $t_i^{(k_a)}$ each satisfy a $\mathfrak{su}(2)$ algebra $[t_i^{(k_a)},t_i^{(k_a)}]=\mathrm{i}\epsilon_{ijk}t_k^{(k_a)}$. In the following we shall refer to the field theory that is dual to this probe brane configuration as the D3⊥D7$_{k_1,k_2}$ field theory.

The $AdS_4\times S^4$ embedding is $SO(5)$-symmetric and corresponds to funnel solution with a fuzzy 4-sphere.

$$\phi_i^{\rm cl}(x_\perp) = \frac{G_i^{(n)}}{\sqrt{8}\,x_\perp}$$

for $i=1,\ldots,5$ and $\phi_6=0$. The matrices have dimension $d_G=\frac{1}{6}(n+1)(n+2)(n+3)$ and are given as [49]

$$G_i^{(n)} = [\gamma_i\otimes 1_{4x4}\cdots\otimes 1_{4x4}+1_{4x4}\otimes\gamma_i\cdots\otimes 1_{4x4}+\ldots]_{\rm sym}\oplus_{(N-d_G)\times(N-d_G)}\,,$$

where each term has a gamma matrix γ^i at the n'th position and we symmetrise the resulting matrix. We shall do any computations in this theory, however we shall reserve the name D3⊥D7$_n$ for comparisons.

2.2 Perturbation theory around fuzzy funnels

Let $\phi_i^{\rm cl}$ be one of the classical solutions discussed above. Then, we can study the perturbative quantisation around this vacuum by writing the scalars as

$$\phi_i = \phi_i^{\rm cl}+\tilde{\phi}_i\,,$$

and inserting into the action (2.1). As per usual, constant terms can be dropped and linear terms vanish by the equations of motion. Expanding the kinetic

term of the scalars gives

$$-\frac{1}{2}D_\mu\tilde{\phi}_i D^\mu\tilde{\phi}_i - \frac{1}{2}A_\mu[\phi_i^{\mathrm{cl}},[\phi_i^{\mathrm{cl}},A^\mu]] - \mathrm{i}A_\mu[\partial^\mu\phi_i^{\mathrm{cl}},\tilde{\phi}_i] - \mathrm{i}[A_\mu,\phi_i^{\mathrm{cl}}]\partial^\mu\tilde{\phi}_i + \dots\,,$$

where the terms in $\dots$ are higher order in fields, i.e. interactions. This is nothing but the Higgs mechanism for $\mathcal{N}=4$, i.e. by assigning VEVs to scalar fields the gauge field acquires a mass term. To do perturbation theory, there are three questions related to quadratic term that must be dealt with. First, the term $[A_\mu,\phi_i^{\mathrm{cl}}]\partial^\mu\tilde{\phi}$ is problematic since it contains a single derivative. However, the term can be cancelled by introducing a gauge-fixing term [50]. The mixing in colour and flavour space, as exhibited by the other two terms, will require diagonalisation of the mass matrix to determine the propagating states for a particular ϕ_i^{cl}. Finally, the mass terms will be depend on $x_\perp$ and we will need to know how to deal with this.

In total, the bosonic mass terms are

$$\begin{aligned}S_{\mathrm{m,b}} = \frac{2}{g_{\mathrm{YM}}^2}\int \mathrm{d}^4x\ \mathrm{Tr}\Big(&\frac{1}{2}[A_\mu,\phi_i^{\mathrm{cl}}][A^\mu,\phi_i^{\mathrm{cl}}] + 2\mathrm{i}[A_\mu,\tilde{\phi}_i]\partial^\mu\phi_i^{\mathrm{cl}}\\ &+\frac{1}{2}[\phi_i^{\mathrm{cl}},\phi_j^{\mathrm{cl}}][\tilde{\phi}_i,\tilde{\phi}_j] + \frac{1}{2}[\phi_i^{\mathrm{cl}},\tilde{\phi}_j]^2 + \frac{1}{2}[\tilde{\phi}_i,\phi_j^{\mathrm{cl}}][\phi_i^{\mathrm{cl}},\tilde{\phi}_j] + \frac{1}{2}[\phi_i^{\mathrm{cl}},\tilde{\phi}_i]^2\Big)\,.\end{aligned}$$

Each term carries a factor $1/x_\perp^2$ and we may write it as

$$S_{\mathrm{m,b}} = \frac{2}{g_{\mathrm{YM}}^2}\int \mathrm{d}^4x\left(-\frac{1}{2x_\perp^2}\Phi_{f',c'}\mathbf{M}_{(f',c'),(f,c)}\Phi_{f,c}\right)$$

where f,f' are flavour indices and c,c' are colour indices corresponding to some particular matrix basis for the colour components.

The bosonic mass terms come with a factor $1/x_\perp^2$. This means that the propagators $K(x,y)$ have to satisfy

$$\Big(-\partial^\mu\partial_\mu + \frac{m^2}{x_\perp^2}\Big)K(x,y) = \frac{g_{\mathrm{YM}}}{2}\delta(x-y)\,.$$

This is equivalent to an AdS_4 propagator with the distance to the defect $x_\perp$ being the radial coordinate. That is, massive modes experience an effective AdS_4 with $x_\perp$ being the boundary. The massive modes are therefore confined to $x_\perp > 0$.

The diagonalisation of D3$\perp$D5$_k$ was accomplished in [43] using fuzzy spherical harmonics $\hat{Y}_\ell^m$ that diagonalise the adjoint action of the $\mathfrak{su}(2)$ generators: $[t_i,[t_i,\hat{Y}_\ell^m] = \ell(\ell+1)\hat{Y}_\ell^m$. We will need the spectrum of scalars and their

propagators in chapter 5 when we compute perturbative corrections to the two-point functions. For the D3⊥D5$_{k_1,k_2}$ the diagonalisation was accomplished in [51].

2.3 The action on the defect for D3⊥D5$_k$

The action of the D3⊥D5$_k$ theory has a bulk part, which is a Higgsed $\mathcal{N}=4$ SYM for $k>2$, and a boundary part containing fundamental degrees of freedom originating from the open $3-5$ strings. For $k=0$ the field theory including boundary interactions is known and explicitly written down [46, 47]. However, for $k\neq 0$ this is no longer the case. In this section I shall recount what we have learned about the spectrum of boundary operators build from bulk operators [42] and I will briefly describe an ongoing project to explicitly obtain the defect action.

Consider a co-dimension one, flat defect located at $x_3=0$. We can think of the delta function as lifting the three-dimensional Lagrangian to four dimensions.

$$S=\int \mathrm{d}^4x\left(\mathcal{L}_4+\delta(x_3)\mathcal{L}_3\right)=\int \mathrm{d}^4x\,\mathcal{L}_4+\int \mathrm{d}^3x\,\tilde{\mathcal{L}}_3\,.$$

There are two types of fields in $\tilde{\mathcal{L}}_3$: Bulk fields evaluated at the defect $\Phi|_{x_3=0}$ and fields that only have support on the defect $\hat{\Phi}$. In the following we shall see how to obtain the spectrum of boundary operators.

Near the boundary the bulk-to-bulk propagator can be expanded as

$$K^\nu(x,y)=\frac{g_{\mathrm{YM}}^2}{2\pi^2}\frac{2^{2\nu}}{\binom{2\nu+1}{\nu+\frac{1}{2}}}\frac{(x_3y_3)^{\nu+\frac{1}{2}}}{(|\boldsymbol{x}-\boldsymbol{y}|^2+y_3^2)^{\nu+\frac{3}{2}}}+\mathcal{O}(x_3^{\nu+\frac{5}{2}})\,.$$

where $\nu=\sqrt{m^2+\frac{1}{4}}$ parametrises the mass matrix eigenvalues (see table 5.1). Near the boundary the propagator behaves as $K^\nu\sim(x_3)^{\nu+\frac{1}{2}}$ hence it satisfy Dirichlet boundary conditions at the defect for $\nu=\frac{1}{2}$, i.e. $K^{\frac{1}{2}}(x,y)|_{x_3=0}=0$. For $\nu=-\frac{1}{2}$ the propagator satisfy Neumann boundary conditions, i.e. $\partial_3K^{-\frac{1}{2}}(x,y)|_{x_3=0}=0$. This follows, since the propagator for $\nu=-\frac{1}{2}$ should be thought as $\lim_{\nu\to-\frac{1}{2}}K^\nu(x,y)$. With this prescription one must use ${}_2F_1(-1,0;0;-\xi^{-1})=1+\frac{1}{2}\xi^{-1}$. Note that the $\frac{1}{2}\xi^{-1}$ piece is important for it to satisfy Neumann boundary conditions. The boundary behaviour of the fields is consistent with D5-like 1/2-BPS boundary conditions [44].

Since the propagator scales as $K^\nu \sim x_\perp^{\nu+\frac{1}{2}}$ near the defect, we may define finite operators at the defect by multiplying with a compensating factor $x_\perp^{-\nu-\frac{1}{2}}$. For $(\phi_1)_{\ell m}$ we have $\nu = \ell - \frac{1}{2}$, therefore

$$(\hat{\phi}_1)_{\ell,m}(\boldsymbol{x}) = \lim_{x_3\to 0}(x_\perp)^{-\ell}(\phi_1)_{\ell m}(x)\,,$$

defines a boundary operator. A further subtlety is that this do not transform in a simple way under the boundary gauge group $U(N-k)$. We can find a basis of operators that are either singlet or fundamental under the gauge group $U(N-k)$ by mixing the field with $(\hat{A}_3)_{\ell m}$ [42].

Lastly, the R-charge is modified. The R-symmetry is broken to $SO(3)_{\rm C} \times SO(3)_{\rm E} \subset SO(6)$ by the defect. The scalar fields ϕ_i for $i = 1, 2, 3$ transform as a vector under $SO(3)_{\rm C}$. We see that the classical solution is not directly invariant under the naive transformation of $\phi_i \to R_{ij}\phi_j$. In the bulk, this can simply be compensated by a gauge transformation U such that $UR_{ij}t_jU^{-1} = t_i$. However, at the defect the gauge symmetry is reduced to $U(N-k)$ and this U becomes a global transformation. Hence, the transformation properties of boundary operators under $SO(3)_{\rm C}$ is determined by both their flavour i and colour ℓ. In particular,

$$(\phi_i)_{\ell m} = \mathrm{tr}(\phi_i(\hat{Y}^m_\ell)^\dagger) \to \mathrm{tr}(R_{ij}\phi_j U^{-1}(\hat{Y}^m_\ell)^\dagger U)\,.$$

The fuzzy spherical harmonics transform in the spin-ℓ representation. For our example: $(\hat{\phi}_1)_{\ell,m}(\boldsymbol{x})$ have scaling dimension $\hat{\Delta} = \ell + 1$, is a singlet under $U(N-k)$ and transforms under R-symmetry as $(1 \otimes \ell, 0) = (\ell - 1, 0) + (\ell, 0) + (\ell + 1, 0)$. By this analysis one can determine the boundary fields coming from the bulk (see table 2.1) [42].

An approach to obtaining the defect action for D3⊥D5$_k$, that we are currently pursuing, consist of folding the theory along the defect to obtain a boundary theory with a gauge group $U(N) \oplus U(N-k)$. This is then one of the D5-like 1/2-BPS boundary conditions classified in [44]. The field theory may then be written, formally, as a three-dimensional theory with a continuum of fields, indexed by $x_\perp > 0$ [44]. Since the theory has 3d $\mathcal{N} = 4$, the possible terms in a Lagrangian is highly restricted. In fact, in terms of 3d $\mathcal{N} = 2$ superfields there are only 6 contributions [44, 47]. Un-doing the folding and squashing, and writing it out in components is technical challenge, however I look forward to report on this soon.

Field	Scaling dim. $(\hat{\Delta})$	$SO(3)_{\rm C}$	$SO(3)_{\rm E}$	$U(N-k)$
$(\hat{\phi}_{1,2,3})_{\ell m}$	$\ell+1$	$1\otimes\ell$	0	sing.
$(\hat{\phi}_{4,5,6})_{\ell m}$	$\ell+2$	ℓ	1	sing.
$(\hat{A}_{\hat{\mu}})_{\ell m}$	$\ell+2$	ℓ	0	sing.
$(\hat{\psi}_{1,2,3,4})_{\ell m}$	$\ell+\frac{3}{2}$	$\frac{1}{2}\otimes\ell$	$\frac{1}{2}$	sing.
$[\hat{\phi}_{1,2,3}]_{n,a}$	$\frac{k+1}{2}$	$1\otimes\frac{k-1}{2}$	0	fund.
$[\hat{\phi}_{1,2,3}]_{n,a}$	$\frac{k+3}{2}$	$\frac{k-1}{2}$	1	fund.
$[\hat{\phi}_{1,2,3}]_{n,a}$	$\frac{k+3}{2}$	$\frac{k-1}{2}$	0	fund.
$[\hat{\phi}_{1,2,3}]_{n,a}$	$\frac{k+2}{2}$	$\frac{1}{2}\otimes\frac{k-1}{2}$	$\frac{1}{2}$	fund.
q_a	$\frac{1}{2}$	$\frac{1}{2}$	0	fund.
χ_a	1	0	$\frac{1}{2}$	fund.

Table 2.1 – The spectrum of boundary fields.

3 Computing one-point functions

> *"Solution to any good problem is given by a determinant."*
>
> — L. D. Faddeev, as remembered by F. Smirnov

In this chapter I shall review the computation of one-point functions in $\mathcal{N} = 4$ SYM with probe brane defects. In particular I will discuss the results on the tree-level one-point functions in D3⊥D7$_{k_1,k_2}$ for non-protected operators in the SU(2) sector.

In a defect conformal field theory the expectation value of a local operator $\mathcal{O}(x)$ takes the form

$$\langle \mathcal{O}(x) \rangle = \frac{C}{x_\perp^\Delta},$$

as a consequence of the residual conformal symmetry. Here Δ is the conformal weight of the local operator and $x_\perp$ denotes the distance to the defect. The constant C depends on normalisation of the operator. We shall fix ambiguities in C by normalising the two-point function of the operator to

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle = |x_1 - x_2|^{-2\Delta}$$

far away from the defect $(x_1)_\perp, (x_2)_\perp \gg |x_1 - x_2|$. We will shall pick the phase such that the one-point structure constant is real and non-negative $C \geq 0$.

The one-point functions of chiral primaries in the D3⊥D5$_k$ setup was shown to agree between a weak-coupling, gauge theory computation and a strong-coupling, supergravity computation, thus initiating a series of precision tests of the AdS/dCFT correspondence [52]. Furthermore, it was noticed that one-point functions at strong coupling $\lambda \gg 1$ for chiral primaries organises itself

into a perturbative series in $\frac{\lambda}{k^2}$ and provides a prediction for the next-to-leading term [52]. These observations generalise to the two D3⊥D7 setups [38].

A determinant formula for one-point functions of $SU(2)$-sector operators at $k = 2$ was discovered based on the coordinate Bethe ansatz expression for the conformal local operators [53]. The determinant formulas extends to any k using an integrability-based recursion relation [54]. A determinant formula for one-point functions continues to work for higher rank sectors [55, 56]. A framework for the perturbative computation of quantum corrections to the one-point functions in D3⊥D5$_k$ in was developed in [43]. The one-loop computation was found to be in perfect agreement with the string theory prediction [57]. An asymptotic formula, i.e. all loop but neglecting wrapping corrections, was conjectured based on the integrability structure [58].

For the D3⊥D7 setups: Closed expressions for one-point functions at tree-level for certain operators with few excitations are known in the SO(5)-symmetric D3⊥D7$_n$ defect theory [59]. In the $SO(3) \times SO(3)$-symmetric D3⊥D7$_{k_1,k_2}$ setup, the one-loop corrections to the BMN vacuum was computed in [51] finding agreement with the double-scaling predictions from supergravity. The D3⊥D5 setup have a counterpart in β-deformed $\mathcal{N} = 4$ SYM. In the β-deformed setup, tree-level one-point functions for operators with few excitations have been computed [60].

3.1 The Bethe Ansatz for $\mathcal{N} = 4$ Operators

The anomalous dilatation operator at one-loop for the local operators of $\mathcal{N} = 4$ SYM constructed from L scalars ϕ_i is given by the Hamiltonian of an integrable spin chain with SO(6) symmetry [15]. In the following, we shall restrict to an SU(2) subsector, i.e. local operators constructed from the complex scalars

$$\Phi_\uparrow := \phi_3 + \mathrm{i}\phi_6\,, \quad \Phi_\downarrow := \phi_1 + \mathrm{i}\phi_4\,.$$

In the SU(2) sector the local operators constructed from M $\Phi_\downarrow$ and $L - M$ $\Phi_\downarrow$ only mix amongst themselves. In fact, the SU(2) sector is an example of a closed sector [61].

In the SU(2) sector, the anomalous dilatation operator takes the form

$$\Gamma(g) = \frac{\lambda}{8\pi^2} \sum_{n=1}^{L} \mathbb{H}_{n,n+1} + \dots\,, \tag{3.1}$$

where $\lambda = g^2 N$ is the t' Hooft coupling and

$$\mathbb{H}_{ab} = \mathbb{1}_{ab} - \mathbb{P}_{ab}\,,$$

is the local Hamiltonian density of the Heisenberg $\mathrm{XXX}_{\frac{1}{2}}$ spin chain. The exact diagonalisation of the $\mathrm{XXX}_{\frac{1}{2}}$ spin chain was obtained in the seminal paper [19] using the coordinate Bethe ansatz. For reviews see [62, 63]. The coordinate Bethe ansatz provides an explicit wave function for the eigenstates

$$|\{x_i\}\rangle = \mathcal{N} \sum_{\{n_i\}} \Big(\sum_{\sigma \in S_M} A_\sigma \prod_{j=1}^{M} x_{\sigma_j}^{n_j} \Big) |\{n_i\}\rangle\ , \tag{3.2}$$

where $\mathcal{N}$ is a normalisation factor, $|\{n_i\}\rangle = \prod_{i=1}^{M} S_{n_i}^{-} |0\rangle$ is the position-basis kets, $\{n_i\}$ is the list of positions of $\downarrow$ in a sea of $\uparrow$s, and the pseudo-vacuum state is $|0\rangle = |\uparrow\uparrow \cdots \uparrow\rangle$. The pseudo-momenta p_i of the excitations, *magnons*, are encoded in the variables $x_i = \mathrm{e}^{ip_i}$. It is beneficial to express quantities in terms of the rapidities u_i. The rapidities are another parametrisation of the momenta related to x_i through

$$x_i = \frac{u_i + \frac{\mathrm{i}}{2}}{u_i - \frac{\mathrm{i}}{2}}\,.$$

The rapidities are constrained by translational invariance to be roots of algebraic equations called the Bethe ansatz equations (BAE). We will discuss the BAE and its solutions in section 3.1.1. To describe an eigenstate of Eq. (3.1) the coefficients A_σ in Eq. (3.2) have to satisfy the condition

$$\frac{A_{(\ldots j \ldots i \ldots)}}{A_{(\ldots i \ldots j \ldots)}} = \frac{u_i - u_j + \mathrm{i}}{u_i - u_j - \mathrm{i}}\,. \tag{3.3}$$

In the expression above, we have written the permutation σ as $\big(\sigma(1)\sigma(2)\ldots\sigma(M)\big)$. This condition relates all the coefficients in the wave function Eq. (3.2), through a decomposition of σ into transpositions, to $A_{\mathbb{1}}$. We choose the normalisation $\mathcal{N}$ such that $A_{\mathbb{1}} = 1$. A hallmark of quantum integrability is that the M-body scattering amplitudes are reduced to products of two-body scatterings is [64].

The algebraic Bethe ansatz or *quantum inverse scattering method* (QISM) provides an alternative method to construct eigenstates of the Hamiltonian Eq. (3.1) [65,66]. I postpone a discussion of the QISM to section 4.1 in relation to open spin chains. However, I shall briefly recall how the algebraic Bethe anstaz constructs Bethe states for Eq. (3.1).

The basic object is the quantum Lax operator $\mathcal{L}_{n,a}(u)$ that acts locally on the n'th site of the spin chain, called the quantum space V_n and an auxiliary space V_a. Importantly, the Lax operator satisfy an intertwining relation

$$R_{ab}(u-v)\mathcal{L}_{n,a}(u)\mathcal{L}_{n,b}(v) = \mathcal{L}_{n,b}(v)\mathcal{L}_{n,a}(u)R_{ab}(u-v)\,. \tag{3.4}$$

Here $R_{ab}(u-v)$ is the R-matrix satisfying the Yang-Baxter equation. From the Lax operators one constructs the monodromy matrix

$$T_a(u) = \mathcal{L}_{a,L}(u)\cdots\mathcal{L}_{a,1}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

which is a 2x2 matrix in the auxiliary space with components being operators that act on the spin chain Hilbert space $\mathcal{H} = \otimes_{n=1}^{L} V_n$. As a consequence of the intertwining relation Eq. (3.4) the monodromy matrix $T_a(u)$ also satisfy the Yang-Baxter algebra. Let $t(u) = \mathrm{tr}_a\, T_a(u) = A(u) + D(u)$. Since the R-matrix is invertible, we have $[t(u), t(v)] = 0$. Therefore, $t(u)$ is an infinite family (indexed by commutes for any value of arbitrary complex parameter $u \in \mathbb{C}$) of commuting matrices. This can be formulated as $\ln t(u)$ being a generating series of conserved charges. In particular the Heisenberg spin chain Hamiltonian is schematically given as

$$H \sim \frac{\partial}{\partial u} \ln t(u)\big|_{u=0}\,.$$

The eigenstates of $t(u)$ can be constructed algebraically as

$$|\{u_i\}\rangle = B(u_M)\cdots B_{u_1}\,|0\rangle\ . \tag{3.5}$$

From the commutation relations the operators of A, B, and D one finds that the state Eq. (3.5) is an eigenstate of $t(u)$, provided that the rapidities satisfy the Bethe ansatz equations [65].

As presented above, there exists methods to analytically construct eigenstates and -values of the matrix Eq. (3.1). For the one-loop operator mixing problem of $\mathcal{N} = 4$ SYM, we can therefore write down the explicit operators with good conformal scaling dimensions, i.e. eigenstates of the dilatation operator [17]. The local operators are obtained from the Bethe states. In particular the local operator in the SU(2) sector with conformal dimension Δ is expressed as

$$\mathcal{O}_\Delta(x) = \Big(\frac{4\pi^2}{\lambda}\Big)^{\frac{L}{2}} \frac{\mathcal{Z}}{L^{\frac{1}{2}}} \sum_{\{s_i\}} \mathrm{tr}(\Phi_{s_1}\cdots\Phi_{s_L})\langle s_1\cdots s_L|\{u_i\}\rangle\!\rangle\,. \tag{3.6}$$

The scaling dimension is then

$$\Delta = L + \frac{\lambda}{8\pi^2}\sum_{i=1}^{M}\frac{1}{u_i^2+\frac{1}{4}}\,,$$

where L is the length of the operator. The pre-factor in Eq. (3.6) is chosen so that the two-point function is normalised according to Eq. (**??**), $\lambda = g_{\mathrm{YM}}^2 N$ is the t' Hooft coupling, and $\mathcal{Z}$ is the multiplicative factor that renders the correlation functions of renormalised operators finite [67]. The $|\{u_i\}\rangle\!\rangle$ is a normalised Bethe eigenstate

$$|\{u_i\}\rangle\!\rangle := \frac{|\{u_i\}\rangle}{\sqrt{\langle\{u_i\}|\{u_i\}\rangle}}\,,$$

where $|\{u_i\}\rangle$ can be either the coordinate or algebraic Bethe states. The norm formula depends on conventions and comes in a variety of forms, with various prefactors corresponding to this. In our conventions, the norm of the coordinate Bethe state Eq. (3.2) is given by

$$\langle\{u_i\}|\{u_i\}\rangle = \prod_{i=1}^{M}\left(u_i^2+\tfrac{1}{4}\right)\det G\,, \tag{3.7}$$

where the Gaudin matrix G is given by

$$G_{ij} = \frac{\partial \Phi_j(\{u_k\})}{\partial u_i} \tag{3.8}$$

and the functions $\Phi_i(\{u_j\})$ are

$$\Phi_i(\{u_j\})) = -\mathrm{i}\ln\Big[\Big(\frac{u_i-\frac{\mathrm{i}}{2}}{u_i+\frac{\mathrm{i}}{2}}\Big)^L\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{u_i-u_j+\mathrm{i}}{u_i-u_j-\mathrm{i}}\Big]\,. \tag{3.9}$$

This norm formula was conjectured by Gaudin [68] and later proven in the framework of the algebraic Bethe ansatz [69].

The local operators should be gauge invariant, and therefore, they are given as a trace of the the adjoint scalar fields [61]. The cyclicity of the trace then implies that the total momentum of the Bethe state should be a $\sum_i p_i = 0$ mod 2π. Therefore, the solutions of Bethe equations that corresponds local operators have to satisfy

$$\prod_{i=1}^{M}\frac{u_i+\frac{\mathrm{i}}{2}}{u_i-\frac{\mathrm{i}}{2}} = 1\,. \tag{3.10}$$

The coordinate Bethe ansatz can also be formulated for integrable spin chains based on higher rank groups through a nesting procedure (e.g. the exposition in [70]). The discussion above could be extended to the various sectors of PSU(2,2|4) where such an ansatz is known [71, 72].

3.1.1 On solving Bethe ansatz equations

The coordinate and algebraic Bethe ansatz approaches have in common that for the states Eq. (3.2) or Eq. (3.5) to be eigenstates of the Hamiltonian Eq. (3.1), the rapidities have to satisfy a set of algebraic equations known as the Bethe ansatz equations (BAE). For the SU(2) spin chain discussed above they take the form

$$\left(\frac{u_i + \frac{\mathrm{i}}{2}}{u_i - \frac{\mathrm{i}}{2}}\right)^L = \prod_{\substack{j=1 \\ j\neq i}}^{M} \frac{u_i - u_j + \mathrm{i}}{u_i - u_j - \mathrm{i}}\,. \tag{3.11}$$

A solution is known as Bethe roots $\{u_i\}$ and they can be encoded in a Baxter polynomial

$$Q(u) = \prod_{i=1}^{M}(u - u_i)\,.$$

Note that the commutativity of the creation operators $[B(u), B(v)] = 0$ means that solutions that are merely permutations of the rapidities produces the same states and should therefore be considered the same solution. The basic idea is that to each solution $\{u_i\}$ of Eq. (3.11) there should correspond an eigenstate $|\{u_i\}\rangle$ of the spin chain. However, this is *not* the case, and choosing admissible solutions is more subtle.

Solutions with coinciding rapidities are believed not to be physical for the $\mathrm{XXX}_{\frac{1}{2}}$ spin chain [73] e.g. one should not consider states Eq. (3.5) of the form

$$\{u, u, u_1, \ldots, u_{M-2}\}\,.$$

This is formulated as the Pauli exclusion completeness conjecture: That the complete spectrum of the Heisenberg spin chain are constructed from states Eq- (3.5) with pairwise distinct rapidities $\{u_i\}$ such that $u_i \neq u_j$ for all i, j with $i \neq j$ [73, 74].

The states given by Eq. (3.2) and Eq. (3.5) are highest weight states[1], i.e. they are annihilated by the total spin raising operator S^+ [65]. The Hilbert

[1]The SU(2) descendents corresponds to solutions with roots at infinity, in particular $B(\infty) \sim S^-$.

space of the $\mathrm{XXX}_{\frac{1}{2}}$ spin chain is the L-fold tensor product of spin-$\mathbf{\frac{1}{2}}$ representations of SU(2). The decomposition in terms of irreducible representation is [73]

$$\left(\mathbf{\frac{1}{2}}\right)^{\otimes L} = \bigoplus_{s=0}^{L/2} \left[\binom{L}{L/2 - s} - \binom{L}{L/2 - s - 1} \right] \boldsymbol{s} \,.$$

In particular, the number of highest weight states in a given subspace of M excitations is

$$\binom{L}{M} - \binom{L}{M-1} \,. \tag{3.12}$$

The Pauli exclusion completeness conjecture requires that there is exactly this number (3.12) of pairwise distinct solutions $\{u_i\}$ of the Bethe equations. This is also *not* the case. There are in general more solutions than necessary. These extra solutions comes from a class of solutions known as *singular solutions* [75, 76]. The singular solutions contains 2-strings $\{\frac{\mathrm{i}}{2}, -\frac{\mathrm{i}}{2}, u_3, \ldots, u_M\}$ and therefore requires a regularisation prescription due to zeros (or infinities) in wavefunctions and energy. Singular solutions are physical when [76]

$$\prod_{k=3}^{M} \left(\frac{u_k + \frac{\mathrm{i}}{2}}{u_k - \frac{\mathrm{i}}{2}} \right)^L = (-1)^L \,. \tag{3.13}$$

Therefore, the spectrum of the Heisenberg spin chain is given by the Bethe eigenstates (plus descendents) where the rapidities $\{u_i\}$ are pairwise distinct and solves the Bethe ansatz equations Eq. (3.11). If the solution is singular it should also satisfy Eq. (3.13) [76, 77]. This was for the simplest spin chain and much less is known in general [77].

For one-point functions we are particularly interested in the unpaired operators [53]. A parity operation on the spin chain states changes the order of the spins [78]

$$\Pi \, |s_1 \ldots s_L\rangle = |s_L \ldots s_1\rangle \ .$$

This Π commutes with the Hamiltonian and flips the sign of the rapidities $u \to -u$ in the Bethe states [67, 78]

$$\Pi \, |\{u_i\}\rangle = (-1)^{M(L+1)} \, |\{-u_i\}\rangle \ .$$

The parity operator Π is therefore block diagonal on the Bethe states $|\{u_i\}\rangle$. We shall distinguish between the unpaired states, that satisfy $|\{u\}\rangle = |-\{u\}\rangle$ and are parity eigenstates, and the 2×2 blocks for paired states $|\{u\}\rangle \neq |-\{u\}\rangle$ [17].

There exists a variety of methods to solve the Bethe equations. An efficient approach for small spin chains is to solve the QQ-relations

$$\begin{vmatrix} Q(u-\frac{i}{2}) & Q(u+\frac{i}{2}) \\ \tilde{Q}(u-\frac{i}{2}) & \tilde{Q}(u+\frac{i}{2}) \end{vmatrix} \propto u^L \,, \tag{3.14}$$

in Baxter polynomials $Q(u), \tilde{Q}(u)$ [79]. Eq. (3.14) is the Wronskian of the two independent polynomial solutions to Baxter's TQ relation

$$\Lambda(u)Q(u) = \left(u+\tfrac{\mathrm{i}}{2}\right)^L Q(u-\mathrm{i}) + \left(u-\tfrac{\mathrm{i}}{2}\right)^L Q(u+\mathrm{i}) \,,$$

where $\Lambda(u)$ is the eigenvalue of the transfer matrix $t(u)$ and $Q(u)$ is the Baxter polynomial. The occurrence of the second Baxter polynomial $\tilde{Q}(u)$ can be understood from solving the BAE for $M > L/2$. No new states arise as the they could have be constructed from the other pseudo-vacuum $|\bar{0}\rangle = |\downarrow \ldots \downarrow\rangle$ by acting with $M \leq L/2$ $C(u)$s [80]. By demanding both $Q(u)$ and $\tilde{Q}(u)$ to be polynomials one finds only the physical solutions, thereby circumventing the problems associated with unphysical solutions [79].

Let me note that we will mainly be interested in unpaired states in the follow. Finding unpaired states can be accomplished by restricting the form of the polynomials. For unpaired states the rapidities comes as pairs $\{u_i, -u_i\}$. Therefore, Q_1-function only contains even powers

$$Q_1(u) = \prod_{j=1}^{M/2} (u-u_j)(u+u_j) = u^M + \sum_{n=0}^{\lfloor (M-1)/2 \rfloor} q_n u^{2n} \,,$$

and as a result, one can choose $Q_2(u)$ to only contain odd powers of u. Furthermore, the requirement for cyclic invariant states Eq. (3.10) takes the form

$$\lim_{u\to 0} \frac{Q(u+\frac{\mathrm{i}}{2})}{Q(u-\frac{\mathrm{i}}{2})} = 1 \,,$$

in terms of the Baxter polynomial.

3.2 Tree level one-point Functions

To compute the one-point functions at tree-level, we need to evaluate the overlap of a Matrix Product State and a Bethe state. In this section, I shall review some general considerations of these computations. In the interest of clarity,

the discussion will be written with the SU(2) sector in mind, though most of the considerations carry over to higher ranks without much modification [56].

The one-point function at leading order is obtained by inserting classical solutions for the scalars into the expression of the operator Eq. (3.6)

$$C^{(0)} = \Big(\frac{4\pi^2}{\lambda}\Big)^{\frac{L}{2}} \frac{\mathcal{Z}}{L^{\frac{1}{2}}} \sum_{\{s_i\}} \mathrm{tr}(T_{s_1} \cdots T_{s_L}) \langle s_1 \cdots s_L | \{u_i\} \rangle\!\rangle$$

where $\phi_i^{\mathrm{cl}}(x) = x_\perp^{-1} T_i$ is one of the solutions to the classical equation of motion discussed in chapter 2. By introducing the spin chain state

$$\langle \mathrm{MPS}| = \mathrm{tr}_a \prod_{n=1}^{L} \Big(\sum_s T_s \otimes \langle s| \Big) , \tag{3.15}$$

we may write the one-point structure constant as

$$C^{(0)} = \Big(\frac{4\pi^2}{\lambda}\Big)^{\frac{L}{2}} \frac{\mathcal{Z}}{L^{\frac{1}{2}}} \frac{\langle \mathrm{MPS}|\{u_i\}\rangle}{\langle \{u_i\}|\{u_i\}\rangle^{\frac{1}{2}}} .$$

The computation of tree-level one-point functions is therefore essentially evaluating the overlap between a matrix product state Eq. (3.16) defined by a classical solution of $\mathcal{N} = 4$ SYM and a Bethe eigenstate describing the operator in question [53].

3.2.1 Matrix product states

The spin chain state that describes the fuzzy funnel solutions in $\mathcal{N} = 4$ Eq. (3.16) is a matrix product state. A Matrix Product State (MPS) is defined as

$$|\mathrm{MPS}\rangle = \sum_{s_1} \cdots \sum_{s_L} \mathrm{tr}\big(t_{s_1} \dots t_{s_L}\big) \, |s_1 \dots s_L\rangle \ , \tag{3.16}$$

Where the states are denoted by s_i in the i'th position and $\{t_s\}$ is a list of matrices, one for each possible state s. Graphically we shall represent a matrix product state as shown in Fig. 3.1. Matrix product states are widely

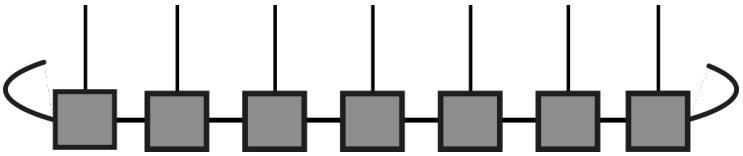

Figure 3.1 – A matrix product state

studied for one-dimensional systems and tensor network computations [81, 82]. They provide excellent approximations to ground and low-lying excited states of gapped Hamiltonians and are therefore interesting in quantum quench problems [83].

In the following, we shall consider the matrix product states corresponding to the D3⊥D5$_k$ and D3⊥D7$_{k_1,k_2}$ fuzzy funnels in $\mathcal{N}=4$ SYM and restrict to the SU(2) sector. The matrix product state representing the D5 defect takes the form

$$\langle \mathrm{D5}_k| = \mathrm{tr}_a \prod_{x=1}^{L} \left(t_3^{(k)} \otimes \langle\uparrow|_x + t_1^{(k)} \otimes \langle\downarrow|_x \right), \tag{3.17}$$

where the sum over $\{s_i\}$ denotes the summation over all possible configurations of the spins $s_i \in \{\uparrow,\downarrow\}$ and the matrices $t_i^{(k)}$ realise a k-dimensional representation of the $\mathfrak{su}(2)$ algebra

$$[t_i^{(k)}, t_j^{(k)}] = \mathrm{i}\epsilon_{ijk} t_k^{(k)} .$$

The matrix product state representing the D7 defect takes the form

$$\langle \mathrm{D7}_{k_1,k_2}(\alpha)| = \mathrm{tr}_a \prod_{n=1}^{L} \left(T_3^{(k_1,k_2)}(\alpha) \otimes \langle\uparrow|_x + T_1^{(k_1,k_2)}(\alpha) \otimes \langle\downarrow|_x \right), \tag{3.18}$$

where the matrices are[2]

$$T_i^{(k_1,k_2)}(\alpha) := t_i^{(k_1)} \otimes \mathbb{1}_{k_2} + \alpha \mathbb{1}_{k_1} \otimes t_i^{(k_2)} .$$

The matrices $T_i^{(k_1,k_2)}(\alpha)$ satisfy an α-dependent $\mathfrak{su}(2)$ algebra

$$[T_i^{(k_1,k_2)}(\alpha), T_i^{(k_1,k_2)}(\beta)] = \mathrm{i}\epsilon_{ijk} T_i^{(k_1,k_2)}(\alpha\beta) .$$

The matrix product states of D3⊥D5$_k$ and D3⊥D7$_n$ are annihilated by the third charge of the spin chain[3] [53, 59]. In fact, they are annihilated by any odd charge [56]. As a consequence only unpaired states have non-zero overlap with the matrix product states.

That the matrix product states are annihilated by the odd charges is a sign of integrability, as we shall discuss in chapter 4. In the following we will just

[2]Note the slight generalisation to arbitrary α. D3⊥D7$_{k_1,k_2}$ corresponds to $\alpha = \mathrm{i}$.

[3]Note that for the SO(5)-symmetric D3⊥D7 configuration one have to consider the full SO(6) sector of scalars to have non-vanishing one-point functions [59]

focus on Q_3. The third charge Q_3 acts on three neighbouring sites and can be expressed in terms of the Hamiltonian density [67]

$$Q_3 = \sum_{j=1}^{L} [\mathbb{H}_{j-1,j}, \mathbb{H}_{j,j+1}] \,.$$

For the SU(2) spin chain with L the Q_3-density is

$$Q^{(3)}_{j-1,j,j+1} = [P_{j-1,j}, P_{j,j+1}] \,.$$

Therefore, for a MPS Eq. (3.16) to be annihilated by Q_3 the matrices must satisfy the conditions

$$\sum_{j=1}^{L} \mathrm{tr}\Big(t_{s_1} \dots \Big(t_{s_j} t_{s_{j+1}} t_{s_{j-1}} - t_{s_{j+1}} t_{s_{j-1}} t_{s_j} \Big) \dots t_{s_L} \Big) = 0 \tag{3.19}$$

for any configuration of the spins $\{s_i\}$. This equation is neatly written in graphics notation (see Fig. 3.2).

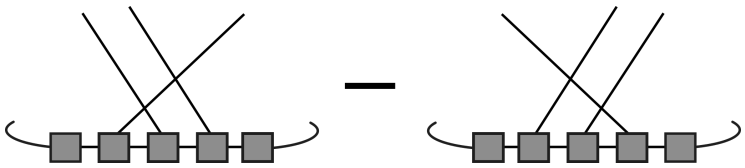

Figure 3.2 – The action of Q_3 on the Matrix Product State.

The MPS for D3⊥D5$_k$ satisfy the conditions Eq. (3.19) as a result of the $\mathfrak{su}(2)$ algebra [53]. Starting at $L = 12$, the MPS for D3⊥D7$_{k_1,k_2}$ do *not* satisfy Eq. (3.19) and is therefore *not* annihilated by Q_3. This have drastic consequences for the selection rules and overlap formulae as we shall explore in section 3.4.

The matrix product states described is in terms are related to the field theory boundary state. Lorentz invariance provides a useful tool to study boundary theories in terms of the bulk theory. In a Lorentz-invariant field theory we can rotate a boundary to a state [84]. A Hamiltonian description of a quantum field theory is essentially chopping up the partition function in a foliation along some chosen time direction. The Hamiltonian describes the evolution along this time direction. When we have a boundary, we can choose to foliate parallel with the boundary or orthogonal to the boundary. This gives two different descriptions of the same system. The one-point functions may therefore be written as

$$\langle \Omega | \, \mathcal{O}(x) \, | \Omega \rangle \ ,$$

where $|\Omega\rangle$ denotes the vacuum in the theory with a boundary or as

$$\langle B|\,\mathcal{O}(x)\,|0\rangle\ ,$$

where $|B\rangle$ is the boundary state i.e. the state in the bulk Hilbert space that describes the boundary in the rotated channel [84, 85]. Therefore, the spin chain matrix product states are the tree-level building blocks of the boundary state $\langle B|$ describing the D-brane in terms of $\mathcal{N}=4$ SYM [53].

3.2.2 Selection Rules

A set if selection rules for the overlaps follows from the properties of the matrix product states defined in Eq. (3.17) and Eq. (3.18). The selection rules restricts the set of Bethe states for which we need to explicitly compute the overlap.

The matrix product states are one-site, or translational invariant. Let $U=\mathrm{e}^{iP}$ be the translation (or shift) operator along the spin chain, then the condition can be written as $U\,|\mathrm{MPS}\rangle=|\mathrm{MPS}\rangle$. In other words, $P=0$ where P is the total lattice momentum. It follows that $\langle\mathrm{MPS}|\{u_i\}\rangle=0$ unless the Bethe state satisfy $\sum_i p_i=0$. In terms of the rapidities this is written in Eq. (3.10). Note that we are simply repeating ourselves as the map between $\mathcal{N}=4$ SYM operators and spins already was phrased as an overlap with a translational invariant matrix product state Eq. (3.6).

Consider the matrix product state Eq. (3.17) for D3$\perp$D5$_k$. The coefficients of a state $\langle s_1\dots s_L|$ is

$$\mathrm{tr}(t_{s_1}\cdots t_{s_L})$$

where t_i are k-dimensional representations of $\mathfrak{su}(2)$. The $\mathfrak{su}(2)$ algebra have the following automorphisms U,V and W [53]

$$\begin{aligned} Ut_1U^{-1}=t_1\,,\ Ut_{2,3}U^{-1}=-t_{2,3}\,,\qquad Vt_3V^{-1}=t_3\,,\ Vt_{1,2}V^{-1}=-t_{1,2}\,,\\ Wt_2W^{-1}=t_2\,,\ Wt_{1,3}W^{-1}=-t_{1,3}\,. \end{aligned} \tag{3.20}$$

As the trace factor is cyclic these similarity transformations provides restrictions on the trace factors, e.g. for U we pick up a minus for each t_3 therefore [53]

$$\mathrm{tr}(t_3^At_1t_3^Bt_1\cdots)=(-1)^{L-M}\,\mathrm{tr}(t_3^At_1t_3^Bt_1\cdots)\,.$$

It follows that $L-M$ must be even if the traces are not to vanish. Similarly for V it follows that M must be even. In total, the overlaps $\langle\mathrm{MPS}|\{u_i\}\rangle=0$ unless both L and M are even [53]. Now consider the D3$\perp$D7$_{k_1,k_2}$ MPS

Eq. (3.18). The similarity transformations Eq. (3.20) naturally lifts to the algebra of $T_i^{(k_1,k_2)}(\alpha)$. E.g. for U and $T_i^{(k_1,k_2)}(\alpha)$ we have

$$U_1 \otimes U_2 T_1^{(k_1,k_2)}(\alpha) U_1^{-1} \otimes U_2^{-1} = T_1^{(k_1,k_2)}(\alpha)\,,$$

and similarly for the rest. This implies that L and M likewise must be even for the D3⊥D7$_{k_1,k_2}$ overlaps to not vanish.

One can derive further identities between the trace factors for D3⊥D7$_{k_1,k_2}$ using the similarity transformations Eq. (3.20). A useful observation is that

$$\begin{aligned}\mathrm{tr}\Big(T_3^{(k_1,k_2)}(\alpha)^A T_1^{(k_1,k_2)}(\alpha) T_3^{(k_1,k_2)}(\alpha)^B \cdots\Big)\\ = \mathrm{tr}\Big(T_3^{(k_1,k_2)}(-\alpha)^A T_1^{(k_1,k_2)}(-\alpha) T_3^{(k_1,k_2)}(-\alpha)^B \cdots\Big)\,,\end{aligned} \quad (3.21)$$

which follows from inserting $(\mathbb{1}\otimes W)(\mathbb{1}\otimes W)^{-1}$ into the trace factor. This leads to a further selection rule for $\alpha = \pm\mathrm{i}$ and $k_1 = k_2 = k$. There exists an invertible matrix S that interchanges the spaces in the Kronecker product, e.g.

$$S\mathbb{1}\otimes t_i S^{-1} = t_i \otimes \mathbb{1}\,.$$

By inserting SS^{-1} in the trace factor we have that

$$\mathrm{tr}\Big(T_{s_1}(\mathrm{i})\cdots T_{s_L}(\mathrm{i})\Big) = \mathrm{i}^L\,\mathrm{tr}\Big(T_{s_1}(-\mathrm{i})\cdots T_{s_L}(-\mathrm{i})\Big)\,.$$

Since the traces in the SU(2) MPS is invariant under $\alpha \to -\alpha$ according to Eq. (3.21), it follows that for $\alpha = \pm\mathrm{i}$ and $k_1 = k_2$ the length L must satisfy $\mathrm{i}^L = 1$, i.e. $L/4 \in \mathbb{N}$, for the traces, and hence the overlaps, to not vanish.

The D3⊥D5$_k$ matrix product state Eq. (3.17) is annihilated by all the odd charges of the spin chain. Therefore, if the overlaps with the on-shell Bethe states are not to vanish the Bethe states must be unpaired [53]. This is not the case for the D3⊥D7$_{k_1,k_2}$.

3.2.3 Organising the overlap computations

Having introduced the Bethe states Eq. (3.2) and the matrix product states Eq. (3.16) we now proceed to evaluate the overlaps. Since the Bethe states are explicitly known, we in principle can compute any the overlap with any state on a case by case basis. However, we are after an integrability-based closed formula for the overlaps expressed in terms of the rapidities, the length of the spin chain L and the number of excitations M.

The overlap between a Bethe states Eq. (3.2) and a matrix product state Eq. (3.16) takes the form

$$\langle \mathrm{MPS}|\{u_i\}\rangle = \mathcal{N} \sum_{n_1<\cdots<n_M} \Big(\sum_{\sigma\in S_M} A_\sigma \prod_{j=1}^{M} x_{\sigma_j}^{n_j} \Big) \langle \mathrm{MPS}|\{n_i\}\rangle \,, \tag{3.22}$$

where $\{n_i\}$ is the positions of the magnons, σ is a permutation of M elements, A_σ is the amplitude given its decomposition into two-body scatterings according to Eq. (3.3), and x_i are the fugacities $x_k = \mathrm{e}^{\mathrm{i}p_k}$. Here $\mathcal{N}$ is a normalisation. In reporting the overlaps we always choose the phase to be positive and real.

We shall denote the wavefunction of the MPS as the *trace factor*. It is given as the trace of distributing the excitations t_1 according to $\{n_i\}$ on the vacuum of t_3's:

$$\langle \mathrm{MPS}|\{n_i\}\rangle = \mathrm{tr}[\cdots \underset{n_1}{t_1} \cdots \underset{n_M}{t_1} \cdots] =: T(\{n_i\}) \,, \tag{3.23}$$

The trace factor is given as

$$T(\{n_i\}) = \mathrm{Tr}\Big(t_3^{L-\delta_{M,1}-1} t_1 \prod_{i=1}^{M-1} t_3^{\delta_{i+1,i}} t_1 \Big) \,, \tag{3.24}$$

where we have introduced the distance between the excitations $\delta_{ij} = n_i - n_j$.

We can think of the overlap computation as integrating the trace factor against a "Bethe measure"

$$\int \mathrm{d}\mu_M \, T(\{n_i\})$$

where $\int \mathrm{d}\mu_M$ denotes the summation over positions weighted by an appropriate Bethe wavefunction in accordance with Eq. (3.22).

The functions that integrates to zero against the Bethe measure can be given by the selection rules for the overlap. Since the Bethe states are highest weight states of weight $\frac{M}{2}$ all wavefunctions of states with different excitation number will result in zero as will their descendents. In particular the vacuum states $\mathcal{A}^{(L)}(S^-)^M |\Omega\rangle$ corresponds to a constant wave function, i.e. independent of $\{n_i\}$. Therefore we can drop any constant term in the trace factors when computing the overlap.

The number of terms in the expression Eq. (3.22) is $\frac{L!}{(L-M)!} \sim L^M + \mathrm{O}(L^{M-1})$ which quickly becomes unwieldy as we increase the size of the spin chain L and the number of excitations M. To organise the computations it is

useful to interchange the sums in the overlap and write

$$\langle \mathrm{MPS}|\{u_i\}\rangle = \mathcal{N} \sum_{\sigma\in S_M} A_\sigma \Sigma_\sigma \,,$$

where we have introduced the sums

$$\Sigma_\sigma := \sum_{n_1<\cdots<n_M} T(\{n_i\}) \prod_{j=1}^{M} x_{\sigma_j}^{n_j} \,.$$

When the trace factors Eq. (3.24) have terms that are exponentials in n_i, the sums Σ_σ become linear combinations of nested geometric sums. The nested geometric sums can be evaluated in general [53]

$$\begin{aligned}\sum_{1\le n_1<\ldots<n_M\le L} q_1^{n_1}\cdots q_M^{n_M} &= \prod_{i=1}^{M} q_i^{L+1} \\ &+ \sum_{a=1}^{M}\left(1-\prod_{i=1}^{a} q_i^{L+1}\right)\left(\prod_{j=1}^{a}\frac{q_j^j}{1-\prod_{i=j}^{a} q_i}\right)\left(\prod_{j=a+1}^{M}\frac{q_j^{L+1}}{\prod_{i=a+1}^{j} q_i - 1}\right). \quad (3.25)\end{aligned}$$

In evaluating the overlaps in `Mathematica` it is useful to have the nested geometric sums pre-computed as in Eq. (3.25) and introduce a replacement rule that transforms terms $\prod_i q_i^{n_i}$ into their summed values.

We are interested in on-shell Bethe states and should therefore restrict the rapidities to satisfy the Bethe equations. For unpaired states, the Bethe equations can be written as

$$x_k^L = x_k \prod_{j=1,j\neq k}^{M/2} \frac{(u_k+i)^2-u_j^2}{(u_k-i)^2-u_j^2} \,.$$

This means that in the overlaps we swap any x_k^L with $x_k\cdots$, thereby decreasing the powers of x_k in the expression.

For unpaired states the rapidities are paired as $u_{i+\frac{M}{2}} = -u_i$. This means that the fugacities are inverses

$$x_{i+\frac{M}{2}} = x_i^{-1} \,.$$

Therefore, the nested geometric sums Eq. (3.25) have spurious poles for unpaired states. One should take care to expand[4] the expressions in ϵ for

[4]For implementations in `Mathematica` it is crucial to `SeriesExpand` as `Limit` too often do not find the limit.

$x_{i+\frac{M}{2}} = x_i^{-1} + \epsilon$ and consider the limit as $\epsilon \to 0$. An important observation is that terms in the geometric sums that have spurious poles are of a form

$$\sim \frac{(xy)^L - 1}{xy - 1} \to L$$

for $y \to x^{-1}$. Hence, pairing the rapidities produce terms that are higher order in L. Thus, the highest power of L produced is $L^{\frac{M}{2}}$ which corresponds to having all the paired rapidities sitting next to each other. The permutations σ that produce the highest power in L are the permutation for which the paired rapidities stay together. For M magnons, the number of permutations that contribute to the leading L term is

$$\left(\tfrac{M}{2}\right)! \, 2^{\frac{M}{2}} \, .$$

We shall apply this trick to compute the leading L overlaps in D3$\perp$D7$_{k_1,k_2}$. However, note that this trick could be very useful for integrable matrix product states where the overlap is expected to take the form

$$\sim \Big[\sum_a \prod_j f_a(u_j)\Big] \det G^+ \, ,$$

and $\det G^+$ carries all the L-dependence [86].

3.3 D3$\perp$D5$_k$

In this section, we shall review what is known about overlaps relevant for the D3$\perp$D5$_k$ one-point functions. We shall focus on the SU(2) sector and the properties that are relevant for a comparison to the overlaps for D3$\perp$D7$_{k_1,k_2}$ which we will discuss in section 3.4.

As described in section 3.2.2, the overlap is only non-zero for on-shell Bethe states for unpaired states with an even number of magnons M and for even length spin chains L. The overlaps between such on-shell Bethe states and the matrix product state Eq. (3.17) for $k = 2$ can be written as a determinant [53]

$$\langle \mathrm{D5}_2 | \{u_i\} \rangle = \frac{1}{2^{L-1}} \prod_{i=1}^{\frac{M}{2}} \frac{u_i^2 + \frac{1}{4}}{u_i} \sqrt{u_i + \tfrac{1}{4}} \det G^+ \, . \tag{3.26}$$

This expression was proven in [87] I shall discuss the idea behind this proof in chapter 4. The matrix G^+ is given by

$$G^+_{ij} = \partial_{u_i} \Phi_j + \partial_{u_{i+M/2}} \Phi_j \, ,$$

where Φ_j are the functions Eq. (3.9). This matrix originates from the block structure of the Gaudin matrix Eq. (3.8) for unpaired states where we have arranged the rapidities as $u_{i+M/2} = -u_i$. For unpaired states the norm formula Eq. (3.7) factorises into

$$\det G = \det G^+ \det G^-$$

where the $\frac{M}{2} \times \frac{M}{2}$ matrices $G^\pm$ are given by [53, 88]

$$G^\pm_{ij} = \Big(\frac{L}{u_i^2 + \frac{1}{4}} - \sum_{k=1}^{\frac{M}{2}} K^+_{ik}\Big)\delta_{ij} + K^\pm_{ij}\,,$$

with

$$K^\pm_{ij} = \frac{2}{1+(u_i-u_j)^2} \pm \frac{2}{1+(u_i+u_j)^2}\,.$$

The overlap for higher dimensional representations k are given in terms of $k = 2$ by the action of a transfer matrix [54]

$$\langle \mathrm{D5}_k|\{u_i\}\rangle = 2^{L-1}\frac{Q(0)}{Q(\frac{\mathrm{i}k}{2})}T_{k-1}(0)\langle \mathrm{D5}_2|\{u_i\}\rangle$$

where $Q(u)$ is the Baxter polynomial and $T_{k-1}(0)$ is the transfer matrix in the k-dimensional irreducible representation [56] (see also Appendix B of [42])

$$T_{k-1}(u) = \sum_{n=\frac{1-k}{2}}^{\frac{k-1}{2}} (u+\mathrm{i}n)^L \frac{Q^{[k]}Q^{[-k]}}{Q^{[2n-1]}Q^{[2n+1]}}$$

where $f^{[n]} := f(u + n\frac{\mathrm{i}}{2})$.

3.3.1 Large k overlaps

For comparison with string theory we are interested in a double scaling limit in which we take $k \to \infty$ [52]. The large k expansion of the D3⊥D5$_k$ one-point functions was studied in [53] where they found that the M magnon overlaps scale as $\sim k^{L-M+1}$. Here I shall present an algebraic approach to compute the necessary trace factors for large representations. While this method seems more laborious, it is well-defined and easily generalises to D3⊥D7$_{k_1,k_2}$.

Consider the trace factor for the vacuum is $\mathrm{tr}(t_3^{(k)})^L$. Where t_3 is the diagonal generator of SU(2) in the k-dimensional representation. The trace is readily evaluated

$$\mathrm{tr}(t_3^{(k)})^L = -2\frac{B_{L+1}(\frac{1-k}{2})}{L+1}\,. \tag{3.27}$$

where $B_n(x)$ is the n'th Bernoulli polynomial given by

$$B_n(x) = \sum_{p=0}^{n} \frac{1}{p+1} \sum_{k=0}^{p} (-1)^k \binom{n}{k} (x+k)^n \,.$$

The trace Eq. (3.27) is a polynomial in k of order $L+1$. We find the first few term

$$\mathrm{tr}(t_3^{(k)})^L = \frac{(-1)^L}{2^L(L+1)} \left(k^{L+1} - \frac{1}{6} L^{(2)} k^{L-1} + \frac{7}{360} (L-2)^{(4)} k^{L-3} + \mathcal{O}(k^{L-5}) \right), \tag{3.28}$$

where $L^{(n)}$ denotes Pochhammer symbols, or factorial powers, e.g. $L^{(n)} = L(L+1)\cdots(L+n-1)$.

Consider the trace factor with two excitations, i.e. $\mathrm{tr}(t_1 t_3^A t_1 t_3^{L-A-2})$ where $A = n_2 - n_1 - 1$ and n_1, n_2 are the positions of the excitations. This trace factor can be rewritten using the $\mathfrak{su}(2)$ algebra

$$\begin{aligned} \mathrm{tr}\Big[t_1 t_3^A t_1 t_3^{L-A-2}\Big] = \frac{1}{2} \sum_{p=0}^{\lfloor A/2 \rfloor} \binom{A}{2p} \mathrm{tr}\Big[(\tfrac{k^2-1}{4} - t_3^2) t_3^{L-2p-2}\Big] \\ - \frac{1}{2} \sum_{p=0}^{\lfloor (A-1)/2 \rfloor} \binom{A}{2p+1} \mathrm{tr}\Big[t_3^{L-2p-2}\Big] \end{aligned}$$

leaving only traces of t_3 to some power.

Since each trace $\mathrm{tr}\, t_3^m$ starts to contribute at order k^{m+1} (see Eq. (3.28)) we can pick out the contributions to the trace factor at each order in k. At $\mathrm{O}(k^{L+1})$ only $p = 0$ from the first term contributes

$$\frac{k^2}{8} \mathrm{tr}\, t_3^{L-2} - \frac{1}{2} \mathrm{tr}\, t_3^L \,.$$

Note that this will also produce $\mathrm{O}(k^L)$ terms when expanding the traces. However, since this contribution to the trace factor is constant it will not contribute to the overlap (cf. the discussion in section 3.2.3).

The next terms from the expansion gives $\mathrm{O}(k^{L-1})$ pieces. Note that there will be constant pieces which we drop. The non-constant terms in the trace factor contributing at order k^{L-1} are

$$-\frac{1}{2} A(A-1) \mathrm{tr}\, t_3^{L-2} + \frac{k^2}{16} A(A-1) \mathrm{tr}\, t_3^{L-4} - \frac{1}{2} A \mathrm{tr}\, t_3^{L-2} \,.$$

We obtain the trace factor at k^{L-1} by inserting the leading terms of the large k expansion of the traces. Computing the overlap with this trace factor $T(\{n_i\})$

as outlined in section 3.2.3, we find

$$\langle \mathrm{D5}_k|u,-u\rangle = \frac{Lu\sqrt{u^2+\frac{1}{4}}}{2^{L-2}(L-3)}k^{L-1}+\mathrm{O}(k^{L-2})$$

in agreement with [53].

3.4 D3⊥D7$_{k_1,k_2}$

In this section, I will discuss the results on overlap formulae relevant for the tree-level one-point functions of SU(2)-sector operators in the D3⊥D7$_{k_1,k_2}$ defect theory. The matrix product state is given in Eq. (3.18). Various selection rules were discussed in section 3.2.2 in particular the restriction is that L and M are even. For $k_1 = k_2$ and $\alpha = \pm\mathrm{i}$ the restriction on the length is enhanced to $L/2$ being even.

3.4.1 Closed expressions for small M

For low excitation numbers ($M = 0, 2, 4$) we may readily evaluate the overlap Eq. (3.22) and obtain closed-form expressions for the overlaps in terms of rapidities $\{u_i\}$ and length of the chain L. This provides an explorative study of the overlaps and will lead us to a conjecture for the leading L term, which we will discuss in section 3.4.2.

The overlap formula for the vacuum $M = 0$ at any α is easily expressed as Bernoulli polynomials, originating from traces of $t_3^{(k_i)}$, by the binomial theorem since $[t_3^{(k_1)}\otimes \mathbb{1}_{k_2}, \mathbb{1}_{k_1}\otimes t_3^{(k_2)}] = 0$. The expression is

$$\langle \mathrm{D7}_{k_1,k_2}(\alpha)|0\rangle = \sum_{q=0}^{L/2}\frac{\alpha^{2q}}{(L-2q+1)(2q+1)}\binom{L}{2q}B_{L-2q+1}(\tfrac{1-k_1}{2})B_{2q+1}(\tfrac{1-k_2}{2}) \tag{3.29}$$

For two magnons, $M = 2$, the trace factor for $\alpha = \mathrm{i}$ and $k_1 = k_2 = 2$ is

$$\mathrm{tr}[T_1T_3^{\delta n-1}T_1T_3^{L-\delta n-1}] = -\frac{1}{2^{\frac{L}{2}}}\mathrm{i}^{\delta n}(1+(-1)^{\delta n})\mathrm{i}^{L/2}(1+\mathrm{i}^L)\,,$$

where $\delta n = n_2 - n_1$. We see that the traces are zero unless $\delta n = 2m$ and $L = 4\ell$ for $m, \ell \in \mathbb{N}$. Computing the overlap as outlined in section 3.2.3, we find

$$\langle \mathrm{D7}_{2,2}(\mathrm{i})|u,-u\rangle = \frac{L}{2^{\frac{L}{2}-2}}\frac{u\sqrt{u^2+\frac{1}{4}}}{\left|u^2-\frac{1}{4}\right|}\,. \tag{3.30}$$

In fact, we may evaluate the overlap for any α and find

$$\langle \mathrm{D7}_{2,2}(\alpha)|u,-u\rangle = \frac{L}{2^{L-2}}\frac{u\sqrt{u^2+\frac{1}{4}}}{8\alpha^2(u^2+\frac{\alpha^2}{4})(u^2+\frac{1}{4\alpha^2})}F(\alpha)\,,$$

where

$$F(\alpha) = (1+4\alpha^2(u^2+\tfrac{1}{4})+\alpha^4)\Big((\alpha-1)^L+(\alpha+1)^L\Big) \\ +(\alpha+\alpha^3)\Big((\alpha-1)^L-(\alpha+1)^L\Big)\,.$$

A few checks of this expression are available from various values of α. For $\alpha=0$ this is proportional to $\langle \mathrm{D5}_2|u,-u\rangle$, for $\alpha=\mathrm{i}$ it gives back Eq. (3.30), and for $\alpha=1$ the matrices are the tensor product representation of two 2-dimensional representations and thus decomposes as $\mathbf{3}\oplus\mathbf{1}$. Accordingly the overlap reduces to the $\langle \mathrm{D5}_3|u,-u\rangle$ overlap.

We have computed $M=2$ overlaps for a selection of small $(k_1,k_2) = (2,3),(3,2),(3,3),(2,4),(4,4)$ and was unable to find a recursion formula that would relate the various overlaps at different (k_1,k_2). The recursion formula for D3$\perp$D5$_k$ was expressed in terms of typical ingredients from integrability, transfer matrix eigenvalues and Baxter polynomials, and can possible be understood from the integrability of the D5$_k$ matrix product state [54,89]. The fact that Q_3 do not annihilate $|\mathrm{D7}_{k_1,k_2}\rangle$ means that is this state is not related to an integrable matrix product boundary state. This may explain the lack of success in finding a recursion relation.

As we increase the number of magnons, the expressions become more cumbersome. At four magnons, that is $M=4$, the trace factor for $\alpha=\mathrm{i}$ and $k_1=k_2=2$ is

$$\mathrm{Tr}\Big[T_1T_3^{\delta_{21}-1}T_1T_3^{\delta_{32}-1}T_1T_3^{\delta_{43}-1}T_1T_3^{L-\delta_{41}-1}\Big] \\ = \frac{(-1)^{\frac{L}{4}}\mathrm{i}^{n_1+n_2+n_3+n_4}}{2^{\frac{L}{2}}}\Big(1+(-1)^{n_1+n_2+n_3+n_4}\Big) \\ \times\Big((-1)^{n_2+n_3}+(-1)^{n_1+n_2}+(-1)^{n_1+n_3}-1\Big)\,,$$

where $\delta_{ij}=n_i-n_j$ and n_i for $i=1,2,3,4$ are the positions of the excitations. Computing the overlap produces

$$\langle \mathrm{D7}_{2,2}(\mathrm{i})|u_1,u_2,-u_1,-u_2\rangle = \frac{1}{2^{\frac{L}{2}}}\frac{u_1u_2\big((u_1^2+\frac{1}{4})(u_2^2+\frac{1}{4})\big)^{\frac{3}{2}}}{|u_1^2-\frac{1}{4}||u_2^2-\frac{1}{4}|}\det G^+ + \mathrm{O}(L)\,. \tag{3.31}$$

Now, the alert reader may have noticed that we have been computing the overlaps for unpaired states even though there is nothing restricting us to unpaired states for D3⊥D7$_{k_1,k_2}$ as the MPS is not annihilated by Q_3. We are studying the course of events from our current vantage point. The fact that there is an order L piece in Eq. (3.31) not captured by the Gaudin-like determinant $\det G^+$ was the first clue that the MPS in this setting behaves qualitatively different from the D3⊥D5$_k$ and D3⊥D7$_n$.

3.4.2 A conjecture for large L overlaps

Since the D7$_{k_1,k_2}$ matrix product state is not integrable, we shall aim a little lower than to find the full overlaps. As we discussed in section 3.2.3 the leading L terms in the overlaps comes from having paired rapidities since each pair produces a factor of L. Having computed the overlaps for unpaired states for $M = 0, 2, 4$, it is tempting to conjecture that the pattern for the leading L contribution continues to hold

$$\langle \mathrm{D7}_{2,2}(\mathrm{i})|\{u_i\}\rangle = \frac{L^{M/2}}{2^{\frac{L}{2}-2}} \prod_{i=1}^{M/2} \frac{u_i \sqrt{u_i^2+\frac{1}{4}}}{\left|u_i^2-\frac{1}{4}\right|} + O(L^{M/2-1})\,. \tag{3.32}$$

This conjectured expression for the overlaps at leading L has further been tested for $M = 6$. The trace factor for $k_1 = k_2 = 2$ and $\alpha = \mathrm{i}$ can be written as

$$\begin{aligned}\mathrm{Tr}\Big[T_1 T_3^{\delta_{21}-1} T_1 T_3^{\delta_{32}-1} T_1 T_3^{\delta_{43}-1} T_1 T_3^{\delta_{54}-1} T_1 T_3^{\delta_{65}-1} T_1 T_3^{L-\delta_{61}-1}\Big] \\ = \frac{(-1)^{\frac{L}{4}}\mathrm{i}^{\sum_i n_i}}{2^{\frac{L}{2}+1}}\Bigg(\sum_i (-1)^{n_i} - \sum_{i<j<k}(-1)^{n_i+n_j+n_k} \\ + \sum_{i<j<k<l<m}(-1)^{n_i+n_j+n_k+n_l+n_m}\Bigg)\,.\end{aligned}$$

In computing the overlap, we make use of all the tricks mentioned in section 3.2.3. The leading L piece of the overlap thus produced exactly agrees with Eq. (3.32) giving further support to the conjecture.

3.4.3 Large k_1, k_2 overlaps

To compare with supergravity computations for D3⊥D7$_{k_1,k_2}$, one would need the the limit of large fluxes k_1 and k_2 on the spheres [38]. In section 3.3.1, we demonstrated an algebraic approach to systematically compute the traces

for large representations in D3⊥D5$_k$. In this section, we will extend this to compute the leading k_1, k_2 terms in the D3⊥D7$_{k_1,k_2}$ overlaps for $M = 0, 2$.

The matrices $T_i(\alpha)$ satisfy the algebra

$$[T_i(\alpha), T_j(\beta)] = \mathrm{i}\epsilon_{ijk} T_k(\alpha\beta)\,.$$

In the following, we shall adhere to the notation $T_i := T_i(\alpha)$. The commutator of powers of $T_3(\alpha)$ with the other generators is then given as

$$[T_3(\alpha)^A, T_1(\beta)] = \mathrm{i}\sum_{p=0}^{\lfloor\frac{A-1}{2}\rfloor}\binom{A}{2p+1} T_2(\alpha^{2p+1}\beta) T_3(\alpha)^{A-2p-1} + \sum_{p=1}^{\lfloor\frac{A}{2}\rfloor}\binom{A}{2p} T_1(\alpha^{2p}\beta) T_3(\alpha)^{A-2p}\,,$$

with a similar expression for $[T_3(\alpha)^A, T_2(\beta)]$.

The trace factor relevant for $M = 2$ overlaps can be reordered by commuting the T_3s to the left and using various relations amongst the trace factors. The final result can be written in terms of traces of T_3s

$$\begin{aligned}&\mathrm{Tr}\,[T_1 T_3^A T_1 T_3^{L-2-A}] = \\ &\frac{1}{2}\sum_{p=0}^{\lfloor\frac{A}{2}\rfloor}\binom{A}{2p}\mathrm{Tr}\left[\left(\left(\tfrac{k_1^2-1}{4} + \alpha^{2p+2}\tfrac{k_2^2-1}{4}\right) - T_3(\mathrm{i}\alpha^{1+p}) T_3(-\mathrm{i}\alpha^{1+p})\right) T_3^{L-2p-2}\right] \\ &\qquad - \frac{1}{2}\sum_{p=0}^{\lfloor\frac{A-1}{2}\rfloor}\binom{A}{2p+1}\mathrm{Tr}\,[T_3(\alpha^{2p+3}) T_3^{L-2p-3}]\,.\end{aligned}$$

In this expression, we can take the large k_1, k_2 limits in a systematic way since the traces that occur are traces of products of T_3's and these are known sums of products of Bernoulli polynomials (see $M = 0$ overlap Eq. (3.29)). The different α's might change the weight slightly but the overall scaling with k_1 and k_2 can easily be estimated.

For the $M = 0$ overlap we expressed the trace in terms of Bernoulli polynomials Eq. (3.29). Here we shall write the trace of powers of $T_3(\alpha)$ in terms of the smaller traces. Expanding $\mathrm{tr}(t_3^{(k_i)})^m$ to leading order in k_i

according to Eq. (3.28) we obtain an expansion with the first few coefficients

$$\begin{aligned}\mathrm{Tr}\,[T_3(\alpha)^L] = &\frac{1}{2^L(L+1)(L+2)}[k_1,k_2]_\alpha^{L+2}\\ &-\frac{1+\alpha^2}{2^L 6}[k_1,k_2]_\alpha^{L}\\ &+\frac{(7+10\alpha^2+7\alpha^4)L(L-1)}{2^L 360}[k_1,k_2]_\alpha^{L-2}+\mathrm{O}(k^{L-4})\,,\end{aligned} \tag{3.33}$$

where we have introduced the notation

$$[k_1,k_2]_\alpha^L := \frac{(k_1+\alpha k_2)^L-(k_1-\alpha k_2)^L}{2\alpha}\,. \tag{3.34}$$

This reduces to the imaginary part of $(k_1+\mathrm{i}k_2)^L$ when $\alpha=\mathrm{i}$

$$\begin{aligned}\frac{(k_1+\mathrm{i}k_2)^{L+2}-(k_1-\mathrm{i}k_2)^{L+2}}{2\mathrm{i}} &= \mathrm{Im}(k_1+\mathrm{i}k_2)^{L+2}\\ &= (k_1^2+k_2^2)^{L/2+1}\sin((m+2)\arctan(\tfrac{k_1}{k_2}))\,.\end{aligned}$$

Note that this generalises to D7$_{k_1,k_2}$, the observation made in [53] that overlaps for $M=0$ only contain odd powers of k. Here it is the combination Eq. (3.34) that only occurs with odd powers. Let me note that a number of large k expressions for traces, including the one above, have also been found in [51] where $\psi_0 := \arctan(\frac{k_1}{k_2})$.

From (3.33) we see that the sum of exponents of k_1 and k_2 for the leading term at large k_1,k_2 is $L+2$. Furthermore this does not depend on the α_is in the T_3s since there will always be a term without αs: $\mathrm{tr}_1(t_3^m)\,\mathrm{tr}_2(\mathbb{1})\sim k_1^{L+1}k_2$. Hence, we can simply count the weight in ks by the number of T_3s.

Returning to the $M=2$ trace factor at large k_1,k_2. From (3.23) we see that the last term have less T_3s than the first and is hence subleading in k_1,k_2. For the first term, the number of T_3s decrease with p, hence we find the leading term in k_1,k_2 for $p=0$. Therefore, the trace is

$$\begin{aligned}&\mathrm{Tr}\,[T_1T_3^AT_1T_3^{L-2-A}] =\\ &\qquad\frac{1}{2}\Big(\tfrac{k_1^2}{4}+\alpha^2\tfrac{k_2^2}{4}\Big)\,\mathrm{Tr}\left[T_3^{L-2}\right]-\frac{1}{2}\,\mathrm{Tr}\left[T_3(\mathrm{i}\alpha)T_3(-\mathrm{i}\alpha)T_3^{L-2}\right]+\mathrm{O}(k^{L+1})\end{aligned}$$

for large k_1,k_2. This do not depend on how the excitations are arranged. That is, the right hand side is independent of $A=n_2-n_1-1$. Therefore, the $M=2$ overlap vanishes at this order in k_1,k_2. We turn to the sub-leading pieces to get a non-trivial contribution to the overlaps. The first non-constant terms in the trace factor are of order k^L

$$\begin{aligned}\mathrm{Tr}\,[T_1T_3^AT_1T_3^{L-2-A}]\sim\\ &\frac{1}{2}\,\mathrm{Tr}\,[(-\tfrac{1}{4}-\tfrac{\alpha^2}{4})T_3^{L-2}]-\frac{1}{2}A\,\mathrm{Tr}\,[T_3(\alpha^3)T_3^{L-3}]\\ &\quad+\frac{1}{4}A(A-1)(\tfrac{k_1^2}{4}+\alpha^4\tfrac{k_2^2}{4})\,\mathrm{Tr}\,[T_3^{L-4}]\\ &\quad-\frac{1}{4}A(A-1)\,\mathrm{Tr}\,[T_3(\mathrm{i}\alpha^2)T_3(-\mathrm{i}\alpha^2)T_3^{L-4}]+\mathrm{O}(k^{L-1})\,.\end{aligned}$$

Evaluating the resulting traces at leading order in k_1, k_2 we find the explicit trace factor

$$\mathrm{Tr}\,[T_1T_3^AT_1T_3^{L-A-2}]=-\frac{\mathrm{i}^L(k_1^2+k_2^2)^{L/2}\sin((L-2)\psi_0)}{2^{L-1}(L-1)(L-2)(L-3)}A(L-2-A)+\mathrm{O}(k^{L-1})$$

where $A=n_2-n_1-1$. The overlap can now be computed along the lines described in section 3.2.3. We find that for $M=2$ and $\alpha=\mathrm{i}$ the overlap is

$$\langle \mathrm{D7}_{k_1,k_2}(\mathrm{i})|u,-u\rangle=\frac{Lu\sqrt{u^2+\frac{1}{4}}}{2^{L-2}(L-2)(L-3)}(k_1^2+k_2^2)^{L/2}\sin((L-2)\psi_0)+\mathcal{O}(k^{L-1})\,.$$

For $k_1=k_2=k$ the overlap simplifies further to

$$\langle \mathrm{D7}_{k,k}(\mathrm{i})|u,-u\rangle=\frac{Lk^Lu\sqrt{u^2+\frac{1}{4}}}{2^{L/2-2}(L-2)(L-3)}+\mathcal{O}(k^{L-1})\,.$$

3.5 Discussion & Outlook

This chapter dealt with the explicit computation of spin chain overlaps relevant for the tree-level one-point functions in $\mathcal{N}=4$ SYM with domain walls. The new results presented in this chapter concern the explicit computation of the overlaps of on-shell Bethe states and the $\mathrm{D7}_{k_1,k_2}$ matrix product state relevant for the tree-level one-point functions of SU(2)-sector operators in the SO(3)×SO(3) symmetric D7 probe brane field theory.

We have seen that while the $\mathrm{D5}_k$ and $\mathrm{D7}_{k_1,k_2}$ matrix product states look very similar, i.e. containing one or two spheres, the resulting overlap formulas are qualitatively different. In contrast to $\mathrm{D5}_k$, the $\mathrm{D7}_{k_1,k_2}$ overlaps of unpaired states have corrections at order $L^{M/2-1}$ to, what may be called, the *integrable piece*, i.e. term proportional to $\det G^+$, as we have explicitly seen for $M=4,6$. Furthermore, paired states have non-zero overlaps this have been verified numerically for $M=6$. At the time, this was contradicting the standard lore;

that the matrix product states for the fuzzy funnel domain walls should be integrable.

In the process of computing overlaps, we have refined some tricks. In particular, the observation that the leading L terms are given by the unpaired states. This means that one can restrict the number of sums Σ_σ needed to determine the overlap at leading order in L to the permutations σ that keeps the paired rapidities together, thereby bringing the number of sums down from $M!$ to $(M/2)!2^{M/2}$. This have lead to a conjecture for the leading L expressions of the $\text{D7}_{2,2}$ overlaps Eq. (3.32). The conjecture have been tested for $M = 0, 2, 4, 6$. It would be interesting to extend this analysis to the full SO(6).

For the eventual comparison to string theory, we have computed the leading piece of the overlaps $M = 0, 2$ for large representations k_1, k_2. To this end we have considered a systematic algebraic approach to finding the expansion of trace factors in k_1, k_2. To test the method we have affirmed the leading k term $M = 2$ in the D5_k setup. Applying this to $\text{D7}_{k_1,k_2}$ gives the leading k_1, k_2 $M = 2$ overlap which for $\alpha = \text{i}$ have a particularly simple form.

The simple form of the overlaps at leading L, for $k_1 = k_2 = 2$ and any M, and large k_1, k_2, for $\alpha = \text{i}$, indicates that we may obtain simple expressions for the overlaps in these limits. In this regard it would be very interesting to extend the analysis of leading k_1, k_2 to higher M to form a conjecture. It is exciting to speculate that the large L leading k_1, k_2 could possibly have a simple form in integrability terms even though the matrix product state itself is not an integrable state.

4 On the integrability of one-point functions

> *"It is well known that a vital ingredient of success is not knowing that what you're attempting can't be done."*
>
> — T. Prachett, Equal Rites

In this chapter, we shall discuss the integrability of one-point functions. In chapter 3 we saw that one-point functions in D3⊥D5 can written formulation in terms of Baxter's Q-functions.The Matrix Product State was annihilated by the parity odd charges of the spin chain. This resulted in a selection rule for overlaps; that the Bethe roots be paired, i.e. $\{u_i\} = \{-u_i\}$. Interestingly, the D3⊥D7 systems has two different behaviours: For the SO(5) symmetric configuration, the Matrix Product State is annihilated by all the odd charges whereas for the SO(3)×SO(3) symmetric system, the Matrix Product State is not. In this chapter we shall contemplate these observations.

The original motivation for the investigations in the current chapter was to generalise, if possible, the proof of the overlap formula in the $SU(2)$-sector based on a 6-vertex model [87] to the sectors $SU(3)$ and $SO(6)$ for which the overlap formulas are conjectured [55, 56]. We shall return to this question in the outlook, however we will initially review the argument for the $SU(2)$ sector in section 4.2.

In the process we will need to review Sklyanin's algebraic construction of integrable open quantum spin chains in section 4.1. The construction is based on solutions of the reflection equation and in particular we shall report on a novel solution describing boundary conditions for an open $SO(6)$ spin chain with $SO(3) \times SO(3)$ boundary symmetry.

By exploiting the underlying integrable structure of $\mathcal{N} = 4$ super Yang-Mills theory we can sometimes compute observables in at loop orders inaccessible by Feynman diagrammatic techniques. To go beyond the asymptotic results for long operators, we need the wrapping corrections. The philosophy of the thermodynamic Bethe ansatz (TBA) is that this corresponds to having the mirror theory at finite temperature. The precise application to one-point functions is still an on-going research project. It is our hope that the open spin chain in the cross-channel for the $SO(6)$ sector will provide useful insights in this regard.

4.1 Integrable open spin chains

In the following I shall review Sklyanin's approach to construct integrable open spin chains [90] based on consistency conditions for factorised scattering on a half-space [91].

The R-matrix $R_{ab}(u-v)$ is a solution of the Yang-Baxter equation Eq. (4.2). We shall graphically denote the R-matrix as in Fig. 4.1 In the Quantum In-

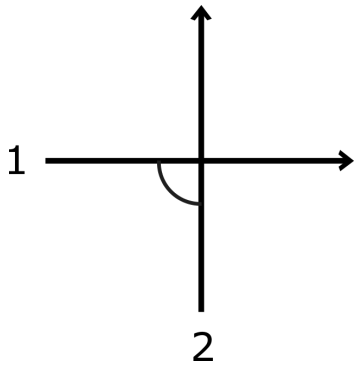

Figure 4.1 – The R-matrix. Note we take time to flow North-East.

verse Scattering Method the R-matrix acts as an intertwiner for local quantum Lax operators $\mathcal{L}_{ai}(u)$ from which one builds a set of commuting matrices $t(u)$ indexed by $u \in \mathbb{C}$. Expanding $t(u)$ in the complex parameter defines a set of commuting observables Q_i amongst which one can choose a Hamiltonian thus obtaining a quantum integrable system [66]. The Yang-Baxter equation is a consistency for scattering amplitudes to factorise into 2-body amplitudes.

In the following we assume the R-matrix to be PT-symmetric

$$R_{21}(u) = R_{12}^{\mathsf{T}}(u)\,.$$

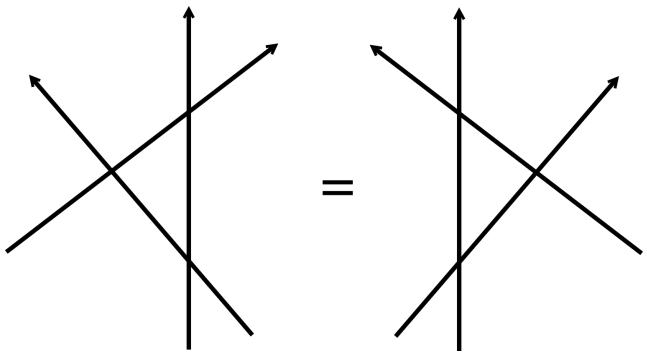

Figure 4.2 – The Yang-Baxter equation.

Note that a 'symmetric' R-matrix is both P- and T-symmetric, $R_{12}(u) = R_{21}(u)$ and $R_{12}(u) = R_{12}^{\mathsf{T}}$. There exists non-symmetric, yet PT-symmetric, R-matrices [92]. A R-matrix is said to be unitary if it satisfies

$$R_{12}(u)R_{21}(-u) = \rho(u)\mathbb{1} \tag{4.1}$$

and cross-unitary if

$$R_{12}^{\mathsf{t}_2}(u)\mathcal{M}_2 R_{21}^{\mathsf{t}_2}(-u-\eta)\mathcal{M}_2^{-1} = \tilde{\rho}(u)\mathbb{1} \tag{4.2}$$

where $\rho(u), \tilde{\rho}(u)$ and η are model-dependent quantities. A large class of known R-matrices fall in to this category [92, 93]

An open spin chain is a quantum system based on the Hilbert space containing a factor $\bigotimes_i V_i$ where V_i corresponds to the Hilbert space of the spin at site i. One can construct an open integrable spin chain from the solutions of the boundary Yang-Baxter equation, also known as the reflection equation [90, 94]. Like the Yang-Baxter equation, the reflection equation also arises from a consistency condition for factorisable scattering on a half space [91]. The reflection equation takes the form

$$\begin{aligned} K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v) \\ = R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) \end{aligned} \tag{4.3}$$

and is depicted in Fig. 4.3

Consider the bulk monodromy matrix $T_a(u)$ obtained from scattering an auxiliary spin along the chain from left to right. This is given as the product of Lax operators

$$T_a(u) = R_{aL}(u-\xi_L)\cdots R_{a1}(u-\xi_1)$$

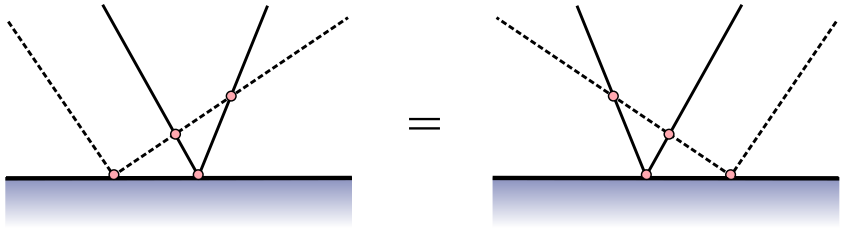

Figure 4.3 – The boundary Yang-Baxter equation. The intersections corresponds to R-matrices and the scattering off of the wall is given by the K-matrix.

where ξ_i are inhomogeneity parameters. Let $\bar{T}$ be the bulk monodromy matrix of the spin chain transversed in opposite order

$$\bar{T}_a(u) = R_{1a}(u+\xi_1)\cdots R_{La}(u+\xi_L)$$

From these we can then construct Sklyanin's monodromy operator $\mathcal{T}$ for the integrable open spin chain according to

$$\mathcal{T}_a^-(u) := T_a(u)K_a^-(u)\bar{T}_a(u) \tag{4.4}$$

where $T_a(u)$ and $\bar{T}_a(u)$ are the monodromies defined above for an auxiliary spin scattering with full spin chain going respectively right and left along the chain. $K_a^-(u)$ is a solution of the boundary Yang-Baxter equation Eq. (4.3) for scattering of the auxiliary spin on the left boundary. A diagrammatic representation is given in Fig. 4.4.

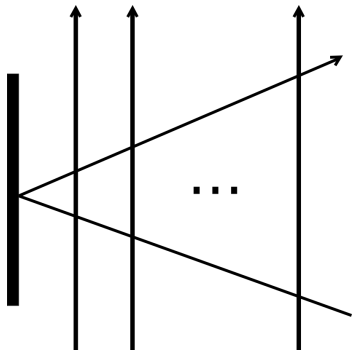

Figure 4.4 – Sklyanin's monodromy matrix for reflection on the left boundary $\mathcal{T}_a^-(u)$ as defined in Eq (4.4).

The operators, i.e. the auxiliary space components, of the open spin chain monodromy matrix $\mathcal{T}_a$ given in Eq. 4.4 satisfy the reflection algebra

$$\mathcal{T}_2^-(v)R_{12}(u+v)\mathcal{T}_1^-(u)R_{21}(u-v) = R_{12}(u-v)\mathcal{T}_1^-(u)R_{21}(u+v)\mathcal{T}_2^-(v)$$

This follows from the reflection equation Eq. (4.3) by inserting quantum spaces $1, 2, \ldots, L$ from the right and moving them towards the boundary using the Yang-Baxter relation Fig. 4.2. Similar to the 'train'-argument for the fundamental commutation relations of the Yang-Baxter algebra in the closed spin chain case this argument is easier drawn than written see Fig. 4.5.

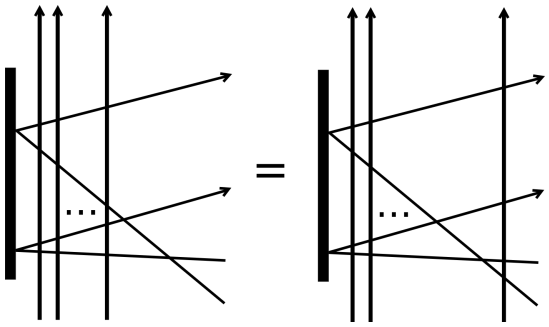

Figure 4.5 – The 'train' argument for lifting the reflection equation to an algebra of quantum operators on the spin chain.

After scattering off of the right end the auxiliary spin have fully transversed the spin chain. Hence we consider the double-row transfer matrix $t(u)$ given by

$$t(u) = \mathrm{tr}_a[K_a^+(u)\mathcal{T}_a^-(u)]$$

where the trace is understood to be taken over the auxiliary space and $K_a^+(u)$ satisfies the boundary Yang-Baxter equation for scattering off of the right end

$$K_b^+(v)R_{ba}^{\mathtt{t}_b,-1,\mathtt{t}_b}(u+v)K_a^+(u)R_{ab}^{-1}(u-v) \\ = R_{ba}^{-1}(u-v)K_a^+(u)R_{ab}^{\mathtt{t}_a,-1,\mathtt{t}_a}(u+v)K_b^+(v)\,. \quad (4.5)$$

Note that due to unitarity Eq. (4.1) and cross-unitarity Eq. (4.2) the various operations on the R-matrix are nothing but matrix notation book-keeping for how the various legs of the linear operators are connected. In particular the right and left reflection algebras are isomorphic in the sense that if $K^-(u)$ satisfy the reflection equation Eq. (4.3) then $K^+(u) = K^-(-u-\eta)$ satisfy the dual reflection equation Eq. (4.5).

The integrability of the spin chain models follows, since the double-row transfer matrices $t(u)$ and $t(v)$ defined according to Eq. 4.1 commutes for arbitrary $u, v \in \mathbb{C}$. That the double-row transfer matrices commute can be seen by an explicit computation [95].

4.2 Vertex models and spin chain overlaps

Integrable 1+1 dimensional quantum systems and exactly solvable statistical mechanics systems in 2 dimensions are intimately related [66, 96, 97]. We can interpret the R-matrix as specifying the local Boltzmann weights of a nearest neighbour lattice model in 2 dimensions [96]. That is, for the four possible configuration of states on the legs in Fig. 4.1 we assign a weight

$$[R]^{\beta\delta}_{\alpha\gamma} \sim \mathrm{e}^{-\beta\epsilon(\alpha,\beta,\gamma,\delta)} ,$$

where $\epsilon(\alpha, \beta, \gamma, \delta)$ is the energy of the local configuration with states $\alpha, \beta, \gamma, \delta$.

Corresponding to the the $\mathrm{XXZ}_{1/2}$ spin chain is the so-called *6-vertex model*. The name derives from the 6 the attainable vertex configurations in the statistical ensemble. The restriction to these vertices follows from spin conservation at the vertices, known as the ice rule[1]. The R-matrix has the expression

$$R_{ab}(u-v) = (u-v)\mathbb{1}_{ab} + \mathrm{i}P_{ab} = \begin{pmatrix} u-v+\mathrm{i} & 0 & 0 & 0 \\ 0 & u-v & \mathrm{i} & 0 \\ 0 & \mathrm{i} & u-v & 0 \\ 0 & 0 & 0 & u-v+\mathrm{i} \end{pmatrix} .$$

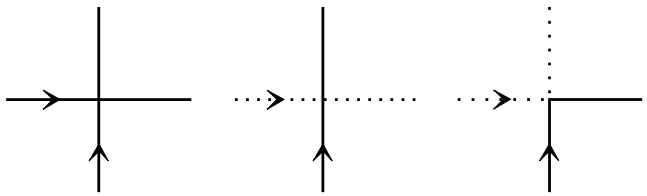

Figure 4.6 – The three different types vertices defined by the rational R-matrix. The types of the vertices are named in order: **a**-type (two lines of the same flavour intersects), **b**-type (wo differently flavoured lines intersect) and, **c**-type (two differently flavoured lines intersect, out going lines are swapped compared to **b**-type).

The weight of a particular configuration of the full lattice is the product of local contribution. To obtain the partition function we sum the weights of all

[1] The 6-vertex model is also known as the *square ice model* as it provides a 2 dimensional approximation of the hydrogen-bonded hexagonal structure of type $\mathrm{I_h}$ water ice crystals [98]. Each vertex then represents a water molecule with the legs signifying the hydrogen bonds to other water molecules of the crystal [99]. Interestingly, square ice has supposedly been observed for water confined to 2 dimensions obtained by squeezing a water droplet between sheets of graphene [100].

configurations. Schematically

$$Z \sim \langle s_i| \langle s_i| \sum RR \cdots R |s_i\rangle |s_i\rangle$$

In the algebraic Bethe ansatz the Bethe eigenstates are given as

$$B(u_M) \cdots B(u_1) |0\rangle \ .$$

Thus, the eigenstates corresponds to a partition function of the 6-vertex model with boundary conditions imposed on three sides. The bottom legs have all spin up corresponding to the pseudo-vacuum $|0\rangle$, while the left side have spin down and the right, spin up. Similarly, the dual states are by given with an unspecified lower boundary and the upper boundary being the pseudo-vacuum.

The overlap of a Bethe state with any configuration of spins $|\{s_i\}\rangle$ can be written as a 6-vertex model with boundary conditions specified by the B-operators, the pseudo-vacuum $|0\rangle$ and the specified configuration $\{s_i\}$. This is simply the statement that the 6-vertex model specifies the wave function of the Bethe state.

We are particularly interested in states that correspond to integrable boundary conditions of the vertex model. Integrable boundary conditions satisfy the boundary Yang-Baxter equation allowing moves of the type shown in Fig. 4.3.

The most general integrable boundary conditions for the (rational) 6-vertex model is given by the K-matrix [101]

$$K^-(u) = \begin{pmatrix} u+\xi & \lambda u \\ \mu u & u-\xi \end{pmatrix}, \tag{4.6}$$

where ξ, λ and μ are free parameters. This can be seen as follows: Consider the most general 2×2 matrix-valued function of u. Note that any solution to the reflection equation Eq. (4.3) implies a family of solutions since an arbitrary u-dependent scalar factor simply cancels out. Thus, let us rescale the first entry to 1 and consider the matrix

$$\begin{pmatrix} 1 & b(u) \\ c(u) & d(u) \end{pmatrix}.$$

Inserting this into the reflection equation Eq. (4.3) implies functional equations for the coefficient functions $b(u), c(u)$ and $d(u)$. Amongst the equations only

one involves $d(u)$:

$$v - u + \big(u+v\big)d(v) + d(u)\Big(-u - v + \big(u-v\big)d(v)\Big) = 0\,. \tag{4.7}$$

From Eq. (4.7) follows that the dependence on u is

$$d(u) = \frac{u-\xi}{u+\xi}\,,$$

for a free parameter ξ. This already fixes the diagonal elements of the solution Eq. (4.6). By inserting this expression for $d(u)$ into the remaining functional equations, the problem reduces to solving system

$$\begin{aligned} b(u)c(v) &= b(u)c(v)\,,\\ v(u+\xi)c(u) &= u(v+\xi)c(v)\,. \end{aligned}$$

The first equation implies that $b(u) \propto c(u)$, while the last equation gives that $c(u)$ takes the form

$$c(u) = \frac{\mu u}{u+\xi}\,,$$

where μ is a free parameter.

The partition function with integrable boundary conditions for $\lambda = \mu = 0$, was computed in [102] using an appropriate generalisation of the method used to prove the Bethe norm [103]. This argument also generalises to $\lambda \neq 0$ [87]. In section 4.2.1, I shall review the computation of the partition function of the six vertex model with a partial reflecting end.

4.2.1 Partition function of the six vertex model with partial reflecting end

The partition function for the six-vertex model on a $N \times 2N$ lattice with a reflecting end boundary condition on one side and domain wall boundary conditions on the other sides was determined in [102] by the Korepin-Izergin method. The Korepin-Izergin methods works by proposing a determinant expression and showing that it satisfy certain analytic properties and recursion formulas. The properties of the partition function follows from the algebraic Bethe Ansatz for open spin chains with a diagonal K-matrix ($\lambda = \mu = 0$ in Eq. (4.6)). In the following we shall review the arguments for the non-diagonal case with $\lambda \neq 0$ based on [87].

The partition function on the $M \times L$ lattice is given as the overlap between an $L/2$-fold product of two-site boundary states $|K(u)\rangle$ and a Bethe state $|\{u_i\}\rangle$ on a L notes long closed spin chain

$$Z(\{u_i\},\{v_a\},\xi,\lambda,\mu) = \langle K(v_1)|_{12} \cdots \langle K(v_N)|_{L-1,L}\, B(u_M) \cdots B(u_1)\, |0\rangle \ . \tag{4.8}$$

Here $B(u_i)$ is the algebraic Bethe ansatz B-operator that creates an excitation with rapidity u_i. Note that we shall not initially require the Bethe state to be on-shell i.e. we let $\{u_i\}$ be arbitrary complex parameters for now. The coefficients ξ, λ and μ are arbitrary complex parameters characterising the couplings at the ends of the open chain. We will consider the case where $\mu = 0$. The partition function can be depicted as Fig. 4.7 in graphic notation.

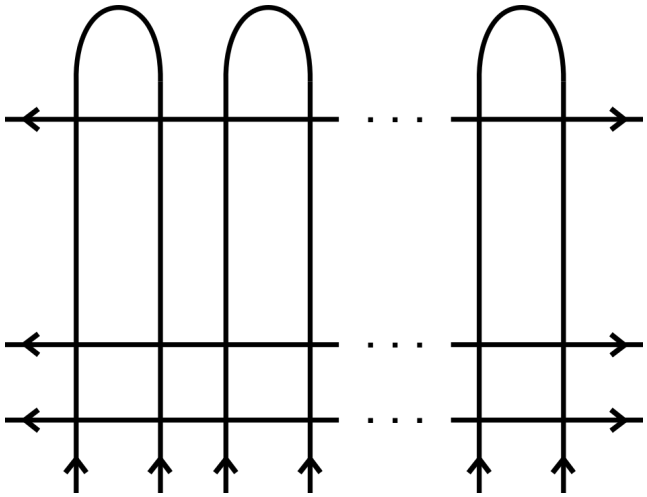

Figure 4.7 – The domain wall partition function for the six-vertex model with a partial reflecting end. Note that it also can be read as the overlap between a two-site product state and a Bethe state. The domain wall boundary conditions (DW) for the horizontal lines picks out the B-operator.

The boundary dual states $\langle K(u)|$ satisfy a reflection equation [87]

$$\begin{aligned}\langle K_{34}(v) \otimes K_{12}(u)|\, R_{14}(u+v) R_{13}(u-v) \\ = \langle K_{12}(u) \otimes K_{34}(v)|\, R_{23}(u+v) R_{24}(u-v)\end{aligned} \tag{4.9}$$

The general boundary state that solves this equation takes the form [104]

$$\langle K(u)|_{12} = \sum_{i,j} [K(-u - \mathrm{i}/2) M]_{i,j}\, \langle i|_1 \otimes \langle j|_2 \tag{4.10}$$

where the K-matrix Eq. (4.6) is a generic solution of the reflection equation Eq. (4.3) and $M = \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right)$. Note also the appearance of the crossing parameter $\mathrm{i}/2$. This will be discussed in section 4.3. Explicitly, the generic boundary state takes the form

$$\langle K(u)|_{12} = \langle 10| \left(u + \frac{\mathrm{i}}{2} + \xi\right) + \langle 01| \left(u + \frac{\mathrm{i}}{2} - \xi\right) + \langle 00| \lambda\left(u + \frac{\mathrm{i}}{2}\right) + \langle 11| \mu\left(u + \frac{\mathrm{i}}{2}\right). \tag{4.11}$$

The partition function Eq. (4.8) is symmetric in $\{u_i\}$, i.e. interchanging $u_i \leftrightarrow u_j$ does not change the value. This follows from the Yang-Baxter algebra for the B-operators since $[B(u), B(v)] = 0$. The partition function is similarly symmetric in $\{v_i\}$ as a result of the boundary Yang-Baxter equation Eq. (4.9). The reflection algebra implies that moving a leg from one $\cap$ across an adjacent $\cap$ one may swap the position. The boundary conditions on the lower end freezes the two intersections and yields an identical overall factor in both diagrams thereby proving symmetry in $\{v_i\}$.

The partition function is a polynomial in u_i of degree $L - 1$. This can be seen as follows: Consider the horizontal line corresponding to $B(u_i)$. The domain wall boundary conditions injects an excitation on the left and require that it does not come out on the right. By spin conservation this excess spin must go somewhere – that is, in any viable configuration along the line labelled by u_i there has to be a least one type **c** vertex. The weight of the other vertices is $\propto u_i$ while the type **c** does not depend on u_i hence the weight of the line is a polynomial in u_i.

The partition functions $Z(\{u_i\}, \{v_a\})$ satisfy a recursion relation. Parts of the configuration "freezes" when evaluated at a specific rapidity. Thereby, equating the partition function on a $M \times L$ lattice at a specific rapidity point to the partition function on a smaller $(M-1) \times (L-1)$ lattice. To establish this recursion relation we shall consider the vertex in the lower left corner.

The lower left vertex have to be a **b**- or **c**-type due to the domain wall boundary conditions. The **b** vertex depends on the difference $u_1 - (-v_1)$ and thus vanishes for $u_1 = -v_1$. For $u_1 = -v_1$, the lower left vertex is fixed to be the appropriate **c**-type vertex. By specifing the vertex in the lower left corner all the vertices freezes along the horizontal line and the first $\cap$ by spin conservation (see Fig. 4.8). Above the freezing line, one has the partition function on a $(M-1) \times (L-2)$ sublattice with partially reflecting domain wall boundary conditions. Note that $\mu = 0$ is crucial for this freezing argument to work.

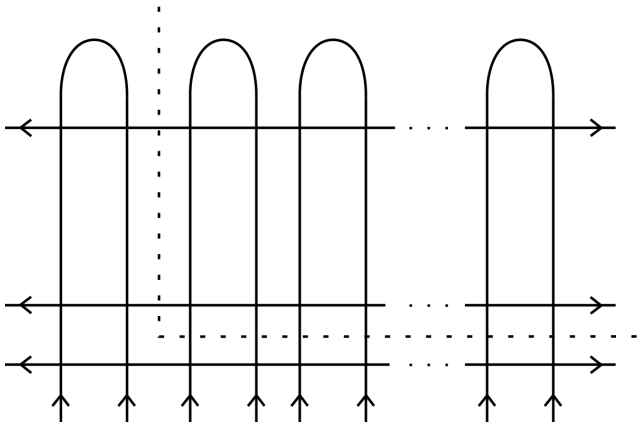

Figure 4.8 – Recursion relation for six-vertex partition function with partial reflecting end.

This extends trivially to any pair of horizontal line and vertical ∩s since the partition function is symmetric in u_i and v_a. Therefore, we have the recursion relation for the partition function

$$Z(\{u_i\},\{v_a\})\big|_{u_j=-v_b} = (-2v_b+\mathrm{i})\mathrm{i}(v_b+\tfrac{\mathrm{i}}{2}+\xi) \times \prod_{i\neq j}\Big((u_i+v_b+\mathrm{i})(u_i-v_b)\Big)\prod_{a\neq b}(v_b^2-v_a^2)Z(\widehat{\{u_i\}},\widehat{\{v_a\}})\,, \quad (4.12)$$

where $\widehat{\{u_i\}}$ is the set of rapidities without u_j. Similarly, $\widehat{\{v_a\}}$ is the inhomogeneities without v_b. Furthermore, a similar argument can be made for the lower right corner when $u_i = v_a$.

For $M = 0$, that is no horizontal lines, the partition function is simply given as a product of the $|00\rangle$ coefficients of the boundary state:

$$Z(\emptyset,\{v_a\};\xi,\lambda) = \lambda^{\frac{L}{2}}\prod_{a=1}^{L/2}(v_a+\tfrac{\mathrm{i}}{2})\,.$$

Together with the properties derived above, this recursively fix the partition function by Lagrange interpolation. For $M > 0$ the partition function is a polynomial in u_i of degree $L-1$ for $i = 1,\ldots,M$. The partition function at $u_i = \pm v_a$ for $a = 1,\ldots,L/2$ is given by Eq. (4.12). For each u_i this gives L points that the degree $L-1$ polynomial should intersect and thus fix it uniquely. The partition function has various determinant forms [87] including as a $M\times M$ determinant. This generalises the Tsuchiya's determinant formula [102] to the case where $\lambda \neq 0$. In the limit where the rapidities $\{u_i\}$ are on-shell the

partition function vanishes unless the rapidities are paired in a parity-invariant way $\{u_i\} = \{-u_i\}$. For paired rapidities the $M \times M$ determinant takes a Gaudin-like form $\propto \det G^+$ [87, 88].

The last step in proving the overlap formula Eq. (3.26) using the partition function discussed above depends on relating the boundary state to the $k = 2$ D3⊥D5 matrix product state. The matrix product state Eq. (3.17) for $k = 2$ is related to the Néel state [54, 105] which in turn is related to the boundary state [106]. The Néel state is the anti-ferromagnetically ordered spin chain state

$$|\text{Néel}\rangle = |\uparrow\downarrow\uparrow\downarrow\cdots\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\cdots\downarrow\uparrow\rangle \ .$$

Note that for even L the number of spins down M is equal to the number of spins up $L - M$ i.e. $M = L/2$. The precise relation is [53]

$$|\text{MPS}_M\rangle = \frac{1}{2^L(\frac{\mathrm{i}}{2})^M}\,|\text{Néel}_M\rangle + S^-\,|\cdots\rangle \ , \tag{4.13}$$

where $|\text{MPS}_M\rangle = P_M\,|\text{MPS}\rangle$ is the projection onto the M magnon sector of the matrix product state and $|\text{Néel}_M\rangle$ is the partial Néel state [87]

$$|\text{Néel}_M\rangle = \sum_{\substack{x_1<\cdots<x_M\\ |x_i-x_j|\text{even}}} |\uparrow\cdots\uparrow\underset{x_1}{\downarrow}\uparrow\cdots\underset{x_2}{\downarrow}\cdots\underset{x_M}{\downarrow}\cdots\uparrow\rangle \ .$$

Consider the boundary state Eq. (4.11) with the parameters $u = 0, \xi = \pm\frac{\mathrm{i}}{2}$ and $\mu = 0, \lambda = -2\mathrm{i}$. This produces the states

$$\begin{aligned} \langle K(0)|\,\big|_{\xi=\frac{\mathrm{i}}{2},\lambda=-2\mathrm{i},\mu=0} &= \langle 10|\,\mathrm{i} + \langle 00|\ , \\ \langle K(0)|\,\big|_{\xi=-\frac{\mathrm{i}}{2},\lambda=-2\mathrm{i},\mu=0} &= \langle 01|\,\mathrm{i} + \langle 00|\ . \end{aligned} \tag{4.14}$$

The $L/2$-fold tensor product of the states in Eq. (4.14) generate states with $M = 0, \ldots, \frac{L}{2}$ spins down in a background of spins up. Note that the states generated by $\xi = \pm\frac{\mathrm{i}}{2}$ differ by a one-site shift of the spins. Therefore, the sum of these states produce all M-partial Néel states from $M = 0, \ldots, \frac{L}{2}$

$$\bigotimes_{i=1}^{L/2}\big(\langle 10|\,\mathrm{i} + \langle 00|\big) + \bigotimes_{i=1}^{L/2}\big(\langle 01|\,\mathrm{i} + \langle 00|\big) = \sum_{M=0}^{L/2} \mathrm{i}^M\,\langle \text{Néel}_M|\ .$$

The on-shell Bethe states $|\{u_i\}\rangle_{i=1}^M$ are highest weight states with exactly M magnons thus the overlap picks out the M-partial Néel state in the above sum. Therefore, the overlap between the partial Néel state and the Bethe states

can be expressed through the partition functions Eq. (4.8) for the partially reflecting six-vertex model as

$$\langle \text{Néel}_M | \{u_i\} \rangle = (-\mathrm{i})^M \Big(Z(\{u_i\}, 0, \tfrac{\mathrm{i}}{2}, -2\mathrm{i}, 0) + Z(\{u_i\}, 0, -\tfrac{\mathrm{i}}{2}, -2\mathrm{i}, 0) \Big) .$$

The partial Néel state is cohomological to the projected matrix product state Eq. (4.13) and hence the overlap between the matrix product state and the on-shell Bethe state can be written as

$$\langle \text{MPS} | \{u_i\} \rangle = \frac{1}{2^{L-M}} \Big(Z(\{u_i\}, 0, \tfrac{\mathrm{i}}{2}, -2\mathrm{i}, 0) + Z(\{u_i\}, 0, -\tfrac{\mathrm{i}}{2}, -2\mathrm{i}, 0) \Big) .$$

Finally, one obtains the determinant expression for the $k = 2$ overlap Eq. (3.26) by taking the limit where the rapidities are parity invariant and on-shell in the determinant representation of the partition function [87].

Note that we did not refer to the algebraic Bethe ansatz in the open spin chain channel. That the partition function is symmetric in the inhomogeneity parameters $\{v_a\}$ follows from the commutativity of the open spin chain B-operators [102, 106]. However, the algebraic Bethe ansatz is more involved for the integrable open spin chains with non-diagonal boundary K-matrices since the number of magnons is not conserved and one lacks an obvious pseudo-vacuum reference state [107]. For the triangular K-matrix there exists an algebraic Bethe ansatz [108, 109]. Furthermore, there exists a proposal for the general boundaries [110]. Presumably, the open spin chain algebraic Bethe ansatz could equally have been used to establish the properties of the partition function based on the algebra in line with [69]. A determinant formula for the trigonometric six-vertex model with arbitrary reflecting end was found in [111]. With the partition function for the general boundary K-matrix one could presumably prove the conjectured overlap formula for arbitrary two-site integrable states proposed in [86].

4.3 The MPS as integrable boundary states

We have seen that integrable boundary conditions on a lattice model corresponds to specifying a two-site invariant state in the quantum spaces. Thus, the overlap between these boundary states and a Bethe state can be interpreted as an exactly solvable vertex model where the bulk weights solve the Yang-Baxter equation and the boundary weights solve the corresponding boundary Yang-Baxter equation.

In general we can express the overlaps between Bethe eigenstates and our matrix product states as a partition function of an exactly solvable vertex model when the projection of the matrix product state $|\text{MPS}\rangle$ for each sector M is equal to a projection of the boundary state at some specified rapidity plus a lower weight piece. That is when the states satisfy

$$P_M |\text{MPS}\rangle \propto P_M |K(0)\rangle + S^- |\cdots\rangle$$

for all $M = 1, \ldots, L/2$ and for some states $|\cdots\rangle$. We shall refer to this as $|\text{MPS}\rangle$ and $|K(0)\rangle$ being cohomologically identical. Note that this implies that the overlaps with Bethe states $\{u_i\}_{i=1}^M$ agree for all M, i.e.

$$\langle \text{MPS}|\{u_i\}_{i=1}^M\rangle \propto \langle K(0)|\{u_i\}_{i=1}^M\rangle \,.$$

This follows immediately from the fact that on-shell Bethe states are highest weight states

$$S^+ |\{u_i\}_{i=1}^M\rangle \ .$$

This have been formulated for $\mathfrak{su}(2)$ but can be formulated in analogy for higher rank spin chains. In section 4.3.2, we shall see how the cohomology condition looks in the case of $\mathfrak{su}(3)$. This is the most general requirement for the overlaps to agree. However, in the following we shall see that the MPS in many cases can be expressed as a boundary state for specified parameters [104].

The matrix product state of both the D3⊥D5$_k$ and the D3⊥D7$_n$ are annihilated by all the odd charges leading to the Gaudin-like determinant structure of the one-point functions. In contrast, the D3⊥D7$_{k_1,k_2}$ non-zero overlaps for paired states which spoil this structure.

That a state is annihilated by the odd charges of the spin chain,

$$Q_{2p+1} |\Psi\rangle = 0 \,, \tag{4.15}$$

has recently been proposed as a condition for integrability of the state in the context of quantum quench dynamics [104] in analogy with the condition for integrable boundary states in 2d field theories [112]. In section 4.3.1 we shall analyse the reflection equation Eq. (4.9) for the dual states. A solution to the reflection equation for states (or dual states) can reasonably be called 'integrable' as they correspond to integrable boundary conditions for open spin chain in the rotated channel [104]. Furthermore, by standard manipulations using the boundary Yang-Baxter equations it can be shown can show that any solution to the reflection equation for states satisfy the proposed integrability condition Eq. (4.15) [104].

4.3.1 The boundary Yang-Baxter for dual states

In the proof of the overlap formula above we exploited that the boundary dual state satisfied a kind of boundary Yang-Baxter equation Eq. (4.9). In the following we shall rewrite this equation into matrix form following [104].

The R-matrix acts as a bilinear map $V_a \otimes V_b \to V_a \otimes V_b$. For the components of the R-matrix in terms of basis states on the vector spaces V_a and V_b of the legs we shall use the convention that

$$\langle i|_a \langle j|_b R_{ab}(u) |l\rangle_a |k\rangle_b = R^{ij}_{lk}(u) .$$

This implies that the transpose of the R-matrix in the vector space V_a has the components

$$\langle i|_a \langle j|_b R^{\mathsf{t}_a}_{ab}(u) |l\rangle_a |k\rangle_b = R^{lj}_{ik}(u) .$$

Note that the transpose $^\intercal$ of R_{ab} written as a $(\dim V_a \dim V_b)^{\times 2}$ matrix is only equal to the transpose in each legs successively $^{\mathsf{t}_a\mathsf{t}_b}$ if the R-matrix is P-symmetric $R_{ab} = R_{ba}$ in which case the transpose in V_a is equal to the transpose in V_b. In case of P-symmetric matrices, we shall denote the transposition by $^{\mathsf{t}}$. The R-matrix is expressed in a component expansion as

$$R_{ab}(u) = R^{ij}_{lk}(u) |i\rangle_a |j\rangle_b \langle l|_a \langle k|_b ,$$

where summation over the basis states specified by repeated indices is implied.

The reflection K-matrix, is a linear operator $V_a \to V_a$. When we write it as a boundary state that means we are interpreting it as a map $V_a \otimes V_a^* \to \mathbb{C}$, i.e. assigning a number to each bra-ket pair. In a component expansion, we have

$$\langle K(u)|_{ab} = \psi_{ij}(u) \langle i|_a \langle j|_b .$$

Here we name the components $\psi_{ij}(u)$ in anticipation of the result.

The reflection equation for the boundary state Eq. (4.9) in components takes the form

$$\langle ijkl| \psi(v)_{\alpha\beta}\psi(u)_{\gamma j} R^{\beta\gamma}_{l\delta}(u+v) R^{\alpha\delta}_{ki}(u-v) = \langle ijkl| \psi_{i\alpha}(u)\psi_{\beta\gamma}(v) R^{\alpha\beta}_{\delta k}(u+v) R^{\delta\gamma}_{jl}(u-v) .$$

The dual state $\langle ijkl|$ is ordered as $\langle i|_1 \langle j|_2 \langle k|_3 \langle l|_4$. Since the dual states $\langle ijkl|$ provide a basis for the equation, we can consider the equation in terms of the components:

$$\psi(v)_{\alpha\beta}\psi(u)_{\gamma j} R^{\beta\gamma}_{l\delta}(u+v) R^{\alpha\delta}_{ki}(u-v) = \psi_{i\alpha}(u)\psi_{\beta\gamma}(v) R^{\alpha\beta}_{\delta k}(u+v) R^{\delta\gamma}_{jl}(u-v) .$$

Let $\psi(u)_{\alpha\beta} = \langle\beta|\,\psi(u)\,|\alpha\rangle$ then we can write the equation in terms of matrix elements. Using completeness $\mathbb{1} = \sum_\alpha |\alpha\rangle\,\langle\alpha|$ this implies an equation in terms of matrices. Therefore, the reflection equation for the states Eq. (4.9) is equal to the the reflection equation

$$\psi_1(u)R^{\mathsf{t}}(u+v)\psi_2(v)R(u-v) = R(u-v)\psi_2(v)R^{\mathsf{t}}(u+v)\psi_1(u) \tag{4.16}$$

where we have assumed P-symmetry.

This reflection equation Eq. (4.16) can be reduced to the standard boundary Yang-Baxter equation if the R-matrix have crossing symmetry [104]. Crossing for an R-matrix means that it satisfy the equation

$$R^{\mathsf{t}}(u) = \gamma(u)M_1^{-1}R(-u-\eta)M_1$$

where $\gamma(u)$ is a scalar function, the matrix M satisfy $[M_1 \otimes M_2, R] = 0$ and η is the crossing parameter. For the $\mathrm{XXX}_{1/2}$ or six-vertex R-matrix we have $M = \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right)$ and $\eta = \mathrm{i}$.

For an R-matrix with crossing symmetry Eq. (4.16) is equal to

$$\begin{aligned}\psi_1(u)M_1^{-1}R(-u-v-\eta)M_1\psi_2(v)R(u-v)& \\ = R(u-v)\psi_2(v)M_2^{-1}R(-u-v-\eta)M_2\psi_1(u)\,.&\end{aligned} \tag{4.17}$$

The equation Eq. (4.17) can be transformed into the standard reflection equation Eq. (4.3) as follows: Insert the identity of the form $M_i^{-1}M_i$ on the right and left hand side of the equation Eq. (4.17). Use the commutativity of $M_1 \otimes M_2$ with the R-matrix and multiply from the right with $(M_1 \otimes M_2)^{-1}$. After redefining the spectral parameters $u \to -v - \eta/2$ and $v \to -u - \eta/2$ we have the equation

$$\begin{aligned}\big(\psi_1(-v-\eta/2)V_1^{-1}\big)R(u+v)\big(\psi_2(-u-\eta/2)M_2^{-1}\big)R(u-v)& \\ = R(u-v)\big(\psi_2(-u-\eta/2)M_2^{-1}\big)R(u+v)\big(\psi_1(-v-\eta/2)M_1^{-1}\big)\,.&\end{aligned} \tag{4.18}$$

Therefore solutions of the equation Eq. (4.18) implies solutions of the reflection equation Eq. (4.3) and vice versa under the isomorphism

$$\psi(u) = K(-u-\eta/2)M\,.$$

This establishes the equation Eq. (4.10) for the boundary state. In particular, for the $\mathrm{XXX}_{1/2}$ R-matrix the generic solutions are given by Eq. (4.6) and

therefore find the states Eq. (4.11) up to overall factors and possible redefinitions of the parameters.

In general, R-matrices are not crossing symmetric and the boundary Yang-Baxter equation Eq. (4.9) for the dual states is therefore not identical to the reflection equation Eq. (4.3) in general. Boundary conditions based on this reflection equation, known as *the twisted boundary Yang-Baxter equation*, are known as 'soliton-non-preserving' (SNP) boundary conditions for the spin chain [94, 113, 114]. For the one-point function this is relevant in the SU(3) sector as the SU(N) R-matrices not crossing symmetric for $N > 2$. As we shall see in section 4.3.2, one can still solve the functional equations that Eq. (4.9) implies for the boundary dual states coefficients.

4.3.2 Towards a proof of the $\mathfrak{su}(3)$ sector overlaps

A determinant formula for the overlap $\langle \mathrm{D5}_2|\{u_i\},\{w_i\}\rangle$ between the D3⊥D5MPS and a on-shell parity invariant Bethe state was conjectured in [55]. In this section, we shall describe an approach towards proving the overlap formula using vertex model-based arguments which where succesful in the $\mathfrak{su}(2)$ sector [87].

To prove the overlap formula by this method, we must overcome two obstacles. First, to diagonalise the rational SU(3) transfer matrix one uses the nested algebraic Bethe ansatz [115]. Therefore, the partition function and boundary conditions that corresponds to Bethe states is more involved [116–118]. Second, and a more imposing problem, is the question of which boundary state would give the overlap between the matrix product state and a Bethe state.

In this section, I shall briefly review the nested Bethe ansatz construction of the Bethe states and address the question of choosing an appropriate boundary state. This defines a particular fifteen-vertex partition function. I shall leave a detailed study of its properties for future investigations.

The SU(3) spin chain is the immediate generalisation of the $\mathrm{XXX}_{\frac{1}{2}}$ Heisenberg spin chain. This is a chain consisting of fundamental SU(3) representations, i.e. a Hilbert space $\mathcal{H} = \bigotimes_{i=1}^{L} \mathbb{C}^3$, and an R-matrix

$$R_{ab}(u-v) = (u-v)\mathbb{1}_{ab} + \mathrm{i}P_{ab} \tag{4.19}$$

where P_{ab} is the permutation operator on $\mathbb{C}^3 \otimes \mathbb{C}^3$. In components, the R-matrix

Eq. (4.19) have fifteen non-zero weights thereby defining a fifteen-vertex model

$$R_{ab}(u-v) = \begin{pmatrix} u-v+\mathrm{i} & & & & & & & & & & & & & & \\ & u-v & & \mathrm{i} & & & & & & & & & & & \\ & & u-v & & & & & & \mathrm{i} & & & & & & \\ & \mathrm{i} & & u-v & & & & & & & & & & & \\ & & & & u-v+\mathrm{i} & & & & & & & & & & \\ & & & & & u-v & & \mathrm{i} & & & & & & & \\ & & \mathrm{i} & & & & u-v & & & & & & & & \\ & & & & & \mathrm{i} & & u-v & & & & & & & \\ & & & & & & & & u-v & & & & & & \\ & & & & & & & & & u-v+\mathrm{i} & & & & & \end{pmatrix},$$

where the empty spaces denotes zeros in the matrix.

The monodromy matrix is

$$T(u) = \begin{pmatrix} A(u) & B_1(u) & B_2(u) \\ C_1(u) & D_{11}(u) & D_{12}(u) \\ C_2(u) & D_{21}(u) & D_{22}(u) \end{pmatrix},$$

when viewed as a 3×3 matrix of operators on the quantum spaces. The generator of conserved charges, the transfer matrix, is the trace of the monodromy

$$\operatorname{tr} T(u) = A(u) + D_{11}(u) + D_{22}(u)\,.$$

The monodromy matrix encodes the fundamental commutation relations through the RTT-relation [117, 119]. In components the commutation relation between the elements of the monodromy matrix $T_{ij}(u)$ is given as

$$[T_{mn}(u), T_{kl}(u)] = \frac{\mathrm{i}}{u-v}\big(T_{ml}(u)T_{kn}(v) - T_{ml}(v)T_{kn}(u)\big)\,.$$

The commutation relations among the $T_{ij}(u)$ for $i < j$ can be written as

$$\boldsymbol{B}_a(u) \otimes \boldsymbol{B}_b(v) = \boldsymbol{B}_b(v) \otimes \boldsymbol{B}_a(u)\mathbb{R}^{(2)}_{ab}(u-v)\,,$$

where $\boldsymbol{B}_a(u) = (B_1(u), B_2(u))$ and $B_i(u) := T_{1,(i+1)}(u)$. The matrix

$$\mathbb{R}^{(2)}_{ab}(u-v) = \frac{1}{u-v+\mathrm{i}}\Big((u-v)\mathbb{1}_{ab} + \mathrm{i}P_{ab}\Big)\,,$$

is the SU(2) R-matrix. The Bethe states are constructed from the B-operators acting on the pseudo-vacuum

$$|\{u_i\}\rangle = \psi_{a_1,\ldots,a_M} B_{a_1}(u_1)\cdots B_{a_M}(u_M)\,|0\rangle\ . \tag{4.20}$$

For Eq. (4.20) to be an eigenstate of the SU(3) transfer matrix the $\psi_{a_1,\ldots,a_M}$ have to be an eigenstate of an auxiliary inhomogeneous SU(2) spin chain [120]. This leads to an auxiliary second set of rapidity variables $\{u_i^{(2)}\}$. Therefore,

when constructing the eigenstates from a vertex model, we have to include an auxiliary lattice that sets up the SU(2) wave function and thereby determines the flavours injected into lattice [116–118].

The SU(3) R-matrix Eq. (4.19) is not crossing symmetric though it satisfy both unitarity and crossing-unitarity

$$R_{ab}(u)R_{ba}(-u) = -(1+u^2)\mathbb{1}_{ab}\,, \quad R^{\mathtt{t}_1}_{ab}(u)R^{\mathtt{t}_1}_{ba}(-u-3\mathrm{i}) = -u(u+3\mathrm{i})\mathbb{1}_{ab}\,.$$

As we saw in the previous section, this implies that the reflection equation for the dual state is not the standard boundary Yang-Baxter equation. This means that we cannot simply promote solutions of the boundary Yang-Baxter equation for the rational SU(3) spin chain [121] to boundary states.

One approach is to solve the reflection equation for the dual states

$$\begin{aligned}\langle K_{34}(v)\otimes K_{12}(u)|\, R_{14}(u+v)R_{13}(u-v)& \\ = \langle K_{12}(u)\otimes K_{34}(v)|\, R_{23}(u+v)R_{24}(u-v)\,.&\end{aligned} \tag{4.21}$$

This circumvents the need to cross R-matrices and directly produce dual states with the algebraic properties that we need. As mentioned in section 4.3.1 this corresponds to solving a twisted boundary Yang-Baxter equation [104, 122].

Looking for the simplest possible states that satisfy Eq. (4.21) we can restrict to various sets of non-zero coefficients and solve the functional equations that arise. In an attempt to capture 'diagonal' states we shall consider only three coefficients turned on. The solutions in this class of states can be subdivided into *embedded SU(2) solutions*. That is, solutions where the non-zero coefficients occur for only two (out of the three) different flavours and solutions for which all the flavours occur in the state decomposition.

Solving the functional equations that Eq. (4.21) implies for the embedded SU(2) ansätze gives states of the form

$$\langle K(u)| = \langle i,i|\,\lambda u^+ + \langle i,j|\,\big(u^+ - \xi\big) + \langle j,i|\,\big(u^+ + \xi\big)$$

for $i \neq j = 1,2,3$. These are exactly the boundary states for SU(2) Eq. (4.11).

The other possibility gives two types of constant states, i.e. with only an overall u-dependent

$$\langle i,i|\,\lambda + \langle j,k| + \langle k,j|\,,$$

for $i \neq j \neq k = 1,2,3$ and a diagonal state

$$\langle 1,1| + \langle 2,2|\,\lambda + \langle 3,3|\,\mu\,. \tag{4.22}$$

This is expected since any invertible solution to the soliton non-preserving reflection equation is constant [123]. In fact, all invertible solutions can be brought into a diagonal form by SU(3) symmetry [122].

The D3⊥D5 MPS in the $\mathfrak{su}(3)$ sector is

$$\langle \mathrm{D}_k| = \mathrm{tr}_a\Big(\prod_{x=1}^{L}\sum_{i=1}^{3} t_i^{(k)} \otimes \langle i|_x\Big)\,, \tag{4.23}$$

where the trace is over the k-dimensional space of the $t_i^{(k)}$ matrices. For $L = 4$ it has been checked that the diagonal state Eq. (4.22) is cohomologically equivalent to the $k = 2$ MPS Eq. (4.23) in the following sense: $\mathfrak{su}(3)$ have three raising operators E_2^1, E_3^1, E_3^2 changing between the flavours. The on-shell Bethe states are highest weight states and therefore annihilated by the raising operators. The $\mathfrak{su}(3)$ algebra implies that $E_3^1 = [E_2^1, E_3^2]$. It follows that if states are related by

$$|A\rangle = |B\rangle + E_2^1\,|\Psi_1\rangle + E_3^2\,|\Psi_2\rangle\ ,$$

where $|\Psi_i\rangle$ $i = 1, 2$ are arbitrary states, they have the same overlap with the on-shell Bethe states

$$\langle A|\{u_i\}, \{w_i\}\rangle = \langle B|\{u_i\}, \{w_i\}\rangle\,.$$

Similar to the case for $\mathfrak{su}(2)$, we shall in fact only require this for projections onto subsectors (M, N) for $M \leq {}^{L}/_{2}$ and $N \leq {}^{M}/_{2}$. To check whether $\mathfrak{su}(3)$ states are cohomological, we need to find two states $|\Psi_i\rangle \in \mathbb{C}^3$. Therefore, checking cohomology by brute force quickly becomes unfeasible for increasing lengths. In the next section, we shall discuss a recent approach to constructing the matrix product states from integrable boundary states circumventing the need for cohomology arguments. This have already been accomplished for the SU(3) MPS for for any k [124].

4.4 A higher spin SO(6)-sector boundary state

In many cases, the matrix product states *are* solutions of the twisted boundary Yang-Baxter equation evaluated at a specific rapidity [104, 122, 124, 125]. The reflection equations Eq. (4.3) and Eq. (4.16) also have matrix solutions in the sense that the components of the K-matrix are operators acting on an

additional quantum space defining boundary degrees of freedom for the spin chain.

Let $\psi_{ab}(u)$ be a matrix solution of the soliton non-preserving reflection equation, then we can construct a two-site invariant integrable matrix product boundary state

$$\mathrm{tr}_b \bigotimes_{k=1}^{L/2} \psi_{ij}(u_k) \langle i| \langle j| = \sum_{\{s_i\}} \langle s_1 \dots s_L| \, \mathrm{tr}_b\big[\psi_{s_1 s_2}(u_1) \cdots \psi_{s_{L-1} s_L}(u_{L/2})\big] \,,$$

where the trace is over the boundary quantum space. Furthermore, if the solution satisfy $\psi_{ij}(0) = t_i^{(k)} t_j^{(k)}$, we recover the D3⊥D5 matrix product state when the inhomogeneities vanish. A solution that corresponds to the matrix product state Eq. (4.23) in the $\mathfrak{su}(3)$-sector for any dimension k of the representation is known [124]. In the $\mathfrak{so}(6)$-sector a solution for $k = 2$ is known in a different, albeit related, context of open spin chains in the D3⊥D5 setup at $k = 0$. Here the boundary degrees of freedom correspond to fundamental operators on the defect [126]. Furthermore for the SO(5) symmetric D3⊥D7 a solution has also been found [124].

In this section, I will describe another solution of the reflection equation for SO(6) that generalises the DeWolfe-Mann solution to higher-dimensional representations for the boundary spin. Hereby, we embed the SO(6) sector matrix product state in a solution of the boundary Yang-Baxter equation.

The R-matrix for the integrable spin chain with SO(6) symmetry is[2] [15]

$$R_{ab}(u) = (u+2)P + u(u+2)I - uK \,.$$

This R-matrix is crossing symmetric satisfying the identity

$$R^{t_1}(u) = R(-u-2) \,.$$

As discussed in section 4.3.1, this implies that solutions of the reflection equations Eq. (4.3) and Eq. (4.16) are related by a simply crossing transformation

$$\psi(u) = K(-u-1) \,.$$

In particular, there is no distinction between K and ψ for the SO(6) equations.

[2]The R-matrix used here differs from [15, 126] by a factor $\frac{1}{2}$ and change of sign in the rapidity $u \to -u$.

The most general ansatz for a $K(u)$-matrix that breaks the $SO(6)$ symmetry of the bulk chain to $SO(3) \times SO(3)$ at the boundary takes the form

$$\begin{aligned} K_{ij}(u) &= f(u)\delta_{ij} + g(u)t_i^{(k)}t_j^{(k)} + \tilde{g}(u)t_j^{(k)}t_i^{(k)}\,, \\ K_{iJ}(u) &= K_{Ij}(u) = 0\,, \\ K_{IJ}(u) &= h(u)\delta_{IJ}\,, \end{aligned} \tag{4.24}$$

where $i, j = 1, 2, 3$ and $I, J = 4, 5, 6$ and $t_i^{(k)}$ are the k-dimensional representation of the $\mathfrak{su}(2)$ algebra.

Inserting the K-matrix ansatz Eq. (4.24) into the reflection equation Eq. (4.3) produces functional equations in u and v that $f(u), g(u), \tilde{g}(u)$ and $h(u)$ are to satisfy. In the following, we shall employ a trick. Since we are after solutions that factorise $K_{ij}(0) = t_i t_j$, we can solve the equation for $v = 0$. Since the matrices factorise, we can simply solve the equation for moving the t_i across the K-matrix (see Fig. 4.9) [124]. In components the square root

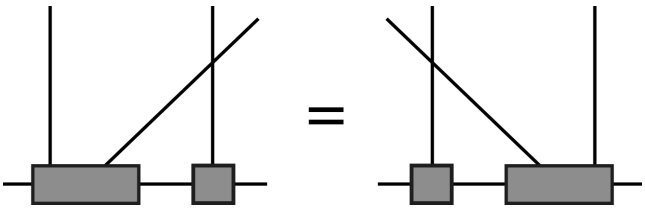

Figure 4.9 – The square root relation for moving the t_i across the K-block.

relation then takes the form

$$\sum_{\alpha,\beta} R^{ab}_{\alpha\beta}(u)K_{c\beta}(u)\omega_\alpha = \sum_{\alpha,\beta} R^{ca}_{\alpha\beta}(u)\omega_\beta K_{\alpha b}(u)\,, \tag{4.25}$$

where the three independent indices can take values $a, b, c \in \{1, 2, 3, 4, 5, 6\}$. We shall refer to $\{1, 2, 3\}$ as $SO(3)_{\mathrm{H}}$ and $\{4, 5, 6\}$ as $SO(3)_{\mathrm{V}}$. The matrices ω_α are given as

$$\omega_\alpha = \begin{cases} t_\alpha & \alpha \in SO(3)_{\mathrm{H}} \\ 0 & \alpha \in SO(3)_{\mathrm{V}} \end{cases}.$$

The equations Eq. (4.25) vanish for certain configurations of the free indices: $a, b, c \in SO(3)_{\mathrm{V}}$, for two of the indices in $SO(3)_{\mathrm{H}}$ and the third in $SO(3)_{\mathrm{V}}$, and for $a \in SO(3)_{\mathrm{H}}$ and $b, c \in SO(3)_{\mathrm{V}}$. This leaves two possibilities.

For either $b \in SO(3)_{\mathrm{H}}$ or $c \in SO(3)_{\mathrm{H}}$ and the rest in $SO(3)_{\mathrm{V}}$ we have the equations

$$\Big[ug(u)\tfrac{k^2-1}{4} + u\tilde{g}(u)\big(\tfrac{k^2-1}{4} - 1\big) + uf(u) + (u+2)h(u)\Big]t_a^{(k)} = 0\,. \tag{4.26}$$

Here $\frac{k^2-1}{4}\mathbb{1} = \sum_i (t_i^{(k)})^2$ is the quadratic Casimir operator and $a = 1, 2, 3$. This conditions involves all four functions.

Three conditions arise for all indices $a, b, c \in SO(3)_{\mathrm{H}}$:

$$\begin{aligned}(u+2)\Big[ug(u) + (u-1)\tilde{g}(u)\Big]\{t_a^{(k)}, [t_a^{(k)}, t_b^{(k)}]\} = 0\,,\\ (u+2)\Big[ug(u) + (u-1)\tilde{g}(u)\Big]\mathrm{i}((t_a^{(k)})^2 - (t_b^{(k)})^2) = 0\end{aligned} \tag{4.27}$$

for $a \neq b = 1, 2, 3$, and

$$\begin{aligned}\Big[\tilde{g}(u)\big(u\tfrac{k^2-1}{4} + u(u+1)\big) + ug(u)\tfrac{k^2-1}{4} + 2(u+1)f(u)\Big]t_c^{(k)}\\ - \mathrm{i}\Big[ug(u) + (u-1)\tilde{g}(u)\Big]t_a^{(k)}t_b^{(k)} = 0\end{aligned} \tag{4.28}$$

for $a \neq b \neq c = 1, 2, 3$. Note that repeated indices on the matrices do not imply summation.

The equations Eq. (4.27) determine the functions $g, \tilde{g}$ up to an arbitrary function of u, i.e. $g(u) = q(u)(u-1)$ and $\tilde{g}(u) = -q(u)u$. Inserting these expressions into Eq. (4.28) determines $f(u)$ and finally Eq. (4.26) fixes $h(u)$. Therefore, we find a solution of the square root relation given by

$$\begin{aligned}f(u) &= u\left(u^2 + u + \frac{k^2-1}{4}\right),\\ h(u) &= -u\left(u^2 + u - \frac{k^2-1}{4}\right),\\ g(u) &= 2(u+1)(u-1)\,,\\ \tilde{g}(u) &= -2u(u+1)\,.\end{aligned} \tag{4.29}$$

We have chosen $q(u) = 2(u+1)$ for aesthetics, since any solution of the reflection equation Eq. (4.3) is determined up to an arbitrary overall function of u .

The square root relation Eq. (4.25) is *not* the boundary Yang-Baxter equation. However, a solution of the square root relation is a solution of the boundary Yang-Baxter equation at $v = 0$. Aided by CAS software, it is now straightforward to verify that Eq. (4.24) with Eq. (4.29) do provide a solution of the boundary Yang-Baxter equation. Furthermore, for $k = 2$ the solutions reduces to the DeWolfe-Mann solution[3] [124, 126].

[3]Note the overall factor of $\frac{1}{4}(2u-1)$ compared to the K-matrix for $k = 2$ in [124, 126].

The K-matrix Eq. (4.24) with Eq. (4.29) can be used to construct an open integrable spin chain with $SO(6)$ bulk symmetry that is broken to $SO(3) \times SO(3)$ at the boundary as discussed in section 4.1. Spin chains of this type have only recently been addressed [127, 128] and this spin chain in particular has, to my knowledge, not been studied before. A novel feature of this solution compared to SU(3) (cf. [124]) is that it depends on the dimension of the representation k through the quadratic Casimir. This is a welcomed feature in the study of wrapping corrections for the one-point functions as we expect the reflection factors off of the boundary to encode this information [58].

In section 4.3.2, we discussed the possibility to use the integrable matrix product state in the vertex model based arguments to prove the overlap formulas. SO(6) vertex models have previously been used for correlation function computation in $\mathcal{N} = 4$ SYM [27, 129]. It will be interesting to investigate whether the partition function of the vertex model described in [27] with a partial reflecting end given by the K-matrix Eq. (4.24) with Eq. (4.29) can be determined by the Izergin-Korepin technique or other methods [130,131]. A determinant expression for the overlaps are known [56] so this amounts to determining appropriate conditions for the partition function and verifying that the known determinant satisfy these.

5 Two-point functions and the conformal bootstrap

In this chapter, I shall review how to compute two-point functions in the D3⊥D5$_k$ defect theory. I shall summarise the results of [42] in which we initiated the computation of two-point functions in the D3⊥D5$_k$ defect theory. We shall adhere to the conventions of the paper [42]. In particular bulk and boundary refers respectively to $x_\perp > 0$ and $x_\perp = 0$ as opposed to its use in holography.

5.1 Defect conformal constraints

The D3⊥D5$_k$ defect theory is believed to be a defect conformal field theory dCFT. Defect or boundary conformal field theories are conformal field theories with a boundary that preserves part of the conformal symmetry of the bulk theory [132–134]. The residual conformal symmetry puts strong constraints on the form of correlation functions [135, 136].

In particular, the spacetime dependence of one-point functions of scalar primaries is completely fixed

$$\langle \mathcal{O}_i(x) \rangle = \frac{a_i}{(2x_\perp)^{\Delta_i}} ,$$

where Δ_i is the scaling dimension of the operator $\mathcal{O}_i$. More generally the two-point function of a bulk and a boundary operator takes the form to be

$$\langle \mathcal{O}_i(x) \hat{\mathcal{O}}_j(\boldsymbol{y}) \rangle = \frac{\mu_{ij}}{(2x_\perp)^{\Delta_i - \Delta_j} (\eta)^{\Delta_j}}$$

where $\eta = x_\perp^2 + (\boldsymbol{y} - \boldsymbol{x})^2$ is between the operators. In a defect conformal field theory, the two-point function between bulk operators of different conformal

dimensions need not vanish. In fact, the defect conformal symmetry allows it to depend on a single conformal cross ratio:

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle = \frac{f_{ij}(\xi)}{(2x_\perp)^{\Delta_i}(2y_\perp)^{\Delta_j}} \tag{5.1}$$

where

$$\xi = \frac{|x-y|^2}{4x_\perp y_\perp}\,.$$

Away from the boundary, the operator product expansion of two bulk operators takes the form

$$\mathcal{O}_i(x)\mathcal{O}_j(y) = \frac{M_{ij}}{|x-y|^{\Delta_{ij}}} + \sum_k \frac{\lambda_{ij}^k}{|x-y|^{\Delta_{ijk}}} C(x-y,\partial_y)\mathcal{O}_k(y) \tag{5.2}$$

where $\Delta_{ij} := \Delta_i + \Delta_j$ and $\Delta_{ijk} := \Delta_i + \Delta_j - \Delta_k$ and λ_{ij}^k are the structure constants. The sum runs over the conformal primary operators, while $C(x-y,\partial_y) = 1 + \ldots$ contains the contributions from the conformal descendents.

There exist defect operators $\hat{\mathcal{O}}(\boldsymbol{x})$ at $x_\perp = 0$. The correlation functions of boundary operators define a conformal field theory on the defect. This means that the spacetime dependence of the two- and three-point functions of boundary operators are determined by their scaling dimensions $\hat{\Delta}_i$ and three-point functions $\hat{\lambda}_{ijk}$. Near the boundary a bulk operator is resolved into boundary operators

$$\mathcal{O}_i(x) = \sum_j \frac{\mu_i^j}{(2x_\perp)^{\Delta_i-\Delta_j}} \hat{C}(x_\perp,\partial_{\boldsymbol{x}})\hat{\mathcal{O}}_j(\boldsymbol{x})\,,$$

where the sum is organised by conformal primaries and where $\hat{C} = 1 + \ldots$ takes the contributions from the full multiplet into account .

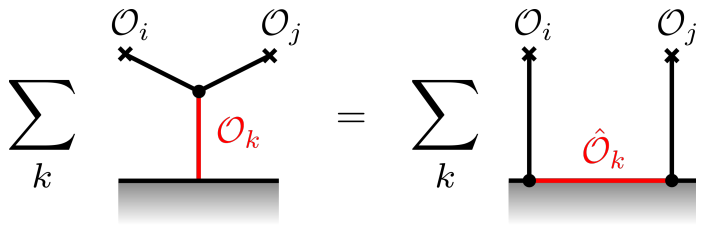

Figure 5.1 – The boundary bootstrap equations.

From the operator expansions we see that the two-point function Eq. (5.1) can be expanded in two channels, one for the bulk and one for the boundary

OPE. Thus $f_{ij}(\xi)$ can be expanded as

$$f_{ij}(\xi) = \xi^{-\frac{\Delta_i+\Delta_j}{2}} \sum_k \lambda_{ijk} a_k F_{\text{bulk}}(\Delta_k, \xi) = \sum_k \mu_{ik}\mu_{jk} F_{\text{bdy}}(\Delta_k, \xi)\,.$$

Here $F_{\text{bulk}}(\Delta_k, \xi)$ and $F_{\text{bdy}}(\Delta_k, \xi)$ are respectively the bulk and boundary conformal blocks. The bulk and boundary conformal blocks are known hypergeometric functions [132].

5.2 Computing two-point functions in D3⊥D5$_k$

In this section we shall discuss the computation of the leading contribution to the connected two-point correlation function. The full two-point correlation functions is decomposed as the connected two-point correlation plus the product of one-point functions. In other words

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle_{\text{c}} = \langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle - \langle \mathcal{O}_i(x)\rangle\langle \mathcal{O}_j(y)\rangle\,.$$

Perturbation theory for D3⊥D5$_k$ was studied in [43] where they found the diagonalisation of the action providing the explicit spectrum and propagators. Table 5.1 shows the spectrum of the scalars [42, 43]. The adjoint fields Φ are written in components as

$$\Phi = \begin{bmatrix} \Phi_{\ell,m}\hat{Y}^m_\ell & [\Phi]_{n,a}E^n_a \\ [\Phi]_{a,n}E^a_n & [\Phi]_{a,a'}E^a_a \end{bmatrix},$$

where E^j_i denotes the unit matrices, i.e. 1 at i, j and 0 elsewhere, and $\hat{Y}^m_\ell$ are the fuzzy spherical harmonics [43]. And the propagator for modes in table 5.1 is [43]

$$K^\nu(x,y) = \frac{g^2_{\text{YM}}}{16\pi^2}\frac{1}{\binom{2\nu+1}{\nu+\frac{1}{2}}}\frac{{}_2F_1(\nu-\frac{1}{2},\nu+\frac{1}{2};2\nu+1;-\xi^{-1})}{(1+\xi)\xi^{\nu+\frac{1}{2}}}\frac{1}{x_\perp y_\perp}\,. \tag{5.3}$$

The massless modes in the $k\times k$ block, i.e. the first row for $\ell = 0$, should not propagate through the defect. The propagator K^ν Eq. (5.3) satisfy Dirichlet boundary conditions for $\nu = \frac{1}{2}$ and Neumann for $\nu = -\frac{1}{2}$. This is consistent with 1/2-BPS boundary conditions for the case when the gauge group is U($N-k$) on the defect [44].

Consider the complex scalar $Z = \phi_3 + \mathrm{i}\phi_6$. The leading contribution to the connected part of the bulk two-point function $\langle \operatorname{tr} Z^J(x) \operatorname{tr} Z^L(y)\rangle$ comes from

#	$\phi_{4,5,6}$ (ν)	$\phi_{1,2,3}$ (ν)
$\ell+1$	$\ell+\frac{1}{2}$	$\ell-\frac{1}{2}$
ℓ	$\ell+\frac{1}{2}$	$\ell+\frac{3}{2}$
$(k+1)(N-k)$	$\frac{k}{2}$	$\frac{k-2}{2}$
$(k-1)(N-k)$	$\frac{k}{2}$	$\frac{k+2}{2}$

Table 5.1 – The spectrum (multiplicity and masses) of scalars in the D3⊥D5$_k$ field theory for $x_\perp > 0$. Here $\ell = 0, \ldots, k-1$ and additionally contains $(N-k)^2$ massless modes that propagates through the defect. The masses are given as $\nu = \sqrt{m^2 + \frac{1}{4}}$.

connecting the two composite operators via a propagator. Each insertion point yields the same contribution since the trace is cyclic. Therefore, we find

$$\langle \operatorname{tr} Z^J(x) \operatorname{tr} Z^J(y)\rangle_{\mathrm{c}} = JL \operatorname{tr}((Z^{\mathrm{cl}})^{J-1}\overbrace{Z)(x) \operatorname{tr}(\overbrace{Z}(Z^{\mathrm{cl}})^{L-1})(y)\,.$$

The Zs that are not contracted gives $Z^{\mathrm{cl}}(x) = -\frac{1}{x_\perp} t_3^{(k)} \oplus \mathbb{1}_{(N-k)\times 2}$. The contraction of complex adjoint scalars gives a propagator for each possible mode. As the classical fields are block diagonal and trivial in the $(N-k)\times(N-k)$ block, we need only the $k \times k$ block propagators. Therefore,

$$\begin{aligned}&\langle \operatorname{tr} Z^J(x) \operatorname{tr} Z^L(y)\rangle_{\mathrm{c}} = \\ &\qquad \frac{(-1)^{J+L} JL}{x_\perp^{J-1} y_\perp^{L-1}} \sum_{\ell,\ell',m,m'} \operatorname{tr}(t_3^{J-1} \hat{Y}_\ell^m) \operatorname{tr}(t_3^{L-1} \hat{Y}_{\ell'}^{m'}) \langle Z_{\ell,m}(x) Z_{\ell',m'}(y)\rangle\,. \quad (5.4)\end{aligned}$$

The expression for the two-point function thus obtained Eq. (5.4) contains two things that we need to address, the complex scalar propagator and the colour trace factors. Let us first consider the propagator.

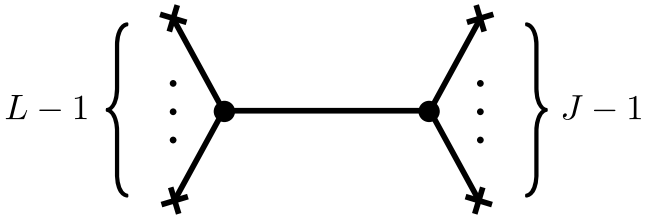

Figure 5.2 – Tree-level diagram for the two-point functions. The local operators are connected by a single propagator.

The propagator can be expressed in terms of scalar propagators found in [43]. Note, there is no mixing between easy and complicated flavours. Therefore,

$$\langle Z_{\ell m}(x) Z_{\ell' m'}(y)\rangle = \langle (\phi_3)_{\ell m}(x)(\phi_3)_{\ell' m'}(y)\rangle - \langle (\phi_6)_{\ell m}(x)(\phi_6)_{\ell' m'}(y)\rangle\,.$$

The scalar propagators for the complicated field ϕ_3 are

$$\langle(\phi_3)_{\ell m}(x)(\phi_3)^\dagger_{\ell' m'}(y)\rangle = \delta_{\ell\ell'}\delta_{mm'}\Big(\frac{\ell+1}{2\ell+1}K^{m^2=\ell(\ell-1)}+\frac{\ell}{2\ell+1}K^{m^2=(\ell+1)(\ell+2)}\Big)\,,$$

while the propagators for the easy field ϕ_6 are

$$\langle(\phi_6)_{\ell m}(x)(\phi_6)^\dagger_{\ell' m'}(y)\rangle = \delta_{\ell\ell'}\delta_{mm'}K^{m^2=\ell(\ell+1)}\,.$$

One should beware of the conjugation properties of the fuzzy spherical harmonics $\hat{Y}^m_\ell$ implies

$$\langle(\phi_i)_{\ell m}(\phi_j)_{\ell' m'}\rangle = (-1)^{m'}\langle(\phi_i)_{\ell m}(\phi_j)^\dagger_{\ell' -m'}\rangle\,.$$

Using a variety of identities relating the hypergeometric function ${}_2F_1$ at various values it is possible to write the complex scalar propagator as a single hypergeometric function

$$\langle Z_{\ell,m}(x)Z_{\ell',m'}(y)\rangle = \delta_{\ell\ell'}\delta_{m+m',0}\frac{g^2_{\rm YM}}{16\pi^2}\frac{(-1)^m}{x_\perp y_\perp}\frac{{}_2F_1(\ell,\ell+1;2\ell+2;-\xi^{-1})}{\binom{2\ell+1}{\ell+1}\xi^{\ell+1}}\frac{\xi}{\xi+1}\,.$$

Colour trace factors involving the matrix structure of the classical solutions $t^{(k)}_i$ for $i=1,2,3$ and the fuzzy spherical harmonics $\hat{Y}^m_\ell$ are ubiquitous in weak coupling expansions of correlation functions in the D3⊥D5$_k$ field theory. For the connected two-point function at leading order Eq. (5.4) we need the traces $\mathrm{tr}(t_3^L\hat{Y}^m_\ell\ )$.

As matrices, the generators t_i can be expressed in terms of the fuzzy spherical harmonics $\hat{Y}^m_\ell$. For the diagonal generator, we have $t_3\sim\hat{Y}^0_\ell$. Therefore, by using that the fuzzy spherical harmonics satisfy an algebra [137]

$$\hat{Y}^{m_1}_{\ell_1}\hat{Y}^{m_2}_{\ell_2} = \sum F^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2}\hat{Y}^{m_2}_{\ell_2}$$

where the fusion coefficients can be expressed in terms of Wigner's $3j$ and $6j$ symbols

$$F^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2} = (-1)^{m_3}\prod_i(-1)^{\ell_i}\sqrt{2\ell_i+1}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & -m_3\end{pmatrix}\begin{Bmatrix}\ell_1 & \ell_2 & \ell_3\\ \frac{k-1}{2} & \frac{k-1}{2} & \frac{k-1}{2}\end{Bmatrix}.$$

Wigner $3j$ symbols are proportional to Clebsh-Gordon coefficients and therefore provides the selection rules for coupling spins ℓ_1 and ℓ_2 to ℓ_3.

Hence the expansion of t_3^L will contain only $\hat{Y}_\ell^0$. Let us introduce the coefficients

$$\alpha_\ell^L := \mathrm{tr}(t_3^L \hat{Y}_\ell^0\ ) .$$

In principle, we can express the trace as sums of products of fusion coefficients by expressing t_3 in terms of $\hat{Y}_\ell^0$ and successively fusing the fuzzy spherical harmonics.

To determine the αs, we proposed an approach based on a numerically more efficient recursion relation [42]. The basic idea is: The traces $\mathrm{tr}\, t_3^{L+m}$ can be written as a Bernoulli polynomial

$$\mathrm{tr}\, t_3^{L+m} = -2\frac{B_{L+m+1}(\frac{k-1}{2})}{L+m+1}$$

and each factor $t_3^L t_3^m$ can individually be expanded in $\hat{Y}_\ell^0$. Using the orthogonality of the fuzzy spherical harmonics, we have that

$$-2\frac{B_{L+m+1}(\frac{k-1}{2})}{L+m+1} = \sum_{\ell=0}^{m} \alpha_\ell^L \alpha_\ell^m ,$$

where we have assumed that $L \geq m$. The last term of the sum, $\ell = m$, produces $\alpha_m^L \alpha_m^m$. It follows that α_m^L can be expressed in terms of αs with lower L, m. To start the recursion we should note that α_m^m and α_0^L are known in general. Furthermore, the recursion relation can be solved to find the αs as linear combinations of Bernoulli polynomials

$$\alpha_m^L \sim \sum_{\ell=0}^{m} \tilde{\beta}_m^{(\ell)} \frac{B_{L+\ell+1}(\frac{k-1}{2})}{L+\ell+1} .$$

For the explicit expressions see Appendix C in [42].

Now that we have determined the propagator and the colour trace factors we may insert them in expression Eq. (5.4) to obtain an expression for the connected two-point function

$$\langle \mathrm{tr}\, Z^J(x)\, \mathrm{tr}\, Z^L(y)\rangle_{\mathrm{c}} = \frac{g_{\mathrm{YM}}^2}{16\pi^2}\frac{JL}{x_\perp^J y_\perp^L}\sum_\ell \frac{\alpha_\ell^{J-1}\alpha_\ell^{L-1}}{\binom{2\ell+1}{\ell+1}} \frac{{}_2F_1(\ell,\ell+1;2\ell+2;-\xi^{-1})}{\xi^\ell(\xi+1)} .$$

Let me remark that the two-point function takes the form that is expected from defect conformal invariance Eq. (5.1). Via similar computations we

have likewise found the leading term of the connected two-point functions: $\langle \operatorname{tr} Z^{J_1} \operatorname{tr} \bar{Z}^{J_2} \rangle$, $\langle \operatorname{tr} Z^{J_1} \operatorname{tr} X^{J_2} \rangle$ and $\langle \operatorname{tr} Z^{J_1} \operatorname{tr} X^{J_2-1} \bar{Z} \rangle$ as well as their large k limit [42].

5.2.1 Conformal data from the OPE

As discussed in section 5.1 the two-point functions in a defect conformal field theory can be decomposed into conformal blocks using the two available operator product expansions. In the paper [42], we studied the constraints imposed by conformal symmetry and what conformal data we could extract from the connected two-point functions of the BPS operators at leading order.

Since the OPE expansion Eq. (5.2) is valid for the full two-point function, we have to include the one-point functions. E.g. for the $\langle \operatorname{tr} Z^{J_1}(x) \operatorname{tr} Z^{J_2}(y) \rangle$ we have

$$\begin{aligned} f(\xi) &= a_{\operatorname{tr} Z^{J_1}} a_{\operatorname{tr} Z^{J_2}} \\ &\quad + 2^{J_1+J_2} J_1 J_2 \frac{g_{\mathrm{YM}}^2}{16\pi^2} \sum_{\ell} \frac{\alpha_\ell^{J_1-1} \alpha_\ell^{J_1-1}}{\binom{2\ell+1}{\ell+1}} \frac{{}_2F_1(\ell, \ell+1; 2\ell+2; -\xi^{-1})}{\xi^\ell(\xi+1)} + \mathrm{O}(g_{\mathrm{YM}}^4) \,. \end{aligned}$$

Note that the one-point functions of the BMN vacuum $a_{\operatorname{tr} Z^J}$ is known to one loop [43]. At leading order, the OPE equates the product of tree-level one-point functions with a sum of tree-point functions and one-point functions. Organising the sum over conformal primaries according to their dimension Δ leads to sum rules for the structure constants at leading order

$$\sum_i \lambda^{(0)}{}_{\operatorname{tr} Z^{J_1} \operatorname{tr} Z^{J_2}}{}^{\mathcal{O}_i} a_i^{(0)} = 2^{J+2} \frac{\binom{\frac{\Delta+\delta J}{2}-1}\binom{\frac{\Delta-\delta J}{2}-1}}{\binom{\Delta-3}{\frac{\Delta-J}{2}}} \frac{B_{J_1+1}(\frac{1-k}{2}) B_{J_2+1}(\frac{1-k}{2})}{(J_1+1)(J_2+1)} \,, \tag{5.5}$$

where $J = J_1 + J_2$ and $\delta J = J_1 - J_2$ are the sum and difference of the dimensions J_i of the BMN vacuum $\operatorname{tr} Z^{J_i}$. The sum runs over the conformal primaries $\mathcal{O}_i$ with dimension Δ.

In the sum rule Eq. (5.5) the k-dependence, related to the defect, must come from the one-point functions since the bulk three-point functions are local properties independent of the defect. In general the one-point functions of $SU(2)$ Bethe states and their descendents are polynomial in k of degree $L - M + 1$. Therefore, only descendents of the vacuum state carries the k^{L+1}. This allowed us to fix various simple three-point functions [42]. At one-loop

we need the connected two-point function. The conformal blocks depends on the conformal data and must likewise be expanded in g_{YM}^2. For small BPS operators, we were able to obtain one-loop one-point functions of multi-trace operators [42]. Similar sum rules can be derived from the boundary OPE. They impose restrictions the coupling of single-trace operators to the defect [42].

5.3 Discussion & Outlook

In this chapter, I have discussed the results published in [42]. This includes computation of the leading contribution to the connected two-point functions in the D3⊥D5$_k$ theory using the perturbative framework of [43]. The constraints imposed by conformal invariance on the operator product expansion allows us to extract conformal data from the two-point function in simple cases. We have not found structure constants that was not already known or as easy to compute using Feynman diagrammatics. However, let me offer another perspective on the computations described in this chapter: They support the assumption that the theory is defect conformal invariant as a quantum field theory. While the theory for $k = 0$ have been shown to preserve defect conformal invariance at the quantum level [47] no similar argument exists for $k \neq 0$. It could potentially be interesting to study the D3⊥D7$_{k_1,k_2}$ theory where the propagators have recently been found [51]. It would certainly be interesting to solve for more involved structure constants e.g. using the analytical bootstrap [138].

In computing the two-point functions we did not make use of the integrability properties of the theory. In a defect conformal field theory the two-point functions are analogous to four-point functions. In $\mathcal{N} = 4$ SYM four-point functions at weak coupling have been computed using integrability [25]. The two-point functions of a SU(2) operator and a length 2 operator are known [139]. It would be interesting to further develop the techniques for computing two-point functions in the D3⊥D5$_k$ theory. Let me note than an immediate obstacle is that the contractions between operators do not 'cut' the spin chains in to local Bethe states.

6 Concluding Remarks

Though the chapters discuss various aspects of different theories, they are all part of a bigger picture. The AdS_5/CFT_4 correspondence, though not yet proven, is widely believed to be true particularly due to our ever-increasing proficiency in computing and comparing observables, especially in the planar limit. Many of these techniques exploit the symmetries of the setup. Introducing defects or boundaries generically breaks symmetries and are therefore interesting for testing the limits of our tools.

The aim of this thesis have been to further the understanding the AdS/dCFT correspondence by computing observables. In this we have succeeded. As a result of the one-point function computations described in chapter 3 we now know that the D3$\perp$D7$_{k_1,k_2}$ field theory differs qualitatively from both the related D3$\perp$D7$_n$ and the supersymmetric D3$\perp$D5$_k$. The D3$\perp$D7$_{k_1,k_2}$ defect have non-zero one-point functions for paired operators, and for the matrix product state this means that it is not an integrable spin chain state. Nevertheless, we have been able to conjecture overlaps for large states and it is interesting to speculate that this is related to integrability.

One-point functions are given by the overlap between a matrix product state and a Bethe eigenstate. These overlap formulae are useful in the study of the dynamics of integrable spin chains. As we discussed in chapter 4, recent developments in spin chain quantum quenches are using these overlaps. This have provided an interesting cross-fertilisation. The definition of integrable initial states for quantum quenches gives us a classification of matrix product states obtained from fuzzy funnel solutions. Using this definition D3$\perp$D7$_{k_1,k_2}$ is not integrable. However, D3$\perp$D5$_k$ certainly is, and in particular in chapter 4 we described a solution of the boundary Yang-Baxter equation that gives the matrix product state of D3$\perp$D5$_k$ for the $SO(6)$-sector. Answers prompts questions, as per usual: Do the characterisation that the matrix product state

is integrable coincide with other characterisations of integrability e.g. does it correspond to the string world-sheet theory having an integrable boundary?

Chapter 5 discussed the results on computation of two-point functions in D3⊥D5$_k$. The aim for this investigation was to clarify how the one-point functions already known in the setup fitted into the OPEs of defect conformal field theories. Here we showed that it was possible to obtain certain conformal data using the two-point functions and a minimal knowledge of the spectrum.

Now, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory may be exactly solvable however there is still a long way ahead before all conformal data is readily available. A current goal in integrability of $\mathcal{N} = 4$ is to write the correlation functions in terms of Q-functions from the quantum spectral curve. One-point functions are potentially amongst the easiest case where one can study this in general. It is my hope that the boundary Yang-Baxter solution obtained in chapter 4 will further the computation of wrapping corrections for the D3⊥D5$_k$ one-point functions.

References

[1] L. Brink, J. H. Schwarz, and J. Scherk, "Supersymmetric Yang-Mills theories," *Nuclear Physics B* **121** no. 1, (1977) 77–92. http://www.sciencedirect.com/science/article/pii/0550321377903285.

[2] M. F. Sohnius and P. C. West, "Conformal Invariance in N=4 Supersymmetric Yang-Mills Theory," *Phys. Lett.* **100B** (1981) 245.

[3] S. Mandelstam, "Light Cone Superspace and the Ultraviolet Finiteness of the N=4 Model," *Nucl. Phys.* **B213** (1983) 149–168.

[4] G. 't Hooft, "A planar diagram theory for strong interactions," *Nuclear Physics B* **72** no. 3, (1974) 461–473.

[5] N. Arkani-Hamed, F. Cachazo, and J. Kaplan, "What is the Simplest Quantum Field Theory?," *JHEP* **09** (2010) 016, arXiv:0808.1446 [hep-th].

[6] A. Hanany and E. Witten, "Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics," *Nucl. Phys.* **B492** (1997) 152–190, arXiv:hep-th/9611230 [hep-th].

[7] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," *Physics Letters B* **428** no. 1, (1998) 105–114. http://www.sciencedirect.com/science/article/pii/S0370269398003773.

[8] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, arXiv:hep-th/9802150 [hep-th].

[9] J. Maldacena, "The large-N limit of superconformal field theories and supergravity," *International journal of theoretical physics* **38** no. 4, (1999) 1113–1133.

[10] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Physics Reports* **323** no. 3, (2000) 183–386. http://www.sciencedirect.com/science/article/pii/S0370157399000836.

[11] G. 't Hooft, "Dimensional reduction in quantum gravity," *arXiv preprint gr-qc/9310026* (1993) .

[12] L. Susskind, "The World as a hologram," *J. Math. Phys.* **36** (1995) 6377–6396, arXiv:hep-th/9409089 [hep-th].

[13] I. Bena, J. Polchinski, and R. Roiban, "Hidden symmetries of the $AdS_5 \times S^5$ superstring," *Phys. Rev.* **D69** (2004) 046002, arXiv:hep-th/0305116 [hep-th].

[14] N. Beisert et al., "Review of AdS/CFT Integrability: An Overview," *Letters in Mathematical Physics* **99** no. 1, (Jan, 2012) 3–32. https://doi.org/10.1007/s11005-011-0529-2.

[15] J. A. Minahan and K. Zarembo, "The Bethe-ansatz for $\mathcal{N} = 4$ super Yang-Mills," *Journal of High Energy Physics* **2003** no. 03, (2003) 013.

[16] N. Beisert, "The complete one-loop dilatation operator of N= 4 super-Yang–Mills theory," *Nuclear Physics B* **676** no. 1, (2004) 3–42.

[17] N. Beisert, C. Kristjansen, and M. Staudacher, "The dilatation operator of conformal N= 4 super-Yang–Mills theory," *Nuclear Physics B* **664** no. 1, (2003) 131–184.

[18] N. Beisert and M. Staudacher, "The N=4 SYM integrable super spin chain," *Nucl. Phys.* **B670** (2003) 439–463, arXiv:hep-th/0307042 [hep-th].

[19] H. Bethe, "Zur theorie der metalle," *Zeitschrift für Physik* **71** no. 3-4, (1931) 205–226.

[20] J. Ambjørn, R. A. Janik, and C. Kristjansen, "Wrapping interactions and a new source of corrections to the spin-chain/string duality," *Nucl. Phys.* **B736** (2006) 288–301, arXiv:hep-th/0510171 [hep-th].

[21] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, "Quantum Spectral Curve for Planar $\mathcal{N} = 4$ Super-Yang-Mills Theory," *Phys. Rev. Lett.* **112** no. 1, (2014) 011602, arXiv:1305.1939 [hep-th].

[22] C. Marboe and D. Volin, "Quantum spectral curve as a tool for a perturbative quantum field theory," *Nucl. Phys.* **B899** (2015) 810–847, arXiv:1411.4758 [hep-th].

[23] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, "Quantum spectral curve for arbitrary state/operator in AdS_5/CFT_4," *JHEP* **09** (2015) 187, arXiv:1405.4857 [hep-th].

[24] J. Escobedo, N. Gromov, A. Sever, and P. Vieira, "Tailoring Three-Point Functions and Integrability," *JHEP* **09** (2011) 028, arXiv:1012.2475 [hep-th].

[25] J. Caetano and J. Escobedo, "On four-point functions and integrability in N=4 SYM: from weak to strong coupling," *JHEP* **09** (2011) 080, arXiv:1107.5580 [hep-th].

[26] B. Basso, S. Komatsu, and P. Vieira, "Structure Constants and Integrable Bootstrap in Planar N=4 SYM Theory," arXiv:1505.06745 [hep-th].

[27] B. Basso, F. Coronado, S. Komatsu, H. T. Lam, P. Vieira, and D.-l. Zhong, "Asymptotic Four Point Functions," arXiv:1701.04462 [hep-th].

[28] A. Cavagli, N. Gromov, and F. Levkovich-Maslyuk, "Quantum spectral curve and structure constants in $\mathcal{N} = 4$ SYM: cusps in the ladder limit," *JHEP* **10** (2018) 060, arXiv:1802.04237 [hep-th].

[29] T. Bargheer, J. Caetano, T. Fleury, S. Komatsu, and P. Vieira, "Handling Handles: Nonplanar Integrability in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory," *Phys. Rev. Lett.* **121** no. 23, (2018) 231602, arXiv:1711.05326 [hep-th].

[30] L. F. Alday and A. Bissi, "Loop Corrections to Supergravity on $AdS_5 \times S^5$," *Phys. Rev. Lett.* **119** no. 17, (2017) 171601, arXiv:1706.02388 [hep-th].

[31] J. M. Drummond, J. M. Henn, and J. Plefka, "Yangian symmetry of scattering amplitudes in N=4 super Yang-Mills theory," *JHEP* **05** (2009) 046, arXiv:0902.2987 [hep-th].

[32] T. Harmark and M. Wilhelm, "The Hagedorn temperature of AdS_5/CFT_4 at finite coupling via the Quantum Spectral Curve," *Phys. Lett.* **B786** (2018) 53–58, arXiv:1803.04416 [hep-th].

[33] D. Mller, H. Mnkler, J. Plefka, J. Pollok, and K. Zarembo, "Yangian Symmetry of smooth Wilson Loops in $\mathcal{N} = 4$ super Yang-Mills Theory," *JHEP* **11** (2013) 081, arXiv:1309.1676 [hep-th].

[34] N. Beisert, "On Yangian Symmetry in Planar N=4 SYM," in *Quantum chromodynamics and beyond: Gribov-80 memorial volume. Proceedings, Memorial Workshop devoted to the 80th birthday of V.N. Gribov, Trieste, Italy, May 26-28, 2010*, pp. 175–203. 2011. arXiv:1004.5423 [hep-th]. [,413(2010)].

[35] N. Beisert, A. Garus, and M. Rosso, "Yangian Symmetry for the Action of Planar $\mathcal{N} = 4$ Super Yang-Mills and $\mathcal{N} = 6$ Super Chern-Simons Theories," *Phys. Rev.* **D98** no. 4, (2018) 046006, arXiv:1803.06310 [hep-th].

[36] A. Karch and L. Randall, "Localized gravity in string theory," *Phys. Rev. Lett.* **87** (2001) 061601, arXiv:hep-th/0105108 [hep-th].

[37] A. Karch and L. Randall, "Open and closed string interpretation of SUSY CFT's on branes with boundaries," *JHEP* **06** (2001) 063, arXiv:hep-th/0105132 [hep-th].

[38] C. Kristjansen, G. W. Semenoff, and D. Young, "Chiral primary one-point functions in the D3-D7 defect conformal field theory," *JHEP* **01** (2013) 117, arXiv:1210.7015 [hep-th].

[39] R. C. Myers and M. C. Wapler, "Transport Properties of Holographic Defects," *JHEP* **12** (2008) 115, arXiv:0811.0480 [hep-th].

[40] C. Kristjansen and G. W. Semenoff, "Giant D5 Brane Holographic Hall State," *JHEP* **06** (2013) 048, arXiv:1212.5609 [hep-th].

[41] C. Kristjansen and G. W. Semenoff, "The D3-probe-D7 brane holographic fractional topological insulator," *JHEP* **10** (2016) 079, arXiv:1604.08548 [hep-th].

[42] M. de Leeuw, A. C. Ipsen, C. Kristjansen, K. E. Vardinghus, and M. Wilhelm, "Two-point functions in AdS/dCFT and the boundary conformal bootstrap equations," *JHEP* **08** (2017) 020, arXiv:1705.03898 [hep-th].

[43] I. Buhl-Mortensen, M. de Leeuw, A. C. Ipsen, C. Kristjansen, and M. Wilhelm, "A Quantum Check of AdS/dCFT," *JHEP* **01** (2017) 098, arXiv:1611.04603 [hep-th].

[44] D. Gaiotto and E. Witten, "Supersymmetric Boundary Conditions in N=4 Super Yang-Mills Theory," *J. Statist. Phys.* **135** (2009) 789–855, arXiv:0804.2902 [hep-th].

[45] D. Gaiotto and E. Witten, "S-Duality of Boundary Conditions In N=4 Super Yang-Mills Theory," *Adv. Theor. Math. Phys.* **13** no. 3, (2009) 721–896, arXiv:0807.3720 [hep-th].

[46] O. DeWolfe, D. Z. Freedman, and H. Ooguri, "Holography and defect conformal field theories," *Phys. Rev.* **D66** (2002) 025009, arXiv:hep-th/0111135 [hep-th].

[47] J. Erdmenger, Z. Guralnik, and I. Kirsch, "Four-dimensional superconformal theories with interacting boundaries or defects," *Phys. Rev.* **D66** (2002) 025020, arXiv:hep-th/0203020 [hep-th].

[48] N. R. Constable, R. C. Myers, and O. Tafjord, "The Noncommutative bion core," *Phys. Rev.* **D61** (2000) 106009, arXiv:hep-th/9911136 [hep-th].

[49] J. Castelino, S. Lee, and W. Taylor, "Longitudinal five-branes as four spheres in matrix theory," *Nucl. Phys.* **B526** (1998) 334–350, arXiv:hep-th/9712105 [hep-th].

[50] L. F. Alday, J. M. Henn, J. Plefka, and T. Schuster, "Scattering into the fifth dimension of N=4 super Yang-Mills," *JHEP* **01** (2010) 077, arXiv:0908.0684 [hep-th].

[51] A. G. Grau, C. Kristjansen, M. Volk, and M. Wilhelm, "A Quantum Check of Non-Supersymmetric AdS/dCFT," arXiv:1810.11463 [hep-th].

[52] K. Nagasaki and S. Yamaguchi, "Expectation values of chiral primary operators in holographic interface CFT," *Phys. Rev.* **D86** (2012) 086004, arXiv:1205.1674 [hep-th].

[53] M. de Leeuw, C. Kristjansen, and K. Zarembo, "One-point Functions in Defect CFT and Integrability," *JHEP* **08** (2015) 098, arXiv:1506.06958 [hep-th].

[54] I. Buhl-Mortensen, M. de Leeuw, C. Kristjansen, and K. Zarembo, "One-point Functions in AdS/dCFT from Matrix Product States," *JHEP* **02** (2016) 052, arXiv:1512.02532 [hep-th].

[55] M. de Leeuw, C. Kristjansen, and S. Mori, "AdS/dCFT one-point functions of the SU(3) sector," *Phys. Lett.* **B763** (2016) 197–202, arXiv:1607.03123 [hep-th].

[56] M. de Leeuw, C. Kristjansen, and G. Linardopoulos, "Scalar one-point functions and matrix product states of AdS/dCFT," *Phys. Lett.* **B781** (2018) 238–243, arXiv:1802.01598 [hep-th].

[57] I. Buhl-Mortensen, M. de Leeuw, A. C. Ipsen, C. Kristjansen, and M. Wilhelm, "One-loop one-point functions in gauge-gravity dualities with defects," *Phys. Rev. Lett.* **117** no. 23, (2016) 231603, arXiv:1606.01886 [hep-th].

[58] I. Buhl-Mortensen, M. de Leeuw, A. C. Ipsen, C. Kristjansen, and M. Wilhelm, "Asymptotic One-Point Functions in Gauge-String Duality

with Defects," *Phys. Rev. Lett.* **119** no. 26, (2017) 261604, arXiv:1704.07386 [hep-th].

[59] M. de Leeuw, C. Kristjansen, and G. Linardopoulos, "One-point functions of non-protected operators in the SO(5) symmetric D3D7 dCFT," *J. Phys.* **A50** no. 25, (2017) 254001, arXiv:1612.06236 [hep-th].

[60] E. Widén, "One-point functions in β-deformed $\mathcal{N} = 4$ SYM with defect," *JHEP* **11** (2018) 114, arXiv:1804.09514 [hep-th].

[61] J. A. Minahan, "Review of AdS/CFT Integrability, Chapter I.1: Spin Chains in $\mathcal{N}$= 4 Super Yang-Mills," *Letters in Mathematical Physics* **99** no. 1-3, (2012) 33–58.

[62] M. Karabach, G. Müller, H. Gould, and J. Tobochnik, "Introduction to the Bethe Ansatz I," *Computers in Physics* **11** no. 1, (1997) .

[63] M. Karbach, K. Hu, and G. Müller, "Introduction to the Bethe Ansatz II," *Computers in Physics* **12** no. 5, (1998) .

[64] B. Sutherland, *Beautiful Models*. WORLD SCIENTIFIC, 2018/12/31, 2004. https://doi.org/10.1142/5552.

[65] L. D. Faddeev, "How algebraic Bethe ansatz works for integrable model," in *Relativistic gravitation and gravitational radiation. Proceedings, School of Physics, Les Houches, France, September 26-October 6, 1995*, pp. pp. 149–219. 1996. arXiv:hep-th/9605187 [hep-th].

[66] E. K. Sklyanin, L. A. Takhtadzhyan, and L. D. Faddeev, "Quantum inverse problem method. I," *Theoretical and Mathematical Physics* **40** no. 2, (Aug, 1979) 688–706. https://doi.org/10.1007/BF01018718.

[67] M. de Leeuw, A. C. Ipsen, C. Kristjansen, and M. Wilhelm, "Introduction to Integrability and One-point Functions in $\mathcal{N} = 4$ SYM and its Defect Cousin," in *Les Houches Summer School: Integrability: From Statistical Systems to Gauge Theory Les Houches, France, June 6-July 1, 2016*. 2017. arXiv:1708.02525 [hep-th].

[68] M. Gaudin, "Diagonalization of a class of spin hamiltonians,". http://inis.iaea.org/search/search.aspx?orig_q=RN:08297187.

[69] V. E. Korepin, "Calculation of norms of Bethe wave functions," *Comm. Math. Phys.* **86** no. 3, (1982) 391–418. https://projecteuclid.org:443/euclid.cmp/1103921777.

[70] J. Escobedo, *Integrability in AdS/CFT: Exact Results for Correlation Functions*. PhD thesis, Waterloo U., 2012. http://uwspace.uwaterloo.ca/handle/10012/6788.

[71] N. Beisert and M. Staudacher, "Long-range psu(2,2|4) Bethe Ansatze for gauge theory and strings," *Nucl. Phys.* **B727** (2005) 1–62, arXiv:hep-th/0504190 [hep-th].

[72] N. Beisert, "The SU(2|2) dynamic S-matrix," *Adv. Theor. Math. Phys.* **12** (2008) 945–979, arXiv:hep-th/0511082 [hep-th].

[73] W. Hao, R. I. Nepomechie, and A. J. Sommese, "Completeness of solutions of Bethe's equations," *Physical Review E* **88** no. 5, (11, 2013) 052113–. https://link.aps.org/doi/10.1103/PhysRevE.88.052113.

[74] A. G. Izergin and V. E. Korepin, "Pauli principle for one-dimensional bosons and the algebraic Bethe ansatz," *Lett. Math. Phys.* **6** (1982) 283–288. [Zap. Nauchn. Semin.120,69(1982)].

[75] L. V. Avdeev and A. A. Vladimirov, "Exceptional solutions to the Bethe ansatz equations," *Theoretical and Mathematical Physics* **69** no. 2, (Nov, 1986) 1071–1079. https://doi.org/10.1007/BF01037864.

[76] R. I. Nepomechie and C. Wang, "Algebraic Bethe ansatz for singular solutions," *Journal of Physics A: Mathematical and Theoretical* **46** no. 32, (2013) 325002. http://stacks.iop.org/1751-8121/46/i=32/a=325002.

[77] Y. Jiang and Y. Zhang, "Algebraic geometry and Bethe ansatz. Part I. The quotient ring for BAE," *Journal of High Energy Physics* **2018** no. 3, (Mar, 2018) 87. https://doi.org/10.1007/JHEP03(2018)087.

[78] A. Doikou and R. I. Nepomechie, "Parity and charge conjugation symmetries and S matrix of the XXZ chain," arXiv:hep-th/9810034 [hep-th].

[79] C. Marboe and D. Volin, "Fast analytic solver of rational Bethe equations," *Journal of Physics A: Mathematical and Theoretical* **50** no. 20, (2017) 204002. http://stacks.iop.org/1751-8121/50/i=20/a=204002.

[80] G. P. Pronko and Y. G. Stroganov, "Bethe Equations "on the Wrong Side of Equator"," *J.Phys.A* **32** (1999) 2333–2340.

[81] F. Verstraete, V. Murg, and J. I. Cirac, "Matrix product states, projected entangled pair states, and variational renormalization group methods for quantum spin systems," *Advances in Physics* **57** no. 2, (03, 2008) 143–224. https://doi.org/10.1080/14789940801912366.

[82] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac, "Matrix Product State Representations," *Quantum Info. Comput.* **7** no. 5, (July,

2007) 401–430.
http://dl.acm.org/citation.cfm?id=2011832.2011833.

[83] L. Piroli, B. Pozsgay, and E. Vernier, "From the Quantum Transfer Matrix to the Quench Action: The Loschmidt echo in XXZXXZ Heisenberg spin chains," *J. Stat.* **Mech.** (2017) 023106, 1611.06126. https://arxiv.org/pdf/1611.06126.

[84] R. M. Konik, *Studies in Boundary and Defect Field Theories*. PhD thesis, Cornell University, 1998.

[85] Z. Bajnok, L. Palla, and G. Takacs, "Boundary one-point function, Casimir energy and boundary state formalism in D+1 dimensional QFT," *Nucl. Phys.* **B772** (2007) 290–322, arXiv:hep-th/0611176 [hep-th].

[86] B. Pozsgay, "Overlaps with arbitrary two-site states in the XXZ spin chain," *Journal of Statistical Mechanics: Theory and Experiment* **2018** no. 5, (2018) 053103. http://stacks.iop.org/1742-5468/2018/i=5/a=053103.

[87] O. Foda and K. Zarembo, "Overlaps of partial Néel states and Bethe states," *J. Stat. Mech.* **1602** no. 2, (2016) 023107, arXiv:1512.02533 [hep-th].

[88] M. Brockmann, J. D. Nardis, B. Wouters, and J.-S. Caux, "A Gaudin-like determinant for overlaps of Néel and XXZ Bethe states," *Journal of Physics A: Mathematical and Theoretical* **47** no. 14, (2014) 145003. http://stacks.iop.org/1751-8121/47/i=14/a=145003.

[89] B. Pozsgay, L. Piroli, and E. Vernier, "Integrable Matrix Product States from boundary integrability,".

[90] E. K. Sklyanin, "Boundary Conditions for Integrable Quantum Systems," *J. Phys.* **A21** (1988) 2375–289.

[91] I. V. Cherednik, "Factorizing particles on a half-line and root systems," *Theoretical and Mathematical Physics* **61** no. 1, (Oct, 1984) 977–983. https://doi.org/10.1007/BF01038545.

[92] L. Mezincescu and R. I. Nepomechie, "Integrability of Open Spin Chains with Quantum Algebra Symmetry," *International Journal of Modern Physics A* **06** no. 29, (1991) 5231–5248. https://doi.org/10.1142/S0217751X91002458.

[93] N. Y. Reshetikhin and M. A. Semenov-Tian-Shansky, "Central extensions of quantum current groups," *Letters in Mathematical Physics* **19** no. 2, (Feb, 1990) 133–142. https://doi.org/10.1007/BF01045884.

[94] A. Doikou, "On reflection algebras and boundary Yangians," *J. Math. Phys.* **46** (2005) 053504, arXiv:hep-th/0403277 [hep-th].

[95] Y. Wang, W.-L. Yang, J. Cao, and K. Shi, "Off-Diagonal Bethe Ansatz for Exactly Solvable Models,". http://dx.doi.org/10.1007/978-3-662-46756-5.

[96] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*. Academic Press, 1982.

[97] L. Faddeev, "Instructive History of the Quantum Inverse Scattering Method," *NATO Sci. Ser. C* **530** (1999) 161–177.

[98] L. Pauling, "The Structure and Entropy of Ice and of Other Crystals with Some Randomness of Atomic Arrangement," *Journal of the American Chemical Society* **57** no. 12, (12, 1935) 2680–2684. https://doi.org/10.1021/ja01315a102.

[99] E. H. Lieb, "Residual Entropy of Square Ice," *Phys. Rev.* **162** (Oct, 1967) 162–172. https://link.aps.org/doi/10.1103/PhysRev.162.162.

[100] G. Algara-Siller, O. Lehtinen, F. C. Wang, R. R. Nair, U. Kaiser, H. A. Wu, A. K. Geim, and I. V. Grigorieva, "Square ice in graphene nanocapillaries," *Nature* **519** (03, 2015) 443 EP –. https://doi.org/10.1038/nature14295.

[101] H. J. de Vega and A. Gonzalez-Ruiz, "Boundary K-matrices for the XYZ, XXZ and XXX spin chains," *Journal of Physics A: Mathematical and General* **27** no. 18, (1994) 6129. http://stacks.iop.org/0305-4470/27/i=18/a=021.

[102] O. Tsuchiya, "Determinant formula for the six-vertex model with reflecting end," *Journal of Mathematical Physics* **39** no. 11, (1998) 5946–5951. https://doi.org/10.1063/1.532606.

[103] A. G. Izergin, D. A. Coker, and V. E. Korepin, "Determinant formula for the six-vertex model," *Journal of Physics A: Mathematical and General* **25** no. 16, (1992) 4315. http://stacks.iop.org/0305-4470/25/i=16/a=010.

[104] L. Piroli, B. Pozsgay, and E. Vernier, "What is an integrable quench?," *Nucl. Phys.* **B925** (2017) 362–402, arXiv:1709.04796 [cond-mat.stat-mech].

[105] M. Brockmann, "Overlaps of q -raised Néel states with XXZ Bethe states and their relation to the Lieb–Liniger Bose gas," *Journal of Statistical Mechanics: Theory and Experiment* **2014** no. 5, (2014) P05006. http://stacks.iop.org/1742-5468/2014/i=5/a=P05006.

[106] B. Pozsgay, "Overlaps between eigenstates of the XXZ spin-1/2 chain and a class of simple product states," 1309.4593. https://arxiv.org/pdf/1309.4593.

[107] A. Doikou, "A note on the boundary spin s XXZ chain," *Physics Letters A* **366** no. 6, (2007) 556–562. http://www.sciencedirect.com/science/article/pii/S0375960107003040.

[108] S. Belliard, N. Crampé, and E. Ragoucy, "Algebraic Bethe Ansatz for Open XXX Model with Triangular Boundary Matrices," *Letters in Mathematical Physics* **103** no. 5, (May, 2013) 493–506. https://doi.org/10.1007/s11005-012-0601-6.

[109] R. A. Pimenta and A. Lima-Santos, "Algebraic Bethe ansatz for the six vertex model with upper triangular K-matrices," *J. Phys.* **A46** (2013) 455002, arXiv:1308.4446 [math-ph].

[110] S. Belliard and N. Cramp, "Heisenberg XXX Model with General Boundaries: Eigenvectors from Algebraic Bethe Ansatz," *SIGMA* **9** (2013) 072, arXiv:1309.6165 [math-ph].

[111] W.-L. Yang, X. Chen, J. Feng, K. Hao, B.-Y. Hou, K.-J. Shi, and Y.-Z. Zhang, "Determinant formula for the partition function of the six-vertex model with a non-diagonal reflecting end," *Nuclear Physics B* **844** no. 2, (2011) 289 – 307. http://www.sciencedirect.com/science/article/pii/S0550321310005936.

[112] S. Ghoshal and A. B. Zamolodchikov, "Boundary S matrix and boundary state in two-dimensional integrable quantum field theory," *Int. J. Mod. Phys.* **A9** (1994) 3841–3886, arXiv:hep-th/9306002 [hep-th]. [Erratum: Int. J. Mod. Phys.A9,4353(1994)].

[113] A. Doikou, "'New' boundary conditions in integrable lattice models," *PoS* **tmr2000** (2000) 054, arXiv:hep-th/0104083 [hep-th].

[114] A. Doikou, "Quantum spin chain with 'soliton nonpreserving' boundary conditions," *J. Phys.* **A33** (2000) 8797–8808, arXiv:hep-th/0006197 [hep-th].

[115] P. P. Kulish and N. Yu. Reshetikhin, "Diagonalization of $GL(N)$ Invariant Transfer Matrices and Quantum N Wave System (Lee Model)," *J. Phys.* **A16** (1983) L591–L596.

[116] N. Y. Reshetikhin, "Calculation of the norm of bethe vectors in models with SU(3)-symmetry," *Journal of Soviet Mathematics* **46** no. 1, (Jul, 1989) 1694–1706. https://doi.org/10.1007/BF01099200.

[117] M. Wheeler, "Scalar Products in Generalized Models with SU(3)-Symmetry," *Communications in Mathematical Physics* **327** no. 3, (May, 2014) 737–777. https://doi.org/10.1007/s00220-014-2019-8.

[118] O. Foda, Y. Jiang, I. Kostov, and D. Serban, "A tree-level 3-point function in the $\mathfrak{su}(3)$-sector of planar $\mathcal{N} = 4$ SYM," *Journal of High Energy Physics* **2013** no. 10, (Oct, 2013) 138. https://doi.org/10.1007/JHEP10(2013)138.

[119] S. Belliard, S. Pakuliak, E. Ragoucy, and N. A. Slavnov, "Bethe vectors of GL(3)-invariant integrable models," *J. Stat. Mech.* **1302** (2013) P02020, arXiv:1210.0768 [math-ph].

[120] F. Levkovich-Maslyuk, "The Bethe ansatz," *J. Phys.* **A49** no. 32, (2016) 323004, arXiv:1606.02950 [hep-th].

[121] J. Abad and M. Rios, "Nondiagonal solutions to reflection equations in su(n) spin chains," *Phys. Lett.* **B352** (1995) 92–95, arXiv:hep-th/9502129 [hep-th].

[122] L. Piroli, E. Vernier, P. Calabrese, and B. Pozsgay, "Integrable quenches in nested spin chains I: the exact steady states," 1811.00432. https://arxiv.org/pdf/1811.00432.

[123] D. Arnaudon, J. Avan, N. Crampe, A. Doikou, L. Frappat, and E. Ragoucy, "General boundary conditions for the sl(N) and sl(M|N) open spin chains," *J.Stat.Mech.* **0408** (2004) P08005, math-ph/0406021. https://arxiv.org/pdf/math-ph/0406021.

[124] B. Pozsgay, "One-point functions in defect CFT, Integrable Matrix Product States, and boundary integrability." Talk at Integrability in Gauge and String Theory (IGST), Copenhagen, 2018.

[125] L. Piroli, E. Vernier, P. Calabrese, and B. Pozsgay, "Integrable quenches in nested spin chains II: the Quantum Transfer Matrix approach," 1812.05330. https://arxiv.org/pdf/1812.05330.

[126] O. DeWolfe and N. Mann, "Integrable open spin chains in defect conformal field theory," *JHEP* **04** (2004) 035, arXiv:hep-th/0401041 [hep-th].

[127] A. Gerrard, N. MacKay, and V. Regelskis, "Nested algebraic Bethe ansatz for open spin chains with even twisted Yangian symmetry," arXiv:1710.08409 [math-ph].

[128] T. Gombor, "Nonstandard Bethe Ansatz equations for open O(N) spin chains," *Nucl. Phys.* **B935** (2018) 310–343, arXiv:1712.03753 [math-ph].

[129] A. Bissi, G. Grignani, and A. V. Zayakin, "The SO(6) Scalar Product and Three-Point Functions from Integrability," arXiv:1208.0100 [hep-th].

[130] W. Galleas and J. Lamers, "Reflection algebra and functional equations," *Nucl. Phys.* **B886** (2014) 1003–1028, arXiv:1405.4281 [math-ph].

[131] J. Lamers, "The Functional Method for the Domain-Wall Partition Function," *SIGMA* **14** (2018) 064.

[132] P. Liendo, L. Rastelli, and B. C. van Rees, "The Bootstrap Program for Boundary CFT$_d$," *JHEP* **07** (2013) 113, arXiv:1210.4258 [hep-th].

[133] F. Gliozzi, P. Liendo, M. Meineri, and A. Rago, "Boundary and Interface CFTs from the Conformal Bootstrap," *JHEP* **05** (2015) 036, arXiv:1502.07217 [hep-th].

[134] M. Billò, V. Gonalves, E. Lauria, and M. Meineri, "Defects in conformal field theory," *JHEP* **04** (2016) 091, arXiv:1601.02883 [hep-th].

[135] J. L. Cardy, "Conformal Invariance and Surface Critical Behavior," *Nucl. Phys.* **B240** (1984) 514–532.

[136] D. M. McAvity and H. Osborn, "Conformal field theories near a boundary in general dimensions," *Nucl. Phys.* **B455** (1995) 522–576, arXiv:cond-mat/9505127 [cond-mat].

[137] S. Kawamoto and T. Kuroki, "Existence of new nonlocal field theory on noncommutative space and spiral flow in renormalization group analysis of matrix models," *JHEP* **06** (2015) 062, arXiv:1503.08411 [hep-th].

[138] A. Bissi, T. Hansen, and A. Söderberg, "Analytic Bootstrap for Boundary CFT," arXiv:1808.08155 [hep-th].

[139] E. Widén, "Two-point functions of SU(2)-subsector and length-two operators in dCFT," *Phys. Lett.* **B773** (2017) 435–439, arXiv:1705.08679 [hep-th].