Tidally deformed extreme mass ratio binary system of Schwarzschild black holes

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Master's thesis
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Date of submission: 22-08-2023

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Abstract

Tides are ubiquitous in nature. Indeed, any extended object in a non-uniform gravitational field will be subject to tidal effects, whether it be the moon giving Earth its tides or entire galaxies giving each other tidal tails. A third example of particular interest in the age of gravitational wave astronomy, is the tidal interaction between black holes. Although this thesis wont go into detail regarding the effects of tides on gravitational wave signals, we will cover the theoretical foundations of tidal interactions in EMR binary systems of Schwarzschild black holes. We consider a large Schwarzschild black hole, referred to as the background black hole, and a much smaller black, referred to as the (tidally) deformed black hole, in orbit around the background black hole. As we will see, a vacuum region of an arbitrary spacetime can be described by a set of tidal moments. In particular, the tidal moments of the background spacetime will serve as building blocks for the metric around the tidally deformed black hole. The resulting metric is referred to as the Poisson-Vlasov metric and will serve as the foundation of much of this thesis. We mainly work under the assumption that the deformed black hole follows a radial geodesic in the background spacetime, implying that all magnetic tidal moments and potentials vanish identically. We compute the tidal shifts in the ISCO parameters of a test-particle orbiting the deformed black hole to quadrupole and octupole order. Furthermore, we compute the specific energy of the test-particle as a function of the Euler angles that specify the orientation of the "deformed black hole + test-particle" binary system with respect to the background black hole. We find that the specific energy is minimized for co-planar orbits, i.e. configurations for which the inclination angle vanishes. Furthermore, we compute the specific energy of the test-particle to octupole order. In particular, the specific energy is found to be increasing as a function of advanced time. Finally, we study the geometry of the deformed horizon. With the precision maintained in this text, the horizon is located at $r = 2m$ as in the unperturbed case. However, the mass of the deformed black hole now acquires a non-trivial time-dependence. Using the approach of Poisson, we compute the change in m to leading order for a radial infall and for a circular orbit.

Acknowledgements

I would like to thank my supervisor, Troels Harmark, for his support and for his enlightening feedback and commentary during the process of writing this thesis.
Additionally, I would like to extend my sincerest gratitude to Daniele Pica for his help and insight during this project. It has been truly invaluable.

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Conventions and notation

A given spacetime is, as always, modelled by some Lorentzian manifold $(\mathscr{M}, g)$ equipped with a metric g. We take the signature of g to be $(-,+,+,+)$. With respect to some coordinate system $\{x^\mu\}_{\mu\in\{0,1,2,3\}}$ on an open neighborhood $\mathscr{O} \subseteq \mathscr{M}$, the components of the metric are written $g_{\mu\nu}$. The spacetime indices on an arbitrary tensor defined on $\mathscr{O}$ may be lowered (raised) using $g_{\mu\nu}$ ($g^{\mu\nu}$). On the other hand, frame indices are raised and lowered using $\eta = \mathrm{diag}[-1,1,1,1]$. We work with the following index convention for the Riemann tensor:

$$R^\rho{}_{\sigma\mu\nu} = \frac{\partial}{\partial x^\nu}\Gamma^\rho{}_{\mu\sigma} - \frac{\partial}{\partial x^\mu}\Gamma^\rho{}_{\nu\sigma} + \Gamma^\rho{}_{\nu\kappa}\Gamma^\kappa{}_{\mu\sigma} - \Gamma^\rho{}_{\mu\kappa}\Gamma^\kappa{}_{\nu\sigma} \tag{1}$$

For the Ricci tensor, we use the following convention:

$$R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu} \tag{2}$$

Finally, we work in geometrical units, such that $G = c = 1$.

Below is a notation key, featuring commonly used symbols and their descriptions.

Notation	Description
$(\mathscr{M}, g)$	Lorentzian manifold
$\mathscr{O}, \mathscr{S}, \mathscr{N}$	Open subsets of $\mathscr{M}$
$(\mathscr{S}, g')$	Hypersurface of $(\mathscr{M}, g)$ where g' is the induced metric on $\mathscr{S}$
D	Levi-Civita connection on $(\mathscr{M}, g)$
$\mu, \nu, \ldots$	(Indices) Spacetime indices, taking values in $\{0, 1, 2, 3\}$
$i, j, \ldots$	(Indices) Spatial indices, taking values in $\{1, 2, 3\}$
$a, b, \ldots$	(Indices) Frame indices, taking values in $\{0, 1, 2, 3\}$
x^μ	Coordinate system on a coordinate patch $\mathscr{O}$
$T_{\mu\nu}$	Components of a $(0, 2)$ tensor T with respect to x^μ
$T_{(\mu\nu)}$	Symmetrization symbol, defined by $T_{(\mu\nu)} := \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$
$T_{[\mu\nu]}$	Antisymmetrization symbol, defined by $T_{[\mu\nu]} := \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$
$T^{\mathrm{STF}}_{\mu\nu}$	The symmetric and trace free part of $T_{\mu\nu}$
$T_{\mu\nu,\sigma} = \partial_\sigma T_{\mu\nu} = \frac{\partial}{\partial x^\sigma} T_{\mu\nu}$	Partial differentiation
$T_{\mu\nu;\sigma} = D_\sigma T_{\mu\nu}$	Covariant differentiation
$\varepsilon_{\alpha\beta\gamma\delta}$	Components of the Levi-Civita tensor with respect to x^μ
$\epsilon_{\alpha\beta\gamma\delta}$	Permutation symbol in four dimensions
ϵ_{ijk}	Permutation symbol in three dimensions
λ^μ_a	Orthonormal tetrad
$e^\mu_{\hat{a}}$	Carter's basis
$\mathcal{E}_{ij}$ $(\mathcal{E}_{ijk})$	Quadrupole (octupole) order electric tidal moments
$\mathcal{B}_{ij}$ $(\mathcal{B}_{ijk})$	Quadrupole (octupole) order magnetic tidal moments
E	Specific energy
L	Specific angular momentum

Chapter 1

Introduction

The overarching theme of this text is that of tidal effects in two-body systems. Tides can be found in a plethora of physical systems, whether it be the tidal interaction between the Earth and the moon, between stars in a binary system, between entire galaxies or between black holes. In this text, we will concern ourselves with the latter of these, specifically the tidal interaction between a pair of extreme mass ratio Schwarzschild black holes. To this end, we review and extensively use the results obtained by Eric Poisson and Igor Vlasov in [15].

Suppose we consider a binary system as described above. The larger of the black holes, referred to as the background black hole, is treated as a background source of a gravitational field. A much smaller black hole, treated as a test-particle with respect to the background black hole, is then placed in this field. We refer to this black hole as the tidally deformed (or simply deformed) black hole. An equivalent formulation would be that we consider a Schwarzschild black hole and then turn on a tidal field, given rise to by a much larger Schwarzschild black hole. As a consequence of turning on this tidal field, the mass of the deformed black hole seizes to be constant in time, an effect known as tidal heating which will be explored in section 7.

Additionally, we will observe tidally induced deformations in the orbits around the deformed black hole. In particular, we consider a third black hole with mass even less than that of the deformed black hole (so that the entire system can be described as a three-body hierarchical system). This black hole is then considered a test-particle with respect to the deformed black hole and we explore how the orbit of this test-particle is effected by the presence of a tidal field. For example, we will observe a tidally induced shift in each of the ISCO parameters of the test-particle.

The tidal environment itself is described by the tidal moments of the background spacetime. Below, we give a brief overview of tides in Newtonian mechanics, closely following the section "Tensors in Newtonian mechanics" from [20]. This overview will result in the introduction of the non-relativistic tidal tensor. With our Newtonian intuition in mind, we go on to consider a relativistic generalization of tides, resulting in the introduction of the relativistic tidal tensors.

1.1 Newtonian tides

In Newtonian mechanics, tides arise when the gravitational force experienced by a body is non-uniform across the extent of the body. For example, the gravitational field from the moon is stronger at a point on Earth that faces the moon, than at a point which faces away from the moon. As a result, Earth is stretched in the direction of the moon. Furthermore, every point on Earth will be pulled toward the center of gravity of the moon. This results in a squishing of the Earth in the direction perpendicular to the direction of stretching. Figure 1.1 gives an illustration of this. We take this squishing (and stretching) as a defining feature of the tidal interaction between the moon and the Earth, and more generally between any two bodies.

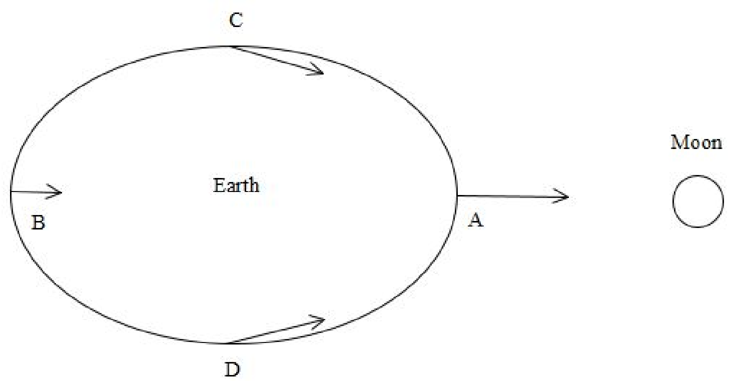


Figure 1.1: Tidal effects on Earth due to the gravitational influence of the moon. The strength of the gravitational attraction from the moon is greater at point A than at point B. Furthermore, points C and D are pulled toward each other.

Suppose we have a spacetime inhabited by two test-particles, call them test-particle A and test-particle B which are at rest relative to each other. Now suppose a non-uniform gravitational field, Φ is introduced. As a result, the two test-particles will begin to move relative to each other, i.e. the separation distance between them becomes a non-trivial function of time. In the following, we set out to determine an evolution equation for this separation distance.

We adopt a coordinate system, (t, x^i) where t is the universal coordinate time of Newtonian mechanics and $x^i, i \in \{1, 2, 3\}$ are spatial coordinates centred on the center of mass of the source of the gravitational field. Now, we would like to describe how the distance between the two test-particles changes as a function of t. We denote by $\boldsymbol{r}_{\mathrm{A}}(t)$ and $\boldsymbol{r}_{\mathrm{B}}(t)$ the respective positions of test-particle A and test-particle B at time t. Their separation vector $\boldsymbol{S}$ at time t is then defined as $\boldsymbol{S}(t) = \boldsymbol{r}_{\mathrm{B}}(t) - \boldsymbol{r}_{\mathrm{A}}(t)$. The acceleration of test-particle B is then given by

$$\frac{d^2\boldsymbol{r}_{\mathrm{B}}}{dt^2} = -\boldsymbol{\nabla}\Phi(\boldsymbol{r}_{\mathrm{B}}) = -\boldsymbol{\nabla}\Phi(\boldsymbol{r}_{\mathrm{A}} + \boldsymbol{S}) \tag{1.1}$$

In many cases, it is appropriate to assume $\|\boldsymbol{S}\| \ll \|\boldsymbol{r}_{\mathrm{A}}\|, \|\boldsymbol{r}_{\mathrm{B}}\|$. In other words, the test-particles are much farther from the source of the gravitational field than they are from each other. In this case, we perform a Taylor expansion of (1.1) to first order, which in component form reads

$$\frac{d^2 r^i_{\mathrm{B}}}{dt^2} = -\frac{\partial\Phi}{\partial x^i}(\boldsymbol{r}_{\mathrm{A}}) - \frac{\partial^2\Phi}{\partial x^j \partial x^i}(\boldsymbol{r}_{\mathrm{A}}) S^j \tag{1.2}$$

The evolution of $\boldsymbol{S}$ is then readily obtained:

$$\frac{d^2 S^i}{dt^2} = -\mathcal{E}^i{}_j S^j \tag{1.3}$$

where

$$\mathcal{E}^i{}_j := \delta^{ik} \frac{\partial^2\Phi}{\partial x^k \partial x^j} \tag{1.4}$$

are the components of the non-relativistic tidal tensor. Notice that $\mathcal{E}_{ij}$ is symmetric and also traceless owing to Φ satisfying Laplace's equation. In conclusion, we see that the tidal environment around a source of gravitation is described by the tidal tensor $\mathcal{E}$. In general relativity, we continue to have a tidal tensor $\mathcal{E}$ in analogy with the above. A new feature, exclusive to general relativity, is that we need also consider a second tidal tensor $\mathcal{B}$. These then serve to describe the tidal environment around a source of gravitation in general relativity.

1.2 Relativistic tides

Guided by our Newtonian intuition, we expect tidal effects to be directly tied to the relative acceleration of test-particles in spacetime. It is clear that a single test-particle is insufficient to detect a gravitational field. Indeed, an observer travelling along a timelike geodesic may adopt a set of Fermi normal coordinates. Let x^μ be such a set of coordinates. Then at a point p in a normal convex neighborhood of the geodesic, the components of the metric for the spacetime at hand evaluate to [16]:

$$g_{00}|_p = -1 - R_{0i0j}|_q x^i x^j + \mathcal{O}(s^3) \tag{1.5a}$$

$$g_{0i}|_p = -\frac{2}{3} R_{0jik}|_q x^j x^k + \mathcal{O}(s^3) \tag{1.5b}$$

$$g_{ij}|_p = \delta_{ij} - \frac{1}{3} R_{ikjl}|_q x^k x^l + \mathcal{O}(s^3) \tag{1.5c}$$

where R is the Riemann tensor and where s measures the spatial distance between the geodesic and p. Notice in particular that the metric reduces to the Minkowski metric along the geodesic. This shows the inadequacy of using a single particle to detect gravity. Appendix A gives a more detailed description of Fermi normal coordinates as well as a derivation of (1.5).

We are thus compelled to instead compare *two* closely separated test-particles and their trajectories. More precisely, we will compare two closely separated (non-intersecting) timelike geodesics. We will refer to them as γ and β. An observer travelling along γ picks a coordinate system $x^\mu, \mu \in \{0,1,2,3\}$ and chooses to parameterize γ by its proper time. For each τ in the domain of γ, we construct a spacelike vector $S(\tau)$ with components given by $S^\mu(\tau) := x^\mu \circ \beta(\tau' = \tau) - x^\mu \circ \gamma(\tau)$ where τ' is proper time along β. We interpret $S(\tau)$ as a separation vector between γ and β along a line of constant time equal to τ. One then computes (see appendix B):

$$\frac{D^2 S^\mu}{d\tau^2} = u^\alpha D_\alpha (u^\beta D_\beta S^\mu) = -R^\mu{}_{\alpha\nu\beta} u^\alpha u^\beta S^\nu \tag{1.6}$$

where u is the four-velocity along γ and D is the Levi-Civita connection on the spacetime. We thus see that the tides of general relativity are completely described by the Riemann curvature tensor of the spacetime. We define the relativistic analogue of (1.4) by

$$\mathcal{E}^{\mu}{}_{\nu} := R^{\mu}{}_{\alpha\nu\beta} u^{\alpha} u^{\beta} \tag{1.7}$$

and refer to it as the gravito-electric tidal tensor. In this text, we exclusively consider vacuum solutions to Einstein's field equations in which case R has ten independent degrees of freedom. However $\mathcal{E}_{\mu\nu}$, being symmetric and trace-free, can at most have nine independent components. The remaining degrees of freedom are encoded in the tensor $\mathcal{B}$ with components

$$\mathcal{B}^{\mu}{}_{\nu} = (R^{*})^{\mu}{}_{\alpha\nu\beta} u^{\alpha} u^{\beta} \tag{1.8}$$

where R^{*} is the dual Riemann tensor (more details can be found in section 2.4). We refer to $\mathcal{B}$ as the gravito-magnetic tidal tensor and it is unique to general relativity, with no analogue in Newtonian mechanics, unlike $\mathcal{E}$. We have already seen how $\mathcal{E}$ is responsible for the stretching and squishing of objects, so a natural question to ask is what physical interpretation $\mathcal{B}$ has. As it turns out, $\mathcal{B}$ describes the "twisting" of objects subject to a tidal field [2]. This effect occurs as a consequence of frame-dragging, which is especially prevalent in the Kerr solution. As we shall see later on, $\mathcal{B}$ vanishes identically for radial geodesics in the Schwarzschild spacetime, leaving only $\mathcal{E}$ non-zero.

1.3 Motivation for studying relativistic two-body systems

We have seen how tidal tensors arise naturally in the study of two-body systems. This allows for a succinct and efficient way of describing the tidal interaction between the two bodies in question. The motivation for studying two-body systems in the first place is plentiful and we mention one of the experimental avenues in which tidal heating, in particular, is likely to play an important role. Suppose we have an astrophysical source of gravitational radiation. In particular, we might imagine a small black hole spiralling around a much larger black hole. The tidal heating of the smaller black hole may then likely be responsible for the generation of low frequency gravitational waves [17], which could be measured by space-based gravitational wave observatories, such as LISA [9]. In fact, for a close encounter between a black hole travelling on a parabolic orbit around a much larger black hole, tidal heating can account for about 5% of the loss in orbital energy, while the remainder of the lost orbital energy is carried away in the form of gravitational waves [11]. In turn, this may then be used as a benchmark for high-precision numerical simulations of gravitational wave sources; one should expect to observe a time dependence in the mass of each black hole [15].

Chapter 2

Preliminaries and the Poisson-Vlasov metric

The main goal of this chapter is to introduce the Poisson-Vlasov metric, the metric that describes the spacetime around a tidally deformed Schwarzschild black hole. To this end, we start by covering some preliminaries. Firstly, the tidal environment around the Schwarzschild black hole is described by a set of tidal moments and corresponding tidal potentials. These are covered in section 2.4, which closely follows the outline given in [15]. As we will see, the aforementioned section relies heavily on the the use of orthonormal tetrads which are introduced in section 2.1, closely following part I of [16]. The Poisson-Vlasov metric will be put forth as an ansatz, motivated in part by the fact that it reduces to the background metric in the appropriate limit. For this reason, we explicitly introduce the background metric in section 2.5, closely following [15]. The construction of the background metric relies on some of the features of Synge's world function and bitensors in general. These are introduced in section 2.2 which closely follows [16]. The components of the metric (both the background metric and the Poisson-Vlasov metric) will be expressed in a set of lightcone coordinates which are introduced in section 2.3, closely following [14]. Finally, the Poisson-Vlasov metric is introduced in section 2.6, closely following [15].

2.1 Orthonormal tetrads

This section covers orthonormal tetrads and their role in decomposing tensors along a world-line. The core idea is to take a geodesic γ, construct a vectorial basis at some point along γ and then parallel transport the basis along γ. The result is a vectorial basis that can be used at each point of the geodesic. In this sense, a tetrad is a convenient choice of basis for a freely falling observer travelling along γ. For this reason, tetrads are intimately related to Fermi normal coordinates as well as lightcone coordinates and allow us to adopt a frame in which the metric of spacetime is locally flat in a neighborhood around γ. The tetrad formalism will prove particularly useful when we compute the tidal moments of the background spacetime. Indeed, these tidal moments will be defined as components of the Riemann tensor with respect to a tetrad erected along a geodesic of the background spacetime. A more precise description of orthonormal tetrads now follows.

Let $(\mathscr{M}, g)$ be a Lorentzian manifold and consider a neighborhood $\mathscr{N} \subseteq \mathscr{M}$ equipped with a coordinate system $x^\mu, \mu \in \{0,1,2,3\}$. Then consider a future-directed timelike geodesic $\gamma : [a,b] \to \mathscr{N}$. The proper time of γ is defined in the usual way:

$$\tau_\gamma := \int_a^b \sqrt{-g_{\mu\nu}(\gamma(t))\frac{d(x^\mu \circ \gamma)}{dt}(t)\frac{d(x^\nu \circ \gamma)}{dt}(t)dt} \tag{2.1}$$

The parameter for γ is chosen to be proper time $\tau \in [0, \tau_\gamma]$ along γ. The components of the four velocity along γ are then defined as

$$u^\mu(\tau) = \frac{d(x^\mu \circ \gamma)}{d\tau}(\tau), \qquad \tau \in [0, \tau_\gamma] \tag{2.2}$$

The four velocity is normalized according to

$$g_{\mu\nu}(\gamma(\tau))u^\mu(\tau)u^\nu(\tau) = -1, \qquad \tau \in [0, \tau_\gamma] \tag{2.3}$$

and of course u is parallel propagated along γ,

$$u^\mu D_\mu u^\nu = 0 \tag{2.4}$$

The first task in establishing an orthonormal tetrad along γ, is to construct an orthonormal basis at some point on γ, consisting of one timelike vector and three spacelike vectors. Without loss of generality, this initial point is taken to be $\gamma(0)$ and the members of the basis at $\gamma(0)$ are denoted by $\lambda_a, a \in \{0,1,2,3\}$. The index a is referred to as a frame index and such indices will be raised and lowered using the matrix $\eta = \text{diag}[-1,1,1,1]$. By convention, $\lambda_0 := u(0)$. Parallel transporting each member of the basis along γ, yields an orthonormal basis along the whole of γ. Hence, λ_a is promoted to be a function of τ for all $a \in \{0, 1, 2, 3\}$. The basis vectors thus constructed are referred to as an orthonormal tetrad on γ and explicitly we have the following defining equations:

$$\lambda_0^\mu(\tau) = u^\mu(\tau), \qquad \frac{D}{d\tau}\lambda_a^\mu(\tau) = 0, \qquad \lambda_a^\mu(\tau)\lambda_b^\nu(\tau)g_{\mu\nu}(\gamma(\tau)) = \eta_{ab}, \qquad \tau \in [0, \tau_\gamma] \tag{2.5}$$

where $a, b \in \{0, 1, 2, 3\}$. Given a tetrad $\{\lambda_a\}_{a\in\{0,1,2,3\}}$, its dual is then defined by

$$\lambda_\mu^a(\tau) := \eta^{ab}g_{\mu\nu}(\gamma(\tau))\lambda_b^\nu(\tau), \qquad \tau \in [0, \tau_\gamma] \tag{2.6}$$

Note that $\lambda_a^\mu\lambda_\nu^a = \delta_\nu^\mu$ and $\lambda_a^\mu\lambda_\mu^b = \delta_a^b$. As a consequence of the last equality in (2.5) together with (2.6), the following two completeness relations hold along γ:

$$g_{\mu\nu} = -\lambda_\mu^0\lambda_\nu^0 + \delta_{ij}\lambda_\mu^i\lambda_\nu^j \tag{2.7}$$

$$g^{\mu\nu} = -u^\mu u^\nu + \delta^{ij}\lambda_i^\mu\lambda_j^\nu \tag{2.8}$$

On γ, we will often want to decompose tensors with respect to $\{\lambda_a\}_{a\in\{0,1,2,3\}}$. Let A be some arbitrary tensor defined on γ with components given by $A^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m}$ where $m, n \in \mathbb{N}$. Then the frame components of A with respect to $\{\lambda_a\}_{a\in\{0,1,2,3\}}$ are defined by

$$A^{a_1\cdots a_n}{}_{b_1\cdots b_m} := A^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m}\lambda_{\mu_1}^{a_1}\cdots\lambda_{\mu_n}^{a_n}\lambda_{b_1}^{\nu_1}\cdots\lambda_{b_m}^{\nu_m} \tag{2.9}$$

where $a_1, \ldots, a_n, b_1, \ldots, b_m \in \{0, 1, 2, 3\}$. We will also take frame components of the first covariant derivative of tensors on γ. To this end, we define

$$A^{a_1\cdots a_n}{}_{b_1\cdots b_m|c} := A^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m;\sigma}\lambda_{\mu_1}^{a_1}\cdots\lambda_{\mu_n}^{a_n}\lambda_{b_1}^{\nu_1}\cdots\lambda_{b_m}^{\nu_m}\lambda_c^\sigma \tag{2.10}$$

These definitions will be central in defining the tidal moments of the background spacetime.

2.2 Bitensors

This section briefly covers bitensors and the notation associated with these. In particular, Synge's world function is introduced and some of its properties are derived. The motivation for doing so is two-fold. Firstly, Synge's world function plays a central role in the construction of the lightcone coordinates to be introduced in chapter 2.3. Secondly, Synge's world function and its derivatives enter in the components g^{vi} and g^{ij} of the inverse background metric which are computed in chapter 2.5. In particular, the components g^{ij} will be computed as an expansion in the radial coordinate r, to be introduced later. In order to accomplish this, we will need to expand the second derivatives of Synge's world function in r. As we will see, the parallel propagator provides an efficient way of computing one of these expansions, which is why it is included in this discussion. It will pay dividends to have the formalities in check, which is why we often start out by discussing bitensors in general and then specializing to Synge's world function. Indeed, we start by defining bitensors in general.

Let $(\mathscr{M}, g)$ be a Lorentzian manifold. A tensor field A which is defined on $\mathscr{M} \times \mathscr{M}$ is known as a bitensor. In other words, A is a tensor field which depends on two points in $\mathscr{M}$. We denote the two points by p' and p. The point p' is referred to as the base point while p is referred to as the field point. It is always assumed that p lies within a normal convex neighborhood $\mathcal{N}_{p'}$ of p' such that the two points can be linked by a unique geodesic. This geodesic will be denoted by β and is taken to be parameterized by some affine parameter $t \in [t_0, t_1]$. By construction, we have $\beta(t_0) = p'$ and $\beta(t_1) = p$.

Suppose now that $z^\mu, \mu \in \{0, 1, 2, 3\}$ is a coordinate system on $\mathcal{N}_{p'}$. Any bitensor defined on $\mathcal{N}_{p'}$ may then be decomposed with respect to z^μ. However, care must be taken in assigning indices to the tensor for the following reason. Generally, a bitensor which transforms as e.g. a vector at p' need not transform as a vector at p. To account for this, p' is assigned indices $\alpha', \beta', \ldots$ while p is assigned indices $\alpha, \beta, \ldots$. An arbitrary point along β will be assigned indices $\mu, \nu, \ldots$.

2.2.1 The parallel propagator

Say we are given a vector (or a tensor in general) at p and wish to parallel transport it to p'. This section covers the parallel propagator which accomplishes this task using the tetrad formalism.

Suppose A is a vector field on β, the unique geodesic connecting p and p', with components A^μ with respect to $z^\mu, \mu \in \{0, 1, 2, 3\}$. Furthermore, suppose an orthonormal tetrad $\{\lambda_a\}_{a \in \{0,1,2,3\}}$ is installed on β. Then A may be decomposed with respect to this tetrad according to $A^\mu = A^a \lambda_a^\mu$. The coefficients A^a are given by $A^a = A^\mu \lambda_\mu^a$. If A is parallel transported along β, then the coefficients A^a must be constant along β since the tetrad is parallel transported along β. We then obtain the components of A at p as

$$A^{\alpha}(p) = A^{\alpha'}(p')\lambda^a_{\alpha'}(p')\lambda^{\alpha}_a(p) = g^{\alpha}{}_{\alpha'}(p,p')A^{\alpha'}(p') \tag{2.11}$$

where $g^{\alpha}{}_{\alpha'}(p,p') := \lambda^{\alpha}_a(p)\lambda^a_{\alpha'}(p') = g_{\alpha'}{}^{\alpha}(p,p')$ is a bitensor known as the parallel propagator. The interpretation is clear from the above. The parallel propagator $g^{\alpha}{}_{\alpha'}(p,p')$ takes a vector at p' and parallel transports it along β to p. Similarly, the components of A at p' can be written

$$A^{\alpha'}(p') = g^{\alpha'}{}_{\alpha}(p,p')A^{\alpha}(p) \tag{2.12}$$

where $g^{\alpha'}{}_{\alpha}(p,p') := \lambda^a_{\alpha}(p)\lambda^{\alpha'}_a(p')$ takes a vector at p and parallel transports it along β to p'. Hence, $g^{\alpha'}{}_{\alpha}(p,p')$ can be interpreted as the inverse of $g^{\alpha}{}_{\alpha'}(p,p')$ and indeed we see that

$$g^{\alpha'}{}_{\alpha}g^{\beta}{}_{\alpha'} = \delta^{\beta}_{\alpha}, \qquad g^{\alpha'}{}_{\alpha}g^{\alpha}{}_{\beta'} = \delta^{\beta'}_{\alpha'} \tag{2.13}$$

The argument can be extended to tensors of arbitrary rank. For instance,

$$A^{\alpha\beta}(p) = g^{\alpha}{}_{\alpha'}(p,p')g^{\beta}{}_{\beta'}(p,p')A^{\alpha'\beta'}(p') \tag{2.14}$$

2.2.2 Synge's world function

Synge's world function σ is an example of a biscalar. With the same setup as above, it is defined by

$$\sigma(p,p') := \frac{1}{2}(t_1 - t_0)\int_{t_0}^{t_1} g_{\mu\nu}(\gamma(t))T^{\mu}(t)T^{\mu}(t)dt \tag{2.15}$$

where T is the tangent vector field to γ, defined by $T^{\mu} := d(z^{\mu} \circ \gamma)/dt$. Since γ is a geodesic, $\varepsilon := g_{\mu\nu}T^{\mu}T^{\nu}$ is constant along γ. Explicitly then,

$$\sigma(p,p') = \frac{1}{2}\varepsilon(t_1 - t_0)^2 \tag{2.16}$$

If γ is timelike, then we may choose t to be proper time τ so that $\sigma(p,p') = -\frac{1}{2}(t_1 - t_2)^2$. If γ is spacelike, then we may choose t to be proper distance s in which case $\sigma(p,p') = \frac{1}{2}(t_1 - t_2)^2$.

If γ is null, then $\sigma(p,p') = 0$. In general, $\sigma(p,p')$ is half the squared geodesic distance between p and p'. The world function may be differentiated with respect to either of its arguments. To this end, we define

$$\sigma_\alpha(p,p') := \frac{\partial \sigma}{\partial z^\alpha}(p,p') \tag{2.17}$$

as the derivative of σ with respect to the first of its arguments. Notice that σ_α transforms as a one-form at p but as a scalar at p'. Similarly, we define

$$\sigma_{\alpha'}(p,p') := \frac{\partial \sigma}{\partial z^{\alpha'}}(p,p') \tag{2.18}$$

as the derivative of σ with respect to its second argument. Notice that $\sigma_{\alpha'}$ transforms as a scalar at p but as a one-form at p'. Continuing, we take the covariant derivative of σ_α and $\sigma_{\alpha'}$ and define

$$\sigma_{\alpha\beta} := D_\alpha \sigma_\beta \tag{2.19}$$

$$\sigma_{\alpha'\beta'} := D_{\alpha'} \sigma_{\beta'} \tag{2.20}$$

Both of these are symmetric in their respective indices. Indeed,

$$\sigma_{\alpha\beta} := D_\alpha \sigma_\beta = \partial_\alpha \partial_\beta \sigma - \Gamma^\lambda{}_{\alpha\beta}\sigma_\lambda = D_\beta \sigma_\alpha = \sigma_{\beta\alpha} \tag{2.21}$$

owing to the symmetry of mixed partial derivatives and the symmetry in the lower indices of the Christoffel symbols. Similarly for $\sigma_{\alpha'\beta'}$. Additionally, we define $\sigma_{\alpha'\beta} := D_{\alpha'}\sigma_\beta$ and $\sigma_{\alpha\beta'} := D_\alpha \sigma_{\beta'}$. Since σ_α transforms as a scalar at p', we have $\sigma_{\alpha'\beta} = \partial_{\alpha'}\sigma_\beta = \sigma_{\beta\alpha'}$. This last identity generalizes to arbitrary bitensors with an arbitrary number of primed and unprimed indices. Indeed if Ω is a bitensor and if $\Omega_{\ldots\alpha\beta'\ldots}$ are its components where $\ldots$ denotes an arbitrary arrangement of both primed and unprimed indices, then

$$\Omega_{\ldots\alpha\beta'\ldots} = \Omega_{\ldots\alpha'\beta\ldots} \tag{2.22}$$

It will prove useful to explicitly compute σ_α which is accomplished in the following. Start by considering the variation of (2.15) as p is varied. In particular we consider a small displacement of p such that the new field point is $p + \delta p$. This results in a corresponding change in σ, described by $\delta\sigma := \sigma(p+\delta p, p') - \sigma(p,p')$. The change in p will also induce a change in β. In particular, we denote by $\beta + \delta\beta$ the unique geodesic that connects $p+\delta p$ and p'. We scale the affine parameter of this new geodesic such that it runs from t_0 to t_1. Since p' is kept fixed, $\delta\beta(t_0) = \delta p' = 0$. We then compute

$$\begin{aligned}
\delta\sigma &= \frac{1}{2}\Delta t \int_{t_0}^{t_1} \left[T^\mu(t)T^\nu(t)\delta g_{\mu\nu}(\beta(t)) + 2g_{\mu\nu}(\beta(t))T^\mu(t)\delta T^\nu(t)\right] dt \\
&= \Delta t \int_{t_0}^{t_1} \left[T^\mu(t)T^\nu(t)g_{\mu\sigma}(\beta(t))\Gamma^\sigma{}_{\rho\nu}(\beta(t))\delta z^\rho(\beta(t)) + g_{\mu\nu}(\beta(t))T^\nu(t)\delta T^\mu(t)\right] dt \\
&= \Delta t \left[g_{\mu\nu}(\beta(t))T^\nu(t)\delta z^\mu(\beta(t))\right]_{t_0}^{t_1} - \Delta t \int_{t_0}^{t_1} \left[\dot{T}_\mu(t) - T_\nu(t)T^\sigma(t)\Gamma^\nu{}_{\mu\sigma}(\beta(t))\right] \delta z^\mu(\beta(t))dt
\end{aligned} \tag{2.23}$$

where $\Delta t := t_1 - t_0$ and where the last equality follows from an application of integration by parts. The integral vanishes since T satisfies the geodesic equation. In particular, since $T^\mu D_\mu T_\nu = 0$ for all $\nu \in \{0,1,2,3\}$. Furthermore, $\delta z^\mu(\beta(t_0)) = 0$ by assumption so we are left with

$$\delta\sigma = \Delta t g_{\alpha\beta}(\beta(t_1))T^\alpha(t_1)\delta z^\beta(\beta(t_1)) \tag{2.24}$$

or equivalently,

$$\sigma_\alpha(p,p') = (t_1 - t_0)g_{\alpha\beta}(\beta(t_1))T^\beta(t_1) \tag{2.25}$$

In particular, $\sigma^\alpha(p,p')$ is simply a rescaled tangent vector of β at p. Similarly,

$$\sigma_{\alpha'}(p,p') = (t_1 - t_0)g_{\alpha'\beta'}(\beta(t_1))T^{\beta'}(t_1) \tag{2.26}$$

From these expressions, we conclude that

$$\sigma_\alpha\sigma^\alpha = \sigma_{\alpha'}\sigma^{\alpha'} = 2\sigma \tag{2.27}$$

Notice also that

$$\sigma_{\alpha\beta}\sigma^\beta = \sigma_\alpha \tag{2.28}$$

The proof is as follows. Firstly we compute

$$\sigma_{\alpha\beta}\sigma^{\beta} = (\partial_\alpha \sigma_\beta)\sigma^\beta - \Gamma^\gamma{}_{\alpha\beta}\sigma_\gamma\sigma^\beta \tag{2.29}$$

Owing to (2.27), the first term is simply σ_α. We expand the second term as follows:

$$\begin{aligned}\Gamma^\gamma{}_{\alpha\beta}\sigma_\gamma\sigma^\beta &= \frac{1}{2}(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta})\sigma^\delta\sigma^\beta \\ &= \frac{1}{2}(\partial_\alpha g_{\beta\delta})\sigma^\delta\sigma^\beta \\ &= \frac{1}{2}[(\partial_\alpha\sigma_\beta)\sigma^\beta - (\partial_\alpha\sigma^\delta)\sigma_\delta] \\ &= 0\end{aligned} \tag{2.30}$$

This finishes the proof. Similarly, $\sigma_{\alpha'\beta}\sigma^\beta = \sigma_{\alpha'}$.

2.2.3 Coincidence limits and the expansion of Synge's world function near coincidence

As mentioned at the beginning of this chapter, we seek to derive expressions for the expansions of the second derivatives of Synge's world function. In particular, we seek to derive expansions for $\sigma_{\alpha'\beta'}(p,p')$ and $\sigma_{\alpha'\beta}(p,p')$. Analogously to the procedure in real analysis, we will treat p' as a base-point and expand $\sigma_{\alpha'\beta'}(p,p')$ ($\sigma_{\alpha'\beta}(p,p')$) around this base-point. The expansion will be carried out near the coincidence limit of $\sigma_{\alpha'\beta'}(p,p')$ ($\sigma_{\alpha'\beta}(p,p')$). The coincidence limit is the limit in which $p \to p'$ and will be the subject of study in this section.

Consider some arbitrary bitensor $\Omega(p,p')$ with components $\Omega_{II'}(p,p')$ where $I = \alpha_1 \dots \alpha_i, i \in \mathbb{N}$ and $I' = \alpha'_1 \dots \alpha'_j, j \in \mathbb{N}$ are multi-indices representing an arbitrary number of unprimed and primed indices respectively. As already mentioned, a primed index and an unprimed index always commute so there is no loss of generality in this notation. It is then reasonable to ask what happens if we let $p \to p'$. This leads us to the definition of coincidence limits. A set of assumptions must be made about Ω before the coincidence limit can be defined in a meaningful way. We include them in the following definition:

- Assume that $\Omega_{II'}(p,p') \to \tilde{\Omega}_{\tilde{I}'}(p')$ as $p \to p'$ where $\tilde{\Omega}_{\tilde{I}'}(p')$ is an ordinary tensor at p' and where $\tilde{I}'$ is a multi-index with $i+j$ primed indices. In other words, Ω approaches an ordinary tensor at p' as $p \to p'$.

- Assume that the limit tensor $\tilde{\Omega}_{\tilde{I}'}(p')$ is unique in the sense that $\Omega_{II'}(p,p') \to \tilde{\Omega}_{\tilde{I}'}(p')$ as $p \to p'$ independent of the direction in which the limit is taken. Explicitly, if $\beta : [t_0, t] \to \mathscr{M}$ is a geodesic connecting $p = \beta(t)$ and $p' = \beta(t_0)$, then $\Omega_{II'}$ can be viewed as a function of t. We then assume that the limit of $\Omega_{II'}$ as $t \to t_0$ is independent of the choice of β.

- If the assumptions above are satisfied, then we define

$$[\Omega_{II'}] := \lim_{p \to p'} \Omega_{II'}(p,p') \tag{2.31}$$

  and refer to $[\Omega_{II'}]$ as the coincidence limit of $\Omega_{II'}(p,p')$.

For future reference, the coincidence limit of Synge's world function and the first few of its derivatives are computed in the following. From the definition of σ, we immediately get $[\sigma] = 0$. From (2.25) and (2.26) we get $[\sigma_\alpha] = [\sigma_{\alpha'}] = 0$. Next, from (2.28), we obtain $\sigma_{\alpha\beta}\sigma^\beta = \sigma_\alpha = g_{\alpha\beta}\sigma^\beta$ or $0 = (\sigma_{\alpha\beta} - g_{\alpha\beta})\sigma^\beta$. Recall that σ^α is simply a rescaled tangent vector to β. Hence, when we take the coincidence limit of $(\sigma_{\alpha\beta} - g_{\alpha\beta})\sigma^\beta$, the dependence on σ^β must drop out. In conclusion, $[\sigma_{\alpha\beta}] = g_{\alpha'\beta'}$. Similarly, $[\sigma_{\alpha\beta'}] = [\sigma_{\alpha'\beta}] = -g_{\alpha'\beta'}$. The procedure can be continued by repeated differentiation of (2.27) and by using the Ricci identity which, in the case of σ_α, reads $\sigma_{\alpha\beta\gamma} - \sigma_{\alpha\gamma\beta} = R^\epsilon{}_{\alpha\beta\gamma}\sigma_\epsilon$. We obtain the following:

$$[\sigma_{\alpha'\beta'\gamma'}] = 0 \tag{2.32}$$

$$[\sigma_{\alpha'\beta'\gamma'\delta'}] = -\frac{1}{3}(R_{\alpha'\gamma'\beta'\delta'} + R_{\alpha'\delta'\beta'\gamma'}) \tag{2.33}$$

$$[\sigma_{\alpha'\beta'\gamma'\delta'\epsilon'}] = -\frac{1}{4}(R_{\alpha'\gamma'\beta'\delta';\epsilon'} + R_{\alpha'\delta'\beta'\gamma';\epsilon'} + R_{\alpha'\delta'\beta'\epsilon';\gamma'} + R_{\alpha'\epsilon'\beta'\delta';\gamma'} + R_{\alpha'\epsilon'\beta'\gamma';\delta'} + R_{\alpha'\gamma'\beta'\epsilon';\delta'}) \tag{2.34}$$

The corresponding coincidence limits with any number of unprimed indices can then be computed by using Synge's rule, which we state without proof (the proof can be found in section 4.2 of [16]):

$$[\sigma_{\ldots\alpha'}] = [\sigma_{\ldots}]_{;\alpha'} - [\sigma_{\ldots\alpha}] \tag{2.35}$$

where the dots can be any combination of primed and unprimed indices.

As before, consider a generic bitensor $\Omega_{\alpha'\beta'}(p,p')$. In real analysis, the expansion of this object would be carried out in powers of the separation between the base-point p' and the free point p. The Lorentzian analogue to this, is an expansion in powers of $-\sigma^{\alpha'}(p,p')$. The expansion will thus take the form

$$\begin{aligned}\Omega_{\alpha'\beta'}(p,p') = A_{\alpha'\beta'}(p') + A^1_{\alpha'\beta'\gamma'}(p')\sigma^{\gamma'} + \frac{1}{2}A^2_{\alpha'\beta'\gamma'\delta'}(p')\sigma^{\gamma'}\sigma^{\delta'} \\ + \frac{1}{6}A^3_{\alpha'\beta'\gamma'\delta'\epsilon'}(p')\sigma^{\gamma'}\sigma^{\delta'}\sigma^{\epsilon'} + \dots\end{aligned} \tag{2.36}$$

where the coefficients $A, A^1, A^2, A^3, \dots$ are all ordinary tensors at p'. Our task is then to compute these coefficients. It follows immediately by taking the coincidence limit of both sides of (2.36) that $[\Omega_{\alpha'\beta'}] = A_{\alpha'\beta'}$. Differentiating (2.36) and taking the coincidence limit of the resulting expression yields $[\Omega_{\alpha'\beta';\gamma'}] = A_{\alpha'\beta';\gamma'} + A^1_{\alpha'\beta'\gamma'}$. Differentiating once more and taking the coincidence limit yields $[\Omega_{\alpha'\beta';\gamma'\delta'}] = A_{\alpha'\beta';\gamma'\delta'} + A^1_{\alpha'\beta'\delta';\gamma'} + A^1_{\alpha'\beta'\gamma';\delta'} + A^2_{\alpha'\beta'\gamma'\delta'}$. Differentiating a third time yields

$$\begin{aligned}[\Omega_{\alpha'\beta';\gamma'\delta'\epsilon'}] = A_{\alpha'\beta';\gamma'\delta'\epsilon'} + A^1_{\alpha'\beta'\kappa'}[\sigma^{\kappa'}{}_{\gamma'\delta'\epsilon'}] + A^1_{\alpha'\beta'\epsilon';\gamma'\delta'} + A^1_{\alpha'\beta'\delta';\gamma'\epsilon'} + A^1_{\alpha'\beta'\gamma';\delta'\epsilon'} \\ + A^2_{\alpha'\beta'\delta'\epsilon';\gamma'} + A^2_{\alpha'\beta'\gamma'\epsilon';\delta'} + A^2_{\alpha'\beta'\gamma'\delta';\epsilon'} + A^3_{\alpha'\beta'\gamma'\delta'\epsilon'}\end{aligned} \tag{2.37}$$

The expressions derived above can then be solved for the expansion coefficients which in turn are substituted into equation (2.36). This gives an expression for $\Omega_{\alpha'\beta'}$ to third order in $-\sigma^{\alpha'}$ near coincidence. For the purposes at hand, this is a sufficient level of precision.

Now consider a bitensor with one primed and one unprimed index, $\Omega_{\alpha'\beta}$. Then the bitensor $\tilde{\Omega}$ with components $\tilde{\Omega}_{\alpha'\beta'} := g^{\beta}{}_{\beta'}\Omega_{\alpha'\beta}$ can be expanded in precisely the same way as above, namely

$$\begin{aligned}\tilde{\Omega}_{\alpha'\beta'}(p,p') = B_{\alpha'\beta'}(p') + B^1_{\alpha'\beta'\gamma'}(p')\sigma^{\gamma'} + \frac{1}{2}B^2_{\alpha'\beta'\gamma'\delta'}(p')\sigma^{\gamma'}\sigma^{\delta'} \\ + \frac{1}{6}B^3_{\alpha'\beta'\gamma'\delta'\epsilon'}(p')\sigma^{\gamma'}\sigma^{\delta'}\sigma^{\epsilon'} + \dots\end{aligned} \tag{2.38}$$

And then the original bitensor Ω can be recovered as

$$\begin{aligned}\Omega_{\alpha'\beta}(p,p') = g^{\beta'}{}_{\beta}\Big(B_{\alpha'\beta'}(p') + B^1_{\alpha'\beta'\gamma'}(p')\sigma^{\gamma'} + \frac{1}{2}B^2_{\alpha'\beta'\gamma'\delta'}(p')\sigma^{\gamma'}\sigma^{\delta'} \\ + \frac{1}{6}B^3_{\alpha'\beta'\gamma'\delta'\epsilon'}(p')\sigma^{\gamma'}\sigma^{\delta'}\sigma^{\epsilon'}\Big) + \dots\end{aligned} \tag{2.39}$$

To compute the expansion coefficients directly in terms of $\Omega_{\alpha'\beta}$, the coincidence limits of the parallel propagator and its derivatives must first be obtained. We start by expanding the definition of the parallel propagator as follows:

$$g^{\alpha}{}_{\beta'}(p,p') = \lambda^{\alpha}_{a}(p)\lambda^{a}_{\beta'}(p') = \lambda^{\alpha}_{a}(p)\eta^{ab}g_{\beta'\gamma'}\lambda^{\gamma'}_{b}(p') \tag{2.40}$$

Taking the coincidence limit and using the completeness relation (2.8), we obtain

$$[g^{\alpha}{}_{\beta'}] = \delta^{\alpha'}_{\beta'} \tag{2.41}$$

Since the tetrad $\{\lambda_a\}_{a\in\{0,1,2,3\}}$ is parallel transported along β, we evidently have $\sigma^{\beta}\lambda^{\alpha}_{a;\beta} = 0$ at p and $\sigma^{\beta'}\lambda^{\alpha'}_{a;\beta'} = 0$ at p'. This then implies

$$g^{\alpha}{}_{\alpha';\beta}\sigma^{\beta} = g^{\alpha}{}_{\alpha';\beta'}\sigma^{\beta'} = 0, \qquad g^{\alpha'}{}_{\alpha;\beta}\sigma^{\beta} = g^{\alpha'}{}_{\alpha;\beta'}\sigma^{\beta'} = 0 \tag{2.42}$$

Repeated differentiation of each expression in (2.42) will yield the coincidence limits of the derivatives of the parallel propagator. For instance, we get from $g^{\alpha}{}_{\alpha';\beta'}\sigma^{\beta'} = 0$ that $g^{\alpha}{}_{\beta';\gamma'\delta'}\sigma^{\gamma'} + g^{\alpha}{}_{\beta';\gamma'}\sigma^{\gamma'}{}_{\delta'} = 0$. Taking the coincidence limit reveals that

$$[g^{\alpha}{}_{\beta';\gamma'}] = 0 \tag{2.43}$$

A second differentiation yields

$$g^{\alpha}{}_{\beta';\gamma'\delta'\epsilon'}\sigma^{\gamma'} + g^{\alpha}{}_{\beta';\gamma'\delta'}\sigma^{\gamma'}{}_{\epsilon'} + g^{\alpha}{}_{\beta';\gamma'\epsilon'}\sigma^{\gamma'}{}_{\delta'} + g^{\alpha}{}_{\beta';\gamma'}\sigma^{\gamma'}{}_{\delta'\epsilon'} = 0 \tag{2.44}$$

Taking the coincidence limit, this reduces to $[g^{\alpha}{}_{\beta';\gamma'\delta'}] + [g^{\alpha}{}_{\beta';\delta'\gamma'}] = 0$. Using the Ricci identity for $g^{\alpha}{}_{\beta';\gamma'\delta'}$, treated as a vector at p, we conclude

$$[g^{\alpha}{}_{\beta';\gamma'\delta'}] = \frac{1}{2}R^{\alpha'}{}_{\beta'\gamma'\delta'} \tag{2.45}$$

Similarly, $[g^{\alpha'}{}_{\beta;\gamma'\delta'}] = -\frac{1}{2}R^{\alpha'}{}_{\beta'\gamma'\delta'}$. Completely analogously, we obtain

$$[g^{\alpha}{}_{\beta';\delta'\gamma'\epsilon'}] = \frac{1}{3}(R^{\alpha'}{}_{\beta'\delta'\gamma';\epsilon'} + R^{\alpha'}{}_{\beta'\delta'\epsilon';\gamma'}) \tag{2.46}$$

and similarly, $[g^{\alpha'}{}_{\beta;\delta'\gamma'\epsilon'}] = -\frac{1}{3}(R^{\alpha'}{}_{\beta'\delta'\gamma';\epsilon'} + R^{\alpha'}{}_{\beta'\delta'\epsilon';\gamma'})$. We are now ready to compute the coefficients in (2.39). Firstly, $[\Omega_{\alpha'\beta}] = B_{\alpha'\beta'}$. Differentiating (2.39) once and taking the coincidence limit gives $[\Omega_{\alpha'\beta;\gamma'}] = B_{\alpha'\beta';\gamma'} + B^1_{\alpha'\beta'\gamma'}$. Differentiating once again, we obtain

$$[\Omega_{\alpha'\beta;\gamma'\delta'}] = \frac{1}{2}R^{\epsilon'}{}_{\beta'\gamma'\delta'}B_{\alpha'\epsilon'} - B_{\alpha'\beta';\gamma'\delta'} - B^1_{\alpha'\beta'\gamma';\delta'} - B^1_{\alpha'\beta'\delta';\gamma'} - B^2_{\alpha'\beta'\gamma'\delta'} \tag{2.47}$$

A final differentiation yields

$$\begin{aligned}
[\Omega_{\alpha'\beta;\phi'\theta'\kappa'}] = {} & B_{\alpha'\beta';\phi'\theta'\kappa'} - \frac{1}{2}R^{\omega'}{}_{\beta'\phi'\theta'}B_{\alpha'\omega';\kappa'} - \frac{1}{2}R^{\omega'}{}_{\beta'\theta'\kappa'}B_{\alpha'\omega';\phi'} - \frac{1}{2}R^{\omega'}{}_{\beta'\phi'\kappa'}B_{\alpha'\omega';\theta'} \\
& - \frac{1}{3}(R^{\omega'}{}_{\beta'\phi'\theta';\kappa'} + R^{\omega'}{}_{\beta'\phi'\kappa';\theta'})B_{\alpha'\omega'} - \frac{1}{3}(R^{\gamma'}{}_{\theta'\phi'\kappa'} + R^{\gamma'}{}_{\kappa'\phi'\theta'})B^1_{\alpha'\beta'\gamma'} \\
& + B^1_{\alpha'\beta'\kappa';\phi'\theta'} + B^1_{\alpha'\beta'\theta';\phi'\kappa'} + B^1_{\alpha'\beta'\phi';\theta'\kappa'} - \frac{1}{2}R^{\omega'}{}_{\beta'\phi'\theta'}B^1_{\alpha'\omega'\kappa'} - \frac{1}{2}R^{\omega'}{}_{\beta'\theta'\kappa'}B^1_{\alpha'\omega'\phi'} \\
& - \frac{1}{2}R^{\omega'}{}_{\beta'\phi'\kappa'}B^1_{\alpha'\omega'\kappa'}B^1_{\alpha'\omega'\kappa'} + B^2_{\alpha'\beta'\theta'\kappa';\phi'} + B^2_{\alpha'\beta'\phi'\kappa';\theta'} + B^2_{\alpha'\beta'\phi'\theta';\kappa'} + B^3_{\alpha'\beta'\phi'\theta'\kappa'}
\end{aligned} \tag{2.48}$$

In the following, we specialize to $\Omega_{\alpha'\beta'} = \sigma_{\alpha'\beta'}$. Putting all the previous results together, the expansion of $\sigma_{\alpha'\beta'}$ near coincidence is given by

$$\sigma_{\alpha'\beta'} = g_{\alpha'\beta'} - \frac{1}{3}R_{\alpha'\gamma'\beta'\delta'}\sigma^{\gamma'}\sigma^{\delta'} + \frac{1}{12}R_{\alpha'\gamma'\beta'\delta';\epsilon'}\sigma^{\gamma'}\sigma^{\delta'}\sigma^{\epsilon'} + \ldots \tag{2.49}$$

Similarly, for $\Omega_{\alpha'\beta} = \sigma_{\alpha'\beta}$ we obtain the expansion

$$\sigma_{\alpha'\beta} = g^{\beta'}{}_{\beta}\left[-g_{\alpha'\beta'} - \frac{1}{6}R_{\alpha'\gamma'\beta'\delta'}\sigma^{\gamma'}\sigma^{\delta'} + \frac{1}{12}R_{\alpha'\gamma'\beta'\delta';\epsilon'}\sigma^{\gamma'}\sigma^{\delta'}\sigma^{\epsilon'}\right] + \ldots \tag{2.50}$$

Equipped with these expansions, we are able to compute the components of the inverse metric in section 2.5. Before then, we introduce the lightcone coordinates, advertised at the beginning of the chapter.

2.3 Lightcone coordinates

This section introduces the coordinates with respect to which the components of the background metric as well as the Poisson-Vlasov metric will later be written. The coordinates are referred to as lightcone coordinates and, as the name suggests, are closely tied to the geometry of (past) lightcones. We start by giving a heuristic overview of the coordinates, followed by a formal definition.

2.3.1 Heuristic overview

Consider a Lorentzian manifold $(\mathscr{M}, g)$ and an open neighborhood $\mathscr{N} \subseteq \mathscr{M}$ of a smooth timelike geodesic $\gamma : [a, b] \to \mathscr{N}$. To each point p on γ, there is a corresponding past lightcone with apex at p. Since such a lightcone is a null hypersurface of $\mathscr{M}$, it is generated by a congruence of null geodesics as discussed in section 7.2. The quasi-spherical lightcone coordinates of $\mathscr{N}$, denoted by (v, r, θ, ϕ), are a set of coordinates specifically tailored to describe the geometry of the generators of the past lightcones of $\mathscr{N}$. In particular, v, r, θ and ϕ are defined such that the following properties hold:

- v is constant on each lightcone. In particular, if a given lightcone has its apex at $\gamma(\tau), \tau \in [a, b]$, then $v = \tau$ on this lightcone.
- θ and ϕ are both constant on the null generators of each lightcone. In this sense, they can be viewed as generator labels.
- $-r$ is an affine parameter of the null generators of each lightcone.

In the case of Schwarzschild, these coordinates correspond to the ingoing Eddington-Finkelstein coordinates. The lightcone coordinates also come in a quasi-Cartesian variant, (v, x^1, x^2, x^3). The construction of both variants will be formally carried out in the following section.

2.3.2 Formal definition

Consider again a Lorentzian manifold $(\mathscr{M}, g)$ and an open neighborhood $\mathscr{N} \subseteq \mathscr{M}$ equipped with a coordinate system $\tilde{x}^\mu, \mu \in \{0, 1, 2, 3\}$. Unless otherwise specified, all components will be taken with respect to this coordinate system. For the construction below to be successful, we demand that $\mathscr{N}$ be a normal convex neighborhood of γ. Consider now a smooth future directed timelike geodesic $\gamma : [a, b] \to \mathscr{N}$ on which an orthonormal tetrad $\{\lambda_a\}_{a \in \{0,1,2,3\}}$ is installed. The goal is to assign to each point $p \in \mathscr{N}$ a set of lightcone coordinates. Initially, we will assign p a set of quasi-Cartesian lightcone coordinates (v, x^1, x^2, x^3).

Consider an arbitrary point $p \in \mathscr{N}$. Since $\mathscr{N}$ is normal convex, there is a unique future directed null geodesic β which starts at p and intersects γ. The point of intersection will be denoted by p'. The advanced time coordinate v of p is then defined indirectly by

$p' = \gamma(\tau = v)$. In other words, v is equal to the value of τ at the point where β and γ intersect. Next, the spatial lightcone coordinates are defined by

$$x^i := -\lambda^i_{\alpha'}\sigma^{\alpha'}(p,p'), \qquad i \in \{1,2,3\} \tag{2.51}$$

Furthermore, since p and p' are linked by a null geodesic, we have $\sigma(p,p') = 0$. The quasi-spherical variant of the lightcone coordinates are defined in the following.

The advanced time coordinate continues to be defined as above. We then make the following definition:

$$r(p) := -\sigma_{\alpha'}(p,p')u^{\alpha'}(p') \tag{2.52}$$

which, for the moment, is simply a scalar field defined on $\mathscr{N}$. In the following, it will be made clear that $-r$ in fact serves as an affine parameter along the null generators of the past lightcone with apex at p'. Using (2.7), we compute

$$\delta_{ij}x^i x^j = (g_{\alpha'\beta'} + \lambda^0_{\alpha'}\lambda^0_{\beta'})\sigma^{\alpha'}\sigma^{\beta'} = u_{\alpha'}u_{\beta'}\sigma^{\alpha'}\sigma^{\beta'} = r^2 \tag{2.53}$$

having used that $\sigma_{\alpha'}\sigma^{\alpha'} = 0$ and $\lambda^0_{\alpha'} = -u_{\alpha'}$. We then define

$$\Omega^i := \frac{x^i}{r} \tag{2.54}$$

which, owing to (2.53), satisfies $\delta_{ij}\Omega^i\Omega^j = 1$. Furthermore, using (2.8), we may decompose $\sigma^{\alpha'}$ in terms of the tetrad $\{\lambda_a\}_{a\in\{0,1,2,3\}}$ as follows:

$$\sigma^{\alpha'} = g^{\alpha'\beta'}\sigma_{\beta'} = (-u^{\alpha'}u^{\beta'} + \delta^{ij}\lambda^{\alpha'}_i\lambda^{\beta'}_j)\sigma_{\beta'} = r(u^{\alpha'} - \Omega^i\lambda^{\alpha'}_i) \tag{2.55}$$

Consider now a small displacement of p so that we end up at a new point $p + \delta p$. This point will have lightcone coordinates $(v + \delta v, x^i + \delta x^i)$. Correspondingly, β is displaced to a new null geodesic, denoted by $\beta + \delta\beta$. This further induces a displacement in p' which becomes $p' + \delta p'$. The coordinates of $p' + \delta p'$ are denoted by $x^{\alpha'} + \delta x^{\alpha'}$ and it follows from the definition of v that $\delta x^{\alpha'} = u^{\alpha'}\delta v$. This is then used in the following computation:

$$0 = \sigma(p+\delta p, p'+\delta p') = \sigma_\alpha \delta x^\alpha + \sigma_{\alpha'}\delta x^{\alpha'} = \sigma_\alpha\delta x^\alpha + \sigma_{\alpha'}u^{\alpha'}\delta v = \sigma_\alpha \delta x^\alpha - r\delta v \tag{2.56}$$

to first order in the displacements. In other words,

$$\partial_\alpha v = -l_\alpha \tag{2.57}$$

where $l_\alpha := -\sigma_\alpha/r$ is future-directed and tangent to β at p. In a similar fashion, we seek an expression for $\partial_\alpha x^i$. Firstly,

$$\delta x^i = -\lambda^i_{\alpha'}\delta\sigma^{\alpha'} = -\lambda^i_{\alpha'}\sigma^{\alpha'}{}_{\beta'}u^{\beta'}\delta v - \lambda^i_{\alpha'}\sigma^{\alpha'}{}_{\beta}\delta x^\beta = \lambda^i_{\alpha'}\sigma^{\alpha'}{}_{\beta'}u^{\beta'}l_\beta\delta x^\beta - \lambda^i_{\alpha'}\sigma^{\alpha'}{}_{\beta}\delta x^\beta \tag{2.58}$$

which is equivalent to

$$\partial_\alpha x^i = \lambda^i_{\alpha'}\sigma^{\alpha'}{}_{\beta'}u^{\beta'}l_\alpha - \lambda^i_{\alpha'}\sigma^{\alpha'}{}_{\alpha} \tag{2.59}$$

Completely analogously, we obtain

$$\partial_\beta r = \sigma_{\alpha'\beta'}u^{\alpha'}u^{\beta'}l_\beta - \sigma_{\alpha'\beta}u^{\alpha'} \tag{2.60}$$

From eq. (2.28), we also obtain

$$\sigma_{\alpha\beta}l^\beta = l_\alpha \tag{2.61}$$

and similarly,

$$\sigma_{\alpha'\beta}l^\beta = -\frac{\sigma_{\alpha'}}{r} \tag{2.62}$$

Using the latter of these in (2.60) yields

$$l^\beta\partial_\beta r = -1 \tag{2.63}$$

having used that $l_\beta l^\beta = 0$.

The next step will be to compute the covariant derivative of l_α. We start by observing how l_α changes under a small displacement as described earlier:

$$r\delta l_\alpha = -\delta\sigma_\alpha - l_\alpha\delta r = [-\sigma_{\alpha\beta} + \sigma_{\alpha\beta'}u^{\beta'}l_\beta - l_\alpha(\sigma_{\alpha'\beta'}u^{\alpha'}u^{\beta'}l_\beta - \sigma_{\alpha'\beta}u^{\alpha'})]\delta x^\beta \tag{2.64}$$

In other words,

$$r D_\beta l_\alpha = -\sigma_{\alpha\beta} + \sigma_{\alpha\beta'} u^{\beta'} l_\beta - \sigma_{\alpha'\beta'} u^{\alpha'} u^{\beta'} l_\alpha l_\beta + \sigma_{\alpha'\beta} u^{\alpha'} l_\alpha \tag{2.65}$$

Contracting with l^β, we find that l satisfies the geodesic equation in affine parameter form:

$$l^\beta D_\beta l^\alpha = 0 \tag{2.66}$$

Taken together with equation (2.63), this implies that $-r$ is an affine parameter along β. Hence a displacement along a given null generator of the past lightcone that converges to p' is described by

$$\delta x^\alpha = -l^\alpha \delta r \tag{2.67}$$

Using this in (2.56) and (2.58), we obtain $\delta v = 0$ and $\delta x^i = \Omega^i \delta r$. Integrating these, we obtain $v =$ constant and $x^i = r\Omega^i(\theta^A)$. The two angles $\theta^A, A \in \{1, 2\}$ are constants with respect to r and serve to parameterize the unit vector Ω.

In conclusion, the geodesics to which l^α is tangent are the generators of the lightcone described by $v =$ constant. A particular generator is chosen by fixing the two angles θ^A and on this generator, $-r$ is an affine parameter. The tuple $(v, r, \theta^1, \theta^2)$ thus constitutes the advertised quasi-spherical lightcone coordinates. See figure 2.1 for an illustration.

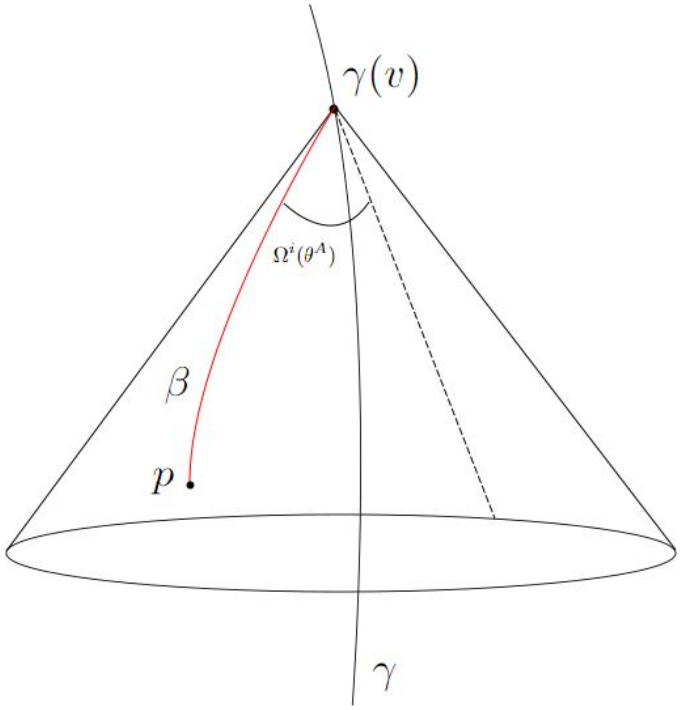


Figure 2.1: A particular lightcone, chosen by setting v equal to some constant. A particular generator on this lightcone is then chosen by specifying the two angles θ^1 and θ^2. Then $-r$ serves as an affine parameter on this generator.

2.4 Tidal moments and tidal potentials

This section covers the set of tidal moments and corresponding tidal potentials that serve to characterise the tidal environment of the background spacetime. The tidal potentials will serve as the building blocks for the Poisson-Vlasov metric.

Let $(\mathscr{M}, g)$ be a Lorentzian manifold and consider a coordinate system $x^\mu, \mu \in \{0,1,2,3\}$ defined on an open neighborhood $\mathscr{O} \subseteq \mathscr{M}$. It is assumed that $\mathscr{O}$ is a vacuum region of spacetime, i.e. the Ricci tensor vanishes identically on $\mathscr{N}$. Now consider a smooth timelike geodesic γ in $\mathscr{O}$, parameterized by proper time. The first step will be to establish an orthogonal tetrad $\{\lambda_a\}_{a\in\{0,1,2,3\}}$ on γ analogously to how we did it for the lightcone coordinates. In particular, we choose $\lambda_0 = u$. The three remaining tetrad vectors will be explicitly constructed for a geodesic in the Schwarzschild spacetime in chapter 3. In four dimensions, the Weyl tensor C has ten independent components. We encode these components in the two symmetric-tracefree tensors whose components are given by

$$\mathcal{E}_{ij} := C_{\alpha\mu\beta\nu}\lambda_i^\alpha u^\mu \lambda_j^\beta u^\nu \tag{2.68}$$

$$\mathcal{B}_{ij} := C^*_{\alpha\mu\beta\nu}\lambda_i^\alpha u^\mu \lambda_j^\beta u^\nu \tag{2.69}$$

where $i, j = \{1, 2, 3\}$ and C^* is the dual Weyl tensor with components given by

$$C^*_{\alpha\mu\beta\nu} = \frac{1}{2}\varepsilon^{\gamma\sigma}{}_{\alpha\mu}C_{\gamma\sigma\beta\nu} \tag{2.70}$$

Here, ε is the Levi-Civita tensor with components given by

$$\varepsilon_{\mu\nu\sigma\rho} = \pm\sqrt{-\det(g)}\epsilon_{\mu\nu\sigma\rho} \tag{2.71}$$

where g is the matrix representation of the metric tensor in the coordinate system $x^\mu, \mu \in \{0, 1, 2, 3\}$ and ϵ is the permutation symbol with convention $\epsilon_{0123} = 1$. The sign in front of the square root depends on the orientation of the coordinate system. We refer to $\mathcal{E}_{ij}$ and $\mathcal{B}_{ij}$ as the quadrupole tidal moments of electric type and magnetic type respectively along γ. Note that since $\mathscr{O}$ is Ricci flat, the Weyl tensor and the Riemann tensor coincide on $\mathscr{O}$. Henceforth, we will therefore simply use the Riemann tensor in the construction of tidal moments. Notice also that (2.68) and (2.69) are related to the tidal tensors (1.7) and (1.8) simply by $\mathcal{E}_{ij} = \hat{\mathcal{E}}_{\alpha\beta}\lambda_i^\alpha\lambda_j^\beta$ and $\mathcal{B}_{ij} = \hat{\mathcal{B}}_{\alpha\beta}\lambda_i^\alpha\lambda_j^\beta$, having introduced hats to distinguish between the the full spacetime tensors of the introduction and the tidal moments introduced here. Next comes the definitions of the octupole tidal moments:

$$\mathcal{E}_{ijk} := \left(R_{\alpha\mu\beta\nu;\sigma}\lambda_i^\alpha u^\mu \lambda_j^\beta u^\nu \lambda_k^\sigma\right)^{\mathrm{STF}} \tag{2.72}$$

$$\mathcal{B}_{ijk} := \left(R^*_{\alpha\mu\beta\nu;\sigma}\lambda_i^\alpha u^\mu \lambda_j^\beta u^\nu \lambda_k^\sigma\right)^{\mathrm{STF}} \tag{2.73}$$

where the STF symbol instructs us to symmetrize in all free indices and remove all traces. For future reference, we also use angled brackets around indices to serve the same purpose. For example, $A_{\langle\mu\nu\rangle\alpha\beta}$ is obtained by symmetrizing $A_{\mu\nu\alpha\beta}$ in μ and ν and removing the trace over μ and ν. We refer to $\mathcal{E}_{ijk}$ and $\mathcal{B}_{ijk}$ as the octupole tidal moments of electric type and magnetic type respectively along γ.

In the following, the tidal moments introduced above will be used to define a set of tidal potentials. To get started, the coordinate system x^μ is assumed to be quasi-Cartesian, namely it is assumed that x^0 is a temporal coordinate and $x^i, i \in \{1,2,3\}$ are Cartesian coordinates (later we will specialize to quasi-Cartesian lightcone coordinates). Then define

$$\Omega^i := \frac{x^i}{r}, \qquad i \in \{1,2,3\} \tag{2.74}$$

to be a radial unit vector where $r := \sqrt{\delta_{ij}x^i x^j}$. The radial direction will be referred to as the longitudinal direction while the orthogonal space will be referred to as the transverse directions. Next, define a projector γ which projects to the transverse space, orthogonal to Ω^i:

$$\gamma^i{}_j := \delta^i{}_j - \Omega^i \Omega_j \tag{2.75}$$

We may transform the Cartesian coordinates $x^i, i \in \{1,2,3\}$ to spherical coordinates, (r, θ, ϕ) by

$$x^i = r\Omega^i(\theta^A) \tag{2.76}$$

where $A \in \{1,2\}$ and $\theta^1 = \theta,\ \theta^2 = \phi$. This implies

$$\frac{\partial x^i}{\partial r} = \Omega^i, \qquad \frac{\partial x^i}{\partial \theta^A} = r\Omega^i_A \tag{2.77}$$

where

$$\Omega^i_A := \frac{\partial \Omega^i}{\partial \theta^A} \tag{2.78}$$

Since $x_i x^i = r^2$ which is independent of θ^A, we have

$$\Omega_i \Omega^i_A = 0 \tag{2.79}$$

In later sections, we will explicitly set $\Omega^1 = \cos\theta$, $\Omega^2 = \sin\theta\sin\phi$ and $\Omega^3 = \sin\theta\cos\phi$.

With this choice, two additional identities hold:

$$\gamma_{ij}\Omega^i_A\Omega^j_B = \Omega_{AB} \tag{2.80a}$$

$$\Omega^{AB}\Omega^i_A\Omega^j_B = \gamma^{ij} \tag{2.80b}$$

where Ω_{AB} is defined through the matrix representation, $\Omega_{AB} \sim \mathrm{diag}[1, \sin^2\theta]$.

The tidal moments can be split into two sectors, namely the even parity sector and the odd parity sector. To see this, we first define a parity transformation as follows. A parity transformation is defined by the following change in tetrad vectors:

$$\lambda_0 \to \lambda_0, \qquad \lambda_i \to -\lambda_i, i \in \{1,2,3\} \tag{2.81}$$

That is, the timelike vector remains unchanged while the three spacelike vectors change sign. Under such a transformation, the tidal moments change as follows:

$$\mathcal{E}_{ij} \to \mathcal{E}_{ij}, \qquad \mathcal{E}_{ijk} \to -\mathcal{E}_{ijk} \tag{2.82}$$

$$\mathcal{B}_{ij} \to -\mathcal{B}_{ij}, \qquad \mathcal{B}_{ijk} \to \mathcal{B}_{ijk} \tag{2.83}$$

Hence, $\mathcal{E}_{ij}$ and $\mathcal{E}_{ijk}$ both transform as Cartesian tensors under a parity transformation. For this reason, they are said to have even parity. Meanwhile, $\mathcal{B}_{ij}$ and $\mathcal{B}_{ijk}$ transform as pseudotensors and are therefore said to have odd parity. The goal is now to construct a set of tidal potentials out of the tidal moments and Ω^i. As a consequence of the parity transformation properties above, the potentials will be divided into an even parity sector and an odd parity sector. We demand that each scalar potential should transform as a scalar under a parity transformation. Likewise, each vector potential should transform as a vector under a parity transformation. Furthermore, each vector potential should be orthogonal to Ω^i. Finally, each tensor potential should transform as a tensor under a parity transformation and should be orthogonal to Ω^i as well as being tracefree. Each tidal potential, regardless of type, should correspond to an irreducible representation of $SO(3)$, labeled by multipole order l. In the following, we spell out this last point in a bit more detail. Generically, the even parity potentials are constructed as follows:

$$\mathcal{E}^{(l)} = \mathcal{E}_{k_1 k_2 \cdots k_l} \Omega^{k_1} \Omega^{k_2} \cdots \Omega^{k_l} \tag{2.84}$$

$$\mathcal{E}^{(l)}_i = \gamma_i{}^j \mathcal{E}_{j k_2 \cdots k_l} \Omega^{k_2} \cdots \Omega^{k_l} \tag{2.85}$$

$$\mathcal{E}^{(l)}_{ij} = 2\gamma_i{}^j \gamma_j{}^m \mathcal{E}_{j m k_3 \cdots k_l} \Omega^{k_3} \cdots \Omega^{k_l} + \gamma_{ij} \mathcal{E}^{(l)} \tag{2.86}$$

where $\mathcal{E}_{k_1 k_2 \cdots k_l}$ are the components of a constant STF tensor of rank l. The potentials satisfy the following eigenvalue equations (see appendix C):

$$r^2 \gamma^{ij} D_i D_j \mathcal{E}^{(l)} + l(l+1)\mathcal{E}^{(l)} = 0 \tag{2.87}$$

$$r^2 \gamma^{ij} D_i D_j \mathcal{E}^{(l)}_m + [l(l+1) - 1]\mathcal{E}^{(l)}_m = 0 \tag{2.88}$$

$$r^2 \gamma^{ij} D_i D_j \mathcal{E}^{(l)}_{mn} + [l(l+1) - 4]\mathcal{E}^{(l)}_{mn} = 0 \tag{2.89}$$

where D_i is a derivative operator defined through

$$D_i T_{j_1 j_2 \cdots j_q} = \gamma_i{}^p \gamma_{j_1}{}^{m_1} \cdots \gamma_{j_q}{}^{m_q} \partial_p T_{m_1 \cdots m_q} \tag{2.90}$$

where $T_{j_1 j_2 \cdots j_q}, q \in \mathbb{N}$ are the components of an arbitrary tensor. The odd-parity sector is constructed in a similar way:

$$\mathcal{B}^{(l)} = \mathcal{B}_{k_1 \cdots k_l} \Omega^{k_1} \cdots \Omega^{k_l} \tag{2.91}$$

$$\mathcal{B}^{(l)}_i = \epsilon_{imn} \Omega^m \mathcal{B}^n{}_{k_2 \cdots k_l} \Omega^{k_2} \cdots \Omega^{k_l} \tag{2.92}$$

$$\mathcal{B}^{(l)}_{ij} = (\epsilon_{imn} \Omega^m \mathcal{B}^n{}_{q k_3 \cdots k_l} \gamma^q_j + \epsilon_{jmn} \Omega^m \mathcal{B}^n{}_{p k_3 \cdots k_l} \gamma^p_i) \Omega^{k_3} \cdots \Omega^{k_l} \tag{2.93}$$

satisfying a set of eigenvalue equations completely analogous to (2.87)-(2.89), simply by replacing $\mathcal{E}$ with $\mathcal{B}$.

As an example, a scalar potential is constructed in the following. For the even-parity sector, the simplest case involves the quadrupole tidal moments $\mathcal{E}_{ij}$. As we have seen, $\mathcal{E}_{ij}$ transforms as a tensor under a parity transformation and so $\mathcal{E}^{\mathrm{q}} := \mathcal{E}_{ij}\Omega^i\Omega^j$ transforms as a scalar under a parity transformation. Since, by construction, $\mathcal{E}_{ij}$ is symmetric and trace-free, it also satisfies the scalar potential eigenvalue equation for $l = 2$. Hence, $\mathcal{E}^{\mathrm{q}}$ satisfies all the criteria demanded of a tidal scalar potential. Similarly, $\mathcal{E}^{\mathrm{q}}_i := \gamma^m_i \mathcal{E}_{mn}\Omega^n$ satisfies all the requirements for a vector tidal potential. Going through this procedure, we end up with the potentials listed in table 2.1. We will primarily work with quasi-spherical

$\mathcal{E}^{\rm q} = \mathcal{E}_{kl}\Omega^k\Omega^l$
$\mathcal{E}^{\rm q}_i = \gamma_i{}^k\mathcal{E}_{kl}\Omega^l$
$\mathcal{E}^{\rm q}_{ij} = 2\gamma_i{}^k\gamma_j{}^l\mathcal{E}_{kl} + \gamma_{ij}\mathcal{E}^{\rm q}$
$\mathcal{E}^{\rm o} = \mathcal{E}_{klm}\Omega^k\Omega^l\Omega^m$
$\mathcal{E}^{\rm o}_i = \gamma_i{}^k\mathcal{E}_{klm}\Omega^l\Omega^m$
$\mathcal{E}^{\rm o}_{ij} = 2\gamma_i{}^k\gamma_j{}^l\mathcal{E}_{klm}\Omega^m + \gamma_{ij}\mathcal{E}^{\rm o}$

$\mathcal{B}^{\rm q}_i = \epsilon_{ikl}\Omega^k\mathcal{B}^l{}_m\Omega^m$
$\mathcal{B}^{\rm q}_{ij} = \epsilon_{ikl}\Omega^k\mathcal{B}^l{}_m\gamma^m{}_j + \epsilon_{jkl}\Omega^k\mathcal{B}^l{}_m\gamma^m{}_i$
$\mathcal{B}^{\rm o}_i = \frac{4}{3}\epsilon_{ikl}\Omega^k\mathcal{B}^l{}_{mn}\Omega^m\Omega^n$
$\mathcal{B}^{\rm o}_{ij} = \frac{4}{3}(\epsilon_{ikl}\Omega^k\mathcal{B}^l{}_{mn}\gamma^m{}_j + \epsilon_{jkl}\Omega^k\mathcal{B}^l{}_{mn}\gamma^m{}_i)\Omega^n$

Table 2.1: Potentials constructed from $\mathcal{E}_{ij}/\mathcal{E}_{ijk}$ (left) and from $\mathcal{B}_{ij}/\mathcal{B}_{ijk}$ (right). Superscript q means quadrupole, while superscript o means octupole.

lightcone coordinates in the chapters to come. For this reason, it will be useful to convert the potentials to spherical coordinates. This is accomplished using Ω^i_A. For example,

$$\mathcal{E}^{\rm q}_A := \mathcal{E}^{\rm q}_i\Omega^i_A, \qquad \mathcal{E}^{\rm q}_{AB} := \mathcal{E}^{\rm q}_{ij}\Omega^i_A\Omega^j_B \tag{2.94}$$

The potentials constructed in this chapter will serve as the building blocks for both the background metric and the Poisson-Vlasov metric.

2.5 The background metric

This section covers the metric of the tidal background. The main goal will be to express the components of the background metric in a set of lightcone coordinates. The notation used closely follows that introduced in section 2.3.2.

Firstly, the components of the inverse metric in quasi-Cartesian lightcone coordinates are given by

$$g^{vv} = g^{\alpha\beta}\partial_\alpha v\partial_\beta v \tag{2.95}$$

$$g^{vi} = g^{\alpha\beta}\partial_\alpha v\partial_\beta x^i \tag{2.96}$$

$$g^{ij} = g^{\alpha\beta}\partial_\alpha x^i\partial_\beta x^j \tag{2.97}$$

By virtue of (2.57) and the fact that l_α is null, $g^{vv} = 0$ identically. Using (2.57) and (2.59), we compute

$$g^{vi} = \lambda^i_{\alpha'}l^\beta\sigma^{\alpha'}{}_\beta = -\frac{1}{r}\lambda^i_{\alpha'}\sigma^{\alpha'} = \frac{x^i}{r} = \Omega^i \tag{2.98}$$

where the second equality follows from (2.62). Similarly,

$$g^{ij} = \lambda^i_{\alpha'}\lambda^j_{\beta'}g^{\alpha\beta}\sigma^{\alpha'}{}_{\alpha}\sigma^{\beta'}{}_{\beta} - \lambda^i_{\alpha'}\sigma^{\alpha'}{}_{\beta'}u^{\beta'}\Omega^j - \lambda^j_{\alpha'}\sigma^{\alpha'}{}_{\beta'}u^{\beta'}\Omega^i \tag{2.99}$$

For these components, we will use (2.49), (2.50) and (2.55) and write the result as a power series in r. For notational convenience, we thus write

$$g^{ij} = G^{ij}_0 + rG^{ij}_1 + r^2G^{ij}_2 + r^3G^{ij}_3 + \dots \tag{2.100}$$

and compute

$$G^{ij}_0 = \delta^{ij} \tag{2.101}$$

$$G^{ij}_1 = 0 \tag{2.102}$$

$$\begin{aligned} G^{ij}_2 = \frac{1}{3}\big[&R^{i\ j}_{\ 0\ 0} - (R^{i\ j}_{\ k\ 0} + R^{j\ i}_{\ k\ 0})\Omega^k + R^{i\ \ j}_{\ m\ k}\Omega^m\Omega^k \\ &+(R^i{}_{0m0}\Omega^m - R^i{}_{km0}\Omega^k\Omega^m)\Omega^j + (R^j{}_{0m0}\Omega^m - R^j{}_{km0}\Omega^k\Omega^m)\Omega^i\big] \end{aligned} \tag{2.103}$$

$$\begin{aligned} G^{ij}_3 = -\frac{1}{12}\Big[&2(\dot{R}^{i\ j}_{\ 0\ 0} - (\dot{R}^{i\ j}_{\ 0\ l} + \dot{R}^{i\ j}_{\ l\ 0})\Omega^l + \dot{R}^{i\ j}_{\ l\ k}\Omega^l\Omega^k) + (\dot{R}^i{}_{0l0}\Omega^l - \dot{R}^i{}_{lk0}\Omega^l\Omega^k)\Omega^j \\ &+(\dot{R}^j{}_{0l0}\Omega^l - \dot{R}^j{}_{lk0}\Omega^l\Omega^k)\Omega^i + 2(-R^{i\ j}_{\ 0\ 0|l}\Omega^l + (R^{i\ \ j}_{\ m\ 0|l} + R^{j\ \ i}_{\ m\ 0|l})\Omega^m\Omega^l - R^{i\ \ j}_{\ m\ k|l}\Omega^k\Omega^m\Omega^l) \\ &+ (-R^i{}_{0m0|l}\Omega^m\Omega^l + R^i{}_{km0|l}\Omega^k\Omega^m\Omega^l)\Omega^j + (-R^j{}_{0m0|l}\Omega^m\Omega^l + R^j{}_{km0|l}\Omega^k\Omega^m\Omega^l)\Omega^i\Big] \end{aligned} \tag{2.104}$$

where the components of the Riemann tensor are frame components with respect to the tetrad $\{\lambda_a\}_{a\in\{0,1,2,3\}}$ and overdots denote differentiation with respect to proper time. For example,

$$\dot{R}^i{}_{lk0} = R^{\alpha'}{}_{\beta'\gamma'\delta';\epsilon'}\lambda^i_{\alpha'}\lambda^{\beta'}_l\lambda^{\gamma'}_k u^{\delta'}u^{\epsilon'} \tag{2.105}$$

We introduce the potentials

$$P_{ij} := R_{i0j0} - (R_{imj0} + R_{jmi0})\Omega^m + R_{imjk}\Omega^m\Omega^k \tag{2.106a}$$

$$P_i := P_{ij}\Omega^j = R_{i0m0}\Omega^m - R_{imk0}\Omega^m\Omega^k \tag{2.106b}$$

$$Q_{ij} := -R_{i0j0|m}\Omega^m + (R_{imj0|k} + R_{jmi0|k})\Omega^m\Omega^k - R_{imjk|l}\Omega^m\Omega^k\Omega^l \tag{2.106c}$$

$$Q_i := Q_{ij}\Omega^j = -R_{i0m0|k}\Omega^m\Omega^k + R_{imk0|l}\Omega^m\Omega^k\Omega^l \tag{2.106d}$$

allowing us to write

$$G_2^{ij} = \frac{1}{3}(P^{ij} + P^i\Omega^j + P^j\Omega^i) \tag{2.107}$$

$$G_3^{ij} = -\frac{1}{12}(2\dot{P}^{ij} + \dot{P}^i\Omega^j + \dot{P}^j\Omega^i) - \frac{1}{12}(2Q^{ij} + Q^i\Omega^j + Q^j\Omega^i) \tag{2.108}$$

Notice that the inverse metric takes the form

$$g^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta} \tag{2.109}$$

where $\eta^{\alpha\beta}$ are the components of the inverse Minkowski metric in lightcone coordinates ($\eta^{vv} = 0, \eta^{vi} = \Omega^i$ and $\eta^{ij} = \delta^{ij}$) and $h^{ij} = r^2 G_2^{ij} + r^3 G_3^{ij} + \mathcal{O}(r^4)$ with all other components vanishing. Hence to order r^3, the background metric is then given by $g_{\alpha\beta} = \eta_{\alpha\beta} - h_{\alpha\beta}$ where indices are lowered using the Minkowski metric. Explicitly, we have

$$g_{vv} = -1 - r^2 P + \frac{1}{3}r^3\dot{P} + \frac{1}{3}r^3 Q + \mathcal{O}(r^4) \tag{2.110a}$$

$$g_{vi} = \Omega_i + \gamma_i^k\left[-\frac{2}{3}r^2 P_k + \frac{1}{4}r^3\dot{P}_k + \frac{1}{4}r^3 Q_k + \mathcal{O}(r^4)\right] \tag{2.110b}$$

$$g_{ij} = \gamma_{ij} + \gamma_i^k\gamma_j^m\left[-\frac{1}{3}r^2 P_{km} + \frac{1}{6}r^3\dot{P}_{km} + \frac{1}{6}r^3 Q_{km} + \mathcal{O}(r^4)\right] \tag{2.110c}$$

where $P := P_i\Omega^i$ and $Q := Q_i\Omega^i$. It will be useful to express (2.110) in terms of the tidal potentials of table 2.1. Firstly, with the notation employed in this section, the tidal moments of quadrupole order are written as

$$\mathcal{E}_{ij} = R_{i0j0} \tag{2.111}$$

$$\mathcal{B}_{ij} = \frac{1}{2}\epsilon_i{}^{mn} R_{mnj0} \tag{2.112}$$

and the tidal moments of octupole order are written as

$$\mathcal{E}_{ijk} = (R_{i0j0|k})^{\mathrm{STF}} \tag{2.113}$$

$$\mathcal{B}_{ijk} = \frac{3}{8}(\epsilon_i{}^{mn} R_{mnj0|k})^{\mathrm{STF}} \tag{2.114}$$

The next step will be to express the Riemann tensor and its derivatives in terms of these tidal potentials. Of course, $R_{i0j0} = \mathcal{E}_{ij}$. Inverting (2.112), we obtain

$$R_{ijk0} = \epsilon_{ijm} \mathcal{B}^{m}{}_{k} \tag{2.115}$$

having made use of the identity $\epsilon_{ijk}\epsilon^{mnk} = \delta_i^m \delta_j^n - \delta_i^n \delta_j^m$. To obtain an expression for R_{imjn}, we start by making the following observations. Firstly, by using the completeness relation of equation (2.8), we have

$$0 = R_{\mu\nu} = -u^\rho u^\sigma R_{\rho\mu\sigma\nu} + \delta^{ij} \lambda_i^\rho \lambda_j^\sigma R_{\rho\mu\sigma\nu} \tag{2.116}$$

Taking frame components then yields

$$\delta^{mn} R_{imjn} = \mathcal{E}_{ij} \tag{2.117}$$

Additionally, the following tracelessness condition holds:

$$\delta^{ij} \delta^{mn} R_{imjn} = 0 \tag{2.118}$$

Hence the number of independent components of R_{imjn} is five, the same number of independent components as that of $\mathcal{E}_{ij}$. This then implies that (2.117) can be inverted to give an expression for R_{imjn} in terms of $\mathcal{E}_{ij}$. From (2.117), we obtain

$$\begin{aligned} &R_{1313} = \mathcal{E}_{11} + \mathcal{E}_{22} = -\mathcal{E}_{33} \quad R_{1213} = \mathcal{E}_{23}, \quad R_{1223} = -\mathcal{E}_{13} \\ &R_{1323} = \mathcal{E}_{12}, \quad R_{2323} = \mathcal{E}_{22} + \mathcal{E}_{33} = -\mathcal{E}_{11} \end{aligned} \tag{2.119}$$

as the independent components of R_{imjn}. These are summarized as

$$R_{ikjl} = \delta_{ij}\mathcal{E}_{kl} + \delta_{kl}\mathcal{E}_{ij} - \delta_{il}\mathcal{E}_{jk} - \delta_{jk}\mathcal{E}_{il} \tag{2.120}$$

Moving on to the derivatives of the Riemann tensor, we start by expanding the definition of $\mathcal{E}_{ijk}$:

$$
\begin{aligned}
\mathcal{E}_{ijk} &= \frac{1}{3}(R_{i0j0|k} + R_{i0k0|j} + R_{j0k0|i}) \\
&= R_{i0j0|k} - \frac{1}{3}(\dot{R}_{jki0} + \dot{R}_{ikj0}) \\
&= R_{i0j0|k} - \frac{1}{3}(\epsilon_{jkm}\dot{\mathcal{B}}^m{}_i + \epsilon_{ikm}\dot{\mathcal{B}}^m{}_j)
\end{aligned} \tag{2.121}
$$

where the second equality follows from the second Bianchi identity. Hence,

$$
R_{i0j0|k} = \mathcal{E}_{ijk} + \frac{1}{3}(\epsilon_{jkm}\dot{\mathcal{B}}^m{}_i + \epsilon_{ikm}\dot{\mathcal{B}}^m{}_j) \tag{2.122}
$$

In a similar fashion, we have

$$
R_{ijk0|l} = \epsilon_{ij}{}^m \left[\frac{4}{3}\mathcal{B}_{mkl} - \frac{1}{3}\left(\epsilon_{mln}\dot{\mathcal{E}}^n{}_k + \epsilon_{kln}\dot{\mathcal{E}}^n{}_m\right)\right] \tag{2.123}
$$

$$
R_{ijkl|m} = -\epsilon_{ij}{}^p\epsilon_{kl}{}^q \left[\mathcal{E}_{pqm} + \frac{1}{3}\left(\epsilon_{pmn}\dot{\mathcal{B}}^n{}_q + \epsilon_{qmn}\dot{\mathcal{B}}^n{}_p\right)\right] \tag{2.124}
$$

Inserting these expressions into (2.106) results in the background metric (2.110) taking the following form:

$$
g_{vv} = -1 - r^2\mathcal{E}^{\mathrm{q}} + \frac{1}{3}r^3\dot{\mathcal{E}}^{\mathrm{q}} - \frac{1}{3}r^3\mathcal{E}^{\mathrm{o}} + \mathcal{O}(r^4) \tag{2.125a}
$$

$$
g_{vi} = \Omega_i - \frac{2}{3}r^2(\mathcal{E}^{\mathrm{q}}_i - \mathcal{B}^{\mathrm{q}}_i) + \frac{1}{3}r^3(\dot{\mathcal{E}}^{\mathrm{q}}_i - \dot{\mathcal{B}}^{\mathrm{q}}_i) - \frac{1}{4}r^3(\mathcal{E}^{\mathrm{o}}_i - \mathcal{B}^{\mathrm{o}}_i) + \mathcal{O}(r^4) \tag{2.125b}
$$

$$
g_{ij} = \gamma_{ij} - \frac{1}{3}r^2(\mathcal{E}^{\mathrm{q}}_{ij} - \mathcal{B}^{\mathrm{q}}_{ij}) + \frac{5}{18}r^3(\dot{\mathcal{E}}^{\mathrm{q}}_{ij} - \dot{\mathcal{B}}^{\mathrm{q}}_{ij}) - \frac{1}{6}r^3(\mathcal{E}^{\mathrm{o}}_{ij} - \mathcal{B}^{\mathrm{o}}_{ij}) + \mathcal{O}(r^4) \tag{2.125c}
$$

For future reference, we also note the components of the background metric in quasi-spherical coordinates. As is always the case, the metric transforms according to

$$
g^{\text{spherical}}_{\mu\nu} = g^{\text{Cartesian}}_{\alpha\beta}\frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\frac{\partial\tilde{x}^\beta}{\partial x^\nu} \tag{2.126}
$$

where $\tilde{x}^\alpha, \alpha \in \{v, x^1, x^2, x^3\}$ are quasi-Cartesian coordinates and $x^\mu, \mu \in \{v, r, \theta^1, \theta^2\}$ are quasi-spherical coordinates. We then use (2.77) as well as (2.94) to arrive at the following non-vanishing components of the background metric in quasi-spherical coordinates:

$$g_{vv} = -1 - r^2\mathcal{E}^{\mathrm{q}} + \frac{1}{3}r^3\dot{\mathcal{E}}^{\mathrm{q}} - \frac{1}{3}r^3\mathcal{E}^{\mathrm{o}} + \mathcal{O}(r^4) \tag{2.127a}$$

$$g_{vr} = 1 \tag{2.127b}$$

$$g_{vA} = -\frac{2}{3}r^3(\mathcal{E}^{\mathrm{q}}_A - \mathcal{B}^{\mathrm{q}}_A) + \frac{1}{3}r^4(\dot{\mathcal{E}}^{\mathrm{q}}_A - \dot{\mathcal{B}}^{\mathrm{q}}_A) - \frac{1}{4}r^4(\mathcal{E}^{\mathrm{o}}_A - \mathcal{B}^{\mathrm{o}}_A) + \mathcal{O}(r^5) \tag{2.127c}$$

$$g_{AB} = r^2\Omega_{AB} - \frac{1}{3}r^4(\mathcal{E}^{\mathrm{q}}_{AB} - \mathcal{B}^{\mathrm{q}}_{AB}) + \frac{5}{18}r^5(\dot{\mathcal{E}}^{\mathrm{q}}_{AB} - \dot{\mathcal{B}}^{\mathrm{q}}_{AB}) - \frac{1}{6}r^5(\mathcal{E}^{\mathrm{o}}_{AB} - \mathcal{B}^{\mathrm{o}}_{AB}) + \mathcal{O}(r^6) \tag{2.127d}$$

Note that $g_{vr} = 1$ is exact.

2.6 The Poisson-Vlasov metric

This section introduces the metric that describes the spacetime around a tidally deformed Schwarzschild black hole, namely the Poisson-Vlasov metric. A detailed derivation of the metric lies outside the scope of this text and we instead give a brief overview of some of the steps taken in deriving it.

To start with, we again consider a smooth timelike geodesic $\gamma : [a, b] \to \mathscr{M}$ in some background spacetime, as described in the previous sections. We then consider a Ricci flat normal convex neighborhood $\mathscr{N} \subseteq \mathscr{M}$ of γ, equipped with a set of quasi-spherical lightcone coordinates (v, r, θ, ϕ). In contrast to the previous sections, we now place a black hole of mass m on γ. This black hole will be referred to as the tidally deformed black hole (or simply, the deformed black hole) and will be the centerpiece of the remainder of the text. The horizon of the deformed black hole will trace out a world tube as depicted in figure 2.2. We demand that this world tube fit well within $\mathscr{N}$, which is achieved by imposing the following:

$$m \ll \mathcal{R} \tag{2.128}$$

where $\mathcal{R}$ is the length scale that characterizes the tidal environment. More precisely, $\mathcal{R}$ is the local radius of curvature of the background spacetime evaluated at the position of the deformed black hole. In particular, $\mathcal{R}$ is defined by

$$\mathcal{R}^2 = \frac{1}{\sqrt{K}} \tag{2.129}$$

where K is the Kretschmann scalar (evaluated at the position of the deformed black hole). It is also required that the neighborhood $\mathscr{N}$ itself should be small as compared to $\mathcal{R}$. In particular, we demand that

$$r \ll \mathcal{R} \tag{2.130}$$

Two further implications of (2.128) and (2.130) are that the black hole is weakly perturbed by the background, and that the world tube is small as viewed on the scale of $\mathcal{R}$. It is in this sense that it is sensible, at least approximately, to speak of the black hole following a worldline.

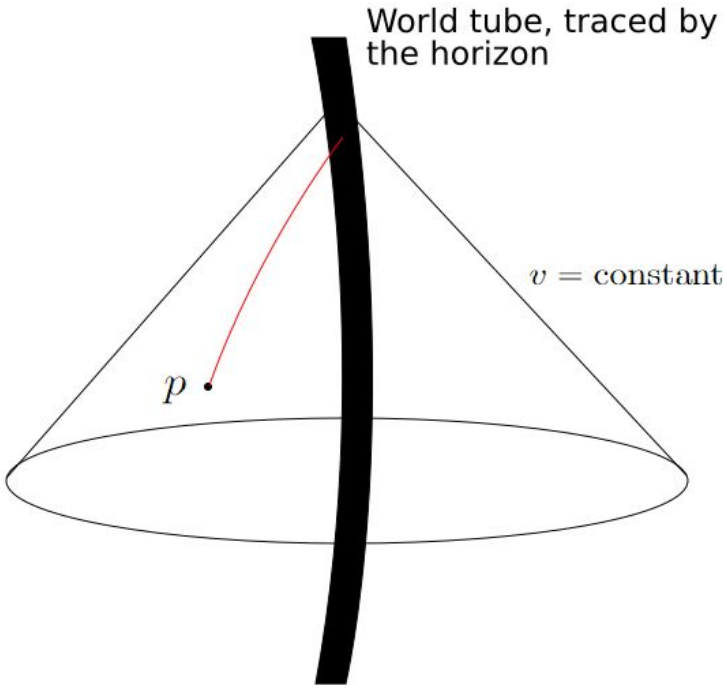


Figure 2.2: The world tube traced by the black hole horizon. The corresponding lightcones are generated by a congruence of null geodesics. On each lightcone, v is constant and along each generator, the angles θ^1 and θ^2 are constant. Furthermore, r still serves as an affine parameter along each generator. The generators now converge toward the world tube and not a worldline.

Despite the fact that there is no longer a worldline for the lightcone coordinates to be calibrated with respect to, each surface of constant v is still a lightcone. The main difference from the background case, is that the generators of each lightcone now converge towards the world tube traced by the deformed black hole, as opposed to a worldline. Nevertheless, we can still make sure that on each lightcone, v is constant (this is true by definition) and on each generator, the angles θ^1 and θ^2 are constant. Furthermore, r can still be made to serve as an affine parameter along each generator. Shortly, we will see that this is accomplished by imposing a set of gauge conditions on the tidal perturbations. Firstly, however, we make a comment on the "rigidity" of the lightcone coordinates in the black hole spacetime. As already mentioned, the lightcone coordinates on each lightcone are no longer as well behaved as we might like. For instance, we are no longer able to say that $r = 0$ corresponds to a point on a worldline γ. We can make up for this, at least partially, by matching the asymptotic behaviour of the black hole spacetime to the behaviour of the background spacetime in the following sense. Far from the deformed black hole in a region where $r \gg m$ (but still $r \ll \mathcal{R}$, of course), the gravitational influence of the deformed black hole will be small compared to that of the background. Namely, light rays will behave (nearly) the same in this region as they would in the background spacetime. This motivates the choice to tune the black hole lightcone coordinates so that the asymptotic description of each generator of a given lightcone coincides with the corresponding description in the background spacetime. This choice will manifest itself in the components of the Poisson-Vlasov metric, which will be required to reduce to those of the background metric in the asymptotic region $r \gg m$.

In going from the spacetime around an unperturbed Schwarzschild black hole to the spacetime around a tidally perturbed Schwarzschild black hole, we would like for the lightcone coordinates around the black hole to retain their geometrical properties, as mentioned above. In the following we will see how this leads to the so-called lightcone gauge. Start by considering the metric of an unperturbed Schwarzschild black hole, denoted g^0. In Eddington-Finkelstein coordinates, the line element corresponding to g^0 is given by

$$ds^2 := g^0_{\mu\nu} dx^\mu dx^\nu = -f dv^2 + 2dvdr + r^2 d\Omega^2 \tag{2.131}$$

where

$$f := 1 - \frac{2m}{r}, \qquad d\Omega^2 := d\theta^2 + \sin^2\theta d\phi^2 \tag{2.132}$$

Furthermore, the advanced time v is related to the usual coordinate time through

$$v = t + r + 2m \ln\left(\frac{r}{2m} - 1\right) \tag{2.133}$$

We then introduce a perturbation with components $p_{\mu\nu}$ and write the full metric of the tidally deformed black hole as

$$g_{\mu\nu} = g^0_{\mu\nu} + p_{\mu\nu} \tag{2.134}$$

In our case, the perturbation components $p_{\mu\nu}$ will be functions of the tidal potentials of section 2.4 as well as a set of radial functions to be introduced later. Furthermore, it is assumed that any v-dependence of the metric is entirely contained in the tidal moments and that this v-dependence is slow in the sense that it has a characteristic time scale of order $\mathcal{R}$. In particular, this implies that any process which occurs over time scales of order $2m$ cannot be described using this metric. The perturbation is constructed as a power series in $r/\mathcal{R}$ and in this text, we only consider terms through order $(r/\mathcal{R})^3$. Terms of order $(r/\mathcal{R})^2$ will be referred to as quadrupole order terms. These terms will contain the quadrupole moments $\mathcal{E}_{ij}$ and $\mathcal{B}_{ij}$. Next, the terms of order $(r/\mathcal{R})^3$ will be referred to as octupole order terms and will, in addition to the quadrupole moments, contain the octupole moments $\mathcal{E}_{ijk}$ and $\mathcal{B}_{ijk}$ as well as the v-derivatives of the quadrupole moments. In the unperturbed case, corresponding to $p_{\mu\nu} = 0$, we recall that the vector field l with components given by

$$l_\mu := -D_\mu v = -\delta_{\mu 0} \tag{2.135}$$

is null. As we saw in section 2.3.2, this implies that each surface of constant v is a null-hypersurface of the spacetime. In particular, these null hypersurfaces are past lightcones. The index of l_μ is raised according to

$$l^\mu = -(g^0)^{\mu 0} = -\delta^{\mu 1} \tag{2.136}$$

making it clear that θ and ϕ are constant on each generator of the null congruence corresponding to the lightcone and that $-r$ is an affine parameter along each generator. The geometrical meaning of the lightcone coordinates is thus encapsulated in equations

(2.135) and (2.136). For this to carry over to the perturbed spacetime, we then require that (2.135) and (2.136) continue to hold when the perturbation is introduced. Allowing for a non-vanishing perturbation, we have

$$l_\mu = (g^0_{\mu\nu} + p_{\mu\nu})l^\nu \tag{2.137}$$

Hence the requirement to be imposed is equivalent to demanding that $p_{\mu\nu}l^\nu = 0$. Written out more explicitly, this amounts to the following:

$$p_{vr} = p_{rr} = p_{r\theta} = p_{r\phi} = 0 \tag{2.138}$$

Collectively, these four conditions are known as the lightcone gauge conditions. As shown by Poisson and Preston, there is in fact some residual gauge freedom which allows one to further impose

$$p_{vv} = p_{vA} = 0 \quad \text{ at } r = 2m \tag{2.139}$$

These are known as the horizon-locking conditions and imply that the black hole horizon is located at

$$r_{\text{horizon}} = 2m + \mathcal{O}\left[(m/\mathcal{R})^5\right] \tag{2.140}$$

In other words, even in the perturbed case, the horizon continues to be described by $r = 2m$ at the level of precision maintained here. The following ansatz is then put forth for the metric around a tidally deformed Schwarzschild black hole, utilizing the lightcone gauge from above (including only terms through octupole order):

$$g_{vv} = -f(1 + r^2 f\mathcal{E}^{\mathrm{q}}) + \frac{1}{3}r^3 e^{\mathrm{q}}_2 \frac{d}{dv}\mathcal{E}^{\mathrm{q}} - \frac{1}{3}r^3 e^{\mathrm{o}}_1 \mathcal{E}^{\mathrm{o}} \tag{2.141a}$$

$$g_{vr} = 1 \tag{2.141b}$$

$$g_{vA} = -\frac{2}{3}r^3(e^{\mathrm{q}}_4\mathcal{E}^{\mathrm{q}}_A - b^{\mathrm{q}}_4\mathcal{B}^{\mathrm{q}}_A) + \frac{1}{3}r^4(e^{\mathrm{q}}_5\frac{d}{dv}\mathcal{E}^{\mathrm{q}}_A - b^{\mathrm{q}}_5\frac{d}{dv}\mathcal{B}^{\mathrm{q}}_A) - \frac{1}{4}r^4\left(e^{\mathrm{o}}_4\mathcal{E}^{\mathrm{o}}_A - b^{\mathrm{o}}_4\mathcal{B}^{\mathrm{o}}_A\right) \tag{2.141c}$$

$$\begin{aligned} g_{AB} &= r^2\Omega_{AB} - \frac{1}{3}r^4\left(e^{\mathrm{q}}_7\mathcal{E}^{\mathrm{q}}_{AB} - b^{\mathrm{q}}_7\mathcal{B}^{\mathrm{q}}_{AB}\right) \\ &\quad + \frac{5}{18}r^5(e^{\mathrm{q}}_8\frac{d}{dv}\mathcal{E}^{\mathrm{q}}_{AB} - b^{\mathrm{q}}_8\frac{d}{dv}\mathcal{B}^{\mathrm{q}}_{AB}) - \frac{1}{6}r^5(e^{\mathrm{o}}_7\mathcal{E}^{\mathrm{o}}_{AB} - b^{\mathrm{o}}_7\mathcal{B}^{\mathrm{o}}_{AB}) \end{aligned} \tag{2.141d}$$

where the radial functions, $e_i^{\mathrm{q}}, e_i^{\mathrm{o}}, b_i^{\mathrm{q}}$ and b_i^{o} are all required to approach unity in the limit $2m/r \to 0$. Note that $g_{vr} = 1$ is exact, owing to the properties of the lightcone coordinates with respect to which the components are given. The motivation for the ansatz above is threefold. Firstly, it reduces to the Schwarzschild metric when the perturbation is switched off, corresponding to setting all tidal moments to zero. Secondly, it reduces to the background metric in the limit $2m/r \to 0$. Thirdly, the expansion in tidal moments as above, amounts to a decomposition of the metric into a basis of spherical harmonic modes. This last point is not one we will dwell further on, but simply mention it for completeness sake. The main task at hand is then to impose Einstein's field equations in order to determine the radial functions. We will not go through the computations here, but simply list the results obtained by Poisson and Vlasov. They can be seen in table (2.2).

$e_1^{\mathrm{q}} = f^2$
$e_2^{\mathrm{q}} = f\left[1 + \frac{1}{4x}(5 + 12\log x) - \frac{1}{4x^2}(27 + 12\log x) + \frac{7}{4x^3} + \frac{3}{4x^4}\right]$
$e_4^{\mathrm{q}} = f$
$e_5^{\mathrm{q}} = f\left[1 + \frac{1}{6x}(13 + 12\log x) - \frac{5}{2x^2} - \frac{3}{2x^3} - \frac{1}{2x^4}\right]$
$e_7^{\mathrm{q}} = 1 - \frac{1}{2x^2}$
$e_8^{\mathrm{q}} = 1 + \frac{2}{5x}(4 + 3\log x) - \frac{9}{5x^2} - \frac{1}{5x^3}(7 + 3\log x) + \frac{3}{5x^4}$
$e_1^{\mathrm{o}} = f^2(1 - \frac{1}{2x})$
$e_4^{\mathrm{o}} = f(1 - \frac{2}{3x})$
$e_7^{\mathrm{o}} = f + \frac{1}{10x^3}$
$b_4^{\mathrm{q}} = f$
$b_5^{\mathrm{q}} = f\left[1 + \frac{1}{6x}(7 + 12\log x) - \frac{3}{2x^2} - \frac{1}{2x^3} - \frac{1}{6x^4}\right]$
$b_7^{\mathrm{q}} = 1 - \frac{3}{2x^2}$
$b_8^{\mathrm{q}} = 1 + \frac{1}{5x}(5 + 6\log x) - \frac{9}{5x^2} - \frac{1}{5x^3}(2 + 3\log x) + \frac{1}{5x^4}$
$b_4^{\mathrm{o}} = f(1 - \frac{2}{3x})$
$b_7^{\mathrm{o}} = f - \frac{1}{10x^3}$

Table 2.2: Radial functions. Here, $x := r/(2m)$ and $f := 1 - 1/x$.

In certain cases, it will prove useful to express the metric in standard Schwarzschild spherical coordinates (t, r, θ, ϕ). Denote by $\tilde{g}$ the metric in these coordinates. Then using

$v := t + r + 2m\ln(r/(2m) - 1)$, we get

$$\tilde{g}_{tt} = g_{vv} \tag{2.142}$$

$$\tilde{g}_{tr} = \frac{1}{f} g_{vv} + 1 \tag{2.143}$$

$$\tilde{g}_{rr} = \frac{1}{f}\left(\frac{1}{f} g_{vv} + 2\right) \tag{2.144}$$

$$\tilde{g}_{tA} = g_{vA} \tag{2.145}$$

$$\tilde{g}_{rA} = \frac{1}{f} g_{vA} \tag{2.146}$$

$$\tilde{g}_{AB} = g_{AB} \tag{2.147}$$

Chapter 3

Computing tidal potentials for a Schwarzschild perturber

In this section, we compute the tidal potentials for a Schwarzschild black hole of mass $M \gg m$. That is, we consider the case in which the Schwarzschild black hole of mass m is perturbed by a much larger, background, Schwarzschild black hole of mass M. The resulting binary system is thus an EMR and the tidally deformed black hole can be viewed as a test-particle orbiting the background black hole.

Sections 3.2 and 3.3 closely follow [10].

3.1 Introducing a second coordinate system

Since we are considering a binary system of black holes, it will prove useful to introduce two coordinates systems, namely one for the background black hole and one for the deformed black hole.

First erect a Schwarzschild coordinate system around the background black hole. With respect to this coordinates system, the coordinates of the tidally deformed black hole will be denoted (t', r', θ', ϕ'). Next, establish a second Schwarzschild coordinate system around the deformed black hole. With respect to this coordinate system, the coordinates of a test-particle orbiting the deformed black hole will be denoted (t, r, θ, ϕ). See figure 3.1 for an illustration.

Furthermore, with reference to the background black hole, we denote the specific energy and specific angular momentum of the deformed black hole as E' and L' respectively.

Similarly, with reference to the deformed black hole, the specific energy and specific angular momentum of the test-particle will be denoted E and L respectively.

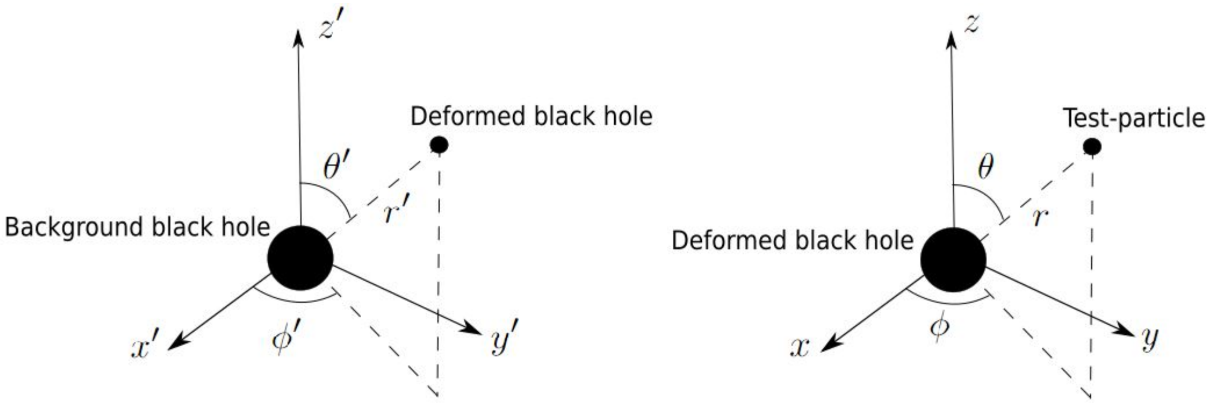


Figure 3.1: Coordinates of the tidally deformed black hole in relation to the background black hole (left). Coordinates of a test-particle in orbit around the deformed black hole (right).

With respect to the coordinate system of the background black hole, the non-vanishing components of the background metric g' are given by [18]:

$$g'_{00} = -\left(1 - \frac{2M}{r'}\right) \tag{3.1}$$

$$g'_{11} = \frac{1}{1 - \frac{2M}{r'}} \tag{3.2}$$

$$g'_{22} = r'^2 \tag{3.3}$$

$$g'_{33} = r'^2 \sin^2 \theta' \tag{3.4}$$

where it is understood that the components are evaluated at the position of the deformed black hole. Furthermore, the independent, non-vanishing components of the Riemann tensor are then easily computed:

$$R'_{0101} = -\frac{2M}{r'^3} \tag{3.5}$$

$$R'_{0202} = \frac{M}{r'}\left(1 - \frac{2M}{r'}\right) \tag{3.6}$$

$$R'_{0303} = \frac{M\sin^2\theta'}{r'}\left(1 - \frac{2M}{r'}\right) \tag{3.7}$$

$$R'_{1212} = \frac{M}{2M - r'} \tag{3.8}$$

$$R'_{1313} = \frac{M\sin^2\theta'}{2M - r'} \tag{3.9}$$

$$R'_{2323} = 2Mr'\sin^2\theta' \tag{3.10}$$

Assuming the deformed black hole follows a geodesic $\gamma : [a, b] \to \mathscr{M}$, the usual integrals of motion apply [10]:

$$\dot{t}' = \frac{E'}{1 - \frac{2M}{r'}} \tag{3.11}$$

$$\dot{r}'^2 = E'^2 - \frac{1}{r'^2}\left(1 - \frac{2M}{r'}\right)(r'^2 + K) \tag{3.12}$$

$$\dot{\theta}'^2 = \frac{1}{r'^4}\left(K - \frac{L'^2_{z'}}{\sin^2\theta'}\right) \tag{3.13}$$

$$\dot{\phi}' = \frac{L'_{z'}}{r'^2\sin^2\theta'} \tag{3.14}$$

where $L'_{z'}$ is the specific angular momentum about the axis of symmetry of the deformed black hole and K is Carter's fourth constant (see appendix D) given by

$$K = p'^2_{\theta'} + \frac{L'^2_{z'}}{\sin^2\theta'} \tag{3.15}$$

where $p'_{\theta'}$ is the latitudinal component of the deformed black hole's specific angular momentum. Finally, overdots denote differentiation with respect to the proper time along γ.

3.2 Constructing an orthonormal tetrad

In accordance with the procedure outlined in section 2.4, we start by constructing an orthonormal tetrad, $\{\lambda_a\}_{a\in\{0,1,2,3\}}$ along γ. Denote the four-velocity of the deformed black hole by u' and set $\lambda_0 := u'$. To construct the next vector of the tetrad, we introduce the Killing-Yano tensor field f for the Schwarzschild geometry (see appendix E for more on Killing-Yano tensors). With respect to the Schwarzschild coordinates (t', r', θ', ϕ'), the components of f may be denoted $f_{\mu\nu}$. The defining equations for f are then $f_{\mu\nu} = -f_{\nu\mu}$ and

$$D_\mu f_{\sigma\nu} + D_\nu f_{\sigma\mu} = 0 \tag{3.16}$$

For the Schwarzschild spacetime, the nonvanishing components of f are $f_{23} = -f_{32} = r'^3 \sin\theta$. Consider then the vector X with components $X^\mu = K^{-1/2} f^\mu{}_\nu u^\nu$. This vector is parallel transported along γ. Indeed, since u obeys the geodesic equation, since K is a constant of motion and since the connection D by assumption is metric compatible, we get

$$\frac{D}{d\tau} X^\mu = K^{-1/2} g^{\mu\sigma} u^\rho u^\nu D_\rho f_{\sigma\nu} = 0 \tag{3.17}$$

having used the anti-symmetry of $D_\rho f_{\sigma\nu}$ in ρ and ν. Furthermore, X is normalized:

$$X_\mu X^\mu = K^{-1} f_{\mu\nu} f^\mu{}_\sigma u^\nu u^\sigma = K^{-1} Q_{\mu\nu} u^\mu u^\nu = 1 \tag{3.18}$$

where $Q_{\mu\nu} = f_{\mu\nu} f^\mu{}_\sigma$ is the Killing tensor corresponding to Carter's constant (see appendices D and E). Finally, X is orthogonal to λ_0:

$$X_\mu \lambda_0^\mu = K^{-1/2} f_{\mu\nu} u^\nu u^\mu = 0 \tag{3.19}$$

having used the antisymmetry of f. In conclusion, we can justifiably set $\lambda_2 := X$. Explicitly,

$$\lambda_2 = \left(0, 0, \frac{L'_{z'}}{K^{1/2} r' \sin\theta'}, -\frac{r'}{K^{1/2}\sin\theta'}\dot{\theta}'\right) \tag{3.20}$$

In order to construct the remaining two members of the tetrad, we put forth two candidates, λ'_1 and λ'_3, solely inspired by the fact that they are normalised and orthogonal to both each other as well as to λ_0 and λ_2. They are given by the following:

$$\lambda_1' = \left(\frac{r'\dot{r}'}{\sqrt{K + r'^2}\left(1 - \frac{2M}{r'}\right)}, \frac{E'r'}{\sqrt{K + r'^2}}, 0, 0 \right) \tag{3.21}$$

$$\lambda_3' = \left(\frac{E'}{1 - \frac{2M}{r'}} \sqrt{\frac{K}{K + r'^2}}, \sqrt{\frac{K}{K + r'^2}}\dot{r}', \sqrt{\frac{K + r'^2}{K}}\dot{\theta}', \sqrt{\frac{K + r'^2}{K}} \frac{L'_{z'}}{r'^2 \sin^2\theta'} \right) \tag{3.22}$$

They are, however, not parallel transported along γ. To remedy this, we introduce a time-dependent rotation angle Ψ. The final two members of the tetrad are then given by

$$\lambda_1 = \lambda_1' \cos\Psi - \lambda_3' \sin\Psi \tag{3.23}$$

$$\lambda_3 = \lambda_1' \sin\Psi + \lambda_3' \cos\Psi \tag{3.24}$$

Owing to the properties of λ_1', λ_3', these vectors are normalized and orthogonal to both each other and to λ_0 and λ_2. Now, we would like to determine a condition on Ψ which ensures that they are parallel transported along γ. Explicitly, we demand that

$$u^\mu D_\mu \lambda_i^\nu = 0, \qquad i \in \{1, 3\} \tag{3.25}$$

for all $\nu \in \{0, 1, 2, 3\}$. It is straightforward, albeit tedious, to show that we must then have:

$$\dot{\Psi} = \frac{K^{1/2}E'}{K + r'^2} \tag{3.26}$$

This finishes the construction of the tetrad. Before moving on to calculate any tidal moments, it will prove useful to express the tetrad in Carter's basis $\{e_{\hat{a}}\}_{a\in\{0,1,2,3\}}$, given by

$$e_{\hat{0}}(\tau) = \left(\frac{1}{\sqrt{1 - \frac{2M}{r'(\tau)}}}, 0, 0, 0 \right) \tag{3.27}$$

$$e_{\hat{1}}(\tau) = \left(0, \sqrt{1 - \frac{2M}{r'(\tau)}}, 0, 0 \right) \tag{3.28}$$

$$e_{\hat{2}}(\tau) = \left(0, 0, \frac{1}{r'(\tau)}, 0 \right) \tag{3.29}$$

$$e_{\hat{3}}(\tau) = \left(0, 0, 0, \frac{1}{r'(\tau)\sin\theta'} \right) \tag{3.30}$$

Explicitly, we decompose each λ_a with respect to Carter's basis according to

$$\lambda_a^\mu = \tilde{\lambda}_a^{\hat{a}} e_{\hat{a}}^\mu \tag{3.31}$$

where $\tilde{\lambda}_a^{\hat{a}}$ are the components of λ_a with respect to Carter's basis. We also decompose tensors on γ with respect to Carter's basis. If T is an arbitrary rank (k,l) tensor on γ with components given by $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$, then we write

$$T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = \tilde{T}^{\hat{a}_1 \cdots \hat{a}_k}{}_{\hat{b}_1 \cdots \hat{b}_l} e_{\hat{a}_1}^{\mu_1} \cdots e_{\hat{a}_k}^{\mu_k} e_{\nu_1}^{\hat{b}_1} \cdots e_{\nu_l}^{\hat{b}_l} \tag{3.32}$$

where

$$e_\mu^{\hat{a}} = \eta^{\hat{a}\hat{b}} g_{\mu\nu} e_{\hat{b}}^\nu \tag{3.33}$$

defines the dual of Carter's basis. For future reference, we explicitly write the components of our tetrad in Carter's basis:

$$\tilde{\lambda}_0 = \left(\frac{E'}{\left(1 - \frac{2M}{r'}\right)^{1/2}}, \frac{\dot{r}'}{\left(1 - \frac{2M}{r'}\right)^{1/2}}, \dot{\theta}' r', \frac{L'_{z'}}{r' \sin\theta'} \right) \tag{3.34}$$

$$\tilde{\lambda}_1 = \tilde{\lambda}'_1 \cos\Psi - \tilde{\lambda}'_3 \sin\Psi \tag{3.35}$$

$$\tilde{\lambda}_2 = \left(0, 0, \frac{L'_{z'}}{K^{1/2} \sin\theta'}, \frac{r'^2 \dot{\theta}'}{K^{1/2}} \right) \tag{3.36}$$

$$\tilde{\lambda}_3 = \tilde{\lambda}'_1 \sin\Psi + \tilde{\lambda}'_3 \cos\Psi \tag{3.37}$$

where

$$\tilde{\lambda}'_1 = \left(\frac{r' \dot{r}'}{\sqrt{\left(K + r'^2\right)\left(1 - \frac{2M}{r'}\right)}}, \frac{E' r'}{\sqrt{\left(K + r'^2\right)\left(1 - \frac{2M}{r'}\right)}}, 0, 0 \right) \tag{3.38}$$

$$\begin{aligned} \tilde{\lambda}'_3 = \Bigg(& \sqrt{\frac{K}{\left(K + r'^2\right)\left(1 - \frac{2M}{r'}\right)}} E', \sqrt{\frac{K}{\left(K + r'^2\right)\left(1 - \frac{2M}{r'}\right)}} \dot{r}', \\ & \sqrt{\frac{K + r'^2}{K}} r' \dot{\theta}', \sqrt{\frac{K + r'^2}{K}} \frac{L'_{z'}}{r' \sin\theta'} \Bigg) \end{aligned} \tag{3.39}$$

Furthermore, the nonvanishing components of the Riemann tensor in Carter's basis are

$$R'_{\hat{0}\hat{1}\hat{0}\hat{1}} = -R'_{\hat{2}\hat{3}\hat{2}\hat{3}} = -\frac{2M}{r'^3} \tag{3.40}$$

$$R'_{\hat{0}\hat{2}\hat{0}\hat{2}} = R'_{\hat{0}\hat{3}\hat{0}\hat{3}} = -R'_{\hat{1}\hat{2}\hat{1}\hat{2}} = -R'_{\hat{1}\hat{3}\hat{1}\hat{3}} = \frac{M}{r'^3} \tag{3.41}$$

3.3 Tidal moments

In this section, we put together the pieces from the previous section in order to compute the tidal moments for a Schwarzschild perturber. Having introduced Carter's basis, we first note that the tidal moments of section 2.4 can be written as follows:

$$\mathcal{E}_{ij} := \tilde{R}_{\hat{a}\hat{b}\hat{c}\hat{d}}\tilde{\lambda}^{\hat{a}}_i\tilde{\lambda}^{\hat{b}}_0\tilde{\lambda}^{\hat{c}}_j\tilde{\lambda}^{\hat{d}}_0 \tag{3.42}$$

$$\mathcal{B}_{ij} := \tilde{R}^*_{\hat{a}\hat{b}\hat{c}\hat{d}}\tilde{\lambda}^{\hat{a}}_i\tilde{\lambda}^{\hat{b}}_0\tilde{\lambda}^{\hat{c}}_j\tilde{\lambda}^{\hat{d}}_0 \tag{3.43}$$

$$\mathcal{E}_{ijk} := \left(\tilde{R}_{\hat{a}\hat{b}\hat{c}\hat{d};\hat{e}}\tilde{\lambda}^{\hat{a}}_i\tilde{\lambda}^{\hat{b}}_0\tilde{\lambda}^{\hat{c}}_j\tilde{\lambda}^{\hat{d}}_0\tilde{\lambda}^{\hat{e}}_k\right)^{\mathrm{STF}} \tag{3.44}$$

$$\mathcal{B}_{ijk} := \left(\tilde{R}^*_{\hat{a}\hat{b}\hat{c}\hat{d};\hat{e}}\tilde{\lambda}^{\hat{a}}_i\tilde{\lambda}^{\hat{b}}_0\tilde{\lambda}^{\hat{c}}_j\tilde{\lambda}^{\hat{d}}_0\tilde{\lambda}^{\hat{e}}_k\right)^{\mathrm{STF}} \tag{3.45}$$

Without loss of generality, it will be assumed that γ lies in the $\theta' = \pi/2$-plane and that $\dot{\phi}' \geq 0$. With this choice, $L'_{z'} = L' = r'^2\dot{\phi}'$ is the total angular momentum of the orbiting body and $K = L'^2$. From this point forth, we will exclusively be working in Carter's basis and so we drop the tildes for notational convenience. The electric quadrupole tidal moments are given by:

$$\begin{aligned}\mathcal{E}_{ij} = \frac{M}{r^3}\Big[&(\lambda^{\hat{0}}_0)^2(-2\lambda^{\hat{1}}_i\lambda^{\hat{1}}_j + \lambda^{\hat{2}}_i\lambda^{\hat{2}}_j + \lambda^{\hat{3}}_i\lambda^{\hat{3}}_j) - (\lambda^{\hat{1}}_0)^2(2\lambda^{\hat{0}}_i\lambda^{\hat{0}}_j + \lambda^{\hat{2}}_i\lambda^{\hat{2}}_j + \lambda^{\hat{3}}_i\lambda^{\hat{3}}_j)\\ &+(\lambda^{\hat{3}}_0)^2(\lambda^{\hat{0}}_i\lambda^{\hat{0}}_j - \lambda^{\hat{1}}_i\lambda^{\hat{1}}_j + 2\lambda^{\hat{2}}_i\lambda^{\hat{2}}_j) + 2\lambda^{\hat{0}}_0\lambda^{\hat{1}}_0(\lambda^{\hat{0}}_i\lambda^{\hat{1}}_j + \lambda^{\hat{1}}_i\lambda^{\hat{0}}_j) - \lambda^{\hat{0}}_0\lambda^{\hat{3}}_0(\lambda^{\hat{0}}_i\lambda^{\hat{3}}_j + \lambda^{\hat{3}}_i\lambda^{\hat{0}}_j)\\ &+\lambda^{\hat{1}}_0\lambda^{\hat{3}}_0(\lambda^{\hat{1}}_i\lambda^{\hat{3}}_j + \lambda^{\hat{3}}_i\lambda^{\hat{1}}_j)\Big]\end{aligned} \tag{3.46}$$

Explicitly, the nonvanishing components are

$$\mathcal{E}_{11} = \left(1 - 3\frac{r'^2 + L'^2}{r'^2}\cos^2\Psi\right)\frac{M}{r'^3} \tag{3.47}$$

$$\mathcal{E}_{22} = \left(1 + 3\frac{L'^2}{r'^2}\right)\frac{M}{r'^3} \tag{3.48}$$

$$\mathcal{E}_{33} = \left(1 - 3\frac{r'^2 + L'^2}{r'^2}\sin^2\Psi\right)\frac{M}{r'^3} \tag{3.49}$$

$$\mathcal{E}_{13} = -3\frac{r'^2 + L'^2}{r'^5}M\cos\Psi\sin\Psi \tag{3.50}$$

with

$$\Psi = \frac{E'L'}{L'^2 + r'^2}\tau \tag{3.51}$$

Furthermore, the nonvanishing electric octupole moments are given by

$$\begin{aligned}
\mathcal{E}_{111} = \frac{15M}{\sqrt{L'^2 + r'^2}r'^6}\Bigg[&\left(L'^2 + r'^2\right)\left(E'r'\cos\Psi - L'\dot{r}'\sin\Psi\right)\cos^2\Psi \\
&-\frac{2r'E'}{5}\left(L'^2 + \frac{3r'^2}{2}\right)\cos\Psi + \frac{L'\dot{r}'r'^2}{5}\sin\Psi\Bigg]
\end{aligned} \tag{3.52}$$

$$\begin{aligned}
\mathcal{E}_{113} = \frac{15M}{\sqrt{L'^2 + r'^2}r'^6}\Bigg[&\left(L'^2 + r'^2\right)\left(E'r'\sin\Psi + L'\dot{r}'\cos\Psi\right)\cos^2\Psi \\
&-\frac{2L'\dot{r}'}{3}\left(L'^2 + \frac{11r'^2}{10}\right)\cos\Psi - \frac{2r'E'}{15}\left(L'^2 + \frac{3r'^2}{2}\right)\sin\Psi\Bigg]
\end{aligned} \tag{3.53}$$

$$\mathcal{E}_{122} = -\frac{7M}{\sqrt{L'^2 + r'^2}r'^6}\left[E'r'\left(L'^2 + \frac{3r'^2}{7}\right)\cos\Psi - \frac{5\dot{r}'L'}{7}\left(L'^2 + \frac{r'^2}{5}\right)\sin\Psi\right] \tag{3.54}$$

$$\begin{aligned}
\mathcal{E}_{133} = -\frac{15M}{\sqrt{L'^2 + r'^2}r'^6}\Bigg[&\left(L'^2 + r'^2\right)\left(E'r'\cos\Psi - L'\dot{r}'\sin\Psi\right)\cos^2\Psi \\
&-\frac{13E'r'\cos(\Psi)}{15}\left(L'^2 + \frac{12r'^2}{13}\right) + \frac{\dot{r}'L'\sin\Psi}{3}\left(L'^2 + \frac{4r'^2}{5}\right)\Bigg]
\end{aligned} \tag{3.55}$$

$$\mathcal{E}_{223} = -\frac{7M}{\sqrt{L'^2 + r'^2}r'^6}\left[\frac{5\dot{r}'L'}{7}\left(L'^2 + \frac{r'^2}{5}\right)\cos\Psi + E'r'\left(L'^2 + \frac{3r'^2}{7}\right)\sin\Psi\right] \tag{3.56}$$

$$\begin{aligned}
\mathcal{E}_{333} = -\frac{15M}{\sqrt{L'^2 + r'^2}r'^6}\Bigg[&\left(L'^2 + r'^2\right)\left(E'r'\sin\Psi + L'\dot{r}'\cos\Psi\right)\cos^2\Psi \\
&-\dot{r}'L'\left(L'^2 + \frac{4r'^2}{5}\right)\cos\Psi - \frac{3E'r'}{5}\left(L'^2 + \frac{2r'^2}{3}\right)\sin\Psi\Bigg]
\end{aligned} \tag{3.57}$$

Similarly, the nonvanishing quadrupole magnetic tidal moments are

$$\mathcal{B}_{12} = -\frac{3ML'}{r'^5}\sqrt{L'^2 + r'^2}\cos\Psi \tag{3.58}$$

$$\mathcal{B}_{23} = -\frac{3ML'}{r'^5}\sqrt{L'^2 + r'^2}\sin\Psi \tag{3.59}$$

whereas the octupole magnetic moments all vanish identically. We will primarily concern ourselves with the case in which γ is radial, in particular we assume that $\phi' \equiv 0$ along γ. This implies $L' = 0$ and $\dot{\Psi} = 0$. Hence, Ψ is an arbitrary constant which we take to be zero. With this choice, only the following tidal moments remain nonvanishing:

$$\mathcal{E}_{11} = -\frac{2M}{r'^3}, \qquad \mathcal{E}_{22} = \mathcal{E}_{33} = \frac{M}{r'^3} \tag{3.60}$$

and

$$\mathcal{E}_{111} = \frac{6ME'}{r'^4}, \qquad \mathcal{E}_{122} = \mathcal{E}_{133} = -\frac{3ME'}{r'^4} \tag{3.61}$$

3.4 Tidal potentials

In this section, we convert the previously computed tidal moments into tidal potentials. We start by placing a test-particle in orbit around the tidally deformed black hole and then assign to this test-particle the coordinates (t, r, θ, ϕ) as described earlier. We will need to decide on an orientation of the resulting binary system with respect to the background black hole. Two common choices are [4]:

- Polar configuration. This amounts to setting $\Omega^1 = \cos\theta$, $\Omega^2 = \sin\theta\sin\phi$ and $\Omega^3 = \sin\theta\cos\phi$.

- Equatorial configuration. This amounts to setting $\Omega^1 = \sin\theta\cos\phi$, $\Omega^2 = \sin\theta\sin\phi$ and $\Omega^3 = \cos\theta$.

We will choose to consider a polar companion (see figure 3.2). In spherical coordinates, the electric quadrupole potentials are then given by

$$\mathcal{E}^{\mathrm{q}} = \frac{M}{r'^3}[1 - 3\cos^2\theta] \tag{3.62}$$
$$\mathcal{E}^{\mathrm{q}}_1 = \frac{3}{2}\frac{M}{r'^3}\sin 2\theta \tag{3.63}$$
$$\mathcal{E}^{\mathrm{q}}_{11} = -3\frac{M}{r'^3}\sin^2\theta \tag{3.64}$$
$$\mathcal{E}^{\mathrm{q}}_{22} = 3\frac{M}{r'^3}\sin^4\theta \tag{3.65}$$

The electric octupole potentials are given by

$$\mathcal{E}^{\mathrm{o}} = \frac{3ME'}{r'^4}\left[5\cos^2\theta - 3\right]\cos\theta \tag{3.66}$$
$$\mathcal{E}^{\mathrm{o}}_1 = -\frac{3ME'}{r'^4}\left[5\cos^2\theta - 1\right]\sin\theta \tag{3.67}$$
$$\begin{aligned}\mathcal{E}^{\mathrm{o}}_{11} = \frac{750ME'}{r'^4}&\left[\frac{1}{4}\left(\sin 2\phi - \sin^2 2\phi + 4\right)\cos^2\theta + \frac{1}{8}\sin 2\theta\sin 2\phi\left(\sin\phi + \cos\phi\right)\cos^4\theta\right.\\ &+\frac{1}{125}\left(52\sin^2 2\phi - 52\sin 2\phi - 198\right)\cos^4\theta\\ &+\frac{73}{500}\sin 2\theta\cos^2\theta\left(\cos^3\phi - \frac{1}{2}\cos\phi\sin 2\phi - \frac{198}{73}\cos\phi - \frac{125}{73}\sin\phi\right)\\ &\left.+\left(\frac{83}{500}\sin 2\phi - \frac{83}{500}\sin^2 2\phi + \frac{271}{250}\right)\cos^2\theta + \frac{73}{500}\sin 2\theta\left(\cos\phi + \sin\phi\right) - \frac{73}{250}\right]\cos\theta\end{aligned} \tag{3.68}$$
$$\begin{aligned}\mathcal{E}^{\mathrm{o}}_{22} = -\frac{375ME'}{r'^4}&\left[\left(\cos^4\theta - \frac{208}{125}\cos^2\theta + \frac{83}{125}\right)\cos^4\phi - \frac{1}{4}\left(\cos^2\theta - \frac{73}{125}\right)\sin 2\theta\cos^3\phi\right.\\ &+\left(\frac{1}{4}\cos^2\theta\sin 2\theta\sin\phi - \cos^4\theta - \frac{73}{500}\sin 2\theta\sin\phi + \frac{208}{125}\cos^2\theta - \frac{83}{125}\right)\cos^2\phi\\ &+\left(\frac{1}{4}\cos^2\theta\sin 2\theta - \frac{1}{2}\cos^4\theta\sin\phi + \frac{104}{125}\cos^2\theta\sin\phi - \frac{73}{500}\sin 2\theta - \frac{83}{250}\sin\phi\right)\cos\phi\\ &\left.-\frac{27}{50}\cos^2\theta + \frac{83}{250}\right]\sin 2\theta\sin\theta\end{aligned} \tag{3.69}$$
$$\begin{aligned}\mathcal{E}^{\mathrm{o}}_{12} = \frac{375ME'}{4r'^4}&\left[\frac{1}{2}\left(\cos\phi - \sin\phi\right)\cos^3\theta + \frac{1}{4}\sin 2\theta\cos 2\phi\cos\theta\right.\\ &\left.+\frac{73}{250}\left(\sin\phi - \cos\phi\right)\cos\theta - \frac{83}{250}\sin\theta\cos 2\phi\right]\sin^2 2\theta\sin 2\phi\end{aligned} \tag{3.70}$$

where E' is the specific energy of the tidally deformed black hole.

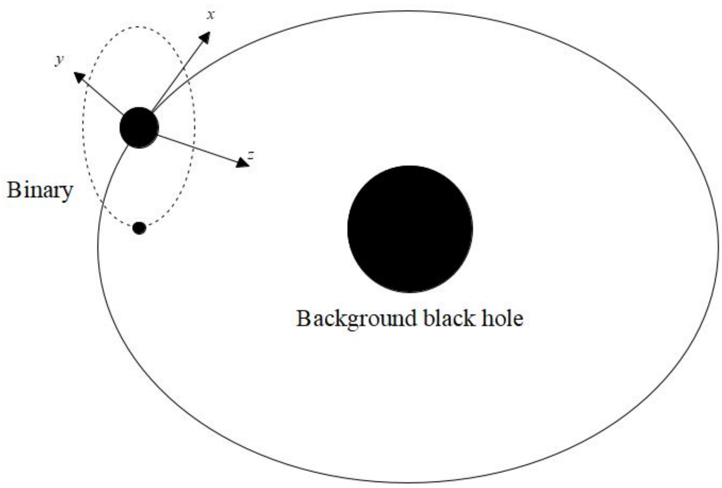


Figure 3.2: Illustration of the binary system in the polar configuration.

Chapter 4

ISCO shifts

In this chapter, we consider the shifts in the radial position, specific energy and specific angular momentum that are brought on by the presence of a tidal field. We start by considering the problem at quadrupole order and then later move on to octupole order.

Sections 4.1 and 4.3 closely follow the methods of [4]. Section 4.2 is based off a conversation with Troels Harmark and Daniele Pica.

4.1 Quadrupole order ISCO shift

Suppose a test-particle is orbiting an unperturbed Schwarzschild black hole of mass m in the $\theta = \pi/2$-plane along a geodesic $\beta : [a, b] \to \mathscr{M}$. Then the innermost stable circular orbit (ISCO) of the test-particle is located at $r_0 = 6m$. While orbiting in the ISCO, the specific energy of the test-particle must be $E_0 = \frac{2\sqrt{2}}{3}$ while the specific angular momentum of the test-particle must be $L_0 = \sqrt{12}m$. In this section, we seek to find the corrections to these quantities, brought on by the presence of a tidal field. We do this perturbatively, first noticing that (2.128) together with $M \gg m$ implies

$$\epsilon := \frac{Mm^2}{r'^3} \ll 1 \tag{4.1}$$

for any $r' \geq 2M$. Hence ϵ is a valid expansion parameter. To quadrupole order, we may write

$$r_{ISCO} = 6m + r_1\epsilon \tag{4.2}$$

$$L_{ISCO} = \sqrt{12}m + L_1\epsilon \tag{4.3}$$

$$E_{ISCO} = \frac{2\sqrt{2}}{3} + E_1\epsilon \tag{4.4}$$

where r_1, L_1 and E_1 are the shifts in the radial coordinate, specific angular momentum and specific energy respectively. Given the current setup, the non-vanishing components of the metric (2.141) are as follows:

$$g_{00} = -f\left(1 + \frac{fMr^2}{r'^3}\right) \tag{4.5}$$

$$g_{01} = -\frac{fMr^2}{r'^3} \tag{4.6}$$

$$g_{11} = \frac{1}{f}\left(1 - \frac{Mfr^2}{r'^3}\right) \tag{4.7}$$

$$g_{22} = r^2\left[1 - \frac{M}{r'^3}(2m^2 - r^2)\right] \tag{4.8}$$

$$g_{33} = r^2\left[1 + \frac{M}{r'^3}(2m^2 - r^2)\right] \tag{4.9}$$

with respect to the usual Schwarzschild spherical coordinates. The metric is independent of both t and ϕ, giving rise to two Killing vectors, ξ_t and ξ_ϕ with components given by $\xi_t^\mu = \delta_0^\mu$ and $\xi_\phi^\mu = \delta_3^\mu$. The two corresponding conserved quantities are the specific energy E and specific angular momentum L:

$$E := -u \cdot \xi_t = -g_{00}\dot{t} - g_{01}\dot{r} = f\left(1 + \frac{r^2 fM}{r'^3}\right)\dot{t} + \frac{r^2 fM}{r'^3}\dot{r} \tag{4.10}$$

$$L := u \cdot \xi_\phi = g_{33}\dot{\phi} = \left[1 + (2m^2 - r^2)\frac{M}{r'^3}\right] r^2\dot{\phi} \tag{4.11}$$

Overdots now denote differentiation with respect to the proper time of the test-particle. Note that by setting $M = 0$, we recover the usual conservation of specific energy and specific angular momentum for the unperturbed Schwarzschild solution.

Inserting E and L into $u^\mu u_\mu = -1$, yields the following differential equation:

$$E^2 = \dot{r}^2 + V(r) \tag{4.12}$$

where the potential V is given by

$$V(r) := -\left(1 + \frac{1}{g_{33}} L^2\right) g_{00} \tag{4.13}$$

Expanding V to quadrupole order, we get the following:

$$V(r) = \frac{2\left(L^2 + r^2\right)\left(\frac{r}{2} - m\right)}{r^3} + \frac{(2m - r)\left(2L^2\left(m^2 + rm - r^2\right) + 2mr^3 - r^4\right)}{r^3 m^2}\epsilon + \mathcal{O}(\epsilon^2) \tag{4.14}$$

Stable circular orbits are characterized by

$$E^2 - V(r) = 0, \qquad \frac{dV(r)}{dr} = 0, \qquad \frac{d^2V(r)}{dr^2} > 0 \tag{4.15}$$

Inserting (4.2)-(4.4) into (4.15), we find

$$r_1 = -1536m \tag{4.16a}$$

$$L_1 = 174\sqrt{3}m \tag{4.16b}$$

$$E_1 = \frac{76\sqrt{2}}{3} \tag{4.16c}$$

Furthermore, the ISCO frequency, $\Omega_{ISCO} := \dot{\phi}/\dot{t}$ evaluated at the ISCO, is then

$$\Omega_{ISCO} = \frac{1 + 491\epsilon}{6\sqrt{6}m} \tag{4.17}$$

A straightforward modification of the above procedure lets us compute the shift in the photon-sphere for the tidally deformed black hole. Indeed, we simply impose $u^\mu u_\mu = 0$ instead of $u^\mu u_\mu = -1$. We obtain (where PS stands for photon sphere):

$$r_{\mathrm{PS}} = 3m(1 + 5\epsilon) \tag{4.18}$$

$$b_{\mathrm{PS}} = 3\sqrt{3}m(1 - 5\epsilon) \tag{4.19}$$

where $b := L/E$ is the impact parameter. The photon sphere frequency is:

$$\Omega_{\text{PS}} = \frac{1+5\epsilon}{3\sqrt{3}m} \tag{4.20}$$

4.2 Time dependence in octupole order computations

This section is an interlude pertaining to the advanced-time parameterization of γ. When we defined the tidal moments in section 2.4, we considered a test-particle travelling on a geodesic γ in the background spacetime. Having parameterized γ in terms of its proper time τ', it naturally follows that the tidal moments are functions of τ'. However, we are really considering a binary system consisting of a tidally deformed black hole and a test-particle. In the region where the binary is far away from the background black hole, these two descriptions match. In other words, we may in this case consider the binary as a point particle and use τ' to parameterize γ. However, as the binary moves closer to the background black hole, the tidal field increases in strength and the structure of the binary becomes important. Hence, we can no longer describe the binary as a point-particle. As mentioned earlier, this means we have to consider a world tube around γ traced by the tidally deformed black hole. For this reason, τ' loses its usefulness as a parameter and we must instead switch to the advanced time coordinate v. The two regimes thus described are illustrated in figure 4.1.

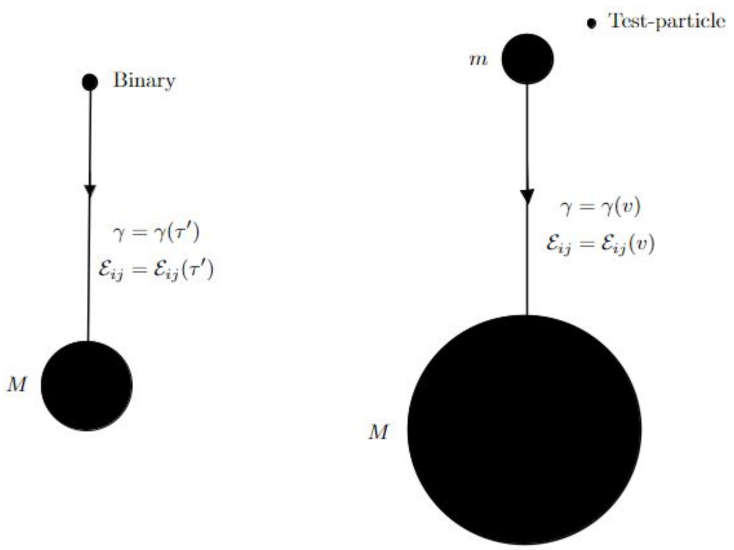


Figure 4.1: Left: Asymptotic region where γ and the tidal moments are parameterized by proper time. Right: Binary region where γ and the tidal moments are parameterized by advanced time, v.

In the intermediary region between the two regions above, both descriptions must be valid, which implies that

$$v = \tau' + r + 2m \log(r/(2m) - 1) \tag{4.21}$$

At quadrupole order, this distinction is not of importance since time-dependence is neglected altogether. However, in the next section we go on to octupole order where it becomes important.

4.3 Octupole order ISCO shift

In this section, we go through the same procedure as in section 4.1 but this time to octupole order. To be precise, the question we are addressing in this section is: If the binary is at some point r' with instantaneous velocity dr'/dv, then what are the ISCO parameters r_{ISCO}, E_{ISCO} and L_{ISCO} at this point. To octupole order, we write

$$r_{ISCO} = r_0 + r_1\epsilon + r_2\epsilon^{4/3} \tag{4.22}$$

$$L_{ISCO} = L_0 + L_1\epsilon + L_2\epsilon^{4/3} \tag{4.23}$$

$$E_{ISCO} = E_0 + E_1\epsilon + E_2\epsilon^{4/3} \tag{4.24}$$

where r_2, L_2 and E_2 are the octupole order shifts to r, L and E respectively. The nonvanishing components of (2.141) with respect to the usual Schwarzschild spherical coordinates are:

$$g_{00} = -\left[f\left(1 + f\frac{r^2 M}{r'^3}\right) + \frac{M r^3}{r'^4}\frac{dr'}{dv}e^{\mathrm{q}}_2\right] \tag{4.25}$$

$$g_{01} = -\left[f\frac{r^2 M}{r'^3} + \frac{M r^3}{f r'^4}\frac{dr'}{dv}e^{\mathrm{q}}_2\right] \tag{4.26}$$

$$g_{11} = \frac{1}{f}\left[1 - f\frac{r^2 M}{r'^3} - \frac{M r^3}{f r'^4}\frac{dr'}{dv}e^{\mathrm{q}}_2\right] \tag{4.27}$$

$$g_{02} = -\frac{3}{4}r^4 f\left(1 - \frac{4m}{3r}\right)\frac{ME'}{r'^4} \tag{4.28}$$

$$g_{12} = -\frac{3}{4}r^4\left(1 - \frac{4m}{3r}\right)\frac{ME'}{r'^4} \tag{4.29}$$

$$g_{22} = r^2\left[1 - \frac{M}{r'^3}(2m^2 - r^2) + \frac{5}{2}r^3\frac{M}{r'^4}\frac{dr'}{dv}e^{\mathrm{q}}_8\right] \tag{4.30}$$

$$g_{33} = r^2\left[1 + \frac{M}{r'^3}(2m^2 - r^2) - \frac{5}{2}r^3\frac{M}{r'^4}\frac{dr'}{dv}e^{\mathrm{q}}_8\right] \tag{4.31}$$

For concreteness, it will be assumed that $E' = 1$. The specific energy and angular momentum of the test-particle are given by

$$E = -g_{00}\dot{t} - g_{01}\dot{r} \tag{4.32}$$

$$L = g_{33}\dot{\phi} \tag{4.33}$$

Similarly to before, we obtain the following differential equation:

$$E^2 = \dot{r}^2 + V(r) \tag{4.34}$$

where

$$V(r) := -\left(1 + \frac{1}{g_{33}}L^2\right)g_{00} \tag{4.35}$$

Expanding the potential to octupole order, yields

$$
\begin{aligned}
V(r) = {} & \frac{2\left(L^2+r^2\right)\left(\frac{r}{2}-m\right)}{r^3} + \frac{(2m-r)\left(2L^2\left(m^2+mr-r^2\right)+2mr^3-r^4\right)}{r^3m^2}\epsilon \\
& + \frac{dr'}{dv}\frac{36\,(r-2m)}{\left(Mm^2\right)^{\frac{1}{3}}m^2r^4}\left[\frac{1}{3}\left(L^2\left(m^2+rm-r^2\right)+mr^3-\frac{1}{2}r^4\right)rm\ln\frac{2m}{r}+\frac{r^6}{36}+\frac{5mr^5}{72}\right. \\
& +\left(\frac{7L^2}{72}-\frac{3m^2}{4}\right)r^4+\left(\frac{7}{24}L^2m+\frac{7}{18}m^3\right)r^3+\left(\frac{1}{3}m^4-\frac{5}{4}L^2m^2\right)r^2 \\
& \left.-\frac{7L^2m^3r}{18}+L^2m^4\right]\epsilon^{\frac{4}{3}}
\end{aligned} \tag{4.36}
$$

Imposing (4.15), we then find

$$r_2 = -72\mathcal{V}\left(128\ln(3)+469\right) \tag{4.37a}$$

$$L_2 = 4\mathcal{V}\left(261\ln(3)+746\right)\sqrt{3} \tag{4.37b}$$

$$E_2 = \frac{8\mathcal{V}}{m}\left(19\ln(3)+45\right)\sqrt{2} \tag{4.37c}$$

where

$$\mathcal{V} := \frac{dr'}{dv}\left(\frac{m^4}{M}\right)^{\frac{1}{3}} \tag{4.38}$$

Notice, in particular, that the octupole order shifts have opposite signs as compared to their quadrupole order counterparts. For instance, r_1 is negative while r_2 is positive. Recall that $\dot{r}'$ satisfies the differential equation (3.12) with $E' = 1$. Differentiating this equation and using that $\dot{r}' \leq 0$ we have $\ddot{r}' \leq 0$ which implies $d^2r'/dv^2 \leq 0$. In other words, the deformed black hole speeds up along it's radial trajectory. We thus see that r_2 is an increasing function of v, while L_2 and E_2 both are decreasing functions of v.

4.4 A lower bound on the energy required to keep test-particle from inspiralling

In this section, we consider the binary system in the polar configuration with $\theta = \pi/2$ to octupole order. Subject to a set assumptions made clear below, we seek to find a lower bound on the energy of the test-particle such that it will not spiral into its deformed companion.

We assume that the binary starts at rest a distance r'_0 from the the background with $E' = 1$. Hence, the results of the previous section apply. For clarity, we switch expansion parameter from ϵ to $\lambda = m/M$ to avoid having time-dependence in the expansion parameter. A quantity A may be expanded with respect to λ as follows:

$$A = A^0 + A^{\rm q}\frac{M^3}{r'^3}\lambda^2 + A^{\rm o}\frac{M^4}{r'^4}\lambda^3 \tag{4.39}$$

to octupole order. Here $A^0, A^{\rm q}$ and $A^{\rm o}$ are the expansion coefficients of A at orders 0, quadrupole and octupole respectively.

We assume the test-particle has energy

$$E = \frac{2\sqrt{2}}{3} + \delta, \qquad \delta \ll 1 \tag{4.40}$$

where δ is a function of v. In accordance with (4.39), we write

$$\delta = E^{\rm q}\frac{M^3}{r'^3}\lambda^2 + E^{\rm o}(v)\frac{M^4}{r'^4}\lambda^3 \tag{4.41}$$

We wish to find a lower bound, δ_0, on δ such that if $\delta \geq \delta_0$ for all v then the radial coordinate of the test-particle will never be smaller than that of the ISCO of the deformed black hole. As we have seen, the quadrupole order ISCO shift, E_1 is positive while the octupole order ISCO shift E_2 is negative. Hence, the ISCO energy will be greatest at the initial distance r'_0 where the positive E_1 dominates over the negative E_2. We thus conclude

$$\delta_0 = \frac{76\sqrt{2}}{3}\frac{Mm^2}{r_0'^3} \tag{4.42}$$

having used that E_2 is zero at r'_0.

Chapter 5

Orbital stability considerations

Thus far, the the deformed black hole and its test-particle companion have been assumed to be in the polar configuration with $\theta = \pi/2$. In this section, we consider the more general situation in which this binary system has some generic orientation with respect to the background black hole. In particular, we seek an expression for the specific energy of the test-particle as a function of the Euler angles that specify the orientation of the binary. In addition, we find that the specific energy of the test-particle is minimal for a co-planar orientation, i.e. one in which the inclination angle of the binary is zero. In this sense, one may refer to co-planar orbits as stable. The approach will be to write down and minimize the Hamiltonian for the test-particle with respect to the inclination angle. We thus start this chapter by giving a brief review of the Hamiltonian formalism, following chapter three of [7].

5.1 The Hamiltonian formalism

Let $(\mathscr{M}, g)$ be a Lorentzian manifold and consider a free test-particle of mass m_*, travelling along a worldline $\gamma : [a, b] \to \mathscr{M}$. Furthermore, let $\mathscr{O}$ be an neighborhood of γ equipped with coordinates $x^\mu, \mu \in \{0, 1, 2, 3\}$. Denoting the parameter of γ by t, the action for γ is then given by

$$\mathcal{S}(\gamma) = \int_a^b dt \mathcal{L}(t, \gamma(t), u(t)) \tag{5.1}$$

where $\mathcal{L} = \mathcal{L}(t, q, u)$ is the Lagrangian for the system, q being the canonical position of the particle and u being the four-velocity of the particle. Explicitly, $\mathcal{L}$ is defined by [3]:

$$\mathcal{L}(t, q, u) := \frac{m_*}{2} g_{\mu\nu}(q) u^\mu u^\nu \tag{5.2}$$

The Hamiltonian for the particle is then obtained as the Legendre transform with respect to u of the Lagrangian. For our purposes this simply amounts to

$$H(t, q, p) = u(t, q, p) \cdot p - \mathcal{L}(t, q, u(t, q, p)) \tag{5.3}$$

where p is the canonical momentum of the particle and where u is given as the unique solution to

$$p_\mu = \frac{\partial \mathcal{L}}{\partial q^\mu}(t, q, v) \tag{5.4}$$

For the Lagrangian (5.2), we compute $p_\mu = m_* g_{\mu\nu} u^\nu$ which is the familiar result for the four-momentum. Hence,

$$H(t, q, p) = \frac{1}{2m_*} g^{\mu\nu}(q) p_\mu p_\nu \tag{5.5}$$

The motion of the test-particle is then determined by Hamilton's equations which read

$$\frac{dq^\mu}{dt}(t) = \frac{\partial H}{\partial p_\mu}(t, q(t), p(t)), \qquad \frac{dp_\mu}{dt}(t) = -\frac{\partial H}{\partial q^\mu}(t, q(t), p(t)) \tag{5.6}$$

Applying (5.6) to (5.3) then yields the following:

$$\frac{dq^\mu}{dt} = \frac{\partial H}{\partial p_\mu} = \frac{1}{m_*} g^{\mu\nu} p_\nu \tag{5.7}$$

$$\frac{dp_\mu}{dt} = \frac{\partial H}{\partial q^\sigma} = -\frac{p_\mu p_\nu}{m_*} \Gamma^\mu{}_{\sigma\lambda} g^{\lambda\nu} \tag{5.8}$$

Using the chain rule, these equations can then be combined to yield

$$0 = p^\mu \left(\frac{\partial p_\nu}{\partial q^\mu} - \Gamma^\sigma{}_{\mu\nu} p_\sigma \right) = p^\mu D_\mu p_\nu \tag{5.9}$$

which is the geodesic equation in affine parameter form. We may take the corresponding affine parameter to be the proper time along γ. In this case,

$$H = -\frac{1}{2}m_* \tag{5.10}$$

owing to the normalization of u. Going forward, we will predominantly be working with the dimensionless Hamiltonian $\tilde{H} := H/m_*$.

For the binary system at hand, g is the Poisson-Vlasov metric and m_* is the mass of the test-particle orbiting the deformed black hole. When considering the dynamics of the binary over timescales much larger than the orbital timescale of the test-particle, it will be of interest to consider the secular orbital average of the Hamiltonian, defined by [19]:

$$\langle \tilde{H} \rangle := \frac{1}{2\pi} \int_0^{2\pi} d\phi \tilde{H}|_\gamma \tag{5.11}$$

where $\tilde{H}|_\gamma$ is $\tilde{H}$ evaluated along γ. We explicitly compute $\langle \tilde{H} \rangle$ in the following section.

5.2 Energy of inclined orbits

We are now in a position to tackle the problem mentioned at the beginning of the chapter. Firstly, we will need to define a set of Euler angles to describe the orientation of the binary. Consider a reference plane, defined as the orbital plane of the binary around the background black hole. Install on this reference plane a Cartesian coordinate system oriented as in the equatorial configuration; the z-axis is perpendicular to the reference plane, the y-axis points from the binary to the background black hole and the x-axis is determined by the right-hand rule. We then perform three rotations in order to arrive at a generic orientation: Firstly, rotate the coordinate system around the z-axis by an angle ϑ. The rotated coordinate system then has axes x', y' and z. Secondly, rotate this new coordinate system around the x'-axis by an angle I, known as the inclination angle. This results in a new coordinate system with axes x', y'' and z''. Finally, perform a rotation around the z'-axis by an angle γ. This then results in a final coordinate system with axes X, Y and $Z := z''$. See figure 5.1 for an illustration.

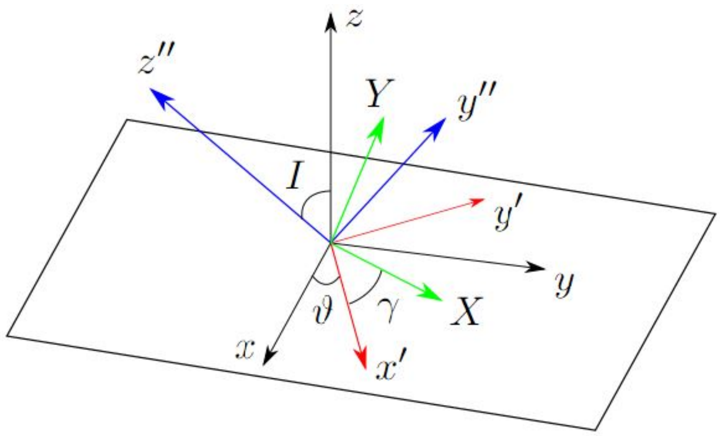


Figure 5.1: Illustration of the Euler angles used to rotate the binary system.

Since the reference plane and the corresponding reference coordinate system is defined as in the equatorial configuration, a generic configuration is given by the vector

$$\Omega = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & \sin I \\ 0 & -\sin I & \cos I \end{pmatrix} \begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \tag{5.12}$$

The Hamiltonian $\tilde{H}$ is then computed using (5.5). We will assume that the test-particle follows a circular orbit in the $\theta = \pi/2$-plane. In this case,

$$\tilde{H} = \frac{1}{2}\left[Lu^{\phi} - Eu^{t}\right] \tag{5.13}$$

where

$$L := g_{\phi\mu}u^{\mu} = g_{\phi t}u^{t} + g_{\phi\phi}u^{\phi} \tag{5.14}$$

$$E := -g_{t\phi}u^{\mu} = -g_{tt}u^{t} - g_{t\phi}u^{\phi} \tag{5.15}$$

with metric components given by (2.141) where the tidal potentials are computed using (5.12). Taking the secular average yields the following:

$$\langle\tilde{H}\rangle = \frac{L^2}{2r^2} - \frac{E^2}{2\left(1-\frac{2m}{r}\right)} + \frac{M}{4r'^3}\left(2 - 3\cos^2 I\sin^2\gamma + 3\cos^2\gamma\right)\left[E^2r^2 + \left(1 - \frac{2m^2}{r^2}\right)L^2\right] \tag{5.16}$$

We then impose (5.10) and find the following expression for E^2:

$$E^2 = \frac{L^2 + r^2}{r^2}\left(1 - \frac{2m}{r}\right) + \frac{M}{2r'^3 r^2}\left[2L^2\left(m^2 + rm - r^2\right) + 2r^3 m - r^4\right]\left(3\cos^2 I \sin^2\gamma + 3\cos^2\gamma - 2\right)\left(1 - \frac{2m}{r}\right) \tag{5.17}$$

Since $E \geq 0$, maximizing (minimizing) E is equivalent to maximizing (minimizing) E^2. It then follows from (5.17) that E has local extrema at $I_1 = 0$ and $I_2 = \pi/2$. The second derivative of E^2 with respect to I is given by

$$\frac{d^2E^2}{dI^2} = -\frac{3M}{r'^3 r^2}\sin^2\gamma\left[2L^2\left(m^2 + rm - r^2\right) + 2r^3 m - r^4\right]\left(1 - \frac{2m}{r}\right)\left(2\cos^2 I - 1\right) \tag{5.18}$$

The term in square parentheses is negative for all $r \geq 2m$ and so

$$\left.\frac{d^2E^2}{dI^2}\right|_{I=0} > 0, \qquad \left.\frac{d^2E^2}{dI^2}\right|_{I=\pi/2} < 0 \tag{5.19}$$

In other words, orbits with zero inclination (i.e. co-planar orbits) are stable. On the other hand, orbits for which the orbital plane of the test-particle is perpendicular to the orbital plane of the binary are unstable.

Chapter 6

Evolution of energy at octupole order

In this section, we return to the polar configuration with $\theta = \pi/2$. At octupole order, the energy E of the test-particle will no longer be conserved as the binary moves towards the background black hole. We seek to find an expression for E as a function of advanced time v, using the Hamiltonian approach. It is assumed that the test-particle starts in a circular orbit. We will be working in an adiabatic approximation, allowing us to assume that the radial velocity of the test-particle can be neglected, $u^r = 0$. The physical reasoning behind this choice is as follows. Recall that there are two distinct timescales in play for the problem at hand. Firstly, there is the short timescale associated with the motion of the test-particle in the binary system. Secondly, there is the long timescale associated with the motion of the binary with respect to the background black hole. A possible change in the radial coordinate r will only enter at octupole order (given an initially circular orbit) and must therefore happen over the long timescale. However, viewed over short time-scales, $u^r = 0$. To a reasonable degree of accuracy, we can thus assume that the test-particle adiabatically moves between circular orbits over long timescales so that we may assume $u^r = 0$, even over long timescales. The test-particle Hamiltonian is computed using (5.5) and we obtain

$$\tilde{H} = \frac{L^2}{2r^2} - \frac{E^2}{2f} + \frac{M}{2r'^3}\left[E^2 r^2 + \left(1 - \frac{2m^2}{r^2}\right)L^2\right] + \frac{rM}{4r'^4}\left[2\left(\frac{Er}{f}\right)^2 e_2^{\mathrm{q}} + 5L^2 e_8^{\mathrm{q}}\right]\frac{dr'}{dv} \quad (6.1)$$

where

$$E := -g_{v\mu}u^{\mu} = -g_{vv}u^{v} - g_{v\phi}u^{\phi} \tag{6.2}$$

$$L := g_{\mu\phi}u^{\mu} = g_{\phi\phi}u^{\phi} \tag{6.3}$$

Imposing (5.10) yields the following expression for E^2:

$$\begin{aligned} E^2 = &\frac{f\,(L^2+r^2)}{r^2} + \frac{M}{r'^3}\left[f\left(L^2+r^2\right) + L^2\left(1-\frac{2m^2}{r^2}\right)\right]f \\ &+ \frac{M}{2r'^4}\left[r\left(5fe^{\mathrm{q}}_{8}+2e^{\mathrm{q}}_{2}\right)L^2 + 2r^3e^{\mathrm{q}}_{2}\right]\frac{dr'}{dv} \end{aligned} \tag{6.4}$$

In the adiabatic approximation, the v-dependence of E comes entirely from the v-dependence in r'. We therefore start by deriving an expression for $r'(v)$. Owing to (4.21), $d\tau'/dv = 1$ in the adiabatic approximation and so from (3.12), we obtain

$$\frac{dr'}{dv} = -\sqrt{\frac{2M}{r'}} \tag{6.5}$$

having set $K = 0$ and assuming $E' = 1$. Integrating from $v = 0$ to v nonzero and imposing the initial condition $r'(0) = r'_0$, we obtain

$$r'(v) = \frac{1}{4}\left(8r'^{\frac{3}{2}}_{0} - 12\sqrt{2M}v\right)^{\frac{2}{3}} \tag{6.6}$$

The time at which the deformed black hole merges with the background black hole will be denoted v_{merge} and is given by $r'(v_{\mathrm{merge}}) = 2M$. This yields the following:

$$v_{\mathrm{merge}} = \frac{2}{3}\sqrt{\frac{r'^3_0}{2M}}\left(1-\left(\frac{2M}{r'_0}\right)^{\frac{3}{2}}\right) \tag{6.7}$$

Using (6.6) and (6.7) in (6.4), yields the following:

$$E^2 = \frac{f\,(L^2+r^2)}{r^2} + \frac{A}{(1-wz)^2} - \frac{B}{(1-wz)^3} \tag{6.8}$$

where $w := v/v_{\text{merge}}$ and

$$A := \frac{Mf}{r^2 {r'}_0^3}\left[fr^4 + L^2(f+1)r^2 - 2L^2m^2\right] \tag{6.9}$$

$$B := \frac{r}{2}\left(\frac{2M}{{r'}_0^3}\right)^{\frac{3}{2}}\left[\frac{5}{2}L^2 e_8^{\mathrm{q}} f + e_2^{\mathrm{q}}\left(L^2 + r^2\right)\right] \tag{6.10}$$

$$z := 1 - \left(\frac{2M}{r'_0}\right)^{\frac{3}{2}} \tag{6.11}$$

This is the general expression for the energy (squared) of the test-particle to octupole order. Note that since $2M < r'_0$, $z < 1$ and so for $v < v_{\text{merge}}$, $W := wz < 1$. In this case, (6.8) has the following series expansion:

$$E^2 = a_0 + \sum_{n=1}^{\infty} a_n W^n \tag{6.12}$$

where

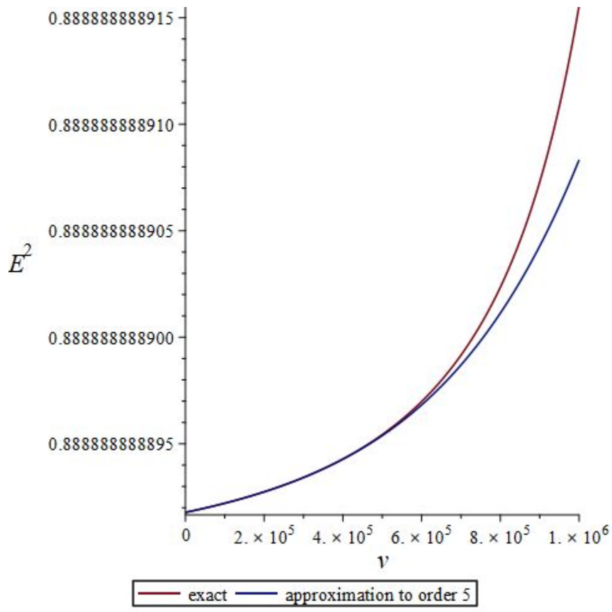
$$a_0 := \frac{f(L^2 + r^2)}{r^2} + A - B, \qquad a_n := A(n+1) - \frac{B}{2}(n+1)(n+2) \tag{6.13}$$

Note that the convergence of the series expansion (6.12) happens very slowly. We illustrate this with a numerical example in which we fix $m = 1$, $r := 6m$, $L := \sqrt{12}m$, $M = 10^2 m$ and $r'_0 = 10^3 M$. This results in a merger time of $v_{\text{merge}} = 1.490578651 \times 10^6$. The corresponding values of a_n for $0 \leq n \leq 5$ are listed in table (6.1). Figure (6.1) shows E^2 as

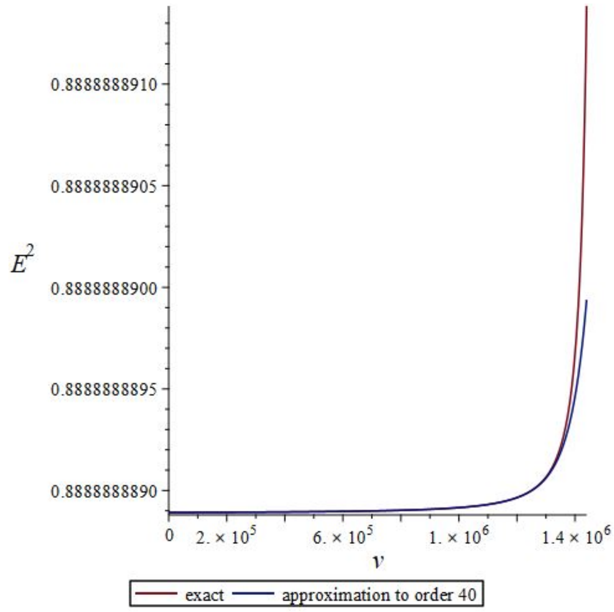
n	a_n
0	0.888888888892
1	$5.777712398 \times 10^{-12}$
2	$8.666535907 \times 10^{-12}$
3	$1.155533763 \times 10^{-11}$
4	$1.155533763 \times 10^{-11}$
5	$1.733287567 \times 10^{-11}$

Table 6.1: Numerical values of a_n for $0 \leq n \leq 5$.

a function of v both for the exact solution and the approximate solution to orders v^5 and v^{40}. Notice that the energy is increasing with v. At order v^5, the approximate solution already starts deviating significantly from the exact solution around $v \sim 10^6$. At order v^{40}, the approximation is accurate for much longer but still differs significantly for $v \sim v_{\text{merge}}$.



(a) Red curve shows exact solution while blue curve shows the approximate solution to order v^5.



(b) Red curve shows exact solution while blue curve shows the approximate solution to order v^{40}.

Figure 6.1: Specific energy (squared) as a function of v. Both the exact solution and the approximate solution have been plotted. The approximate solution is plotted at two different orders, namely 5 and 40.

Chapter 7

Dynamics of the tidally deformed Schwarzschild black hole

This section is restricted to the background black hole and its tidally deformed companion, thus leaving out the test-particle in orbit around the deformed black hole. The deformed black hole is subject to a number of dynamical effects, not present in the unperturbed Schwarzschild solution. In particular, we will see that the surface gravity of the deformed black hole is non-uniform over its horizon with tidal contributions starting at order $(r/\mathcal{R})^3$. Furthermore, we will see that the mass of the deformed black hole acquires a non-trivial time-dependence, an effect which is known as tidal heating. Much of this chapter boils down to giving a description of the geometry of the horizon of the tidally deformed black hole. For this reason, it will prove useful to start by covering some general preliminaries. In particular, section 7.1 covers null geodesic congruences, closely following [13]. Associated with a given null geodesic congruence is an expansion scalar which will be central in the study of how the geometry of the horizon evolves. As we will see, the horizon of the deformed black hole is a null hypersurface of the background spacetime and is generated by a null geodesic congruence. For this reason, it will prove useful to review hypersurfaces and some of their properties in general. This is accomplished in section 7.2, which closely follows [13].

Having gone through the preliminaries, we then consider the geometry of the deformed horizon in section 7.3. Subsequently, we present a derivation of the surface gravity of the deformed black hole in section 7.4. Both sections closely follow [15]. Finally, we cover tidal heating in section 7, closely following both [15] and [12].

7.1 Null geodesic congruences

Let $(\mathscr{M}, g)$ be a Lorentzian manifold and consider an open subset $\mathscr{O} \subseteq \mathscr{M}$. Then a family of curves in $\mathscr{O}$ is called a congruence in $\mathscr{O}$, if for every $p \in \mathscr{O}$, exactly one member of the family passes through p. Clearly a congruence always gives rise to a vector field on $\mathscr{O}$. In particular, the tangent vectors to the members of the congruence yield a vector field on $\mathscr{O}$. The converse is also true, in the sense that given a smooth vector field v on $\mathscr{O}$, one can construct a corresponding congruence on $\mathscr{O}$. Indeed the integral curves of v exactly yield a congruence on $\mathscr{O}$ (section 2.2 of [18]). If every member of the family is a null geodesic, then the congruence is called a null geodesic congruence. In this section, we exclusively work with null geodesic congruences. The goal will be to determine how such null congruences evolve in time. In particular, we will determine how the separation between neighboring members of the congruence changes in time.

Consider a null geodesic congruence on $\mathscr{O}$. Then pick two geodesics γ_0 and γ_1 belonging to the null geodesic congruence. By the defining property of congruences, each point between γ_0 and γ_1 will have exactly one null geodesic going through it. This then gives rise to a two-parameter family of null geodesics which we denote by $\gamma(\lambda, s)$ where $s \in [0, 1]$ specifies a null geodesic and λ is a parameter for the given null geodesic. We choose s such that $\gamma(\lambda, 0) = \gamma_0$ and $\gamma(\lambda, 1) = \gamma_1$. See figure 7.1 for an illustration of the setup.

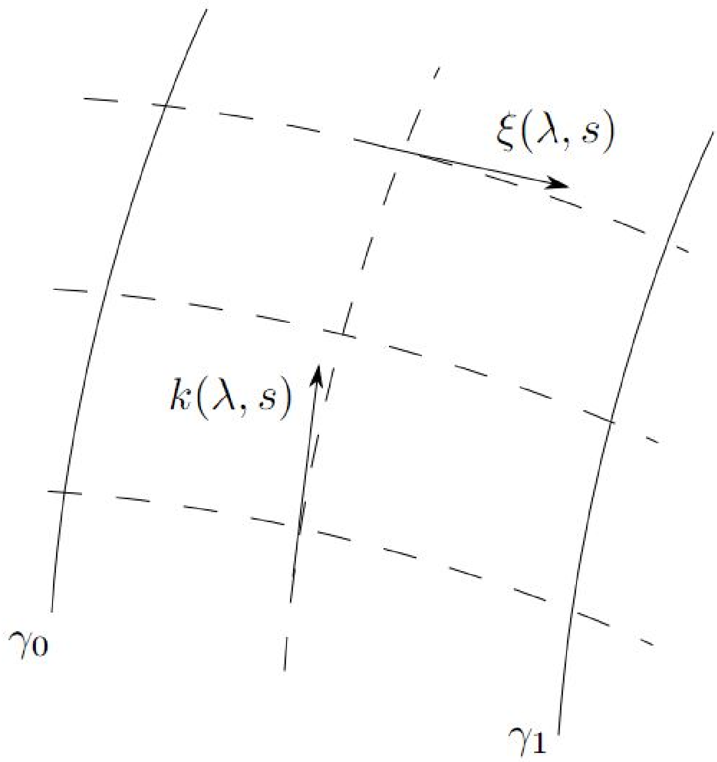


Figure 7.1: Two-parameter family of geodesics $\gamma(\lambda, s)$ labeled by s and parameterised by λ. The vector k is tangent to the null geodesics, while ξ is tangent to the curves connecting γ_0 and γ_1.

For a given value of s, we define the tangent to the corresponding null geodesic by

$$k^\alpha(\lambda, s) := \frac{\partial(x^\alpha \circ \gamma)}{\partial\lambda}(\lambda, s) \tag{7.1}$$

with respect to some coordinate system $x^\mu, \mu \in \{0, 1, 2, 3\}$ on $\mathscr{O}$. Of course, this implies that k satisfies the geodesic equation in its general form:

$$k^\beta k^\alpha{}_{;\beta} = \kappa k^\alpha \tag{7.2}$$

where κ is some scalar. For a given value of λ, we can interpret $s \mapsto \gamma(\lambda, s)$ as describing a curve (generically this will not be a geodesic) going from γ_0 to γ_1. The tangent to this curve is defined as

$$\xi^\alpha(\lambda, s) := \frac{\partial(x^\alpha \circ \gamma)}{\partial s}(\lambda, s) \tag{7.3}$$

We interpret $\xi(\lambda, 0)$ as a deviation vector between γ_0 and γ_1 for $\lambda \in [a, b]$. It is this deviation vector which will be the object of study in the following.

As already mentioned, since k is tangent to the members of a congruence, k constitutes a vector field on $\mathscr{O}$. This then allows a decomposition of the metric g into a transverse part and a longitudinal part on $\mathscr{O}$:

$$g_{\alpha\beta} = h_{\alpha\beta} - (k_\alpha N_\beta + N_\alpha k_\beta) \tag{7.4}$$

where h is the transverse part and N is an auxiliary null vector field chosen such that $k_\alpha N^\alpha = -1$. By construction, $h_{\alpha\beta} N^\alpha = h_{\alpha\beta} k^\alpha = 0$ showing that h is indeed purely transverse in the sense that it is orthogonal to both k and N. We will mainly be concerned with the transverse behaviour of the null geodesic congruence which is why we went through the trouble of introducing h. Next, define a tensor B with components

$$B_{\alpha\beta} = k_{\alpha;\beta} \tag{7.5}$$

and note that it measures the extent to which ξ fails to be parallel transported along the congruence, since

$$\xi^\alpha{}_{;\beta} k^\beta = k^\alpha{}_{;\beta} \xi^\beta = B^\alpha{}_\beta \xi^\beta \tag{7.6}$$

The first equality follows directly from the definitions of k and ξ. For later use, note that B satisfies the following evolution equation:

$$\begin{aligned} k^\gamma B_{\alpha\beta;\gamma} &= k_{\alpha;\beta\gamma} k^\gamma \\ &= (k_{\alpha;\gamma\beta} - R_{\delta\alpha\gamma\beta} k^\delta) k^\gamma \\ &= (k_{\alpha;\gamma} k^\gamma)_{;\beta} - k_{\alpha;\gamma} k^\gamma{}_{;\beta} - R_{\delta\alpha\gamma\beta} k^\delta k^\gamma \\ &= \kappa B_{\alpha\beta} + \kappa_{;\beta} k_\alpha - B_{\alpha\gamma} B^\gamma{}_\beta - R_{\delta\alpha\gamma\beta} k^\delta k^\gamma \end{aligned} \tag{7.7}$$

where we recall that κ is given by $k^\alpha{}_{;\beta} k^\beta = \kappa k^\alpha$. The transverse part of B will be denoted $\tilde{B}$ and has components

$$\tilde{B}_{\alpha\beta} = h^\mu{}_\alpha h^\nu{}_\beta B_{\mu\nu} = B_{\alpha\beta} + k_\alpha N^\mu B_{\mu\beta} + k_\beta B_{\alpha\mu} N^\mu + k_\alpha k_\beta B_{\mu\nu} N^\mu N^\nu \tag{7.8}$$

We then decompose $\tilde{B}$ into its irreducible parts, i.e. its trace, its symmetric trace free part and its antisymmetric part:

$$\tilde{B}_{\alpha\beta} = \frac{1}{2}\Theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \tag{7.9}$$

where $\Theta := \tilde{B}^{\alpha}{}_{\alpha}$ is the expansion scalar, $\sigma_{\alpha\beta} := \tilde{B}_{(\alpha\beta)} - \frac{1}{2}\Theta h_{\alpha\beta}$ are the components of the shear tensor and $\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]}$ are the components of the rotation tensor. We note without proof that Θ measures the fractional rate of change of the cross-sectional area of the congruence (see section 2.4.8 of [13]). In the following, we derive an evolution equation for Θ. Firstly, expanding the definition of Θ yields

$$\Theta = k^{\alpha}{}_{;\alpha} + k^{\alpha} N^{\mu} B_{\mu\alpha} + k^{\alpha} B_{\alpha\mu} N^{\mu} = k^{\alpha}{}_{;\alpha} - \kappa \tag{7.10}$$

Taking the trace of (7.7) and using (7.10) then yields

$$\frac{\partial \Theta}{\partial \lambda} + \frac{\partial \kappa}{\partial \lambda} = \kappa^2 + \kappa\Theta + \frac{\partial \kappa}{\partial \lambda} - B_{\alpha\gamma} B^{\gamma\alpha} - R_{\alpha\beta} k^{\alpha} k^{\beta} \tag{7.11}$$

A straightforward computation reveals that

$$B_{\alpha\gamma} B^{\gamma\alpha} = \tilde{B}_{\alpha\gamma} \tilde{B}^{\gamma\alpha} + \kappa^2 \tag{7.12}$$

and furthermore,

$$\tilde{B}_{\alpha\gamma} \tilde{B}^{\gamma\alpha} = \frac{1}{2}\Theta^2 + \sigma_{\alpha\beta}\sigma^{\alpha\beta} - \omega_{\alpha\beta}\omega^{\alpha\beta} \tag{7.13}$$

Hence, (7.11) reduces to the following:

$$\frac{\partial \Theta}{\partial \lambda} = \kappa\Theta - \frac{1}{2}\Theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - R_{\alpha\beta} k^{\alpha} k^{\beta} \tag{7.14}$$

This is known as Raychaudhuri's equation for null geodesic congruences. Before moving on, we note some of the implications of (7.14). Firstly, if λ is an affine parameter, then $\kappa = 0$. Furthermore, since σ is purely transverse, $\sigma_{\alpha\beta}\sigma^{\alpha\beta} \geq 0$. Additionally, in the next section, we will see that the hypersurfaces of relevance to us have vanishing rotation. Finally, assume that the spacetime in question satisfies the null energy condition then $R_{\alpha\beta} k^{\alpha} k^{\beta} \geq 0$. This will, in particular, hold for vacuum spacetimes. In conclusion, we then observe that:

$$\frac{\partial \Theta}{\partial \lambda} \leq 0 \tag{7.15}$$

Say the members of the congruence are initially converging, so that $\Theta < 0$. Then (7.15) implies that this convergence will happen ever more rapidly into the future, serving to focus the members of the congruence. Notice that under the assumptions above, we in fact have a stronger bound:

$$\frac{\partial \Theta}{\partial \lambda} \leq -\frac{1}{2}\Theta^2 \tag{7.16}$$

Integrating this inequality yields the following:

$$\frac{1}{\Theta(\lambda)} \geq \frac{1}{\Theta(0)} + \frac{\lambda}{2} \tag{7.17}$$

showing that if $\Theta(0) < 0$ (i.e. the congruence is converging at $\lambda = 0$), then $\Theta(\lambda) \to -\infty$ as $\lambda \to \frac{2}{|\Theta(0)|}$ from the left. This usually happens when a so-called caustic forms in the congruence, a caustic being a point at which the members of the congruence cross each other. See figure 7.2 for an illustration.

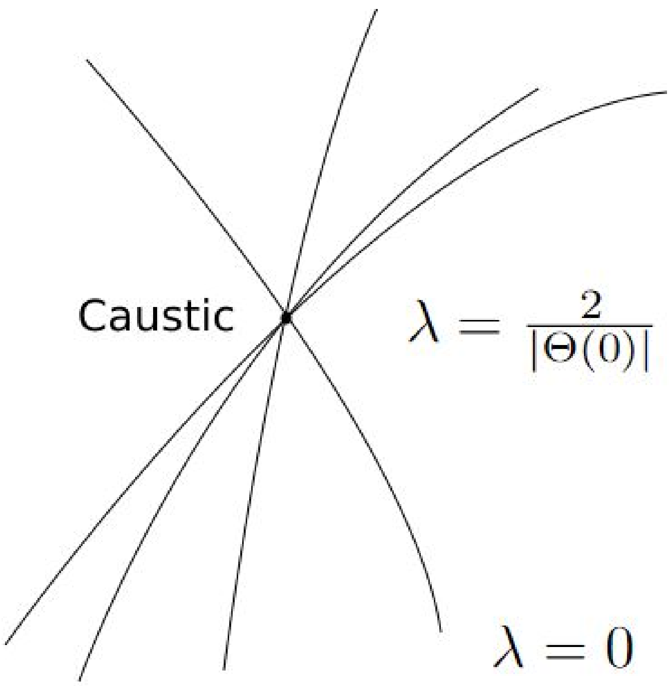


Figure 7.2: A caustic where the members of a congruence cross each other.

In the next section, it will be made clear how null-hypersurfaces are generated by null-geodesic congruences.

7.2 Embedded submanifolds

Let $(\mathscr{M}, g)$ be a Lorentzian manifold of dimension n and consider a second Lorentzian manifold $(\mathscr{S}, g')$ of dimension p where $p \in \{1, \ldots, n\}$ and $\mathscr{S} \subseteq \mathscr{M}$. The manifold $(\mathscr{S}, g')$ is called an embedded submanifold of $(\mathscr{M}, g)$ if there exists a diffeomorphism $\varphi : \mathscr{S} \to \varphi(\mathscr{S}) \subseteq \mathscr{M}$. Furthermore, the structure of $(\mathscr{S}, g')$ is inherited from that of $(\mathscr{M}, g)$ by choosing g' to be the pullback by φ of g to $\mathscr{S}$. We will also refer to g' as the induced metric of the embedded submanifold, with the understanding that it is induced by the metric g. We compute the induced metric later in this section. If the codimension of $(\mathscr{S}, g')$ in $(\mathscr{M}, g)$ is one, then $(\mathscr{S}, g')$ is called a hypersurface of $(\mathscr{M}, g)$. In particular, a hypersurface $(\mathscr{S}, g')$ is called a null-hypersurface of $(\mathscr{M}, g)$ if g', as defined above, is degenerate. Equivalently, a hypersurface is a null-hypersurface if its normal is everywhere null.

In practical applications, one typically opts for a local description of a given hypersurface $(\mathscr{S}, g')$. Indeed, suppose $\mathscr{S}$ is contained in an open subset $\mathscr{O} \subseteq \mathscr{M}$ equipped with a coordinate system $x^\mu, \mu \in \{0, 1, 2, 3\}$. Then $\mathscr{S}$ can be specified by imposing a condition on the coordinate functions $x^\mu \circ p$ for $p \in \mathscr{O}$. Concretely,

$$\mathscr{S} = \{p \in \mathscr{O} : \Phi(x^\mu \circ p) = 0\} \tag{7.18}$$

where $\Phi : \mathbb{R}^4 \to \mathbb{R}$ is a smooth function. Hence, in this local description, $\mathscr{S}$ can be viewed as a level set of some smooth scalar function Φ. This implies that the gradient of Φ is everywhere normal to $\mathscr{S}$. In the following, we take $(\mathscr{S}, g')$ to be null. Inspired by the observation above, we define a normal vector k to $\mathscr{S}$ by $k_\alpha = -\partial_\alpha \Phi$. The sign is chosen such that k is future-directed when Φ is an increasing function of time. We then compute

$$k^\beta D_\beta k_\alpha = \frac{1}{2} D_\alpha (k_\beta k^\beta) \tag{7.19}$$

Since $k_\beta k^\beta$ is identically zero (and hence constant) on $\mathscr{S}$, its gradient must point in the direction normal to $\mathscr{S}$. In other words, $D_\alpha(k_\beta k^\beta) = 2\kappa k_\alpha$ for some scalar κ. This then implies

$$k^{\beta} D_{\beta} k^{\alpha} = \kappa k^{\alpha} \tag{7.20}$$

which is the geodesic equation in its general form. Since $(\mathscr{S}, g')$ is null, we have $k_{\alpha} k^{\alpha} = 0$ and so k is also tangent to $\mathscr{S}$, meaning that the geodesics whose tangents are given by k lie within $\mathscr{O}$. In light of this, we say that $\mathscr{S}$ is generated by null geodesics and k serves as a tangent to the geodesic generators. Since k as given above serves to define a vector field on $\mathscr{O}$, we know that there is a corresponding congruence on $\mathscr{O}$. Hence, what we have shown is that every null hypersurface is generated by a corresponding null geodesic congruence (which might have caustics). Since the tangent vector field k is proportional to the normal of the null hypersurface, the corresponding congruence is called hypersurface orthogonal. As we will see below, it is a general result that such congruences have vanishing rotation.

The setup is the same as above, except we now assume the more general statement that k simply be proportional to the normal of the null hypersurface. In particular, we write

$$k_{\alpha} = -\mu \Phi_{;\alpha} \tag{7.21}$$

for some scalar μ. Explicitly writing out ω, using (7.8) yields the following:

$$\omega_{\alpha\beta} = B_{[\alpha\beta]} - B_{\mu[\alpha} k_{\beta]} N^{\mu} - k_{[\alpha} B_{\beta]\mu} N^{\mu} \tag{7.22}$$

We note that

$$k_{[\alpha;\beta} k_{\gamma]} = \frac{1}{3!}(k_{\alpha;\beta} k_{\gamma} - k_{\alpha;\gamma} k_{\beta} + k_{\gamma;\alpha} k_{\beta} - k_{\gamma;\beta} k_{\alpha} + k_{\beta;\gamma} k_{\alpha} - k_{\beta;\alpha} k_{\gamma}) = 0 \tag{7.23}$$

having used that $\Phi_{;\alpha\beta} = \Phi_{;\beta\alpha}$ owing to the symmetry of the mixed partial derivatives and the symmetry of the lower indices of the Christoffel symbols. Using the definition of B and contracting with N^{γ}, this implies

$$0 = -B_{[\alpha\beta]} + B_{\gamma[\alpha} k_{\beta]} N^{\gamma} + k_{[\alpha} B_{\beta]\gamma} N^{\gamma} \tag{7.24}$$

Inserting this into (7.22), we obtain $\omega_{\alpha\beta} = 0$ as desired.

Finally, we compute the induced metric of the null hypersurface. On $\mathscr{S}$, we may choose to use a coordinate system $y^a, a \in \{1,2,3\}$ which is intrinsic to $\mathscr{S}$. It will prove useful to construct these coordinates in a way that is well suited to the behaviour of the generators of the hypersurface. To this end, we choose one of the coordinates to be the parameter λ. The two remaining coordinates will be denoted $\theta^A, A \in \{2,3\}$ and serve to label the generators of the congruence, in the sense that they are constant on each generator.

When restricted to $\mathscr{S}$, the ambient coordinates $x^\mu, \mu \in \{0,1,2,3\}$ can be viewed as functions of the intrinsic coordinates $y^a, a \in \{1,2,3\}$. This then defines a coordinate transformation with corresponding Jacobian given by

$$J^\alpha_a := \left.\frac{\partial x^\alpha}{\partial y^a}\right|_{\mathscr{S}} \tag{7.25}$$

We also introduce the notation

$$J^\alpha_A := \left.\frac{\partial x^\alpha}{\partial \theta^A}\right|_{\mathscr{S}} \tag{7.26}$$

The induced metric g' is then given by

$$g'_{ab} = g_{\alpha\beta}|_{\mathscr{S}} J^\alpha_a J^\beta_b, \qquad a,b \in \{1,2,3\} \tag{7.27}$$

So far, the discussion applies to any hypersurface. We now impose the condition that the hypersurface be null. In this case, we have $J^\alpha_1 = \partial x^\alpha/\partial\lambda = k^\alpha$ by definition. Hence,

$$g'_{11} = g_{\alpha\beta}|_{\mathscr{S}} k^\alpha k^\beta = 0 \tag{7.28}$$

since k is null. Furthermore,

$$g'_{1A} = g_{\alpha\beta}|_{\mathscr{S}} k^\alpha J^\beta_A = -\left.\frac{\partial\Phi}{\partial x^\beta}\frac{\partial x^\beta}{\partial\theta^A}\right|_{\mathscr{S}} = -\left.\frac{\partial\Phi}{\partial\theta^A}\right|_{\mathscr{S}} = 0 \tag{7.29}$$

since Φ only changes in the direction normal to $\mathscr{S}$. Hence, the induced metric is degenerate and effectively two-dimensional as expected. Motivated by this observation, we write

$$\gamma_{AB} = g_{\alpha\beta}|_{\mathscr{S}} J^\alpha_A J^\beta_B \tag{7.30}$$

for the non-vanishing components of the induced metric.

7.3 Geometry of the deformed horizon

Once again, we consider the spacetime around a tidally deformed Schwarzschild black hole. Let $(\mathscr{M}, g)$ be the corresponding Lorentzian manifold where g is the Poisson-Vlasov metric. In lightcone coordinates, the condition $r = 2m$ defines a hypersurface of $(\mathscr{M}, g)$ in accordance with the discussion of section 7.2. This hypersurface has a corresponding induced metric g', given by (7.27). Since we are simply restricting the value of r and leaving the other coordinates unchanged, $J^\alpha_A = \delta^\alpha_A$ and $J^\alpha_1 = \delta^{0\alpha}$. Evaluating (2.141) at $r = 2m$ and using (7.27), we see that the only non-vanishing components of g' are $g'_{AB} = g_{AB}|_{r=2m}$. In particular, the metric is degenerate and effectively two-dimensional. Hence, the hypersurface defined by $r = 2m$ is null and we can justifiably refer to it as the horizon of the deformed black hole. As encapsulated by equation (2.140), this conclusion holds through order $(r/\mathcal{R})^4$. In light of the observations made above, we can use the notation reserved for null-hypersurfaces in section 7.2. The non-vanishing components of g' are thus denoted by γ_{AB}:

$$\gamma_{AB} := g_{AB}|_{r=2m} = 4m^2\Omega_{AB} - \frac{8}{3}m^4(\mathcal{E}^{\mathrm{q}}_{AB} + \mathcal{B}^{\mathrm{q}}_{AB}) - \frac{8}{15}m^5(\mathcal{E}^{\mathrm{o}}_{AB} + \mathcal{B}^{\mathrm{o}}_{AB}) \tag{7.31}$$

We refer to γ as the horizon metric. We emphasize that $k^\alpha = J^\alpha_1 = \delta^{0\alpha}$ is null on the horizon and tangent to the generators of the horizon. This will be used in the next section.

Since the horizon is a null-hypersurface of $(\mathscr{M}, g)$, it is generated by a (hypersurface orthogonal) congruence of null geodesics. Following the procedure outlined in section 7.1, we define a tensor field B with components given by

$$B_{AB} = k_{\alpha,\beta} J^\alpha_A J^\beta_b \tag{7.32}$$

Note that this corresponds to the $\tilde{B}$ of section 7.1, but we have omitted the tilde for notational convenience. Next, B is decomposed into its irreducible parts:

$$B_{AB} = \frac{1}{2}\Theta\gamma_{AB} + \sigma_{AB} \tag{7.33}$$

having used that the null congruence has vanishing rotation as argued in section 7.2. Notice that by (7.29), J^α_A is transverse to k^α and so the metric γ is in fact transverse to the generators of the horizon. This justifies the appearance of γ as the transverse metric in (7.33).

7.4 Surface gravity

In this section, we will see how the surface gravity of the deformed horizon becomes non-uniform at octupole order.

Firstly, recall the defining equation for κ:

$$k^\beta k^\alpha{}_{;\beta} = \kappa k^\alpha \tag{7.34}$$

Completely analogously to the unperturbed case, this is also the defining equation for the surface gravity of the black hole. Hence, we are justified in identifying κ as the surface gravity of the deformed black hole. Using the metric (2.141) and $k^\alpha = \delta^{0\alpha}$, we explicitly compute,

$$\kappa = \Gamma^0{}_{00}\Big|_{r=2m} = -\frac{1}{2}\left(g^{01}\partial_1 g_{00}\right)\Big|_{r=2m} \tag{7.35}$$

having used that the metric only implicitly depends on time through the time-dependence of r', θ' and ϕ'. Now by definition, the metric g and its inverse satisfy $g^{\alpha\gamma}g_{\beta\gamma} = \delta^\alpha_\beta$ and using (2.141), we immediately get $g^{01} = 1$. We thus obtain

$$\kappa = \frac{1}{4m}\left[1 + \frac{16}{3}M^3\frac{d\mathcal{E}_{ij}}{dv}\Omega^i\Omega^j\right] \tag{7.36}$$

to octupole order. To quadrupole order, the surface gravity of the deformed black hole is uniform across the horizon with the same value as in the unperturbed case. However at octupole order, this uniformity no longer holds.

7.5 Tidal heating

As a consequence of the tidal interaction between the two black holes, the mass m of the tidally deformed black hole should be regarded as a function of time, $m = m(v)$. We only consider long-term changes in m, such that the black hole starts in some initial stationary state with mass m and then after a time Δv settles into another stationary state with mass δm. The averaged change of mass over the time period Δv is then defined by

$$\left\langle \frac{dm}{dv} \right\rangle = \frac{\delta m}{\Delta v} \tag{7.37}$$

To leading order, Poisson has shown that [12]:

$$\left\langle \frac{dm}{dv} \right\rangle = \frac{16}{45} m^6 \left(\frac{d}{dv}\mathcal{E}_{ij} \frac{d}{dv}\mathcal{E}^{ij} + \frac{d}{dv}\mathcal{B}_{ij} \frac{d}{dv}\mathcal{B}^{ij} \right) \tag{7.38}$$

The main objective of this section is to give an outline of the proof for this equation. Having done this, we finish the section with two applications of the equation.

7.5.1 Outline of Poisson's proof

Firstly, we set out to find an evolution equation for the horizon metric. To this end, we compute the following:

$$\begin{aligned}
\frac{\partial \gamma_{AB}}{\partial v} &= k^\alpha \gamma_{AB;\alpha} \\
&= k^\alpha (g_{\alpha\beta}|_{\mathscr{S}} J^\alpha_A J^\beta_B)_{;\alpha} \\
&= g_{\alpha\beta}|_{\mathscr{S}} J^\alpha_{A;\gamma} k^\gamma J^\beta_B + g_{\alpha\beta}|_{\mathscr{S}} J^\alpha_A J^\beta_{B;\gamma} k^\gamma \\
&= 2 g_{\alpha\beta}|_{\mathscr{S}} J^\alpha_A J^\beta_{B;\gamma} k^\gamma \\
&\overset{(\dagger)}{=} 2 g_{\alpha\beta}|_{\mathscr{S}} J^\beta_B k^\alpha{}_{;\gamma} J^\gamma_A \\
&= 2 k_{\beta;\gamma} J^\gamma_A J^\beta_B \\
&= 2 B_{AB}
\end{aligned} \tag{7.39}$$

where (†) uses

$$J^\alpha_{A;\beta} k^\beta = k^\alpha{}_{;\beta} J^\beta_A \tag{7.40}$$

which follows from a straightforward computation using $J^\alpha_A = \delta^\alpha_A$ and $k^\alpha = \delta^{0\alpha}$. Hence, we arrive at the following evolution equation for γ:

$$\frac{\partial \gamma_{AB}}{\partial v} = \Theta \gamma_{AB} + 2\sigma_{AB} \tag{7.41}$$

Now contract with the inverse of γ to arrive at an expression for the expansion scalar:

$$\Theta = \frac{1}{2}\gamma^{AB}\frac{\partial \gamma_{AB}}{\partial v} = \frac{1}{\sqrt{\gamma}}\frac{\partial\sqrt{\gamma}}{\partial v} \tag{7.42}$$

where γ is the determinant of the matrix representation of γ_{AB} in some basis. Explicitly,

$$\sqrt{\gamma} = 4m^2 \sin\theta \left[1 + \mathcal{O}\left(\frac{1}{\mathcal{R}^4}\right)\right] \tag{7.43}$$

which is shown in appendix F. Using this in (7.42), we conclude that

$$\Theta = \mathcal{O}\left(\frac{1}{\mathcal{R}^5}\right) \tag{7.44}$$

This will be important later, when we decide which terms should be included and which should be omitted given our level of precision.

We turn now to the shear tensor. Using the previous results, eq. (7.41) implies

$$\sigma_{AB} = \frac{1}{2}\partial_v \gamma_{AB} + \mathcal{O}\left(\frac{1}{\mathcal{R}^5}\right) = -\frac{4}{3}m^4\left(\frac{d}{dv}\mathcal{E}^{\mathrm{q}}_{AB} + \frac{d}{dv}\mathcal{B}^{\mathrm{q}}_{AB}\right) + \mathcal{O}\left(\frac{1}{\mathcal{R}^4}\right) \tag{7.45}$$

The indices on σ_{AB} should be raised using the inverse of the induced metric. Given the present level of precision, it will prove sufficient to take

$$\gamma^{AB} = \frac{1}{4m^2}\Omega^{AB} + \mathcal{O}\left(\frac{1}{\mathcal{R}^2}\right) \tag{7.46}$$

This then results in

$$\sigma^{AB} = -\frac{1}{12}\left(\frac{d}{dv}\mathcal{E}^{\mathrm{q}\,AB} + \frac{d}{dv}\mathcal{B}^{\mathrm{q}\,AB}\right) \tag{7.47}$$

where indices on $\mathcal{E}^{\mathrm{q}}_{AB}$ and $\mathcal{B}^{\mathrm{q}}_{AB}$ have been raised with Ω^{AB}.

Next, we turn to Raychaudhuri's equation. As was argued in section 7.2, the congruence of generators of the horizon will have vanishing rotation. Furthermore, we are considering a vacuum solution to Einstein's field equations so the Ricci tensor vanishes. In this case, Raychaudhuri's equation simplifies to

$$\frac{\partial\Theta}{\partial v} = \kappa\Theta - \frac{1}{2}\Theta^2 - \sigma_{AB}\sigma^{AB} \tag{7.48}$$

We write $\kappa = \kappa_0 + \kappa_{\text{correction}}$ where $\kappa_0 := (4m)^{-1}$ is the surface gravity of the unperturbed black hole and $\kappa_{\text{correction}}$ is the tidally induced correction to κ_0, as given by (7.36). Owing to (7.44), $\kappa\Theta = \kappa_0\Theta + \mathcal{O}(1/\mathcal{R}^8)$. Since the desired equation (7.38) is of order $1/\mathcal{R}^6$, we can justifiably set $\kappa = \kappa_0$ in (7.48). Similarly, the Θ^2-term in (7.48) can be completely neglected. We are thus left with the following:

$$\frac{\partial\Theta}{\partial v} = \kappa_0\Theta - \sigma_{AB}\sigma^{AB} \tag{7.49}$$

Below, we will see that the evolution of the area of the deformed horizon is intimately tied to the expansion scalar Θ.

On the horizon, the area element $\sqrt{\gamma}d\theta d\phi$ is given in terms of the induced metric. The area $\mathcal{A}(v)$ of the deformed horizon as a function of v is then defined by

$$\mathcal{A}(v) := \int \sqrt{\gamma}d\theta d\phi \tag{7.50}$$

Using (7.42), we then compute the advanced time derivative of $\mathcal{A}$:

$$\frac{d}{dv}\mathcal{A}(v) = \int \Theta\sqrt{\gamma}d\theta d\phi \tag{7.51}$$

Furthermore,

$$\frac{d^2}{dv^2}\mathcal{A}(v) = \int \frac{\partial\Theta}{\partial v}\sqrt{\gamma}d\theta d\phi + \int \Theta^2\sqrt{\gamma}d\theta d\phi \tag{7.52}$$

Once again, we neglect the term containing Θ^2. Using (7.49) and (7.43), we then have

$$\begin{aligned}\kappa_0\frac{d}{dv}\mathcal{A} - \frac{d^2}{dv^2}\mathcal{A} &= \int \sigma_{AB}\sigma^{AB}\sqrt{\gamma}d\theta d\phi \\ &= \frac{4}{9}m^6 \int \left[\frac{d}{dv}\mathcal{E}^{\text{q}}_{AB}\frac{d}{dv}\mathcal{E}^{\text{q}AB} + \frac{d}{dv}\mathcal{B}^{\text{q}}_{AB}\frac{d}{dv}\mathcal{B}^{\text{q}AB} + 2\frac{d}{dv}\mathcal{E}^{\text{q}}_{AB}\frac{d}{dv}\mathcal{B}^{\text{q}AB}\right] d\Omega\end{aligned} \tag{7.53}$$

where $d\Omega = \sin\theta d\theta d\phi$. In the following, we evaluate these angular integrals and express them in terms of their corresponding derivatives of quadrupole tidal moments.

Firstly, by definition

$$\frac{d}{dv}\mathcal{E}^{\mathrm{q}}_{AB} := \frac{d}{dv}\mathcal{E}^{\mathrm{q}}_{ij}\Omega^i_A\Omega^j_B, \qquad \frac{d}{dv}\mathcal{B}^{\mathrm{q}}_{AB} := \frac{d}{dv}\mathcal{B}^{\mathrm{q}}_{ij}\Omega^i_A\Omega^j_B \tag{7.54}$$

which implies

$$\frac{d}{dv}\mathcal{E}^{\mathrm{q}}_{AB}\frac{d}{dv}\mathcal{E}^{\mathrm{q}AB} = \frac{d}{dv}\mathcal{E}^{\mathrm{q}}_{ij}\Omega^i_A\Omega^j_B\frac{d}{dv}\mathcal{E}^{\mathrm{q}AB} = \frac{d}{dv}\mathcal{E}^{\mathrm{q}}_{ij}\frac{d}{dv}\mathcal{E}^{\mathrm{q}\,ij} \tag{7.55}$$

and similarly for $\mathcal{B}$. We then use the expressions for the tidal potentials as listed in table (2.1) to compute the following:

$$\begin{aligned}
\frac{d}{dv}\mathcal{E}^{\mathrm{q}}_{ij}\frac{d}{dv}\mathcal{E}^{\mathrm{q}ij} &= (2\gamma_i{}^k\gamma_j^l\frac{d}{dv}\mathcal{E}_{kl} + \gamma_{ij}\frac{d}{dv}\mathcal{E}_{kl}\Omega^k\Omega^l)(2\gamma^{im}\gamma^{jn}\frac{d}{dv}\mathcal{E}_{mn} + \gamma^{ij}\frac{d}{dv}\mathcal{E}_{mn}\Omega^m\Omega^n)\\
&= \frac{d}{dv}\mathcal{E}_{kl}\frac{d}{dv}\mathcal{E}_{mn}(4\gamma^{km}\gamma^{ln} + 2\gamma^{kl}\Omega^m\Omega^n + 2\gamma^{mn}\Omega^k\Omega^l + 2\Omega^m\Omega^n\Omega^k\Omega^l)\\
&= \frac{d}{dv}\mathcal{E}_{kl}\frac{d}{dv}\mathcal{E}_{mn}\left[4(\delta^{km}\delta^{ln} - \delta^{km}\Omega^l\Omega^n - \delta^{ln}\Omega^k\Omega^m + \Omega^m\Omega^n\Omega^k\Omega^l)\right.\\
&\left.+2(\delta^{kl}\Omega^m\Omega^n - \Omega^k\Omega^l\Omega^m\Omega^n) + 2(\delta^{mn}\Omega^k\Omega^l - \Omega^m\Omega^n\Omega^k\Omega^l) + 2\Omega^m\Omega^n\Omega^k\Omega^l\right]\\
&= \frac{d}{dv}\mathcal{E}_{kl}\frac{d}{dv}\mathcal{E}_{mn}[4(\delta^{km}\delta^{ln} - \delta^{km}\Omega^l\Omega^n - \delta^{ln}\Omega^k\Omega^m) + 2\Omega^m\Omega^n\Omega^k\Omega^l]
\end{aligned} \tag{7.56}$$

Note that γ in the above is not the induced metric, but rather the projector defined in (2.75). We also made use of the fact that the tidal moments are traceless. We then wish to integrate the above with respect to the surface measure $d\Omega$. To do so, note the following two identities, which can be established by straightforward computation:

$$\int_0^{2\pi}\int_0^{\pi}\sin\theta\Omega^m\Omega^n\Omega^k\Omega^l d\theta d\phi = \frac{4\pi}{15}(\delta^{lk}\delta^{mn} + \delta^{lm}\delta^{kn} + \delta^{ln}\delta^{km}) \tag{7.57}$$

$$\int_0^{2\pi}\int_0^{\pi}\sin\theta\Omega^m\Omega^n d\theta d\phi = \frac{4\pi}{3}\delta^{mn} \tag{7.58}$$

We thus obtain

$$\int \frac{d}{dv}\mathcal{E}^{\mathrm{q}}_{AB}\frac{d}{dv}\mathcal{E}^{\mathrm{q}AB}d\Omega = \frac{32\pi}{5}\frac{d}{dv}\mathcal{E}_{ij}\frac{d}{dv}\mathcal{E}^{ij} \tag{7.59}$$

Completely analogously, we compute

$$\int \frac{d}{dv}\mathcal{B}^{\mathrm{q}}_{AB}\frac{d}{dv}\mathcal{B}^{\mathrm{q}AB}d\Omega = \frac{32\pi}{5}\frac{d}{dv}\mathcal{B}_{ij}\frac{d}{dv}\mathcal{B}^{ij} \tag{7.60}$$

and

$$\int \frac{d}{dv}\mathcal{B}^{\mathrm{q}}_{AB}\frac{d}{dv}\mathcal{E}^{\mathrm{q}AB}d\Omega = 0 \tag{7.61}$$

Putting the preceding results together, we have

$$\frac{\kappa_0}{8\pi}\frac{d}{dv}\mathcal{A} - \frac{1}{8\pi}\frac{d^2}{dv^2}\mathcal{A} = \frac{16}{45}m^6\left(\frac{d}{dv}\mathcal{E}_{ij}\frac{d}{dv}\mathcal{E}^{ij} + \frac{d}{dv}\mathcal{B}_{ij}\frac{d}{dv}\mathcal{B}^{ij}\right) \tag{7.62}$$

To continue, it will be useful to introduce the flux function, $\mathcal{F}$ defined by

$$\mathcal{F}(v) := \frac{16}{45}m^6\left(\frac{d}{dv}\mathcal{E}_{ij}\frac{d}{dv}\mathcal{E}^{ij} + \frac{d}{dv}\mathcal{B}_{ij}\frac{d}{dv}\mathcal{B}^{ij}\right) \tag{7.63}$$

Then the general solution to (7.62) is

$$\frac{d}{dv}\mathcal{A} = e^{\kappa_0 v}\frac{d\mathcal{A}}{dv}(0) - 8\pi\int_0^v \mathcal{F}(v')e^{\kappa_0(v-v')}dv' \tag{7.64}$$

Integrating by parts, we get:

$$\frac{d}{dv}\mathcal{A} = \frac{8\pi}{\kappa_0}\mathcal{F}(v) + e^{\kappa_0 v}\left[\frac{d\mathcal{A}}{dv}(0) - \frac{8\pi}{\kappa_0}\mathcal{F}(0)\right] - 8\pi\int \frac{d\mathcal{F}(v')}{dv'}e^{\kappa_0(v-v')}dv' \tag{7.65}$$

The last term can be neglected as it is of order $1/\mathcal{R}^7$. Furthermore, the term in brackets, proportional to $e^{\kappa_0 v}$, grows exponentially over time scales $v \sim 1/\kappa_0 = 4m$ which is unphysical for our setup. Rather, we would expect small changes in the surface area over long time scales. Hence, we impose the initial condition that

$$\frac{d\mathcal{A}}{dv}(0) = \frac{8\pi}{\kappa_0}\mathcal{F}(0) \tag{7.66}$$

whereby

$$\frac{d}{dv}\mathcal{A} = \frac{8\pi}{\kappa_0}\mathcal{F}(v) \tag{7.67}$$

Identifying $\kappa_0/(8\pi)\frac{d}{dv}\mathcal{A}$ with $\langle\frac{dm}{dv}\rangle$, we get the result in (7.38). Notice that the shear tensor entered quadratically in (7.53). In other words, tidal moments will only appear quadratically in $\langle\delta m/\Delta v\rangle$. This is the justification for using a metric of octupole order to derive (7.38), which starts at order $1/\mathcal{R}^6$.

7.5.2 Tidal heating for a radial infall

In the case of a radial infall, we simply plug (3.60) into (7.38) and obtain:

$$\left\langle\frac{dm}{dv}\right\rangle = \frac{96m^6M^2}{5r'^8}\left(\frac{dr'}{dv}\right)^2 \tag{7.68}$$

7.5.3 Tidal heating for circular orbit

For a circular orbit, we use (3.47)-(3.50) and (3.58)-(3.59) with $r' > 2M$ equal to some constant. Note that by the chain rule,

$$\frac{d\Psi}{dv} = \frac{\dot{\Psi}}{\dot{t}'} = \frac{1}{L'^2 + r'^2}\left(1 - \frac{2M}{r'}\right) \tag{7.69}$$

We then compute

$$\left\langle\frac{dm}{dv}\right\rangle = \frac{32}{5}\left(\frac{m}{M}\right)^6 V^{18}\frac{\left(1-2V^2\right)^2(1+2V^2)}{(1+V^2)} \tag{7.70}$$

where $V = \dot{\phi}'r' = \sqrt{M/r'}$ is the orbital speed.

Chapter 8

Conclusion and outlook

We have seen how the metric of a vacuum region of an arbitrary spacetime can be constructed in terms of a set of tidal moments and corresponding tidal potentials. This is, in particular, true for the Poisson-Vlasov metric which was introduced to describe the spacetime around a tidally deformed Schwarzschild black hole. We have seen how the components of the Poisson-Vlasov metric are conveniently written with respect to a set of lightcone coordinates. The advanced-time coordinate plays an especially important role in octupole order computations and above, where it replaces the usual proper time coordinate.

In chapter 3.4, we computed the tidal potentials for a Schwarzschild perturber. This was done by taking frame components of the Riemann tensor along a geodesic in the Schwarzschild spacetime. These moments were subsequently converted to potentials which could then be substituted into the Poisson-Vlasov metric. For many of the practical applications in this text, we chose to work with a radially infalling deformed black hole. With this choice, we have seen that all magnetic tidal moments vanish identically.

In chapter 4, we used the Poisson-Vlasov metric to compute the tidally induced shifts in the ISCO parameters of a test-particle orbiting the deformed black hole. This was first done to quadrupole order where the deformed black hole could be assumed stationary with respect to the background black hole. This allowed us to utilize two Killing vectors to identify the specific energy and specific angular momentum of the test-particle. The quadrupole order ISCO shifts are listed in (4.16). At octupole order, the distance between the deformed black hole and the background black hole has to be regarded as a function of advanced time, v. At a given point along the trajectory of the deformed black hole, the octupole order ISCO shifts were computed and are listed in (4.37). Having computed the ISCO shifts, we computed a lower bound on the specific energy of the test-particle in orbit around the deformed black hole such that its radial coordinate would never decrease below the ISCO radius. This specific energy threshold is given in (4.42).

In chapter 5, we considered the specific energy of the test-particle as a function of the Euler angles that specify the orientation of the binary system with respect to the background black hole. In particular, we found that co-planar orbits are stable. That is to say, configurations for which the inclination angle of the binary is zero, minimize the specific energy of the test-particle. The specific energy of the test-particle was computed by first determining the Hamiltonian of the test-particle and then imposing four-velocity normalization. The specific energy (squared) of the test-particle to quadrupole order is given in (5.17). In the subsequent chapter, we computed the Hamiltonian of the test-particle to octupole order. The expression for the specific energy (squared) to octupole order is given in (6.4). This allowed us to determine the evolution of the specific energy of the test-particle as the binary moved closer to the background black hole. In particular, we found that the specific-energy increases as a function of advanced time.

In chapter 7, we considered the dynamics of the tidally deformed black hole itself. We found that the horizon of the deformed black hole is a null-hypersurface of the background spacetime, generated by a null-geodesic congruence. The horizon was found to still be located at $r = 2m$ at the level of precision maintained in this text. However at octupole order, the surface gravity of the deformed horizon is no longer uniform as can be seen from (7.36). Furthermore, we have seen how the mass of the deformed black hole seizes to be constant in time, with changes in mass arising at order $(m/\mathcal{R})^6$. In particular, we computed $\langle dm/dv \rangle$ to leading order for a radial infall and for a circular orbit. The results are given in (7.68) and (7.70), respectively.

Further study, based off this thesis may include the following: Firstly, many of the results presented in this thesis may be generalized further. For example, one might consider the deformed black travelling along an arbitrary geodesic in the background spacetime. One might also consider a Kerr perturber instead of a Schwarzschild perturber as has been done in [1]. Secondly, higher order terms of the Poisson-Vlasov metric may be included. Namely one might include contributions at hexadecapole order, the order to which Poisson and Vlasov originally expressed their metric.

Appendix A

Fermi normal coordinates

In this appendix, we introduce the Fermi normal coordinates for a free-falling observer, closely following [16].

Let $(\mathscr{M}, g)$ be a Lorentzian manifold. Now, an observer at some point $p \in \mathscr{M}$ can construct a local inertial coordinate system, $x^\mu, \mu \in \{0, 1, 2, 3\}$ around p. In these coordinates, we write the components of the metric as $g_{\mu\nu}$ and

$$g_{\mu\nu}|_p = \eta_{\mu\nu} \tag{A.1}$$

$$(\partial_\rho g_{\mu\nu})|_p = 0 \tag{A.2}$$

In other words, at p (and in a small neighborhood around p) spacetime looks like flat Minkowski spacetime. The core idea behind the construction of Fermi normal coordinates, is to take an observer in free fall and then assign to that observer an inertial system which applies to their entire worldline instead of just a single point. In other words, the goal is to construct a coordinate system such that spacetime in a small tube around the observer's worldline looks like flat Minkowski spacetime. In this section, we go through the construction of the Fermi normal coordinates for an observer following a geodesic in $\mathscr{M}$.

Let $[a, b] \subseteq \mathbb{R}$ be an interval and let $\gamma : [a, b] \to \mathscr{M}$ be a (smooth) timelike geodesic on $\mathscr{M}$. Consider now a coordinate system $x^\mu, \mu \in \{0, 1, 2, 3\}$ defined on a neighborhood $\mathscr{O} \subseteq \mathscr{M}$ with $\gamma \subseteq \mathscr{O}$. Since γ is timelike, we refer to its arc length as the proper time of γ. We denote the proper time of γ by τ_γ and it is defined in the usual way:

$$\tau_\gamma = \int_a^b \sqrt{-g_{\mu\nu}(\gamma(t)) \frac{d(x^\mu \circ \gamma)}{dt}(t) \frac{d(x^\nu \circ \gamma)}{dt}(t)} dt \tag{A.3}$$

In the following, we parameterize γ by letting the proper time $\tau \in [0, \tau_\gamma]$ along γ serve as an affine parameter for the curve. With respect to x^μ, the relativistic velocity on γ is defined as

$$u^\mu(\tau) = \frac{d(x^\mu \circ \gamma)}{d\tau}(\tau), \qquad \tau \in [0, \tau_\gamma] \tag{A.4}$$

Since γ is timelike, we have

$$g_{\mu\nu}(\gamma(\tau))u^\mu(\tau)u^\nu(\tau) = -1, \qquad \tau \in [0, \tau_\gamma] \tag{A.5}$$

By assumption, u also satisfies the geodesic equation:

$$\frac{D}{d\tau}u^\mu = 0 \tag{A.6}$$

for all $\mu \in \{0, 1, 2, 3\}$.

Now, let $\mathscr{O}$ be a normal convex neighborhood of γ and take a point $p \in \mathscr{O}$. Then there is a unique space-like geodesic $\beta : [c, d] \to \mathscr{M}, [c, d] \subseteq \mathbb{R}$ which intersects γ orthogonally and ends at p. The point of intersection will be labelled q and we define τ_0 to be the value of τ at the intersection point. That is, $q := \gamma(\tau_0)$. Denote by s_β the geodesic distance between q and p measured along β, or in other words, the arc length of β. We shall parameterize β by the geodesic distance $s \in [0, s_\beta]$ along β. Furthermore, we define the tangent to β with respect to x^μ as

$$t^\mu(s) = \frac{d(x^\mu \circ \beta)}{ds}(s), \qquad s \in [0, s_\beta] \tag{A.7}$$

However, we will primarily work with the rescaled tangent, $v^\mu := s_\beta t^\mu$. The requirement that β intersect γ orthogonally then reads

$$g_{\mu\nu}(\gamma(\tau_0))u^\mu(\tau_0)v^\nu(0) = 0 \tag{A.8}$$

Now define the Fermi normal coordinates $\tilde{x}^\mu$ for p as follows:

$$\tilde{x}^0 = \tau_0, \qquad \tilde{x}^i = \lambda^i_\mu(\tau_0)v^\mu(0) \tag{A.9}$$

where $i \in \{1,2,3\}$ and $\{\lambda_a\}_{a\in\{0,1,2,3\}}$ is an orthonormal tetrad on γ with $\lambda_0 := u$. Inverting the last equation in (2.5) and using the definition of the Fermi normal coordinates, we see that

$$\begin{aligned}
\delta_{ij}\tilde{x}^i\tilde{x}^j &= \delta_{ij}\lambda^i_\mu(\tau_0)v^\mu(0)\lambda^j_\nu(\tau_0)v^\nu(0) \\
&= [\eta_{ab}\lambda^a_\mu(\tau_0)\lambda^b_\nu(\tau_0) + \lambda^0_\mu(\tau_0)\lambda^0_\nu(\tau_0)]v^\mu(0)v^\nu(0) \\
&= [g_{\mu\nu}(\gamma(\tau_0)) + u_\mu(\tau_0)u_\nu(\tau_0)]v^\mu(0)v^\nu(0) \\
&= g_{\mu\nu}(\gamma(\tau_0))v^\mu(0)v^\nu(0) \\
&= s_\beta^2
\end{aligned} \tag{A.10}$$

showing that s_β is simply the spatial distance between p and q measured along β. Generically, for a point $\beta(s)$ on the geodesic connecting $q = \gamma(\tau)$ and $p = \beta(s_\beta)$, the Fermi normal coordinates are $\tilde{x}^0 = \tau$ and $\tilde{x}^i = s\Omega^i(\tau), i \in \{1,2,3\}$ where $\Omega^i(\tau) := \lambda^i_\mu(\tau)t^\mu(0), i \in \{1,2,3\}$.

We still need to show that these coordinates indeed exhibit the local flatness property mentioned at the start of the section. Indeed, we will see that the metric expressed in Fermi normal coordinates and evaluated at p is given by the following:

$$\tilde{g}_{00}|_p = -1 - \tilde{R}_{0i0j}|_q\tilde{x}^i\tilde{x}^j + \mathcal{O}(s^3) \tag{A.11}$$

$$\tilde{g}_{0i}|_p = -\frac{2}{3}\tilde{R}_{0jik}|_q\tilde{x}^j\tilde{x}^k + \mathcal{O}(s^3) \tag{A.12}$$

$$\tilde{g}_{ij}|_p = \delta_{ij} - \frac{1}{3}\tilde{R}_{ikjl}|_q\tilde{x}^k\tilde{x}^l + \mathcal{O}(s^3) \tag{A.13}$$

where the components of the Riemann tensor are evaluated in Riemann normal coordinates centred on q. In order to show (A.11)-(A.13), we start by considering a general series expansion of $\tilde{g}$ around $q = \gamma(\tau_0), \tau \in [0, \tau_\gamma]$, evaluated at $p = \beta(s), s \in [0, s_\beta]$:

$$\tilde{g}_{\mu\nu}|_p = \tilde{g}_{\mu\nu}(\gamma(\tau_0)) + \tilde{g}_{\mu\nu,\alpha}(\gamma(\tau_0))\tilde{x}^\alpha + \frac{1}{2}\tilde{g}_{\mu\nu,\alpha\beta}(\gamma(\tau_0))\tilde{x}^\alpha\tilde{x}^\beta + \mathcal{O}(s^3) \tag{A.14}$$

We start by computing $\tilde{g}_{\mu\nu}(\gamma(\tau_0))$. We make the following observation:

$$
\begin{aligned}
\tilde{x}^i \lambda_i^\mu(\tau_0) &= v^\mu(0) \\
&= s\frac{d(x^\mu \circ \beta)}{ds}(0) \\
&= s \left. \frac{\partial(x^\mu \circ \beta)}{\partial \tilde{x}^i} \right|_{s=0} \frac{d\tilde{x}^i}{ds}(0) \\
&= s \left. \frac{\partial(x^\mu \circ \beta)}{\partial \tilde{x}^i} \right|_{s=0} \lambda_\mu^i(\tau_0) t^\mu(0) \\
&= \tilde{x}^i \left. \frac{\partial(x^\mu \circ \beta)}{\partial \tilde{x}^i} \right|_{s=0}
\end{aligned} \tag{A.15}
$$

Hence,

$$
\frac{\partial(x^\mu \circ \beta)}{\partial \tilde{x}^i}(0) = \lambda_i^\mu(\tau_0) \tag{A.16}
$$

This construction works for any $\tau_0 \in [0, \tau_\gamma]$, showing that

$$
\left. \frac{\partial x^\mu}{\partial \tilde{x}^i} \right|_\gamma = \lambda_i^\mu \tag{A.17}
$$

Together with the definition of λ_0, we have

$$
\left. \frac{\partial x^\mu}{\partial \tilde{x}^a} \right|_\gamma = \lambda_a^\mu \tag{A.18}
$$

for all $a \in \{0, 1, 2, 3\}$. Using the usual transformation rule for rank two tensors, we then find that on γ

$$
\tilde{g}_{ab}|_\gamma = g_{\alpha\beta}|_\gamma \left. \frac{\partial x^\alpha}{\partial \tilde{x}^a} \right|_\gamma \left. \frac{\partial x^\beta}{\partial \tilde{x}^b} \right|_\gamma = g_{\alpha\beta}|_\gamma \lambda_a^\alpha \lambda_b^\beta = \eta_{ab} \tag{A.19}
$$

Hence, the metric on γ is everywhere Minkowski. Next, we turn to the derivatives of the metric on γ. Since β is a geodesic and parameterized by proper time, we have

$$
\frac{d^2(\tilde{x}^a \circ \beta)}{ds^2} + \tilde{\Gamma}^a_{bc} \circ \beta \frac{d(\tilde{x}^b \circ \beta)}{ds} \frac{d(\tilde{x}^c \circ \beta)}{ds} = 0 \tag{A.20}
$$

with $a, b, c \in \{0, 1, 2, 3\}$. The first term vanishes and simplifying the second term yields

$$\tilde{\Gamma}^a_{ij} \circ \beta \Omega^i(\tau_0)\Omega^j(\tau_0) = 0 \tag{A.21}$$

Generically, this requires the Christoffel symbols to vanish on β. In particular at q. Similarly to before, this construction can be repeated for any point on γ. Hence,

$$\tilde{\Gamma}^a_{ij}|_\gamma = 0 \tag{A.22}$$

Furthermore, the tetrad is parallel transported along γ and so

$$\frac{d\lambda^a_\mu}{d\tau} + \tilde{\Gamma}^a_{bc} \circ \gamma \lambda^b_\mu \lambda^c_0 = 0 \tag{A.23}$$

By (A.18), in Fermi normal coordinates, we have $\lambda^a_\mu = \delta^a_\mu$, so the above implies $\tilde{\Gamma}^a_{b0}|_\gamma = 0$. Hence, all the Christoffel symbols vanish on γ. This of course implies $\tilde{g}_{\mu\nu,\alpha}|_\gamma = 0$.

Now, as for the second derivatives of the metric on γ. Since the Christoffel symbols are all zero on γ, we get

$$\tilde{\Gamma}^a_{\mu\nu,0}|_\gamma = 0 \tag{A.24}$$

Then, by the coordinate expression for the Riemann tensor components, we have

$$\tilde{\Gamma}^\alpha_{\mu 0,\nu}|_\gamma = \tilde{R}^\alpha_{\mu\nu 0}|_\gamma \tag{A.25}$$

Considering the derivative of the geodesic equation and permuting some indices we also find

$$0 = \tilde{\Gamma}^\alpha_{ij,k}|_\gamma + \tilde{\Gamma}^\alpha_{jk,i}|_\gamma + \tilde{\Gamma}^\alpha_{ki,j}|_\gamma \tag{A.26}$$

From the coordinate expression for the Riemann tensor, it then follows that

$$\tilde{\Gamma}^\alpha_{ij,k}|_\gamma = -\frac{1}{3}(\tilde{R}^\alpha_{ijk} + \tilde{R}^\alpha_{jik})|_\gamma \tag{A.27}$$

We wish to convert these expressions into statements regarding the second derivatives of the metric. Recall the coordinate expression for the Christoffel symbols:

$$\tilde{\Gamma}^{\alpha}_{\mu\nu} = \frac{1}{2}\tilde{g}^{\alpha\lambda}(\tilde{g}_{\mu\lambda,\nu} + \tilde{g}_{\lambda\nu,\mu} - \tilde{g}_{\mu\nu,\lambda}) \tag{A.28}$$

On γ, the metric is simply the Minkowski metric. Differentiating this and setting $\mu = \nu = 0$, yields

$$\tilde{\Gamma}^{\alpha}_{00,\sigma}|_{\gamma} = -\frac{1}{2}\delta^{\alpha i}\tilde{g}_{00,i\sigma}|_{\gamma} \tag{A.29}$$

Using (A.25), we thus get

$$\tilde{g}_{00,kj}|_{\gamma} = -2\tilde{R}_{k0j0}|_{\gamma} \tag{A.30}$$

Next, we see that

$$-\frac{2}{3}(\tilde{R}_{0jik} + \tilde{R}_{0kij})|_{\gamma} = 2\eta_{0\alpha}(\tilde{\Gamma}^{\alpha}_{ij,k} + \tilde{\Gamma}^{\alpha}_{ik,j})|_{\gamma} = (2\tilde{g}_{0i,jk} + \tilde{g}_{0j,ik} + \tilde{g}_{0k,ij})|_{\gamma} = \tilde{g}_{0i,jk}|_{\gamma} \tag{A.31}$$

where the last equality follows from a derivative of Gauss' lemma. Similarly,

$$\tilde{g}_{ij,kl}|_{\gamma} = -\frac{1}{3}(\tilde{R}_{ikjl} + \tilde{R}_{iljk})|_{\gamma} \tag{A.32}$$

Since the metric on γ is everywhere Minkowski, all the temporal derivatives of the metric vanish. Plugging the above expressions into (A.14), we arrive at the desired result.

Appendix B

Derivation of the geodesic deviation equation

In this appendix, we derive the geodesic deviation equation (1.6). The derivation closely follows that presented in [13]. We consider the same setup as in section 7.1, although this time we consider $\gamma(\lambda, s)$ with λ varying to be timelike and we set λ equal to τ, namely the proper time along γ for fixed s. As in the aforementioned section, we have

$$\xi^{\alpha}{}_{;\beta}u^{\beta} = u^{\alpha}{}_{;\beta}\xi^{\beta} \tag{B.1}$$

We then carry out a computation, using many of the same tricks as employed in (7.7):

$$\begin{aligned}
\frac{D^2\xi^{\alpha}}{dt^2} &= (\xi^{\alpha}{}_{;\beta}u^{\beta})_{;\gamma}u^{\gamma} \\
&= (u^{\alpha}{}_{;\beta}\xi^{\beta})_{;\gamma}u^{\gamma} \\
&= u^{\alpha}{}_{;\beta\gamma}\xi^{\beta}u^{\gamma} + u^{\alpha}{}_{;\beta}\xi^{\beta}{}_{;\gamma}u^{\gamma} \\
&= u^{\alpha}{}_{;\gamma\beta}\xi^{\beta}u^{\gamma} - R^{\alpha}{}_{\mu\beta\gamma}u^{\mu}\xi^{\beta}u^{\gamma} + u^{\alpha}{}_{;\beta}u^{\beta}{}_{;\gamma}\xi^{\gamma} \\
&= (u^{\alpha}{}_{;\gamma}u^{\gamma})_{;\beta}\xi^{\beta} - u^{\alpha}{}_{;\gamma}u^{\gamma}{}_{;\beta}\xi^{\beta} - u^{\alpha}{}_{;\beta}u^{\beta}{}_{;\gamma}\xi^{\gamma} - R^{\alpha}{}_{\mu\beta\gamma}u^{\mu}\xi^{\beta}u^{\gamma} \\
&= -R^{\alpha}{}_{\beta\gamma\delta}u^{\beta}u^{\delta}\xi^{\gamma}
\end{aligned} \tag{B.2}$$

as desired.

Appendix C

Eigenvalue equations for tidal potentials

In this Appendix, we show that eq. (2.87) holds with $\mathcal{E}^{(l)}$ given by (2.84). Using the definitions, we get

$$\begin{aligned}
\gamma^{ij} D_i D_j \mathcal{E}^{(l)} &= \gamma^{ij} \gamma_i^p \gamma_j^q \partial_p [D_q \mathcal{E}^{(l)}] \\
&= \mathcal{E}^{(l)}_{k_1 \cdots k_l} \gamma^{pq} \partial_p [\gamma_q^m \partial_m (\Omega^{k_1} \cdots \Omega^{k_l})] \\
&= \mathcal{E}^{(l)}_{k_1 \cdots k_l} \gamma^{pq} [\partial_p \gamma_q^m \partial_m (\Omega^{k_1} \cdots \Omega^{k_l}) + \gamma_q^m \partial_p \partial_m (\Omega^{k_1} \cdots \Omega^{k_l})]
\end{aligned} \tag{C.1}$$

We note that

$$\partial_p \gamma_q^m = -\frac{1}{r} (\Omega_q \delta_p^m + \Omega^m \delta_{pq}) \tag{C.2}$$

and

$$\partial_m (\Omega^{k_1} \cdots \Omega^{k_l}) = \frac{1}{r} [\delta^{k_1} \Omega^{k_2} \cdots \Omega^{k_l} + \delta^{k_2} \Omega^{k_1} \Omega^{k_3} \cdots \Omega^{k_l} + \ldots + \delta_m^{k_l} \Omega^{k_1} \cdots \Omega^{k_{l-1}}] \tag{C.3}$$

Furthermore,

$$\begin{aligned}\partial_p\partial_m(\Omega^{k_1}\cdots\Omega^{k_l}) = \frac{1}{r^2}[&\delta_m^{k_1}\delta_p^{k_2}\Omega^{k_3}\cdots\Omega^{k_l}+\ldots+\delta_m^{k_1}\delta_p^{k_l}\Omega^{k_2}\cdots\Omega^{k_{l-1}} \\ &+\delta_m^{k_2}\delta_p^{k_1}\Omega^{k_3}\cdots\Omega^{k_l}+\ldots+\delta_m^{k_2}\delta_p^{k_l}\Omega^{k_1}\cdots\Omega^{k_{l-1}} \\ &\vdots \\ &+\delta_m^{k_l}\delta_p^{k_1}\Omega^{k_2}\cdots\Omega^{k_{l-1}}+\ldots+\delta_m^{k_l}\delta_p^{k_{l-1}}\Omega^{k_1}\Omega^{k_{l-2}}]\end{aligned} \tag{C.4}$$

From (2.75), it is clear that

$$\gamma_i^i = \gamma^{ij}\delta_{ij} = 2, \qquad \gamma_j^i\Omega^j = 0 \tag{C.5}$$

Collecting all the pieces, we compute the first term in (C.1):

$$\mathcal{E}^{(l)}_{k_1\cdots k_l}\gamma^{pq}\partial_p\gamma_q^m\partial_m(\Omega^{k_1}\cdots\Omega^{k_l}) = -\frac{l}{r^2}\mathcal{E}^l\gamma_{cp}\gamma_d^p\gamma^{cd} = -\frac{2l}{r^2}\mathcal{E}^{(l)} \tag{C.6}$$

The second term in (C.1) is computed as follows:

$$\begin{aligned}\mathcal{E}^{(l)}_{k_1\cdots k_l}\gamma^{pq}\gamma_q^m\partial_p\partial_m(\Omega^{k_1}\cdots\Omega^{k_l}) = \frac{1}{r^2}\mathcal{E}_{k_1\cdots k_l}[&\gamma^{k_1k_2}\Omega^{k_3}\cdots\Omega^{k_l}+\ldots+\gamma^{k_1k_l}\Omega^{k_2}\cdots\Omega^{k_{l-1}} \\ &+\gamma^{k_2k_1}\Omega^{k_3}\cdots\Omega^{k_l}+\ldots+\gamma^{k_2k_l}\Omega^{k_1}\Omega^{k_3}\cdots\Omega^{k_{l-1}} \\ &\vdots \\ &+\gamma^{k_lk_1}\Omega^{k_2}\cdots\Omega^{k_{l-1}}+\ldots+\gamma^{k_lk_{l-1}}\Omega^{k_1}\cdots\Omega^{k_{l-2}}]\end{aligned} \tag{C.7}$$

Owing to the tracelessness of $\mathcal{E}_{k_1\cdots k_l}$, we are justified in replacing $\gamma^{k_mk_n}$ with $-\Omega^{k_m}\Omega^{k_n}$ in the above, where $1\leq m,n\leq l$. Furthermore, the sum in brackets has a total of $l(l-1)$ terms, so we end up with

$$\mathcal{E}^{(l)}_{k_1\cdots k_l}\gamma^{pq}\gamma_q^m\partial_p\partial_m(\Omega^{k_1}\cdots\Omega^{k_l}) = -\frac{1}{r^2}l(l-1)\mathcal{E}^{(l)} \tag{C.8}$$

In conclusion,

$$\gamma^{ij}D_iD_j\mathcal{E}^{(l)} = -\frac{2l}{r^2}\mathcal{E}^{(l)} - \frac{1}{r^2}l(l-1)\mathcal{E}^{(l)} = -\frac{1}{r^2}l(l+1)\mathcal{E}^{(l)} \tag{C.9}$$

which yields the desired result. Equations (2.88) and (2.89) are shown similarly.

Appendix D

Killing tensors and conserved quantities

In this appendix, we review the concept of Killing tensors and their associated conserved quantities. We start by reviewing Killing vectors, closely following chapter 8.2 of [8].

Let $(\mathscr{M}, g)$ be a Lorentzian manifold and let $\gamma : [a, b] \to \mathscr{M}$ be a timelike geodesic. We will work in a neighborhood $\mathscr{O} \subseteq \mathscr{M}$ with a coordinate system $x^\mu, \mu \in \{0, 1, 2, 3\}$ and we denote by t the parameter of γ. The action for γ is then given as in (5.1). For the purposes of varying this action, one usually uses a slightly different Lagrangian than that introduced in chapter 5.1, namely

$$\mathcal{L}(t, q, u) = -m\sqrt{-g_{\mu\nu}(q)u^\mu u^\nu} \tag{D.1}$$

Of course the two produce equivalent Euler-Lagrange equations [3], which read

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial u^\mu}(t, \gamma(t), u(t))\right) - \frac{\partial \mathcal{L}}{\partial q^\mu}(t, \gamma(t), u(t)) = 0 \tag{D.2}$$

If g is independent of x^a for some particular $a \in \{0, 1, 2, 3\}$, then we have a corresponding Killing vector, ξ with components $\xi^\mu = \delta^\mu_a$. We then compute

$$\frac{\partial \mathcal{L}}{\partial u^a}(t, \gamma(t), u(t)) = \frac{m}{\sqrt{-g_{\mu\nu}(\gamma(t))u^\mu(t)u^\nu(t)}}\xi(\gamma(t)) \cdot u(t) \tag{D.3}$$

In conclusion, we see that $\xi \cdot u$ is conserved along γ.

As Carter discovered [5], this procedure doesn't produce all the conserved quantities for a given physical setup. Indeed, Carter's constant doesn't have a corresponding Killing vector. Instead it has a corresponding Killing tensor field. Let Q be a symmetric tensor field of rank k on $\mathscr{M}$ whose components with respect to x^μ are denoted $Q_{\mu_1\cdots\mu_k}$. Then Q is called a Killing tensor field if

$$D_{(\nu}Q_{\mu_1\cdots\mu_k)} = 0 \tag{D.4}$$

thus generalising the Killing equation to symmetric tensor fields [18]. Suppose $\mathscr{M}$ admits a Killing tensor field and suppose u is the four velocity on γ. Then the scalar

$$K := Q_{\mu_1\cdots\mu_k}u^{\mu_1}\cdots u^{\mu_k} \tag{D.5}$$

is conserved along γ. Indeed,

$$\begin{aligned}\frac{D}{d\tau}K &= \frac{D}{d\tau}(Q_{\mu_1\cdots\mu_k}u^{\mu_1}\cdots u^{\mu_k})\\ &= u^{\mu_1}\cdots u^{\mu_k}u^\nu D_\nu Q_{\mu_1\cdots\mu_k}\\ &= u^{\mu_1}\cdots u^{\mu_k}u^\nu D_{(\nu}Q_{\mu_1\cdots\mu_k)}\\ &= 0\end{aligned} \tag{D.6}$$

In the third equality, we simply used that $u^{\mu_1}\cdots u^{\mu_k}u^\nu$ is symmetric in all indices and so it picks out the symmetric part of $D_\nu Q_{\mu_1\cdots\mu_k}$. Finally, we made use of (D.4).

The Schwarzschild spacetime admits the following Killing tensor field (see page 321 of [18]):

$$Q_{\mu\nu} = 2r^2 l_{(\mu}n_{\nu)} + r^2 g_{\mu\nu} \tag{D.7}$$

where l and n are two null vectors given by

$$l^\mu = \frac{1}{1-\frac{2m}{r}}\delta^{\mu 0} + \delta^{\mu 1} \tag{D.8}$$

$$n^\mu = \frac{1}{2}\delta^{\mu 0} - \frac{1}{2}\left(1-\frac{2m}{r}\right)\delta^{\mu 1} \tag{D.9}$$

The non-vanishing components of Q come out to be

$$Q_{22} = r^4, \quad Q_{33} = r^4 \sin^2\theta \tag{D.10}$$

Hence, the corresponding conserved quantity is

$$K = r^4\dot{\theta}^2 + r^4 \sin^2\theta\dot{\phi}^2 \tag{D.11}$$

This exactly reproduces the expression given in (3.15).

Appendix E

Killing-Yano tensors

In this appendix, we review Killing-Yano tensors and their connection to Killing tensors, closely following [6].

Let $(\mathscr{M}, g)$ be a Lorentzian manifold and let $\mathscr{O} \subseteq \mathscr{M}$ be a neighborhood with coordinates x^μ. Let f be a totally antisymmetric tensor field of rank k on $\mathscr{M}$ whose components with respect to x^μ are denoted $f_{\mu_1 \cdots \mu_k}$. Then f is called a Killing-Yano tensor field if

$$D_{(\nu} f_{\mu_1)\mu_2 \cdots \mu_k} = 0 \tag{E.1}$$

Killing-Yano tensors are related to Killing tensors through the following proposition: If f is a Killing-Yano tensor of rank k, then

$$Q_{\alpha\beta} := f_{\alpha\mu_2\cdots\mu_k} f_\beta{}^{\mu_2\cdots\mu_k} \tag{E.2}$$

is a Killing tensor of rank 2. That $Q_{\alpha\beta}$ is symmetric in α and β is obvious. Using this symmetry, we then compute

$$\begin{aligned}
3D_{(\sigma} Q_{\alpha\beta)} &= D_\sigma Q_{\alpha\beta} + D_\alpha Q_{\beta\sigma} + D_\beta Q_{\sigma\alpha} \\
&= f_\beta{}^{\mu_2\cdots\mu_k} D_\sigma f_{\alpha\mu_2\cdots\mu_k} + f_{\alpha\mu_2\cdots\mu_k} D_\sigma f_\beta{}^{\mu_2\cdots\mu_k} \\
&+ f_\sigma{}^{\mu_2\cdots\mu_k} D_\alpha f_{\beta\mu_2\cdots\mu_k} + f_{\beta\mu_2\cdots\mu_k} D_\alpha f_\sigma{}^{\mu_2\cdots\mu_k} \\
&+ f_\alpha{}^{\mu_2\cdots\mu_k} D_\beta f_{\sigma\mu_2\cdots\mu_k} + f_{\sigma\mu_2\cdots\mu_k} D_\beta f_\alpha{}^{\mu_2\cdots\mu_k}
\end{aligned}$$

Using (E.1), we see that the terms cancel pairwise and hence, $D_{(\sigma} Q_{\alpha\beta)} = 0$. We conclude that given a Killing-Yano tensor on a spacetime, one can always construct a corresponding

Killing tensor. The converse is not true in general. However, we note that the Schwarzschild solution admits a Killing-Yano tensor whose only non-vanishing components are $f_{23} = -f_{32} = r^3 \sin\theta$ [10]. Then $f_{\sigma\mu} f^{\sigma}{}_{\nu}$ exactly yields the Killing tensor in (D.10).

Appendix F

Determinant of the horizon metric

In this appendix, we compute the (square root of the) determinant γ of the horizon metric, following appendix B of [15]. First, write

$$\gamma_{AB} = 4m^2\Omega_{AB} + p_{AB} \tag{F.1}$$

where

$$p_{AB} = -\frac{8}{3}m^4(\mathcal{E}^{\rm q}_{AB} + \mathcal{B}^{\rm q}_{AB}) - \frac{8}{15}m^5(\mathcal{E}^{\rm o}_{AB} + \mathcal{B}^{\rm o}_{AB}) \tag{F.2}$$

To minimize cluttering, let Ω_M and p_M be the matrix representations, in some basis, of $4m^2\Omega_{AB}$ and p_{AB} respectively. Then,

$$\begin{aligned}
\sqrt{\gamma} &= \sqrt{\det(\Omega_M + p_M)} \\
&= \sqrt{\det(\Omega_M)}\sqrt{\det\big(\mathbb{1} + \Omega_M^{-1}p_M\big)} \\
&= \sqrt{\det(\Omega_M)}\exp\left[\frac{1}{2}\ln\det\big(\mathbb{1} + \Omega_M^{-1}p_M\big)\right] \\
&= \sqrt{\det(\Omega_M)}\exp\left[\frac{1}{2}\operatorname{Tr}\ln\big(\mathbb{1} + \Omega_M^{-1}p_M\big)\right] \\
&= \sqrt{\det(\Omega_M)}\exp\left[\frac{1}{2}\operatorname{Tr}\big(\Omega_M^{-1}p_M\big) + \mathcal{O}\left(\frac{1}{\mathcal{R}^4}\right)\right] \\
&= \sqrt{\det(\Omega_M)}\left[1 + \frac{1}{2}\operatorname{Tr}\big(\Omega_M^{-1}p_M\big) + \mathcal{O}\left(\frac{1}{\mathcal{R}^4}\right)\right] \\
&= 4m^2\sin\theta\left[1 + \mathcal{O}\left(\frac{1}{\mathcal{R}^4}\right)\right]
\end{aligned} \tag{F.3}$$

since, by construction of the tidal potentials,

$$\mathrm{Tr}\big(\Omega_M^{-1} p_M\big) = \Omega^{AB} p_{AB} = 0 \tag{F.4}$$

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