Gravitational Waves and Quasinormal Modes of Black Holes

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Abstract: The detection of gravitational waves provides us a new way to explore the universe since the gravitational waves travel in the universe uninfluenced by the objects they encounter thus bring information far away to us. We are particularly interested in the gravitational waves emitted from black hole merger events. Black holes are losing energy since they are emitting gravitational waves which carry energy to infinity. We would expect quasinormal modes for the oscillation of such a dissipate system which reveal the intrinsic nature of this system. In a black hole merger event, the orbit of black holes shrinks then two black holes merge into one. After the merger there left a single black hole. The oscillation of this black hole sends ringdown gravitational waves, providing us parameters of the black hole. In this thesis we first introduce the concept of gravitational waves and quasinormal modes and introduce the quasinormal modes for Schwarzschild black holes.

It is also suggested that general relativity breaks down near the event horizon of a black hole since it is not a quantum theory. The boundary conditions at the event horizon get changed. Waves don't perfectly falling into the horizon. Instead, part of them get reflected near the horizon, this leads to the generation of black hole echoes. We will introduce the generation of black hole echoes and set a new boundary condition near the horizon, the robin boundary condition, which is suggested to dominate at low energies.

Contents

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1 Introduction

A hundred years ago, Albert Einstein came up with the theory of general relativity, relating energy-momentum tensor to the curvature tensor of spacetime. It states that the energy and momentum of matter or radiation are the sources of gravitational fields and change the geometry of spacetime. The variations of energy momentum tensor leads to variation of spacetime geometry. That predicts the existence of gravitational waves. Imagine that there are two black holes merging into one. It would change the geometry of spacetime, generating ripples in spacetime as known by gravitational waves. Just like the situation when you toss a stone into a lake, the surface of water gets perturbed and the ripples would spread around, the gravitational waves is spreading through the universe. It would cause the distortion of spacetime where it travels by. The gravitational waves would stretch the spacetime in one direction and squeeze it in another direction. In 2015, scientists made the first detection of gravitational waves by the LIGO detectors[28]. LIGO made the detection by setting two perpendicular tunnels and compare how long it takes for a signal to travel in each tunnel. The detection of gravitational waves provides us a new perspective to study the universe. We know there are four fundamental forces of nature and gravity is the weakest one. The gravitational waves can propagate through the universe, hardly interacting with the objects they pass by and keep the information about its source. Therefore it is an good object for observation, bringing information from deep universe to our view.

We will first introduce the definition of gravitational waves as perturbations around background spacetime. We will show how gravitational waves arise from the linearized Einstein equation. By solving the linearized perturbed Einstein equations and by fixing the gauge properly we find a simple expression for gravitational waves, which have only two degrees of freedom (two polarizations). We can check the effect of gravitational waves by studying how do they interact with freely falling particles. We would find the gravitational waves don't change the coordinate of the test particle in TT gauge but the proper distance of two freely falling particles varies in time periodically. That shows how gravitational waves stretch and squeeze spacetime. We know from general relativity that the spacetime is curved by the energy and momentum of matter and radiation.

We have known the gravitational waves carry energy and momentum since the geometry of spacetime is changed under the effect of gravitational waves. How could we know how much energy and momentum do they carry? This is done by studying how is the curvature of spacetime get changed. We redefine the gravitational field as perturbations around a dynamical background spacetime. We split Einstein equations into high-frequency part and low-frequency part. By studying the low-frequency part of

Einstein equations we get a effective "coarse-grained" expression for energy-momentum tensor instead of a local expression. There is no local expression for energy-momentum tensor of gravitational field. The reason will be discussed in Chapter 3, where we introduce gravitational field as a classical field living in the Minkowski spacetime. We then study gravitational field using tools in classical field theory. We introduce Noether's theorem and find the energy-momentum tensor is the Noether current associated with symmetry of spacetime translation. We will consider the application of Noether theorem to scalar field, electromagnetic field and to the gravitational field. We will find that the Noether's theorem also doesn't give us a local expression for energy-momentum tensor of gravitational field.

Then we may ask, where do these gravitational waves come from and how do they influence their source since they are taking energy away? In Chapter 4 we will show that the primary contribution to gravitational waves comes from the quadrupole oscillation of the mass distribution of the source. We know the primary term in electromagnetic waves is electric dipole radiation. The monopole radiation don't exist by the restriction of charge conservation. We will make analogous analysis to gravitational waves. Perform the multipole expansion to the radiation and to the energy-momentum tensor of the source, we find there is no monopole radiation or dipole radiation in gravitational waves. The monopole radiation is forbidden by the conservation of total mass. The dipole radiation is forbidden by the conservation of momentum.

As it is said above, gravity is the weakest fundamental force in the universe. The interaction between gravitational waves and matters is very small. We will show that the amplitude of gravitational waves (or more precisely, the quadrupole moment of the radiation) decreases proportional to the distance the radiation travels. Therefore when gravitational waves reach earth from the far away source, it is very hard to detect them. Therefore to make the detection, we need the source of the gravitational waves to be a strong field. They must be generated by violent events in the universe such that we can detect them when they reach the earth. One example is the coalescence of a binary system with two compact (massive and dense) bodies, for example, black holes or neutron stars. Actually, LIGO's first detection of gravitational waves was based on a binary black hole merger event[28]. We will study such a binary system in Newtonian approximation. We will derive the expression for quadrupole radiation emitted by such a system and find out the energy and angular momentum it carried away from the binary system. And due to the emission of gravitational waves, the orbit of the system shrinks. During the inspiral phase, the radiations it emits are also changing. We would derive the relation between the frequency of the radiation and the angular velocity of the orbit and see how the waveform evolves.

After the two black holes in a binary system merge into one, the new black hole will oscillate and emit ringdown gravitational waves to reach a equilibrium state. We are particularly interested in the ringdown signals. We know for a closed system, when it is perturbed, the wave function is a superposition of normal modes, which is purely determined by the intrinsic natures of the system. What would happen when a black hole is perturbed? The situation is little different since black hole is an open system. It is losing energy to spatial infinity due to emission of gravitational waves. Thus we will obtain quasi-normal modes from perturbation of black holes. Quasinormal modes also depend purely on the intrinsic natures of black holes, therefore can help us specify the parameters of black holes (its mass and spin). In Chapter 5 we will introduce black holes by solving vacuum Einstein equations for spacetime with certain matter distributions. We will introduce Schwarzschild black holes and Kerr rotating black holes since the black hole after merger is generically rotating. In Chapter 6 we introduce quasinormal modes as poles of Green's functions located on complex plane. We then introduce quasinormal modes of Schwarzschild black holes.

The above discussion is based on the theory of general relativity. But there are hypotheses stating that general relativity is not a valid theory passing the event horizon. It is suggested the curvature of spacetime behind the event horizon is so large that it requires some new theories like quantum gravity rather than general relativity. What would the possiblly existed new physics bring to us? We have introduced black holes with event horizons that waves can only fall into the horizon with no reflection. With new physics, we expect the boundary condition changes at the horizon. It is like there is a reflecting mirror near the horizon then may be reflected back at the horizon. Such waves may get reflected by the potential when propagating towards infinity and get reflected again at the horizon. This procedure can be repeated again and again thus black hole echoes arise. We will study the generation of black hole echoes in Chapter 7.

The behaviour of the black hole echoes is related to the boundary conditions and the boundary conditions are determined by the new physics near the horizon. Then we may ask, what the new physics exactly is? How is general relativity modified near the horizon? We can investigate how would certain boundary conditions affect the wave function of black hole echoes and therefore affect the observables. We can compare the prediction with the observation data then adjust the theory to fit the data. We will breifly introduce a theory possibly dominating at low energies and gives robin boundary conditions at the horizon. The study of boundary conditions is based on the construction of renormalization group. We put the reflecting mirror on a surface near the horizon. The location where we put in has nothing to do with physics then shouldn't affect the observables. That means, the reflection coefficient is required to be independent with the location of the mirror. This requirement determines how the coupling coefficient depend on the location of the mirror, giving a renormalization group flow. We will see how the fixed points of the flow lead to black holes and white holes.

2 Gravitational waves

In 1915, Albert Einstein published the theory of general relativity, with the famous field equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.1}$$

It relates the geometry of spacetime (the curvature tensor) to the distribution of matter and radiation (the energy-momentum tensor). One of the predictions of general relativity is the gravitational waves. The distribution of matter and radiation is not static. For example, consider a black hole. The black hole curves the spacetime, working as source of gravitational fields. When the black hole is perturbed, the curvature of spacetime would also change. There will a perturbation to the spacetime geometry and the perturbation will travel across the universe and deform the spacetime where it travels by. In this Chapter we will see how gravitational waves arise from perturbation of Einstein equation and how it propagates.

2.1 Description of GWs

We now study the perturbation of Einstein equations. Consider the gravitational waves generated by excitation of black holes, we as far away observer are interested in the wave zone far from the source. The wave zone can be viewed as nearly flat and only deformed slightly by the gravitational waves. Therefore the spacetime can be taken as a background Minkowski space plus a perturbation. The metric would be

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1 \tag{2.2}$$

The inversion of the metric is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2) \tag{2.3}$$

where $h_{\mu\nu}$ is the perturbation we added on spacetime. We will see how it behaves in the Einstein equations. Since $h_{\mu\nu}$ is already very small, we will keep only the first-order terms.

We expand the Ricci tensor in our metric, its first order term in $h_{\mu\nu}$ is (The calculation is shown in Appendix A)

$$R^{(1)}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(\partial^\alpha\partial_\nu h_{\mu\alpha} + \partial_\mu\partial^\alpha h_{\alpha\nu} - \partial_\mu\partial_\nu h - \partial^\alpha\partial_\alpha h_{\mu\nu}) \tag{2.4}$$

Then we can find the linearization of Einstein equations. To make it written in a more compact form, we first define

$$h = \eta^{\mu\nu}h_{\mu\nu} \tag{2.5}$$

and

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{2.6}$$

The linearization of Einstein equation becomes

$$\Box\overline{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma\overline{h}_{\rho\sigma} - \partial^\rho\partial_\nu\overline{h}_{\mu\rho} - \partial^\rho\partial_\mu\overline{h}_{\nu\rho} = -\frac{16\pi G}{c^4}T_{\mu\nu} \tag{2.7}$$

where $\Box$ is the d'Alembert operator. In flat spacetime, $\Box = \eta_{\mu\nu}\partial_\mu\partial_\nu$.
So far we find the equation of motion for $h_{\mu\nu}$. The solution of Eq.(2.7) would give us the expression for the perturbation $h_{\mu\nu}$ (the gravitational waves). This equation seems complicate and the solution is not clear. Notice that we haven't fixed the gauge by now. We can impose a proper gauge and will find the equation becomes more compact and easier to solve.
We now introduce the Lorentz gauge. Consider a coordinate transformation

$$x'^\mu = x^\mu + \xi^\mu(x) \tag{2.8}$$

where $\partial_\mu\xi_\nu$ are at most of the same order of smallness as $|h_{\mu\nu}|$, then $h_{\mu\nu}$ would change as

$$h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu) \tag{2.9}$$

$$\overline{h}'_{\mu\nu} = \overline{h}_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho) \tag{2.10}$$

Therefore,

$$(\partial^\nu\overline{h}_{\mu\nu})' = \partial^\nu\overline{h}_{\mu\nu} - \Box\xi_\mu \tag{2.11}$$

We have no restrictions for $\partial^\nu\overline{h}_{\mu\nu}$ so far. If it is not zero, we can choose appropriate gauge factor ξ_μ to make $\partial^\nu\overline{h}_{\mu\nu}$ vanish after coordinate transformation. The gauge we are choosing is the Lorentz gauge (also called the harmonic gauge)

$$\partial^\nu\overline{h}_{\mu\nu} = 0 \tag{2.12}$$

By imposing the Lorentz gauge so that Eq.(2.7) looks more simple and easy to find its solution

$$\Box\overline{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu} \tag{2.13}$$

Also it makes the conservation of energy momentum tensor more obvious

$$\partial^\mu T_{\mu\nu} = 0 \tag{2.14}$$

Eq.(2.13) shows the generation of GWs in linearized theory. When studying its propagation, we can set $T_{\mu\nu} = 0$(the space outside the source), therefore the equation of motion becomes

$$\Box\overline{h}_{\mu\nu} = 0 \qquad \text{(outside the source)} \tag{2.15}$$

We find this wave equation is similar to the wave equation of a scalar field

$$\Box\psi = 0 \tag{2.16}$$

whose solution is simply the superposition of plane waves

$$\psi(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} A(\mathbf{k}) e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} + B(\mathbf{k}) e^{-i\omega t - i\mathbf{k}\cdot\mathbf{x}} \tag{2.17}$$

where $\omega = |\mathbf{k}|$. Similarly we can write down the solution to Eq.(2.15).

$$h_{\mu\nu}(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} A_{\mu\nu}(\mathbf{k}) e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} \tag{2.18}$$

(Here we only consider the wave propagating towards us.)
We notice that there still remain degrees of freedom. To get more information about $h_{\mu\nu}$ We can fix the gauge completely by introducing the transverse-traceless gauge (the TT gauge). Consider a coordinate transformation

$$x^\mu \to x^\mu + \xi_\mu \tag{2.19}$$

with the gauge factor ξ^μ satisfies

$$\Box\xi_\mu = 0 \tag{2.20}$$

which means we can choose the functions ξ_μ so as to impose four functions on $h_{\mu\nu}$. We can choose ξ^0 to make the trace $\overline{h} = 0$. When $\overline{h} = 0$, $\overline{h}_{\mu\nu} = h_{\mu\nu}$. The Lorentz gauge condition with $\mu = 0$ becomes

$$\partial^0 h_{00} + \partial^i h_{0i} = 0 \tag{2.21}$$

Then we choose ξ^i to make $h^{0i} = 0$.

$$\partial^0 h_{00} = 0 \tag{2.22}$$

Here we can see that h_{00} is time-independent while the gravitational wave is the time-dependent part in the gravatational interaction. Therefore, as far as the GW is concerned, $\partial^0 h_{00} = 0$ means that $h_{00} = 0$. In conclusion, we get

$$h_{0\mu} = 0,\ h^i_i = 0,\ \partial^j h_{ij} = 0 \tag{2.23}$$

In momentum space, the gauge condition $\partial^j h_{ij} = 0$ reads

$$ik^j h_{ij} = 0 \tag{2.24}$$

stating that the wave is oscillating perpendicularly to the direction of propagation, i.e., the wave is transverse. That is why we call it by transverse-traceless gauge.
The general metric $h_{\mu\nu}$ is a symmetric matrix therefore has 10 degrees of freedom. By setting the Lorentz gauge we reduce 4 degrees of freedom. The TT gauge reduce another four degrees of freedom so we are left with two degrees of freedom now. We now are going to see how it appears in $h_{\mu\nu}$.
Consider the wave is propagating along z-direction, its wave vector $\mathbf{k}$ would be

$$\mathbf{k} = (0, 0, \omega) \tag{2.25}$$

Recall $h_{\mu\nu}$ has the form Eq.(2.18)

$$h_{\mu\nu}(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} A_{\mu\nu}(\mathbf{k}) e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} \tag{2.26}$$

$h_{0\mu}$ =0 implies $A_{0\mu} = 0$. $ik^j h_{ij} = 0$ implies $ik^j A_{ij} = 0$. Also $h^i_i = 0$ implies $A^i_i = 0$. Thus in TT gauge $h_{\mu\nu}$ can be written as

$$h^{TT}_{\mu\nu}(t, z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{-i\omega(t-z)} \tag{2.27}$$

where the plus and cross sign denote two polarizations. We can find the effect of gravitational waves more clearly by taking a look at the line element

$$ds^2 = -dt^2 + (1 + h_+ e^{-i\omega(t-z)})dx^2 + (1 - h_+ - e^{-i\omega(t-z)})dy^2 + 2h_\times e^{-i\omega(t-z)} dxdy + dz^2 \tag{2.28}$$

It clearly shows how the gravitational waves stretch the spacetime periodically in one direction and squeeze it in another direction.

2.2 Interaction of GWs with test particles

We have seen in the former section how gravitational waves arise from linearization of Einstein equation. But what effect does it have? In this section we are going to study how gravitational waves interact with a detector (viewed as a test particle). We will see how the test particle behaves under the effect of gravitational wave. We first need to specify which reference frame we are in. We have introduced the TT gauge in the last section. We denote the corresponding frame as TT frame. Then we consider the motion of the test particle in this frame. We can set the test particle at rest initially. To study its motion, recall that from general relativity we can derive the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\nu} \frac{dx^\rho}{d\tau} \frac{dx^\nu}{d\tau} = 0 \tag{2.29}$$

Consider the $\mu = i$ component of the equations. The initial condition we set on the test particle ensures $dx^i/d\tau = 0$

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\rho\nu}\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau} = \Gamma^i_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} \tag{2.30}$$

The Christoffel symbol is given by

$$\Gamma^\mu_{\rho\nu} = \frac{1}{2}g^{\mu\sigma}(g_{\rho\sigma,\nu} + g_{\nu\sigma,\rho} - g_{\rho\nu,\sigma}) \tag{2.31}$$

With the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.32}$$

we have $\Gamma^i_{00} = 0$. That leads to

$$\frac{d^2x^i}{d\tau^2} = \frac{d^2x^i}{dt^2} = 0 \tag{2.33}$$

It states that the acceleration of the test particle is zero. The test particle remains at rest under the effect of gravitational waves in TT frame.The gravitational waves will not change the coordinates of the test particle. However, this doesn't mean gravitational waves have no effect. By observing the line elementEq.(2.28), we find the metric is changed when gravitational waves are passing by. The coordinates of TT frame stretch or squeeze themselves so that coordinates of test particles remain the same. We can take a look at the proper distance between two test particles with coordinates (t,x_1,0,0) and (t,x_2,0,0). The proper distance ds^2 is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{2.34}$$

Then we have the proper distance between the two particles

$$D_p = \int_{x_1}^{x_2}\sqrt{g_{11}}dx = \int_{x_1}^{x_2}\sqrt{1 + h_+e^{-i\omega(t)}}dx = (x_2 - x_1)\sqrt{1 + h_+e^{-i\omega t}} \tag{2.35}$$

We know the amplitude of gravitational wave is very small, thus we can make the approximation

$$D_p \approx (x_2 - x_1)(1 + \frac{1}{2}h_+e^{-i\omega t}) \tag{2.36}$$

The coordinates of rest particles don't change under the effect of gravitational waves in TT frame. But the distance between them changes with time periodically. That shows how gravitational waves stretch or squeeze the spacetime. It gives a way for gravitational waves detection. That is what LIGO (the Laser Interferometer Gravitational-Wave Observatory) does in the past years. LIGO was designed based on a Michelson

Figure 1. Redefinition of Gravitational waves as perturbations over dynamical background metric

interferometer. It has two arms perpendicular to each other with mirrors at each end of the arms. We have illustrated the proper distance between two rest particles varies with time when gravitational waves passing by. Therefore the time it takes for a laser signal to travel between the two fixed points changes. By measuring that we can detect gravitation waves.

2.3 The Energy-Momentum Tensor of GWs

In this section we are going to study the energy and momentum carried by gravitational waves. From general relativity we know that there is a direct relation between energy and the curvature of spacetime. To get the energy and momentum of gravitational waves, we can study how do they change the curvature of spacetime. In the previous sections we expanded the Einstein equations to the first order in $h_{\mu\nu}$. We treat GWs as perturbations around a fixed flat background spacetime with metric $\eta_{\mu\nu}$. Since we have set the background spacetime as flat, of course we can derive nothing about how gravitational waves change the geometry of spacetime. It doesn't change at all. Therefore we need to define gravitational waves on a new background, a dynamical spacetime whose geometry can be changed. Still gravitational waves are perturbations to the background spacetime. We illustrate this with Figure 1. The total metric can be written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1 \tag{2.37}$$

The problem now is to decide which part of the spacetime belongs to the background and which part belongs to gravitational waves. When we consider the linearization of Einstein equations, the background metric is fixed and clearly distinguished from perturbations. Here the background spacetime is dynamical and not determined. The perturbations may get mixed up with the background spacetime. The distinction would be clear if there is a separation of scales. Imagine there is a coordinate system where the background spacetime looks quite smooth. It is not flat but the scale of variation is much larger than the wavelength of gravitational waves. In other words, the frequency of background metric is much smaller than that of the perturbation. Then we can easily separate the perturbation from the background metric by considering their scale. Now we have redefined the gravitational waves. To derive the energy-momentum tensor, we first expand the Einstein equation around the background metric $\bar{g}_{\mu\nu}$ to quadratic order in $h_{\mu\nu}$. The Einstein equation can be written as

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\big) \tag{2.38}$$

where $T_{\mu\nu}$ is the energy-momentum tensor and T is its trace.
We have illustrated how could we distinguish gravitational waves from the background metric by considering their wavelengths or frequencies. Therefore, it would be useful if we expand tensors with terms in different order of $h_{\mu\nu}$, to specify if its frequency is high or low. Here we expand the Ricci tensor

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \cdots\ , \tag{2.39}$$

where $\bar{R}_{\mu\nu}, R^{(1)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$ are zero-order, first-order and second-order in $h_{\mu\nu}$ respectively. With our metric Eq.(2.37), we write down the terms of Ricci tensor and pick out the first-order and second-order term in $h_{\mu\nu}$.

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\big(\bar{D}^\alpha\bar{D}_\mu h_{\nu\alpha} + \bar{D}^\alpha\bar{D}_\nu h_{\mu\alpha} - \bar{D}^\alpha\bar{D}_\alpha h_{\mu\nu} - \bar{D}_\mu\bar{D}_\nu h\big) \tag{2.40}$$

$$\begin{aligned} R^{(2)}_{\mu\nu} =& \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\big[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma}) \\ &+ h_{\rho\sigma}(\bar{D}_\mu\bar{D}_\nu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma}) \\ &+ (\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\big] \end{aligned} \tag{2.41}$$

The detailed calculation is given in Appendix A.
Having expanded the Ricci tensor, we can check the frequency of each term. The zero-order of Ricci tensor $\bar{R}_{\mu\nu}$ contains low-frequency modes only. $R^{(1)}_{\mu\nu}$ is linear in $h_{\mu\nu}$ therefore contains high-frequency modes only. $R^{(2)}_{\mu\nu}$ is quadratic in $h_{\mu\nu}$, therefore contains both low-frequency and high-frequency modes. The low-frequency modes appear in $R^(2)_{\mu\nu}$ since if there is a term $\sim h_1h_2$, the wave vectors in h_1 and h_2 may be in opposite direction and have same value therefor cancel each other $\mathbf{k}_1 = \mathbf{k}_2$. Having separated the Ricci tensor into terms with different scales of frequencies, we can divide the Einstein equations into low-frequency and high-frequency parts.

$$\bar{R}_{\mu\nu} + \big[R^{(2)}_{\mu\nu}\big]^{Low} = \frac{8\pi G}{c^4}\big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\big)^{Low} \tag{2.42}$$

$$R^{(1)}_{\mu\nu} + \left[R^{(2)}_{\mu\nu}\right]^{High} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{High} \tag{2.43}$$

For the spacetime outside the source (the region of spacetime we are interested in since we observers on the earth are far away from the source), $T_{\mu\nu} = 0$. therefore the last term in Eq.(2.42) vanishes. Eq.(2.42) becomes

$$\bar{R}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^{Low} \tag{2.44}$$

From the expression of $R^{(2)}_{\mu\nu}$ Eq.(2.41) we find it is made up of terms of order $(\partial h)^2$ and $h\partial^2 h$ (they are of the same order). From Eq.(2.44) we know $\bar{R}$ and $R^{(2)}_{\mu\nu}$ must be of the same order.

$$\bar{R}_{\mu\nu} \sim (\partial h)^2 \tag{2.45}$$

Therefore we can analyze the scale of variation on each hand side of Eq.(2.44) and can estimate the order of magnitude of gravitational waves. We denote the scale of variation of background metric as L_B

$$\partial\bar{g}_{\mu\nu} \sim 1/L_B \tag{2.46}$$

The length scale of h is λ, which is the reduced wavelength of gravitational waves. We use the reduced wavelength rather than wavelength sicne if we consider a wave of form e^{ikx}, which is the case for gravitational waves, the length scale can be obtained by considering $|\mathrm{d}e^{ikx}/\mathrm{dx}| = \mathrm{k}e^{ikx} = 1/\lambda \mathrm{e}^{ikx}$. Together with Eq.(2.18) we find

$$\partial h \sim \frac{h}{\lambda} \tag{2.47}$$

$\bar{R}_{\mu\nu}$ is of order $\partial^2 g$ (Set $\mathrm{O}(\bar{g}_{\mu\nu}) = 1$), then

$$\bar{R}_{\mu\nu} \sim \partial^2\bar{g}_{\mu\nu} \sim \frac{1}{L_B^2} \tag{2.48}$$

Then we have

$$\frac{1}{L_B^2} \sim \left(\frac{h}{\lambda}\right)^2 \tag{2.49}$$

We then find the relation between the amplitude of the gravitational waves and the scale of variation of the gravitational waves and the metric.

$$h \sim \frac{\lambda}{L_B} \tag{2.50}$$

We find when we are outside the source, $T_{\mu\nu} = 0$, the amplitude of the gravitational waves is proportional to the ratio of the length scale of the gravitational waves to the length scale of the background metric. The amplitude of the gravitational waves is very small since we have made the assumption the length scale of the gravitational waves is much smaller than that of the background metric (That is how we define the gravitational waves, as perturbations to the background metric). The curvature of the spacetime is mostly determined by the background metric. If there is a massive object acting as an external source of energy, $T_{\mu\nu} \neq 0$, its contribution to the geometry of spacetime will be much larger than that of the gravitational radiation, then the curvature would be mainly determined by this massive object. Consider the scale of variation on each hand side of Eq.(2.44) again

$$\frac{1}{L_B^2} \sim \left(\frac{h}{\lambda}\right)^2 + (\text{external source}) \ll \left(\frac{h}{\lambda}\right)^2 \tag{2.51}$$

which gives

$$h \ll \frac{\lambda}{L_B} \tag{2.52}$$

The magnitude of $\frac{\lambda}{L_B}$ is restricted since we need it to be large enough to separate the background spacetime and GWs. This indicates that the amplitude of GWs h satisfies $h \ll 1$. Now we consider the low-frequency part of Einstein equations Eq.(2.42). We notice there are both high-frequency and low-frequency modes in $R^{(2)}_{\mu\nu}$ and possibly in the energy-momentum tensor of the source. To pick out the low-frequency modes, we can introduce a scale $\bar{l}$ such that $\lambda \ll \bar{l} \ll L_B$, and take average over a spatial volume with side $\bar{l}$, the low-frequency modes remain the same while the high-frequency modes oscillate fast and average to 0. In this way we can get the low-frequency projection of the Einstein equation

$$\bar{R}_{\mu\nu} = -\langle R^{(2)}_{\mu\nu} \rangle + \frac{8\pi G}{c^4} \langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle \tag{2.53}$$

where $\langle \cdots \rangle$ denotes a spatial average over $\bar{l}$. Instead of writing tensors in the form of spatial average, we can define the effective quantity that contributes in low-frequency modes. We define the effective energy-momentum tensor $\bar{T}_{\mu\nu}$ such that

$$\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle = \bar{T}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{T} \tag{2.54}$$

where $\bar{T} = \bar{g}_{\mu\nu} \bar{T}^{\mu\nu}$ is its trace. And define the quantity $t_{\mu\nu}$ as

$$t_{\mu\nu} = -\frac{c^4}{8\pi G} \langle R^{(2)}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} R^{(2)} \rangle, \tag{2.55}$$

where $R^{(2)} = \bar{g}^{\mu\nu} R^{(2)}_{\mu\nu}$. And we define its trace as

$$\begin{aligned} t &= \bar{g}^{\mu\nu} t_{\mu\nu} \\ &= \frac{c^4}{8\pi G} \langle R^{(2)} \rangle \end{aligned} \tag{2.56}$$

Notice here $\bar{g}^{\mu\nu}{}^{(2)}_{\mu\nu}\rangle = \langle\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}\rangle$ since $\bar{g}^{\mu\nu}$ is low-frequency and therefore can be view as a constant when we take the average. So the spatial average of $R^{(2)}_{\mu\nu}$ can be written in terms of $t_{\mu\nu}$ and its trace,

$$-\langle R^{(2)}_{\mu\nu}\rangle = \frac{8\pi G}{c^4}\big(t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t\big) \tag{2.57}$$

The effective tensors we defined are known as the "coarse-grained" quantities. It gives the same effect as the real tensor considering the average over a spatial volume without bothering with the structure inside the volume. Replacing the quantities in the low-frequency part of Einstein equation Eq.(2.53) with the coarse-graianed ones, we obtain

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{8\pi G}{c^4}\big(\bar{T}_{\mu\nu} + t_{\mu\nu}\big) \tag{2.58}$$

which is called the "coarse-grained" form of the Einstein equations. It describes how the background metric changes under the effect of energy-momentum tensor, not on scales as tiny as the length scale of the gravitational waves, but describes the averaged behaviour over a spatial volume whose size is much larger than the scale of variation of gravitational waves, but also much smaller than the scale of variation of gravitational waves.

Here we notice that there are two terms on the right-hand side of Eq.(2.58). By definition, $\bar{T}_{\mu\nu}$ is a smoothed form of the matter energy-momentum tensor $T_{\mu\nu}$ therefore is a low-frequency term. The one left, $t_{\mu\nu}$, would naturally be the energy-momentum tensor of gravitational waves, and is defined to be quadratic in $h_{\mu\nu}$. The energy-momentum tensor of the gravitational waves is a coarse-grained quantity, which is restricted by its definition Eq.(2.55). Is there any local expression for the energy-momentum density for gravitational waves? The answer is no. Only the total amount of energy and momentum is determined. We will disscuss the reason behind in Chapter 3.

We now are going to find the expression for energy-momentum tensor of the gravitational waves $t_{\mu\nu}$ in terms of $h_{\mu\nu}$. We have defined $t_{\mu\nu}$ in terms of $R^{(2)\mu\nu}$ and recall we have find the expression for $R^{(2)\mu\nu}$ in Eq.(2.41). In Eq.(2.41) it is written in a general case where we use the covariant derivatives since the background spacetime is dynamical and not fixed to flat. Here we consider the situation where we we detect the gravitational waves is far away from the source so that the background spacetime can be taken as flat. Therefore we can replace the covariant derivatives with partial derivatives $\bar{D}^\mu \to \partial^\mu$ in Eq.(2.41),

$$\begin{aligned}R^{(2)}_{\mu\nu} =& \frac{1}{2}\big[\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} - h^{\alpha\beta}\partial_\nu\partial_\beta h_{\alpha\nu} - h^{\alpha\beta}\partial_\mu\partial_\beta h_{\alpha\nu} \\ &+ h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \partial^\beta h^\alpha_\nu\partial_\beta h_{\alpha\mu} - \partial^\beta h^\alpha_\nu\partial_\alpha h_{\beta\mu} - \partial_\beta h^{\alpha\beta}\partial_\nu h_{\alpha\mu} \\ &+ \partial_\beta h^{\alpha\beta}\partial_\alpha h_{\mu\nu} - \partial_\beta h^{\alpha\beta}\partial_\mu h_{\alpha\nu} - \frac{1}{2}\partial^\alpha h\partial_\alpha h_{\mu\nu} + +\frac{1}{2}\partial^\alpha h\partial_\nu h_{\alpha\mu} \\ &+ \frac{1}{2}\partial^\alpha h\partial_\mu h_{\alpha\nu}\big]\end{aligned} \tag{2.59}$$

We have discussed before that $h_{\mu\nu}$ is a symmetric 4×4 matrix therefore has 10 degrees of freedom. After imposing the Lorentz gauge $\partial^\mu\bar{h}_{\mu\nu} = 0$ we have 6 degrees of freedom left. Then we impose TT gauge to reduce it to 2, which are physical degrees of freedom contained in h^{TT}_{ij}. We can also set h = 0 (without spoiling the gauge choices) so that the Lorentz gauge becomes $\partial^\mu h_{\mu\nu} = 0$.
Since we are taking spatial average, the spacetime derivative ∂^{mu} can be integrated by parts. After choosing the gauge and using the method of integration by parts, we have

$$\langle R^{(2)}_{\mu\nu}\rangle = -\frac{1}{4}\langle\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle \tag{2.60}$$

and $\langle R^{(2)}\rangle$ vanishes. Then we get

$$t_{\mu\nu} = \frac{c^4}{32\pi G}\langle\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle \tag{2.61}$$

When we look at Eq.(2.58), we find that on the left-hand side there are physical contributions and coordinate-dependent contributions. So will be the right-hand side. We don't want the energy-momentum tensor be gauged away so we are looking for the physical contributions. We have made some gauge choices and get the expression for $t_{\mu\nu}$. To illustrate $t_{\mu\nu}$ has nothing to do with the gauge factor ξ_μ used to fix the gauge, we can check how $t_{\mu\nu}$ changes under coordinate transformation $x'^\mu = x^\mu + \xi^\mu$.

$$\begin{aligned}\delta t_{\mu\nu} &= \frac{c^4}{32\pi G}\big[\langle\partial_\mu h_{\alpha\beta}\partial^\nu(\delta h^{\alpha\beta})\rangle + (\mu\leftrightarrow\nu)\big] \\ &= \frac{c^4}{32\pi G}\big[\langle\partial_\mu h_{\alpha\beta}\partial^\nu(\partial^\alpha\xi^\beta + \partial^\beta\xi^\alpha)\rangle + (\mu\leftrightarrow\nu)\big] \\ &= \frac{c^4}{16\pi G}\big[\langle\partial_\mu h_{\alpha\beta}\partial^\nu\partial^\alpha\xi^\beta\rangle + (\mu\leftrightarrow\nu)\big]\end{aligned} \tag{2.62}$$

Perform integration by parts we will find $\delta t_{\mu\nu}$ vanishes. The energy-momentum tensor we derived for gravitational waves is invariant under coordinate transformation $x'^\mu = x^\mu + \xi^\mu$. This proves that $t_{\mu\nu}$ depends only on physical modes h^{TT}_{ij}. So we don't need to worry if the effect of gravitational waves can be gauged away. From the Bianchi identity we have

$$\bar{D}^{\mu}(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}) = 0 \tag{2.63}$$

therefore, with Eq.(2.58),

$$\bar{D}^{\mu}(\bar{T}_{\mu\nu} + t_{\mu\nu}) = 0 \tag{2.64}$$

This shows that the total energy-momentum tensor is conserved while there is in general exchange of energy and momentum between the matter sources and gravitational waves. When we are far away from the source, the background spacetime is approximately flat and $\bar{T}_{\mu\nu} = 0$, then we have

$$\partial^{\mu} t_{\mu\nu} = 0 \tag{2.65}$$

which shows the conservation of energy-momentum tensor of the gravitational waves. We have derived the expression for the energy-momentum tensor of the gravitational waves. We can now compute the energy flux of GWs. From the conservation of the energy-momentum tensor Eq.(2.65) we can write (the $\mu = 0$ component of Eq.(2.65))

$$\int_V d^3x(\partial_0 t^{00} + \partial_i t^{i0}) = 0 \tag{2.66}$$

where V is a spatial region far away from the source (to ensure the background spacetime is flat so we can use partial derivatives), bounded by a surface S. The energy of gravitational waves inside the volume V is

$$E_V = \int_V d^3x t^{00} \tag{2.67}$$

Then we can rewrite Eq.(2.66)

$$\begin{aligned} \frac{1}{c}\frac{dE_V}{dt} &= -\int_V d^3x \partial_i t^{0i} \\ &= -\int_S \mathrm{n}_i t^{0i} \end{aligned} \tag{2.68}$$

where n^i is the outer normal to the surface and dA is the surface. Let S be a spherical surface at a large distance r from the source. Its surface element is $dA = r^2 d\Omega$, and its normal $\hat{n} = \hat{r}$ is the unit vector in the radial direction. Then Eq.(2.68) gives

$$\frac{dE_V}{dt} = -c\int dA t^{0r} \tag{2.69}$$

We already derived the expression for $t_{\mu\nu}$ in Eq.(2.61)

$$t^{0r} = \frac{c^4}{32\pi G}\langle \partial^0 h_{ij}^{TT} \partial^r h_{ij}^{TT} \rangle \tag{2.70}$$

We will show in Section 4.1 the amplitude of gravitational waves at large distance should be inversely proportional to r, therefore h_{ij}^{TT} can be written in this form,

$$h_{ij}^{TT} = \frac{1}{r} f_{ij}(t - r/c) \tag{2.71}$$

where $f_{ij}(t - r/c)$ is some function of retarded time. We use retarded time here as the same reason as we use retard time in electromagnetism. Since we are considering the problem for radiations, they propagate at speed of light. The radiation we detect is the radiation emitted by the source at time $t_{ret} = t - r/c$. Therefore

$$\begin{aligned}
\frac{\partial}{\partial r} h_{ij}^{TT} &= -\frac{1}{r^2} f_{ij}(t - r/c) + \frac{1}{r}\frac{\partial}{\partial r} f_{ij}(t - r/c) \\
&= -\frac{1}{r}\frac{1}{c\partial t} f_{ij}(t - r/c) + \mathcal{O}(\frac{1}{r^2}) \\
&= \partial^0 h_{ij}^{TT} + \mathcal{O}(\frac{1}{r^2})
\end{aligned} \tag{2.72}$$

We would find

$$t^{0r} = \frac{c^4}{32\pi G} \langle \partial^0 h_{ij}^{TT} \partial^0 h_{ij}^{TT} \rangle = t^{00} \tag{2.73}$$

therefore we find the energy changed in a spatial volume V per unit time is

$$\frac{dE_V}{dt} = -c \int dA t^{00} \tag{2.74}$$

It is negative since the gravitational waves are propagating outwards from the source to us and taking away the energy. It is straightforward to get t^{00} with Eq.(2.61)

$$t^{00} = \frac{c^2}{32\pi G} \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle \tag{2.75}$$

where the dot denotes $\partial_t = c\partial_0$. It can also be written in terms of the amplitudes h_+ and $h_\times$,

$$t^{00} = \frac{c^2}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \tag{2.76}$$

Then we derive the energy changed in a spatial volume V per unit time in terms of the wave functions of gravitational waves (in TT gauge)

$$\frac{dE}{dt} = \frac{c^3 r^2}{32\pi G} \int d\Omega \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle \tag{2.77}$$

Or in terms of the two polarizations of gravitational waves h_+ and $h_\times$,

$$\frac{dE}{dAdt} = \frac{c^3}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \tag{2.78}$$

The total energy passing through dA from $t = -\infty$ to $t = \infty$ is

$$\frac{dE}{dA} = \frac{c^3}{16\pi G}\int_{-\infty}^{\infty} dt\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle \tag{2.79}$$

The angular brackets here are to take temporal average. However, we are taking integral from $t = -\infty$ to $t = \infty$, then it doesn't matter if we take the average or not.

$$\frac{dE}{dA} = \frac{c^3}{16\pi G}\int_{-\infty}^{\infty} dt(\dot{h}_+^2 + \dot{h}_\times^2) \tag{2.80}$$

The momentum that gravitational waves carries can be computed in the same way. Inside a volume V at large distance from the source

$$P_v^k = \frac{1}{c}\int d^3x t^{0k} \tag{2.81}$$

$$\begin{aligned} c\partial_0 P_v^k &= \int_V d^3x \partial_0 t^{0k} \\ &= -\int_S dA t^{0k} \end{aligned} \tag{2.82}$$

$$\frac{dP^k}{dt} = \frac{c^3}{32\pi G} r^2 \int d\Omega \langle \dot{h}_{ij}^{TT}\partial^k h_{ij}^{TT}\rangle \tag{2.83}$$

In this section we found out the expression for energy-momentum tensor of the gravitational waves and calculated the energy and momentum carried by the gravitational waves. The way we make it is to split the high-frequency (varying fast in spacetime) part and the low-frequency part (varying slowly) of tensors in Einstein equation. We consider the low-frequency part of the Einstein equations and using the effective "coarse-grained" quantity instead of using the original tensors. The effect of the high-frequency part of the tensors would disappear when we take average over a spatial volume if we set this spatial volume to be large enough compared to the length scale of gravitational waves and small enough compared to the length scale of the background metric. In this way we find how the background metric of spacetime get changed by the gravitational wave (plus the effect of external sources, if there is any) in an average over a spatial volume with proper size. Therefore we derived the expression for energy-momentum tensor.

2.4 Propagation in flat and curved spacetime

In the last section we spilt the Einstein equations in to high-frequency part Eq.(2.43) and low-frequency part Eq.(2.42). We analyzed the low-frequency equations by taking average over a spatial volume to cancel the high-frequency part in the tensors (both the quadratic Ricci tensor and the energy-momentum tensor of possible external sources) then derived the energy-momentum tensor of gravitational waves. Now we are going to study the high-frequency Einstein equations. We will find it reveals how the gravitational waves propagate in a curved spacetime.

When study the propagation of gravitational waves, obviously they are in the region outside the source, where the energy-momentum tensor of the source vanishes and we assume there are no external sources. The right-hand side of Eq.(2.43) vanishes, the high-frequency equation becomes

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\text{High}} \tag{2.84}$$

The equation can be complicated since there are so many terms in $R^{(1)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$, and we need to pick out the high-frequency term in $R^{(2)}_{\mu\nu}$. The work can be done in a much simpler way if we expand each hand side in power series and compare terms of the same order. We will find there is no need to get the expression for the high-frequency part of $R^{(2)}_{\mu\nu}$. We find there are two small parameters h and $\frac{\lambda}{L_B}$. We find there are two small parameters h and $\frac{\lambda}{L_B}$. When we estimate the order of magnitude, which parameter should we consider? It is not a problem here in the case without external sources since they are of the same order. Remember that we are considering the case without external sources. In last section we have analyzed the order of magnitude of low-frequency part of Einstein equations and find $h \sim \frac{\lambda}{L_B}$ (Eq.(2.50)).

Now we estimate the order of magnitude of each hand side of Eq.(2.84). Remember the expression for $h_{\mu\nu}$ we derived in Eq.(2.18) $h \sim e^{ikx}$. The order of $R^{(1)}_{\mu\nu}$ is

$$R^{(1)}_{\mu\nu} \sim \partial^2 h \sim \frac{h}{\lambda^2} \sim \frac{1}{\lambda L_B} \tag{2.85}$$

while

$$R^{(2)}_{\mu\nu} \sim (\partial h)^2 \sim \frac{h^2}{\lambda^2} \sim \frac{1}{L_B^2} \tag{2.86}$$

The leading order term in $R^{(1)}_{\mu\nu}$ is of order $\frac{1}{\lambda L_B}$ there is no such term in $R^{(2)}_{\mu\nu}$ having the same order. Terms in $R^{(2)}_{\mu\nu}$ are at most of order $\frac{1}{L_B^2}$ which is much small than $\frac{1}{\lambda L_B}$ since we have restricted $\frac{\lambda}{L_B}$ to be very small to distinguish gravitational waves from the background metric. So the leading order of the left-hand side must be zero.

$$[R^{(1)}_{\mu\nu}]_{\text{leading order}} = 0 \tag{2.87}$$

Extract the leading order term from Eq.(2.40), we get

$$\eta^{\rho\sigma}(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\nu\partial_\mu h_{\rho\sigma} - \partial_\rho\partial_\sigma h_{\mu\nu}) = 0 \tag{2.88}$$

This gives how gravitational waves propagates in flat background spacetime since we are considering the situation where there are no external sources. This coincides with the linearized Einstein equations we derived in Eq.(2.7) since here we are considering the equation for the leading order term of $R^{(1)}_{\mu\nu}$. We haven't fix the gauge yet. As what we have done to the linearized Einstein equations, if we introduce $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ and impose the Lorentz gauge, we will find the propagation equation can be written in a simple form

$$\Box\bar{h}_{\mu\nu} = 0 \tag{2.89}$$

We have showed how the high-frequency part of Einstein equation determines the propagation of gravitational waves in flat spacetime without external sources. Now we consider the case where there are external sources, that is, we are in curved spacetime now. We have analyzed this situation for low-frequency equations in the last section, where we estimated the order of magnitude and find $h \ll \frac{\lambda}{L_B}$. Therefore the two small parameters h and $\frac{\lambda}{L_B}$ are of different orders. l is much smaller than $\frac{\lambda}{L_B}$, so When we expand the equations into series, we keep only the leading order term in h (terms linear in h). And we keep only the leading order and next-to-leading order in λ/L_B. The high-frequency equation becomes

$$R^{(1)}_{\mu\nu} = 0 \tag{2.90}$$

Remember here we are considering the case with external sources. The background spacetime is curved, so we use covariant derivatives here

$$\bar{g}^{\rho\sigma}(\bar{D}_\rho\bar{D}_\nu h_{\mu\sigma} + \bar{D}_\rho\bar{D}_\mu h_{\nu\sigma} - \bar{D}_\nu\bar{D}_\mu h_{\rho\sigma} - \bar{D}_\rho\bar{D}_\sigma h_{\mu\nu}) = 0 \tag{2.91}$$

This is the propagation equation for gravitational waves in curved background spacetime. Again we use some tricks to make the equation more compact. We introduce the quantity

$$h = \bar{g}^{\mu\nu}h_{\mu\nu} \qquad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}h \tag{2.92}$$

and impose the Lorentz gauge

$$\bar{D}^\mu\bar{h}_{\mu\nu} = 0 \tag{2.93}$$

The propagation equation in Lorentz gauge is therefore

$$\bar{D}^\rho\bar{D}_\rho\bar{h}_{\mu\nu} + 2\bar{R}_{\mu\rho\nu\sigma}\bar{h}^{\rho\sigma} - \bar{R}_{\mu\rho}\bar{h}^\rho_\nu - \bar{R}_{\nu\rho}\bar{h}^\rho_\mu = 0 \tag{2.94}$$

It describes how the gravitational waves propagates in general in curved spacetime with the existence of external sources. If we are interested in the region outside the external sources, we would the propagation equation more simpler. In the region outside the external sources, the energy-momentum tensor of external sources doesn't contribute, $\bar{T}_{\mu\nu} = 0$. Eq.(2.94) becomes

$$\bar{D}^{\rho}\bar{D}_{\rho}\bar{h}_{\mu\nu} = 0 \tag{2.95}$$

It is the equation describes how do the gravitational waves propagates in curved background but in the particular region without the present of external sources.
In this section we analyzed the high-frequency part of the Einstein equations by expanding it into series and compare the order of each term. The high-frequency part of the Einstein equations determines the propagation of gravitational waves in flat spacetime and in curved spacetime, depending on if there are external sources or not.

3 Field-theoretical approach

In the last section we treated gravitational wave as a perturbation to the dynamical background spacetime $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. We decomposed the Einstein equations into high frequency part and low frequency part. By studying the low-frequency part of Einstein equations we checked how gravitational waves curve the background spacetimewe defined the "coarse-grained" energy-momentum tensor of gravitational waves. We then studied the high-frequency part of Einstein equations and derived their propagation equation both in flat spacetime and in curved spacetime. To understand GWs better, we introduce another perspective. We treat GWs as a field $h_{\mu\nu}$ in a flat spacetime with metric $\eta_{\mu\nu}$ similarly as other fields, for example, electromagnetic field.

3.1 Noether Theorem

We consider a field theory living in Minkowski spacetime, with fields ϕ_i. Its action S is the integral of its Lagrangian density over spacetime

$$S = \int dtd^3x\mathcal{L}(\phi_i, \partial\phi_i) \tag{3.1}$$

Since we are interested in physical problems, it is naturally to make the assumption that the fields vanishes on infinities. For gravitational waves, ϕ_i would be the independent components in $h_{\mu\nu}$. But here we don't restrict ourselves to a particular case so that we can compare the gravitational field with other fields, for example, electromagnetic field.

We consider an small coordinate transformation (small enough so that the transformation can be taken as continuous) which will also lead to a transformation of the fields

$$x'^{\mu} = x^{\mu} + \epsilon^a A^{\mu}_a(x) \tag{3.2}$$

$$\phi'_i(x') = \phi_i(x) + \epsilon^a F_{i,a}(\phi, \partial\phi) \tag{3.3}$$

The transformation is determined by $A^{\mu}_a(x)$ and $F_{i,a}(\phi, \partial\phi)$, and parameterized by infinitesimal parameters ϵ^a with $a = 1, 2, , N$.There may be cases where the action of the field is invariant after transformation. We call the transformation a symmetry transformation if it leaves the action invariant. Noether's theorem tells us that if there is a coordinate transformation parameterized by a small parameter and keeps the action of the field invariant, that is, if there is a continous symmetry, the current corresponding to this parameter, which is know as Noether current, will be conserved when the equations of motion are satisfied.

$$(\partial_{\mu} j^{\mu}_a) = 0 \tag{3.4}$$

The corresponding charge Q to such a current is defined as

$$Q_a \equiv \int d^3x j_a^0(\mathbf{x}, t) \tag{3.5}$$

Then we have

$$\begin{aligned} \partial_0 Q_a &= \int d^3x \partial_0 j_a^0(\mathbf{x}, t) \\ &= \int d^3x \nabla \cdot \mathbf{j}(\mathbf{x}, t) \end{aligned} \tag{3.6}$$

Again since we are considering practical physical problem, we won't expect there is any current at the spatial infinity. That means there is no currents coming in or going out at infinity. Therefore the total charge is conserved

$$\partial_0 Q_a = 0 \tag{3.7}$$

We can express the Noether current in terms of the Lagrangian density and the functions that parameterizing the transformation

$$j_a^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}[A_a^\nu(x)\partial_\nu \phi_i - F_{i,a}(\phi, \partial\phi)] - A_a^\mu(x)\mathcal{L} \tag{3.8}$$

One example of Noether's theorem is the conservation of energy-momentum tensors, which is the quantity we want to find for gravitational fields. The energy-momentum tensor is the Noether current corresponding to the constant spacetime translation $x'^\mu = x^\mu + \epsilon^\mu \Longrightarrow x^\mu + \epsilon^\nu \delta_\nu^\mu$. The spacetime translation doesn't change the value of the fields at a ceterin point, therefore we have $\phi'(x') = \phi(x)$. Comparing with Eq.(3.2) and Eq.(3.3) we fix the functions which determine the transformation. $A_\nu^\mu(x) = \delta_\nu^\mu$, $F_{i,a} = 0$. We have expressed the Noether current in terms of these functions and Lagrangian. Then we can determine the Noether current for space-time translation, i.e., the energy-momentum tensor $\theta_\nu^\mu \equiv -j_\nu^\mu$ in terms of Lagrangian. With Eq.(3.8), we have

$$\theta^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial^\nu \phi_i + \eta_{\mu\nu}\mathcal{L} \tag{3.9}$$

When the equations of motion are satisfied, the Noether current is conserved. That means, the energy-momentum tensor is conserved. Eq.(3.4) becomes

$$\partial_\mu \theta^{\mu\nu} = 0 \tag{3.10}$$

The conserved charge associated to space-time translations is the four-momentum P^v. The symmetry of time translation $t \to t + \epsilon^0$ gives the conservation of energy. The symmetry of space translation $x^i \to x^i + \epsilon^i$ gives the conservation of momentum.

$$cP^0 \equiv \int d^3x \theta^{00} \tag{3.11}$$

$$cP^i \equiv \int d^3x\theta^{0i} \tag{3.12}$$

This is how we defined the four-momentum in classical field theory: they are Noether currents associate to symmetry of spacetime translations. Before searching for the energy-momentum tensor for gravitational waves, we can first check the application of Noether's theorem in electrodynamics. The Lagrangian in electrodynamics is

$$\mathcal{L}_{em} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{3.13}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This field theory also has a symmetry under constant spacetime translations. This symmetry gives the energy-momentum tensor of the field. The energy-momentum tensor is given by

$$\theta^{\mu\nu}_{em} = -\frac{\partial\mathcal{L}_{em}}{\partial(\partial_\mu A_\rho)}\partial^\nu A_\rho + \eta_{\mu\nu}\mathcal{L}_{em} \tag{3.14}$$

It is easy to compute

$$\begin{aligned}
\frac{\partial\mathcal{L}_{em}}{\partial(\partial_\mu A_\rho)} &= \frac{\partial}{\partial(\partial_\mu A_\rho)}\big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\big) \\
&= -\frac{1}{2}F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\rho)} \\
&= -F^{\alpha\beta}\frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\mu A_\rho)} \\
&= -F^{\mu\rho}
\end{aligned} \tag{3.15}$$

Therefore we get energy-momentum tensor for electromagnetic field,

$$\theta^{\mu\nu}_{em} = F^{\mu\rho}\partial^\nu A_\rho - \frac{1}{4}\eta^{\mu\nu}F^2 \tag{3.16}$$

We know that classical electrodynamics is invariant under gauge transformations

$$A_\mu \to A_\mu - \partial_\mu\theta \tag{3.17}$$

It is easily seen that $F_{\mu\nu}$ is invariant, therefore Lagrangian is invariant. This means the field theory should remain the same as before after transformation. We then would expect the energy-momentum tensor keeps invariant under this transformation. However, the energy-momentum tensor transforms as

$$\theta^{\mu\nu}_{em} \to \theta^{\mu\nu}_{em} - F^{\mu\rho}\partial^\nu\partial_\rho\theta \tag{3.18}$$

It seems that the energy-momentum tensor is not invariant under this transformation. This problem can be solved if we rewrite the the energy-momentum tensor

$$\begin{aligned}\theta^{\mu\nu}_{em} &= F^{\mu\rho}(\partial^\nu A_\rho - \partial_\rho A^\nu + \partial_\rho A^\nu) - \frac{1}{4}\eta^{\mu\nu}F^2 \\ &= (F^{\mu\rho}F^\nu_\rho - \frac{1}{4}\eta^{\mu\nu}F^2) + F^{\mu\rho}\partial_\rho A^\nu \\ &= (F^{\mu\rho}F^\nu_\rho - \frac{1}{4}\eta^{\mu\nu}F^2) + \partial_\rho(F^{\mu\rho}A^\nu)\end{aligned} \tag{3.19}$$

In the last step we used the equations of motion $\partial_\mu F^{\mu\nu} = 0$. We split the energy-momentum tensor derived before into two parts. The first term is obviously variant under gauge transformation Eq.(3.17) while the second part is not gauge-invariant.

$$\theta^{\mu\nu}_{em} = T^{\mu\nu}_{em} + \partial_\rho C^{\rho\mu\nu} \tag{3.20}$$

where $C^{\rho\mu\nu} = F^{\mu\rho}A^\nu$ and

$$T^{\mu\nu}_{em} = F^{\mu\rho}F^\nu_\rho - \frac{1}{4}\eta^{\mu\nu}F^2 \tag{3.21}$$

which is the energy-momentum tensor we commonly use in electrodynamics. Its 00 component gives the energy density

$$T^{\mu\nu}_{00} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \tag{3.22}$$

Now we are going to explain why could we drop the gauge invariant term. For $C^{\rho\mu\nu}$, we find that it is antisymmetric under $\rho \leftrightarrow \mu$. Therefore when $\theta^{\mu\nu}_{em}$ is conserved, $T^{\mu\nu}_{em}$ is also conserved since

$$\partial_\mu\partial_\rho C^{\rho\mu\nu} = 0 \tag{3.23}$$

The difference between the charges given by $T^{\mu\nu}_{em}$ and $\theta^{\mu\nu}_{em}$ is given by

$$\int d^3x \partial_\rho C^{\rho 0\nu} = \int d^3x \partial_i C^{i0\nu} \tag{3.24}$$

where we use the fact that $C^{00\nu} = 0$ since it is antisymmetric under $\rho \leftrightarrow \mu$. This spatial integral vanishes since we expect no currents coming in or going out at spatial infinity. The electromagnetic field vanishes at the boundaries of the integral. Therefore the four-momenta given by $T^{\mu\nu}_{em}$ and $\theta^{\mu\nu}_{em}$ would be the same. We could see that the Noether theorem cannot give us a physical expression for the energy-momentum tensor since it is not gauge-invariant. However, when integrated over space, it gives the total energy and momentum as long as the boundary terms can be neglected.
From Eq.(3.20) we have

$$\langle\theta^{00}_{em}\rangle = \langle T^{00}_{em}\rangle + \langle\partial_i C^{i00}\rangle \tag{3.25}$$

where the bracket represents the average. When we take the average over a spatial volume such that the boundary terms can be neglected, we have

$$\langle \theta^{00}_{em} \rangle = \langle T^{00}_{em} \rangle = \frac{1}{2} \langle \mathbf{E}^2 + \mathbf{B}^2 \rangle \tag{3.26}$$

We find Noether's theorem can't fix the definition for energy-momentum tensor of electromagnetic field since it may give a term which is not gauge-invariant. We need to find a gauge-invariant tensor that has physical meanings.

3.2 Energy-momentum tensor of gravitational field

In the last section we introduced the Noether theorem and introduced energy-momentum tensor as the Noether current associated to the symmetry of spacetime translation. We derive the energy-momentum tensor to scalar field and electromagnetic field. Now we are going to derive the energy-momentum tensor for gravitational field.
Similarly to the derivation of energy-momentum tensor of scalar field and electromagnetic field, we find the Noether current associated with symmetry of spacetime translation. With Eq.(3.9) we can write down the energy-momentum tensor for gravitational field

$$t^{\mu\nu} = \langle - \frac{\partial \mathcal{L}}{\partial(\partial_\mu h^{\alpha\beta})} \partial^\nu h^{\alpha\beta} + \eta_{\mu\nu} \mathcal{L} \rangle \tag{3.27}$$

To get the Lagrangian $\mathcal{L}$ for gravitational fields we start from the full Einstein Action and expand it to quadratic order in $h_{\mu\nu}$.

$$S_E = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} R \tag{3.28}$$

Here we are consider linearized theory. Remember We take the gravitational waves as a field living in the flat spacetime. So the metric is written as

$$g_{\mu\mu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3.29}$$

We calculate the Ricci scalar and keep the terms quadratic in $h_{\mu\nu}$,

$$R = g^{\mu\nu} R_{\mu\nu} = (\eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2))(R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \mathcal{O}(h^2)) \tag{3.30}$$

where $R^{(1)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$ represent the first-order term and second-order term in $R_{\mu\nu}$, respectively. Their expressions have been given in Eq.(2.40) and Eq.(2.41). Since we are expanding around flat spacetime, the zero-order term vanishes. To compute $\sqrt{-g}$, we rewirte it as

$$g_{\mu\nu} = \eta_{\mu\nu} g^{rho}_{\nu} \tag{3.31}$$

Therefore,

$$\begin{aligned} det(g_{\mu\nu}) &= det(\eta_{\mu\nu})det(g^{rho}_{\nu}) \\ &- det(g^{\rho}_{\nu}) \end{aligned} \tag{3.32}$$

$$g^{\rho}_{\nu} = \delta^{\rho}_{\nu} + h^{\rho}_{\nu} \equiv (I + H)^{\rho}_{\nu} \tag{3.33}$$

where I is the identity matrix and H a matrix whose elements are h^{ρ}_{ν}. Use the identity log(det A) = Tr(logA), we have

$$\begin{aligned} -g &= det(I + H) \\ &= exp[log\ det(I + H)] \\ &= exp[Tr\ log(I + H)] \\ &= exp[Tr(H + \mathcal{O}(H^2))] \\ &= 1 + TrH + \mathcal{O}(H^2) \\ &= 1 + h + \mathcal{O}(h^2_{\mu\nu}) \end{aligned} \tag{3.34}$$

where h is the trace of $h_{\mu\nu}$. Then we can compute the action, using integration by parts, and get

$$S_E = \frac{c^3}{64\pi G} \int d^4x[\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \partial_\mu h \partial^\mu h + 2\partial_\mu h^{\mu\nu}\partial_\nu h - 2\partial_\mu h^{\mu\nu}\partial_\rho h^{\rho}_{\nu}] \tag{3.35}$$

We then obtain the Lagrangian density for gravitational field,

$$\mathcal{L} = \frac{c^4}{64\pi G}[\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \partial_\mu h \partial^\mu h + 2\partial_\mu h^{\mu\nu}\partial_\nu h - 2\partial_\mu h^{\mu\nu}\partial_\rho h^{\rho}_{\nu}] \tag{3.36}$$

Then we can calculate the energy-momentum tensor $t_{\mu\nu}$ of the gravitational field. To make our calculation simpler, we can fix the gauge

$$\partial_\mu h^{\mu\nu} = 0, \quad h = 0 \tag{3.37}$$

Then we have

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu h_{\alpha\beta})} = -\frac{c^4}{32\pi G}\partial^\mu h^{\alpha\beta} \tag{3.38}$$

To compute $\langle \mathcal{L} \rangle$, we preform integration by parts and find this term vanishes

$$\langle \mathcal{L} \rangle = 0 \tag{3.39}$$

Therefore we obtain the expression for energy-momentum tensor of gravitational field.

$$t^{\mu\nu} = \frac{c^4}{32\pi G}\langle \partial^\mu h^{\alpha\beta}\partial^\nu h_{\alpha\beta} \rangle \tag{3.40}$$

We get the same result as before in Eq.(2.61), as expected.
We find that the expression for energy-momentum tensor of GWs is written in terms of average. We didn't get any expressions for local energy-momentum tensor. We get the same result in Chpater 1. In Chapter 1, by analyzing the low-frequency part of Einstein equations we obtained the "coarse-grained" energy-momentum tensor which takes effect over a spatial volume with proper size.
The Noether's theorem itself cannot fix the energy momentum tensor uniquely. We have seen that when looking for the energy-momentum tensor of electromagnetic field. Suppose for a field theory, we add a total divergence to the Lagrangian density,

$$\mathcal{L}' = \mathcal{L} + \partial_\mu K^\mu \tag{3.41}$$

The equation of motion is derived from variation of the action. Since the total divergence will only give a boundary term after integration, we will derive the same equation of motion from this two action $S' = \int d^4x\mathcal{L}'$ and $S' = \int d^4x\mathcal{L}$. The two Lagrangian density define the same field theory. However, they give different Noether current and energy-momentum tensor. In electromagnetism, it is reasonable to define $T^{em}_{\mu\nu}$ as the local energy-momentum tensor since it is gauge-invariant and when taking average over a proper spatial volume, it gives the same behaviour as $t_{\mu\nu}$, the energy-momentum tensor we derived from Noether theorem. For GWs, we find that $\partial^\mu h^{\alpha\beta}\partial^\nu h_{\alpha\beta}$ is not gauge-invariant therefore cannot be the local energy-momentum tensor. Actually there is no such local expression. Consider the equivalence principle, at any given point, we can always find a local inertial frame where the gravitational field vanishes. The energy then becomes zero, and will be always zero if there exists a gauge-invariant term to describe it.

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4 Generation of GWs

In the previous sections we have introduced gravitational waves as perturbations to the background spacetime. We have seen how do they arise from linearized Einstein equations. We have studied the the effect of gravitational waves, how do they interact with test particles. Also we have derived the energy and momentum the gravitational waves carries. We have known all these properties of gravitational waves, but how are they generated in the universe? That would be the topic of this section.

4.1 Quadrupole formula

Here we still restrict ourselves to linearized theory. We assume the source of the gravitational waves is weak enough so that the background spacetime is approximately Minkowski spacetime and the gravitational waves it generated can be taken as small perturbations. In Chapter 2 we have derived the equation for perturbations from linearization of Einstein equations, Eq.(2.13), which we write down here again

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \tag{4.1}$$

Before thinking about the solution to this equation, we may recall what we have learned in electromagnetism to help us understand gravitational waves better.
We know that the electromagnetic waves arise when the electromagnetic field varies, which may caused by, for example, an accelerating charge. The monopole radiation, if there were any, should give the most contribution to the radiation. But the electric monopole moment is $\int \rho d^3x$, the total charge. From Maxwell equations we know the charge is conserved. Therefore there is no monopole radiation. Then the most primary contribution to the electromagnetic radiation comes from the electric dipole moment, which is given by

$$p_i = \int \rho x_i d^3x \tag{4.2}$$

where ρ denotes the density of charge.
We now turn back to gravitational waves. As same as how we solve the wave equations for electromagnetic waves, we solve Eq.(**??**) by using the Green's function techniques,

$$\Box G(x - x') = \delta^4(x - x') \tag{4.3}$$

The radiation takes time to propagate towards us, therefore we use the retarded Green's function

$$G(x - x') = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}\delta(x^0_{ret} - x'^0) \tag{4.4}$$

where $x'^0 = ct'$, $x^0_{ret} = ct_{ret}$ and

$$t_{ret} = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c} \tag{4.5}$$

With the retarded Green's function, we can write down the solution to Eq.(**??**)

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{\mu\nu}\big(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}'\big) \tag{4.6}$$

Remember how we fix the gauge in Chapter 2. Here we haven't apply any constraint on gauge choice yet. $\bar{h}_{\mu\nu}$ has 10 degrees of freedom now. For the purpose of simplicity we would like to reduce the gauge freedom. We impose the TT gauge and this can be done by introducing a projector, which can act on a tensor to project it into TT gauge

$$\Lambda_{ij,kl}(\hat{\mathbf{n}}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}, \quad P_{ij}(\hat{\mathbf{n}}) = \delta_{ij} - n_i n_j \tag{4.7}$$

Act this projector on Eq.(4.6) we project the solution into TT gauge

$$h^{TT}_{ij}(t, \mathbf{x}) = \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{kl}\big(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}'\big) \tag{4.8}$$

where $\hat{\mathbf{n}} = \hat{\mathbf{x}}$ denotes the direction from theh source to the detector. Here we take two limits to simplify the expression of the perturbation and to see its nature more clear. The first is the weak-field limit that we assume we the observers are at sitting at far away enough from the source, which area is known as far zone such that the distance between the source and us can be taken as

$$|\mathbf{x} - \mathbf{x}'| \approx r \tag{4.9}$$

Another one is the slow-motion limit that we assume the source is non-relativistic. That is, the speed of motion inside the source is much smaller than the speed of light. That ensures the wavelength of gravitational waves is much larger than the scale of the source. Therefore we neglect the effect arising due to the inner structure of the source. Taking the weak-field limit and the slow-motion limit, the wave function in TT gauge Eq.(4.8) can be written as

$$h^{TT}_{ij}(t, \mathbf{x}) = \frac{1}{r}\frac{4G}{c^4} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \int d^3x' T_{kl}\big(t - \frac{r}{c}, \mathbf{x}'\big) \tag{4.10}$$

Perform Fourier transform of the energy-momentum tensor of the source T_{kl}, and expand the exponential into series, we are going to find the leading term is proportional to the second time derivative of the quadrupole moment of the source

$$h^{TT}_{ij}(t, \mathbf{x}) = \frac{2}{r}\frac{G}{c^4}\frac{\partial^2}{\partial_t^2} I^{TT}_{ij}(t - \frac{r}{c}) \tag{4.11}$$

where I_{ij} is the quadrupole moment of the source

$$I_{ij} = \int \rho x_i x_j d^x \tag{4.12}$$

This ρ here comes from the 00-component of the energy-momentum tensor $\rho = T_{00}/c^2$. Eq.(4.11) tells us a lot of information about gravitational waves. We find h_{ij} is proportional to $\frac{1}{r}$, which implies the amplitude of gravitational waves decreases as $\frac{1}{r}$, which is the same as electromagnetic waves.

Also from Eq.(4.11) we can estimate the order of magnitude of gravitational waves

$$h_{ij} \sim 10^{-21} \left(\frac{ML^2\omega^2}{M_\odot c^2} \right) \left(\frac{100Mpc}{r} \right) \tag{4.13}$$

where M denotes the mass of the source, L denotes the scale of the source and ω the frequency of quadrupole oscillation of the source. Actually $ML^2\omega^2$ is just the no spherical part of kinetic energy of the source.

We find that the leading term in gravitational waves is quadrupole radiation, coming from the quadrupole oscillation of mass of the source. Where is the monopole radiation and the dipole radiation? How is it different from the electromagnetic waves? The monopole moment is $\int \rho d^3x$. In the case of gravitational waves it is the total mass of the source. The mass of the source is conserved, restricted by the conservation law $\partial_\mu T^{\mu\nu} = 0$. Therefore there is no monopole radiation. The dipole moment is given by $\int \rho x^i d^3x$. Its first derivative $\int \rho v^i d^3x$ is the total momentum of the source which is constant since there is no external force to make it accelerate. The source is static. Therefore the second derivative of the dipole moment is zero. There is no dipole radiation in gravitational waves. The primary contribution in gravitational waves comes from quadrupole radiation, generated by quadrupole oscillations of mass. Of course there are other higher order terms for example, the mass octupole radiation and current quadrupole radiation, whose magnitude are negligible.

In section 2.1 we have introduced the TT gauge and reduced the degree of freedom in $h_{\mu\nu}$ from 10 to 2. We find the two left degrees of freedom represent 2 polarizations. Now we will show how do the polarizations depend on the quadrupole moment. Recall we have introduced the projector $\Lambda_{ij,kl}$ in Eq.(4.7). We act it on the quadrupole moments and find

$$\Lambda_{ij,kl}\ddot{I}_{kl} = \begin{pmatrix} (\ddot{I}_{xx} - \ddot{I}_{yy})/2 & \ddot{I}_{xy} & 0 \\ \ddot{I}_{yx} & -(\ddot{I}_{xx} - \ddot{I}_{yy})/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \tag{4.14}$$

Compared to Eq.(2.27) we find expression for the quadrupole part of the two polarizations

$$\begin{aligned} h_+ &= \frac{1}{r}\frac{G}{c^4}(\ddot{I}_{xx} - \ddot{I}_{yy}) \\ h_\times &= \frac{2}{r}\frac{G}{c^4}\ddot{I}_{xy} \end{aligned} \tag{4.15}$$

In Section 2.3, we have calculated the energy and momentum carried by gravitational waves. The power of quadrupole radiation is given by

$$P = \frac{32}{5}\mu^2 M^{4/3}\Omega^{10/3} \tag{4.16}$$

4.2 Effect of GWs on their source

We have illustrate in Chapter 2 that the gravitational waves are taking energy and angular momentum away from the source to spatial infinity. The source must get affected by the emission of gravitational waves. Imagine there is a binary system with two massive and dense objects, for example, black holes or neutron stars. As the gravitational waves taking energy and angular momentum away, the orbit of this binary system must decay. In this section we are going to study how does the loss of energy affect such a binary system.

We take the binaries as two point particles with masses m_1 and m_2 and spatial coordinates $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. And we assume the orbits of the binary system are circular. When considering a two body problem, it is natural to use the center of mass coordinate system. The total mass is $M = m_1 + m_2$. The reduced mass is $\mu = (m_1 + m_2)/M$. When analyzing this binary system, we apply Newtonian approximation, that is, we neglect the relativistic effects, assuming the velocity of the binaries are much smaller than the speed of light. The Kepler's law gives

$$\Omega^2 = \frac{GM}{a^3} \tag{4.17}$$

where Ω denotes the orbital angular velocity and a denotes the distance between the two binaries. It is easy to get the quadrupole moment of the mass distribution

$$\begin{aligned} I_{xx} &= \mu a^2 \cos^2\Omega t \\ I_{yy} &= \mu a^2 \sin^2\Omega t \\ I_{xy} &= I_{yx} = -\mu a^2 \cos\Omega t \sin\Omega t \end{aligned} \tag{4.18}$$

We have derived how the quadrupole radiation of gravitational waves depends on the second derivative of quadrupole moment of the mass distribution of the source Eq.(4.11).

$$\begin{aligned} \ddot{I}_{xx} &= -\ddot{I}_{yy} = 2\mu a^2\Omega^2 \cos 2\Omega t \\ \ddot{I}_{xy} &= 2\mu a^2\Omega^2 \sin 2\Omega t \end{aligned} \tag{4.19}$$

Notice here the quadrupole moments are given without fixing the gauge while Eq.(4.11) is written in TT gauge. Therefore we need to project the quadrupole moment into TT gauge. Now we are in Cartesian coordinate system, it will be easier for us to project the quadrupole moment into TT gauge if we transform the coordinate system to a spherical coordinate system. Substitute this in Eq.(4.15) we find the two polarizations of the quadrupole radiation emitted by this binary system

$$\begin{aligned} h_+ &= \frac{2}{r}\frac{G\mu\Omega^2 a^2}{c^4}(1+\cos^2\theta)\cos[2(\Omega t+\phi)] \\ h_\times &= \frac{4}{r}\frac{G\mu\Omega^2 a^2}{c^4}\cos\theta\sin[2(\Omega t+\phi)] \end{aligned} \tag{4.20}$$

Observing the expression we find that the frequency of quadrupole moment of gravitational waves is twice the orbital angular velocity of the binary system.

$$\omega = 2\pi f = \frac{\Omega}{2} \tag{4.21}$$

We find the amplitude of the quadrupole radiation depend on the angular velocity of the system Ω and the separation between the two stars a. The angular velocity Ω is the parameter we can observe directly while a is not. We can use the Kepler's law Eq.(4.17) to represent a in terms of Ω, and introduce the chirp mass to make the expression more compact. The chirp mass is defined as

$$\mathcal{M} = \mu^{3/5} M^{2/5} \tag{4.22}$$

Then Eq.(4.20) becomes

$$\begin{aligned} h_+ &= \frac{2}{r}\frac{(G\mathcal{M})^{5/3}}{c^4}\Omega^{2/3}(1+\cos^2\theta)\cos[2(\Omega t+\Phi)] \\ h_\times &= \frac{4}{r}\frac{(G\mathcal{M})^{5/3}}{c^4}\Omega^{2/3}\cos\theta\sin[2(\Omega t+\Phi)] \end{aligned} \tag{4.23}$$

So far we have calculated the quadrupole radiation emitted by a binary system in Newtonian approximation. We would like to know how this emission of radiation reacts on the binary system. We write down the orbit energy of the binary system in terms of period $T = 2\pi/\Omega$

$$\begin{aligned} E_{\text{orbit}} &= \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\phi}^2) \\ &= -\frac{\mu}{2}(2\pi)^{2/3} M^{2/3} T^{-2/3} \end{aligned} \tag{4.24}$$

The change of orbit energy per unit time would be the power of the emitted radiation,

$$\frac{dE_{\text{orbit}}}{dt} = -P \tag{4.25}$$

Together with Eq.(4.16) and Eq.(4.24), we obtain how the period of the binary system change with time

$$\frac{dT}{dt} = -\frac{96}{5}(2\pi)^{8/3} M^{2/3} \mu T^{-5/3} \tag{4.26}$$

We find the period is decreasing faster and faster and finally when T goes to 0, the two particles coalescence. We introduce the two-body system as a simple model for binary black holes. A binary black hole will shrink its orbit (we call this phase inspiral) and merge into one single black hole. Then it enters the ringdown phase: the new perturbed black hole oscillates and emits gravitational waves to reach equilibrium state. The ringdown signals are superposition of quasinormal modes of this black hole. Here we use a theoretical model for a black hole merger GW150914 (Figure.2) to show the evolution of waveform in the merger event.

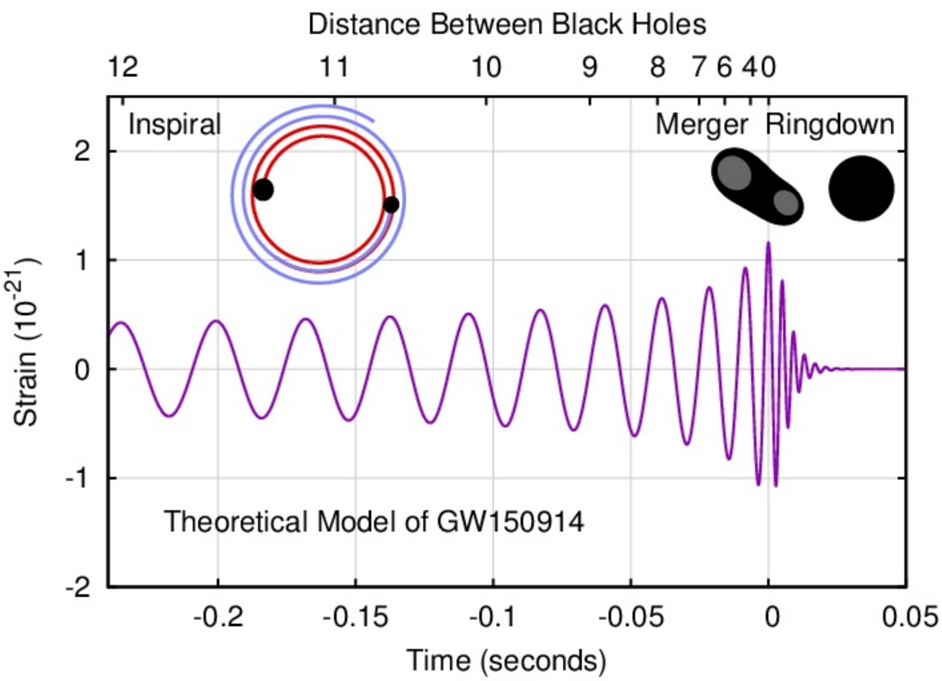

Figure 2. Theoretical model Source: http://ccrg.rit.edu/GW150914

We have shown in Eq.(4.21) the frequency of gravitational waves is twice as the orbital angular velocity, therefore increases in the inspiral phase. In the ringdown phase, the system becomes a perturbed single black hole. The gravitational waves are the superposition of quasinormal modes of the black hole. We will study that in Chapter 6.

5 Black Holes

In the former chapters we introduced gravitational waves as perturbations around background spacetime. The perturbations are generated due to the change of gravitational field. Black holes are one of the sources of gravitational field. The existence of black holes is also one of the predictions of Einstein's general relativity. It is a region of spacetime where the spacetime is curved so much that even light can't escape. The hypersurface separating this region from the rest of spacetime is called event horizon. There is no way for a signal to escape once entering into the event horizon, therefore there is no direct way for us to observe black holes. However, some events of black holes (for example, black hole merger or black hole excitation) lead to change in gravitational fields and thus generate gravitational waves. As we said before, gravitational waves travel in universe at speed of light and interact very weakly with objects they encounter, retaining the information of their sources. Therefore we can study black holes by observing gravitational waves. In this chapter we introduce the concept of black holes and solve Einstein's equation to find two types of black holes.

5.1 Schwarzschild metric

Before illustrating how does Einstein's equation predicts the existence of black holes, we first consider the spacetime with a static and spherically symmetric distribution of matter, with total mass M. We are going to find the metric for such a spacetime.
First we would like to find the line-element for such a distribution. The matter distribution is spherically symmetric, we then expect the geometry of gravitational field it generates also to be spherically symmetric. So the line-element should be spherically symmetric. One example for a spherically symmetric line-element is the line-element for Minkowski space

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{5.1}$$

Or if we express it in spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{5.2}$$

where $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$. However, it is for the flat spacetime with no source to the gravitational field. We would like to find a general expression for spherically symmetric line-element. It should have the form

$$ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + r^2 d\Omega^2 \tag{5.3}$$

Terms of the form $dtdr$ can also exist but can be disappeared by coordinate transformation. We are interested in the spacetime outside the distribution of matter, there the energy-momentum tensor of the source $T_{\mu\nu} = 0$. Therefore the metric of spacetime outside the matter distribution obeys the vacuum Einstein equations.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{5.4}$$

where $R_{\mu\nu}$ is the Ricci tensor

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} \tag{5.5}$$

and R the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu} \tag{5.6}$$

The Christoffel symbol is given by

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu.\sigma}) \tag{5.7}$$

We can now compute the Christoffel symbols with our metric components

$$g_{tt} = -A, g_{rr} = B, g_{\theta\theta} = r^2, g_{\phi\phi} = r^2 sin^2\theta \tag{5.8}$$

and their inversions

$$g^{tt} = -\frac{1}{A}, g_{rr} = \frac{1}{B}, g_{\theta\theta} = \frac{1}{r^2}, g_{\phi\phi} = \frac{1}{r^2 sin^2\theta} \tag{5.9}$$

Here we write down the non-vanishing terms of Christoffle symbols

$$\begin{aligned}
&\Gamma^{t}_{tr} = \Gamma^{t}_{rt} = \frac{A'}{2A}, \\
&\Gamma^{r}_{tt} = \frac{A'}{2B}, \Gamma^{r}_{rr} = \frac{B'}{2B}, \Gamma^{r}_{\theta\theta} = \frac{r}{B}, \Gamma^{r}_{\phi\phi} = \frac{r sin^2\theta}{B} \\
&\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r}, \Gamma^{\theta}_{\phi\phi} = -cos\theta sin\theta \\
&\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{r}, \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \frac{cos\theta}{sin\theta}
\end{aligned} \tag{5.10}$$

Then we can compute the Ricci tensor

$$\begin{aligned}
R_{tt} &= -\frac{A"}{2B} + \frac{A'B'}{4B^2} + \frac{(A")^2}{4AB} - \frac{A'}{rB} \\
R_{rr} &= \frac{A"}{2A} - \frac{(B')^2}{4B^2} - \frac{A'B'}{4AB} - \frac{B'}{Br} \\
R_{\theta\theta} &= \frac{rA'}{2AB} + \frac{1}{B} - \frac{rB'}{2B^2} - 1 \\
R_{\phi\phi} &= (\frac{rA'}{2AB} + \frac{1}{B} - \frac{rB'}{2B^2} - 1) sin^2\theta
\end{aligned} \tag{5.11}$$

The Ricci scalar is

$$\begin{aligned} R &= g^{\mu\nu}R_{\mu\nu} \\ &= -\frac{A"}{AB} + \frac{A'B'}{2AB^2} + \frac{(A')^2}{2A^2B} - \frac{2A'}{rAB} + \frac{2B'}{rB^2} + \frac{2}{r^2}(1 - \frac{1}{B}) \end{aligned} \tag{5.12}$$

Then we can solve the vacuum Einstein Equation Eq.(5.4). It gives us three independent equations

$$\frac{B'}{rB^2} + \frac{1}{r^2}(1 - \frac{1}{B}) = 0 \tag{5.13}$$

$$-\frac{A'}{rAB} + \frac{1}{r^2}(1 - \frac{1}{B}) = 0 \tag{5.14}$$

$$-\frac{A'}{A} + \frac{B'}{B} - \frac{rA"}{A} + \frac{rA'B'}{2AB} + \frac{r(A')^2}{2B^2} = 0 \tag{5.15}$$

Eq.(5.13) gives

$$\frac{dB}{B^2 - B} = -\frac{dr}{r} \tag{5.16}$$

Integrating both sides we find

$$B = \frac{1}{1 - \frac{c}{r}} \tag{5.17}$$

where c is a constant of integration. With Eq.(5.14) we can solve A

$$A = 1 - \frac{c}{r} \tag{5.18}$$

Then the metric is determined

$$ds^2 = -(1 - \frac{c}{r})dt^2 + \frac{1}{1 - \frac{c}{r}}dr^2 + r^2 d\Omega^2 \tag{5.19}$$

We then find the expression for a spherically symmetric metric. Since we are considering a physical problem, we would expect the constant c to have some physical meanings instead of an arbitrary number. To find the value of c we can consider the Newton limit of this metric and compare it to what we have known in Newtonian Physics. We can take the Newton limit when the gravitational field is very weak, that is where we are far away from the source. We find that when $r \gg c$, i.e., at large distance from the matter distribution, the metric is close to Minkowski metric. So we write the metric as the Minkowski metric plus a perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{5.20}$$

where $|h_{\mu\nu}| \ll 1$. Consider the motion of a freely falling particle in such a metric, we have the Geodesics equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \tag{5.21}$$

In Newton limit, it is natural to assume the speed of the freely falling particle is much smaller that light, otherwise we need to consider relativistic effect.

$$|\frac{dx^i}{d\tau}| \ll 1 \tag{5.22}$$

With this assumption, Eq.(5.21) becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{tt}\big(\frac{dt}{d\tau}\big)^2 = 0 \tag{5.23}$$

To first order in $h_{\mu\nu}$, we have

$$\Gamma^\mu_{tt} = -\frac{1}{2}\eta^{\mu\sigma}\frac{\partial h_{tt}}{\partial x^\sigma} \tag{5.24}$$

Assume the metric is static, then

$$\Gamma^t_{tt} = 0 \tag{5.25}$$

Therefore

$$\frac{d^2t}{d\tau^2} = 0 \tag{5.26}$$

which means$dt/d\tau$ is costant. Therefore

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{00} = \frac{1}{2}\frac{\partial h_{tt}}{\partial x^i} \tag{5.27}$$

So far we have find how do a freely falling particle moves in the perturbed Minkowski metric using geodesic equations. We now consider the motion of the freely falling particles from another perspective, that is, the Newtonian mechanism, then compare the results we derived. Newtonian mechanism tells us the acceleration of a freely falling particle in a gravitational field is determined by the gravitational potential,

$$\frac{d^2\mathbf{x}}{dt^2} = -\nabla\phi \tag{5.28}$$

where ϕ is the gravitational potential. In Newtonian physics, the gravitational potential is given by the Poisson's equation $\hat{\nabla}^2\phi = 4\pi G\rho$, where ρ is the mass density.

$$\phi = -\frac{GM}{r} \tag{5.29}$$

We have considered the motion of a freely particle both with geodesic equations and with Newtonian mechanism. They should give the same result. Compare Eq.(5.27) and Eq.(5.28) we will find

$$h_{tt} = -2\phi = \frac{2GM}{r} \tag{5.30}$$

We then find the tt-component of the metric for gravitational fields far away from the source. With this we can fix the expression of c in Eq.(5.19)

$$c = 2GM \tag{5.31}$$

which is known as the Schwarzschild radius. The metric Eq.(5.19) becomes

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{1}{1 - \frac{2GM}{r}}dr^2 + r^2 d\Omega^2 \tag{5.32}$$

This is the metric we derived for the spacetime outside a spherically symmetric distribution of matter. It is known as Schwarzschild metric. According to Brikhoff's theorem, this is the only solution of Einstein equations for a spherically symmetric spacetime.

5.2 Schwarzschild Black Hole

In the last section we have derived to solution of Einstein equations for a spherically symmetric distribution of matter. Remember the equations we employed are the vacuum Einstein equations. This means the solution we derived is only valid for region of spacetime outside the matter distribution. In this section we are going to study what would happen if the mass distribution is very compact and introduce the concept of black holes.

Starting with the Schwarzschild metric we derived before, we find that when r goes to 2GM or 0, some of the metric components blow up, indicating there might be singularities at $r = 0$ and $r = 2GM$. But we don't know if the singularities are real physical singularities or just arise because of our bad choice of coordinate system. To check the existence of singularity we may consider the value of a scalar to see if it blows up at those singularities Since a scalar is invariant under coordinate transformation, it won't become singular because of choice of coordinate system. Here we consider the Ricci scalar and find

$$R = \frac{48G^2M^2}{r^6} \tag{5.33}$$

It is easily seen that $r = 0$ is a real singularity while the singularity at $r = 2GM$ is not. It indicates the singularity at $r = 2GM$ can be removed by coordinate transformation, which we will introduce later.

The Schwarzschild metric is the solution for vacuum Einstein equation, valid in the region outside the matter distribution. For a star, it would be valid only for the spacetime outside the sphere of the star. Take our Sun for example, its radius is

$$R = 10^6 GM \tag{5.34}$$

which is much larger than its Schwarzschild radius $r_0 = 2GM$. For such a case, the Schwarzschild metric only applies to the exterior spacetime outside the sphere of the sun while for the spacetime inside the star we need another metric.

There is nothing special if the size of matter distribution is large than its Schwarzschild radius. But what if the matter is distributed very compactly such that its radius is smaller than its Schwarzschild radius? We will study the motion of a freely falling particle moving towards the matter distribution and will find it has an interesting behaviour. We then introduce the concept of black holes and event horizons.

Consider a freely falling massive particle moving towards the center of the Schwarzschild metric. We denote its proper time by τ. We the observers far away use the coordinate time t since the metric for r $\gg r_0$ is approximately Minkowski metric. For the purpose of simplicity we assume that the particle is moving along a time-like radial geodesic, with fixed angles θ and ϕ.

$$\frac{d\theta}{d\tau} = \frac{d\phi}{d\tau} = 0 \tag{5.35}$$

For a timelike curve, we have

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1 \tag{5.36}$$

We know that each of the symmetry of the metric is associated with a conserved quantity. The time translation symmetry gives a timelike Killing vector

$$K^\mu = (\partial_t)^\mu = (1, 0, 0, 0) \tag{5.37}$$

Lower the index we have

$$K_\mu = \big(-\big(1 - \frac{2GM}{r}), 0, 0, 0) \tag{5.38}$$

The associated conserved quantity is the energy (per unit mass)

$$E = -K_\mu \frac{dx^\mu}{d\tau} = (1 - \frac{2GM}{r})\frac{dt}{d\tau} \tag{5.39}$$

When r goes to infinity, the proper time τ that the test particle measures should be the same with coordinate time since it is approximately in the Minkowski metric. And since the energy E is a conserved quantity, we have

$$E = 1 \tag{5.40}$$

Therefore

$$\frac{dt}{d\tau} = \frac{1}{1 - \frac{2GM}{r}} \tag{5.41}$$

From Eq.(5.36) we have

$$-(1 - \frac{2GM}{r})(\frac{dt}{d\tau})^2 + (1 - \frac{2GM}{r})^{-1}(\frac{dr}{d\tau})^2 + r^2(\frac{d\theta}{d\tau})^2 + r^2 sin^2\theta(\frac{d\phi}{d\tau})^2 = -1 \tag{5.42}$$

We have already fixed θ and ϕ, therefore

$$\frac{dr}{d\tau} = -\sqrt{\frac{2GM}{r}} \tag{5.43}$$

This gives the velocity of a test particle with respect to the proper time that the test particle itself measures. We choose minus sign here since the particle is moving towards the center of the metric. We then obtain the velocity of test particles at the perspective of far away observers who measure coordinate time t

$$\frac{dr}{dt} = -\sqrt{\frac{2GM}{r}}(1 - \frac{2GM}{r}) \tag{5.44}$$

In the point of view of the far away observer, the test particle is moving more and more slowly when it is approaching $r = 2GM$. The series expansion at $r = 2GM$ gives

$$-\sqrt{\frac{2GM}{r}}(1 - \frac{2GM}{r}) = -\frac{r - 2GM}{2GM} + \mathcal{O}((r - 2GM)^2) \tag{5.45}$$

Therefore, near $r = 2GM$ we have

$$r = 2GM + ce^{-t/2GM} \tag{5.46}$$

where c is a constant. The second term will always be posiitive. This implies that as the far away observer can see, the test particle will get closer and closer to $r = 2GM$ but never get through it. However the test particle itself don't use the coordinate time. There is nothing special about its velocity with respect to its proper time. From Eq.(5.43) we derive

$$\frac{2}{3}(r^{\frac{3}{2}} - 2GM^{\frac{3}{2}}) = \sqrt{2GM}(\tau_0 - \tau) \tag{5.47}$$

where $r(\tau_0) = 2GM$. It shows that although in the point of view of far away observers a test particle can never reach $r = 2GM$, the particle is able to get through $r = 2GM$ in finite time. $r = 2GM$ determines a sphere which we call it by event horizon. Analogously to a horizon on Earth, we can't see further beyond the horizon but if we move towards it we can get through it in finite time. As we have said before, the existence of event horizon prevents us from observing inside the black hole directly. We will show that a light signal once entered the event horizon can never escape. Consider a radial null curve inside black hole (behind event horizon), we have

$$g_{\mu\nu}dx^\mu dx^\nu = 0 \tag{5.48}$$

which gives

$$-(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1}dr^2 = 0 \tag{5.49}$$

Suppose the light is moving outwards, we find

$$\frac{dr}{dt} = 1 - \frac{2GM}{r} \tag{5.50}$$

The light ray at $r = 2GM$ would have zero velocity thus sit still at the event horizon forever. Outside the event horizon $dr > 0$, the light ray is moving away from the center. Inside the horizon, $dr < 0$, which means the outgoing null curve is actually pointing towards the center. The light ray inside the event horizon would never escape. This means we can never know what happens behind the event horizon. We call the region of spacetime behind the event horizon a black hole. In the Schwarzschild metric, we call it Schwarzschild black hole.

We have introduced the interesting properties of event horizon by considering the motion of a test particle moving towards it and the motion of a light ray trying to escape from behind the event horizon. However, this hypersurface is not special in spacetime. There is no singularity at the event horizon. We can illustrate this by makeing a coordinate transformation and will find no metric components blow up in the new coordinate system at $r = 2GM$. Here we define the tortoise radial coordinate

$$r^* = r + 2GM\log(\frac{r}{2GM} - 1) \tag{5.51}$$

for $r > 2GM$. Then

$$dr^* = \frac{dr}{1-\frac{2GM}{r}} \tag{5.52}$$

We can define new quantities

$$u = t - r^*, \quad v = t + r^* \tag{5.53}$$

and make the coordinate transformation for the spherical coordinate system (t, r, θ, ϕ) to the Eddington-Finkelstein coordinates (v, r, θ, ϕ), we find in the Eddington-Finkelstein coordinates, the Schwarzschild metric becomes

$$ds^2 = -(1-\frac{2GM}{r})dv^2 + 2dvdr + r^2d\Omega^2 \tag{5.54}$$

It is obviously seen that nothing blows up at $r = 2GM$, indicating that there is no singularity at $r = 2GM$.

5.3 Kerr Black Hole

We have find the solution to vacuum Einstein equations for a spherically symmetric distribution of matter. And we find a special case where the matter is distributed only inside the Schwarzschild radius, which leads to a black hole. But Schwarzschild black holes is not the only solution of the vacuum Einstein equations. There are also other structures of spacetime and therefore leads to other type of black holes. One example is the Kerr black hole, a solution to vacuum Einstein equations but is not spherically symmetric. The absence of spherical symmetry means that the Kerr metric describes a rotating black hole, therefore a Kerr black hole possesses angular momentum J. The Kerr metric is

$$\begin{aligned} ds^2 = &-\big(1 - \frac{2GMr}{\rho^2}\big)dt^2 - \frac{4GMarsin^2\theta}{\rho^2}sin^2\theta dtd\phi \\ &+ \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{sin^2\theta}{\rho^2}[(r^2+a^2)^2 - a^2\Delta sin^2\theta]d\phi^2 \end{aligned} \tag{5.55}$$

where

$$\rho^2(r,\theta) = r^2 + a^2cos^2\theta, \quad \Delta(r) = r^2 - 2GMr + a^2 \tag{5.56}$$

while a is the angular momentum per unit mass

$$a = \frac{J}{M} \tag{5.57}$$

The event horizon for Kerr black hole is at where $g^{rr} = 0$, that is

$$\frac{\Delta}{\rho^2} = 0 \tag{5.58}$$

Since $\rho^2 > 0$, this leads to

$$\Delta(r) = r^2 - 2GMr + a^2 = 0 \tag{5.59}$$

When $GM < a$, Δ will always be positive. We find that there is no event horizon in such a case. When $GM = a$, the spacatime will be unstable. We are only interested in the case $GM > a$,

$$r_\pm = GM \pm \sqrt{G^2M^2 - a^2} \tag{5.60}$$

Therefore we get two event horizons. A rotating black hole can be stationary but not static. Consider the Killing vector ∂_t,

$$K^\mu K_\mu = -\frac{1}{\rho^2}(\Delta - a^2sin^2\theta) \tag{5.61}$$

We find that at event horizons, where $\Delta = 0$,

$$K^{\mu}K_{\mu} = \frac{1}{\rho^2}a^2 sin^2\theta \tag{5.62}$$

The surface where the Killing vector ∂_t becomes null is the stationary limit surface. It is given by

$$(r - GM)^2 = G^2M^2 - a^2 cos^2\theta \tag{5.63}$$

The region between stationary limit surface and the outer event horizon is called the ergoregion. An observer entering the ergoregion may leave it again but can't stay stationary.

6 Quasinormal modes of BHs

We know a classical object would have "characteristic sound". No matter what perturbation we impose on such an objectthe response of the object will always be a superposition of normal modes. The normal modes are not determined by the perturbations we impose but purely depend on the intrinsic properties of the object. We expect there are also such characteristic modes for black holes that reveal the parameters of black holes. But since gravitational waves are taking energy away, there is no normal modes. Instead, we have quasinormal modes for such a dissipate system. In this Chapter we introduce the concept of quasinormal modes as poles on Green's functions and find the quasinormal modes for Schwarzschild black holes.

6.1 Quasinormal modes

We know that if there is no lose of energy, when an object is perturbed, its oscillation can be described by a superposition of normal modes. For example, if we vibrate a guitar string, with both ends fixed, we could always hear the same note (frequency). This is the fundamental frequency of the string. It doesn't depend on how and where we vibrate the string, but only depend on the intrinsic properties of the string (the length of the string, for example). For such a system, the wave equation would be

$$\partial_t^2\psi - \partial_x^2\psi = 0 \tag{6.1}$$

Its solutions depend only on its boundary conditions. The general solution will be a superposition of normal modes,

$$\psi(t,x) = \sum_{n=1}^{\infty} a_n e^{i\omega_n t}\psi_n(x) \tag{6.2}$$

where ψ_n are the normal modes of this object. The normal modes of a system depend on its intrinsic nature. They are determined only by the properties of the system rather than the external excitation we put on to it.
However, things change when the system is dissipate. If we add a potential in the system, the wave equation becomes

$$\partial_t^2\psi - \partial_x^2\psi + V(x)\psi = 0 \tag{6.3}$$

with initial value $\psi(0,x)$ and $\partial_t\psi(0,x)$ fixed. Suppose the system is unbounded, that is, the potential vanishes at infinity, which is the physical situation we are interested in

$$V \to 0, \quad x \to \pm\infty \tag{6.4}$$

then the frequencies will be continuous. At infinity the potential vanishes, the situation would be the same as the normal system without energy loss, the waves will behave as plane waves at infinity.

$$\psi \sim e^{\pm i\omega x}, \quad x \to \pm\infty \tag{6.5}$$

What about the region where the potential is present? To solve the wave equation with potential, we first perform the Laplace transform to make the wave equation an ordinary differential equation.

$$\hat{\psi}(s,x) = \int_0^\infty \psi(t,x)e^{-st}dt \tag{6.6}$$

where s is complex, $s = \alpha + i\beta$.
The wave equation Eq.(6.3) becomes an ODE,

$$s^2\hat{\psi} - \partial_x^2\hat{\psi} + V\hat{\psi} = s\psi(0,x) + \partial_t\psi(0,x) \equiv j(x) \tag{6.7}$$

which is an inhomogeneous equation and j(x) denotes its inhomogeneity. The homogeneous equation is

$$s^2\hat{\psi} - \partial_x^2\hat{\psi} + V\hat{\psi} = 0 \tag{6.8}$$

We can use Green's function to solve this equation. The Green's function can be constructed by two independent solutions of the homogeneous equation ψ_1 and ψ_2

$$G(s,x,x') = \begin{cases} \frac{1}{W}\psi_1(s,x')\psi_2(s,x) & (x' < x) \\ \frac{1}{W}\psi_1(s,x)\psi_2(s,x') & (x' > x) \end{cases} \tag{6.9}$$

where W(s) is the Wronskian

$$W(s) = \psi_1\psi_2' - \psi_1'\psi_2 \tag{6.10}$$

The solution to the wave equation Eq.(6.7) will be given by

$$\hat{\psi}(s,x) = \int_{-\infty}^{+\infty} G(s,x,x')j(s,x')dx' \tag{6.11}$$

Now to derive the Green's function for this wave equation, we need to find two independent solutions of the wave equation. One simple choice is

$$\begin{aligned} \psi_1 &= e^{-sx}, \quad x \to +\infty \\ \psi_2 &= e^{+sx}, \quad x \to -\infty \end{aligned} \tag{6.12}$$

Then the Wronskian is

$$W = -2s \tag{6.13}$$

The Green's function is therefore

$$G(s,x,x') = \begin{cases} -\dfrac{1}{2s}e^{-s(x'-x)} & (x' < x) \\ -\dfrac{1}{2s}e^{-s(x-x')} & (x' > x) \end{cases} \tag{6.14}$$

$$G(t,x,x') = \frac{1}{2\pi i}\int e^{st}G(s,x,x')ds \tag{6.15}$$

Then we obtain the wave function

$$\begin{aligned} \hat{\psi}(s,x) &= \int_{-\infty}^{x'} G(s,x,x')j(s,x')dx' + \int_{x'}^{+\infty} G(s,x,x')j(s,x')dx' \\ &= -\frac{1}{2s}\int_{-\infty}^{x'} e^{-s(x-x')}j(s,x')dx' - \frac{1}{2s}\int_{x'}^{+\infty} e^{s(x-x')}j(s,x')dx' \end{aligned} \tag{6.16}$$

The wave function ψ can be obtained by taking the inverse Laplace transformation

$$\psi(t,x) = \frac{1}{2\pi i}\int e^{st}\hat{\psi}(s,x)ds \tag{6.17}$$

The integral can be evaluated by contour integration. The function becomes singular when the Wronskian $W(s_n) = 0$. In such case, ψ_1 and ψ_2 are no longer independent

$$\hat{\psi}_1(s_n,x) = c_n\hat{\psi}_2(s_n,x) \equiv c_n\hat{\psi}_n(x) \tag{6.18}$$

With the residue theorem

$$\oint f(z)dz = 2\pi i\sum_n Res(f,a_n) \tag{6.19}$$

We have

$$G(t,x,x') = \sum_n e^{s_n t}\frac{c_n\hat{\psi}_n(x)\hat{\psi}_n(x')}{W'(s_n)} \tag{6.20}$$

$$\begin{aligned} \psi(t,x) &= \sum_n e^{s_n t}c_n\hat{\psi}_n(x)\int j(x')\hat{\psi}_n(x')dx' \\ &= \sum_n e^{s_n(t-x)}c_n\int j(x')e^{s_n x'}dx' \end{aligned} \tag{6.21}$$

s_n that make the Green's function singular are defined as the quasinormal frequencies. The corresponding ψ_n are called the quasi eigenfunctions.

Mathematically we have illustrated that Quasinormal modes are poles of Green's functions. And it is shown in [**?**] for positive potentials with compact support there are infinite number of quasinormal modes. What is the physcial meaning of quasinormal modes? Let's consider a Green's function

$$\tilde{G}(\omega) = \frac{1}{\omega - (\alpha - i\beta)} \tag{6.22}$$

Apparently there is a pole at $\omega = \alpha - i\beta$. Preform the inverse Fourier transformation we find

$$G(t) = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega - (\alpha - i\beta)} \tag{6.23}$$

The integral gives

$$G(t) = ie^{-i\alpha t - \beta t}\Theta(t) \tag{6.24}$$

where Θ is an Heaviside step-function. We could see that the real part in quasinormal frequencies gives an oscillating behavior while the imaginary part determines how fast the quasinormal modes decay.

6.2 Quasinormal Modes of Schwarzschild Black Holes

In the previous sections we have introduced the concept of gravitational waves. We know that a black hole is a special region of spacetime.When there is a perturbation to the black hole, it lose energy at both ends: the event horizon and spatial infinity, in terms of gravitational waves. Therefore we are suppose to get quasinormal modes rather than normal modes from the perturbation of the black hole. Such gravitational waves are determined only by the parameters of the black hole: mass, charge and angular momentum, if there is any.

We now start from the simplest case, Schwarzschild black hole. In sperical coordinate system, the wave function will be $\psi(t, r, \theta, \phi)$. We can decompose it into spherical tensor harmonics

$$\psi(t, r, \theta, \phi) = \sum_{lm} \frac{\psi_{lm}(r, t)}{r} Y_{lm}(\theta, \phi) \tag{6.25}$$

Since the spacetime is spherically symmetric, we can get rid of indice m. The wave function for the radial components is

$$\partial_t^2 \psi_l - \partial_{r*}^2 \psi_l + V(r)\psi_l = 0 \tag{6.26}$$

r^* is the tortoise coordinate

$$r^* = r + 2GM\log(\frac{r}{2GM} - 1) \tag{6.27}$$

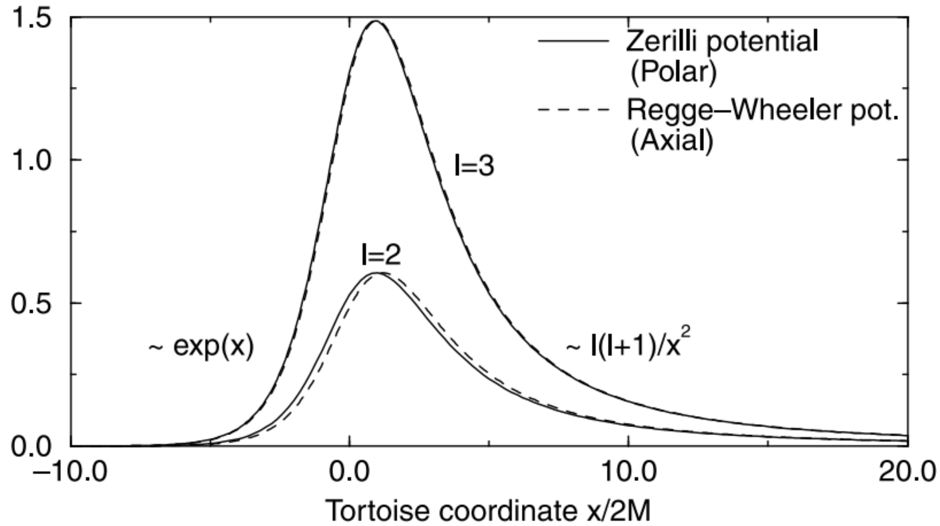

Figure 3. ReggeWheeler and Zerilli potentials for $l = 2$ and 3.[17]

When we are close to the event horizon, $r \to 0$, $r^* \to -\infty$. When $r \to \infty$, $r^* \to \infty$. For axial perturbations, the potential is given by (The calculation is shown in Appendix B.)

$$V_l(r) = \big(1 - \frac{2GM}{r}\big)\big[\frac{l(l+1)}{r^2} + \frac{2\sigma GM}{r^3}\big] \tag{6.28}$$

which is called the Regge-Wheele potential[12]. The parameter $\sigma = 1 - s^2$ where s denotes the spin of the perturbation field. When we consider scalar perturbations, s=0, $\sigma = 1$. When we consider electromagnetic perturbations, s=1, $\sigma = 0$. For gravitational perturbations, $s = 2$, $\sigma = -3$. For polar perturbations the potential is given by Zerilli[13]

$$V_l(r) = \big(1 - \frac{2GM}{r}\big)\frac{2n^2(n+1)r^3 + 6n^2GMr^2 + 18nG^2M^2r + 18G^3M^3}{r^3(nr+3GM)^2} \tag{6.29}$$

where

$$2n = (l-1)(l+2) \tag{6.30}$$

Although the expressions for these two potential look very different, we can plot the potentials (Figure.3) and would find their values quite close. Actually Chandrasekhar [18] has shown that the two potentials can transform to each other, which means the quasinormal frequencies of polar and axial perturbations have to be identical. We may would like to derive the exact value of quasinormal frequency to get the parameter of black holes. However, so far it is not possible to solve the wave equations analytically. Calculations of quasinormal frequencies can be done numerically. Table 1 shows some

n	l=2	l=3	l=4
0	0.37367-0.08896i	0.59944-0.09270i	0.80918-0.09416i
1	0.34671-0.27391i	0.58264-0.28130i	0.79669-0.28449i
2	0.30105-0.47828i	0.55168-0.47909i	0.77271-0.47991i
3	0.25150-0.70514i	0.51196-0.69034i	0.73984-0.68392i

Table 1. Quasinormal modes of a Schwarzschild Black hole for l = 2,3 and 4, measured in units of the black hole mass M

quasinormal modes of Schwarzschild black hole for $l = 2, 3$ and 4.[16]
Here we list only the modes whose real part is positive. Actually the poles of Green's function lie symmetrically on the half complex plane $Re(s) < 0$. The quasinormal modes comes up in pairs, with both positive and negative real part. We find that as the order of the modes increase, the real part varies slowly and finally remains constant, while the imaginary part grows very fast. From Eq.(6.24) we could see that the mode with larger imaginary part (larger β) decays faster, indicating that the higher order modes don't contribute a lot to the gravitational waves. The fundamental mode ($n = 0$) would be our interest. The list of quasinormal modes begins with $l = 2$ terms indicating there is no monopole or dipole radiation in gravitational waves, which we have illustrated in Chapter 4.

7 Black Hole Echoes

Suppose general relativity is a valid theory everywhere, the solution of Einstein Equation can give us a standard black hole. For such a black hole, the wave propagating towards the event horizon will totally fall in the horizon without any reflection. However, there are hypotheses suggesting that GR may break down inside the black hole since the curvature becomes very large (to Planck scale), quantum effects should not be ignored. Therefore the structure of black hole need to be modified.

We don't know much about things inside a black hole. But we can consider the event horizon. For a standard black hole we have a perfect in-fall boundary condition. Once there are some modifications to the black hole, we may expect the boundary condition near the event horizon get changed and there will be outgoing wave near the horizon that we can observe far away from the center of the black hole to check the hypothesis. Imagine that when a wave falls into the event horizon, there exists reflection at the event horizon. The reflected wave may propagate to infinity or it can be reflected back to the event horizon. The reflected wave could be reflected again at the horizon. This process would be repeated again and again and we are expected to detect a series of decaying echoes.

How would a scalar wavepacket propagate in such a case compared to a standard black hole case? We study the sourced wave equation

$$\Box\Psi = -\rho \tag{7.1}$$

We separate the variables to get the radial wave equation. We decompose Ψ into spherical harmonics

$$\psi(t,r,\theta,\phi) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{l,m} \tilde{\psi}_{lm}(\omega,r) Y_{lm}(\theta,\phi) e^{-i\omega t} \tag{7.2}$$

We also expand the source

$$\rho(t,r,\theta,\phi) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{l,m} \tilde{\rho}_{lm}(\omega,r) Y_{lm}(\theta,\phi) e^{-i\omega t} \tag{7.3}$$

We don't know the t-dependence g(t) but for any functions it can be written in

$$g(t) = \int \frac{d\omega}{2\pi} \tilde{g}(\omega) e^{-i\omega t} \tag{7.4}$$

So we can write $\psi(t,r,\theta,\phi)$ as Eq.(7.2) without losing any generality.

The radial wave function is given by (The calculations are shown in Appendix B)

$$\frac{d^2\tilde{\psi}_{lm}}{dr^{*2}} + (\omega^2 - fV)\tilde{\psi}_{lm} = \tilde{S}(\omega, r^*) \tag{7.5}$$

where $\tilde{S}$ is defined as

$$\tilde{S}(\omega, r^*) = -r(r^*) f \rho_{lm}(\omega, r^*) \tag{7.6}$$

and r^* the tortoise coordinate

$$\frac{dr^*}{dr} = \frac{1}{f(r)} \tag{7.7}$$

Both $f(r)$ and $V(r)$ depend on background geometry. We will give some examples in Appendix C. Here we write down the case for Schwarzschild spacetime,

$$f(r) = 1 - \frac{2GM}{r}, \quad V(r) = \frac{l(l+1)}{r^2} + \frac{2GM}{r^3} \tag{7.8}$$

For a standard black hole, there are only in-falling waves $e^{-i\omega r^*}$ at event horizon and outgoing waves $e^{+i\omega r^*}$ at spatial infinity. When there is a reflecting mirror, there are both infalling waves and outgoing waves near the horizon,

$$\tilde{\psi} \propto e^{-i\omega(r^* - r_s)} + \tilde{R}(\omega) e^{i\omega(r^* - r_s)} \tag{7.9}$$

where r_s denotes the radius of event horizon and $\tilde{R}$ is the reflection coefficient. We now consider how is the waveform with the reflecting boundary condition different from the one propagating outside a standard black hole. The radial wave function Eq.(7.5) can be solved by using the green's function techniques

$$\frac{d^2 \tilde{g}_{ref}(x, x')}{dr^{*2}} + (\omega^2 - fV)\tilde{g}_{ref}(x, x') = \delta(x - x') \tag{7.10}$$

Then $\tilde{\psi}_{lm}$ is given by

$$\tilde{\psi}_{lm}(r^*) = \int_{-\infty}^{\infty} dr^{*\prime} \tilde{g}_{ref}(r^*, r^{*\prime}) \tilde{S}(r^{*\prime}) \tag{7.11}$$

Recall how we calculate the QNMs for Schwarzschild black holes. Since the potential is vanishing at the event horizon and at spatial infinity, we have

$$\tilde{\psi}_{BH} \propto e^{-i\omega r^*} \tag{7.12}$$

as $r \to r_s$ and $r \to \infty$. We can write down a pair of solutions satisfying the boundary conditions, which Chrzanowski and Misner denote as the IN- and OUT-modes

$$\tilde{\psi}^{in}_{BH}(\omega, r^*) \sim \begin{cases} e^{-i\omega r^*} & (r^* \to -\infty) \\ A_{out}(\omega) e^{i\omega r^*} + A_{in}(\omega) e^{-i\omega r^*} & (r^* \to +\infty) \end{cases} \tag{7.13}$$

where ω is positive and real. The OUT-mode is just the complex conjugate of IN-mode,

$$\tilde{\psi}^{out}_{BH}(\omega, r^*) \sim \begin{cases} e^{+i\omega r^*} & (r^* \to -\infty) \\ A^*_{out}(\omega) e^{-i\omega r^*} + A^*_{in}(\omega) e^{+i\omega r^*} & (r^* \to +\infty) \end{cases} \tag{7.14}$$

The IN- and OUT-modes are two independent solutions to the homogeneous wave equation. We know that the Wronskian of two independent solutions must be a constant. This gives

$$|A_{out}|^2 + |A_{in}|^2 = 1 \tag{7.15}$$

The reflection and transmission amplitude are then given by

$$R_{BH} = \frac{A_{out}}{A_{in}}, \quad T_{BH} = \frac{1}{A_{in}} \tag{7.16}$$

We also have

$$|T_{BH}|^2 + |R_{BH}|^2 = 1 \tag{7.17}$$

where $|T_{BH}|^2$ is the transmission coefficient and $|R_{BH}|^2$ the reflection coefficient. It's easy to find another pair of solution to the homogeneous wave equation: the UP- and DOWN-modes

$$\tilde{\psi}^{up}_{BH}(\omega, r^*) \sim \begin{cases} B_{out}(\omega)e^{i\omega r^*} + B_{in}(\omega)e^{-i\omega r^*} & (r^* \to -\infty) \\ e^{+i\omega r^*} & (r^* \to +\infty) \end{cases} \tag{7.18}$$

and its complex conjugate

$$\tilde{\psi}^{down}_{BH}(\omega, r^*) \sim \begin{cases} B^*_{out}(\omega)e^{-i\omega r^*} + B^*_{in}(\omega)e^{+i\omega r^*} & (r^* \to -\infty) \\ e^{-i\omega r^*} & (r^* \to +\infty) \end{cases} \tag{7.19}$$

With the fact that the Wronskian should be a constant, we have[20]

$$B_{out} = A_{in}, \;\; B_{in} = -A^*_{out} \tag{7.20}$$

With any two of the modes we can construct the Green's function. Here we choose the IN- and UP-modes.

$$W(IN, UP) = \tilde{\psi}^{in}\frac{d}{dr^*}\tilde{\psi}^{up} - \tilde{\psi}^{up}\frac{d}{dr^*}\tilde{\psi}^{in} = 2i\omega A_{in} \tag{7.21}$$

The Green's function is given by

$$\tilde{g}_{BH}(r_*, r'_*) = \begin{cases} \dfrac{1}{W_{BH}}\tilde{\psi}^{in}(r'_*)\tilde{\psi}^{up}(r_*) & (r'_* < r_*) \\ \dfrac{1}{W_{BH}}\tilde{\psi}^{in}(r_*)\tilde{\psi}^{up}(r'_*) & (r'_* > r_*) \end{cases} \tag{7.22}$$

The Green's function for the reflecting case is given by[21]

$$\tilde{g}_{ref}(r_*, r'_*) = \tilde{g}_{BH}(r_*, r'_*) + \tilde{K}\frac{\tilde{\psi}^{up}(r_*)\tilde{\psi}^{up}(r'_*)}{W_{BH}} \tag{7.23}$$

where

$$\tilde{K}(\omega) = \frac{\tilde{R}_{BH}\tilde{R}e^{-2i\omega r_s}}{1-\tilde{R}_{BH}\tilde{R}e^{-2i\omega r_s}} \tag{7.24}$$

We now compare the amplitude of waves propagating in the reflecting situation with the one in the standard black hole case. We already obtained the Green's function Eq.(7.22), the wave function is then given by

$$\begin{aligned}\tilde{\psi}_{BH}(\omega, r_*) &= \int_{-\infty}^{+\infty} dr'_* \tilde{g}_{BH}(r_*, r'_*)\tilde{S}(r'_*) \\ &= \int_{-\infty}^{r_*} dr'_* \frac{\tilde{S}(r'_*)}{W_{BH}}\tilde{\psi}^{in}(r'_*)\tilde{\psi}^{up}(r_*) + \int_{r_*}^{+\infty} dr'_* \frac{\tilde{S}(r'_*)}{W_{BH}}\tilde{\psi}^{in}(r_*)\tilde{\psi}^{up}(r'_*)\end{aligned} \tag{7.25}$$

Consider its behavior at boundaries

$$\psi_{BH}(r^*) \sim \begin{cases} \int_{-\infty}^{\infty} dr'_* \frac{\tilde{\psi}_{in}(r'_*)\tilde{S}(r'_*)}{W_{BH}} e^{i\omega r_*} & r_* \to \infty \\ \int_{-\infty}^{\infty} dr'_* \frac{\tilde{\psi}_{up}(r'_*)\tilde{S}(r'_*)}{W_{BH}} e^{-i\omega r_*} & r_* \to -\infty \end{cases} \tag{7.26}$$

We use Z^{∞}_{BH} and Z^{H}_{BH} to denote the amplitude of wave at infinity and at event horizon respectively

$$Z^{\infty}_{BH} = \int_{-\infty}^{\infty} dr'_* \frac{\tilde{\psi}_{in}(r'_*)\tilde{S}(r'_*)}{W_{BH}} \tag{7.27}$$

$$Z^{H}_{BH} = \int_{-\infty}^{\infty} dr'_* \frac{\tilde{\psi}_{up}(r'_*)\tilde{S}(r'_*)}{W_{BH}} \tag{7.28}$$

Similarly we find the amplitude of waves with the reflecting boundary condition. With Green's function given in Eq.(7.23) we have

$$\begin{aligned}\tilde{\psi}(\omega, r_*) &= \int_{-\infty}^{+\infty} dr'_* \tilde{g}_{BH}(r_*, r'_*)\tilde{S}(r'_*) \\ &= \int_{-\infty}^{+\infty} dr'_* \tilde{g}_{ref}(r_*, r'_*)\tilde{S}(r'_*) + \int_{-\infty}^{+\infty} dr'_* \tilde{K}\frac{\tilde{\psi}^{up}(r_*)\tilde{\psi}^{up}(r'_*)}{W_{BH}}(r_*, r'_*)\tilde{S}(r'_*)\end{aligned} \tag{7.29}$$

Consider its behaviour at spatial infinity

$$\begin{aligned}\tilde{\psi} &\sim \int_{-\infty}^{+\infty} dr'_* \frac{\tilde{\psi}_{in}(r'_*)\tilde{S}(r'_*)}{W_{BH}} e^{i\omega r_*} + \int_{-\infty}^{\infty} dr'_* \frac{\tilde{K}\tilde{\psi}_{up}(r'_*)\tilde{S}(r'_*)}{W_{BH}} e^{i\omega r_*} \\ &\sim (Z^{\infty}_{BH} + \tilde{K}Z^{H}_{BH})e^{i\omega r_*}\end{aligned} \tag{7.30}$$

Denote its amplitude as Z^{∞}, this gives

$$Z^{\infty} = Z^{\infty}_{BH} + \tilde{K}Z^{H}_{BH} \tag{7.31}$$

This implies that in the reflecting case, the waves at infinity can be viewed as a combination of two parts: the normal one in the standard black holes case and the one appeared due to the reflecting mirror. Expanding $\tilde{K}(\omega)$ in series we find

$$\tilde{K}(\omega) = \tilde{T}_{BH}\tilde{R}e^{-2i\omega r_s}\sum_{n=1}^{\infty}(\tilde{R}_{BH}\tilde{R})^{n-1}e^{-2i(n-1)\omega r_s} \tag{7.32}$$

when n=1, we get a term $\tilde{T}_{BH}\tilde{R}e^{-2i\omega r_s}$, that is the wave reflected back at the mirror towards infinity ($\tilde{R}$), and is partly transmitted over the potential ($\tilde{T}_{BH}$). The phase of the wave is changed by $e^{-2i\omega r_s}$. When reflected back to infinity, the wave can get reflected by the potential. It may be reflected again at the mirror or fall into event horizon. The part that being reflected again and finally escape to infinity is the case $n = 2$. The phase is now changed by $e^{-4i\omega r_s}$. The change of phase arises since the wave travels between the potential and the event horizon. It denotes the time delay between echoes. It is one of the two observables we can detect to get information about the boundary conditions. The other one is the amount of damping between each echo.

We see that the reflecting process can be repeated again and again. As there are infinity terms in $\tilde{K}(\omega)$, there are infinity number of echoes, just as we expected.

8 Robin Boundary Condition

In the last section we discussed how the modification of boundary conditions at the event horizon gives rise to black hole echoes, without specifying what the modification is. Since we don't know the new physics near the horizon, there is no way for us to determine directly what boundary condition we should impose at the horizon. However, we can set models with certain boundary conditions at the horizon and predict the behaviour of black hole echoes. There are two observables for black hole echoes: the time delay between each echo and the amount of damping. We will find how the boundary conditions affect the observables and compare the predicted behaviour to observational data. Observational data helps us determine the boundary conditions at the horizon therefore helps us investigate the new physics near the horizon.
In this thesis we discuss the black hole echoes for Kerr black holes since observations are focus on black hole merger events whose products are rotating black holes. We have introduced Kerr metric in Section 5.3.

We know the equation of motion for a field is derived from the principle of least action. To determine the behaviour of black hole echoes, we first consider what the Lagrangian would be with the new physics employed near the horizon. The Lagrangian will be the Lagrangian for free field theory plus an interacting term. We put the interacting term at a surface near the horizon with $r = r_+\epsilon, 0 < \epsilon \ll r_s$, where r_+ is the outer event horizon and $r_s = 2GM$. We don't impose this interacting term exactly at the horizon since we notice that with our new boundary condition, many quantities become divergent at the event horizon. To avoid this, we set the boundary condition not at the event horizon but at a hypothetical reflecting mirror very close to the horizon. The total action therefore can be wrriten as

$$S = \int d^4x\sqrt{-g}\mathcal{L} + \oint_{r=r_++\epsilon} d^3x\sqrt{-\gamma}\mathcal{L}_\epsilon \tag{8.1}$$

where $\mathcal{L}$ is the Lagrangian density for free field theory and $\mathcal{L}_\epsilon$ the interacting term. Here ϵ is a fake parameter. The exact location of the mirror doesn't really matter. We just know it is very close to the event horizon and there is nothing physical about its location. That is, nothing observable should depend on the exact value of ϵ.
Now we consider theory for massless scalar fields. For the interacting term, the leading order contribution of Lagrangian is from the term with fewest fields and derivatives, which should be given by

$$\mathcal{L}_\epsilon = -\frac{h(\theta)}{2}\psi^2 \tag{8.2}$$

where $h(\theta)$ is the coupling constant. Then the total action is given by

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}\partial^\mu\psi\partial_\mu\psi + \oint_{r=r_+-\frac{1}{2}\epsilon} d^3x\sqrt{-\gamma}h\psi^2 \tag{8.3}$$

We impose this action on the surface of the reflecting mirror which is located at the hypersurface of $r = r_+ + \epsilon$, thus $d^3x = dtd\theta d\phi$, $\sqrt{-\gamma}$ denotes the surface metric. We obtain the boundary condition from the principle of least action. We ask the variation of the total action to be zero then find the boundary condition we derived is the robin boundary condition

$$N_\mu\partial^\mu\psi - h(\theta)\psi = 0 \tag{8.4}$$

where N_μ is the normal to surface $r = r_+ + \epsilon$. We find the boundary condition we derived is the robin boundary condition. In Kerr spacetime, we apply the Kerr metricEq.(5.55), then the boundary condition near the horizon becomes

$$\partial_r \mathrm{ln}\psi = \frac{\rho(\theta)}{\sqrt{\Delta}}h(\theta) \tag{8.5}$$

For simplicity, we set $h(\theta)$ as a particular function of θ so that the right-hand side of Eq.(8.5) is independent on θ.

$$h(\theta) = \frac{h_0}{\rho(\theta)} \tag{8.6}$$

where $f(\theta)$ cancels the θ-dependence in ρ. therefore the wave function for the field can be decomposed it $\psi(t, r, \theta, \phi) = \varphi(r)S(\theta)e^{-i\omega t+im\phi}$. Then the boundary condition can be written in terms of radial wave function $\varphi(r)$

$$\frac{d\mathrm{ln}\varphi}{dr} = \frac{h_0}{\sqrt{\Delta}} \equiv \lambda \tag{8.7}$$

We then want to find the expression for the radial wave function $\varphi_(r)$. The equation for perturbations of a Kerr black hole is given by the Teukolsky equation. Decomposing the wave function we obtain the radial wave equation. The solution of this radial equation can be found in a simple way if we write $\varphi(r) = F(r)\chi(r)$ and set F(r) in a particular way to make $\chi(r)$ satisfy a equation of Schrodinger-like form

$$-\nabla^2\chi + V(r)\chi = 0 \tag{8.8}$$

The potential V(r) can be derived from the radial Teukolsky equation. Since we are interested in the boundary conditions, we analyze the solution of radial wave functions in the limit $r \to r_+$ and find two solutions

$$\chi^\pm \sim (r - r_+)^{\pm i\xi-(d-2)/2} \tag{8.9}$$

$$\psi^{\pm} \sim \varphi^{\pm} e^{-i\omega t} = (r - r_+)^{-s/2} e^{-i(\omega t \mp \xi \ln(r - r_+))} \quad (r - r_+ \to 0) \tag{8.10}$$

where ξ is defined as

$$\xi = \frac{\omega r_s r_+ - am}{\sqrt{r_s^2 - 4a^2}} \tag{8.11}$$

We have already derived the two solutions of the radial wave equation. A general solution can be written as a superposition of the two solutions. We also noticed that the two solutions represents purely infalling waves and outgoing waves, respectively. Therefore the ratio of amplitudes of the outgoing wave to infalling wave is the reflection coefficient R.

$$\psi = \psi^+ + R\psi^- \tag{8.12}$$

From Eq.(8.7) we determine the relation between φ and h_0 (or λ). ϵ is a fake parameter so that nothing physical should depends on. That requires the reflection coefficient R should not depend on ϵ. This requirement gives restriction on the ϵ-dependence of h_0 (or λ), which is a renormalization group flow. Substitute the expression we have found for radial wave functions φ in Eq.(8.7), we find the evolution equation for λ with respect to ϵ.

$$\epsilon \frac{d\lambda}{d\epsilon} = -\frac{1}{2}(4\xi^2 + \lambda^2) \tag{8.13}$$

There are two fixed points in the evolution equation $\lambda(\epsilon) = \pm 2i\xi$. To understand their physcial meanings, we may first derive the relation between the near-horizon reflection coefficient and λ, which parameterizes the boundary condition,

$$R = \frac{2i\xi + \lambda(\epsilon)}{2i\xi - \lambda(\epsilon)} \epsilon^{-2i\xi} \tag{8.14}$$

We notice when $\lambda = -2i\xi$, the reflection coefficient is zero There is no reflection at the horizon at all. It represents a standard black hole. When $\lambda = +2i\xi$, the reflection coefficient goes to infinity. There is only outgoing waves, which leads to a white hole.

9 Conclusion and Outlook

In this thesis we first introduce the definition and properties of gravitational waves. We showed how gravitational waves arises as perturbations around background spacetime by analyzing linearized Einstein equations. We study the energy-momentum tensor of gravitational waves with the low-frequency projection of Einstein equations and find there is no local expression for gravitational waves. And the high-frequency projection of Einstein equations gives us the propagation equations for gravitational waves in curved spacetime. We also introduced gravitational waves from another perspective: we treat gravitational waves as a field living in Minkowski spacetime as same as other fields. We use classical field theory to analyze the gravitational field. The Noether's theorem gives us the definition of energy-momentum tensor related to the symmetry of the field under spacetime translations. With classical field theory we derived the energy-momentum tensor for gravitational field again.

Gravitational waves from black hole merger events have been detected and analyzed. A binary black hole system would lose energy and angular momentum due to the emission of gravitational waves. There will be three phase of the merger event: inspiral, merger and ring down. The orbit of the binary system becomes smaller and smaller then the two black holes into one single black hole. The oscillation of the single black hole gives the post-merger ring down signals. The ring down gravitational waves will be a superposition of quasinormal modes of the black hole, which reveal the intrinsic properties of the black hole (its mass and spin). By solving quasinormal modes of black hole and compare it to the observation of ring down signals, black hole merger events provides us the way to test general relativity. In this thesis we analyze the coalescence of a binary system with two point particles, as the simplified model for black hole merger. We study how the emission of gravitational waves react back to this system and show how does the waveform of gravitational waves evolves. Then we study the concept of quasinormal modes as poles on Green's functions and introduced the solution to Schwarzschild black hole perturbations.

There are hypothesis that general relativity is not valid near the event horizon of a black hole since the black hole curved spacetime so much, we need to consider quantum effects. Therefore there are new physics to consider near the horizon, it is natural to think the boundary conditions at the event horizon is modified. In general relativity waves fall into the event horizon completely. When new physics appears, we would assume there are waves reflected near the horizon. That is, the reflection coefficient at event horizon is not 0 anymore. The reflected wave may get reflected back when propagating towards infinity and get reflected again near the horizon. The procedure can be repeated again and again therefore generate black hole echoes.

There is no restriction for the new boundary conditions near the horizon we discuss the robin boundary condition as a possible candidate dominating at low energies. We derive the evolution equation of λ and find the two fixed points related to black holes and white holes, respectively. We also derive the expression of reflection coefficient. Also there are other models giving different boundary conditions to study.
The study of gravitational waves emitted from black holes gives us a new opportunity to study the black holes and test general relativity. The observation of black merger events may reveal the existence of new physics for gravity.

A Expansion of Ricci tensor in curved background space

We expand the Ricci tensor around a curved background spacetime to linear order in $h_{\mu\nu}$. The metric is

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{A.1}$$

Its inversion is

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2) \tag{A.2}$$

The Christoffel symbol is

$$\begin{aligned} \Gamma^{\mu}_{\rho\nu} &= \frac{1}{2} g^{\mu\sigma}(D_\nu g_{\sigma\rho} + D_\rho g_{\sigma\nu} - D_\sigma g_{\rho\nu}) \\ &= \frac{1}{2}(\bar{g}^{\mu\nu} - h^{\mu\nu})(D_\nu \bar{g}_{\sigma\rho} + D_\rho \bar{g}_{\sigma\nu} - D_\sigma \bar{g}_{\rho\nu} + D_\nu h_{\sigma\rho} + D_\rho h_{\sigma\nu} - D_\sigma h_{\rho\nu}) \\ &= \bar{\Gamma}^{\mu}_{\nu\rho} + \frac{1}{2}\bar{g}^{\mu\sigma}(\bar{D}_\nu h_{\sigma\rho} + \bar{D}_\rho h_{\sigma\nu} - \bar{D}_\sigma h_{\rho\nu}) - \mathcal{O}(h^2) \end{aligned} \tag{A.3}$$

To simplify our computation, We can choose a coordinate system where $\bar{\Gamma}^{\mu}_{\rho\nu} = 0$. The Ricci tensor is

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} \tag{A.4}$$

Since we have taken $\bar{\Gamma}^{\mu}_{\rho\nu} = 0$, the last two terms will not contribute to linear order terms. Therefore, the linear order term in $R_{\mu\nu}$ is

$$\begin{aligned} R^{(1)}_{\mu\nu} &= \bar{D}_\nu \bar{\Gamma}^{\alpha}_{\mu\alpha} - \bar{D}_\alpha \bar{\Gamma}^{\alpha}_{\mu\nu} - \frac{1}{2}\bar{g}^{\alpha\beta}\bar{D}_\nu(\bar{D}_\alpha h_{\beta\nu} + \bar{D}_\mu h_{\beta\alpha} - \bar{D}_\beta h_{\mu\alpha}) \\ &\quad + \frac{1}{2}\bar{g}^{\alpha\beta}\bar{D}_\alpha(\bar{D}_\nu h_{\beta\mu} + \bar{D}_\mu h_{\beta\nu} - \bar{D}_\beta h_{\mu\nu}) \\ &= \frac{1}{2}\bar{g}^{\alpha\beta}(\bar{D}^\alpha \bar{D}_\nu h_{\mu\alpha} + \bar{D}_\mu \bar{D}^\alpha h_{\alpha\nu} - \bar{D}_\mu \bar{D}_\nu h - \bar{D}^\alpha \bar{D}_\alpha h_{\mu\nu}) \end{aligned} \tag{A.5}$$

Although we use a coordinate system such that $\bar{\Gamma}^{\mu}_{\rho\nu} = 0$, $R_{\mu\nu}$ is a tensor and made up of covariant derivatives, the result would be same under coordinate transformation.
If the background metric is flat, $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, and the covariant derivatives would be partial derivatives. If we impose Lorentz gauge

$$\partial^\mu \bar{h}_{\mu\nu} = 0 \tag{A.6}$$

Eq.(A.5) becomes

$$R^{(1)}_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} \tag{A.7}$$

B Radial wave equation in spherically symmetric spacetime

Consider a scalar wave packet Φ propagating freely in a spherically symmetric spacetime with metric $g_{\mu\nu}$, the wave equation is known as the Klein-Gordon equation

$$\Box\Phi = 0 \tag{B.1}$$

The d'Alembert operator $\Box$ is given by

$$\Box = \sqrt{-g}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) \tag{B.2}$$

where is g is the determinant of $g_{\mu\nu}$. The wave equation Eq.(B.1) becomes

$$\Box\Phi = (-g)g^{\mu\nu}\partial_\mu\partial_\nu\Phi + \sqrt{-g}g^{\mu\nu}\partial_\mu\sqrt{-g}\partial_\nu\Phi \tag{B.3}$$

Since the background spacetime is spherically symmetric, the linear element can be written as

$$ds^2 = -F(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2 \tag{B.4}$$

Actually according to Birkhoffs theorem, the only spherically symmetic solution to the Einstein equation is the Schwarzchild metric, where

$$F = B = 1 - \frac{2GM}{r} \tag{B.5}$$

But we may have other models where the metric is not continous, we use the more general form Eq.(B.4). Having the metric, we can calculate

$$g = -\frac{F}{B}r^4 sin^2\theta \tag{B.6}$$

We first calculate the second term in Eq.(B.3) We know that

$$\partial_\mu\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\partial_\mu g_{\alpha\beta} \tag{B.7}$$

Then

$$\begin{aligned} I &\equiv \sqrt{-g}g^{\mu\nu}\partial_\mu\sqrt{-g}\partial_\nu\Phi \\ &= -\frac{1}{2}gg^{\mu\nu}g^{\alpha\beta}\partial_\mu g_{\alpha\beta}\partial_\nu\Phi \\ &= -\frac{1}{2}gB\left(\frac{F'}{F} + \frac{B'}{B} + \frac{4}{r}\right)\partial_r\Phi - \frac{1}{2}g\frac{2cos\theta}{r^2 sin\theta}\partial_\theta\Phi \end{aligned} \tag{B.8}$$

We decompose the wave function Φ into spherical harmonics

$$\Phi = \sum_{lm} Y_{lm}(\theta,\phi)\frac{\psi_{lm}(t,r)}{r} \tag{B.9}$$

Substituting it into Eq.(B.8) we have

$$I = -\frac{1}{2} g B Y_{lm} \left(\frac{F'}{f} + \frac{B'}{B} + \frac{4}{r} \right) \partial_r \left(\frac{\psi_{lm}}{r} \right) - \frac{1}{2} g \left(\frac{\psi_{lm}}{r} \right) \frac{2cos\theta}{r^2 sin\theta} \partial_\theta Y_{lm} \tag{B.10}$$

The first term in Eq.(B.3) is

$$\begin{aligned} U &\equiv (-g) g^{\mu\nu} \partial_\mu \partial_\nu \Phi \\ &= (-g) \left[-\frac{1}{F} \partial_t^2 \Phi + B \partial_r^2 \Phi + \frac{1}{r^2} \partial_\theta^2 \Phi + \frac{1}{r^2 sin^2\theta} \partial_\phi^2 \Phi \right] \\ &= (-g) \left[-\frac{1}{F} \frac{Y_{lm}}{r} \partial_t^2 \psi_{lm} + B Y \partial_r^2 \left(\frac{\psi_{lm}}{r} \right) + \frac{1}{r^2} \frac{\psi_{lm}}{r} \partial_\theta^2 Y + \frac{1}{r^2 sin^2\theta} \frac{\psi_{lm}}{r} \partial_\phi Y \right] \end{aligned} \tag{B.11}$$

For spherical harmonics we have

$$\frac{1}{sin\theta} \frac{\partial}{\partial\theta} \left(sin\theta \frac{\partial Y}{\partial\theta} \right) + \frac{1}{sin^2\theta} \frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)Y \tag{B.12}$$

Therefore

$$U = (-g) \left[-\frac{1}{F} \frac{Y_{lm}}{r} \partial_t^2 \psi_{lm} + B Y \partial_r^2 \left(\frac{\psi_{lm}}{r} \right) + \frac{1}{r^2} \frac{\psi_{lm}}{r} \left(-l(l+1) Y_{lm} - \frac{cos\theta}{sin\theta} \partial_\theta Y_{lm} \right) \right] \tag{B.13}$$

We now change the coordinate r to tortoise coordinate r_*

$$\frac{dr_*}{dr} = \frac{1}{\sqrt{FB}} \tag{B.14}$$

$$\frac{d^2 r_*}{dr^2} = -\frac{F'B + FB'}{2(FB)^{3/2}} \tag{B.15}$$

then we have

$$\partial_r \psi = \frac{dr_*}{dr} \partial_{r_*} \psi = \frac{1}{\sqrt{FB}} \partial_{r_*} \psi \tag{B.16}$$

$$\begin{aligned} \partial_r^2 \psi &= \left(\frac{dr_*}{dr} \right)^2 \partial_{r_*}^2 \psi + \frac{d^2 r_*}{dr} \partial_{r_*} \psi \\ &= \frac{1}{FB} \partial_{r_*}^2 \psi - \frac{F'B + B'F}{2(FB)^{3/2}} \partial_{r_*} \psi \end{aligned} \tag{B.17}$$

With Eq.(B.13), Eq.(B.16) and Eq.(B.17), we have

$$\begin{aligned} U = (-g) \Bigg[&-\frac{1}{F} \frac{Y}{r} \partial_t^2 \psi + Y B \Bigg[\frac{1}{r} \left(\frac{1}{FB} \partial_{r_*}^2 \psi - \frac{F'B + B'F}{2(FB)^{3/2}} \partial_{r_*} \psi \right) \\ &- \frac{2}{r^2} \frac{1}{\sqrt{FB}} \partial_{r_*} \psi + \frac{2}{r^3} \psi \Bigg] + \frac{\psi}{r} \frac{1}{r^2} \left(-l(l+1) Y - \frac{cos\theta}{sin\theta} \partial_\theta Y \right) \Bigg] \end{aligned} \tag{B.18}$$

With Eq.(B.10), Eq.(B.16) and Eq.(B.17) we have

$$
\begin{aligned}
I = & -\frac{1}{2} gYB\left[\frac{1}{r}\frac{1}{\sqrt{FB}}\left(\frac{F'}{F}+\frac{B'}{B}+\frac{4}{r}\right)\partial_{r_*}\psi - \left(\frac{F'B-B'F}{BF}+\frac{4}{r}\right)\frac{\psi}{r^2}\right] \\
& -\frac{1}{2} g\frac{\psi}{r}\frac{2cos\theta}{r^2 sin\theta}\partial_\theta Y
\end{aligned}
\tag{B.19}
$$

The wave equation gives

$$
I + U = 0 \tag{B.20}
$$

Therefore we have

$$
\left[-\partial_t^2 + \partial_{r_*}^2 - V(r)\right]\psi(r,t) = 0 \tag{B.21}
$$

where

$$
V(r) = \frac{Fl(l+1)}{r^2} + \frac{F'B + FB'}{2r} \tag{B.22}
$$

We can further decompose $\psi(r,t)$ into

$$
\psi(r,t) = \int \frac{d\omega}{2\pi}\tilde{\psi}(\omega,r)e^{-i\omega t} \tag{B.23}
$$

This allows us write Eq.(B.21) into an ordinary differential equation

$$
\frac{d^2\tilde{\psi}(\omega,r)}{dr_*^2} + (\omega^2 - V(r))\tilde{\psi}(\omega,r) = 0 \tag{B.24}
$$

This is the radial wave equation. The potential V depends on the background spacetime where the wave propagates. In Schwarzschild spacetime, it is

$$
V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2GM}{r^3}\right) \tag{B.25}
$$

References

[1] Maggiore M 2007 Gravitational Waves. Vol. 1: Theory and Experiments Oxford Master Series in Physics (Oxford University Press) ISBN 9780198570745, 9780198520740

[2] Creighton J D E and Anderson W G 2011 Gravitational-wave physics and astronomy: An introduction to theory, experiment and data analysis

[3] V. P. Frolov and I. D. Novikov, "Black hole physics: Basic concepts and new developments," Fundam. Theor. Phys. 96 (1998)

[4] Carroll, S. M. (2019). Spacetime and geometry. Cambridge University Press.

[5] Schwartz, M. D. (2014). Quantum field theory and the standard model. Cambridge University Press.

[6] Banados, Max, and Ignacio Reyes. "A short review on Noethers theorems, gauge symmetries and boundary terms." International Journal of Modern Physics D 25.10 (2016): 1630021.

[7] Babak, S. V., Grishchuk, L. P. (1999). Energy-momentum tensor for the gravitational field. Physical Review D, 61(2), 024038.

[8] Kokkotas, K. D. (2002). Gravitational wave physics. Encyclopedia of Physical Science and Technology, 7(3), 67-85.

[9] Nielsen, A. (2019, June). Lecture notes on gravitational waves. In Journal of Physics: Conference Series (Vol. 1263, No. 1, p. 012008). IOP Publishing.

[10] Bellorín, Jorge, and Alvaro Restuccia. "Einsteins quadrupole formula from the kinetic-conformal Hoava theory." International Journal of Modern Physics D 27.01 (2018): 1750174.

[11] Maleknejad, A. MPA Lectures on Gravitational Waves in Cosmology. Gravitational waves, 5, 2.

[12] Regge, T., and Wheeler, J.A., Stability of a Schwarzschild singularity, Phys. Rev., 108, 10631069, (1957).

[13] Zerilli, F.J., Gravitational field of a particle falling in a Schwarzschild geometry analysed in tensor harmonics, Phys. Rev. D, 2, 21412160, (1970).

[14] arXiv:0905.2975 [gr-qc]

[15] arXiv:gr-qc/9909058

[16] Leaver, E.W., An analytic representation for the quasi-normal modes of Kerr black holes, Proc. R. Soc. London, Ser. A, 402, 285298, (1985).

[17] Hans-Peter, Nollert1 Quasinormal modes: the characteristic 'sound' of black holes and neutron stars.

[18] Chandrasekhar, S., On the equations governing the perturbations of the Schwarzschild black hole, Proc. R. Soc. London, Ser. A, 343, 289298, (1975)

[19] Gwyneth Allwright and David M. Jacobs, "Robin boundary conditions are generic in quantum mechanics ," [arXiv:1610.09581 [quant-ph]].

[20] Thorne, K.S., 1980. Multipole expansions of gravitational radiation. Reviews of Modern Physics, 52(2), p.299.

[21] Thorne, K. S., Kovács, S. J. (1975). The generation of gravitational waves. I-Weak-field sources. The Astrophysical Journal, 200, 245-262.

[22] Pestoni, B., Bortolas, E., Capelo, P. R., Mayer, L. (2021). Generation of gravitational waves and tidal disruptions in clumpy galaxies. Monthly Notices of the Royal Astronomical Society, 500(4), 4628-4638.

[23] Alex Nielsen 2019 J. Phys.: Conf. Ser. 1263 012008

[24] V Pustovoit et al 2020 J. Phys.: Conf. Ser. 1557 012034

[25] Journal of Mathematical Physics 40, 4474 (1999); doi: 10.1063/1.532980

[26] Cardoso V, Hopper S, Macedo C F B, Palenzuela C and Pani P 2016 Phys. Rev. D94 084031, [arXiv:1608.08637]

[27] Capano, Collin, "Searching for Gravitational Waves from Compact Binary Coalescence Using LIGO and Virgo Data" (2011). Physics - Dissertations. 114.

[28] B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration) (2016). "Observation of Gravitational Waves from a Binary Black Hole Merger". Phys. Rev. Lett. 116 (6): 061102. [arXiv:1602.03837]
B. P. Abbott et al. (LIGO Scientific and Virgo Collaborations), GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence, Phys. Rev. Lett. 116 (2016) 241103 [arXiv:1606.04855 [gr-qc]]
B. P. Abbott et al. (LIGO Scientific and VIRGO Collaborations), GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2, Phys. Rev. Lett. 118 (2017) 221101 [arXiv:1706.01812 [gr-qc]].

[29] Kanibolotsky, V. (2010, January). A nature of gravitation and the problem of the laboratory gravitational waves generation. In AIP Conference Proceedings (Vol. 1208, No. 1, pp. 523-537). American Institute of Physics.

[30] Liu, Tan, et al. "Waveforms of compact binary inspiral gravitational radiation in screened modified gravity." Physical Review D 98.8 (2018): 083023.

[31] Anile, A. M., Hunter, J. K., Truong, B. (1999). The generation of gravitational waves by the resonant interaction of sound waves. Journal of Mathematical Physics, 40(9), 4474-4494.

[32] Blanchet, L. (1998). Quadrupole-quadrupole gravitational waves. Classical and Quantum Gravity, 15(1), 89.

[33] Cutler, C., Flanagan, E. E. (1994). Gravitational waves from merging compact binaries: How accurately can one extract the binarys parameters from the inspiral waveform?. Physical Review D, 49(6), 2658.

[34] Capano, C. (2011). Searching for gravitational waves from compact binary coalescence using LIGO and virgo data.

[35] Berti, E., Cardoso, V., Starinets, A. O. (2009). Quasinormal modes of black holes and black branes. Classical and Quantum Gravity, 26(16), 163001.

[36] Cardoso, V. (2004). Quasinormal modes and gravitational radiation in black hole spacetimes. arXiv preprint gr-qc/0404093.

[37] Kokkotas, K. D., Schmidt, B. G. (1999). Quasi-normal modes of stars and black holes. Living Reviews in Relativity, 2(1), 1-72.

[38] Cardoso, V., Lemos, J. P. (2001). Scalar, electromagnetic, and Weyl perturbations of BTZ black holes: Quasinormal modes. Physical Review D, 63(12), 124015.

[39] Peters, P. C. (1966). Perturbations in the Schwarzschild metric. Physical Review, 146(4), 938.

[40] Giesler, M., Isi, M., Scheel, M. A., Teukolsky, S. A. (2019). Black hole ringdown: the importance of overtones. Physical Review X, 9(4), 041060.

[41] Nollert, H. P. (1999). Quasinormal modes: the characteristicsound'of black holes and neutron stars. Classical and Quantum Gravity, 16(12), R159.

[42] Horowitz, G. T., Hubeny, V. E. (2000). Quasinormal modes of AdS black holes and the approach to thermal equilibrium. Physical Review D, 62(2), 024027.

[43] Banados, M., Teitelboim, C., Zanelli, J. (1992). Black hole in three-dimensional spacetime. Physical Review Letters, 69(13), 1849.

[44] Corichi, A., Sudarsky, D. (2002). When is S D A= 4. Mod. Phys. Lett. A, 17, 1431.

[45] Vishveshwara, C.V., Scattering of gravitationa radiation by a Schwarzschild black-hole, Nature, 227, 936938, (1970).

[46] Pani, Paolo, et al. "Gravitational wave signatures of the absence of an event horizon: Nonradial oscillations of a thin-shell gravastar." Physical Review D 80.12 (2009): 124047.

[47] V. Cardoso, E. Franzin and P. Pani, "Is the gravitational-wave ringdown a probe of the event horizon?," Phys. Rev. Lett. 116, no. 17, 171101 (2016) Erratum: [Phys. Rev. Lett. 117, no. 8, 089902 (2016)] [arXiv:1602.07309 [gr-qc]].

[48] V. Cardoso, S. Hopper, C. F. B. Macedo, C. Palenzuela and P. Pani,

"Gravitational-wave signatures of exotic compact objects and of quantum corrections at the horizon scale," Phys. Rev. D 94, no. 8, 084031 (2016) [arXiv:1608.08637 [gr-qc]].

[49] A. Maselli, S. H. Vlkel and K. D. Kokkotas, "Parameter estimation of gravitational wave echoes from exotic compact objects," Phys. Rev. D 96, no. 6, 064045 (2017) [arXiv:1708.02217 [gr-qc]].

[50] R. S. Conklin, B. Holdom and J. Ren, "Gravitational wave echoes through new windows," [arXiv:1712.06517 [gr-qc]].

[51] H. Nakano, N. Sago, H. Tagoshi and T. Tanaka, "Black hole ringdown echoes and howls," PTEP 2017, no. 7, 071E01 (2017) [arXiv:1704.07175 [gr-qc]]

[52] Z. Mark, A. Zimmerman, S. M. Du and Y. Chen, "A recipe for echoes from exotic compact objects", Phys. Rev. D96, 8, (2017), [arXiv:1706.06155].

[53] P. Bueno, P. A. Cano, F. Goelen, T. Hertog and B. Vercnocke, Echoes of Kerr-like wormholes," Phys. Rev. D 97, no. 2, 024040 (2018) [arXiv:1711.00391 [gr-qc]].

[54] Burgess, C. P. (2004). Quantum gravity in everyday life: General relativity as an effective field theory. Living Reviews in Relativity, 7(1), 1-56.

[55] Burgess, C. P., Plestid, R., Rummel, M. (2018). Effective field theory of black hole echoes. Journal of High Energy Physics, 2018(9), 1-31.

[56] C. P. Burgess, P. Hayman, M. Williams and L. Zalavari, "Point-Particle Effective Field Theory I: Classical Renormalization and the Inverse-Square Potential," JHEP 1704, 106 (2017) [arXiv:1612.07313 [hep-ph]].

[57] C. P. Burgess, P. Hayman, M. Rummel, M. Williams and L. Zalavari, "Point-Particle Effective Field Theory II: Relativistic Effects and Coulomb/Inverse-Square Competition," JHEP 1707 (2017) 072 [arXiv:1612.07334 [hep-ph]].

[58] C. P. Burgess, P. Hayman, M. Rummel and L. Zalavari, "Point-Particle Effective Field Theory III: Relativistic Fermions and the Dirac Equation," JHEP 1709 (2017) 007 [arXiv:1706.01063 [hep-ph]].

[59] R. Plestid, C. P. Burgess and D. H. J. ODell, "Fall to the Centre in Atom Traps and Point-Particle EFT for Absorptive Systems," [arXiv:1804.10324 [hep-ph]] electromagnetic and neutrino field perturbations," Astrophys. J. 185, 635 (1973). doi:10.1086/152444

[60] Dappiaggi, Claudio, Hugo RC Ferreira, and Carlos AR Herdeiro. "Superradiance in the BTZ black hole with Robin boundary conditions." Physics Letters B 778 (2018): 146-154.

[61] Israel, W. (1966). Singular hypersurfaces and thin shells in general relativity. Il Nuovo Cimento B (1965-1970), 44(1), 1-14.

[62] S. A. Teukolsky, Perturbations of a Rotating Black Hole. I. Fundamental Equations for

Gravitational, Electromagnetic, and Neutrino-Field Perturbations, Astrophys. J. 185, 635 (1973).

[63] W. H. Press and S. A. Teukolsky, Perturbations of a Rotating Black Hole. II. Dynamical Stability of the Kerr Metric, Astrophys. J. 185, 649 (1973).

[64] A. Buonanno, G. B. Cook, and F. Pretorius, Inspiral, Merger and Ring-Down of Equal-Mass Black-Hole Binaries, Phys. Rev. D 75, 124018 (2007).