UNIVERSITY OF COPENHAGEN
FACULTY OF SCIENCE

Master's Thesis

Taro V. Brown

Massive Exponential BCJ Numerators from CHY

Niels Bohr International Academy

Advisor: N.E.J. Bjerrum-Bohr

Handed in: 7th August 2020

Abstract

In this thesis, an exponential form of DDM-basis BCJ master numerators is obtained through CHY. The numerators are dimensionally agnostic and can be represented by half-ladder diagrams. They are crossing symmetric in $n-2$ of the external particles and the two remaining can have any spin $s = 0, 1/2, 1$ and be massive $k_1^2 = k_n^2 = m^2$. We also present a diagrammatic algorithm, to obtain the numerators, which improves the efficiency of the state of the art. Using the exponential form of the numerators we calculate the three-point amplitude for two massive spin s particles and a graviton $\mathcal{M}(\mathbf{1}^s, 2^{s=2}, \mathbf{3}^s)$. The expression obtained matches recent work on minimally-coupled massive particles and the results here could provide an avenue to obtain general spin results at higher n, since double-copy constructions are very natural in the CHY formalism.

Acknowledgments

There are a great many people I would like to thank and without whom this thesis wouldn't have been possible. First of all, my supervisor N.E.J. Bjerrum-Bohr who has taught me an incredible amount about scattering amplitudes and about the physics field as a whole and who has supported me in every step of my PhD application. You have been a great inspiration. Furthermore Humberto Gomez has been an incredible partner in every calculation and has been kind to answer question after question. I look forward to continue our work.

My fiancée Frederikke has also been incredible throughout this whole process and was especially fantastic when the corona-crises was at it's peak in Denmark. Her patience knows no bounds and she has supported me through long nights of calculation and was the one who suggested that I pursue a PhD in the United States. A big thanks to my father-in-law as well, since his house provided me with an office during the 3 months we were in lock down.

I want to thank Johannes Agerskov for pointing me towards the amplitudes field and in particular towards Emil. You have been a great inspiration for me throughout my studies and without the Mathematica program you provided I would have been set back months.

Finally I want to thank Fei Teng for a discussion about BCJ numerators which has formed a large part of my thesis.

Contents

Chapter 1

Introduction

With powerful colliders such as the LHC producing increasingly precise data on particle scattering, theorist are faced with an intriguing proposition: Should we push our current calculational methods to its limits to obtain ever more accurate theoretical predictions, or should we instead focus on developing new techniques and hope that they help us make a computational leap that would have been unattainable using the old methods?

One of the first hints that the standard Feynman diagram approach used for over half a century is not the most efficient way to calculate amplitudes was provided by Parke and Taylor [1] through Maximally Helicity-Violating (MHV) amplitudes. Using the spinor helicity formalism these amplitudes can be written in one line even when considering a large number of particles, n, in the scattering process. The result is astonishing when considering that for just $n = 7$ one would have to draw 154 (planar) Feynman diagrams and go through a very large number of calculations. Feynman diagrams still present a great amount of upside, e.g. by being very intuitive in their construction and they have an enormous amount of applications to this day, but the prospect of developing new methods is fascinating.

Since the detection of gravitational waves by the LIGO collaboration in 2016 [2] a part of the amplitudes community has raced to produce theoretical predictions for gravitational scattering of massive objects using modern amplitude methods, considering for instance the interaction between two massive spinning black holes [3–9].

One framework which has not been used much yet in this application was developed by Cachazo, He and Yuan (CHY) [10–15] using the *Scattering Equations* with generalization to massive particles being proposed in [16–19] as well as loop-level constructions in [20–22]. The formalism is very double-copy friendly so producing tree level gravitational amplitudes can be done easily. An example of this is the generalization of the KLT relations [23–27] in the CHY integrand [12]. Using this natural double-copy construction the scattering amplitudes for massive scalar particles interacting gravitationally have been studied using CHY [18, 19, 28].

Until recently however, the formalism has been constrained to integer spin particles and so a general spin result has not been obtainable. Calculations including fermions have been performed using ambi-twistor strings, however. The CHY formalism's relation to string theory has been explored by many, see for instance [29], and was recently exploited by [30,31] to introduce a way to incorporate fermions. The spin 1/2 particles are included by modifying an algorithm developed in a recent series of papers [32,33] which constructs BCJ master numerators in the Del-Duca, Dixon and Maltoni (DDM) basis [34]. These numerators can then be used in the CHY formalism to compute amplitudes with external fermions.

In this thesis we rewrite the half-ladder BCJ numerators by combining terms into an exponential. This exponential representation is very compact and by providing an algorithm to compute the numerators we take the computational efficiency from $\mathcal{O}[(n-1)!]$ [31] to $\mathcal{O}[(n-2)!]$. The numerators can be used to build amplitudes with 2 massive external particles and spin $s = 0, 1/2, 1$ while the remaining $n-2$ particles are massless and have spin 1. We provide various direct checks for the $s = 1/2$ case using massive Dirac spinors, and since we do not use the $10d$ Majorana-Weyl spinors or the dimensional reduction procedures of [31], the results in this thesis provide the first direct calculations of amplitudes with Dirac spinors in the CHY formulation. Furthermore the general spin three-point gravitational amplitude $\mathcal{M}(\mathbf{1}^s, 2^{s=2}, \mathbf{3}^s)$ obtained using the numerators matches the minimally coupled amplitude obtained in [35].

Thesis outline

The structure of this thesis will be as follows. First, we introduce the notion of color-ordered Feynman diagrams in chapter 2. This is then followed by chapter 3 on both massless and massive spinor-helicity. We then introduce the CHY formalism in chapter 4 and extend it to include massive particles in chapter 5 while calculating examples along the way. In chapter 6 we then discuss the color/kinematics duality and describe how to obtain half-ladder BCJ numerators using the algorithm of [32,33]. In the final stages of this chapter we introduce the massless fermion numerators and in chapter 7 we extend them to include masses and check their validity, finding agreement with the literature. Finally in chapter 8 we introduce the exponential numerators and provide the algorithm that computes them.

Chapter 2

Scattering amplitudes and Feynman diagrams

In this chapter, we present the subject of scattering amplitudes and review how these are calculated using traditional Feynman diagram methods. This will serve as a gateway into new amplitude methods such as the spinor helicity formalism and the approach developed by Cachazo, He, and Yuan (CHY). Furthermore we present the basics of color ordered amplitudes, which will become pivotal later in this thesis. The material covered in this chapter can be found in a plethora of textbooks and where no references are provided we refer to the great texts by H. Elvang and Y. Huang, M. Srednicki, L. Dixon, and M. Schwartz [36–39].

2.1 Scattering amplitude and color-ordering

When looking at particle collisions, be it at the LHC in collider experiments or analyzing binary black hole systems by considering them as fundamental particles, the object used to calculate the quantum mechanical probability for a specific process is called the differential cross section. While many other types computations and contributions have to be made before one could analyze the outcome of e.g. collisions at the LHC, with especially computer algorithms doing the brunt of the work, theorists focus on the scattering amplitude from which the scattering cross-section can be computed.

As a simple example we could look at a scattering process $1 + 2 \to 3 + \cdots + n$ with all momenta treated as outgoing and momentum conservation dictating $\sum_i p_i = 0$. This scattering process can be described by evolving the initial (asymptotic) state evaluated at $t_i = -\infty$ and computing the overlap with the final (asymptotic) state at $t_f = \infty$. The evolution operator is referred to as the S-Matrix and it can be further deconstructed into a non-interaction part $\mathbb{1}$ and an interaction part T.

$$S = \mathbb{1} + iT. \tag{2.1.1}$$

The scattering amplitude is calculated using the interacting part. Denoting the final state by ψ_f and the initial state by ψ_i

$$\langle \psi_f | T | \psi_i \rangle \equiv (2\pi)^4 \delta^{(4)} \Big(\sum_{i=1} p_i \Big) i\mathcal{A}_n, \tag{2.1.2}$$

where the 3+1 dimensional δ-function serves to impose the momentum conservation constraint.

Calculating scattering amplitudes in quantum field theories cannot be done exactly so traditionally one uses Feynman diagrams to calculate contributions order by order and sum over all contributions. The leading order is called tree level, since there are no self-interactions and these would be drawn using loops. In this thesis the main objective will be tree-level amplitudes and furthering techniques to calculate them, especially focusing on going beyond standard Feynman diagrams. In Yang-Mills theories one further has to consider a degree of freedom, color. The color gauge group is $\mathrm{SU}(N_c)$ and the generators of $\mathrm{SU}(N_c)$ in the fundamental representation are $N_c \times N_c$ matrices T^a obeying

$$\begin{aligned} \mathrm{Tr}\Big(T^a T^b\Big) &= \delta^{ab} \\ [T^a, T^b] &= i\sqrt{2} f^{abc} T^c, \end{aligned} \tag{2.1.3}$$

with $a = 1 \dots N_c^2 - 1$, and f^{abc} are the structure constants which we can obtain from the generators

$$f^{abc} = -\frac{i}{\sqrt{2}} \mathrm{Tr}\Big[T^a, [T^b, T^c]\Big]. \tag{2.1.4}$$

When converting Feynman diagrams to mathematical expressions, combinations of these color factors show up at every vertex and so the color degree of freedom seems to add a lot of complexity to the calculations. Luckily the full scattering amplitude can be completely decomposed into color- and kinematic dependent parts[1]

$$\mathcal{A}_n^{\mathrm{tree}} = \sum_{\sigma \in S_n / \mathbb{Z}_n} \Big(\mathrm{Tr}[T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}] A_n^{\mathrm{tree}}(\sigma(1), \dots, \sigma(n)) \Big), \tag{2.1.5}$$

where the A_n's are gauge invariant objects known as *partial* or *color ordered amplitudes*. This decomposition helps our search for simplicity by greatly reducing the amount of work required to calculate the full amplitude since most of the work can be focused on

[1]Note that we have suppressed powers of the coupling here. We will do this in the remainder of the thesis.

obtaining color-ordered amplitudes, for which the Feynman rules are easier. Similarly one can show that at leading order, only the planar graphs contribute, further simplifying the calculations. For convenience we note that the color ordered amplitudes obey the following properties

- Cyclic:
$$A_n(1,2,\ldots,n) = A_n(2,\ldots,n,1). \tag{2.1.6}$$

- Reflection:
$$A_n(1,2,\ldots,n) = (-1^n)A_n(n,\ldots,2,1). \tag{2.1.7}$$

- Photon decoupling identity:
$$0 = A_n(1,2,3,\ldots,n)+A_n(2,1,3,\ldots,n)+A_n(2,3,1,\ldots,n)+\cdots+A_n(2,3,\ldots,1,n). \tag{2.1.8}$$

- Kleiss-Kuijf (KK) relations
$$A_n(1,\{\alpha\},n,\{\beta\}) = (-1)^{|\beta|} \sum_{\sigma\in\ \mathrm{OP}(\{\alpha\},\{\beta^T\})} A_n(1,\sigma,n), \tag{2.1.9}$$
where OP denotes ordered permutations, meaning permutations of the joined set $\{\alpha\}\cup\{\beta\}^T$, which preserves the original individual orderings of $\{\alpha\}$ and $\{\beta\}^T$.

- Linearized gauge invariance
$$A_n(k,\epsilon)\Big|_{\epsilon_i\to ak_i} = 0, \tag{2.1.10}$$
with a an arbitrary constant.

We can also define color factors [40]

$$c_{\{1,\sigma,n\}} \equiv \left(f^{a_1 a_{\sigma(2)} b_1} f^{b_1 a_{\sigma(3)} b_2} \cdots f^{b_{n-3} a_{\sigma(n-2)} a_n}\right), \tag{2.1.11}$$

which obey Jacobi relations because of the their group theoretical structure. Taking e.g. $n=4$ we have [41]

$$c_s + c_t + c_u \sim (f^{12a}f^{a34} + f^{23a}f^{a14} + f^{31a}f^{a24}) = 0. \tag{2.1.12}$$

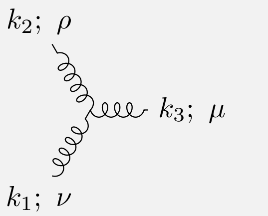
Del Duca, Dixon, and Maltoni (DDM) [34] showed that the KK-relations and the Jacobi identities reduce the basis necessary for the color-ordered amplitudes, so that (2.1.5) can be expressed as

$$\mathcal{A}_n = \sum_{\sigma\in S_{n-2}} f^{a_1 a_{\sigma(2)} b_1} f^{b_1 a_{\sigma(2)} b_2} \cdots f^{b_{n-3} a_{\sigma(n-2)} a_n} A_n(1,\sigma,n). \tag{2.1.13}$$

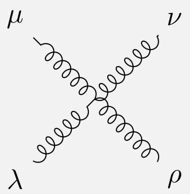
In section 6 this basis is going to be important when calculating *BCJ numerators*. We will now use the color ordered Feynman rules to calculate an example of four-point scattering.

2.2 Example in pure Yang-Mills

First we are going to need the color ordered Feynman rules for gluons. We will work in Feynman gauge, take all momenta to be outgoing and use the rules stated in [38]



$$= \frac{i}{\sqrt{2}} \left\{ \eta_{\nu\rho}(k_1-k_2)_\mu + \eta_{\rho\mu}(k_2-k_3)_\nu + \eta_{\mu\nu}(k_3-k_1)_\rho \right\},$$



$$= i\eta_{\mu\rho}\eta_{\nu\lambda} - \frac{i}{2}\left(\eta_{\mu\nu}\eta_{\rho\lambda} + \eta_{\mu\lambda}\eta_{\nu\rho}\right),$$

$$= -i\frac{\eta^{\mu\nu}}{k^2}.$$

The diagrams contributing to $A(1,2,3,4)$ at lowest order are shown in Figure 2.1 with the contribution to the amplitude for each individual diagram obtained by gluing together appropriate vertex factors and propagators and then contracting with the polarization vectors of the external particles.

For the s-channel diagram the contribution to the amplitude is

$$\begin{aligned} iA_s =& \frac{i}{\sqrt{2}} \left\{ (\epsilon_1\cdot\epsilon_2)(k_1-k_2)_\alpha + \epsilon_{2,\alpha}(\epsilon_1\cdot(k_2-k)) + \epsilon_{1,\alpha}(\epsilon_2\cdot(k-k_1)) \right\} \\ & \times \left(-i\frac{\eta^{\alpha\beta}}{s_{12}}\right) \\ & \times \frac{i}{\sqrt{2}} \left\{ (\epsilon_3\cdot\epsilon_4)(k_3-k_4)_\beta + \epsilon_{4,\beta}(\epsilon_3\cdot(k_4+k)) + \epsilon_{3,\beta}(\epsilon_4\cdot(-k-k_3)) \right\}. \end{aligned}$$

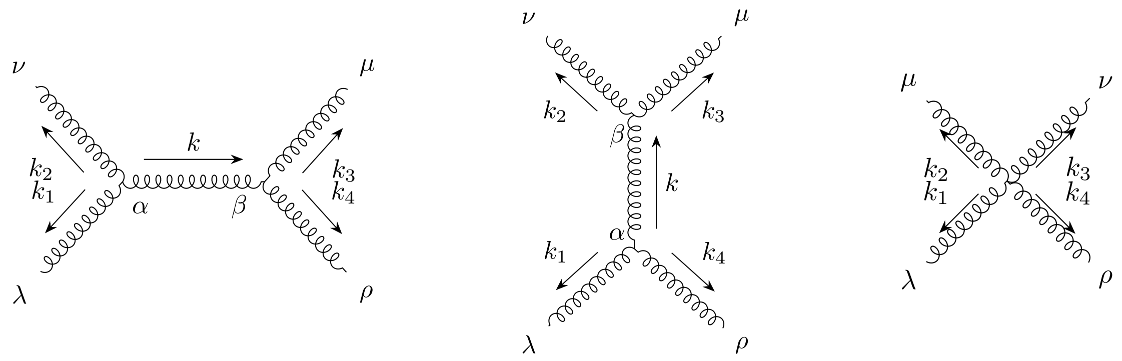


Figure 2.1: s and t channels plus the contact diagram

Which we can simplify slightly

$$\begin{aligned} iA_s =& \frac{i}{2s_{12}} \Big\{ (\epsilon_1 \cdot \epsilon_2)(k_1 - k_2)^\beta + \epsilon_2^\beta(\epsilon_1 \cdot (k_2 - k)) + \epsilon_1^\beta(\epsilon_2 \cdot (k - k_1)) \Big\} \\ & \times \{ (\epsilon_3 \cdot \epsilon_4)(k_3 - k_4)_\beta + \epsilon_{4,\beta}(\epsilon_3 \cdot (k_4 + k)) + \epsilon_{3,\beta}(\epsilon_4 \cdot (-k - k_3)) \} \\ =& \frac{i}{2s_{12}} \Big\{ (\epsilon_1 \cdot \epsilon_2)(k_1 - k_2)^\beta + 2\epsilon_2^\beta(\epsilon_1 \cdot k_2) - 2\epsilon_1^\beta(\epsilon_2 \cdot k_1) \Big\} \\ & \times \{ (\epsilon_3 \cdot \epsilon_4)(k_3 - k_4)_\beta + 2\epsilon_{4,\beta}(\epsilon_3 \cdot k_4) - 2\epsilon_{3,\beta}(\epsilon_4 \cdot k_3) \} \,. \end{aligned} \tag{2.2.1}$$

where we have used transversality $\epsilon_i \cdot k_i = 0$ and momentum conservation at the vertices in the following way

$$k_1 + k_2 + k = 0 \Rightarrow \begin{cases} k_2 - k & = 2k_2 + k_1, \\ k - k_1 & = -2k_1 - k_2. \end{cases} ,$$

$$k_3 + k_4 - k = 0 \Rightarrow \begin{cases} k_4 + k & = 2k_4 + k_3, \\ -k - k_3 & = -2k_3 - k_4. \end{cases} .$$

Similarly we can construct the s-channel amplitude by taking $2 \leftrightarrow 4$

$$\begin{aligned} iA_t =& \frac{i}{2s_{14}} \Big\{ (\epsilon_1 \cdot \epsilon_4)(k_1 - k_4)^\beta + 2\epsilon_4^\beta(\epsilon_1 \cdot k_4) - 2\epsilon_1^\beta(\epsilon_4 \cdot k_1) \Big\} \\ & \times \{ (\epsilon_3 \cdot \epsilon_2)(k_3 - k_2)_\beta + 2\epsilon_{2,\beta}(\epsilon_3 \cdot k_2) - 2\epsilon_{2,\beta}(\epsilon_2 \cdot k_3) \} \,. \end{aligned} \tag{2.2.2}$$

Finally the contact diagram is

$$iA_c = i \left((\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot \epsilon_4) - \frac{1}{2}(\epsilon_1 \cdot \epsilon_4)(\epsilon_2 \cdot \epsilon_3) - \frac{1}{2}(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4) \right) . \tag{2.2.3}$$

We will not perform further simplifications in this section, but just note that the color-ordered amplitude then can be calculated through $A = A_s + A_t + A_c$. At this point we want to emphasize that even though the simplifications made here were somewhat straight forward it quickly gets more complicated as the number of particles increases [36]:

$$\begin{aligned}
&g+g \to g+g && 3 \text{ diagrams}\\
&g+g \to g+g+g && 10 \text{ diagrams}\\
&g+g \to g+g+g+g && 38 \text{ diagrams}\\
&g+g \to g+g+g+g+g && 154 \text{ diagrams.}
\end{aligned}$$

Similarly the simplifications of the expressions become increasingly tedious. In the following chapters we will proceed to look at methods to remedy these problems such as spinor helicity and CHY.

Chapter 3

Spinor-Helicity

In the previous chapter we saw the capabilities of the classic Feynman diagram approach to calculating scattering amplitudes. The expressions obtained are often large and unwieldy and the spinor helicity (SH) approach does away with some of these limitations by choosing a specific (helicity) basis for the amplitudes. This will enable us to efficiently get results for a number of scattering cases and will in particular be useful later on in the thesis for doing numerical checks. First we will go through conventional massless SH methods as presented in [36, 38] and then extend this to massive particles following the approaches described in [7, 9, 42].

3.1 Massless

The usual approach to SH is by looking at the massless Dirac equation $i\partial\!\!\!/\Psi = 0$ and to then solve this using a plane wave expansion of Ψ, see for instance [36]. We will use a different approach here and instead only assume four dimensions, on-shellness and momentum conservation i.e.

$$d = 4, \qquad p_i^\mu p_{i\mu} = 0, \qquad \sum_i p_i = 0, \tag{3.1.1}$$

This is the method used by [42]. We can then go on to define the Pauli matrices

$$\begin{aligned} \sigma^\mu_{\alpha\dot\beta} &= (\mathbb{1}_{\alpha\dot\beta}, \sigma^1_{\alpha\dot\beta}, \sigma^2_{\alpha\dot\beta}, \sigma^3_{\alpha\dot\beta}), \\ (\bar\sigma^\mu)^{\dot\alpha\beta} &= (\mathbb{1}^{\dot\alpha\beta}, -(\sigma^1)^{\dot\alpha\beta}, -(\sigma^2)^{\dot\alpha\beta}, -(\sigma^3)^{\dot\alpha\beta}), \end{aligned} \tag{3.1.2}$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1.3}$$

The spinor indices α and $\dot{\beta}$ can be raised and lowered using the two-dimensional Levi-Civita symbol $\epsilon_{\alpha\beta}$ satisfying $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$. We can use this to define momentum bi-spinors from the four-momenta p_μ

$$p_{\alpha\dot{\beta}} \equiv p_\mu \sigma^\mu_{\alpha\dot{\beta}}, \qquad p^{\dot{\alpha}\beta} \equiv p_\mu (\sigma^\mu)^{\dot{\alpha}\beta}. \tag{3.1.4}$$

Here it can be constructive to actually write out e.g. $p_{\alpha\dot{\beta}}$ as a matrix

$$\begin{aligned} p_{\alpha\dot{\beta}} =& p_0 \mathbb{1}_{\alpha\dot{\beta}} - p_1\sigma^1_{\alpha\dot{\beta}} - p_2\sigma^2_{\alpha\dot{\beta}} - p_3\sigma^3_{\alpha\dot{\beta}} \\ =& \begin{pmatrix} p_0 - p_3 & -p_1 + ip_2 \\ -p_1 - ip_2 & p_0 + p_3 \end{pmatrix}. \end{aligned}$$

We are also going to work with slashed momenta, so let us define them in terms of the gamma matrices

$$\not{p} = p_\mu \gamma^\mu = p_\mu \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot{\beta}} \\ (\sigma^\mu)^{\dot{\alpha}\beta} & 0 \end{pmatrix} = \begin{pmatrix} 0 & p_{\alpha\dot{\beta}} \\ p^{\dot{\alpha}\beta} & 0 \end{pmatrix}. \tag{3.1.5}$$

In the massless case, the on-shell condition means that the degenerate momentum bi-spinor matrix can be decomposed as an outer product of two Weyl spinors. This is just a standard linear algebra statement known as rank factorization, which goes as follows: an $m \times n$ matrix M of rank r can be decomposed into $M = AB$ where A is an $m \times r$ matrix and B is an $r \times n$ matrix.

Example

Take the 2×2 matrix

$$M = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}.$$

It has $\det(M) = 0$ and $\operatorname{rank}(M) = 1$ so we should be able to deconstruct it into an outer product of two vectors a and b. This can be done by taking $a = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ since

$$a \otimes b = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} = M.$$

Again, since taking the determinant of the momentum bispinors gives

$$\begin{aligned}\det p_{\alpha\dot\beta} &= \left|\begin{pmatrix} p_0 - p_3 & -p_1 + ip_2 \\ -p_1 - ip_2 & p_0 + p_3 \end{pmatrix}\right| \\ &= (p_0 - p_3)(p_0 + p_3) + (p_1 + ip_2)(-p_1 + ip_2) \\ &= p_0^2 - p_1^2 - p_2^2 - p_3^2 = p^2 = m^2. \end{aligned} \tag{3.1.6}$$

$p_{\alpha\dot\beta}$ has rank 1 and we can decompose it into an outer product of the two Weyl spinors λ_a and $\tilde\lambda_{\dot\beta}$[1]: Furthermore we can also define a bra-ket notation for the Weyl spinors.

$$\begin{aligned} p_{\alpha\dot\beta} &= \lambda_\alpha \tilde\lambda_{\dot\beta} \equiv |p\rangle_\alpha \, [p|_{\dot\beta}, \\ p^{\dot\alpha\beta} &= \tilde\lambda^{\dot\alpha}\lambda^\beta \equiv |p]^{\dot\alpha} \, \langle p|^\beta \, . \end{aligned} \tag{3.1.7}$$

To make things a little clearer we can again write out the first line explicitly

$$\begin{pmatrix} p_0 - p_3 & -p_1 + ip_2 \\ -p_1 - ip_2 & p_0 + p_3 \end{pmatrix} = \begin{pmatrix} \lambda_{p1}\tilde\lambda_{p1} & \lambda_{p1}\tilde\lambda_{p2} \\ \lambda_{p2}\tilde\lambda_{p1} & \lambda_{p2}\tilde\lambda_{p2} \end{pmatrix}.$$

This can then in turn be solved to give explicit representations of the massless spinors. In numerical calculation it is easy to first create massless four-momenta satisfying momentum conservation and then from there calculate the spinors

$$\lambda_\alpha = \begin{pmatrix} \sqrt{p^0 + p^3} \\ \frac{p^1 + ip^2}{\sqrt{p^0 + p^3}} \end{pmatrix}, \qquad \tilde\lambda_{\dot\alpha} = \begin{pmatrix} \sqrt{p^0 + p^3} \\ \frac{p^1 - ip^2}{\sqrt{p^0 + p^3}} \end{pmatrix}. \tag{3.1.8}$$

We can use the bra-ket notation to define spinor (bracket) products $\langle kp\rangle \equiv \langle k|^\alpha|p\rangle_\alpha$ and $[kp] \equiv [k|_{\dot\alpha}|p]^{\dot\alpha}$. From eq (3.1.7) we can see that applying the Levi-Civita symbol on a ket turns it into a bra $\epsilon^{\alpha\beta}\,|p\rangle_\beta = \epsilon^{\alpha\beta}\lambda_\beta = \lambda^\alpha = \langle p|^\alpha$. The reverse applies to the kets as well and this implies antisymmetri of the bracket products e.g.

$$\langle kp\rangle = \langle k|^\alpha|p\rangle_\alpha = \epsilon^{\alpha\beta}\,|k\rangle_\beta\,|p\rangle_\alpha = -\epsilon^{\beta\alpha}\,|k\rangle_\beta\,|p\rangle_\alpha = -\langle p|^\beta|k\rangle_\beta = -\langle pk\rangle\,. \tag{3.1.9}$$

Let's summarize

[1] We have suppressed an index refering to the momentum p for clarity

$$\epsilon^{\alpha\beta}\,|p\rangle_\beta = \langle p|^\alpha\,, \qquad \epsilon_{\alpha\beta}\,\langle p|^\beta = |p\rangle_\alpha\,,$$
$$\epsilon^{\dot\alpha\dot\beta}[p|_{\dot\beta} = |p]^{\dot\alpha}, \qquad \epsilon_{\dot\alpha\dot\beta}|p]^{\dot\beta} = [p|_{\dot\alpha}, \tag{3.1.10}$$
$$\left.\begin{aligned}\langle kp\rangle &= -\langle pk\rangle\\ [kp] &= -[pk]\end{aligned}\right\} \Rightarrow \langle pp\rangle = [pp] = 0.$$

The above identities can then be used to see what happens when we act on the spinors $|p\rangle_\alpha$, $|p]^{\dot\alpha}$ with the momentum bispinor

$$\begin{aligned} p^{\dot\alpha\beta}\,|p\rangle_\beta &= |p]^{\dot\alpha}\,\langle p|^\beta\,|p\rangle_\beta = 0, \qquad & [p|_{\dot\alpha}p^{\dot\alpha\beta} &= [p|_{\dot\alpha}|p]^{\dot\alpha}\,\langle p|^\beta = 0,\\ \langle p|^\alpha\,p_{\alpha\dot\beta} &= \langle p|^\alpha\,|p\rangle_\alpha\,[p|_{\dot\beta} = 0, & p_{\alpha\dot\beta}|p]^{\dot\beta} &= a\,|p\rangle_\alpha\,[p|_{\dot\beta}|p]^{\dot\beta} = 0. \end{aligned} \tag{3.1.11}$$

since $\not{p}$ was expressed purely in terms of the momentum bispinors the identities correspond to

$$\begin{pmatrix} 0 & p_{\alpha\dot\beta}\\ p^{\dot\alpha\beta} & 0\end{pmatrix}\begin{pmatrix}|p]^{\dot\beta}\\ 0\end{pmatrix} = \not{p}v_+(p) = 0, \qquad \begin{pmatrix}\langle p|^\alpha & 0\end{pmatrix}\begin{pmatrix} 0 & p_{\alpha\dot\beta}\\ p^{\dot\alpha\beta} & 0\end{pmatrix} = \bar u_+(p)\not{p} = 0, \tag{3.1.12}$$

$$\begin{pmatrix} 0 & p_{\alpha\dot\beta}\\ p^{\dot\alpha\beta} & 0\end{pmatrix}\begin{pmatrix}0\\ |p\rangle_\beta\end{pmatrix} = \not{p}v_-(p) = 0, \qquad \begin{pmatrix}0 & [p|_{\dot\alpha}\end{pmatrix}\begin{pmatrix} 0 & p_{\alpha\dot\beta}\\ p^{\dot\alpha\beta} & 0\end{pmatrix} = \bar u_-(p)\not{p} = 0, \tag{3.1.13}$$

where we have defined the incoming and outgoing (anti) four-spinors $v_\pm$, $\bar u_\pm$ in terms of the two-spinors [36]. This is of course nothing more than the usual Dirac equation in the momentumbasis for the four-spinors, but using the SH notation there is no reason to work with the four-spinors and for a lot of the work done in this thesis it is more convenient to just use the Weyl spinors. When translating amplitude expressions such as (2.2.1) we are going to need expression for the polarization vectors as well as a way to express the momenta in terms of SH variables. To start we note that we never see products of the kind $[p|_{\dot\alpha}\,|k\rangle^\beta$ unless there is a Pauli matrix contracting with the spinor indices $[p|_{\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\beta}\,|k\rangle_\beta$, so a common way to write the slashed momenta (eq. (3.1.5)) is by abusing notation slightly

$$\not{p} = p_{\alpha\dot\beta} + p^{\dot\alpha\beta} = |p\rangle_\alpha\,[p|_{\dot\beta} + |p]^{\dot\alpha}\,\langle p|^\beta\,. \tag{3.1.14}$$

It will always be obvious how the spinors should be contracted. We can use this to form expressions for the momenta, since contracting both sides of the above with the

σ and $\bar{\sigma}$ matrices and using $\mathrm{Tr}[\sigma_\mu \sigma_\nu] = 2\eta_{\mu\nu}$

$$p_\mu = \frac{1}{2}[p|_{\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\alpha}\beta}\,|p\rangle_\beta \equiv \frac{1}{2}[p|\sigma_\mu|p\rangle, \qquad p^\mu = \frac{1}{2}\,\langle p|^\alpha\,(\sigma^\mu)_{\alpha\dot{\beta}}|p]^{\dot{\beta}} \equiv \frac{1}{2}\langle p|\sigma^\mu|p]. \tag{3.1.15}$$

Similarly we can construct polarization vectors related to particles with momentum p and using a reference momentum $q \neq p$ satisfying $q^2 = 0$ [7]

$$\epsilon_+^\mu(p) = \frac{1}{\sqrt{2}}\frac{\langle q|\sigma^\mu|p]}{\langle qp\rangle}, \qquad \epsilon_-^\mu(p) = -\frac{1}{\sqrt{2}}\frac{[q|\sigma^\mu|p\rangle}{[qp]}, \qquad \not{\epsilon}_+(p) = \sqrt{2}\frac{|q\rangle[p| + |p]\langle q|}{\langle qp\rangle}, \qquad \not{\epsilon}_-(p) = -\sqrt{2}\frac{|p\rangle[q| + |q]\langle p|}{[qp]}. \tag{3.1.16}$$

We note that these polarization vectors automatically obey transversality $p_i \cdot \epsilon_i^\pm = 0$. Also, taking the difference between two polarization vectors we find

$$\begin{aligned}\epsilon_+^\mu(p,q') - \epsilon_+^\mu(p,q) =& \frac{1}{\sqrt{2}}\left(\frac{\langle q'|\sigma^\mu|p]}{\langle q'p\rangle} - \frac{\langle q|\sigma^\mu|p]}{\langle qp\rangle}\right)\\ =& \sqrt{2}\frac{\langle q'q\rangle}{\langle q'p\rangle\,\langle pq\rangle}p^\mu,\end{aligned}$$

which vanishes in the final amplitude because of gauge invariance, i.e. $A\Big|_{\epsilon_i \to \epsilon_i + ak_i} = A$. Since the spinors are two-dimensional, three spinors can never be linearly independent and we will always be able to form one spinor as a linear combination of the two others

$$|k\rangle = \alpha\,|i\rangle + \beta\,|j\rangle\,, \tag{3.1.17}$$

we can find the coeffecients α and β by multiplying both sides of the above by $\langle i|$ and $\langle j|$:

$$\left.\begin{aligned}\langle ik\rangle = \beta\,\langle ij\rangle\\ \langle jk\rangle = \alpha\,\langle ji\rangle\end{aligned}\right\} \Rightarrow -\,|k\rangle\,\langle ij\rangle = |i\rangle\,\langle jk\rangle + |j\rangle\,\langle ik\rangle\,.$$

This of course holds for the square brackets as well. If we then group all the terms on side of the equals sign and multiply by a fourth spinor $\langle l|$, we obtain a very useful equation known as the *Schouten identity*

$$\begin{aligned}\langle li\rangle\langle jk\rangle + \langle lj\rangle\langle ik\rangle + \langle lk\rangle\langle ij\rangle = 0,\\ [li][jk] + [lj][ik] + [lk][ij] = 0,\end{aligned} \tag{3.1.18}$$

Lastly we note the following identities that often come in handy[2]

$$\begin{aligned} s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j = \langle i|j|i] = \langle ij\rangle\,[ji], \\ \langle i|\sigma^\mu|j] = [j|\sigma^\mu|i\rangle, \\ [i\,|nm|\,k] = [in]\,\langle nm\rangle\,[mk]. \end{aligned} \tag{3.1.19}$$

3.2 The little group

First note that under the rescalings $|p\rangle_\alpha \to w^{-1}\,|p\rangle_\alpha$ and $[p|_{\dot\beta} \to w[p|_{\dot\beta}$ the momentum bispinor is left invariant $p_{\alpha\dot\beta} \to p_{\alpha\dot\beta}$.

The subgroup of the Lorentz group that leaves the momentum invariant is known as the *little group*. For a massive particle, one can always transform to a center of mass frame where the particle lies still, i.e. $p_\mu = (m\ 0\ \cdots\ 0\ 0)$. This vector is invariant to SO$(D-1)$ rotations, hence the little group for massive particles in four dimensions is SO(3) $\sim$ SU(2) [9].

For massless particles we cannot go to a center of mass frame since the particles are traveling at speed $v = 1$ (for $c = 1$), and so the best we can do is go to a frame where the particle is invariant to rotations around some axis, say z.

Example

Take the null vector $p_\mu = \begin{pmatrix}2E\ 0\ 0\ 2E\end{pmatrix}$. From this we can form explicit representations of the SH variables

$$p_{\alpha\dot\beta} = \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}, \qquad |p\rangle_\alpha = \sqrt{2E}\begin{pmatrix}1\\0\end{pmatrix}, \qquad [p|_{\dot\beta} = \sqrt{2E}\begin{pmatrix}1\\0\end{pmatrix}.$$

The momentum is obviously invariant to rotations in the xy-plane. Performing a rotation around z-axis should leave this momentum invariant. Using the $SL(2,\mathbb{C})$ represenation of the Lorentz group we can rotate using the transformations

$$\Lambda^\alpha_\beta = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix}, \qquad \tilde\Lambda^{\dot\beta}_{\dot\alpha} = \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix},$$

from which we get

$$|p\rangle_\alpha \to |p\rangle'_\alpha = e^{i\frac{\phi}{2}}\,|p\rangle_\alpha\,, \qquad [p|_{\dot\beta} \to [p|'_{\dot\beta} = e^{-i\frac{\phi}{2}}[p|_{\dot\beta}, \qquad p_{\alpha\dot\beta} \to p_{\alpha\dot\beta}.$$

The rotations in the above example are nothing but U(1) $\sim$ SO(2) transformations and so they exactly form the little group for massless particles in four dimensions, while

[2] Here the commonly used notation $\langle i|\sigma^\mu|j] \equiv \langle p_i|\sigma^\mu|p_j]$ has been adopted

we in general can identify the little group in D dimensions for massless particles to be SO$(D-2)$.

3.3 Yang-Mills example

In this section we will demonstrate how to use the massless spinor helicity variables to simplify scattering amplitudes by looking at the amplitude $A(1^+, 2^+, 3^-, 4^-)$ with the following reference momenta $q_1 = 4,\ q_2 = 4,\ q_3 = 1,\ q_4 = 1$. The only non-zero momentum/polarization products are

$$
\begin{aligned}
k_2 \cdot \epsilon_1^+ &= -\frac{1}{\sqrt{2}} \frac{\langle 24 \rangle [12]}{\langle 14 \rangle}, & \epsilon_2^+ \cdot \epsilon_3^- &= -\frac{\langle 34 \rangle [12]}{\langle 24 \rangle \langle 13 \rangle}, \\
k_3 \cdot \epsilon_1^+ &= -\frac{1}{\sqrt{2}} \frac{\langle 34 \rangle [13]}{\langle 14 \rangle}, & k_1 \cdot \epsilon_2^+ &= \frac{1}{\sqrt{2}} \frac{\langle 14 \rangle [12]}{\langle 24 \rangle}, \\
k_3 \cdot \epsilon_2^+ &= -\frac{1}{\sqrt{2}} \frac{\langle 34 \rangle [23]}{\langle 24 \rangle}, & k_2 \cdot \epsilon_3^- &= -\frac{1}{\sqrt{2}} \frac{\langle 23 \rangle [12]}{[13]}, \\
k_4 \cdot \epsilon_3^- &= \frac{1}{\sqrt{2}} \frac{\langle 34 \rangle [14]}{[13]}, & k_2 \cdot \epsilon_4^- &= -\frac{1}{\sqrt{2}} \frac{\langle 24 \rangle [12]}{[14]}, \\
k_3 \cdot \epsilon_4^- &= -\frac{1}{\sqrt{2}} \frac{\langle 34 \rangle [13]}{[14]}.
\end{aligned}
$$

First we consider the s-channel expression (2.2.1)

$$
\begin{aligned}
A_s =& \frac{1}{2s_{12}} \left\{ (\epsilon_1 \cdot \epsilon_2)(k_1 - k_2)^\beta + 2\epsilon_2^\beta (\epsilon_1 \cdot k_2) - 2\epsilon_1^\beta (\epsilon_2 \cdot k_1) \right\} \\
& \times \left\{ (\epsilon_3 \cdot \epsilon_4)(k_3 - k_4)_\beta + 2\epsilon_{4,\beta} (\epsilon_3 \cdot k_4) - 2\epsilon_{3,\beta} (\epsilon_4 \cdot k_3) \right\}.
\end{aligned}
$$

Since most of the polarization products vanish, this then reduces to

$$
\begin{aligned}
A_s &= -\frac{2}{s_{12}} (\epsilon_2^+ \cdot \epsilon_3^-)(k_2 \cdot \epsilon_1^+)(k_3 \cdot \epsilon_4^-) \\
&= \frac{1}{s_{12}} \frac{\langle 34 \rangle^2 [12]^2}{\langle 14 \rangle [14]} \\
&= \frac{\langle 34 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}.
\end{aligned}
$$

where we have used momentum conservation $\langle 34 \rangle [14] = \langle 23 \rangle [12]$. Both the t-channel and contact diagrams vanish since all the products are zero

$$
A_t = 0, \qquad A_c = 0.
$$

The total color ordered amplitude is then obtained by summing over all contributions coming from the 3 channels

$$A_4^{\text{tree}}(1^+,2^+,3^-,4^-) = \sum_{\text{planar diagrams}} A_4 = \frac{\langle 34\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{3.3.1}$$

This results is known as an *MHV*- or *Parke-Taylor* amplitude [1]. In general one has for MHV and anti-MHV[3] amplitudes [36, 38]

$$\begin{aligned} MHV &\equiv A_4^{\text{tree}}(1^+,2^+,3^+,\ldots,i^-,\ldots,j^-,\ldots,n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}, \\ \overline{MHV} &\equiv A_4^{\text{tree}}(1^-,2^-,3^-,\ldots,i^+,\ldots,j^+,\ldots,n^-) = \frac{[ij]^4}{[12][23]\cdots[n1]}. \end{aligned} \tag{3.3.2}$$

The simplicity of these results is astonishing given the computational complexity it took to get here through Feynman diagrams. This results is one of the many reasons physicists search for new and more efficient methods to compute scattering amplitudes.

3.4 Massive spinor-helicity

Let us now turn our attention to massive particles. Assuming the massive momenta are on shell this in turn corresponds to the condition

$$\det\left(p_{\alpha\dot\beta}\right) = m^2, \tag{3.4.1}$$

whichs means $p_{\alpha\dot\beta}$ has full rank and the decomposition in terms of two Weyl spinors breaks down. Instead we can expand the momentum in terms of two matrices each with rank 1[4] $p_{\alpha\dot\beta} = M_1 + M_2$ where $M_1 = \lambda_\alpha^1\tilde\lambda_{\dot\beta,1}$ and $M_2 = \lambda_\alpha^2\tilde\lambda_{\dot\beta,2}$ or more concisely

$$\begin{aligned} p_{\alpha\dot\beta} &= \lambda_\alpha^a\tilde\lambda_{\dot\beta,a} = \lambda_\alpha^a\epsilon_{ab}\tilde\lambda_{\dot\beta}^b = -\lambda_{\alpha,a}\tilde\lambda_{\dot\beta}^a, \\ p^{\dot\alpha\beta} &= \tilde\lambda_b^{\dot\alpha}\lambda^{\beta,b} = -\tilde\lambda^{\dot\alpha,a}\epsilon_{ab}\lambda^{\beta,b} = -\tilde\lambda^{\dot\alpha,b}\lambda_b^\beta, \end{aligned} \tag{3.4.2}$$

where the $a = 1,2$ and $b = 1,2$ are $SU(2)$ little group indices that are raised and lowered using the 2 dimensional Levi-Civita symbol just like the spinor indices. Under

[3] 2 postive helicity particles

[4] We have again suppressed an index p on the Weyl spinors for clarity

an $SU(2)$ transformation we see that the momentum is invariant

$$\begin{aligned} \lambda^{\beta,a} &\to U^a_b \lambda^{\beta,b}, \qquad \tilde{\lambda}^{\dot{\alpha},c} \to U^c_d \tilde{\lambda}^{\dot{\alpha},d}, \\ p^{\dot{\alpha}\beta} = -\epsilon_{ca}\tilde{\lambda}^{\dot{\alpha},K}\lambda^{\beta,a} &\to -\epsilon_{ca}U^c_d U^a_b \tilde{\lambda}^{\dot{\alpha},d}\lambda^{\beta,b} = -\epsilon_{db}\tilde{\lambda}^{\dot{\alpha},d}\lambda^{\beta,b} = p^{\dot{\alpha}\beta}. \end{aligned} \tag{3.4.3}$$

where we have used one of the defining proporties of $SU(2)$, namely the invariance of ϵ_{ca} under $SU(2)$ transformations, $\epsilon_{ca}U^c_d U^a_b = \epsilon_{db}$. We now turn to establish the fundamental properties from the new massive SH variables. Since for square matrices $\det(AB) = \det(A)\det(B)$ we can extract

$$\det(\lambda) = M, \qquad \det\left(\tilde{\lambda}\right) = \tilde{M}, \quad \text{with } M\tilde{M} = m^2, \tag{3.4.4}$$

where we in the remaining will set $M = \tilde{M} = m$. This can the be used in conjunction with the transformation property of the Levi Civita symbol $\epsilon_{i_1 i_2 \dots i_n} B^{i_1 i_2 \dots i_n}_{j_1 j_2 \dots j_n} = \epsilon_{j_1 j_2 \dots j_n} \det(B)$ to show that

$$\begin{aligned} \lambda_{\alpha,b}\lambda^{\beta,b} &= \lambda^a_\alpha \epsilon_{ab}\lambda^{\beta,b} = -\epsilon^{\beta\gamma}\epsilon_{ab}\lambda^a_\alpha\lambda^b_\gamma = \epsilon^{\beta\gamma}\epsilon_{\gamma\alpha}\det(\lambda) = m\delta^\beta_\alpha, \\ \tilde{\lambda}^{\dot{\alpha},b}\tilde{\lambda}_{\dot{\beta},b} &= -\tilde{\lambda}^{\dot{\alpha},b}\epsilon_{ba}\tilde{\lambda}^a_{\dot{\beta}} = -\epsilon_{\dot{\beta}\dot{\gamma}}\epsilon_{ba}\tilde{\lambda}^{\dot{\alpha},b}\tilde{\lambda}^{\dot{\gamma},a} = \epsilon_{\dot{\beta}\dot{\gamma}}\epsilon^{\dot{\gamma}\dot{\alpha}}\det\left(\tilde{\lambda}\right) = m\delta^{\dot{\alpha}}_{\dot{\beta}}, \\ \lambda^{\beta,a}\lambda^c_\beta &= -\epsilon^{\alpha\beta}\lambda^a_\alpha\lambda^c_\beta = -\det(\lambda)\epsilon^{ac} = -m\epsilon^{ac}, \\ \tilde{\lambda}^a_{\dot{\beta}}\tilde{\lambda}^{\dot{\beta},c} &= \epsilon^{\dot{\beta}\dot{\alpha}}\tilde{\lambda}^a_{\dot{\beta}}\tilde{\lambda}^c_{\dot{\alpha}} = \det\left(\tilde{\lambda}\right)\epsilon^{ac} = m\epsilon^{ac}. \end{aligned} \tag{3.4.5}$$

From which the two-dimensional Dirac equation can be recovered

$$\begin{aligned} p^{\dot{\alpha}\beta}\lambda^d_\beta &= \tilde{\lambda}^{\dot{\alpha}}_a\lambda^{\beta,a}\lambda^d_\beta = -m\epsilon^{ad}\tilde{\lambda}^{\dot{\alpha}}_a = m\tilde{\lambda}^{\dot{\alpha},d}, \\ p_{\alpha\dot{\beta}}\tilde{\lambda}^{\dot{\beta},d} &= -\lambda_{\alpha,a}\tilde{\lambda}^a_{\dot{\beta}}\tilde{\lambda}^{\dot{\beta},d} = -m\epsilon^{ad}\lambda_{\alpha,a} = m\lambda^d_\alpha, \\ \lambda^{\alpha,d}p_{\alpha\dot{\beta}} &= \lambda^{\alpha,d}\lambda^a_\alpha\tilde{\lambda}_{\dot{\beta},a} = -m\epsilon^{da}\tilde{\lambda}_{\dot{\beta},a} = -m\tilde{\lambda}^d_{\dot{\beta}}, \\ \tilde{\lambda}^d_{\dot{\alpha}}p^{\dot{\alpha}\beta} &= \tilde{\lambda}^d_{\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_a\lambda^{\beta,a} = \epsilon_{ab}\tilde{\lambda}^d_{\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha},b}\lambda^{\beta,a} = m\epsilon_{ab}\epsilon^{db}\lambda^{\beta,a} = -m\lambda^{\beta,d}. \end{aligned} \tag{3.4.6}$$

Since we are at some point going to convert between two and four dimensional representations of the spinors, let us compare the above with the four-dimensional Dirac equation

$$\begin{aligned} \left(\not{p} - m\right) u^a(p) = \bar{u}^a(p)\left(\not{p} - m\right) = 0, \\ \left(\not{p} + m\right) v^a(p) = \bar{v}^a(p)\left(\not{p} + m\right) = 0, \end{aligned} \tag{3.4.7}$$

where m is shorthand for mass matrix $m \otimes \begin{pmatrix} \delta^\beta_\alpha & 0 \\ 0 & \delta^{\dot{\alpha}}_{\dot{\beta}} \end{pmatrix}$. If we identify

$$u^a(p) = \begin{pmatrix} \lambda^a_\alpha \\ \tilde{\lambda}^{\dot{\alpha},a} \end{pmatrix}, \qquad \bar{u}^a(p) = \begin{pmatrix} -\lambda^{\alpha,a} & \tilde{\lambda}^a_{\dot{\alpha}} \end{pmatrix},$$
$$v^a(p) = \begin{pmatrix} -\lambda^a_\alpha \\ \tilde{\lambda}^{\dot{\alpha},a} \end{pmatrix}, \qquad \bar{v}^a(p) = \begin{pmatrix} \lambda^{\alpha,a} & \tilde{\lambda}^a_{\dot{\alpha}} \end{pmatrix}, \tag{3.4.8}$$

we have e.g. for $u^a(p)$ and $\bar{u}^a(p)$

$$\begin{pmatrix} -m\delta^\beta_\alpha & p_{\alpha\dot{\beta}} \\ p^{\dot{\alpha}\beta} & -m\delta^{\dot{\alpha}}_{\dot{\beta}} \end{pmatrix} \begin{pmatrix} \lambda^a_\beta \\ \tilde{\lambda}^{\dot{\beta},a} \end{pmatrix} = \begin{pmatrix} p_{\alpha\dot{\beta}}\tilde{\lambda}^{\dot{\beta},a} - m\lambda^a_\alpha \\ p^{\dot{\alpha}\beta}\lambda^a_\beta - m\tilde{\lambda}^{\dot{\alpha},a} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.4.9}$$

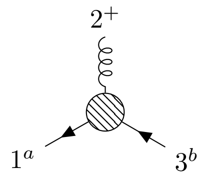
$$\begin{pmatrix} -\lambda^{\alpha,a} & \tilde{\lambda}^a_{\dot{\alpha}} \end{pmatrix} \begin{pmatrix} -m\delta^\beta_\alpha & p_{\alpha\dot{\beta}} \\ p^{\dot{\alpha}\beta} & -m\delta^{\dot{\alpha}}_{\dot{\beta}} \end{pmatrix} = \begin{pmatrix} m\lambda^{\beta,a} + \tilde{\lambda}^a_{\dot{\alpha}} p^{\dot{\alpha}\beta} & -m\tilde{\lambda}^a_{\dot{\beta}} - \lambda^{\alpha,a} p_{\alpha\dot{\beta}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}. \tag{3.4.10}$$

Just like for the massless spinors, it is often more convenient (and nicer) to work in a bra-ket notation, so let us summarize the above results using it.

$$\begin{aligned}
|p^a\rangle_\alpha &\equiv \lambda^a_\alpha, & \langle p_a|^\beta &\equiv \lambda^\beta_a, & |p^a]^{\dot{\alpha}} &\equiv \tilde{\lambda}^{\dot{\alpha},a}, \\
p_{\alpha\dot{\beta}} &= |p^a\rangle_\alpha [p_a|_{\dot{\beta}}, & p^{\dot{\alpha}\beta} &= |p_a]^{\dot{\alpha}} \langle p^a|^\beta, & m\delta^{\dot{\alpha}}_{\dot{\beta}} &= |p^a]^{\dot{\alpha}} [p_a|_{\dot{\beta}}, \\
p^{\dot{\alpha}\beta}|p^a\rangle_\beta &= m|p^a]^{\dot{\alpha}}, & p_{\alpha\dot{\beta}}|p^a]^{\dot{\beta}} &= m|p^a\rangle_\alpha, & \langle p^a|^\alpha p_{\alpha\dot{\beta}} &= -m[p^a|_{\dot{\beta}}, \\
\langle p^a k^b\rangle &\equiv \langle p^a|^\beta |k^b\rangle_\beta, & [p^a k^b] &\equiv [p^a|_{\dot{\beta}} |k^b]^{\dot{\beta}}, & \langle p^a p^b\rangle &= -m\epsilon^{ab}, \\
[p_a|_{\dot{\beta}} &\equiv \tilde{\lambda}_{\dot{\beta},a}, & m\delta^\beta_\alpha &= |p_a\rangle_\alpha \langle p^a|^\beta, & [p^a|_{\dot{\alpha}} p^{\dot{\alpha}\beta} &= -m\langle p^a|^\beta, \\
[p^a p^b] &= m\epsilon^{ab}.
\end{aligned} \tag{3.4.11}$$

As opposed to the spinor index which we saw we could suppress in the massless case, since it was always clear how to contract the spinors, the little group index follows along in all our calculations. The introduction of all of this notation warrants an example. Since it will be useful to us later on, let us look at the color ordered three-point quark-quark-gluon amplitude. The three-vertex rule is just γ_μ [7] so denoting quarks and antiquarks with underscores and bars respectively, the amplitude with one positive helicity gluon is

$$A(\underline{1}^a, 2^+, \overline{3}^b) =$$



$$= \bar{u}_1^a \not{\epsilon}_2^+ v_3^b. \tag{3.4.12}$$

Inserting the expression for our massive spinors as well as the gluon polarization we obtain

$$\begin{aligned} A(\underline{1}^a, 2^+, \overline{3}^b) &= \left(-\langle 1^a| + [1^a|\right) \left(\sqrt{2}\frac{|q\rangle[2| + |2]\langle q|}{\langle q2\rangle}\right) \left(-|3^b\rangle + |1^b]\right) \\ &= -\frac{\sqrt{2}}{\langle q2\rangle} \left(\langle 1^a q\rangle[23^b] + [1^a 2]\langle q3^b\rangle\right). \end{aligned}$$

This can be put into an even more compact form if we use the identities just established. Let us go through the calculations as an exercise

$$\begin{aligned} A(\underline{1}^a, 2^+, \overline{3}^b) &= -\frac{\sqrt{2}}{\langle q2\rangle\, m} \left(m\delta_d^b\langle 1^a q\rangle[23^d] + m\delta_c^a[1^c 2]\langle q3^b\rangle\right) \\ &= -\frac{\sqrt{2}}{\langle q2\rangle\, m} \left(m\epsilon_{dc}\epsilon^{cb}\langle 1^a q\rangle[23^d] + m\epsilon_{cd}\epsilon^{da}[1^c 2]\langle q3^b\rangle\right) \\ &= -\frac{\sqrt{2}}{\langle q2\rangle\, m} \left(\epsilon_{cd}[23^d]\langle 3^c 3^b\rangle\langle 1^a q\rangle + \epsilon_{cd}\langle 1^a 1^d\rangle[1^c 2]\langle q3^b\rangle\right), \end{aligned}$$

where we have used the identities $\epsilon_{ab}\epsilon^{bc} = \delta_a^c$ and $\langle p^a p^b\rangle = -m\epsilon^{ab}$ as well as the antisymmetry of the Levi-Civita symbol and the spinor products. Using the definition of the slashed massive momenta we can combine the first term into an angle-square bracket, on which we can then employ momentum conservation, the schouten identity and then expand back into bra-ket products

$$\begin{aligned} A(\underline{1}^a, 2^+, \overline{3}^b) &= -\frac{\sqrt{2}}{\langle q2\rangle\, m} \left([2|3|3^b\rangle\langle 1^a q\rangle + \epsilon_{cd}\langle 1^a 1^d\rangle[1^c 2]\langle q3^b\rangle\right) \\ &= -\frac{\sqrt{2}}{\langle q2\rangle\, m}\epsilon_{cd}[1^c 2] \left(\langle 3^b 1^d\rangle\langle 1^a q\rangle + \langle 1^a 1^d\rangle\langle q3^b\rangle\right) \\ &= \frac{\sqrt{2}}{\langle q2\rangle\, m}\epsilon_{cd}[1^c 2]\langle 1^a 3^b\rangle\langle 1^d q\rangle \\ &= -\sqrt{2}\frac{\langle 1^a 3^b\rangle[2|1|q\rangle}{m\,\langle 2q\rangle}. \end{aligned}$$

We now have an expression that explicitly contains a mass term and the result matches the one obtained in [7].

Having established both massless and massive spinor helicity techniques we will now turn to another method the *CHY formalism*. The spinor helicity methods will prove useful when we want to check the results we obtain later in the thesis, especially for numerical checks.

Chapter 4

Massless CHY

In this section we present the approach to scattering amplitudes developed by Cachazo, He and Yuan (CHY) in an impressive series of papers [10–15]. The method is based on a set of relations known as the *scattering equations* and provide the scattering amplitudes for a large number of different theories, but have until just recently been reserved for particles with integer spin $s \leq 2$. The formalism provides an incredible efficient way to calculate d-dimensional covariant scattering amplitudes. In this thesis the focus is on tree level scattering but it should be mentioned that extensions to loop level exist [20, 21]. Furthermore a double cover approach has been developed in [43–45] and it is exceptionally suited for phrasing the CHY amplitudes in a factorized form, see for instance [46–49].

We focus on the single cover in this thesis and the chapter will go as follows. First we present the formalism and the scattering equations. We then go through the proof of the diagrammatic rules obtained in [50] and combine them with the CHY monodromy relations found in [51]. Following this we present the integrands for various theories and perform explicit calculations of different amplitudes. The checks performed in Mathematica are based of the algorithm used in [21].

4.1 The scattering equations

Cachazo, He and Yuan found that d-dimensional scattering amplitudes can be calculated through the integral

$$\mathcal{A}(1, \ldots, n) = \int \mathrm{d}\Omega_{\mathrm{CHY}}\, \mathcal{I}_L \times \mathcal{I}_R \tag{4.1.1}$$

where the left and right handed integrands $\mathcal{I}_L$ and $\mathcal{I}_R$ depend on the field theory in question and the CHY measure is

$$\mathrm{d}\Omega_{\mathrm{CHY}} = (z_r - z_s)^2 (z_s - z_t)^2 (z_t - z_r)^2 \prod_{\substack{i=1 \\ i \neq r,s,t}} \mathrm{d}z_i \, \delta(S_i). \tag{4.1.2}$$

The S_i's in the deltafunction are the scattering equations and they encode n on shell-momenta $\{k_1^\mu, \ldots, k_n^\mu\}$ in terms of auxiliary variables $z_i \in \mathbb{CP}^1$ that corresponds to punctures on the Riemann sphere.

$$S_i(z) \equiv \sum_{\substack{j=1 \\ j \neq i}} \frac{2k_i \cdot k_j}{z_i - z_j} = 0, \quad i = 1, ..., n. \tag{4.1.3}$$

In the massless case the mandelstam variables are $s_{ij} = 2k_i \cdot k_j$ so we can express it in a somewhat neater form

$$S_i(z) \equiv \sum_{\substack{j=1 \\ j \neq i}} \frac{s_{ij}}{z_{ij}} = 0, \tag{4.1.4}$$

where we have also defined the shorthand $z_{ij} \equiv z_i - z_j$. At a first glance, we have n equations, but because of $\mathrm{SL}(2, \mathbb{C})$ invariance of S_i we get three extra constraints, reducing the number of independent equations to $n - 3$:

$$0 = \sum_i S_i, \qquad 0 = \sum_i S_i z_i, \qquad 0 = \sum_i S_i z_i^2. \tag{4.1.5}$$

First let us show that the scattering equations $S_i = 0$ are invariant under a Möbius transformation of the auxiliary variables $z_i \to \frac{az_i+b}{cz_i+d}$. The transformation takes

$$S_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{s_{ij}}{z_i - z_j} \to S_i' = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{s_{ij}}{\frac{az_i+b}{cz_i+d} - \frac{az_j+b}{cz_j+d}}.$$

Putting the fractions in the denominator on a common divisor we get

$$\begin{aligned} S_i' &= \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{s_{ij}}{\frac{(az_i+b)(cz_j+d)-(az_j+b)(cz_i+d)}{(cz_i+d)(cz_j+d)}} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{s_{ij}}{\frac{(ad-bc)(z_i-z_j)}{(cz_i+d)(cz_j+d)}} \\ &= \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{s_{ij}(cz_i + d)(cz_j + d)}{(z_i - z_j)}, \end{aligned}$$

where we have used the fact that for a Möbius transformation $ad - bc = 1$ in the last line. We then have

$$\begin{aligned}S_i' =&(cz_i+d)\sum_{\substack{j=1\\j\neq i}}^{n}\frac{s_{ij}(cz_j+d)}{(z_i-z_j)}\\ =&(cz_i+d)\left(dS_i+c\sum_{\substack{j=1\\j\neq i}}^{n}\frac{s_{ij}(z_j)}{(z_i-z_j)}\right)\\ =&(cz_i+d)\left(dS_i+c\sum_{\substack{j=1\\j\neq i}}^{n}\frac{s_{ij}(z_j-z_i)}{(z_i-z_j)}+cz_i\sum_{\substack{j=1\\j\neq i}}^{n}\frac{s_{ij}}{(z_i-z_j)}\right)\\ =&(cz_i+d)^2S_i.\end{aligned} \tag{4.1.6}$$

In the second line we expanded $z_j = z_j - z_i + z_i$ and in the last line we used the definition of S_i and $\sum_j s_{ij} = 0$ for the last and middle term respectively. The constraints on the solutions 4.1.5 follows from this. The first one can also easily be shown through symmetry of the numerator and antisymmetry of the denominator

$$\sum_{i=1}^{n}S_i=\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\frac{s_{ij}}{z_{ij}}=-\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\frac{s_{ji}}{z_{ji}}=-\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\frac{s_{ij}}{z_{ij}}=0.$$

The second constraint follows from the first

$$\begin{aligned}\sum_{i=1}^{n}z_iS_i&=\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\frac{z_is_{ij}}{z_{ij}}=\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\frac{(z_{ij}+z_j)s_{ij}}{z_{ij}}\\ &=\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\left(\frac{z_{ij}s_{ij}}{z_{ij}}\right)+\sum_{\substack{j=1\\j\neq i}}^{n}z_j\sum_{i=1}^{n}\left(\frac{s_{ij}}{z_{ij}}\right)\\ &=-\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\frac{z_js_{ji}}{z_{ji}}=-\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\frac{z_is_{ij}}{z_{ij}}=0,\end{aligned}$$

where the first term in the second line vanishes due to momentum conservation and the last line vanish from symmetry/antisymmetry of the numerator/denominator. Lastly the third condition is

$$\sum_{i}^{n}z_i^2S_i=\sum_{i}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\frac{z_iz_js_{ij}}{z_{ij}}=0,$$

which is zero, again by use of antisymmetry. Having now shown the extra conditions relating the scattering equations to each other, we end up with $n-3$ independent relations. The delta function in the CHY measure means that the integral is completely localized under the support of the scattering equations and just turns into a sum over solutions [40] however in this thesis we will not explicitly solve the scattering equations, but instead employ a diagrammatic expansion which we will derive in section 4.2. It might however, provide some insight to look at how finding the solutions could be done by e.g. looking at the case with $n = 4$.

Since we only have 1 independent solution, let us set $z_1 = \infty$, $z_2 = 1$ and $z_4 = 0$. For this specific choice of variables we clearly have $S_1 = \sum_{j=2,3,4} \frac{s_{1j}}{z_1 - z_j} = 0 \; \forall \, z_j$ since $z_1 = \infty$ and we don't get any relations that let us determine z_3 from this. We just need one of the remaining equations to determine z_3, e.g. S_4:

$$\begin{aligned} S_4 &= \sum_{j=2,3} \frac{s_{4j}}{z_4 - z_j} = \frac{s_{42}}{z_4 - z_2} + \frac{s_{43}}{z_4 - z_3} = -s_{42} - \frac{s_{43}}{z_3} = 0 \\ &\Rightarrow z_3 = -\frac{s_{43}}{s_{42}}, \end{aligned}$$

which indeed provides a solution to the remaining scattering equations

$$\begin{aligned} S_2 &= = \frac{s_{23}}{z_2 - z_3} + \frac{s_{24}}{z_2 - z_4} = \frac{s_{23}}{1 + \frac{s_{43}}{s_{42}}} + s_{24} = \frac{s_{23} s_{24}}{s_{42} + s_{43}} + s_{24} = -\frac{s_{14} s_{24}}{s_{14}} + s_{24} = 0 \\ S_3 &= \frac{s_{32}}{z_3 - z_2} + \frac{s_{34}}{z_3 - z_4} = -\frac{s_{32}}{\frac{s_{43}}{s_{42}} + 1} - \frac{s_{34} s_{42}}{s_{43}} = \frac{s_{14} s_{42}}{s_{14}} - s_{42} = 0. \end{aligned}$$

This turns cumbersome quickly as n becomes large and as already mentioned we will instead employ a diagrammatic method. For numerical calculations finding solutions through the above method is however very feasible. We will now turn to explore the integral used to calculate the amplitudes and then derive integration rules for it.

4.2 CHY integration rules

In this section we go through the derivation in [50] which presents a powerful way of obtaining scattering amplitudes from the CHY integral that has a diagrammatical formulation in terms of four-regular n-gons. Some of the expansions were already done in [16, 17], but the great insight of C. Baadsgaard et al. was to notice the diagrammatic nature of this expansion.

Using the integration rules found in this section we avoid having to explicitly sum over the solutions to the scattering equations. Since solving the scattering equations used to be the bottleneck of the theory [40] this partly solves that problem, especially since the integration rules are easily implemented in a computer program such as Mathematica.

At large n however, the procedure still becomes cumbersome and the resulting expressions become large and unwieldy and as we will see later in this thesis, methods suchs as expressing the result in terms of BCJ-numerators can greatly simplify the calculations.

Möbius invariance of the integrand

Before looking at the integration rules themselves we will quickly show what consequence requiring Möbius invariance of the CHY integral has. First note that since $S_i \to (cz_i+d)^2 S_i$ under a SL$(2,\mathbb{C})$ transformation (see (4.1.6)) the CHY integration measure transforms as

$$\begin{aligned}
\mathrm{d}\Omega_{\mathrm{CHY}} = z_{rs}^2 z_{st}^2 z_{tr}^2 \prod_{\substack{i=1\\ i\neq r,s,t}} \mathrm{d}z_i\, \delta(S_i) \to& \frac{z_{rs}^2 z_{st}^2 z_{tr}^2}{(cz_r+d)^4(cz_s+d)^4(cz_t+d)^4} \prod_{\substack{i=1\\ i\neq r,s,t}} \frac{\mathrm{d}z_i}{(cz_i+d)^2} \frac{\delta(S_i)}{(cz_i+d)^2} \\
&= \mathrm{d}\Omega_{\mathrm{CHY}} \prod_i \frac{1}{(cz_i+d)^4},
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
z_{rt} = (z_r - z_t) \to& \frac{az_r+b}{cz_r+d} - \frac{az_t+b}{cz_t+d} = \frac{(az_r+b)(cz_t+d)-(az_t+b)(cz_r+d)}{(cz_r+d)(cz_t+d)} \\
&= \frac{acz_r z_t + adz_r + cbz_t + bd - acz_t z_r - adz_t - cbz_r - bd}{(cz_r+d)(cz_t+d)} \\
&= \frac{1}{(cz_r+d)(cz_t+d)} z_{rt},
\end{aligned}$$

as well as $\delta(f(z)\alpha) = |\alpha|^{-2}\delta(f(z))$ on the Riemann sphere and the Jacobian $\frac{\mathrm{d}z_i'}{\mathrm{d}z_i} = \frac{\mathrm{d}}{\mathrm{d}z_i}\left(\frac{az_i+b}{cz_i+d}\right) = \frac{a}{cz_i+d} - \frac{c(az_i+b)}{(cz_i+d)^2} = \frac{1}{(cz_i+d)^2}$. So for the amplitude to remain invariant under SL(2,$\mathbb{C}$) transformations we need the integrand $\mathcal{I} = \mathcal{I}_L \times \mathcal{I}_R$ to transform according to

$$\mathcal{I} \to \mathcal{I} \prod_i (cz_i+d)^4. \tag{4.2.1}$$

All CHY amplitudes can be put into the form [50]

$$\mathcal{A}_n\left[H(z)\right] = \int \mathrm{d}\Omega_{\mathrm{CHY}}\, H(z), \tag{4.2.2}$$

where $H(z)$ is a function of the punctures z_i

$$H(z) = \prod_{\substack{i,j\\ i<j}} z_{ij}^{c_{ij}}, \qquad \text{with } c_{ij} \in \mathbb{Z} \text{ and } c_{ij} = c_{ji}. \tag{4.2.3}$$

The Möbius transformation of the integrand (4.2.1) dictates that every variable has to come with weight -4, meaning we require $\sum_{\substack{j=1 \\ j\neq i}} c_{ij} = -4$. For this reason, we will be able to represent integrands as 4-regular n-gons.

Derivation of the rules

We will now derive the integration rules. Working with integrands of the form (4.2.3) and using the constraints (4.1.5) to gauge fix three of the punctures

$$\begin{aligned} z_r &= z_1 = \infty, \\ z_s &= z_2 = 1, \\ z_t &= z_n = 0, \end{aligned} \tag{4.2.4}$$

while also defining

$$G(z) \equiv \lim_{z_1 \to \infty} H(z) z_1^4 \qquad . \tag{4.2.5}$$

Combining (4.2.2) with (4.2.4) and (4.2.5) we end up with an integral of the form

$$\begin{aligned} \mathcal{A}_n\left[G(z)\right] &= \int G(z) \prod_{i=3}^{n-1} \mathrm{d}z_i \delta(S_i) \\ &= \frac{1}{(2\pi i)^{n-3}} \oint_{S_3=S_4=\cdots S_{n-1}=0} \frac{G(z)}{\prod_{i=3}^{n-1} S_i} \prod_{i=3}^{n-1} \mathrm{d}z_i, \end{aligned} \tag{4.2.6}$$

where the δ-function integrals can be viewed as contour integrations around the residues of S_i. We can then use the global residue theorem which states that the residues at the solutions to the scattering equations are equal to minus the sum of the remaining residues.

The remaining poles lie in the function $G(z)$ when factors $(z_i - z_j)^{-1}$ tend to zero, so finding the sum of theses residues, gives us the residues of S_i. Let us consider the case where $z_i \to z_n = 0$ for $i \in \tau$ where τ is a subset of $\mathbb{Z}_n/\{1,2\}$. We will assume that τ has at least one member, a, other than n. We then define

$$z_a = \epsilon, \qquad z_i = \epsilon x_i \qquad \text{for } \; i \in \tau,$$

so that to order ϵ

$$z_{ij} = \begin{cases} z_i\left(1 + \mathcal{O}(\epsilon)\right) & \text{for } i \notin \tau \;\wedge\; j \in \tau \\ -z_j\left(1 + \mathcal{O}(\epsilon)\right) & \text{for } i \in \tau \;\wedge\; j \notin \tau \; , \\ \epsilon x_{ij} & \text{for } i, j \in \tau \end{cases} \tag{4.2.7}$$

where $x_{ij} \equiv (x_i - x_j)$. Since $G(z)$ consisted of products of z_{ij}, we see that it now factors into a function of the x_i's with $i \in \tau$ as well as a function of the remaining z_i's (with $i \notin \tau$ of course). We also get a factor of ϵ from every factor x_{ij} when both $i, j \in \tau$

$$G(z) = \epsilon^g \hat{G}(z)\tilde{G}(x)\left(1+\mathcal{O}(\epsilon)\right)(-1)^{n_{\text{inv}}}. \tag{4.2.8}$$

Where g counts the number of ϵ factors we get from the $z_{ij} = \epsilon x_{ij}$

$$g = \sum_{\substack{i,j\in\tau \\ i<j}} c_{ij}. \tag{4.2.9}$$

$\tilde{G}(x)$ includes the products of x_{ij} while $\hat{G}(z)$ includes the remaining products z_{ij} not included in τ

$$\begin{aligned} \tilde{G}(x) &= \prod_{\substack{i,j\in\tau \\ i<j}} x_{ij}^{c_{ij}}, \\ \hat{G}(z) &= \lim_{z_i\to 0\forall i} \frac{G(z)}{\prod_{\substack{i,j\in\tau \\ i<j}} z_{ij}^{c_{ij}}}. \end{aligned} \tag{4.2.10}$$

Lastly the factor n_{inv} counts the number of z_{ij}'s where $i \in \tau$ and $j \notin \tau$. The measure can be written in terms of the new variables

$$\prod_{i=3}^{n-1} \mathrm{d}z_i = \mathrm{d}\epsilon\, \epsilon^{|\tau|-2} \prod_{i\in\tau^{\complement}/\{1,2\}} \mathrm{d}z_i \prod_{i\in\tau/\{a,n\}} \mathrm{d}x_i\,, \tag{4.2.11}$$

with $\tau^{\complement}$ meaning the complement of τ and the $\mathrm{d}\epsilon$ comes from setting $z_a = \epsilon$. Similarly the scattering equations can be expanded in the new variables as well. Here we again have two cases

$$\begin{aligned} S_i &= \frac{\tilde{S}_i}{\epsilon}\left(1+\mathcal{O}(\epsilon)\right), & \tilde{S}_i &= \sum_{\substack{j\in\tau \\ j\neq i}} \frac{s_{ij}}{x_{ij}}, && \text{for } i\in\tau, \\ S_i &= \hat{S}_i\left(1+\mathcal{O}(\epsilon)\right), & \hat{S}_i &= \sum_{\substack{j\in\tau \\ j\neq i}} \frac{s_{ij}}{z_{ij}} + \sum_{j\in\tau} \frac{s_{ij}}{z_i}, && \text{for } i\notin\tau, \end{aligned} \tag{4.2.12}$$

where the last sum in $\hat{S}_i$ comes from the fact that $z_j \to 0$ for $j \in \tau$. Just as with $G(z)$ we can now express S_i in terms of the new functions. We get a prefactor, since every term where $i \in \tau$ comes with a $\frac{1}{\epsilon}$. Notice here that getting τ factors of $\frac{1}{\epsilon}$ is overcounting (since $n \in \tau$ but we are excluding it from the product), so we have to subtract one, leaving $\frac{1}{\epsilon^{\tau-1}} = \epsilon^{1-\tau}$

$$\prod_{i=3}^{n-1} S_i = \epsilon^{1-|\tau|}(1+\mathcal{O}(\epsilon))\tilde{S}_a \prod_{i\in\tau^{\complement}/\{1,2\}} \hat{S}_i \prod_{i\in\tau/\{a,n\}} \tilde{S}_i. \tag{4.2.13}$$

Combining (4.2.8), (4.2.11) and (4.2.13) to leading order in ϵ

$$\oint_{S_3=\cdots S_{n-1}=0} \mathrm{d}\epsilon \, \frac{(-1)^{n_{\text{inv}}}}{(2\pi i)^{n-3}} \prod_{i\in\tau^{\complement}/\{1,2\}} \mathrm{d}z_i \prod_{i\in\tau/\{a,n\}} \mathrm{d}x_i \, \frac{\hat{G}(z)\tilde{G}(x)}{\tilde{S}_a \prod_{i\in\tau^{\complement}/\{1,2\}} \hat{S}_i \prod_{i\in\tau/\{a,n\}} \tilde{S}_i} \epsilon^{g+2|\tau|-3}. \tag{4.2.14}$$

Letting $\epsilon \to 0$ we can read of the pole dependence on g by looking at the last term. For higher order poles the Monodromy relations will be used and so for the integration rules we will only focus on first order poles which shows up at $g = 2 - 2|\tau|$. One can show that[1]

$$\sum_{i\in\tau} \tilde{S}_i x_i = s_\tau .$$

We can now rewrite the integral using the global residue theorem. Letting $\epsilon \to 0$ the integral factorizes and we will have one of the residues being at $\tilde{S}_i = 0 \;\; \forall \;\; i \in \tau/\{a,n\}$. For this specific residue we have

$$s_\tau = \sum_{i\in\tau} \tilde{S}_i x_i = \underbrace{\sum_{i\in\tau/\{a,n\}} \tilde{S}_i x_i}_{0} + \underbrace{\tilde{S}_n x_n}_{0} + \tilde{S}_a x_a = \tilde{S}_a .$$

This in turn means that the factor of $\frac{1}{\tilde{S}_a}$ turns into a propagator connecting the lines in the subset. Denoting the integrand by I, the residue is then

$$\mathrm{Res}(I) = \frac{(-1)^{n_{\text{inv}}}}{(2\pi i)^{n-4}} \frac{1}{s_\tau} \oint \prod_{i\in\tau^{\complement}/\{1,2\}} \mathrm{d}z_i \, \frac{\hat{G}(x)}{\hat{S}_i} \oint \prod_{i\in\tau/\{a,n\}} \mathrm{d}x_i \, \frac{\tilde{G}(z)}{\tilde{S}_i} . \tag{4.2.15}$$

In the above, the rescaled variables x have seperated completely from the non-rescaled ones, z, so if we iterate this procedure until all the $n-3$ integrations have been performed, the propagators will either be nested or disjoint. Every time the global residue theorem is used we pick up a factor (-1) leaving us with a $(-1)^{n-3}$ on the final result. Lastly we sum over all residues. From this we are now ready to write down the integration rules

[1] For an inductive proof see Appendix A

Integration rules

- For an n point amplitude represent the integrand by placing the points in an n-gon pattern.
- For each factor $(z_{ij})^{-1}$ draw a line between points i and j.
- For each factor (z_{ij}) draw a dashed line between points i and j.
- Find all subsets of vertices $\tau \subset \{1, 2, \ldots, n\}$ that are connected by $2\tau - 2$ lines. Due to momentum conservation, complimentary subsets are equivalent.
- Identify all compatible subsets that are remaining. Form groups comprised each of $n-3$ of the subsets in every way possible. Compatible is here taken to mean subsets that are either disjoint or nested.
- The above groups containing subsets $\{i, j, \ldots, k\}$ now each contribute a factor $\frac{1}{s_{ij\ldots k}}$.
- The integral is then the sum over the contribution of all the groups multiplied by a factor $(-1)^{n_{\mathrm{inv}}+n-3}$.

4.3 Integrands and examples

Having now derived a diagrammatic way to obtain expressions from the CHY integral it is time to explore the rules for different theories. We will start off by going through φ^3-theory since it turns out to be remarkably simple. In this case the number of diagrams needed does in fact not depend on the number of points, which is in in stark contrast to the Feynman diagrammatic expansion.

For the n-point tree level amplitude $A_n^{\varphi^3}(1, 2, 3, \ldots, n)$ the left and right integrands are

$$\mathcal{I}_L^{\varphi^3} = \mathrm{PT}(1, 2, 3 \ldots, n), \qquad \mathcal{I}_R^{\varphi^3} = \mathrm{PT}(1, 2, 3, \ldots, n), \tag{4.3.1}$$

where $\mathrm{PT}(1, 2, 3 \ldots, n)$ denotes a *Parke-Taylor factor*

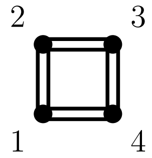
$$\mathrm{PT}(1, \ldots, n) \equiv \frac{1}{(z_1 - z_2)(z_2 - z_3)\cdots(z_n - z_1)} = \frac{1}{z_{12}z_{23}\cdots z_{n1}}. \tag{4.3.2}$$

So the amplitude is obtained through the integral

$$A_n^{\varphi^3}(1, 2, 3, \ldots, n) = \int \mathrm{d}\Omega_{\mathrm{CHY}} \frac{1}{z_{12}^2 z_{23}^2 \cdots z_{n1}^2}. \tag{4.3.3}$$

As an example, let us consider $n = 4$. Using the integrations rules just established, the integrand $\mathcal{I} = \mathcal{I}_L \times \mathcal{I}_R$ then corresponds the diagram

$$\mathcal{I}(z) = \frac{1}{z_{12}^2 z_{23}^2 z_{34}^2 z_{41}^2} \rightarrow \tag{4.3.4}$$



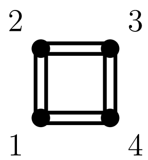
Taking the set $\{1, 2, 3, 4\}$ we now look at subsets containing τ number of points, where the weight of lines in the subset equals

$$w = 2\tau - 2. \tag{4.3.5}$$

The smallest possible subset contains two points and then needs weight +2. For the four-point diagram in 4.3.4 the subsets are

$$\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}.$$

The subsets now have to be combined. They are *compatible* if they are *disjoint* (no point is present in both subsets) or if they are *completely joint/nested* (one subset is contained in the other). Before we combine them we also have to take momentum conservation into account so {1,2} is for instance complementary to {3,4} and only one of them has to be counted. Now the subsets left are $\{1, 2\}$ and $\{4, 1\}$. None of them are compatible so the integral can then be evaluated by summing the compatible subsets according to the rule $\{i, j\} \rightarrow \frac{1}{s_{ij}}$. We get

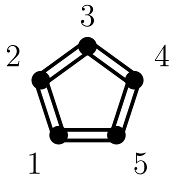


$$\rightarrow -\frac{1}{s_{12}} - \frac{1}{s_{14}}, \tag{4.3.6}$$

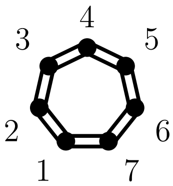
which under support of the scattering equations is also the d-dimensional four-point tree-level amplitude

$$A_4^{\varphi^3}(1, 2, 3, 4) = -\frac{1}{s_{12}} - \frac{1}{s_{14}}. \tag{4.3.7}$$

As already mentioned, these rules are easily implemented in Mathematica and so we can e.g. similarly obtain the five and seven-point amplitudes



$$A_5^{\varphi^3}(1,2,3,4,5) = \quad = \frac{1}{s_{12}s_{34}} + \frac{1}{s_{15}s_{34}} + \frac{1}{s_{12}s_{45}} + \frac{1}{s_{15}s_{23}} + \frac{1}{s_{45}s_{23}}.$$



$$\begin{aligned} A_7^{\varphi^3}(1,2,3,4,5,6,7) = \quad &= \frac{1}{s_{12}s_{34}s_{56}s_{127}} + \frac{1}{s_{17}s_{34}s_{56}s_{127}} + \frac{1}{s_{17}s_{23}s_{45}s_{167}} \\ &+ \frac{1}{s_{23}s_{45}s_{67}s_{167}} + \frac{1}{s_{17}s_{23}s_{56}s_{234}} + \frac{1}{s_{17}s_{34}s_{56}s_{234}} \\ &+ \frac{1}{s_{17}s_{23}s_{167}s_{234}} + \frac{1}{s_{17}s_{34}s_{167}s_{234}} + \frac{1}{s_{23}s_{67}s_{167}s_{234}} \\ &+ \frac{1}{s_{34}s_{67}s_{167}s_{234}} + \frac{1}{s_{12}s_{34}s_{67}s_{345}} + \frac{1}{s_{12}s_{45}s_{67}s_{345}} \\ &+ \frac{1}{s_{12}s_{34}s_{127}s_{345}} + \frac{1}{s_{17}s_{34}s_{127}s_{345}} + \frac{1}{s_{12}s_{45}s_{127}s_{345}} \\ &+ \frac{1}{s_{17}s_{45}s_{127}s_{345}} + \frac{1}{s_{17}s_{34}s_{167}s_{345}} + \frac{1}{s_{17}s_{45}s_{167}s_{345}} \\ &+ \frac{1}{s_{34}s_{67}s_{167}s_{345}} + \frac{1}{s_{45}s_{67}s_{167}s_{345}} + \frac{1}{s_{17}s_{23}s_{45}s_{456}} \\ &+ \frac{1}{s_{17}s_{23}s_{56}s_{456}} + \frac{1}{s_{12}s_{45}s_{127}s_{456}} + \frac{1}{s_{17}s_{45}s_{127}s_{456}} \\ &+ \frac{1}{s_{12}s_{56}s_{127}s_{456}} + \frac{1}{s_{17}s_{56}s_{127}s_{456}} + \frac{1}{s_{12}s_{34}s_{56}s_{567}} \\ &+ \frac{1}{s_{12}s_{34}s_{67}s_{567}} + \frac{1}{s_{23}s_{56}s_{234}s_{567}} + \frac{1}{s_{34}s_{56}s_{234}s_{567}} \\ &+ \frac{1}{s_{23}s_{67}s_{234}s_{567}} + \frac{1}{s_{34}s_{67}s_{234}s_{567}} + \frac{1}{s_{12}s_{45}s_{67}s_{123}} \\ &+ \frac{1}{s_{23}s_{45}s_{67}s_{123}} + \frac{1}{s_{12}s_{45}s_{456}s_{123}} + \frac{1}{s_{23}s_{45}s_{456}s_{123}} \\ &+ \frac{1}{s_{12}s_{56}s_{456}s_{123}} + \frac{1}{s_{23}s_{56}s_{456}s_{123}} + \frac{1}{s_{12}s_{56}s_{567}s_{123}} \\ &+ \frac{1}{s_{23}s_{56}s_{567}s_{123}} + \frac{1}{s_{12}s_{67}s_{567}s_{123}} + \frac{1}{s_{23}s_{67}s_{567}s_{123}}. \end{aligned}$$

The number of Feynman diagrams needed to compute the n-point amplitude for bi-adjoint scalar field theory is given by the function [22]

$$C(n) = \frac{(2n-5)!!2^{n-2}}{(n-1)!} \tag{4.3.8}$$

meaning we would have had to draw 42 Feynman diagrams instead of just one CHY diagram, to calculate the seven-point amplitude. The above is of course the simplest example imaginable and in [52] all the known left and right CHY integrands were presented. The ones relevant for this thesis are shown in Table 4.1.

Theory	$\mathcal{I}_L$	$\mathcal{I}_R$
pure bi-adjoint scalar	$\mathrm{PT}(\alpha)$	$\mathrm{PT}(\beta)$
pure color-ordered Yang-Mills	$\mathrm{PT}(\alpha)$	$\mathrm{Pf}'\Psi$
gravity theory	$\mathrm{Pf}'\Psi$	$\mathrm{Pf}'\Psi$

Table 4.1: Form of the integrands for various theories.

Note that the Parke-Taylor factor $\mathrm{PT}(\alpha)$ implies the color order $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of the partial Yang-Mills amplitude. Even without knowing what the quantities in table mean, the double copy approach of obtaining graviton amplitudes from Yang-Mills by replacing the factors containing color information, $\mathrm{PT}(\alpha)$, with kinematic factors, $\mathrm{Pf}'\Psi$, is seen to come out naturally in the CHY formalism. We will explore this in detail in chapter 6.

The Pfaffian of an $(2N \times 2N)$ matrix Ψ is defined as

$$\mathrm{Pf}\,\Psi = \frac{1}{2^N N!} \sum_{\sigma \in S_{2N}} \mathrm{sgn}(\sigma) \prod_{i=1}^{N} \Psi_{\sigma(2i-1),\sigma(2i)}, \tag{4.3.9}$$

and $\mathrm{Pf}'(\Psi)$ is the reduced Pfaffian, obtained by removing the i and j'th rows and columns from the matrix Ψ and dividing by the punctures corresponding to those rows and columns[2]:

$$\mathrm{Pf}' \equiv \frac{(-1)^{i+j}}{z_{ij}} \mathrm{Pf}\,\Psi_{i,j}. \tag{4.3.10}$$

Lastly, Ψ is a skew-symmetric matrix

$$\Psi = \left(\begin{array}{c|c} A & -C^T \\ \hline C & B \end{array}\right), \tag{4.3.11}$$

[2]If e.g. $j > n$ then $z_{ij} \to z_{i(j-n)}$. This will be the case when reducing Ψ in order to obtain scalar-gluon amplitudes

with entrances

$$A_{ij} = \begin{cases} \frac{s_{ij}}{z_{ij}}, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases},$$
$$B_{ij} = \begin{cases} \frac{\epsilon_{ij}}{z_{ij}}, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases}, \tag{4.3.12}$$
$$C_{ij} = \begin{cases} \frac{\epsilon k_{ij}}{z_{ij}}, & \text{if } i \neq j \\ -\sum_{l\neq} \frac{\epsilon k_{il}}{z_{il}}, & \text{if } i = j \end{cases},$$

where $\epsilon_{ij} \equiv \epsilon_i \cdot \epsilon_j$ and $\epsilon k_{ij} \equiv \sqrt{2}\epsilon_i \cdot k_j$. We note that this definition of the dot products is consistent with the one used in [28] and gives the correct normalization of the CHY amplitudes when comparing them to expressions obtained through Feynman diagrams. One problem with this way of casting the matrix elements, is that the integrand as stated, is not term-wise Möbius invariant. This problem was solved in [51] by rewriting the diagonal elements of the C matrix. First notice that for $i \neq l$ we have

$$\frac{\epsilon k_{il}}{z_{ai}} + \frac{\epsilon k_{il} z_{la}}{z_{ai} z_{il}} = \frac{\epsilon k_{il} z_{il}}{z_{ai} z_{il}} + \frac{\epsilon k_{il} z_{la}}{z_{ai} z_{il}} = \frac{\epsilon k_{il}(z_i - z_l) + \epsilon k_{il}(z_l - z_a)}{z_{ai} z_{il}}$$
$$= -\frac{\epsilon k_{il} z_{ai}}{z_{ai} z_{il}} = -\frac{\epsilon k_{il}}{z_{il}}.$$

So the diagonal terms of C can be rewritten

$$C_{ii} = \sum_{l\neq i} \frac{\epsilon k_{il}}{z_{ai}} + \frac{\epsilon k_{il} z_{la}}{z_{ai} z_{il}} = \sum_{l\neq i} \frac{\epsilon k_{il} z_{la}}{z_{ai} z_{il}}, \tag{4.3.13}$$

where the first term vanishes because of momentum conservation.

$$\sum_{l\neq i} \epsilon k_{il} = -\epsilon k_{ii} = 0.$$

This form of the C matrix is manifestly Möbius invariant and the reduced Pfaffian is guaranteed to be term-wise Möbius invariant. When actually calculating amplitudes we have to make choices for a. Since $a \neq i$, we can for instance choose $a = 2$ when $i = 1$ and $a = 1$ when $i > 1$, giving us a concrete Möbius invariant representation of the diagonal elements:

$$C_{ii} = \begin{cases} \sum_{l=3}^{n} \frac{\epsilon k_{1l} z_{l2}}{z_{21} z_{1l}}, & i = 1 \\ \sum_{l\notin\{1,i\}} \frac{\epsilon k_{il} z_{l1}}{z_{1i} z_{il}}, & i > 1 \end{cases}. \tag{4.3.14}$$

In the Mathematica code used for a large part of the thesis this implementation has been chosen, but we want to stress that taking a different value of a, while not changing the end result, can have a large effect on the simplicity of the intermediate steps. For the sake of concreteness let us consider the simple example of $n = 3$. Here we have a 6×6 matrix Ψ. The elements are

$$
\begin{gathered}
A_{11} = A_{22} = A_{33} = 0, \quad A_{12} = -A_{21} = \frac{s_{12}}{z_{12}}, \quad A_{13} = -A_{13} = \frac{s_{13}}{z_{13}}, \quad A_{23} = -A_{32} = \frac{s_{23}}{z_{23}} \\
B_{11} = B_{22} = B_{33} = 0, \quad B_{12} = -B_{21} = \frac{\epsilon_{12}}{z_{12}}, \quad B_{13} = -B_{13} = \frac{\epsilon_{13}}{z_{13}}, \quad B_{23} = -B_{32} = \frac{\epsilon_{23}}{z_{23}} \\
C_{12} = \frac{\epsilon k_{12}}{z_{12}}, \quad C_{13} = \frac{\epsilon k_{13}}{z_{13}}, \quad C_{23} = \frac{\epsilon k_{23}}{z_{23}}, \quad C_{21} = \frac{\epsilon k_{21}}{z_{21}}, \quad C_{32} = \frac{\epsilon k_{32}}{z_{32}}, \quad C_{31} = \frac{\epsilon k_{31}}{z_{31}} \\
C_{11} = \frac{\epsilon k_{13} z_{32}}{z_{31} z_{13}}, \qquad C_{22} = \frac{\epsilon k_{23} z_{31}}{z_{12} z_{23}}, \qquad C_{33} = \frac{\epsilon k_{32} z_{21}}{z_{13} z_{32}}.
\end{gathered}
\tag{4.3.15}
$$

We can then explicitly write out the matrix Ψ and show its reduction. If we reduce Ψ by e.g. $\{i, j\} = \{1, 3\}$ this corresponds to.

$$
\Psi = \begin{pmatrix}
0 & \frac{s_{12}}{z_{12}} & \frac{s_{13}}{z_{13}} & -\frac{z_{23}\epsilon k_{13}}{z_{12}z_{13}} & \frac{\epsilon k_{21}}{z_{12}} & \frac{\epsilon k_{31}}{z_{13}} \\
-\frac{s_{12}}{z_{12}} & 0 & \frac{s_{23}}{z_{23}} & -\frac{\epsilon k_{12}}{z_{12}} & \frac{z_{13}\epsilon k_{23}}{z_{12}z_{23}} & \frac{\epsilon k_{32}}{z_{23}} \\
-\frac{s_{13}}{z_{13}} & -\frac{s_{23}}{z_{23}} & 0 & -\frac{\epsilon k_{13}}{z_{13}} & -\frac{\epsilon k_{23}}{z_{23}} & -\frac{z_{12}\epsilon k_{32}}{z_{13}z_{23}} \\
\frac{z_{23}\epsilon k_{13}}{z_{12}z_{13}} & \frac{\epsilon k_{12}}{z_{12}} & \frac{\epsilon k_{13}}{z_{13}} & 0 & \frac{\epsilon_{12}}{z_{12}} & \frac{\epsilon_{13}}{z_{13}} \\
-\frac{\epsilon k_{21}}{z_{12}} & -\frac{z_{13}\epsilon k_{23}}{z_{12}z_{23}} & \frac{\epsilon k_{23}}{z_{23}} & -\frac{\epsilon_{12}}{z_{12}} & 0 & \frac{\epsilon_{23}}{z_{23}} \\
-\frac{\epsilon k_{31}}{z_{13}} & -\frac{\epsilon k_{32}}{z_{23}} & \frac{z_{12}\epsilon k_{32}}{z_{13}z_{23}} & -\frac{\epsilon_{13}}{z_{13}} & -\frac{\epsilon_{23}}{z_{23}} & 0
\end{pmatrix},
$$

$$
\to \Psi^{1,3} = \begin{pmatrix}
0 & -\frac{\epsilon k_{12}}{z_{12}} & \frac{z_{13}\epsilon k_{23}}{z_{12}z_{23}} & \frac{\epsilon k_{32}}{z_{23}} \\
\frac{\epsilon k_{12}}{z_{12}} & 0 & \frac{\epsilon_{12}}{z_{12}} & \frac{\epsilon_{13}}{z_{13}} \\
-\frac{z_{13}\epsilon k_{23}}{z_{12}z_{23}} & -\frac{\epsilon_{12}}{z_{12}} & 0 & \frac{\epsilon_{23}}{z_{23}} \\
-\frac{\epsilon k_{32}}{z_{23}} & -\frac{\epsilon_{13}}{z_{13}} & -\frac{\epsilon_{23}}{z_{23}} & 0
\end{pmatrix}.
$$

Taking the reduced Pfaffian of this 4×4 matrix

$$
\mathrm{Pf}' \Psi^{1,3} = \frac{(-1)^4}{z_{13}} \frac{1}{8} \sum_{\sigma \in S_4} \mathrm{sgn}(\sigma) \prod_{i=1}^{2} \Psi^{1,3}_{\sigma(2i-1),\sigma(2i)}, \tag{4.3.16}
$$

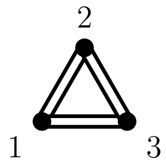
the group S_4 has 24 elements, but amazingly there are only three distinct elements in the sum (since factors like ϵ_{ij} and ϵk_{lm} commute), and each element shows up exactly eight times, canceling the prefactor in front so we are left with

$$
\begin{aligned}
\mathrm{Pf}' \Psi^{1,3} &= \frac{1}{z_{13}} \left(\Psi^{1,3}_{14} \Psi^{1,3}_{23} - \Psi^{1,3}_{13} \Psi^{1,3}_{24} + \Psi^{1,3}_{12} \Psi^{1,3}_{34} \right) \\
&= \frac{\epsilon_{12} \epsilon k_{32}}{z_{12} z_{23} z_{13}} - \frac{\epsilon_{13} \epsilon k_{23}}{z_{12} z_{23} z_{13}} - \frac{\epsilon_{23} \epsilon k_{12}}{z_{12} z_{23} z_{13}}.
\end{aligned}
\tag{4.3.17}
$$

The amplitude is then (under support of the scattering equations)

$$\begin{aligned} A_3^{\text{YM}}(1,2,3) &= \text{PT}(1,2,3)\text{Pf}'\Psi^{1,3} \\ &= \frac{1}{z_{12}^2 z_{23}^2 z_{13}^2}\left\{-\epsilon_{12}\epsilon k_{32} + \epsilon_{13}\epsilon k_{23} + \epsilon_{23}\epsilon k_{12}\right\} \end{aligned}$$

$$= $$



$$\left\{-\epsilon_{12}\epsilon k_{32} + \epsilon_{13}\epsilon k_{23} + \epsilon_{23}\epsilon k_{12}\right\} \tag{4.3.18}$$

$$= -\epsilon_{12}\epsilon k_{32} - \epsilon_{13}\epsilon k_{21} + \epsilon_{23}\epsilon k_{12},$$

where we have used momentum conservation $\epsilon k_{23} = -\epsilon k_{21} - \epsilon k_{22}$ and transversality $\epsilon k_{22} = 0$ in the last line. As a simple check we can compare this to the three-gluon amplitude calculated through Feynman diagrams. This can be obtained by taking the gluon three-vertex from chapter 2 and contracting with the external polarization vectors

$$\begin{aligned} iA_3^{\text{YM}}(1,2,3) &= \frac{i}{\sqrt{2}}\left\{(\epsilon_{12})\epsilon_3\cdot(k_1-k_2) + (\epsilon_{23})\epsilon_1\cdot(k_2-k_3) + (\epsilon_{13})\epsilon_2\cdot(k_3-k_1)\right\} \\ &= i\left\{-\epsilon_{12}\epsilon k_{32} - \epsilon_{13}\epsilon k_{21} + \epsilon_{23}\epsilon k_{12}\right\}, \end{aligned} \tag{4.3.19}$$

where we have used momentum conservation in the following way

$$k_1 + k_2 + k_3 = 0 \Rightarrow \begin{cases} k_1 - k_2 = -2k_2 - k_3 \\ k_3 - k_1 = -2k_1 - k_2 \\ k_2 - k_3 = -2k_3 - k_1 \end{cases}. \tag{4.3.20}$$

The CHY result matches the Feynman diagram approach as expected. We again note that dot product convention used, $\epsilon k_{ij} \equiv \sqrt{2}(\epsilon_i \cdot k_j)$, is exactly the one needed to get the same normalization as the amplitude obtained through Feynman diagrams. At three points we do not encounter any higher order poles in the integrands, but as they arise already at four points, we will now give a prescription to remove them.

4.4 Monodromy relations

As already mentioned, we sometimes have to deal with higher order poles in the integrands. The trick is to multiply the integrand by something that is equal to 1 (under integration) which removes the pole. The identities used for this are known as the monodromy relations and were first described in CHY by [51]:

$$1 = \mathrm{Id}_{\{1,\ldots,k\}} \equiv \frac{-1}{\mathrm{PT}(1,\ldots,n)}\frac{1}{s_{1\cdots k}} \times \sum_{\sigma\in\left(\{2,\ldots,k\}\tilde{\sqcup\!\sqcup}\{k+1,\ldots,n-1\}\right)} \mathrm{PT}(1,\sigma_1,\ldots,\sigma_{n-2},n)\left(s_{1\cdots k} + \sum_{\{i,j\}|\sigma_i>\sigma_j} s_{\sigma_i\sigma_j}\right). \tag{4.4.1}$$

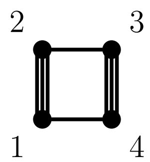
with $\tilde{\sqcup\!\sqcup}$ denoting the shuffle product exluding the identity. By shuffles, we mean all permutations which preserve the relative ordering of the original sets. The last sum depends on the number of elements that trade places.

A higher order pole is defined (see the derivation of the rules) to be subset with k points that have weight $w > 2k-2$ and are called problematic k-tuples. It will be instructive to look at examples of how the identity is applied, so we will go through both a four- and five-point example before we proceed with calculating Yang-Mills and gravity amplitudes.

Four-point example

One diagram that shows up in the calculation of the four-point pure gluon amplitude is also an example of a problematic integrand:

$$\frac{1}{z_{12}^3 z_{23} z_{34}^3 z_{14}} \to$$



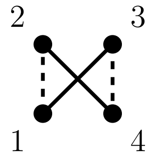
Here we have a higher order pole in the subsets $\{1,2\}$ and $\{3,4\}$ since they have weight $w = 3 > 2$. This statement is equivalent to the points 1 and 2 being connected by three lines and likewise for the points 3 and 4. The two subsets $\{1,2\}$ and $\{3,4\}$ are complimentary though, so only one of the poles has to be removed. The monodromy identity here reads:

$$\mathrm{Id}_{\{1,2\}} = -\frac{1}{\mathrm{PT}(1,2,3,4)}\frac{1}{s_{12}} \sum_{\sigma\in\left(\{2\}\tilde{\sqcup\!\sqcup}\{3\}\right)} \mathrm{PT}(1,\sigma_1,\sigma_2,4)\left(s_{12} + \sum_{\{i,j\}|\sigma_i>\sigma_j} s_{\sigma_i\sigma_j}\right).$$

Since the shuffling excludes the identity, the only Parke-Taylor factor in the sum is the one with 2 and 3 mixed: $\mathrm{PT}(1,3,2,4)$. Also, since 3 and 2 trade places, the mandelstam variable in the last sum is $\sum_{\{i,j\}|\sigma_i>\sigma_j} s_{\sigma_i\sigma_j} = s_{23}$. All in all the identity reads

$$\mathrm{Id}_{\{1,2\}} = -\left(-z_{12}z_{23}z_{34}z_{14}\right)\left(\frac{1}{s_{12}}\right)\left(\frac{1}{z_{13}z_{23}z_{24}z_{14}}\right)(s_{12}+s_{23})$$

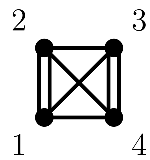
$$= \left(\frac{s_{12}+s_{23}}{s_{12}}\right)$$



.

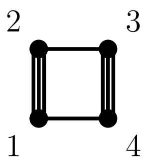
Notice how we have adopted canonical ordering for all the z_{ij}'s and have adjusted the signs accordingly. We have drawn the punctures in terms of a diagram, since we can then easily read of where to remove and add lines when multiplying this with the problematic diagram. It is clear that this factor removes the "bad" lines and introduces two new ones at the expense of some mandelstam factors. Applying the identity gives

$$\mathrm{Id}_{\{1,2\}}\frac{1}{z_{12}^3 z_{23} z_{34}^3 z_{14}} = \left(\frac{s_{12}+s_{23}}{s_{12}}\right)\frac{1}{z_{12}^2 z_{13} z_{23} z_{34}^2 z_{24} z_{14}} \to \left(\frac{s_{12}+s_{23}}{s_{12}}\right)$$

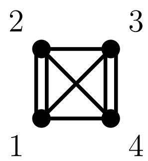


.

The remaining diagram is easily evaluated using the derived integration rules since the only contributing set (again using the complementary condition as well as needing weight $w = 2$) is $\{1, 2\}$ so we end up with the following diagrammatic reduction



$$\to \left(\frac{s_{12}+s_{23}}{s_{12}}\right)$$



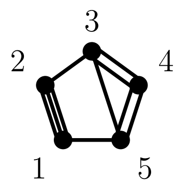
$$\to -\left(\frac{s_{12}+s_{23}}{s_{12}^2}\right) = \frac{s_{13}}{s_{12}^2}. \tag{4.4.2}$$

The application of the identity is very algorithmic and makes us able to use the integration rules for any integrand. Before we proceed with the full four-point Yang-Mills let us apply the monodromy relations to a harder problematic diagram.

Five-point example

An example of a problematic five-point diagram could for instance be

$$\frac{1}{z_{12}^3 z_{23} z_{34}^2 z_{45}^2 z_{15} z_{35}} \to$$



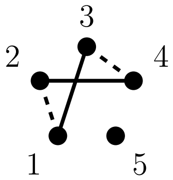
.

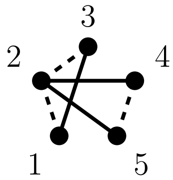
We again have a problematic 2-tuple for the subset $\{1,2\}$. The identity here reads

$$\mathrm{Id}_{\{1,2\}} = -\frac{1}{\mathrm{PT}(1,2,3,4,5)}\frac{1}{s_{12}} \sum_{\sigma\in\left(\{2\}\tilde{\sqcup}\{3,4\}\right)} \mathrm{PT}(1,\sigma_1,\sigma_2,\sigma_3,5)\left(s_{12}+\sum_{\{i,j\}|\sigma_i>\sigma_j} s_{\sigma_i\sigma_j}\right).$$

Explicitly writing out the sum we get

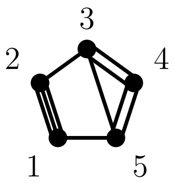
$$\begin{aligned}\mathrm{Id}_{\{1,2\}} &= -\frac{1}{\mathrm{PT}(1,2,3,4,5)}\frac{1}{s_{12}}\Big(\mathrm{PT}(1,3,2,4,5)\left(s_{12}+s_{23}\right)+\mathrm{PT}(1,3,4,2,5)\left(s_{12}+s_{2(34)}\right)\Big)\\ &= \left(-z_{12}z_{23}z_{34}z_{45}z_{51}\right)\left(\frac{1}{z_{13}z_{32}z_{24}z_{45}z_{51}}\frac{s_{12}+s_{23}}{s_{12}}+\frac{1}{z_{13}z_{34}z_{42}z_{25}z_{51}}\frac{s_{12}+s_{234}}{s_{12}}\right)\end{aligned}$$

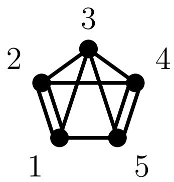
$$= \frac{s_{12}+s_{23}}{s_{12}}$$ 

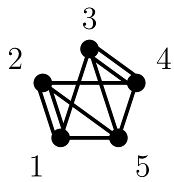
$$+ \frac{s_{12}+s_{234}}{s_{12}}$$ 

,

where we have denoted the two sets that we shuffle by coloring them. Using the identity on the five-point diagram we can now use the last line of the above to again easily see where to remove and add lines:

$\mathrm{Id}_{\{1,2\}}$ 

$$= \frac{s_{12}+s_{23}}{s_{12}}$$ 

$$+ \frac{s_{12}+s_{234}}{s_{12}}$$ 

.

Again we see that employing the identity, exactly cancels the higher order pole in exchange of an appropriate combination mandelstam variables. If any of the remaining diagrams had contained higher order poles after applying the identity, the procedure would just have to be repeated until all the problematic poles are removed.

4.5 Four-point Yang-Mills

Having shown how the procedure works we will now compute a few examples. One benefit of the monodromy relations is that the identity is fairly simple to employ in a computer program and in the remaining we're going to use a Mathematica code to compute most of the CHY diagrams needed. The first example we will go through is the color ordered four-point pure gluon amplitude. The CHY integral we need to compute is

$$A_4^{\mathrm{YM}}(1,2,3,4) = \int \mathrm{d}\Omega_{\mathrm{CHY}}\,\mathrm{PT}(1,2,3,4)\mathrm{Pf}'(\Psi_{i,j}). \tag{4.5.1}$$

We choose to reduce Ψ by $\{i,j\}=\{1,2\}$ and for the sake of concreteness we write out $\Psi_{1,2}$

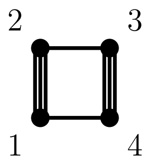
$$\Psi=\begin{pmatrix} 0 & \frac{s_{12}}{z_{34}} & -\frac{\epsilon k_{13}}{z_{13}} & -\frac{\epsilon k_{23}}{z_{23}} & \frac{\epsilon k_{34}z_{14}}{z_{13}z_{34}}-\frac{\epsilon k_{32}z_{12}}{z_{13}z_{23}} & \frac{\epsilon k_{43}}{z_{34}} \\ -\frac{s_{12}}{z_{34}} & 0 & -\frac{\epsilon k_{14}}{z_{14}} & -\frac{\epsilon k_{24}}{z_{24}} & -\frac{\epsilon k_{34}}{z_{34}} & -\frac{\epsilon k_{42}z_{12}}{z_{14}z_{24}}-\frac{\epsilon k_{43}z_{13}}{z_{14}z_{34}} \\ \frac{\epsilon k_{13}}{z_{13}} & \frac{\epsilon k_{14}}{z_{14}} & 0 & \frac{\epsilon_{12}}{z_{12}} & \frac{\epsilon_{13}}{z_{13}} & \frac{\epsilon_{14}}{z_{14}} \\ \frac{\epsilon k_{23}}{z_{23}} & \frac{\epsilon k_{24}}{z_{24}} & -\frac{\epsilon_{12}}{z_{12}} & 0 & \frac{\epsilon_{23}}{z_{23}} & \frac{\epsilon_{24}}{z_{24}} \\ \frac{\epsilon k_{32}z_{12}}{z_{13}z_{23}}-\frac{\epsilon k_{34}z_{14}}{z_{13}z_{34}} & \frac{\epsilon k_{34}}{z_{34}} & -\frac{\epsilon_{13}}{z_{13}} & -\frac{\epsilon_{23}}{z_{23}} & 0 & \frac{\epsilon_{34}}{z_{34}} \\ -\frac{\epsilon k_{43}}{z_{34}} & \frac{\epsilon k_{42}z_{12}}{z_{14}z_{24}}+\frac{\epsilon k_{43}z_{13}}{z_{14}z_{34}} & -\frac{\epsilon_{14}}{z_{14}} & -\frac{\epsilon_{24}}{z_{24}} & -\frac{\epsilon_{34}}{z_{34}} & 0 \end{pmatrix}.$$

The amplitude resulting from the Pfaffian of this can be expanded as [40]

$$A_4=\alpha_1\epsilon_{12}\epsilon_{34}+\alpha_2\epsilon_{13}\epsilon_{24}+\beta_1\epsilon_{12}+\beta_2\epsilon_{13}+\textit{distinct}\text{ cyclic},\tag{4.5.2}$$

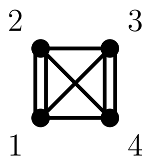
where the coefficients can e.g. be found by differentiating the integrand w.r.t. the specific polarization-products in front, so obtaining the first two

$$\alpha_1=\partial_{\epsilon_{12}}\partial_{\epsilon_{34}}A_4=s_{12}$$



$$=\frac{s_{13}}{s_{12}},$$

$$\alpha_2=\partial_{\epsilon_{13}}\partial_{\epsilon_{24}}A_4=-s_{12}\tag{4.5.3}$$

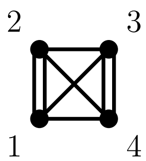


$$=1,$$

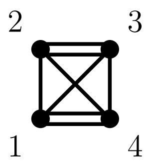
where the monodromy relations have been applied to the problematic diagram α_1. For the two remaining terms we only differentiate once, so we have to subtract the terms that would contribute to α_1 and α_2,

$$\beta_1=\partial_{\epsilon_{12}}A_4-\alpha_1\epsilon_{34}$$

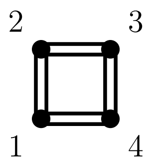
$$=\epsilon k_{34}\epsilon k_{42}$$



$$-\epsilon k_{32}\epsilon k_{42}$$

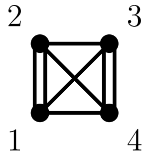
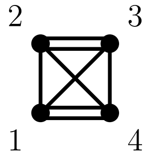
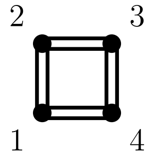


$$-\epsilon k_{32}\epsilon k_{43}\tag{4.5.4}$$



$$=-\frac{\epsilon k_{32}\epsilon k_{41}}{s_{14}}+\frac{\epsilon k_{32}\epsilon k_{43}-\epsilon k_{34}\epsilon k_{42}}{s_{12}},$$

$$\beta_2 = \partial_{\epsilon_{13}} A_4 - \alpha_2 \epsilon_{24}$$

$= \epsilon k_{24} \epsilon k_{43}$  $+ \epsilon k_{23} \epsilon k_{42}$  $+ \epsilon k_{23} \epsilon k_{43}$  (4.5.5)

$$= \frac{\epsilon k_{21} \epsilon k_{43}}{s_{12}} + \frac{\epsilon k_{23} \epsilon k_{41}}{s_{14}}.$$

So that in total we have the amplitude

$$A_4^{\text{YM}} = \Bigg[\epsilon_{13}\epsilon_{24} + \frac{1}{s_{12}} \left\{ \epsilon_{12}\epsilon_{34} s_{13} + \epsilon_{12} \left(\epsilon k_{32} \epsilon k_{43} - \epsilon k_{34} \epsilon k_{42} \right) + \epsilon_{13} \epsilon k_{21} \epsilon k_{43} \right\} + \frac{1}{s_{14}} \left\{ \epsilon_{13} \epsilon k_{23} \epsilon k_{41} - \epsilon_{12} \epsilon k_{32} \epsilon k_{41} \right\} \Bigg] + \textit{ distinct} \text{ cyclic.} \tag{4.5.6}$$

Taking the above and expressing it in massless SH variables with reference vectors $q_1 = 4, q_2 = 4, q_3 = 1$ and $q_4 = 1$ we obtain an exact match with the anti-MHV result from section 3.3.

$$A_4^{\text{YM}}(1^-, 2^-, 3^+, 4^+) = \frac{[34]^4}{[12][23][34][41]}.$$

4.6 Four-point scalar Yang-Mills

One could also have considered mix of particles e.g, two scalars interacting with two gluons, i.e. $A(1_\varphi, 2_g, 3_g, 4_\varphi)$. Since the scalars have no polarization this amounts to setting $\epsilon_{14} = 1$ and $\epsilon_{12} = \epsilon_{13} = \epsilon_{24} = \epsilon_{34} = 0$. One way this could be obtained is by embedding a D-dimensional space into $D + 1$ dimensions, with the D-dimensional polarization denoted by ϵ and $(D + 1)$-dimensional denoted by $\tilde{\epsilon}$. Then setting $\tilde{\epsilon}_1 = (0 \cdots 0, 1)$, $\tilde{\epsilon}_2 = (\epsilon_2, 0)$, $\tilde{\epsilon}_3 = (\epsilon_3, 0)$ and $\tilde{\epsilon}_4 = (0 \cdots 0, 1)$. Note that the momenta all lie in the D-dimensional subspace and so $k_i \cdot \epsilon_1 = k_i \cdot \epsilon_4 = 0 \;\; \forall\, i$.

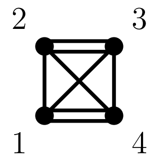
Another interpretation is that it reduces the complexity of the Pfaffian even more since the polarizations of the 1'st and n'th particles all lie in the $n+1$ and $2n$ rows and columns of Ψ [18, 28]. So we can just reduce Ψ further by removing rows and columns $n + 1$ and $2n$ and multiply by $\frac{1}{z_{1n}}$. The general n-point scalar-gluon amplitude is then

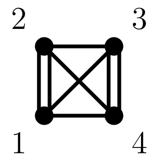
$$A(1_\varphi, 2_g, \ldots, (n-1)_g, n_\varphi) = \int \mathrm{d}\Omega_{\text{CHY}}\, \text{PT}(1, \ldots, n) \text{Pf}'(\Psi_{i,j,n+1,2n}),$$

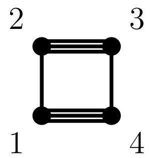
where the four subscripts on Ψ mean that we have reduced it by the i'th, j'th, $(n+1)$'st and $(2n)$'th rows and columns.

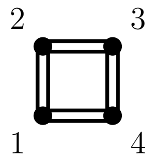
$$\mathrm{Pf}'(\Psi_{i,j,n+1,2n}) = \frac{(-1)^{i+j}}{z_{ij} z_{1n}} \mathrm{Pf}\,(\Psi_{i,j,n+1,2n}).$$

So at four points, taking $\{i, j\} = \{1, 4\}$ we find

$A(1_\varphi, 2_g, 3_g, 4_\varphi) = \epsilon k_{24} \epsilon k_{32}$


$- \epsilon k_{24} \epsilon k_{34}$


$\epsilon_{23} s_{14}$


$- \epsilon k_{23} \epsilon k_{34}$


$$= -\frac{\epsilon k_{21} \epsilon k_{34}}{s_{12}} - \frac{\epsilon k_{24} \epsilon k_{32} - \epsilon k_{23} \epsilon k_{34} + \epsilon_{23}\,(s_{12} + s_{14})}{s_{14}}.$$

This can be put into a form which will be useful for us later on by using momentum conservation on the polarization products, as well as on the mandelstam variables $s_{12} + s_{13} + s_{14} = 0$

$$\begin{aligned} A(1_\varphi, 2_g, 3_g, 4_\varphi) &= -\frac{\epsilon k_{21} \epsilon k_{34}}{s_{12}} - \frac{\epsilon k_{24} \epsilon k_{32} - \epsilon k_{23} \epsilon k_{34} - \epsilon_{23} s_{13}}{s_{14}} \\ &= \frac{\epsilon k_{21} \epsilon k_{34} s_{13}}{s_{12} s_{14}} + \frac{\epsilon k_{24} \epsilon k_{31} + \epsilon_{23} s_{13}}{s_{14}}. \end{aligned}$$

From which we obtain

$$A(1_\varphi, 2_g, 3_g, 4_\varphi) = \frac{s_{13}}{s_{14}} \left\{ \frac{\epsilon k_{21} \epsilon k_{34}}{s_{12}} + \frac{\epsilon k_{24} \epsilon k_{31}}{s_{13}} + \epsilon_{23} \right\}. \tag{4.6.1}$$

Next let us turn to the more involved example of four gravitons.

4.7 Four-point gravity

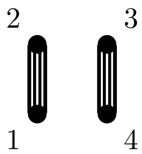
Since gravity amplitudes come from the product of two reduced pfaffians (or a single determinant if the reduced legs are the same for both pfaffians), we can again write the amplitude in terms of polarization products and coefficients, but this time we have to include all permutations and not just cyclic ones.

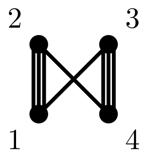
$$\begin{aligned}\mathcal{M}_4 =&\alpha_1\epsilon_{12}^2\epsilon_{34}^2+\alpha_2\epsilon_{12}\epsilon_{13}\epsilon_{34}\epsilon_{24}+\alpha_3\epsilon_{12}^2\epsilon_{34}+\alpha_4\epsilon_{12}\epsilon_{23}\epsilon_{14}\\ &+\alpha_5\epsilon_{12}^2+\alpha_6\epsilon_{12}\epsilon_{13}+\alpha_7\epsilon_{12}\epsilon_{34}+\ \textit{distinct}\text{ permutations.}\end{aligned} \tag{4.7.1}$$

Here it is important to note that the gravitons are decomposed according to $h_i^{\mu\nu}\equiv\epsilon_i^\mu\epsilon_i^\nu$. The coefficients can again be found by differentiating and we have to remember to subtract contributions of the first coefficients on the latter ones.

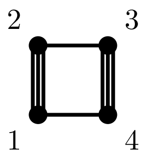
$$\begin{aligned}\alpha_1 =&\frac{1}{4}\partial_{\epsilon_{12}}^2\partial_{\epsilon_{34}}^2\mathcal{M},\\ \alpha_2 =&\partial_{\epsilon_{12}}\partial_{\epsilon_{34}}\partial_{\epsilon_{13}}\partial_{\epsilon_{24}}\mathcal{M},\\ \alpha_3 =&\frac{1}{2}\partial_{\epsilon_{12}}^2\partial_{\epsilon_{34}}\mathcal{M},\\ \alpha_4 =&\partial_{\epsilon_{12}}\partial_{\epsilon_{23}}\partial_{\epsilon_{14}}\mathcal{M}-\text{Perm}[\alpha_2,(1324)]\epsilon_{34},\\ \alpha_5 =&\frac{1}{2}\partial_{\epsilon_{12}}^2\mathcal{M}-\alpha_1\epsilon_{34}^2-\alpha_3\epsilon_{34},\\ \alpha_6 =&\partial_{\epsilon_{12}}\partial_{\epsilon_{13}}\mathcal{M}-\text{Perm}[\alpha_4,(12)]\epsilon_{24}-\text{Perm}[\alpha_4,(132)\epsilon_{34}]+\alpha_2\epsilon_{34}\epsilon_{24},\\ \alpha_7 =&\partial_{\epsilon_{12}}\partial_{\epsilon_{34}}\mathcal{M}-\text{Perm}[\alpha_4,(132)]\epsilon_{13}-\text{Perm}[\alpha_4,(24)]\epsilon_{14}+\alpha_2\epsilon_{13}\epsilon_{24},\\ &-\text{Perm}[\alpha_4,(124)]\epsilon_{24}-\text{Perm}[\alpha_4,(1234)]\epsilon_{23}+\text{Perm}[\alpha_2,(12)]\epsilon_{23}\epsilon_{14}.\end{aligned} \tag{4.7.2}$$

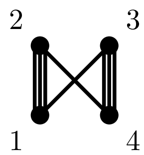
Here we have adopted the notation Perm$[a,b]$ to denote the permutation of a by the cycle b. The coeffecients result in the following CHY-diagrams

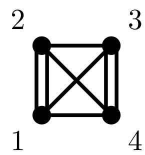
$\alpha_1 = s_{12}^2$ 

, $\alpha_2 = -2s_{12}^2$ 

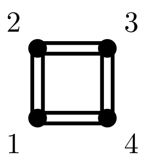
,

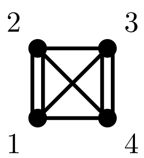
$\alpha_3 = -2\epsilon k_{32}\epsilon k_{43}s_{34}$ 

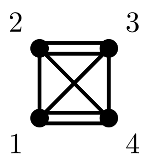
$+2\epsilon k_{34}\epsilon k_{42}s_{34}$ 

$-2\epsilon k_{32}\epsilon k_{42}s_{34}$ 

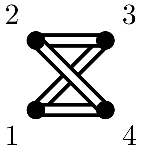
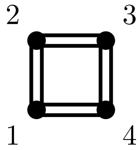
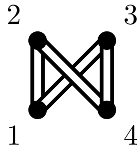
,

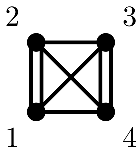
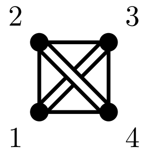
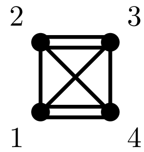
$\alpha_4 = -2\epsilon k_{32}\epsilon k_{43}s_{34}$ 

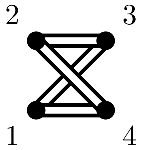
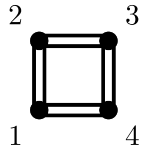
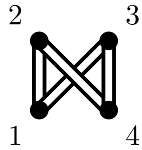
$2\epsilon k_{34}\epsilon k_{42}s_{34}$ 

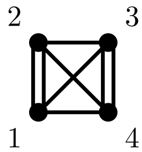
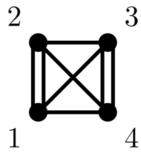
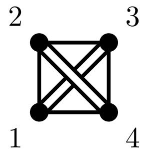
$-2\epsilon k_{32}\epsilon k_{42}s_{34}$ 

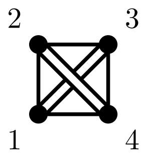
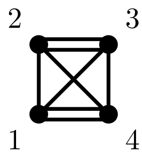
,

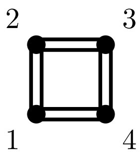
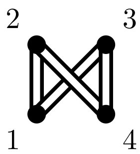
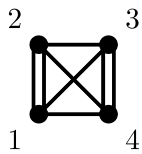
$\alpha_5 = \epsilon k_{32}^2 \epsilon k_{42}^2$  $+ \epsilon k_{32}^2 \epsilon k_{43}^2$  $+ \epsilon k_{34}^2 \epsilon k_{42}^2$ 

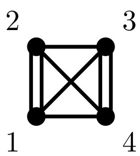
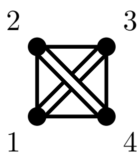
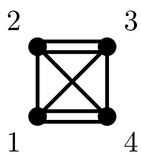
$- 2\epsilon k_{32} \epsilon k_{34} \epsilon k_{42} \epsilon k_{43}$  $- 2\epsilon k_{32} \epsilon k_{34} \epsilon k_{42}^2$  $+ 2\epsilon k_{32}^2 \epsilon k_{42} \epsilon k_{43}$  ,

$\alpha_6 = - 2\epsilon k_{23} \epsilon k_{32} \epsilon k_{42}^2$  $- 2\epsilon k_{23} \epsilon k_{32} \epsilon k_{43}^2$  $+ 2\epsilon k_{24} \epsilon k_{34} \epsilon k_{42} \epsilon k_{43}$ 

$+ 2\epsilon k_{23} \epsilon k_{34} \epsilon k_{42} \epsilon k_{43}$  $- 2\epsilon k_{24} \epsilon k_{32} \epsilon k_{43}^2$  $+ 2\epsilon k_{23} \epsilon k_{34} \epsilon k_{42}^2$ 

$- 2\epsilon k_{24} \epsilon k_{32} \epsilon k_{42} \epsilon k_{43}$  $- 4\epsilon k_{23} \epsilon k_{32} \epsilon k_{42} \epsilon k_{43}$  ,

$\alpha_7 = - 2\epsilon k_{14} \epsilon k_{23} \epsilon k_{32} \epsilon k_{43}$  $- 2\epsilon k_{13} \epsilon k_{24} \epsilon k_{34} \epsilon k_{42}$  $+ 2\epsilon k_{14} \epsilon k_{23} \epsilon k_{34} \epsilon k_{42}$ 

$+ 2\epsilon k_{13} \epsilon k_{24} \epsilon k_{32} \epsilon k_{43}$  $2\epsilon k_{13} \epsilon k_{24} \epsilon k_{32} \epsilon k_{42}$  $- 2\epsilon k_{14} \epsilon k_{23} \epsilon k_{32} \epsilon k_{42}$  .

Evaluation of the diagrams using the integration rules and the appropriate monodromy relations then gives the following coefficients in terms of polarizations and momenta

$$\begin{aligned}
\alpha_1 =&\frac{s_{13}s_{23}}{s_{34}}, \qquad \alpha_2 = 2s_{23},\\
\alpha_3 =&\frac{2(\epsilon k_{32}(\epsilon k_{42}s_{34} - \epsilon k_{43}s_{24}) - \epsilon k_{34}\epsilon k_{42}s_{23})}{s_{34}},\\
\alpha_4 =&\frac{2(\epsilon k_{32}(\epsilon k_{42}s_{34} + \epsilon k_{43}s_{42}) - \epsilon k_{34}\epsilon k_{42}s_{14})}{s_{14}},\\
\alpha_5 =&-\frac{(\epsilon k_{31})^2(\epsilon k_{42})^2}{s_{24}} - \frac{(\epsilon k_{32})^2(\epsilon k_{41})^2}{s_{14}} - \frac{(\epsilon k_{34}\epsilon k_{42} - \epsilon k_{32}\epsilon k_{43})^2}{s_{34}},\\
\alpha_6 =&2\left(\frac{\epsilon k_{23}\epsilon k_{32}(\epsilon k_{41})^2}{s_{14}} - \frac{\epsilon k_{21}\epsilon k_{43}(\epsilon k_{32}\epsilon k_{43} - \epsilon k_{34}\epsilon k_{42})}{s_{34}} - \frac{\epsilon k_{31}\epsilon k_{42}(\epsilon k_{23}\epsilon k_{42} - \epsilon k_{24}\epsilon k_{43})}{s_{24}}\right),\\
\alpha_7 =&2\left(-\frac{\epsilon k_{13}\epsilon k_{24}\epsilon k_{31}\epsilon k_{42}}{s_{24}} - \frac{\epsilon k_{14}\epsilon k_{23}\epsilon k_{32}\epsilon k_{41}}{s_{14}} + \frac{(\epsilon k_{13}\epsilon k_{24} - \epsilon k_{14}\epsilon k_{23})(\epsilon k_{34}\epsilon k_{42} - \epsilon k_{32}\epsilon k_{43})}{s_{34}}\right).
\end{aligned}$$

To check this result, we can compare it to the MHV amplitude. Experimenting with different helicity configurations we find the configuration that simplifies the result the most is the same one we used for the four-point Yang-Mills amplitude. The full expression has more than 300 terms, but simplifies in Mathematica to

$$\mathcal{M}(1^-, 2^-, 3^+, 3^+) = -\frac{\langle 12\rangle \langle 13\rangle [34]^6}{\langle 14\rangle [12][13][14][24]^2}.$$

Using $\langle 14\rangle[24] = -\langle 13\rangle[23]$ and $s_{12} = \langle 12\rangle [21]$ this can be written in a more suggestive form

$$\mathcal{M}(1^-, 2^-, 3^+, 3^+) = -s_{12}\frac{[43]^4}{[12][24][43][31]}\frac{[34]^4}{[12][23][34][41]},$$

which is in accordance with the expected KLT relation [36]

$$\mathcal{M}(1^-, 2^-, 3^+, 3^+) = -s_{12}A(1^-, 2^-, 4^+, 3^+)A(1^-, 2^-, 3^+, 4^+).$$

The fact that we can get the tree level four-point graviton amplitude in a covariant form so easily is quite remarkable since just writing the contribution from each of the four Feynman diagrams for the same amplitude would take up about 1 page per diagram [36].

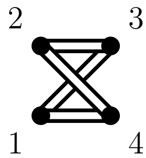
4.8 Four-point scalar graviton amplitude

Our final example is calculating the amplitude for two scalars interacting with two gravitons. This is done by repeating the steps of section 4.6 but this time with the gravity integrand. Taking $\{i, j\} = \{1, 4\}$ and again also removing the parts of Ψ containing information about the scalar polarization "vectors" the amplitude is calculated through

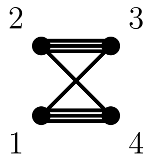
$$\mathcal{M}(1_s, 2_h, 3_h, 1_s) = \int \mathrm{d}\Omega_{\text{CHY}} \, \frac{1}{z_{14}^4} \det(\Psi_{1,4,5,8}).$$

And we find the following contributions under integration and support of the scattering equations

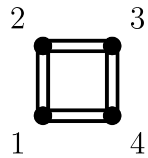
$\mathcal{M}(1_s, 2_h, 3_h, 1_s) =$



$\epsilon k_{24}^2 \epsilon k_{32}^2 +$

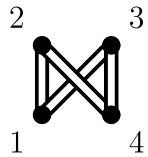


$2\epsilon_{23}\epsilon k_{24}\epsilon k_{32} s_{23} +$

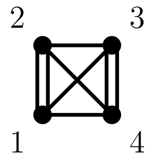


$\epsilon k_{23}^2 \epsilon k_{34}^2$

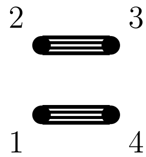
$+$



$\epsilon k_{24}^2 \epsilon k_{34}^2 +$

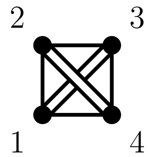


$2\epsilon k_{23}\epsilon k_{24}\epsilon k_{34}^2 +$

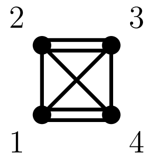


$\epsilon_{23}^2 s_{23}^2$

$-$

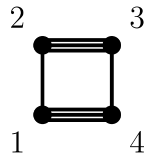


$2\epsilon k_{24}^2 \epsilon k_{32}\epsilon k_{34} -$



$2\left(\epsilon k_{23}\epsilon k_{24}\epsilon k_{32}\epsilon k_{34} + \epsilon_{23}\epsilon k_{24}\epsilon k_{34} s_{23}\right)$

$-$



$2\epsilon_{23}\epsilon k_{23}\epsilon k_{34} s_{23}$

.

After expanding the diagrams using the integration rules and the necessary monodromy relations we find the full scalar-graviton amplitude

$$\begin{aligned}
\mathcal{M}(1_s, 2_h, 3_h, 1_s) =& 2\epsilon_{23}\left[\epsilon k_{24}\epsilon k_{32} + \epsilon k_{23}\epsilon k_{34} + \epsilon k_{24}\epsilon k_{34}\right] + \epsilon_{23}^2\left(s_{14} + s_{13} + s_{12}\right) \\
&+ \frac{1}{s_{14}}\Big(2\epsilon k_{23}\epsilon k_{24}\epsilon k_{32}\epsilon k_{34} - (\epsilon k_{24})^2(\epsilon k_{32})^2 - (\epsilon k_{23})^2(\epsilon k_{34})^2 \\
&+ 2\epsilon_{23}\epsilon k_{24}\epsilon k_{32} s_{13} + 2\epsilon_{23}\epsilon k_{23}\epsilon k_{34} s_{12} + \epsilon_{23}^2 s_{13} s_{12}\Big) \\
&- \frac{1}{s_{13}}\Big(2(\epsilon k_{24})^2\epsilon k_{32}\epsilon k_{34} + (\epsilon k_{24})^2(\epsilon k_{32})^2 + (\epsilon k_{24})^2(\epsilon k_{34})^2\Big) \\
&- \frac{1}{s_{12}}\Big(2\epsilon k_{23}\epsilon k_{24}(\epsilon k_{34})^2 + (\epsilon k_{23})^2(\epsilon k_{34})^2 + (\epsilon k_{24})^2(\epsilon k_{34})^2\Big),
\end{aligned}$$

which we have checked numerically in four dimensions to be equal to

$$\mathcal{M}(1_s, 2_h, 3_h, 1_s) = \frac{s_{13}s_{12}}{s_{14}}\left[\frac{\epsilon k_{21}\epsilon k_{34}}{s_{12}} + \frac{\epsilon k_{24}\epsilon k_{31}}{s_{13}} + \epsilon_{23}\right]^2. \tag{4.8.1}$$

This concludes our chapter on massless CHY. As we have seen the diagrammatic expansion greatly eases the calculation of amplitudes, especially since the rules can be implemented in Mathematica. The amount of diagrams needed is still quite large and the expressions are somewhat unwieldy. In Chapter 6 we will see how to simplify the calculations even further by introducing *BCJ numerators*. So far all the particles in our amplitudes have been massless, but now we turn to constructing amplitudes with massive external particles.

Chapter 5

Adding massive particles

Over the past years, there has been some development to include massive particles in the CHY formalism. S. Naculich, L. Dolan, and P. Goddard pioneered early formulations [17–19] and these were discussed and elaborated in [28]. We will take the approach by Naculich and discuss how to incorporate masses into the diagrammatic rules described in chapter 4. The focus will mostly be on 2 external massive particles but we will include a small discussion on the consequences of including more than 2 masses. After introducing the modified massive scattering equations we show Möbius invariance and calculate some examples using the results obtained with scalars in chapter 4.

5.1 Massive scattering equations

Since the scattering equations are the defining elements of the formalism, an obvious approach is to redefine them in a way that includes masses, while still retaining Möbius invariance. The following discussion is inspired by [28]. Take the massive equations to be

$$E_i(z) \equiv \sum_{\substack{j=1 \\ j\neq i}} \frac{2k_i \cdot k_j + 2\Delta_{ij}}{z_{ij}} = 0, \quad i = 1, ..., n, \tag{5.1.1}$$

where for n massive particles Δ matrix is symmetric and related to the masses of the particles

$$\Delta_{ij} = \Delta_{ji}, \quad \sum_{\substack{i=1 \\ i\neq j}}^{n} \Delta_{ij} = m_j^2. \tag{5.1.2}$$

For two massive particles (m_1, m_2) and four total particles this would look like

$$\begin{aligned} m_1^2 &= \Delta_{12} + \Delta_{13} + \Delta_{14}, \\ m_2^2 &= \Delta_{12} + \Delta_{23} + \Delta_{24} \end{aligned}$$

and for three (m_1, m_2, m_3) with five particles we get

$$\begin{aligned} m_1^2 &= \Delta_{12} + \Delta_{13} + \Delta_{14} + \Delta_{15}, \\ m_2^2 &= \Delta_{12} + \Delta_{23} + \Delta_{24} + \Delta_{25}, \\ m_3^2 &= \Delta_{13} + \Delta_{23} + \Delta_{34} + \Delta_{35}. \end{aligned}$$

One problem is now that we have to many matrix elements to fix them all and we have to make a choice for some of them. A convenient choice is obviously to set some of them to zero, so taking $\Delta_{14} = \Delta_{15} = 0$ we get the simple invertible relations

$$\begin{aligned} m_1^2 &= \Delta_{12} + \Delta_{13}, \\ m_2^2 &= \Delta_{12} + \Delta_{23}, \\ m_3^2 &= \Delta_{13} + \Delta_{23}, \end{aligned}$$

with solutions

$$\begin{aligned} \Delta_{12} &= \frac{m_1^2 + m_2^2 - m_3^4}{2}, \\ \Delta_{13} &= \frac{m_1^2 - m_2^2 + m_3^4}{2}, \\ \Delta_{23} &= \frac{-m_1^2 + m_2^2 + m_3^4}{2}. \end{aligned}$$

As pointed out by [28] this has the interesting consequence that for one massive particle e.g. $m_1^2 \neq 0$ and $m_2 = m_3 = 0$ we get three non-zero Δ's, namely $\Delta_{12} = \frac{m_1^2}{2}$, $\Delta_{13} = \frac{m_1^2}{2}$ and $\Delta_{23} = \frac{-m_1^2}{2}$. The situation gets a lot easier when just looking at 2 massive particles with equal mass, e.g. $m_1 = m_n = m$. In this case it is sufficient to consider $\Delta_{1n} = \Delta_{n1} = m^2$ and set the remaining Δ's to zero. Furthermore the usual sub-matrices of Ψ are all unchanged except for the matrix A (4.3.12) [18].

$$A_{ij} = \begin{cases} \frac{2k_i \cdot k_j}{z_{ij}}, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases} \quad \rightarrow \quad A_{ij} = \begin{cases} \frac{2k_i \cdot k_j + 2\Delta_{ij}}{z_{ij}}, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases}.$$

The trick is now to remove the 1st and nth row and column of Ψ when taking the reduced Pfaffian. Since these entrances are the only ones where the Δ's show up we obtain

$$\mathrm{Pf}'_{\text{massive}} \Psi_{i,n} = \mathrm{Pf}'_{\text{massless}} \Psi_{i,n}. \tag{5.1.3}$$

The solutions to the scattering equations themselves are of course changed. In the diagrammatic expansion we have used in this paper, this corresponds to the propagators stemming from the massless diagrams being replaced in the following way

$$\begin{aligned} s_{1n} &\to 2k_1 \cdot k_n + 2m^2 = s_{1n}, & \\ s_{1i} &\to 2k_1 \cdot k_i = s_{1i} - m^2, & \text{for } i \neq \{1, n\}, \\ s_{in} &\to 2k_i \cdot k_n = s_{in} - m^2, & \text{for } i \neq \{1, n\}, \\ s_{ij} &\to 2k_i \cdot k_j = s_{ij}, & \text{for } i, j \neq \{1, n\}. \end{aligned} \tag{5.1.4}$$

Since the massive version of the scattering equations has to hold under Möbius transformations, just as the massless ones, let us prove that the massive scattering equations indeed are invariant, before we calculate a few amplitude examples. The proof follows analogously to the one performed for the massless version.

5.2 Proof of Möbius invariance for the massive scattering equations

Here we show that the massive scattering equations $E_i = 0$ hold under a $\mathrm{SL}(2, \mathbb{C})$ transformation of the auxiliary variables. The transformation takes

$$E_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{s_{ij} + \Delta_{ij}}{z_i - z_j} \to E_i' = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{s_{ij} + \Delta_{ij}}{\frac{az_i + b}{cz_i + d} - \frac{az_j + b}{cz_j + d}}.$$

This is similar to the Möbius tranformation of the massless scattering equations and so can be similarly rewritten.

$$\begin{aligned} E_i' =& \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{(s_{ij} + \Delta_{ij})(cz_i + d)(cz_j + d)}{(z_i - z_j)} \\ =& (cz_i + d) \left(dE_i + c \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{(s_{ij} + \Delta_{ij})(z_j - z_i + z_i)}{(z_i - z_j)} \right) \\ =& (cz_i + d) \left(dE_i + c \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{(s_{ij} + \Delta_{ij})(z_j - z_i)}{(z_i - z_j)} + cz_i \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{(s_{ij} + \Delta_{ij})}{(z_i - z_j)} \right) \\ =& (cz_i + d)^2 E_i, \end{aligned}$$

where we have used the definition of the scattering equations in the first line and in the end term on the last line as well as momentum conservation on the middle term $\sum_{\substack{j=1 \\ j \neq i}}^{n} k_i \cdot k_j = k_i^2 = -m_i^2$ and the definition of the Δ-matrix $\sum_{\substack{j=1 \\ j \neq i}}^{n} \Delta_{ij} = m_i^2$. Since $E_i = 0$ then $E_i' = 0$ and so the transformation leaves the equations invariant. We also see that the Möbius weight of this version of the scattering equations is the same as in the massless case.

5.3 Massive four-point scalar Yang-Mills and scalar graviton amplitudes

In the following we will take the results from section 4.6 and 4.8 and do the replacements (5.1.4) to obtain amplitudes with massive scalars. We will denote the massive legs by **boldface**. The massless amplitudes were

$$\begin{aligned} A(1_s, 2_g, 3_g, 4_s) &= \frac{s_{13}}{s_{14}} \left\{ \frac{\epsilon k_{21} \epsilon k_{34}}{s_{12}} + \frac{\epsilon k_{24} \epsilon k_{31}}{s_{13}} + \epsilon_{23} \right\}, \\ \mathcal{M}(1_s, 2_h, 3_h, 4_s) &= \frac{s_{13} s_{12}}{s_{14}} \left[\frac{\epsilon k_{21} \epsilon k_{34}}{s_{12}} + \frac{\epsilon k_{24} \epsilon k_{31}}{s_{13}} + \epsilon_{23} \right]^2, \end{aligned}$$

and doing the replacements $s_{12} \to s_{12} - m^2$, $s_{13} \to s_{13} - m^2$ and $s_{14} \to s_{14}$ we obtain the amplitudes with massive scalars

$$\begin{aligned} A(\mathbf{1}_s, 2_g, 3_g, \mathbf{4}_s) &= \frac{u - m^2}{t} \left[\frac{\epsilon k_{21} \epsilon k_{34}}{s - m^2} + \frac{\epsilon k_{24} \epsilon k_{31}}{u - m^2} + \epsilon_{23} \right], \\ \mathcal{M}(\mathbf{1}_s, 2_h, 3_h, \mathbf{4}_s) &= \frac{(u - m^2)(s - m^2)}{t} \left[\frac{\epsilon k_{21} \epsilon k_{34}}{s - m^2} + \frac{\epsilon k_{24} \epsilon k_{31}}{u - m^2} + \epsilon_{23} \right]^2, \end{aligned} \tag{5.3.1}$$

where $s \equiv s_{12}$, $u \equiv s_{13}$ and $t \equiv s_{14}$. This result is the same obtained by [18, 28] and here it was pointed out that the graviton-scalar amplitude obtained obeys the KLT squaring relation from [12, 24–27]

$$M(1_s, \rho_h, n_s) = \sum_{\substack{\alpha \in S_{n-3} \\ \beta \in S_{n-3}}} A_n(1_s, \alpha, (n-2)_g, n_s) \times m_n[1, \alpha, (n-2), n | 1, \beta, (n-1), n]^{-1} \times A_n(1_s, \beta, (n-1)_g, n_s), \tag{5.3.2}$$

where $m_n[1, \alpha, (n-2), n | 1, \beta, (n-1), n]$ is the matrix array obtained by integrating the product of two Parke-Taylor factors with orderings $(1, \alpha, (n-2), n)$ and $(1, \beta, (n-1), n)$.

In the four point case it is just $m_n(1,3,2,4|1,2,3,4) = \frac{1}{s_{14}}$, so

$$\begin{aligned}
\mathcal{M}(\mathbf{1}_s, 2_h, 3_h, \mathbf{4}_s) =& s_{14} A(\mathbf{1}_s, 2_g, 3_g, \mathbf{4}_s) A(\mathbf{1}_s, 3_g, 2_g, \mathbf{4}_s) \\
=& t \times \frac{(u-m^2)}{t} \left[\frac{\epsilon k_{21} \epsilon k_{34}}{s-m^2} + \frac{\epsilon k_{24} \epsilon k_{31}}{u-m^2} + \epsilon_{23} \right] \\
& \times \frac{(s-m^2)}{t} \left[\frac{\epsilon k_{31} \epsilon k_{24}}{u-m^2} + \frac{\epsilon k_{34} \epsilon k_{21}}{s-m^2} + \epsilon_{23} \right] \\
=& \frac{(u-m^2)(s-m^2)}{t} \left[\frac{\epsilon k_{21} \epsilon k_{34}}{s-m^2} + \frac{\epsilon k_{24} \epsilon k_{31}}{u-m^2} + \epsilon_{23} \right]^2 ,
\end{aligned}$$

which exactly reproduces the graviton result from (5.3.1). In the remainder of this thesis we will turn to computing amplitudes through BCJ numerators since these will reduce the amount of CHY diagrams needed for each calculation. The procedure described in this chapter will be important later on when we want to consider massive BCJ numerators.

Chapter 6

Color/Kinematics duality and BCJ numerators

In this chapter we discuss the color/kinematics duality [53–55] and review a powerful algorithm to calculate BCJ numerators presented in [31–33] that turns out to work, not only for gluons, but for two external fermions as well. The approach comes from an expanded Pfaffian and is the first CHY formulation of fermions. We want to stress that [30,31] use $10d$ Majorana-Weyl spinors, and that much of the work in this chapter is focused on showing that we obtain correct numerators for the Dirac spinors. This extension to fermions is very promising since a lot of papers lately have been focused on scattering of massive general spin particles with gluons and gravitons [3–5,7], and it is our hope that the CHY formalism might provide new insights on these calculations.

The BCJ numerators greatly reduce the number of CHY diagrams needed for calculations but also introduces a new type of diagram needed to calculate the numerators themselves. In the last chapter of this thesis we will present a novel algorithm designed by us to compute the numerators which is more efficient than the one presented in this chapter. Our brief introduction of the color/kinematics duality is based on the great review [41] as well as the texts [36,39,56].

6.1 Review of the color/kinematics duality

Through the usual Feynman construction of scattering amplitudes we know that at tree level, the full n-point color dressed amplitude can be written as a sum over all cubic trees with n external legs[1].

$$\mathcal{A}_n = \sum_{i \in cubic} \frac{c_i n_i}{D_i}. \tag{6.1.1}$$

[1] Many processes would have quartic interactions as well. This is usually dealt with by turning the contact diagrams into diagrams with two cubic vertices multiplied by appropriate propagators. We will not use this approach however, and so won't worry about this in the remainder of the thesis.

Here D_i denotes the propagators for the corresponding diagram, the $n_i = n_i(\epsilon, k)$ contain all the kinematic information, while the c_i are the color factors described in chapter 2. As an example the four-point pure gluon amplitude can be organized as[2] [41]

$$\mathcal{A}(g,g,g,g) = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}. \tag{6.1.2}$$

The kinematic numerators aren't completely unique because of the Jacobi identity obeyed by the color factors. This can be seen by performing the transformation $n_i \to n_i + s_i\Delta$

$$\begin{aligned}\mathcal{A} \to \mathcal{A}' =& \frac{c_s\left(n_s + s\Delta\right)}{s} + \frac{c_t\left(n_t + t\Delta\right)}{t} + \frac{c_u\left(n_u + t\Delta\right)}{u} \\ =& \mathcal{A} + \left(c_s + c_t + c_u\right)\Delta = \mathcal{A}.\end{aligned} \tag{6.1.3}$$

The Δ can be an arbitrary function and so it is referred to as *generalized gauge transformation* [36]. Since the numerators themselves aren't physical observables, the fact that they aren't unique is not a concern and as we will see later when we actually calculate the numerators, we always obtain the correct amplitude even if the numerators don't exactly match the literature. The great insight of Bern, Carassco, and Johansson (BCJ) was to suggest that because of the non-uniqueness of the kinematic numerators one will always be able to write them in a form which satisfies a Jacobi identity similar to the one for the color factors

$$n_s + n_t + n_u = 0 \Leftrightarrow c_s + c_t + c_u = 0. \tag{6.1.4}$$

This is known as the color/kinematics duality. Furthermore this implies that since the kinematic and color numerators obey the same identities we can exchange one set of numerators with the other. This exercise leads to the "*double copy*", where one can obtain the tree-level gravity amplitude from the gluon amplitude through a simple replacement:

$$\mathcal{M}_n = \mathcal{A}_n\big|_{c_i \to \tilde{n}_i} = \sum_{i \in cubic} \frac{\tilde{n}_i n_i}{D_i}, \tag{6.1.5}$$

at four points taking $\tilde{n}_i = n_i$ we for instance get

$$\mathcal{M}_4 = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}. \tag{6.1.6}$$

It is apparent that finding these kinematic numerators could have the potential to ease many amplitude calculations, especially with regards to gravitational interactions.

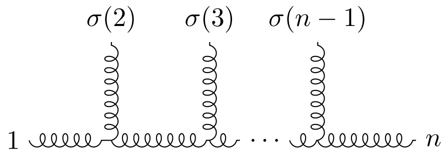
[2] We have suppressed powers of the coupling here.

6.2 DDM expansion

As already explained in chapter 2, Del Duca, Dixon, and Maltoni (DDM) showed that because of the relations between the color factors, the full amplitude can be expanded into a KK basis of color factors and partial amplitudes

$$\mathcal{A}_n = \sum_{\sigma \in S_{n-2}} f^{a_1 a_{\sigma(2)} b_1} f^{b_1 a_{\sigma(2)} b_2} \cdots f^{b_{n-3} a_{\sigma(n-2)} a_n} A_n(1, \sigma, n). \tag{6.2.1}$$

The $A_n(1, \sigma, n)$'s are color ordered Yang-Mills amplitudes and the color factors in this basis can be represented by the half-ladder (multi-peripheral) diagrams



.

Exchanging the color factors with kinematic numerators in (6.2.1) gives an expression for the graviton amplitude

$$\mathcal{M}_n = \sum_{\beta \in S_{n-2}} N(1, \beta, n) A_n(1, \beta, n). \tag{6.2.2}$$

The $(n-2)!$ numerators $N(1, \beta, n)$ can then also be associated with the half-ladder diagrams and are referred to as *DDM-basis numerators* [31–33] and we will use capital N throughout the remainder of this thesis to denote them. The remaining numerators can be found from these through Jacobi identities and so we can think of them as master numerators. As proposed by Cachazo, He and Yuan in [12] the CHY formalism naturally extends this. For instance we can consider the pure graviton amplitude obtained by reducing rows 1 and n in the CHY integral

$$\mathcal{M}_n = \int d\Omega_{\text{CHY}} \, \text{Pf}'(\Psi_{1,n}) \times \text{Pf}'(\Psi_{1,n}). \tag{6.2.3}$$

Similarly using (6.2.2) and taking the Yang-Mills integrand

$$\mathcal{M}_n = \int d\Omega_{\text{CHY}} \sum_{\sigma \in S_{n-2}} N(1, \beta, n) \text{PT}(1, \sigma, n) \times \text{Pf}'(\Psi_{1,n}). \tag{6.2.4}$$

This points to the fact that we can expand the Pfaffian into kinematic (numerator) parts and Parke-Taylor (z_i dependent) parts. Having 1 and n fixed corresponds to reducing Ψ by $\{i, j\} = \{1, n\}$ i.e.

$$\mathrm{Pf}'(\Psi_{1,n}) = \sum_{\beta \in S_{n-2}} N(1, \beta, n)\mathrm{PT}(1, \beta, n), \tag{6.2.5}$$

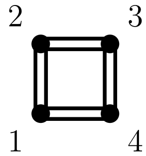
where the equality only holds under the support of the scattering equations, momentum conservation, and transversality. Using superscripts to denote the particle spin, we have for spin 1 the Pfaffian

$$\mathrm{Pf}'(\Psi) = N(1^1, 2^1, 3^1, 4^1)\mathrm{PT}(1, 2, 3, 4) + N(1^1, 3^1, 2^1, 4^1)\mathrm{PT}(1, 3, 2, 4). \tag{6.2.6}$$

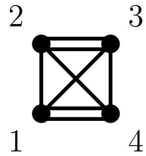
Multiplying by the Parke-Taylor factor needed go obtain the usual YM integrand we have an expression for the four-point color ordered gluon amplitude which doesn't involve actually calculating the Pfaffian

$$A^{\mathrm{YM}}(1, 2, 3, 4) = N(1^1, 2^1, 3^1, 4^1)\mathrm{PT}(1, 2, 3, 4)^2 + N(1^1, 3^1, 2^1, 4^1)\mathrm{PT}(1, 3, 2, 4)\mathrm{PT}(1, 2, 3, 4)$$

$$= N(1^1, 2^1, 3^1, 4^1)$$



$$- N(1^1, 3^1, 2^1, 4^1)$$

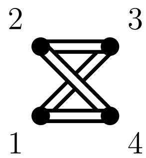


$$\begin{aligned}
&= -N(1^1, 2^1, 3^1, 4^1)\left(\frac{1}{s_{12}} + \frac{1}{s_{14}}\right) + N(1^1, 3^1, 2^1, 4^1)\left(\frac{1}{s_{14}}\right) \\
&= n_s\left(\frac{1}{s} + \frac{1}{t}\right) + n_u\left(\frac{1}{t}\right) \\
&= \frac{n_s}{s} - \frac{n_t}{t},
\end{aligned}$$

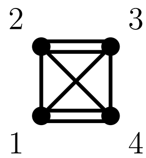
where we have defined $n_s \equiv -N(1^1, 2^1, 3^1, 4^1)$ and $n_u \equiv N(1^1, 3^1, 2^1, 4^1)$ and used the Jacobi identity. The other color ordered amplitude is calculated likewise

$$A^{\mathrm{YM}}(1, 3, 2, 4) = N(1^1, 3^1, 2^1, 4^1)\mathrm{PT}(1, 3, 2, 4)^2 + N(1^1, 2^1, 3^1, 4^1)\mathrm{PT}(1, 3, 2, 4)\mathrm{PT}(1, 2, 3, 4)$$

$$= N(1^1, 3^1, 2^1, 4^1)$$



$$- N(1^1, 2^1, 3^1, 4^1)$$



$$\begin{aligned}
&= -N(1^1, 3^1, 2^1, 4^1)\left(\frac{1}{s_{13}} + \frac{1}{s_{14}}\right) + N(1^1, 2^1, 3^1, 4^1)\left(\frac{1}{s_{14}}\right) \\
&= -n_u\left(\frac{1}{u} - \frac{1}{t}\right) - n_s\left(\frac{1}{t}\right) \\
&= \frac{n_t}{t} - \frac{n_u}{u}.
\end{aligned}$$

This gives us the full color dressed amplitude:

$$\begin{aligned}\mathcal{A}^{\mathrm{YM}}(1,2,3,4) &= c_s A^{\mathrm{YM}}(1,2,3,4) - c_u A^{\mathrm{YM}}(1,3,2,4)\\ &= c_s\left(\frac{n_s}{s}-\frac{n_t}{t}\right) - c_u\left(\frac{n_t}{t}-\frac{n_u}{u}\right)\\ &= c_s\frac{n_s}{s}+c_u\frac{n_u}{u}+c_t\frac{n_t}{t},\end{aligned}$$

as expected from eq (6.1.2). Seeing how to expand the amplitudes in terms of the numerators we will now turn to actually calculating them.

6.3 Graphical evaluation of BCJ numerators

In [57] an algorithm describing a way to obtain BCJ numerators in the DDM-basis was presented. A graphical version of the algorithm was later put forth in [30–33]. Here we review the graphical rules and show how to calculate simple three- and four-point spin 1 numerators. The advantage of this approach is that the numerators turn out to easily be extended to particles 1 and n being fermions.

One major disadvantage of this method is that we not only have to consider the color order (CO) but also the reference order (RO). Although we will not go further into the proof of these rules, we note that they were obtained by using a Laplace expansion of the Pfaffian. When doing the expansion one has to choose a row along which to expand. This choice breaks the symmetry between the chosen row and the remaining and is the origin of the reference order [31].

The RO breaks the crossing symmetry between the numerators e.g. using the notation $N_{\{\mathrm{RO}\}}(1^s,\mathrm{CO},n^s)$, with s being the spin of the particle,

$$N_{\{\beta\}}(1^s,2^1,3^1,4^s)\Big|_{2\leftrightarrow 3} \neq N_{\{\beta\}}(1^s,3^1,2^1,4^s).$$

To restore it one could for instance average over all RO's, so a crossing symmetric numerator would be

$$N(1^s,2^1,3^1,4^s) = \sum_{\beta\in S_{n-2}} N_{\{\beta\}}(1^s,2^1,3^1,4^s). \tag{6.3.1}$$

Let us now present the graphical rules used to obtain the numerators from [31–33]

Graphical Rules

<u>Step 1: Draw increasing trees</u>

- Draw all the increasing trees according to the color order $1 \prec \alpha(2)\cdots \prec \alpha(n-$

1) $\prec n$, where an increasing tree is a tree where all edges $i \to j$ satisfy $i \prec j$ in $(1, \alpha, n)$.

- The "baseline" $(1, \rho, n)$ of the individual trees is the path connecting points 1 and n.
- For each tree, draw the "ordered splitting paths" given a reference order:
  1. Draw a path from the first element of RO towards the baseline, which will end on the baseline or a previously traversed ordered splitting path.
  2. Repeat until all vertices are traversed.

Step 2: Read of contributions from the diagrams

- The numerator is then found by taking the baseline and ordered splittings path contributions to be $(-1)^{n-|\rho|} B_s(1, \rho, n)$ and $\epsilon_i f_j \cdots f_k \cdot k_l$.
- Sum all trees to get the full numerator.

where the linearized (momentum space) field strength tensor f_i is given by

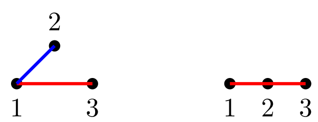
$$f_i \equiv f_i^{\mu\nu} = p_i^\mu \epsilon_i^\nu - p_i^\nu \epsilon_i^\mu, \tag{6.3.2}$$

and the baseline factor for spin 1 particles is[3]

$$B_{s=1}(1, \rho, n) = \epsilon_1 \cdot f_{\rho(b_1)} \cdot f_{\rho(b_2)} \cdots f_{\rho(b_{|\mathrm{B}|})} \cdot \epsilon_n. \tag{6.3.3}$$

Three-point spin 1 numerator

The rules are most easily learned by applying them to examples. Taking the three point spin 1 numerator which only has one color order and one reference order $N_{\{2\}}(1^1, 2^1, 3^1)$. Here the following trees contribute



,

with the factors

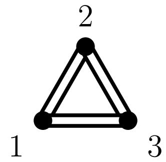
$$\begin{aligned} N_{\{2\}}(1^1, 2^1, 3^1) &= -(\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot k_1) + (\epsilon_1 \cdot f_2 \cdot \epsilon_3) \\ &= -(\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot k_1) + (\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot k_2) - (\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot k_2). \end{aligned} \tag{6.3.4}$$

[3] In section 6.4 we look at the fermionic baseline factor

Under the support of the scattering equations we can the calculate the color ordered three-point gluon amplitude

$$A(1_g, 2_g, 3_g) = N_{\{2\}}(1^1, 2^1, 3^1)\mathrm{PT}(1,2,3)^2$$

$$= N_{\{2\}}(1^1, 2^1, 3^1)$$

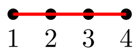


$$= -(\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot k_1) + (\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot k_2) - (\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot k_2),$$

which is in accordance with the amplitudes calculated both through the Feynman diagram and CHY approaches, see equations (4.3.19) and (4.3.18).

Four-point spin 1 numerator

For the $n = 4$ case the simplest tree contributing to $N_{\{2,3\}}(1^1, 2^1, 3^1, 4^1)$ is the one where the baseline is just (1,2,3,4):



. (6.3.5)

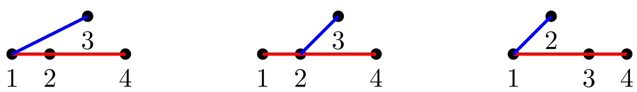
All vertices have been traversed so the only contribution we get is from the baseline:

$$N_{\{2,3\}}[T_1] = (-1)^{4-4}\epsilon_1 \cdot f_2 \cdot f_3 \cdot \epsilon_4.$$

For a baseline of length three we could have either (1,2,4) or (1,3,4). Let us draw them below

.

Since our color order is $\{2, 3\}$ we need the edge to satisfy $2 \prec 3$, meaning the 2 on the second diagram can only be connected to 1, while the point 3 on the first diagram can be connected to either 2 or 1, leaving us with three trees:



.

The contributions from the three trees are then (in the order they have been drawn):

$$(-1)^{4-3}\,(\epsilon_1\cdot f_2\cdot\epsilon_4)(\epsilon_3\cdot k_1),\qquad (-1)^{4-3}\,(\epsilon_1\cdot f_2\cdot\epsilon_4)(\epsilon_3\cdot k_2),\qquad (-1)^{4-3}\,(\epsilon_1\cdot f_3\cdot\epsilon_4)(\epsilon_2\cdot k_1).$$

The remaining trees have the simplest baseline of just (1,4), but here the RO comes in to play when connecting the two remaining points. With a RO of $\{2,3\}$ we have to look at 2 first and connect it to 1. The 3 can then be connected to either 1 or 2, giving us 2 different diagrams:

.

The important thing to note is that because of the RO there are two *distinct* paths on the second diagram. The contributions are

$$(-1)^{4-2}\,(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot k_1)(\epsilon_3\cdot k_1),\qquad (-1)^{4-2}\,(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot k_1)(\epsilon_3\cdot k_2).$$

Had we used the RO $\{3,2\}$ we would have the diagrams,

contributing

$$(-1)^{4-2}\,(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot k_1)(\epsilon_3\cdot k_1),\qquad (-1)^{4-2}\,(\epsilon_1\cdot\epsilon_4)(\epsilon_3\cdot f_2\cdot k_1).$$

The three point numerator with a specific reference order is then obtained by summing all the contributions

$$\begin{aligned}N_{\{2,3\}}(1^1,2^1,3^1,4^1) =&\epsilon_1\cdot f_2\cdot f_3\cdot\epsilon_4-(\epsilon_1\cdot f_2\cdot\epsilon_4)\,(\epsilon_3\cdot k_1)-(\epsilon_1\cdot f_2\cdot\epsilon_4)\,(\epsilon_3\cdot k_2)\\&+(\epsilon_1\cdot\epsilon_4)\,(\epsilon_2\cdot k_1)\,(\epsilon_3\cdot k_1)+(\epsilon_1\cdot\epsilon_4)\,(\epsilon_2\cdot k_1)\,(\epsilon_3\cdot k_2)\\&-(\epsilon_1\cdot f_3\cdot\epsilon_4)\,(\epsilon_2\cdot k_1)\,,\end{aligned}$$

which can be simplified slightly to a form which will be convenient for us later

$$N_{\{2,3\}}(1^1,2^1,3^1,4^1) = \epsilon_1\cdot f_2\cdot f_3\cdot\epsilon_4 - (\epsilon_1\cdot f_2\cdot\epsilon_4)(\epsilon_3\cdot k_{12}) - (\epsilon_1\cdot f_3\cdot\epsilon_4)(\epsilon_2\cdot k_1) + (\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot k_1)(\epsilon_3\cdot k_{12})\,, \tag{6.3.6}$$

where $k_{ij\ldots n}\equiv(k_i+k_j+\cdots+k_n)$. We have similarly written the 6 diagrams contributing to the numerator $N_{\{2,3\}}(1^1,3^1,2^1,4^1)$ in Figure 6.1. Reading of the diagrams

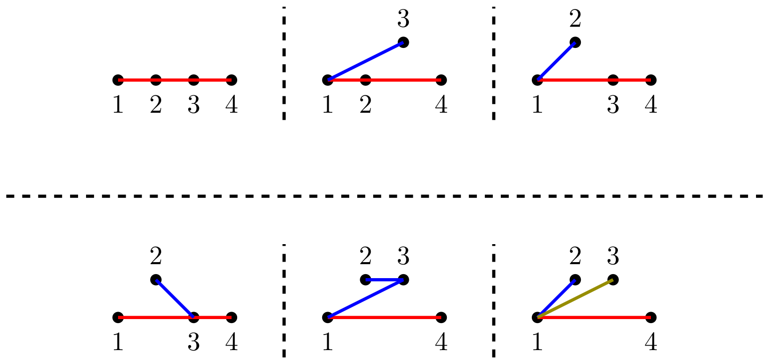


Figure 6.1: Trees contributing to $N_{\{2,3\}}(1^1,3^1,2^1,4^1)$.

we have

$$\begin{aligned}N_{\{2,3\}}(1^1,3^1,2^1,4^1) =&\epsilon_1\cdot f_3\cdot f_2\cdot\epsilon_4 - (\epsilon_1\cdot f_2\cdot\epsilon_4)(\epsilon_3\cdot k_1) - (\epsilon_1\cdot f_3\cdot\epsilon_4)(\epsilon_2\cdot k_1)\\ &+ (\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot f_3\cdot\epsilon_1) + (\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot k_1)(\epsilon_3\cdot k_1)\\ &- (\epsilon_1\cdot f_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)\,,\end{aligned}$$

and again simplifying

$$\begin{aligned}N_{\{2,3\}}(1^1,3^1,2^1,4^1) =&\epsilon_1\cdot f_2\cdot f_3\cdot\epsilon_4 - (\epsilon_1\cdot f_2\cdot\epsilon_4)(\epsilon_3\cdot k_1) - (\epsilon_1\cdot f_3\cdot\epsilon_4)(\epsilon_2\cdot k_{13})\\ &+ (\epsilon_1\cdot\epsilon_4)\left[(\epsilon_2\cdot f_3\cdot k_1) + (\epsilon_2\cdot k_1)(\epsilon_3\cdot k_1)\right].\end{aligned} \tag{6.3.7}$$

One could of course also have chosen the other RO. To get a result which is crossing symmetric $2\leftrightarrow 3$ an average over the two RO's can be performed. For an approach which doesn't include averaging over the RO's we refer to [31] or chapter 8 of this thesis. We now turn to external fermions.

6.4 Fermionic BCJ numerators

In [30, 31] it is proposed that to extend numerators to include two external fermions, we have to change the baseline factor. In that paper 10 dimensional Majorana-Weyl spinors are used, but in this section we show that the generalization to d-dimensional Dirac spinors works as well, with the only other condition being the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. We will use the baseline factor

$$B_{s=1/2}(1, \rho, n) = \bar{u}_1 \not{f}_{\rho(2)} \cdots \not{f}_{\rho|B|} \xi_n, \tag{6.4.1}$$

where $\bar{u}_1$ is a 4 dimensional incoming Dirac spinor and ξ_n is related to the outgoing spinor v_n,

$$\bar{u}_1 \not{k}_1 = 0, \qquad \not{k}_n v_n = 0, \qquad v_n = \not{k}_n \xi_n. \tag{6.4.2}$$

The fermionic field strength tensor is given by

$$\not{f}_i = \frac{1}{2} f_{i,\mu\nu} \Sigma^{\mu\nu} = \frac{i}{8} f_{i,\mu\nu} [\gamma^\mu, \gamma^\nu] = \frac{i}{2} \not{k}_i \not{\epsilon}_i. \tag{6.4.3}$$

where this form of the spin matrix $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ will be useful later on. In the following we are going to check that this constructions works by calculating the three- and four-point numerators and comparing them to the results found in [7, 56] while also checking that they give the correct final amplitude. We will be very explicit in our calculations since to our knowledge this is the only place where these are available.

Three-point massless numerator

Since the only change compared to the gluons is to the baseline factor, the replacement is simply

$$N_{\{2\}}(1^{\frac{1}{2}}, 2^1, 3^{\frac{1}{2}}) = \bar{u}_1 \not{f}_2 \xi_3 - \bar{u}_1 \xi_3 (\epsilon_2 \cdot k_1). \tag{6.4.4}$$

This is of course not the standard way of expressing the numerator but it can be put in to a form that matches the expression obtained in equation (3.4.12) through repeated use of the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, momentum conservation and the equations of motion. Note in particular how the Clifford algebra leads to the frequently used identity

$$k_i \cdot \epsilon_j = k_{i,\mu} \epsilon_{j,\nu} \eta^{\mu\nu} = \frac{1}{2} k_{i,\mu} \epsilon_{j,\nu} \{\gamma^\mu, \gamma^\nu\} = \frac{1}{2} \not{k}_i \not{\epsilon}_j + \frac{1}{2} \not{\epsilon}_j \not{k}_i. \tag{6.4.5}$$

For the three point numerator we then obtain

$$\begin{aligned}
N_{\{2\}}(1^{\frac{1}{2}},2^1,3^{\frac{1}{2}}) &= \frac{1}{2}\bar{u}_1 \not{k}_2 \not{\epsilon}_2 \xi_3 - \bar{u}_1\xi_3(\epsilon_2\cdot k_1) \\
&= -\frac{1}{2}\bar{u}_1 \not{\epsilon}_2 \not{k}_2 \xi_3 - \bar{u}_1\xi_3(\epsilon_2\cdot k_1) \\
&= \frac{1}{2}\bar{u}_1 \not{\epsilon}_2 \not{k}_1 \xi_3 + \frac{1}{2}\bar{u}_1 \not{\epsilon}_2 \not{k}_3 \xi_3 - \bar{u}_1\xi_3(\epsilon_2\cdot k_1) \\
&= -\frac{1}{2}\bar{u}_1 \not{k}_1 \not{\epsilon}_2 \xi_3 + \bar{u}_1\xi_3(\epsilon_2\cdot k_1) + \frac{1}{2}\bar{u}_1 \not{\epsilon}_2 \not{k}_3 \xi_3 - \bar{u}_1\xi_3(\epsilon_2\cdot k_1) \\
&= \frac{1}{2}\bar{u}_1 \not{\epsilon}_2 v_3,
\end{aligned}$$

where we have used momentum conservation $\sum_i \not{k}_i = 0$ and the identity (6.4.5) repeatedly. As already stated, this matches the color ordered qgq amplitude obtained through a Feynman diagram calculation in (3.4.12).

Four-point massless numerator

The four point numerator is more involved and so we will go through it slowly. Choosing RO $\{2,3\}$ the numerators can similarly be obtained by reading of the diagrams from the previous section

$$\begin{aligned}
N_{\{2,3\}}(1^{\frac{1}{2}},2^1,3^1,4^{\frac{1}{2}}) =& \bar{u}_1 \not{\epsilon}_2 \not{\epsilon}_3 \xi_4 - (\bar{u}_1 \not{\epsilon}_2 \xi_4)(\epsilon_3\cdot k_{12}) - (\bar{u}_1 \not{\epsilon}_3 \xi_4)(\epsilon_2\cdot k_1) \\
&+ (\bar{u}_1\xi_4)(\epsilon_2\cdot k_1)(\epsilon_3\cdot k_{12}) \\
=& \bar{u}_1 \not{\epsilon}_2 \not{\epsilon}_3 \xi_4 + (\bar{u}_1 \not{\epsilon}_2 \xi_4)(\epsilon_3\cdot k_4) - (\bar{u}_1 \not{\epsilon}_3 \xi_4)(\epsilon_2\cdot k_1) \\
&- (\bar{u}_1\xi_4)(\epsilon_2\cdot k_1)(\epsilon_3\cdot k_4)\,.
\end{aligned} \tag{6.4.6}$$

Again, this is in a non-standard version of the four-point numerator so we want to check that the result can be put into a form like the one in [7, 56]. Here the massless numerator was written as $N(1^{\frac{1}{2}},2^1,3^1,4^{\frac{1}{2}}) = -\frac{1}{2}\bar{u}_1 \not{\epsilon}_2 \not{k}_{12} \not{\epsilon}_3 v_4$. In order compare this to (6.4.6), we will look at a term of the form $\bar{u}_1 \not{\epsilon}_2 \not{k}_1 \not{\epsilon}_3 v_4$. By the identity (6.4.5) and the equations of motion

$$\begin{aligned}
\bar{u}_1 \not{\epsilon}_2 \not{k}_1 \not{\epsilon}_3 v_4 &= \bar{u}_1 \not{\epsilon}_2 \not{k}_1 \not{\epsilon}_3 \not{k}_4 \xi_4 \\
&= \bar{u}_1 \{\not{\epsilon}_2, \not{k}_1\} \not{\epsilon}_3 \not{k}_4 \xi_4 - \underbrace{\bar{u}_1 \not{k}_1}_{0} \not{\epsilon}_2 \not{\epsilon}_3 \not{k}_4 \xi_4 \\
&= 2\bar{u}_1 \not{\epsilon}_3 \not{k}_4 \xi_4 \,(\epsilon_2\cdot k_1)\,.
\end{aligned}$$

Again applying (6.4.5), momentum conservation $-\not{k}_4 = \not{k}_1 + \not{k}_2 + \not{k}_3$ and the equation of motion

$$\begin{aligned}\bar{u}_1 \not\epsilon_2 \not{k}_1 \not\epsilon_3 v_4 &= 4\bar{u}_1 \xi_4 \left(\epsilon_3 \cdot k_4\right) \left(\epsilon_2 \cdot k_1\right) - 2\bar{u}_1 \not{k}_4 \not\epsilon_3 \xi_4 \left(\epsilon_2 \cdot k_1\right) \\ &= 4\bar{u}_1 \xi_4 \left(\epsilon_3 \cdot k_4\right) \left(\epsilon_2 \cdot k_1\right) + 4\bar{u}_1 \not{f}_3 \xi_4 \left(\epsilon_2 \cdot k_1\right) + 2\bar{u}_1 \not{k}_2 \not\epsilon_3 \xi_4 \left(\epsilon_2 \cdot k_1\right).\end{aligned}$$

Focusing on the last term we get

$$\begin{aligned}2\bar{u}_1 \not{k}_2 \not\epsilon_3 \xi_4 \left(\epsilon_2 \cdot k_1\right) &= 2\bar{u}_1 \not{k}_2 \not\epsilon_3 \xi_4 \left(\epsilon_2 \cdot k_{12}\right) \\ &= \bar{u}_1 \not{k}_2 \not\epsilon_2 \not{k}_{12} \not\epsilon_3 \xi_4 + \bar{u}_1 \not{k}_2 \not{k}_{12} \not\epsilon_2 \not\epsilon_3 \xi_4 \\ &= \bar{u}_1 \not{k}_2 \not\epsilon_2 \{\not{k}_{12}, \not\epsilon_3\} \xi_4 - \bar{u}_1 \not{k}_2 \not\epsilon_2 \not\epsilon_3 \not{k}_{12} \xi_4 + \bar{u}_1 \not{k}_2 \not{k}_1 \not\epsilon_2 \not\epsilon_3 \xi_4 \\ &= -4\bar{u}_1 \not{f}_2 \xi_4 \left(\epsilon_3 \cdot k_4\right) + 2\bar{u}_1 \not{f}_2 \not\epsilon_3 \not{k}_{34} \xi_4 + \bar{u}_1 \not{k}_2 \not{k}_1 \not\epsilon_2 \not\epsilon_3 \xi_4 \\ &= -4\bar{u}_1 \not{f}_2 \xi_4 \left(\epsilon_3 \cdot k_4\right) - 4\bar{u}_1 \not{f}_2 \not{f}_3 \xi_4 - \bar{u}_1 \not\epsilon_2 \not{k}_2 \not\epsilon_3 \not{k}_4 \xi_4 + \bar{u}_1 \not{k}_2 \not{k}_1 \not\epsilon_2 \not\epsilon_3 \xi_4 \\ &= -4\bar{u}_1 \not{f}_2 \xi_4 \left(\epsilon_3 \cdot k_4\right) - 4\bar{u}_1 \not{f}_2 \not{f}_3 \xi_4 - \bar{u}_1 \not\epsilon_2 \not{k}_2 \not\epsilon_3 v_4 + s_{12} \bar{u}_1 \not\epsilon_2 \not\epsilon_3 \xi_4,\end{aligned}$$

where we have used transversality $\epsilon_2 \cdot k_2 = 0$ in the first line and $\bar{u}_1 \not{k}_2 \not{k}_1 = -\bar{u}_1 \not{k}_1 \not{k}_2 + 2\bar{u}_1 (k_2 \cdot k_1) = s_{12} \bar{u}_1$ in the last line. This leads to

$$\begin{aligned}\frac{1}{4} \bar{u}_1 \not\epsilon_2 \not{k}_{12} \not\epsilon_3 v_4 =& \bar{u}_1 \xi_4 \left(\epsilon_3 \cdot k_4\right) \left(\epsilon_2 \cdot k_1\right) + \bar{u}_1 \not{f}_3 \xi_4 \left(\epsilon_2 \cdot k_1\right) - \bar{u}_1 \not{f}_2 \xi_4 \left(\epsilon_3 \cdot k_4\right) \\ & - \bar{u}_1 \not{f}_2 \not{f}_3 \xi_4 + \frac{1}{4} s_{12} \bar{u}_1 \not\epsilon_2 \not\epsilon_3 \xi_4.\end{aligned}$$

So the numerator we found can be expressed in a way which looks more like what we would expect from a Feynman diagram calculation

$$N_{\{2,3\}}(1^{\frac{1}{2}}, 2^1, 3^1, 4^{\frac{1}{2}}) = -\frac{1}{4} \bar{u}_1 \not\epsilon_2 \not{k}_{12} \not\epsilon_3 v_4 - \frac{1}{4} s_{12} \bar{u}_1 \not\epsilon_2 \not\epsilon_3 \xi_4. \tag{6.4.7}$$

It matches the expression in [7,56] except for the last term. For the $N_{\{2,3\}}(1^{\frac{1}{2}}, 3^1, 2^1, 4^{\frac{1}{2}})$ numerator the calculation is similar. Using the result from the spin 1 calculation, we can readily read of the contribution

$$\begin{aligned}N_{\{2,3\}}(1^{\frac{1}{2}}, 3^1, 2^1, 4^{\frac{1}{2}}) =& \bar{u}_1 \not{f}_2 \not{f}_3 \xi_4 - (\bar{u}_1 \not{f}_2 \xi_4) \left(\epsilon_3 \cdot k_1\right) - (\bar{u}_1 \not{f}_3 \xi_4) \left(\epsilon_2 \cdot k_{13}\right) \\ & + (\bar{u}_1 \xi_4) \left(\epsilon_2 \cdot f_3 \cdot k_1\right) + (\bar{u}_1 \xi_4) \left(\epsilon_2 \cdot k_1\right) \left(\epsilon_3 \cdot k_1\right) \\ =& \bar{u}_1 \not{f}_2 \not{f}_3 \xi_4 - (\bar{u}_1 \not{f}_2 \xi_4) \left(\epsilon_3 \cdot k_1\right) + (\bar{u}_1 \not{f}_3 \xi_4) \left(\epsilon_2 \cdot k_4\right) \\ & + (\bar{u}_1 \xi_4) \left(\epsilon_2 \cdot f_3 \cdot k_1\right) + (\bar{u}_1 \xi_4) \left(\epsilon_2 \cdot k_1\right) \left(\epsilon_3 \cdot k_1\right).\end{aligned}$$

Just like before, we will look at a term of the form $\bar{u}_1 \not\epsilon_3 \not{k}_1 \not\epsilon_2 v_4$ to relate this to the results from [7, 56], $N(1^{\frac{1}{2}}, 3^1, 2^1, 4^{\frac{1}{2}}) = -\frac{1}{2} \bar{u}_1 \not\epsilon_3 \not{k}_{13} \not\epsilon_2 v_4$. Following the procedure from above and using the same identities we obtain

$$
\begin{aligned}
\bar{u}_1 \not{\epsilon}_3 \not{k}_1 \not{\epsilon}_2 v_4 &= \bar{u}_1 \not{\epsilon}_3 \not{k}_1 \not{\epsilon}_2 \not{k}_4 \xi_4 \\
&= 2\bar{u}_1 \not{\epsilon}_2 \not{k}_4 \xi_4 \left(\epsilon_3 \cdot k_1\right) \\
&= 4\bar{u}_1 \xi_4 \left(\epsilon_2 \cdot k_4\right)\left(\epsilon_3 \cdot k_1\right) - 2\bar{u}_1 \not{k}_4 \not{\epsilon}_2 \xi_4 \left(\epsilon_3 \cdot k_1\right) \\
&= 4\bar{u}_1 \xi_4 \left(\epsilon_2 \cdot k_4\right)\left(\epsilon_3 \cdot k_1\right) + 4\bar{u}_1 \not{f}_2 \xi_4 \left(\epsilon_3 \cdot k_1\right) + 2\bar{u}_1 \not{k}_3 \not{\epsilon}_2 \xi_4 \left(\epsilon_3 \cdot k_1\right).
\end{aligned}
$$

Focusing on the last term through similar considerations we get

$$
\begin{aligned}
2\bar{u}_1 \not{k}_3 \not{\epsilon}_2 \xi_4 \left(\epsilon_3 \cdot k_1\right) &= 2\bar{u}_1 \not{k}_3 \not{\epsilon}_2 \xi_4 \left(\epsilon_3 \cdot k_{13}\right) \\
&= \bar{u}_1 \not{k}_3 \not{\epsilon}_3 \not{k}_{13} \not{\epsilon}_2 \xi_4 + \bar{u}_1 \not{k}_3 \not{k}_{13} \not{\epsilon}_3 \not{\epsilon}_2 \xi_4 \\
&= \bar{u}_1 \not{k}_3 \not{\epsilon}_3 \{\not{k}_{13}, \not{\epsilon}_2\} \xi_4 - \bar{u}_1 \not{k}_3 \not{\epsilon}_3 \not{\epsilon}_2 \not{k}_{13} \xi_4 + \bar{u}_1 \not{k}_3 \not{k}_1 \not{\epsilon}_3 \not{\epsilon}_2 \xi_4 \\
&= -4\bar{u}_1 \not{f}_3 \xi_4 \left(\epsilon_2 \cdot k_4\right) - 4\bar{u}_1 \not{f}_3 \not{f}_2 \xi_4 - \bar{u}_1 \not{\epsilon}_3 \not{k}_3 \not{\epsilon}_2 v_4 + \bar{u}_1 \not{k}_3 \not{k}_1 \not{\epsilon}_3 \not{\epsilon}_2 \xi_4.
\end{aligned}
$$

We can use this to obtain the numerator by noticing that the last term can be rewritten

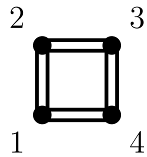
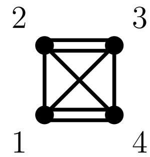
$$
\bar{u}_1 \not{k}_3 \not{k}_1 \not{\epsilon}_3 \not{\epsilon}_2 \xi_4 = -\,4\bar{u}_1 \xi_4 (\epsilon_2 \cdot k_{14}) \left(\epsilon_3 \cdot k_1\right) - 4\bar{u}_1 \xi_4 (\epsilon_2 \cdot f_3 \cdot k_1) - s_{13} \bar{u}_1 \not{\epsilon}_2 \not{\epsilon}_3 \xi_4,
$$

So that the numerator can be expressed as

$$
N_{\{2,3\}}(1^{\frac{1}{2}}, 3^1, 2^1, 4^{\frac{1}{2}}) = -\frac{1}{4} \bar{u}_1 \not{\epsilon}_3 \not{k}_{13} \not{\epsilon}_2 v_4 + \frac{1}{4} s_{13} \bar{u}_1 \not{\epsilon}_2 \not{\epsilon}_3 \xi_4. \tag{6.4.8}
$$

Both of the numerators obtained here are in agreement with the ones found in [7,56] except for an overall normalization and the factors proportional to the propagators s_{12} and s_{13}. We see however that when calculating the color ordered $qggq$ amplitude using the expansion (6.2.5) the extra terms vanish. Under support of the scattering equations:

$$
A(1_q, 2_g, 3_g, 4_q) = \mathrm{PT}(1,2,3,4) \sum_{\beta \in S_2} N_{\{2,3\}}(1^{\frac{1}{2}}, \beta, 4^{\frac{1}{2}}) \mathrm{PT}(1, \beta, 4)
$$

$= N_{\{2,3\}}(1^{\frac{1}{2}}, 2^1, 3^1, 4^{\frac{1}{2}})$  $+ N_{\{2,3\}}(1^{\frac{1}{2}}, 3^1, 2^1, 4^{\frac{1}{2}})$ 

$$
\begin{aligned}
&= n_s^{2\mathrm{q}} \left(\frac{1}{s_{12}} + \frac{1}{s_{14}}\right) + n_u^{2\mathrm{q}} \frac{1}{s_{14}} + \frac{1}{4} \bar{u}_1 \not{\epsilon}_2 \not{\epsilon}_3 \xi_4 \left\{ s_{12} \left(\frac{1}{s_{12}} + \frac{1}{s_{14}}\right) + \frac{s_{13}}{s_{14}} \right\} \\
&= \frac{n_s^{2\mathrm{q}}}{s_{12}} - \frac{n_t^{2\mathrm{q}}}{s_{14}},
\end{aligned}
$$

where we have defined $n_s^{2\mathrm{q}} \equiv -N_{\{2,3\}}(1^{\frac{1}{2}}, 2^1, 3^1, 4^{\frac{1}{2}}) - 2 s_{12} \bar{u}_1 \not{\epsilon}_2 \not{\epsilon}_3 \xi_4$ and $n_u^{2\mathrm{q}} \equiv N_{\{2,3\}}(1^{\frac{1}{2}}, 3^1, 2^1, 4^{\frac{1}{2}}) - 2 s_{13} \bar{u}_1 \not{\epsilon}_2 \not{\epsilon}_3 \xi_4$ and used the Jacobi identity $n_t^{2\mathrm{q}} = -n_s^{2\mathrm{q}} - n_u^{2\mathrm{q}}$ as well momentum conservation. This behavior was expected since in general the numerators aren't unique.

Chapter 7

Including massive fermions in CHY

As described in chapter 5 we know that the reduced Pfaffian is not affected when including two massive particles if the rows and columns we reduce the Pfaffian by matches the labels of the massive particles. Since the DDM-expansion of the Pfaffian (6.2.5) used to get the numerators fixes the 1^{st} and n^{th} particle, this means that the expansion should hold for $k_1^2 = k_n^2 = m^2$, which in turn points to the numerators obtained through the graphical rules also being correct, even when the external particles are massive. Since this is the first time this has been done we will be careful and go through every calculation slowly.

In this section we provide various checks that this is in fact the case for the fermionic numerators. Unlike the approach in [31] where $10d$ Majorana-Weyl spinors are used, everything here will be d-dimensional and because of the Pfaffian reduction there is no need to use dimensional reduction to describe massive particles. At a low number of points analytical checks are feasible, but going to $n \geq 5$ we instead turn to the massive spinor helicity methods described in chapter 3. The spinor helicity basis is a powerful way to provide numerical checks since all-multiplicity results exist for the specific amplitudes we look at in this thesis [7].

7.1 Analytical checks of the massive BCJ numerators

In this section we check that the three- and four-point numerators obtained from the graphical rules match those of [7, 56]. We find that the numerators are the same up to a factor which vanishes when calculating the amplitude just like for the massless case. This behavior is expected since in general the kinematic numerators are not unique. As already stated, the massive numerators are equal to the massless ones. However when rewriting the expressions to compare them with the literature, the equations of motion change to the massive

$$\bar{u}_1(\not{k}_1 - m) = 0, \qquad (\not{k}_n + m)v_n = 0, \qquad v_n = (\not{k}_n - m)\xi_n. \tag{7.1.1}$$

We will use **boldface** to denote the massive legs. The identity (6.4.5) obtained using the Clifford algebra still holds and will be used frequently.

Three-point massive

At three-points the massive calculation is only slightly more complicated than the massless case. Since we only have to change $\epsilon_1 \to \bar{u}_1$, $\epsilon_3 \to v_3$ and $f \to \not{f}$ the initial expression for the master numerator is easily obtained.

$$\begin{aligned} N(\mathbf{1}^{\frac{1}{2}}, 2^1, \mathbf{3}^{\frac{1}{2}}) &= \bar{u}_1 \not{f}_2 \xi_3 - \bar{u}_1 \xi_3 (\epsilon_2 \cdot k_1) \\ &= -\frac{1}{2}\bar{u}_1 \not{\epsilon}_2 \not{k}_2 \xi_3 - \bar{u}_1 \xi_3 (\epsilon_2 \cdot k_1). \end{aligned}$$

Similarly to the massless expression, we then rewrite this using momentum conservation and the Clifford algebra (6.4.5)

$$\begin{aligned} N(\mathbf{1}^{\frac{1}{2}}, 2^1, \mathbf{3}^{\frac{1}{2}}) &= \frac{1}{2}\bar{u}_1 \not{\epsilon}_2 \not{k}_1 \xi_3 + \frac{1}{2}\bar{u}_1 \not{\epsilon}_2 \not{k}_3 \xi_3 - \bar{u}_1 \xi_3 (\epsilon_2 \cdot k_1) \\ &= -\frac{1}{2}\bar{u}_1 \not{k}_1 \not{\epsilon}_2 \xi_3 + \bar{u}_1 \xi_3 (\epsilon_2 \cdot k_1) + \frac{1}{2}\bar{u}_1 \not{\epsilon}_2 \not{k}_3 \xi_3 - \bar{u}_1 \xi_3 (\epsilon_2 \cdot k_1). \end{aligned}$$

The two terms proportional to $(\epsilon_2 \cdot k_1)$ cancel and the equations of motion $\bar{u}_1 \not{k}_1 = \bar{u}_1 m$ can be applied to the first term which yielding

$$\begin{aligned} N(\mathbf{1}^{\frac{1}{2}}, 2^1, \mathbf{3}^{\frac{1}{2}}) &= \frac{1}{2}\bar{u}_1 \not{\epsilon}_2 \left(\not{k}_3 - m\right) \xi_3 \\ &= \frac{1}{2}\bar{u}_1 \not{\epsilon}_2 v_3. \end{aligned}$$

Since the Parke-Taylor factor integrates to 1 in the three point case, this also gives the expression for the color ordered qgq amplitude just like in the massless case. The result is again in agreement with the corresponding Feynman diagram calculation.

Four-point massive

Turning to the four-point case the computations become increasingly tedious but follow the same pattern: use momentum conservation, the Clifford algebra and the equations of motion. Just like for the massless case, we want to compare the results obtained through the graphical rules with the expression from [7, 56], $N(\mathbf{1}^{\frac{1}{2}}, 2^1, 3^1, \mathbf{4}^{\frac{1}{2}}) = -\frac{1}{2}\bar{u}_1 \not{\epsilon}_2 (\not{k}_{12} +$

$m)\not\epsilon_3 v_4$ and $N(\mathbf{1}^{\frac{1}{2}}, 3^1, 2^1, \mathbf{4}^{\frac{1}{2}}) = -\frac{1}{2}\bar{u}_1\not\epsilon_3(\not k_{13}+m)\not\epsilon_2 v_4$, where these BCJ numerators were found by reading of the Feynman diagram computations. For the first numerator this means that we are looking for an expression similar to

$$\bar{u}_1\not\epsilon_2(\not k_{12}+m)\not\epsilon_3 v_4.$$

Since this is the first place the checks are performed, we will be very deliberate in showing every step of the calculations. First we split up the above expression into

$$\bar{u}_1\not\epsilon_2(\not k_{12}+m)\not\epsilon_3 v_4 = \bar{u}_1\not\epsilon_2(\not k_1+m)\not\epsilon_3\not k_4\xi_4 + \bar{u}_1\not\epsilon_2\not k_2\not\epsilon_3\not k_4\xi_4 - m\bar{u}_1\not\epsilon_2(\not k_{12}+m)\not\epsilon_3\xi_4.$$

We will start out by looking at the first term.

$$\begin{aligned}
\bar{u}_1\not\epsilon_2(\not k_1+m)\not\epsilon_3\not k_4\xi_4 =&\bar{u}_1\not\epsilon_2\not k_1\not\epsilon_3\not k_4\xi_4 + m\bar{u}_1\not\epsilon_2\not\epsilon_3\not k_4\xi_4\\
=&2\bar{u}_1\not\epsilon_3\not k_4\xi_4\,(\epsilon_2\cdot k_1)\\
=&4\bar{u}_1\xi_4\,(\epsilon_3\cdot k_4)\,(\epsilon_2\cdot k_1) - 2\bar{u}_1\not k_4\not\epsilon_3\xi_4\,(\epsilon_2\cdot k_1)\\
=&4\bar{u}_1\xi_4\,(\epsilon_3\cdot k_4)\,(\epsilon_2\cdot k_1) + 4\bar{u}_1\not f_3\xi_4\,(\epsilon_2\cdot k_1) + 2\bar{u}_1\not k_2\not\epsilon_3\xi_4\,(\epsilon_2\cdot k_1)\\
&+ 2m\bar{u}_1\not\epsilon_3\xi_4\,(\epsilon_2\cdot k_1)\,,
\end{aligned}$$

where we have used the Clifford algebra and the equations of motion in the first line and momentum conservation in the third. The expression is already reminiscent of the one obtained through the graphical rules. Focusing on the two last terms $A(m) \equiv 2\bar{u}_1\not k_2\not\epsilon_3\xi_4\,(\epsilon_2\cdot k_1) + 2m\bar{u}_1\not\epsilon_3\xi_4\,(\epsilon_2\cdot k_1)$

$$\begin{aligned}
A(m) =&2\bar{u}_1\not k_2\not\epsilon_3\xi_4\,(\epsilon_2\cdot k_1) + m\bar{u}_1\{\not\epsilon_2,\not k_1\}\not\epsilon_3\xi_4\\
=&\bar{u}_1\not k_2\{\not\epsilon_2,\not k_{12}\}\not\epsilon_3\xi_4 + m\bar{u}_1\not\epsilon_2\not k_1\not\epsilon_3\xi_4 + m^2\bar{u}_1\not\epsilon_2\not\epsilon_3\xi_4\\
=&\bar{u}_1\not k_2\not\epsilon_2\{\not k_{12},\not\epsilon_3\}\xi_4 - \bar{u}_1\not k_2\not\epsilon_2\not\epsilon_3\not k_{12}\xi_4 + \bar{u}_1\not k_2\not k_1\not\epsilon_2\not\epsilon_3\xi_4 + m\bar{u}_1\not\epsilon_2\not k_1\not\epsilon_3\xi_4 + m^2\bar{u}_1\not\epsilon_2\not\epsilon_3\xi_4\\
=&-4\bar{u}_1\not f_2\xi_4\,(\epsilon_3\cdot k_4) - 4\bar{u}_1\not f_2\not f_3\xi_4 - \bar{u}_1\not\epsilon_2\not k_2\not\epsilon_3\not k_4\xi_4 + \bar{u}_1\not k_2\not k_1\not\epsilon_2\not\epsilon_3\xi_4 + m\bar{u}_1\not\epsilon_2\not k_1\not\epsilon_3\xi_4\\
&+ m^2\bar{u}_1\not\epsilon_2\not\epsilon_3\xi_4\\
=&-4\bar{u}_1\not f_2\not f_3\xi_4 - \bar{u}_1\not\epsilon_2\not k_2\not\epsilon_3\not k_4\xi_4 + s_{12}\bar{u}_1\not\epsilon_2\not\epsilon_3\xi_4 + m\bar{u}_1\not\epsilon_2\not k_{12}\not\epsilon_3\xi_4,
\end{aligned}$$

where we have used the transversality condition and the Clifford algebra in the first line. In the second line we used the Clifford algebra twice on the first term and then finally in the last line we have $s_{12} = 2k_1\cdot k_2 + m^2$. Using this we can find an expression for $\bar{u}_1\not\epsilon_2(\not k_{12}+m)\not\epsilon_3\not k_4\xi_4$ in terms of the numerators obtained through the increasing tree algorithm:

$$\begin{aligned}
\bar{u}_1\not\epsilon_2(\not k_{12}+m)\not\epsilon_3\not k_4\xi_4 =&4N_{\{2,3\}}(\mathbf{1}^{\frac{1}{2}}, 2^1, 3^1, \mathbf{4}^{\frac{1}{2}}) + s_{12}\bar{u}_1\not\epsilon_2\not\epsilon_3\xi_4 + m\bar{u}_1\not\epsilon_2\not k_{12}\not\epsilon_3\xi_4\\
=&4N_{\{2,3\}}(\mathbf{1}^{\frac{1}{2}}, 2^1, 3^1, \mathbf{4}^{\frac{1}{2}}) + \left(s_{12}-m^2\right)\bar{u}_1\not\epsilon_2\not\epsilon_3\xi_4 + m\bar{u}_1\not\epsilon_2\left(\not k_{12}+m\right)\not\epsilon_3\xi_4.
\end{aligned}$$

So that finally

$$\bar{u}_1 \not{\epsilon}_2 \left(\not{k}_{12} + m\right) \not{\epsilon}_3 v_4 = 4N_{\{2,3\}}(\mathbf{1}^{\frac{1}{2}}, 2^1, 3^1, \mathbf{4}^{\frac{1}{2}}) + \left(s_{12} - m^2\right) \chi_1 \not{\epsilon}_2 \not{\epsilon}_3 \xi_4.$$

From this we can get the massive numerator

$$\begin{aligned} N_{\{2,3\}}(\mathbf{1}^{\frac{1}{2}}, 2^1, 3^1, \mathbf{4}^{\frac{1}{2}}) =& \bar{u}_1 \not{f}_2 \not{f}_3 \xi_4 + \left(\bar{u}_1 \not{f}_2 \xi_4\right)\left(\epsilon_3 \cdot k_4\right) - \left(\bar{u}_1 \not{f}_3 \xi_4\right)\left(\epsilon_2 \cdot k_1\right) \\ & - \left(\bar{u}_1 \xi_4\right)\left(\epsilon_2 \cdot k_1\right)\left(\epsilon_3 \cdot k_4\right) \\ =& -\frac{1}{4} \bar{u}_1 \not{\epsilon}_2 (\not{k}_{12} + m) \not{\epsilon}_3 v_4 - \frac{1}{4}(s_{12} - m^2) \bar{u}_1 \not{\epsilon}_2 \not{\epsilon}_3 \xi_4. \end{aligned} \tag{7.1.2}$$

The calculation is very similar for the other numerator here the expression is similar to the one for the already calculated, except having $2 \to 3$ and $3 \to 2$. We then note that performing this replacement in the expansion for first other numerator only changes the final answer by replacing $2 \to 3$ and $3 \to 2$, so we can use the expression just obtained, i.e.

$$\begin{aligned} \frac{1}{4} \bar{u}_1 \not{\epsilon}_3 (\not{k}_{13} + m) \not{\epsilon}_2 v_4 =& \frac{1}{4} \bar{u}_1 \not{\epsilon}_2 (\not{k}_{12} + m) \not{\epsilon}_3 v_4 \Big|_{\{2,3\} \to \{3,2\}} \\ =& -\bar{u}_1 \not{f}_3 \not{f}_2 \xi_4 - \left(\bar{u}_1 \not{f}_3 \xi_4\right)\left(\epsilon_2 \cdot k_4\right) + \left(\bar{u}_1 \not{f}_2 \xi_4\right)\left(\epsilon_3 \cdot k_1\right) \\ & + \left(\bar{u}_1 \xi_4\right)\left(\epsilon_3 \cdot k_1\right)\left(\epsilon_2 \cdot k_4\right) - \frac{1}{4}(s_{13} - m^2) \bar{u}_1 \not{\epsilon}_3 \not{\epsilon}_2 \xi_4. \end{aligned}$$

Then using momentum conservation on the fourth term and noticing that the last term can be manipulated slightly

$$\frac{1}{4}(s_{13} - m^2) \bar{u}_1 \not{\epsilon}_3 \not{\epsilon}_2 \xi_4 = -\frac{1}{4}(s_{13} - m^2) \bar{u}_1 \not{\epsilon}_2 \not{\epsilon}_3 \xi_4 - (\bar{u}_1 \xi_4)(k_1 \cdot k_3)(\epsilon_2 \cdot \epsilon_3).$$

This leads to

$$\frac{1}{4} \bar{u}_1 \not{\epsilon}_3 (\not{k}_{13} + m) \not{\epsilon}_2 v_4 = -N_{\{2,3\}}(\mathbf{1}^{\frac{1}{2}}, 3^1, 2^1, \mathbf{4}^{\frac{1}{2}}) + \frac{1}{4}(s_{13} - m^2) \bar{u}_1 \not{\epsilon}_2 \not{\epsilon}_3 \xi_4.$$

So the expression for our massive numerator is

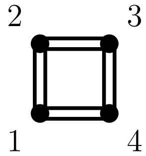
$$\begin{aligned} N_{\{2,3\}}(\mathbf{1}^{\frac{1}{2}}, 3^1, 2^1, \mathbf{4}^{\frac{1}{2}}) =& \bar{u}_1 \not{f}_3 \not{f}_2 \xi_4 - \left(\bar{u}_1 \not{f}_2 \xi_4\right)\left(\epsilon_3 \cdot k_1\right) + \left(\bar{\chi}_1 \not{f}_3 \xi_4\right)\left(\epsilon_2 \cdot k_4\right) \\ & + \left(\bar{u}_1 \xi_4\right)\left(\epsilon_2 \cdot f_3 \cdot k_1\right) + \left(\bar{u}_1 \xi_4\right)\left(\epsilon_2 \cdot k_1\right)\left(\epsilon_3 \cdot k_1\right) \\ =& -\frac{1}{4} \bar{u}_1 \not{\epsilon}_3 (\not{k}_{13} + m) \not{\epsilon}_2 v_4 + \frac{1}{4}(s_{13} - m^2) \bar{u}_1 \not{\epsilon}_2 \not{\epsilon}_3 \xi_4. \end{aligned}$$

The expressions for both numerators match the literature [7, 56] apart from the normalization and the extra factors proportional to the massive propagators e.g. $s_{12} -$

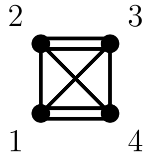
$m^2 = (k_1+k_2)^2 - m^2 = 2k_1 \cdot k_2$, which vanishes when calculating the color ordered $qggq$ amplitude

$$A(\mathbf{1}_{\bar{q}}, 2_g, 3_g, \mathbf{4}_q) = \mathrm{PT}(1,2,3,4) \sum_{\beta \in S_2} N_{\{2,3\}}(\mathbf{1}^{\frac{1}{2}}, \beta, \mathbf{4}^{\frac{1}{2}}) \mathrm{PT}(1,\beta,4)$$

$$= N_{\{2,3\}}(\mathbf{1}^{\frac{1}{2}}, 2^1, 3^1, \mathbf{4}^{\frac{1}{2}})$$



$$+ N_{\{2,3\}}(\mathbf{1}^{\frac{1}{2}}, 3^1, 2^1, \mathbf{4}^{\frac{1}{2}})$$



$$= n_s^{2\mathrm{q}} \left(\frac{1}{s_{12} - m^2} + \frac{1}{s_{14}} \right) + n_u^{2\mathrm{q}} \frac{1}{s_{14}}$$
$$+ \frac{1}{4} \bar{u}_1 \not{\epsilon}_2 \not{\epsilon}_3 \xi_4 \left\{ (s_{12} - m^2) \left(\frac{1}{s_{12} - m^2} + \frac{1}{s_{14}} \right) + \frac{(s_{13} - m^2)}{s_{14}} \right\}$$
$$= \frac{n_s^{2\mathrm{q}}}{s_{12} - m^2} - \frac{n_t^{2\mathrm{q}}}{s_{14}}.$$

Here we have again defined $n_s^{2\mathrm{q}} \equiv -N_{\{2,3\}}(\mathbf{1}^{\frac{1}{2}}, 2^1, 3^1, \mathbf{4}^{\frac{1}{2}}) - \frac{1}{4}(s_{12} - m^2)\bar{u}_1 \not{\epsilon}_2 \not{\epsilon}_3 \xi_4$ and $n_u^{2\mathrm{q}} \equiv N_{\{2,3\}}(\mathbf{1}^{\frac{1}{2}}, 3^1, 2^1, \mathbf{4}^{\frac{1}{2}}) - \frac{1}{4}(s_{13} - m^2)\bar{u}_1 \not{\epsilon}_2 \not{\epsilon}_3 \xi_4$ and used the Jacobi identity $n_t^{2\mathrm{q}} = -n_s^{2\mathrm{q}} - n_u^{2\mathrm{q}}$ and massive momentum conservation $s_{12} + s_{13} + s_{14} = 2m^2$. Having shown that the numerators found reproduce QCD amplitudes at four points, we will not turn to massive spinor helicity to obtain results for amplitudes containing more particles.

7.2 Checking using massive spinor-helicity

Not only does calculating the six numerators themselves turn into a lot of work at five points, but checking them all using the methods already discussed becomes a large task. We then instead turn to the massive spinor-helicity techniques developed by [42] and presented in section 3.4. Here we will briefly discuss the techniques and then use them to numerically compare our result with the all-multiplicity result obtained by [7]. The spinors $\bar{u}, v$ can be formed from the massive spinor-helicity variables

$$\begin{aligned} v_n^b &= -|k_n^b\rangle + |k_n^b], \\ \bar{u}_1^a &= -\langle k_1^a| + [k_1^a|, \end{aligned} \tag{7.2.1}$$

while the auxiliary spinor ξ_n has to obey $(\not{k}_n - m)\xi_n^c = v_n^c$. There are different choices for the representation of the spinor, but we will work with

$$\xi_n^b = -\left(\frac{|k_n^b\rangle}{m} + 2\frac{|k_n^b]}{m} \right), \tag{7.2.2}$$

since

$$\begin{aligned}(\not{k}_n - m)\xi_n^c &= -\frac{2\epsilon_{ab}\,|n^a\rangle[n^b n^c]}{m} + \frac{\epsilon_{ab}|n^a]\langle n^b n^c\rangle}{m} + |n^c\rangle + 2|n^c] \\ &= -2\epsilon_{ab}\epsilon^{bc}\,|n^a\rangle - \epsilon_{ab}\epsilon^{bc}|n^a] + |n^c\rangle + 2|n^c] \\ &= -|n^c\rangle + |n^c] = v_n^c.\end{aligned}$$

Now we can calculate the helicity basis expression for the numerator obtained by the graphic rules. First we plug in the SH variables

$$\begin{aligned}N(\underline{1}^a, 2^+, \overline{3}^b) =& \bar{u}_1 \not{\epsilon}_2 \xi_3 - \bar{u}_1 \xi_3 (\epsilon_2 \cdot k_1) \\ =& \frac{1}{\sqrt{2}\,\langle q2\rangle\, m}\left\{-2[1^a 2]\,\langle 2q\rangle\,[23^b] - \langle 1^a 3^b\rangle\langle q|1|2] + 2[1^a 3^b]\langle q|1|2]\right\}.\end{aligned}$$

We can then collect the first term in an angle-bracket product $\langle i|j|k]$, use momentum conservation and then expand it again, using $\not{k}_i = |i\rangle[i| + |i]\langle i|$ for massless particles and $\not{k}_i = |i^a\rangle[i_a| + |i_a]\langle i^a| = \epsilon_{ab}|i^a\rangle[i^b| + \epsilon_{ab}|i^b]\langle i^a|$ for the massive particles,

$$\begin{aligned}N(\underline{1}^a, 2^+, \overline{3}^b) =& \frac{1}{\sqrt{2}\,\langle q2\rangle\, m}\left\{-2[1^a|2|q\rangle[23^b] + 2[1^a 3^b]\langle q|1|2] - \langle 1^a 3^b\rangle\langle q|1|2]\right\} \\ =& \frac{1}{\sqrt{2}\,\langle q2\rangle\, m}\left\{-2\epsilon_{cd}[1^a 1^c]\langle 1^d q\rangle[23^b] - 2\epsilon_{cd}[1^a 3^c]\langle 3^d q\rangle[23^b]\right. \\ & \left. + 2[1^a 3^b]\langle q|1|2] - \langle 1^a 3^b\rangle\langle q|1|2]\right\}.\end{aligned}$$

Then using $[i^a j^b] = m\epsilon^{ab}$ as well as $\epsilon^{ab}\epsilon_{bc} = \delta^a_c$ on the first term and the Schouten identity on the second term

$$\begin{aligned}N(\underline{1}^a, 2^+, \overline{3}^b) =& \frac{1}{\sqrt{2}\,\langle q2\rangle\, m}\left\{-2\epsilon^{ac}\epsilon_{cd} m\langle 1^d q\rangle[23^b] - 2\epsilon_{cd}[1^a 3^c]\langle 3^d q\rangle[23^b]\right. \\ & \left. + 2[1^a 3^b]\langle q|1|2] - \langle 1^a 3^b\rangle\langle q|1|2]\right\} \\ =& \frac{1}{\sqrt{2}\,\langle q2\rangle\, m}\left\{-2m\langle 1^a q\rangle[23^b] - 2\epsilon^{bc}\epsilon_{cd} m[1^a 2]\langle q3^d\rangle + 2[1^a 3^b]\langle q|3|2]\right. \\ & \left. + 2\,[1^a 3^b]\langle q|1|2] - \langle 1^a 3^b\rangle\langle q|1|2]\right\} \\ =& \frac{1}{\sqrt{2}\,\langle q2\rangle\, m}\left\{-2m\langle 1^a q\rangle[23^b] - 2m[1^a 2]\langle q3^b\rangle - \langle 1^a 3^b\rangle\langle q|1|2]\right\}.\end{aligned}$$

Lastly we notice that the first two terms correspond exactly to the calculation in section 3.4. Reusing the result obtained there, we find agreement with [7]

$$N(\underline{1}^a, 2^+, \overline{3}^b) = \frac{\langle 1^a 3^b\rangle\langle q|1|2]}{\sqrt{2}\,\langle q2\rangle\, m}. \tag{7.2.3}$$

Let us try the same thing for four point, $N_{\{2,3\}}(1^a, 2^-, 3^+, 4^b)$. We need an expression for the field strength tensor. Depending on the helicity of the particle we find

$$\not{f}_i^- = \frac{1}{2}\not{k}_i\not{\epsilon}_i^- = \frac{|i\rangle\langle i|}{\sqrt{2}}, \qquad \not{f}_i^+ = \frac{1}{2}\not{k}_i\not{\epsilon}_i^+ = -\frac{|i][i|}{\sqrt{2}}.$$

Using this we can find spinor helicity expressions for all the terms contributing to the four-point numerators

$$\begin{aligned}
\bar{u}_1\not{f}_2^-\not{f}_3^+\xi_4 &= 0,\\
\bar{u}_1\not{f}_2^-\xi_4\left(\epsilon_3^+\cdot k_4\right) &= \frac{\langle 1^a2\rangle\,\langle 24^b\rangle\langle q_3|4|3]}{2m\langle q_33\rangle}\\
\bar{u}_1\not{f}_3^+\xi_4\left(\epsilon_2^-\cdot k_1\right) &= -\frac{[1^a3][34^b][q_2|1|2\rangle}{m[q_22]}
\end{aligned}$$

$$\begin{aligned}
\bar{u}_1\xi_4\left(\epsilon_2^-\cdot k_1\right)\left(\epsilon_3^+\cdot k_4\right) &= \frac{1}{m[q_22]\langle q_33\rangle}\left\{[1^a4^b]\langle q_3|4|3][q_2|1|2\rangle - \frac{1}{2}\langle 1^a4^b\rangle\langle q_3|4|3][q_2|1|2\rangle\right\},\\
(s_{12}-m^2)\bar{u}_1\epsilon_2^-\epsilon_3^+\xi_4 &= \frac{2\epsilon_{cd}[21^c]\langle 1^d2\rangle}{m[q_22]\langle q_33\rangle}\left\{\langle 1^a2\rangle[q_23]\langle q_34\rangle - 2[1^aq_2]\langle 2q_3\rangle[34^b]\right\}.
\end{aligned}$$

As described in chapter 3, the reference vectors aren't unique and so we can use this to set $q_2 = 3$ and $q_3 = 2$. Then after using momentum conservation and putting everything on a common denominator the BCJ numerator can be written as

$$\begin{aligned}
N_{\{2,3\}}(\underline{1}^a,2^-,3^+,\overline{4}^b) =& \frac{[3|1|2\rangle}{ms_{23}}\left\{-\frac{1}{2}\langle 1^a2\rangle\,\langle 24^b\rangle[32] + [1^a3][34^b]\langle 23\rangle\right.\\
&\left. - [1^a4^b]\langle 2|4|3] + \frac{1}{2}\langle 1^a4^b\rangle\langle 2|4|3]\right\}\\
=& \frac{[3|1|2\rangle}{ms_{23}}\left\{\frac{1}{2}\langle 24^b\rangle\langle 1^a|2|3] - [34^b]\langle 2|3|1^a]\right.\\
&\left. - [1^a4^b]\langle 2|4|3] + \frac{1}{2}\langle 1^a4^b\rangle\langle 2|4|3]\right\}.
\end{aligned}$$

Now we note that

$$\begin{aligned}
\frac{1}{2}\langle 1^a4^b\rangle\langle 2|4|3] &= -\frac{m}{2}\left\{[4^b3]\langle 1^a2\rangle + [1^a3]\langle 4^b2\rangle\right\} - \frac{1}{2}\langle 24^b\rangle\langle 1^a|2|3]\\
-[1^a4^b]\langle 2|4|3] &= m\left\{[4^b3]\langle 1^a2\rangle + [1^a3]\langle 4^b2\rangle\right\} + [34^b]\langle 2|3|1^a].
\end{aligned} \tag{7.2.4}$$

After plugging in the above and using momentum conservation this leads to the numerator

$$N_{\{2,3\}}(\underline{1}^a,2^-,3^+,\overline{4}^b) = \frac{[3|1|2\rangle}{2s_{23}}\left\{[4^b3]\langle 1^a2\rangle + [1^a3]\langle 4^b2\rangle\right\}. \tag{7.2.5}$$

This again matches the expression from [7]. To check the numerators at higher n, we will use the all-multiplicity result, also from [7] obtained using BCFW recursion. Here the all plus amplitude with 2 quarks is given by

$$A(\underline{1}^a, 3^+, 4^+, \ldots, n^+, \overline{2}^b) = \frac{im\langle 1^a 2^b\rangle [3| \prod_{j=3}^{n-2} \left\{ \not{p}_{13\ldots j} \not{p}_{j+1} + (s_{13\ldots j} - m^2) \right\} |n]}{(s_{13} - m^2)(s_{134} - m^2) \ldots (s_{13_{n-1}} - m^2) \langle 34\rangle \langle 45\rangle \ldots \langle (n-1)n\rangle}, \tag{7.2.6}$$

where $\not{p}_{ij\ldots k} = \not{p}_i + \not{p}_j + \cdots + \not{p}_k$. For 3 gluons, since $n = 5$, this is

$$\begin{aligned} A(\underline{1}^a, 3^+, 4^+, 5^+, \overline{2}^b) =& \frac{im\langle 1^a 2^b\rangle [3| \prod_{j=3}^{3} \left\{ \not{p}_{13\ldots j} \not{p}_{j+1} + (s_{13\ldots j} - m^2) \right\} |5]}{(s_{13} - m^2)(s_{134} - m^2) \langle 34\rangle \langle 45\rangle} \\ =& \frac{im\langle 1^a 2^b\rangle [3| \left\{ \not{p}_{13} \not{p}_4 + (s_{13} - m^2) \right\} |5]}{(s_{13} - m^2)(s_{134} - m^2) \langle 34\rangle \langle 45\rangle}. \end{aligned}$$

We can then do the relabeling $1 \to 1$, $2 \to 5$, $3 \to 2$, $4 \to 3$, $5 \to 4$ to get the result in the notation that we have used throughout

$$A(\underline{1}^a, 2^+, 3^+, 4^+, \overline{5}^b) = \frac{im\langle 1^a 5^b\rangle [2| \left\{ \not{p}_{12} \not{p}_3 + (s_{12} - m^2) \right\} |4]}{(s_{12} - m^2)(s_{123} - m^2) \langle 23\rangle \langle 34\rangle}.$$

The all minus helicity configuration is easily obtained through the exchange $\langle\rangle \leftrightarrow []$

$$A(\underline{1}^a, 2^-, 3^-, 4^-, \overline{5}^b) = \frac{im[1^a 5^b] \langle 2| \left\{ \not{p}_{12} \not{p}_3 + (s_{12} - m^2) \right\} |4\rangle}{(s_{12} - m^2)(s_{123} - m^2)[23][34]}. \tag{7.2.7}$$

We have implemented the massive spinor helicity variables in Mathematica and numerically checking against (7.2.7), we find agreement by using our numerators and calculating the color ordered $qgggq$ amplitude through

$$A_5 = \int \mathrm{d}\Omega_{\mathrm{CHY}} \sum_{\beta \in S_3} N(1^a, \beta, n^b) \mathrm{PT}(1, \beta, n)$$

Finally we want to note that in the paper where these massive spinor helicity variables were introduced [42], a Compton amplitude between two massive spin $s \leq 1$ particles was introduced:

$$M(\mathbf{1}^s, 2^+, 3^-, \mathbf{4}^s) = \frac{\langle 3|1|2]^{2-2s}}{(s - m^2)(u - m^2)} \left(\langle \mathbf{4}3\rangle [\mathbf{1}2] + \langle \mathbf{1}3\rangle [\mathbf{4}2] \right)^{2s}, \tag{7.2.8}$$

where the **bold** notation here just means that the little group indices of the massive particles are symmetrized. Using the numerators from this section, we would expect this expression to hold. Since we already know the numerators in the spin 1/2 case, a simple calculation gives

$$
\begin{aligned}
M(1^a, 2^{+1}, 3^{-1}, 4^b) &= \int \mathrm{d}\Omega_{\mathrm{CHY}} \sum_{\alpha \in S_2} \mathrm{PT}(1, \alpha, n) \sum_{\beta \in S_2} N(1, \beta, n)\mathrm{PT}(1, \beta, n) \\
&= -\frac{N(1^a, 2^+, 3^-, 4^b)}{s - m^2} - \frac{N(1^a, 3^+, 2^-, 4^b)}{u - m^2} \\
&= \frac{\langle 3|1|2]}{2(s - m^2)(u - m^2)} \Big(\langle 4^b 3\rangle [1^a 2] + \langle 1^a 3\rangle [4^b 2] \Big) ,
\end{aligned}
$$

which matches 7.2.8. For the two other cases numerical checks find agreement as well. Interestingly, in [9] using similar considerations, they give an amplitude which also works for non-fundamental spin $s > 2$. We haven't been able to cast the numerators in a form which reproduces that result and it seems like non-fundamental spin is out of reach using CHY for now.

Feeling assured that the numerators obtained reproduce QCD amplitudes in accordance with the literature, we still face some issues. Particularly the subject of the reference order, which makes the calculation much more tedious. If we instead could have obtained a crossing symmetric numerator much work would be saved. In the next chapter we present a novel algorithm that produces the numerators much more efficiently since it is manifestly crossing symmetric in the $n - 2$ massless gluons.

Chapter 8

Exponential BCJ numerators

In this chapter we present a powerful diagrammatic algorithm for calculating n point DDM-basis master numerators which only requires $(n-2)!$ diagrams (plus relabeling), with the current state of the art being $(n-1)!$. The algorithm was constructed by noticing that the numerators discussed in chapters 6-7 can be put in to an exponential form. Then by performing an average over all reference orders, we obtain a numerator which is crossing symmetric.

Furthermore, the generalization to external massive particles of spin $s = 0, 1/2, 1$ still holds. This thesis provides a first presentation of the algorithm, so we want to make every aspect of the construction very clear. We start off by motivating the exponential numerators at three-point. Then the algorithm is presented and we show how it is used at three, four, five, and six points checking the results as we go. At the end we provide a brief analysis of the numerators obtained and discuss their relation to current topics in the amplitudes field.

8.1 Three-point motivation

We begin by looking at the three-point numerator, since they only have one reference order and so are automatically crossing symmetric. First of all, let us recall the three-point numerators obtained in chapter 6 and add the numerator with scalars at the end points

$$\begin{aligned} N(1^0, 2^1, 3^0) =& (\epsilon_2 \cdot k_1), \\ N(1^{\frac{1}{2}}, 2^1, 3^{\frac{1}{2}}) =& \bar{u}_1 \not{f}_2 \xi_3 - \bar{u}_1 \xi_3 (\epsilon_2 \cdot k_1), \\ N(1^1, 2^1, 3^1) =& (\epsilon_1 \cdot f_2 \cdot \epsilon_3) - (\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot k_1). \end{aligned} \tag{8.1.1}$$

The structures of all three numerators are very similar. Note in particular how we can rewrite the last two

$$\begin{aligned} N(1^{\frac{1}{2}},2^1,3^{\frac{1}{2}}) =&(\epsilon_2\cdot k_1)\left\{\bar{u}_1\left[\mathbb{1}-\frac{\not{f}_2}{(\epsilon_2\cdot k_1)}\right]\xi_3\right\},\\ N(1^1,2^1,3^1) =&(\epsilon_2\cdot k_1)\left\{\epsilon_1\cdot\left[\mathbb{1}-\frac{f_2}{(\epsilon_2\cdot k_1)}\right]\cdot\epsilon_3\right\}, \end{aligned} \tag{8.1.2}$$

with $\mathbb{1}\equiv\eta_{\mu\nu}$. We can take this further by writing this in terms of the spin matrix Σ. In this way we can even include the scalar numerator by defining the scalar polarizations through the dimensional reduction procedure described in section 4.6 with $\tilde{\epsilon}_1\cdot\tilde{\epsilon}_3=1$

$$\begin{aligned} N(1^0,2^1,3^0) =&(\epsilon_2\cdot k_1)\left\{\tilde{\epsilon}_1\cdot\left[\mathbb{1}+i\frac{f_{2,\mu\nu}\Sigma^{\mu\nu}_{s=0}}{2(\epsilon_2\cdot k_1)}\right]\cdot\tilde{\epsilon}_3\right\},\\ N(1^{\frac{1}{2}},2^1,3^{\frac{1}{2}}) =&(\epsilon_2\cdot k_1)\left\{\bar{u}_1\left[\mathbb{1}+i\frac{f_{2,\mu\nu}\Sigma^{\mu\nu}_{s=\frac{1}{2}}}{2(\epsilon_2\cdot k_1)}\right]\xi_3\right\},\\ N(1^1,2^1,3^1) =&(\epsilon_2\cdot k_1)\left\{\epsilon_1\cdot\left[\mathbb{1}+i\frac{f_{2,\mu\nu}\Sigma^{\mu\nu}_{s=1}}{2(\epsilon_2\cdot k_1)}\right]\cdot\epsilon_3\right\}, \end{aligned} \tag{8.1.3}$$

where

$$\begin{aligned} \Sigma^{\mu\nu}_{s=0} =&i\eta^{\mu\nu},\\ \Sigma^{\mu\nu}_{s=\frac{1}{2}} =&\frac{i}{4}[\gamma^\mu,\gamma^\nu],\\ \Sigma^{\mu\nu}_{s=1} =&i\left[\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}\right], \end{aligned} \tag{8.1.4}$$

and the energy-momentum tensor is traceless $f_{\mu\nu}\eta^{\mu\nu}=0$. As explained in the introduction of this thesis a lot of interest has been cast on gravitational scattering lately [3–6], so in light of this we will assume $\epsilon_i\cdot\epsilon_i=0$, since for gravitational amplitude obtained using double copy the polarization vectors obey $h_i^{\mu\nu}=\epsilon_i^\mu\epsilon_i^\nu$ and $h^\mu_{i,\mu}=\epsilon_i\cdot\epsilon_i=0$. In this setting we then also have $f^2=f_{i,\mu\nu}f_i^{\nu\sigma}=0$ leading to an even more compact rewriting of the numerators in terms of an exponential. Finally we can define spin dependent variables $\zeta_i(s)$ so that

$$N(1^s,2^1,3^s) =(\epsilon_2\cdot k_1)\left\{\zeta_1\times\exp\left[i\frac{f_{2,\mu\nu}\Sigma^{\mu\nu}_s}{2(\epsilon_2\cdot k_1)}\right]\times\zeta_3\right\}, \tag{8.1.5}$$

with $\zeta(s)$ in general behaving as

$$\begin{array}{rlll} \text{spin } 0: & \zeta_1 = \tilde{\epsilon}_1, & \zeta_n = \tilde{\epsilon}_2, & \times \to \cdot, \\ \text{spin } 1/2: & \zeta_1 = \bar{u}_1, & \zeta_n = \xi_n, & \times \to \times, \\ \text{spin } 1: & \zeta_1 = \epsilon_1, & \zeta_n = \epsilon_2, & \times \to \cdot. \end{array} \tag{8.1.6}$$

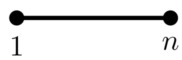
To be explicit, let's insert the ζ's and spin operator definitions

$$\begin{aligned} N(1^0, 2^1, 3^0) =& (\epsilon_2 \cdot k_1) \left\{ \tilde{\epsilon}_1 \cdot \tilde{\epsilon}_3 \right\}, \\ N(1^{\frac{1}{2}}, 2^1, 3^{\frac{1}{2}}) =& (\epsilon_2 \cdot k_1) \left\{ \bar{u}_1 \exp\left[-\frac{\not{f}_2}{(\epsilon_2 \cdot k_1)} \right] \xi_3 \right\}, \\ N(1^1, 2^1, 3^1) =& (\epsilon_2 \cdot k_1) \left\{ \epsilon_1 \cdot \exp\left[-\frac{f_2}{(\epsilon_2 \cdot k_1)} \right] \cdot \epsilon_3 \right\}, \end{aligned} \tag{8.1.7}$$

This exponential form is the main reason the algorithm is so efficient and using equation (8.1.5) as motivation we will in the next section present an algorithm which produces n point exponential numerators. The rules work for $n-2$ external massless gluons and 2 scalars or fermions with $k_1^2 = k_n^2 = m^2$.

8.2 Algorithm for exponential BCJ numerators

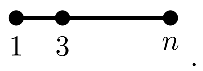
- Draw a *baseline* containing points 1 and n



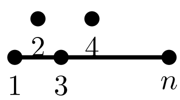
- At n points we now have $(n-3)$ levels in the diagram and the remaining points have to be placed by filling each level from the bottom and up. Diagrams are build in 3 steps.

- <u>Step 1: Inserting points into the diagram</u>

  1. The baseline can contain $m_0 = 0, 1, 2 \ldots, (n-4)$ points. Place the points corresponding to each particle's place on the line. Placing e.g. the third particle:



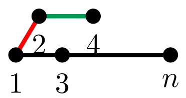
  2. The first level *has to contain at least 2 points*. It is the only level with this restriction. Placing e.g. the 4$^{\text{th}}$ and 2$^{\text{nd}}$ particles:



3. Continue to fill the levels until reaching the last level. The $(n-3)^{\text{rd}}$ level is the last. Only the $(n-1)^{th}$ particle can be in this level and it can only be there when the $(n-2)^{\text{nd}}$ particle is on the $(n-4)^{\text{th}}$ level.

- Step 2: Connecting the points

1. Connect all points on the same level with a horizontal line.
2. Take the far left point on each level and connect it to one of the levels below by drawing an angled line to the left and hitting the first point showing up on the lower level. If there are more points on this lower level, continue the line to the point farthest to the left on the level.



3. Only points from the first level can connect to the points $2,\ldots,(n-3)$ on the baseline.
4. A level is not allowed to contain a single point if it is connected to a level above it which has two or more points.

- Step 3: Read of contributions and multiply them

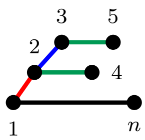
1. Contribution from baseline
   - For r points, excluding 1 and n, with $i<j<...<r$, the contribution is:

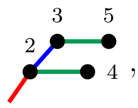
$$(\epsilon_i\cdot k_1)(\epsilon_j\cdot k_{1i})\ldots(\epsilon_r\cdot k_{1ij\ldots(r-1)})\left\{\zeta_1\times\exp\left[i\frac{f_{i,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_i\cdot k_1}\right]\times\exp\left[i\frac{f_{j,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_j\cdot k_{1i}}\right]\times\cdots\times\exp\left[i\frac{f_{r,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_r\cdot k_{1ij\ldots(r-1)}}\right]\times\zeta_n\right\}$$

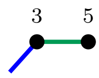
2. Contribution from other levels.
   - All horizontal lines with points $i<j<\cdots<r$ contribute a lower point crossing symmetric numerator $N(i^1,j^1,\ldots,r^1)$, with $N(i^1,j^1)\equiv(\epsilon_i\cdot\epsilon_j)$.
   - Starting at the first level and working up, all the remaining lines consisting of $i<j<\cdots<r<s$ points contribute $\epsilon_s\cdot k_{ij\ldots r}$ unless the s^{th} point has already been traversed, then the contribution is $k_s\cdot k_{ij\ldots r}$.
3. To find the numerical coefficient, consider each diagram as as tree with roots: the baseline is the root and the connected points on the levels form the tree. A subtree or a *branch* is then a horizontal line that is connected to another horizontal line. The overall numerical coefficient is given by $\prod_i\frac{1}{\mathcal{T}}\frac{1}{\mathcal{B}_i}$, with $\mathcal{T}$ being the number of points in the full tree and $\mathcal{B}_i$ is the number of points in the branches. If there are no branches on a tree then $\mathcal{B}_i\equiv 1$.

4. Taking m_l as the number of points in the l^{th} level, the overall sign of the diagram is given by $\left(\prod_{l=1}(-1)^{m_l+1}\right)$.

As an example we have the following decomposition of a diagram into trees and branches

Diagram: 

, Tree: 

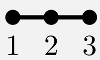
, Branch: 

so the overall numerical factor is $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$ and the sign is positive $(-1)^{2+1} \times (-1)^{2+1} = 1$.

- The numerator is the sum over all the ways of constructing the diagrams. Note that the diagram with all points on the baseline isn't constructed by the rules but still contributes.

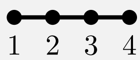
8.3 Three- and four-point checks

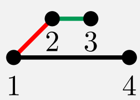
To show how the algorithm works, we will work through three and four points and then perform checks that we reproduce the master numerators found in chapters 6 and 7. At three points we only have one diagram since the baseline takes particles 1 and 3 and the first level needs at least 2 particles to be a valid contribution.

$N(1^s, 2^1, 3^s) =$ 

$$= (\epsilon_2 \cdot k_1) \left\{ \zeta_1 \times \exp\left[i \frac{f_{2,\mu\nu}\Sigma_s^{\mu\nu}}{2(\epsilon_2 \cdot k_1)} \right] \times \zeta_3 \right\}. \qquad (8.3.1)$$

This is easily checked to be the correct numerator, since it clearly matches the results (8.1.5). At four points we get one more diagram, specifically the diagram with two points in the first level. The diagrams and expression is:

$N(1^s, 2^1, 3^1, 4^s) =$ 

$+$ 

$$= (\epsilon_2 \cdot k_1)(\epsilon_3 \cdot k_{12}) \left\{ \zeta_1 \times \exp\left[i \frac{f_{2,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_2 \cdot k_1} \right] \times \exp\left[i \frac{f_{3,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_3 \cdot k_{12}} \right] \times \zeta_4 \right\}$$
$$- \frac{1}{2}(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_3)(\zeta_1 \times \zeta_4). \qquad (8.3.2)$$

To prove that this gives the correct numerator we choose a specific spin, e.g. $s = 1/2$ and notice that the four-point numerator (6.4.6) can be put into an exponential form as well

$$\begin{aligned} N_{\{2,3\}}(1^{\frac{1}{2}}, 2^1, 3^1, 4^{\frac{1}{2}}) =& \bar{u}_1 \not{f}_2 \not{f}_3 \xi_4 + (\bar{u}_1 \not{f}_2 \xi_4)\,(\epsilon_3 \cdot k_{12}) - (\bar{u}_1 \not{f}_3 \xi_4)\,(\epsilon_2 \cdot k_1) \\ & - (\bar{u}_1 \xi_4)\,(\epsilon_2 \cdot k_1)\,(\epsilon_3 \cdot k_{12}) \\ =& (\epsilon_2 \cdot k_1)(\epsilon_3 \cdot k_{12}) \left\{ \bar{u}_1 \exp\left[-\frac{\not{f}_2}{\epsilon_2 \cdot k_1} \right] \exp\left[-\frac{\not{f}_3}{\epsilon_3 \cdot k_{12}} \right] \xi_4 \right\}. \end{aligned} \tag{8.3.3}$$

The numerator with R.O. $\{3, 2\}$ is identical to the above expression apart from a small correction which we can find through

$$N_{\{3,2\}}(1^{\frac{1}{2}}, 2^1, 3^1, 4^{\frac{1}{2}}) - N_{\{2,3\}}(1^{\frac{1}{2}}, 2^1, 3^1, 4^{\frac{1}{2}}) = -(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_3)(\bar{u}_1 \xi_4),$$

which means the full crossing symmetric numerator is $N = \frac{1}{2} \sum_{\beta \in S_2} N_{\{\beta\}}$

$$\begin{aligned} N(1^{\frac{1}{2}}, 2^1, 3^1, 4^{\frac{1}{2}}) =& (\epsilon_2 \cdot k_1)(\epsilon_3 \cdot k_{12}) \left\{ \bar{u}_1 \exp\left[-\frac{\not{f}_2}{\epsilon_2 \cdot k_1} \right] \exp\left[-\frac{\not{f}_3}{\epsilon_3 \cdot k_{12}} \right] \xi_4 \right\} \\ & - \frac{1}{2}(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_3)(\bar{u}_1 \xi_4), \end{aligned}$$

finding agreement with equation (8.3.2). These numerators were very easy to come by and the checks were similarly simple. Before moving to $n = 5$, let us employ the exponential numerators to calculate the four point scalar-gluon amplitude from chapter 5. Inserting the spin definitions for scalar particles we obtain the following numerator

$$N(\mathbf{1}^0, 2^1, 3^1, \mathbf{4}^0) = (\epsilon_2 \cdot k_1)(\epsilon_3 \cdot k_{12}) - \frac{1}{2}(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_3),$$

from which we can get the other numerator just from exchanging $2 \leftrightarrow 3$

$$N(\mathbf{1}^0, 3^1, 2^1, \mathbf{4}^0) = (\epsilon_3 \cdot k_1)(\epsilon_2 \cdot k_{13}) - \frac{1}{2}(k_1 \cdot k_3)(\epsilon_2 \cdot \epsilon_3).$$

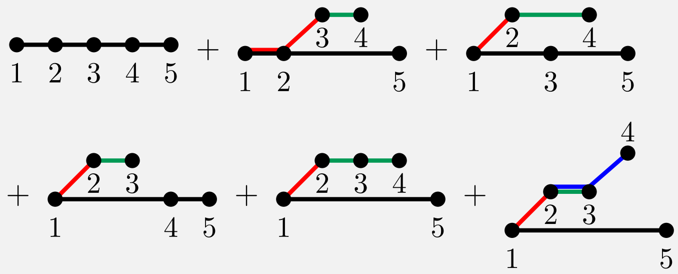
The amplitude calculated under support of the massive scattering equations is easily obtained and matches equation (5.3) as expected

$$\begin{aligned} A(1_\varphi, 2_g, 3_g, 4_\varphi) =& \frac{(-1)}{2} \mathrm{PT}(1, 2, 3, 4) \sum_{\beta \in S_2} N(\mathbf{1}^0, \beta, \mathbf{4}^0) \mathrm{PT}(1, \beta, 4) \\ =& N(\mathbf{1}^0, 2^1, 3^1, \mathbf{4}^0) \left(\frac{1}{s - m^2} + \frac{1}{t} \right) - N(\mathbf{1}^0, 3^1, 2^1, \mathbf{4}^0) \frac{1}{s - m^2} \\ =& \frac{(u - m^2)}{t} \left[\frac{2(\epsilon_3 \cdot k_4)(\epsilon_3 \cdot k_1)}{(u - m^2)} + \frac{2(\epsilon_2 \cdot k_1)(\epsilon_3 \cdot k_4)}{(s - m^2)} + (\epsilon_2 \cdot \epsilon_3) \right]. \end{aligned} \tag{8.3.4}$$

8.4 Five-point check

At five-points the diagrams begin to become more complicated. Here we have the following diagrams contributing to the numerator

$N(1^s, 2^1, 3^1, 4^1, 5^s) =$



$$
\begin{aligned}
=&(\epsilon_2\cdot k_1)(\epsilon_3\cdot k_{12})(\epsilon_4\cdot k_{123})\\
&\left\{\zeta_1\times\exp\left[i\frac{f_{2,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_2\cdot k_1}\right]\times\exp\left[i\frac{f_{3,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_3\cdot k_{12}}\right]\times\exp\left[i\frac{f_{4,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_4\cdot k_{123}}\right]\times\zeta_5\right\}\\
&-\frac{1}{2}\left\{\zeta_1\times\exp\left[i\frac{f_{2,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_2\cdot k_1}\right]\times\zeta_5\right\}(\epsilon_2\cdot k_1)(k_3\cdot k_{12})(\epsilon_3\cdot\epsilon_4)\\
&-\frac{1}{2}\left\{\zeta_1\times\exp\left[i\frac{f_{3,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_3\cdot k_1}\right]\times\zeta_5\right\}(\epsilon_3\cdot k_1)(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)\\
&-\frac{1}{2}\left\{\zeta_1\times\exp\left[i\frac{f_{4,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_4\cdot k_1}\right]\times\zeta_5\right\}(\epsilon_4\cdot k_1)\,(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)\\
&+\frac{1}{3}(\zeta_1\times\zeta_5)(k_1\cdot k_2)N(2^1,3^1,4^1)\\
&-\frac{1}{3}(\zeta_1\times\zeta_5)(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_{23}).
\end{aligned}
\tag{8.4.1}
$$

The result is of course crossing symmetric, so e.g. the numerator $N(1^s, 3^1, 2^1, 4^1, 5^s)$ is obtained through relabeling $2 \leftrightarrow 3$. This is very compact compared to the $24\times 6 = 144$ diagrams needed to compute all the numerators through the rules of chapter 6.

To show that this indeed gives the correct referenced ordered numerator, we will do the same exercise we did for four points, namely pick a specific spin and calculate the exponential numerator with color order $\{2, 3, 4\}$ and then look at different reference orders. Finally we will sum over the R.O's to obtain a crossing symmetric result. Taking $s = 1$ and reference order $\{2, 3, 4\}$ and using the notation $N^{5|\mathrm{YM}}_{\{RO\}} \equiv N_{\{RO\}}(1^1, 2^1, 3^1, 4^1, 5^1)$ we find

$$
N^{5|\mathrm{YM}}_{\{2,3,4\}} = (\epsilon_2{\cdot}k_1)(\epsilon_3{\cdot}k_{12})(\epsilon_4{\cdot}k_{123})\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{123}}\right]\cdot\epsilon_5\right\}.
$$

The numerator $N^{5|\mathrm{YM}}_{\{2,4,3\}}$ is very similar to this, so to compute the correction obtained from changing the reference order we take

$$N^{5|\mathrm{YM}}_{\{2,4,3\}} - N^{5|\mathrm{YM}}_{\{2,3,4\}} = -\left\{\epsilon_1 \cdot \exp\left[-\frac{f_2}{\epsilon_2 \cdot k_1}\right] \cdot \epsilon_5\right\} (\epsilon_2 \cdot k_1)(\epsilon_4 \cdot \epsilon_3)(k_3 \cdot k_{12}).$$

A similar prescription works for the one other of the reference orders

$$N^{5|\mathrm{YM}}_{\{3,2,4\}} - N^{5|\mathrm{YM}}_{\{2,4,3\}} = -\left\{\epsilon_1 \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_{123}}\right] \cdot \epsilon_5\right\} (\epsilon_4 \cdot k_{123})\,(\epsilon_3 \cdot \epsilon_2)(k_2 \cdot k_1).$$

The remaining have even more corrections. We can express them in a compact way using the two above numerators already calculated

$$N^{5|\mathrm{YM}}_{\{3,4,2\}} - N^{5|\mathrm{YM}}_{\{3,2,4\}} = -\left\{\epsilon_1 \cdot \exp\left[-\frac{f_3}{\epsilon_3 \cdot k_1}\right] \cdot \epsilon_5\right\} (\epsilon_3 \cdot k_1)\,(\epsilon_4 \cdot \epsilon_2)(k_2 \cdot k_1),$$

$$\begin{aligned} N^{5|\mathrm{YM}}_{\{4,2,3\}} - N^{5|\mathrm{YM}}_{\{2,4,3\}} = &-\left\{\epsilon_1 \cdot \exp\left[-\frac{f_3}{\epsilon_3 \cdot k_{12}}\right] \cdot \epsilon_5\right\} (k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_4)(\epsilon_3 \cdot k_{12}) \\ &+ (k_1 \cdot k_2)(\epsilon_1 \cdot \epsilon_5)\left((\epsilon_3 \cdot \epsilon_4)(\epsilon_2 \cdot k_3) - (\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot k_3)\right) \end{aligned},$$

$$N^{5|\mathrm{YM}}_{\{4,3,2\}} - N^{5|\mathrm{YM}}_{\{4,2,3\}} = -\left\{\epsilon_1 \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_1}\right] \cdot \epsilon_5\right\} (k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot k_1).$$

So that an expression for the five-point RO averaged numerators $N^{5|\mathrm{YM}} = \frac{1}{6}\sum_{\sigma \in S_3} N^{5|\mathrm{YM}}_{\{\sigma\}}$ can be obtained by combining all the terms

$$\begin{aligned} N^{5|\mathrm{YM}} =& (\epsilon_2 \cdot k_1)(\epsilon_3 \cdot k_{12})(\epsilon_4 \cdot k_{123})\left\{\epsilon_1 \cdot \exp\left[-\frac{f_2}{\epsilon_2 \cdot k_1}\right] \cdot \exp\left[-\frac{f_3}{\epsilon_3 \cdot k_{12}}\right] \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_{123}}\right] \cdot \epsilon_5\right\} \\ &- \frac{1}{2}\left\{\epsilon_1 \cdot \exp\left[-\frac{f_2}{\epsilon_2 \cdot k_1}\right] \cdot \epsilon_5\right\} (\epsilon_2 \cdot k_1)(\epsilon_4 \cdot \epsilon_3)(k_3 \cdot k_{12}) \\ &- \frac{1}{3}\left\{\epsilon_1 \cdot \exp\left[-\frac{f_3}{\epsilon_3 \cdot k_{12}}\right] \cdot \epsilon_5\right\} (k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_4)(\epsilon_3 \cdot k_{12}) \\ &- \frac{1}{3}\left\{\epsilon_1 \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_{123}}\right] \cdot \epsilon_5\right\} (\epsilon_4 \cdot k_{123})\,(\epsilon_3 \cdot \epsilon_2)(k_2 \cdot k_1) \\ &- \frac{1}{6}\left\{\epsilon_1 \cdot \exp\left[-\frac{f_3}{\epsilon_3 \cdot k_1}\right] \cdot \epsilon_5\right\} (\epsilon_3 \cdot k_1)\,(\epsilon_4 \cdot \epsilon_2)(k_2 \cdot k_1) \\ &- \frac{1}{6}\left\{\epsilon_1 \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_1}\right] \cdot \epsilon_5\right\} (k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot k_1) \\ &+ \frac{1}{3}(k_1 \cdot k_2)(\epsilon_1 \cdot \epsilon_5)\left((\epsilon_3 \cdot \epsilon_4)(\epsilon_2 \cdot k_3) - (\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot k_3)\right). \end{aligned}$$

This is however, not in the form given by the algorithm. We can show that it can be rewritten by employing the easily checkable identity

$$(\epsilon_i \cdot k_{1,abc\cdots z})\left\{\epsilon_1 \cdot \exp\left[-\frac{f_i}{\epsilon_i \cdot k_{1,abc\cdots z}}\right] \cdot \epsilon_6\right\} = (\epsilon_i \cdot k_1)\left\{\epsilon_1 \cdot \exp\left[-\frac{f_i}{\epsilon_i \cdot k_1}\right] \cdot \epsilon_6\right\} + (\epsilon_i \cdot k_{abc\cdots z})(\epsilon_1 \cdot \epsilon_6),$$

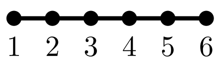
Using this we can rewrite the numerator into a form which is almost the same as the one obtained through the algorithm

$$\begin{aligned} N^{5|\mathrm{YM}} =& (\epsilon_2 \cdot k_1)(\epsilon_3 \cdot k_{12})(\epsilon_4 \cdot k_{123})\left\{\epsilon_1 \cdot \exp\left[-\frac{f_2}{\epsilon_2 \cdot k_1}\right] \cdot \exp\left[-\frac{f_3}{\epsilon_3 \cdot k_{12}}\right] \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_{123}}\right] \cdot \epsilon_5\right\} \\ & - \frac{1}{2}\left\{\epsilon_1 \cdot \exp\left[-\frac{f_2}{\epsilon_2 \cdot k_1}\right] \cdot \epsilon_5\right\}(\epsilon_2 \cdot k_1)(\epsilon_3 \cdot \epsilon_4)(k_3 \cdot k_{12}) \\ & - \frac{1}{2}\left\{\epsilon_1 \cdot \exp\left[-\frac{f_3}{\epsilon_3 \cdot k_1}\right] \cdot \epsilon_5\right\}(\epsilon_3 \cdot k_1)(\epsilon_2 \cdot \epsilon_4)(k_1 \cdot k_2) \\ & - \frac{1}{2}\left\{\epsilon_1 \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_1}\right] \cdot \epsilon_5\right\}(\epsilon_4 \cdot k_1)(\epsilon_2 \cdot \epsilon_3)(k_1 \cdot k_2) \\ & + \frac{1}{3}(k_1 \cdot k_2)(\epsilon_1 \cdot \epsilon_5)\left[(\epsilon_3 \cdot \epsilon_4)(\epsilon_2 \cdot k_3) - (\epsilon_2 \cdot \epsilon_4)(\epsilon_3 \cdot k_2) - (\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot k_3) \right. \\ & \left. \qquad - (\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot k_{23})\right] \end{aligned}$$

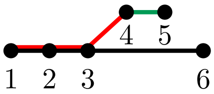
This matches the result (8.4.1) if we notice that $N(2^1, 3^1, 4^1) = (\epsilon_3 \cdot \epsilon_4)(\epsilon_2 \cdot k_3) - (\epsilon_2 \cdot \epsilon_4)(\epsilon_3 \cdot k_2) - (\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot k_3)$. The result have been slightly tedious to check, which points to the efficiency of our algorithm compared to the one described in chapter 6. For completeness we will now present the diagrams and terms obtained at $n = 6$.

8.5 Six-point diagrams

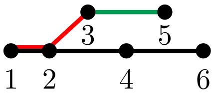
Below we have listed the 24 diagrams contributing at six points. As one might imagine, the check here is even more tedious than at $n = 5$ and since not much new insight is gained from it, we have left it for Appendix B but we note that the expression also has been checked in Mathematica. The diagrams and their corresponding contributions are:



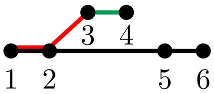
$$= (\epsilon_2 \cdot k_1)(\epsilon_3 \cdot k_{12})(\epsilon_4 \cdot k_{123})(\epsilon_5 \cdot k_{1234}) \left\{\zeta_1 \times \exp\left[i\frac{f_{2,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_2 \cdot k_1}\right] \times \exp\left[i\frac{f_{3,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_3 \cdot k_{12}}\right] \times \exp\left[i\frac{f_{4,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_4 \cdot k_{123}}\right] \times \exp\left[i\frac{f_{5,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_5 \cdot k_{1234}}\right] \times \zeta_6\right\},$$



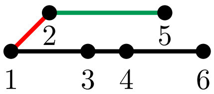
$$= -\frac{1}{2}(k_4 \cdot k_{123})(\epsilon_4 \cdot \epsilon_5)(\epsilon_2 \cdot k_1)\,(\epsilon_3 \cdot k_{12}) \left\{\zeta_1 \times \exp\left[i\frac{f_{2,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_2 \cdot k_1}\right] \times \exp\left[i\frac{f_{3,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_3 \cdot k_{12}}\right] \times \zeta_6\right\},$$



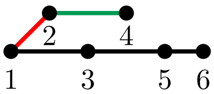
$$= -\frac{1}{2}(k_3 \cdot k_{12})(\epsilon_3 \cdot \epsilon_5)(\epsilon_2 \cdot k_1)(\epsilon_4 \cdot k_{12}) \left\{\zeta_1 \times \exp\left[i\frac{f_{2,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_2 \cdot k_1}\right] \times \exp\left[i\frac{f_{4,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_4 \cdot k_{12}}\right] \times \zeta_6\right\},$$



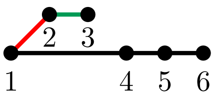
$$= -\frac{1}{2}(k_3 \cdot k_{12})(\epsilon_3 \cdot \epsilon_4)(\epsilon_2 \cdot k_1)\,(\epsilon_5 \cdot k_{12}) \left\{\zeta_1 \times \exp\left[i\frac{f_{2,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_2 \cdot k_1}\right] \times \exp\left[i\frac{f_{5,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_5 \cdot k_{12}}\right] \times \zeta_6\right\},$$



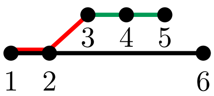
$$= -\frac{1}{2}(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_5)(\epsilon_3 \cdot k_1)\,(\epsilon_4 \cdot k_{13}) \left\{\zeta_1 \times \exp\left[i\frac{f_{3,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_3 \cdot k_1}\right] \times \exp\left[i\frac{f_{4,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_4 \cdot k_{13}}\right] \times \zeta_6\right\},$$



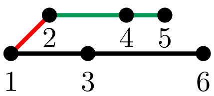
$$= -\frac{1}{2}(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_4)(\epsilon_3 \cdot k_1)(\epsilon_5 \cdot k_{13}) \left\{\zeta_1 \times \exp\left[i\frac{f_{3,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_3 \cdot k_1}\right] \times \exp\left[i\frac{f_{5,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_5 \cdot k_{13}}\right] \times \zeta_6\right\},$$



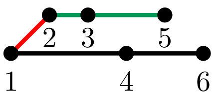
$$= -\frac{1}{2}(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot k_1)\,(\epsilon_5 \cdot k_{14}) \left\{\zeta_1 \times \exp\left[i\frac{f_{4,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_4 \cdot k_1}\right] \times \exp\left[i\frac{f_{5,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_5 \cdot k_{14}}\right] \times \zeta_6\right\},$$



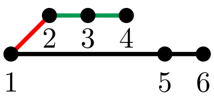
$$=\frac{1}{3}(k_3 \cdot k_{12})N(3^1,4^1,5^1)(\epsilon_2 \cdot k_1)\left\{\zeta_1 \times \exp\left[i\frac{f_{2,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_2 \cdot k_1}\right] \times \zeta_6\right\},$$



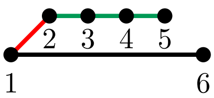
$$=\frac{1}{3}(k_1 \cdot k_2)N(2^1,4^1,5^1)(\epsilon_3 \cdot k_1)\left\{\zeta_1 \times \exp\left[i\frac{f_{3,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_3 \cdot k_1}\right] \times \zeta_6\right\},$$



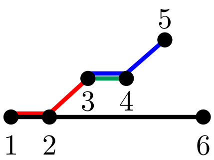
$$=\frac{1}{3}(k_1 \cdot k_2)N(2^1,3^1,5^1)(\epsilon_4 \cdot k_1)\left\{\zeta_1 \times \exp\left[i\frac{f_{4,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_4 \cdot k_1}\right] \times \zeta_6\right\},$$



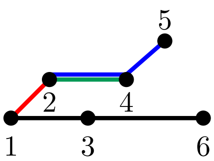
$$=\frac{1}{3}(k_1 \cdot k_2)N(2^1,3^1,4^1)(2\epsilon_5 \cdot k_1)\left\{\zeta_1 \times \exp\left[i\frac{f_{5,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_5 \cdot k_1}\right] \times \zeta_6\right\},$$



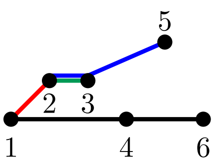
$$= -\frac{1}{4}(k_1 \cdot k_2)N(2^1,3^1,4^1,5^1)(\zeta_1 \times \zeta_6),$$



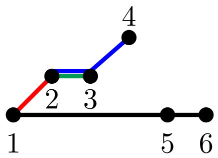
$$= -\frac{1}{3}(k_3 \cdot k_{12})(\epsilon_3 \cdot \epsilon_4)(\epsilon_5 \cdot k_{34})(\epsilon_2 \cdot k_1)\left\{\zeta_1 \times \exp\left[i\frac{f_{2,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_2 \cdot k_1}\right] \times \zeta_6\right\},$$



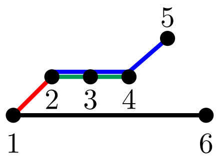
$$= -\frac{1}{3}(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_4)(\epsilon_5 \cdot k_{24})(\epsilon_3 \cdot k_1)\left\{\zeta_1 \times \exp\left[i\frac{f_{3,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_3 \cdot k_1}\right] \times \zeta_6\right\},$$



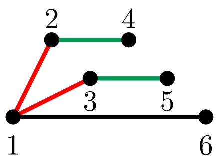
$$= -\frac{1}{3}(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_3)(\epsilon_5 \cdot k_{23})(\epsilon_4 \cdot k_1)\left\{\zeta_1 \times \exp\left[i\frac{f_{4,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_4 \cdot k_1}\right] \times \zeta_6\right\},$$



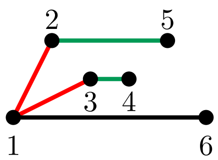
$$= -\frac{1}{3}(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot k_{23})(\epsilon_5 \cdot k_1)\left\{\zeta_1 \times \exp\left[i\frac{f_{5,\mu\nu}\Sigma_s^{\mu\nu}}{2\epsilon_5 \cdot k_1}\right] \times \zeta_6\right\},$$



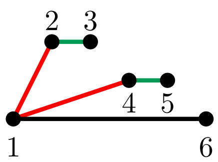
$$=\frac{1}{4}(k_1 \cdot k_2)N(2^1, 3^1, 4^1)(\epsilon_5 \cdot k_{234})(\zeta_1 \times \epsilon_6),$$



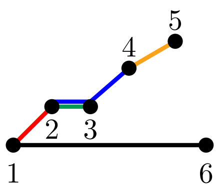
$$=\frac{1}{4}(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_4)(k_1 \cdot k_3)(\epsilon_3 \cdot \epsilon_5)(\zeta_1 \times \zeta_6),$$



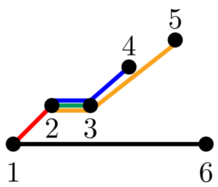
$$=\frac{1}{4}(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_5)(k_1 \cdot k_3)(\epsilon_3 \cdot \epsilon_4)(\zeta_1 \times \zeta_6),$$



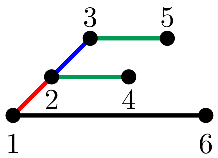
$$=\frac{1}{4}(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_3)(k_1 \cdot k_4)(\epsilon_4 \cdot \epsilon_5)(\zeta_1 \times \zeta_6),$$



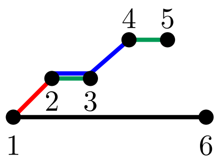
$$= -\frac{1}{4}(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot k_{23})(\epsilon_5 \cdot k_4)(\zeta_1 \times \zeta_6),$$



$$= -\frac{1}{4}(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_3)(\epsilon_4 \cdot k_{23})(\epsilon_5 \cdot k_{23})(\zeta_1 \times \zeta_6),$$



$$=\frac{1}{8}(k_1 \cdot k_2)(\epsilon_3 \cdot \epsilon_5)(\epsilon_2 \cdot \epsilon_4)(k_3 \cdot k_2)(\zeta_1 \times \zeta_6),$$

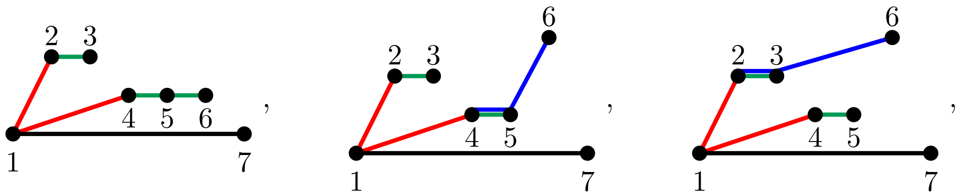


$$=\frac{1}{8}(k_1 \cdot k_2)(\epsilon_4 \cdot \epsilon_5)(\epsilon_2 \cdot \epsilon_3)(k_4 \cdot k_{23})(\zeta_1 \times \zeta_6).$$

We want to again emphasize the efficiency of this approach with the 24 diagrams being less than the 120 needed using the method in [31] and being much less than the $120 \times 24 = 2880$ diagrams one would have to draw using the algorithm from section 6.

8.6 Analysis and further checks

The diagrams used so far have all been colored to make each contribution to the numerators clear. We want to emphasize however, that colors are not needed since each diagram represents a unique expression. To provide further checks of the diagrammatic rules we have sampled multiple expressions and diagrams at 7 points and the algorithm produces correct results each time. Examples of 7 point diagrams that we have checked are for instance



(8.6.1)

Note that just performing a small permutation, in this case moving the 6'th point, gives vastly different contributions. The above diagrams have the following expressions (in the order drawn)

$$\begin{aligned}
1: &\quad -\frac{1}{6}(k_1 \cdot k_2)(k_1 \cdot k_4)N(4^1, 5^1, 6^1)(\epsilon_2 \cdot \epsilon_3)(\zeta_1 \times \zeta_7), \\
2: &\quad \frac{1}{6}(k_1 \cdot k_2)(k_1 \cdot k_4)(\epsilon_4 \cdot \epsilon_5)(\epsilon_2 \cdot \epsilon_3)(\epsilon_6 \cdot k_{45})(\zeta_1 \times \zeta_7), \\
3: &\quad \frac{1}{6}(k_1 \cdot k_2)(k_1 \cdot k_4)(\epsilon_4 \cdot \epsilon_5)(\epsilon_2 \cdot \epsilon_3)(\epsilon_6 \cdot k_{23})(\zeta_1 \times \zeta_7),
\end{aligned} \tag{8.6.2}$$

It would be interesting to examine exactly what type of gravitational amplitude one gets from performing double copy computations using the numerators presented in this chapter. The general spin s three point amplitude for instance can be calculated using an expansion similar to equation (6.2.4) [1]

$$\begin{aligned}
\mathcal{M}(1^s, 2^2, 3^s) &= \int \mathrm{d}\Omega_{\mathrm{CHY}} N(1^s, 2^1, 3^s)\mathrm{PT}(1,2,3)\frac{1}{z_{13}^2}\mathrm{Pf}\,\Psi_{1,3,4,6} \\
&= -(\epsilon_2 \cdot k_1)^2 \left\{ \zeta_1 \times \exp\left[i\frac{f_{2,\mu\nu}\Sigma_s^{\mu\nu}}{(\epsilon_2 \cdot k_1)} \right] \times \zeta_3 \right\}.
\end{aligned} \tag{8.6.3}$$

[1]Notice this notation for the reduced matrix Ψ which was established in section 4.6. One can think of the integral here as the double copy computation $M(1^s, 2^2, 3^s) \cong A(1^s, 2^1, 3^s) \times A(1^0, 2^1, 3^0)$.

This exactly matches the covariant result from [5,35] which is calculated in the minimal coupling framework. Exploring the connection to these results would be an interesting topic of future research. Furthermore, so far we have only provided a prescription for obtaining the exponential numerators and checked that they provide the correct amplitudes, so another point to explore would be to explain on a phenomenological level why the exponential terms arise (and why some of the terms don't include exponentials).

Chapter 9

Conclusion and discussion

In this thesis we have provided a novel algorithm that produces BCJ DDM-basis master numerators in an exponential form using the CHY formalism. The algorithm builds on the great work by [32, 33] and improves the efficiency of the current state of the art, see Table 9.1. We have also shown that the numerators obtained work for the external particles having $s = 1, 1/2, 1$ and also with $k_1^2 = k_n^2 = m^2$.

Source	[32, 33]	[31]	This thesis
Efficiency	$\mathcal{O}[(n-1)! \times (n-2)!]$	$\mathcal{O}[(n-1)!]$	$\mathcal{O}[(n-2)!]$
Fermions	No	$10d$ Majorana-Weyl spinors	d-dimensional Dirac spinors
Masses	No	By dimensional reduction	Yes

Table 9.1: Comparing the numerators from this thesis to the literature

The fermion baseline factor used in this thesis was presented in [30, 31] and the calculations made there used 10 dimensional Majorana-Weyl spinors. In this thesis we have shown that up to at least 5 points the generalization to Dirac spinors is straightforward[1]. Analytical checks were performed up to 4 points by comparing with QCD amplitudes and the remaining checks were numerical using the massive spinor helicity formalism developed in [42].

Both the inclusion of Dirac spinors and the extension to massive particles were made without using a dimensional reduction procedure since the reduction of the CHY Pfaffian makes sure the numerators hold for massive external particles as well. To our knowledge this thesis is the first time amplitudes containing (massive) Dirac spinors have been calculated explicitly using CHY. Furthermore we have not been able to find any other works describing exponential versions of BCJ-numerators.

So far we have checked our algorithm up to 6 points, since going above this would require implementing it in a computer program which we have not had time to do yet.

[1] 6 points has also been checked numerically in 4 dimensions by a collaborator.

We note that using the exponential numerators and the double cover formalism, the massive factorization procedure from [28] has been generalized to fermions [58].

In the works [9,42] expressions for general spin amplitudes were bootstrapped using massive spinor helicity. These expressions also work for non-fundamental spin, i.e. $s > 2$. Using CHY the amplitudes for fundamental spin can be replicated, but the expressions do not generalize to $s > 2$ and so it seems like more modifications are needed before CHY can be used in these applications. The fact that one can obtain amplitudes that work in general for $s = 0, \frac{1}{2}, 1$ using the BCJ numerators from this thesis is however very promising.

It would be interesting to examine exactly what type of gravitational amplitude one gets from performing double copy computations using these numerators. The fact that the general spin s three point graviton amplitude exactly matches the work by [5, 35] is intriguing. These general spin amplitudes are of particular interest to many in the amplitudes community at the time of writing because of the observation of gravitational waves at LIGO and the need for predictions of spinning black hole scattering. We hope that the results obtained here will be useful in these applications since the CHY formalism provides an angle of attack which has not been explored much.

Appendix A

Identity used in derivation of CHY integration rules

We want to show that $\sum_{i\in\tau}\tilde{S}_i x_i = \sum_{\substack{i,j\in\tau\\ j\neq i}} s_{ij}$. First we can generally write:

$$\begin{aligned}\sum_{i\in\tau}\tilde{S}_i x_i &= \sum_{\substack{i,j\in\tau\\ j\neq i}}\frac{s_{ij}}{x_{ij}}x_i\\ &= \sum_{\substack{i,j\in\tau\\ j\neq i}}\frac{s_{ij}}{x_{ij}}\left(\frac{x_i+x_i}{2}\right)\\ &= \sum_{\substack{i,j\in\tau\\ j>i}} s_{ij}\\ &\equiv s_\tau\end{aligned}$$

The second to last equality can be seen inductively by assuming $\tau=\{1,2,3\}$:

$$\sum_{\substack{i,j\in\tau\\ j\neq i}}\frac{s_{ij}}{x_{ij}}\left(\frac{x_i+x_i}{2}\right)=\left\{\frac{s_{12}}{x_{12}}\left(\frac{x_1+x_1}{2}\right)+\frac{s_{13}}{x_{13}}\left(\frac{x_1+x_1}{2}\right)+\frac{s_{21}}{x_{21}}\left(\frac{x_2+x_2}{2}\right)+\right.$$
$$\left.\frac{s_{23}}{x_{23}}\left(\frac{x_2+x_2}{2}\right)+\frac{s_{31}}{x_{31}}\left(\frac{x_3+x_3}{2}\right)+\frac{s_{32}}{x_{32}}\left(\frac{x_3+x_3}{2}\right)\right\}$$

we then use $x_{ij}=-x_{ji}$ and $s_{ij}=s_{ji}$.

$$\sum_{\substack{i,j\in\tau\\ j\neq i}}\frac{s_{ij}}{x_{ij}}\left(\frac{x_i+x_i}{2}\right)=\left\{\frac{s_{12}}{x_{12}}\left(\frac{x_1+x_1}{2}\right)+\frac{s_{13}}{x_{13}}\left(\frac{x_1+x_1}{2}\right)-\frac{s_{12}}{x_{12}}\left(\frac{x_2+x_2}{2}\right)+\right.$$
$$\left.\frac{s_{23}}{x_{23}}\left(\frac{x_2+x_2}{2}\right)-\frac{s_{13}}{x_{13}}\left(\frac{x_3+x_3}{2}\right)-\frac{s_{23}}{x_{23}}\left(\frac{x_3+x_3}{2}\right)\right\}$$

and combine terms with the same factors of $\frac{s_{ij}}{x_{ij}}$:

$$
\begin{aligned}
\sum_{\substack{i,j\in\tau \\ j\neq i}} \frac{s_{ij}}{x_{ij}}\left(\frac{x_i + x_i}{2}\right) = &\left\{\frac{s_{12}}{x_{12}}\left(\frac{x_1 - x_2}{2}\right) + \frac{s_{12}}{x_{12}}\left(\frac{x_1 - x_2}{2}\right) + \frac{s_{13}}{x_{13}}\left(\frac{x_1 - x_3}{2}\right) + \right. \\
&\left. \frac{s_{13}}{x_{13}}\left(\frac{x_1 - x_3}{2}\right) - \frac{s_{23}}{x_{23}}\left(\frac{x_2 - x_3}{2}\right) - \frac{s_{23}}{x_{23}}\left(\frac{x_2 - x_3}{2}\right)\right\} \\
&= s_{12} + s_{13} + s_{23} \\
&= \sum_{\substack{i,j\in\tau \\ j>i}} s_{ij}
\end{aligned}
$$

We see that every terms shows up twice and, since $x_{ij} \equiv (x_i - x_j)$, all the factors except the s_{ij} cancel

Appendix B

Six-point exponential check

In the following we show how the analytical checks were performed for $n = 6$ and $s = 1$. The expressions are very large (both in width and in length) so to accommodate them on the page we have used a smaller math font. The final expression was checked in Mathematica as well. To make the notation easier let it be implicit that the CO is $\{2, 3, 4, 5\}$ and we will just the denote the RO from now on. The starting point is the numerator

$$N_{\{2,3,4,5\}} = (\epsilon_2 \cdot k_1)(\epsilon_3 \cdot k_{12})(\epsilon_4 \cdot k_{123})(\epsilon_5 \cdot k_{1234}) \left\{ \epsilon_1 \cdot \exp\left[-\frac{f_2}{\epsilon_2 \cdot k_1} \right] \cdot \exp\left[-\frac{f_3}{\epsilon_3 \cdot k_{12}} \right] \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_{123}} \right] \cdot \exp\left[-\frac{f_5}{\epsilon_5 \cdot k_{1234}} \right] \cdot \epsilon_6 \right\}$$

Just like for the 4 and 5 point numerators we can then find corrections from changing the reference order. This has to be done 24 times

$$\begin{aligned} N_{\{2,3,5,4\}} - N_{\{2,3,4,5\}} &= (k_4 \cdot k_{123})\,(\epsilon_4 \cdot \epsilon_5)\,((\epsilon_3 \cdot k_{12})\,((\epsilon_1 \cdot f_2 \cdot \epsilon_6) - (\epsilon_1 \cdot \epsilon_6)(\epsilon_2 \cdot k_1)) + (\epsilon_1 \cdot f_3 \cdot \epsilon_6)(\epsilon_2 \cdot k_1) - \epsilon_1 \cdot f_2 \cdot f_3 \cdot \epsilon_6) \\ &= -\,(k_4 \cdot k_{123})\,(\epsilon_4 \cdot \epsilon_5)(\epsilon_2 \cdot k_1)\,(\epsilon_3 \cdot k_{12}) \left\{ \epsilon_1 \cdot \exp\left[-\frac{f_2}{\epsilon_2 \cdot k_1} \right] \cdot \exp\left[-\frac{f_3}{\epsilon_3 \cdot k_{12}} \right] \cdot \epsilon_6 \right\} \\ N_{\{2,4,3,5\}} - N_{\{2,3,4,5\}} &= (k_3 \cdot k_{12})\,(\epsilon_3 \cdot \epsilon_4)\,((\epsilon_5 \cdot k_{1234})\,(\epsilon_1 \cdot f_2 \cdot \epsilon_6 - (\epsilon_1 \cdot \epsilon_6)(\epsilon_2 \cdot k_1)) + (\epsilon_1 \cdot f_5 \cdot \epsilon_6)(\epsilon_2 \cdot k_1) - \epsilon_1 \cdot f_2 \cdot f_5 \cdot \epsilon_6) \\ &= -\,(k_3 \cdot k_{12})\,(\epsilon_3 \cdot \epsilon_4)(\epsilon_2 \cdot k_1)\,(\epsilon_5 \cdot k_{1234}) \left\{ \epsilon_1 \cdot \exp\left[-\frac{f_2}{\epsilon_2 \cdot k_1} \right] \cdot \exp\left[-\frac{f_5}{\epsilon_5 \cdot k_{1234}} \right] \cdot \epsilon_6 \right\} \\ N_{\{3,2,4,5\}} - N_{\{2,3,4,5\}} &= (k_2 \cdot k_1)\,(\epsilon_2 \cdot \epsilon_3)\,((\epsilon_5 \cdot k_{1234})\,(\epsilon_1 \cdot f_4 \cdot \epsilon_6 - (\epsilon_1 \cdot \epsilon_6)\,(\epsilon_4 \cdot k_{123})) + \epsilon_1 \cdot f_5 \cdot \epsilon_6\,(\epsilon_4 \cdot k_{123}) - \epsilon_1 \cdot f_4 \cdot f_5 \cdot \epsilon_6) \\ &= -\,(k_2 \cdot k_1)\,(\epsilon_2 \cdot \epsilon_3)\,(\epsilon_4 \cdot k_{123})\,(\epsilon_5 \cdot k_{1234}) \left\{ \epsilon_1 \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_{123}} \right] \cdot \exp\left[-\frac{f_5}{\epsilon_5 \cdot k_{1234}} \right] \cdot \epsilon_6 \right\} \\ N_{\{2,4,5,3\}} - N_{\{2,4,3,5\}} &= (k_3 \cdot k_{12})\,(\epsilon_3 \cdot \epsilon_5)\,((\epsilon_4 \cdot k_{12})\,(\epsilon_1 \cdot f_2 \cdot \epsilon_6 - (\epsilon_1 \cdot \epsilon_6)(\epsilon_2 \cdot k_1)) + (\epsilon_1 \cdot f_4 \cdot \epsilon_6)(\epsilon_2 \cdot k_1) - \epsilon_1 \cdot f_2 \cdot f_4 \cdot \epsilon_6) \\ &= -\,(k_3 \cdot k_{12})\,(\epsilon_3 \cdot \epsilon_5)\,(\epsilon_4 \cdot k_{12})\,(\epsilon_2 \cdot k_1) \left\{ \epsilon_1 \cdot \exp\left[-\frac{f_2}{\epsilon_2 \cdot k_1} \right] \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_{12}} \right] \cdot \epsilon_6 \right\} \\ N_{\{2,5,3,4\}} - N_{\{2,3,5,4\}} &= (k_3 \cdot k_{12})\,(((\epsilon_3 \cdot \epsilon_5)\,(\epsilon_4 \cdot k_{123}) - (\epsilon_4 \cdot \epsilon_5)(\epsilon_3 \cdot k_4) + (\epsilon_3 \cdot \epsilon_4)(\epsilon_5 \cdot k_4))\,(\epsilon_1 \cdot f_2 \cdot \epsilon_6 - (\epsilon_1 \cdot \epsilon_6)(\epsilon_2 \cdot k_1)) \\ &\quad +(\epsilon_3 \cdot \epsilon_5)(\epsilon_1 \cdot f_4 \cdot \epsilon_6)(\epsilon_2 \cdot k_1) - (\epsilon_3 \cdot \epsilon_5)(\epsilon_1 \cdot f_2 \cdot f_4 \cdot \epsilon_6)) \\ &= (k_3 \cdot k_{12})\,(\epsilon_3 \cdot \epsilon_5)\,\{(\epsilon_4 \cdot k_{123})\,[\epsilon_1 \cdot f_2 \cdot \epsilon_6 - (\epsilon_1 \cdot \epsilon_6)(\epsilon_2 \cdot k_1)] + (\epsilon_1 \cdot f_4 \cdot \epsilon_6)(\epsilon_2 \cdot k_1) - \epsilon_1 \cdot f_2 \cdot f_4 \cdot \epsilon_6\} \\ &\quad + (k_3 \cdot k_{12})\,\{\epsilon_1 \cdot f_2 \cdot \epsilon_6 - (\epsilon_1 \cdot \epsilon_6)(\epsilon_2 \cdot k_1)\}\,[(\epsilon_3 \cdot \epsilon_4)(\epsilon_5 \cdot k_4) - (\epsilon_4 \cdot \epsilon_5)(\epsilon_3 \cdot k_4)] \\ &= -\,(k_3 \cdot k_{12})\,(\epsilon_3 \cdot \epsilon_5)(\epsilon_2 \cdot k_1)(\epsilon_4 \cdot k_{123}) \left\{ \epsilon_1 \cdot \exp\left[-\frac{f_2}{\epsilon_2 \cdot k_1} \right] \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_{123}} \right] \cdot \epsilon_6 \right\} \\ &\quad -\,(k_3 \cdot k_{12})(\epsilon_2 \cdot k_1) \left\{ \epsilon_1 \cdot \exp\left[-\frac{f_2}{\epsilon_2 \cdot k_1} \right] \cdot \epsilon_6 \right\} [(\epsilon_3 \cdot \epsilon_4)(\epsilon_5 \cdot k_4) - (\epsilon_4 \cdot \epsilon_5)(\epsilon_3 \cdot k_4)] \end{aligned}$$

$$\begin{aligned} N_{\{2,5,4,3\}} - N_{\{2,5,4,3\}} &= (k_3 \cdot k_{12})\,(\epsilon_3 \cdot \epsilon_4)\,((\epsilon_5 \cdot k_{12})\,(\epsilon_1 \cdot f_2 \cdot \epsilon_6 - (\epsilon_1 \cdot \epsilon_6)(\epsilon_2 \cdot k_1)) + (\epsilon_1 \cdot f_5 \cdot \epsilon_6)(\epsilon_2 \cdot k_1) - \epsilon_1 \cdot f_2 \cdot f_5 \cdot \epsilon_6) \\ &= -\,(k_3 \cdot k_{12})\,(\epsilon_3 \cdot \epsilon_4)\,(\epsilon_5 \cdot k_{12})\,(\epsilon_2 \cdot k_1) \left\{ \epsilon_1 \cdot \exp\left[-\frac{f_2}{\epsilon_2 \cdot k_1} \right] \cdot \exp\left[-\frac{f_5}{\epsilon_5 \cdot k_{12}} \right] \cdot \epsilon_6 \right\} \end{aligned}$$

$$\begin{aligned}N_{\{3,2,5,4\}}-N_{\{3,2,4,5\}} =&(k_4\cdot k_{123})\,(\epsilon_4\cdot\epsilon_5)\,((\epsilon_3\cdot k_{12})\,(\epsilon_1\cdot f_2\cdot\epsilon_6-(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot k_1))+(\epsilon_1\cdot f_3\cdot\epsilon_6)(\epsilon_2\cdot k_1)-\epsilon_1\cdot f_2\cdot f_3\cdot\epsilon_6)\\&+(k_4\cdot k_{123})\,(\epsilon_4\cdot\epsilon_5)\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_3)\\=&-(k_4\cdot k_{123})\,(\epsilon_4\cdot\epsilon_5)\,(\epsilon_2\cdot k_1)\,(\epsilon_3\cdot k_{12})\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\epsilon_6\right\}\\&+(k_4\cdot k_{123})\,(\epsilon_4\cdot\epsilon_5)\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_3)\end{aligned}$$

$$\begin{aligned}N_{\{3,4,2,5\}}-N_{\{3,2,4,5\}} =&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)\,((\epsilon_5\cdot k_{1234})\,(\epsilon_1\cdot f_3\cdot\epsilon_6-(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot k_1))+(\epsilon_1\cdot f_5\cdot\epsilon_6)(\epsilon_3\cdot k_1)-\epsilon_1\cdot f_3\cdot f_5\cdot\epsilon_6)\\=&-(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)\,(\epsilon_5\cdot k_{1234})\,(\epsilon_3\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\right]\cdot\epsilon_6\right\}\end{aligned}$$

$$\begin{aligned}N_{\{3,4,5,2\}}-N_{\{3,4,2,5\}} =&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)\,((\epsilon_4\cdot k_{13})\,(\epsilon_1\cdot f_3\cdot\epsilon_6-(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot k_1))+(\epsilon_1\cdot f_4\cdot\epsilon_6)(\epsilon_3\cdot k_1)-\epsilon_1\cdot f_3\cdot f_4\cdot\epsilon_6)\\=&-(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)\,(\epsilon_4\cdot k_{13})\,(\epsilon_3\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{13}}\right]\cdot\epsilon_6\right\}\end{aligned}$$

$$\begin{aligned}N_{\{3,5,2,4\}}-N_{\{3,2,5,4\}} =&k_1\cdot k_2\,(\epsilon_1\cdot f_3\cdot\epsilon_6\,(\epsilon_2\cdot\epsilon_5\,(\epsilon_4\cdot k_{123})-\epsilon_4\cdot\epsilon_5\epsilon_2\cdot k_4+\epsilon_2\cdot\epsilon_4\epsilon_5\cdot k_4)-\epsilon_2\cdot\epsilon_5\epsilon_1\cdot f_3\cdot f_4\cdot\epsilon_6\\&+\epsilon_3\cdot k_1\,(\epsilon_2\cdot\epsilon_5\epsilon_1\cdot f_4\cdot\epsilon_6+\epsilon_1\cdot\epsilon_6\,(-(\epsilon_2\cdot\epsilon_5)\,(\epsilon_4\cdot k_{123})+\epsilon_4\cdot\epsilon_5\epsilon_2\cdot k_4-\epsilon_2\cdot\epsilon_4\epsilon_5\cdot k_4)))\\=&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)\,\{(\epsilon_4\cdot k_{123})\,[\epsilon_1\cdot f_3\cdot\epsilon_6-(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot k_1))]+(\epsilon_1\cdot f_4\cdot\epsilon_6)(\epsilon_3\cdot k_1)-\epsilon_1\cdot f_3\cdot f_4\cdot\epsilon_6\}\\&+(k_1\cdot k_2)\,\{(\epsilon_1\cdot f_3\cdot\epsilon_6)\,[(\epsilon_2\cdot\epsilon_4)(\epsilon_5\cdot k_4)-(\epsilon_4\cdot\epsilon_5)(\epsilon_2\cdot k_4)]\}\\&-(k_1\cdot k_2)\,\{(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot k_1)\,[(\epsilon_2\cdot\epsilon_4)(\epsilon_5\cdot k_4)-(\epsilon_4\cdot\epsilon_5)(\epsilon_2\cdot k_4)]\}\\=&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)\,\{(\epsilon_4\cdot k_{123})\,[\epsilon_1\cdot f_3\cdot\epsilon_6-(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot k_1))]+(\epsilon_1\cdot f_4\cdot\epsilon_6)(\epsilon_3\cdot k_1)-\epsilon_1\cdot f_3\cdot f_4\cdot\epsilon_6\}\\&+(k_1\cdot k_2)\,\{[(\epsilon_1\cdot f_3\cdot\epsilon_6)-(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot k_1)]\,[(\epsilon_2\cdot\epsilon_4)(\epsilon_5\cdot k_4)-(\epsilon_4\cdot\epsilon_5)(\epsilon_2\cdot k_4)]\}\\=&-(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)(\epsilon_4\cdot k_{123})(\epsilon_3\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{123}}\right]\cdot\epsilon_6\right\}\\&-(k_1\cdot k_2)(\epsilon_3\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\epsilon_6\right\}[(\epsilon_2\cdot\epsilon_4)(\epsilon_5\cdot k_4)-(\epsilon_4\cdot\epsilon_5)(\epsilon_2\cdot k_4)]\end{aligned}$$

$$\begin{aligned}N_{\{3,5,4,2\}}-N_{\{3,5,2,4\}} =&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)\,((\epsilon_5\cdot k_{13})\,(\epsilon_1\cdot f_3\cdot\epsilon_6-(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot k_1))+(\epsilon_1\cdot f_5\cdot\epsilon_6)(\epsilon_3\cdot k_1)-\epsilon_1\cdot f_3\cdot f_5\cdot\epsilon_6)\\=&-(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)\,(\epsilon_5\cdot k_{13})\,(\epsilon_3\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{13}}\right]\cdot\epsilon_6\right\}\end{aligned}$$

$$\begin{aligned}N_{\{4,2,3,5\}}-N_{\{2,4,3,5\}} =&(k_1\cdot k_2)\,((\epsilon_5\cdot k_{1234})\,((\epsilon_2\cdot\epsilon_4)(\epsilon_1\cdot f_3\cdot\epsilon_6)+(\epsilon_1\cdot\epsilon_6)\,((\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)-(\epsilon_2\cdot\epsilon_4)\,(\epsilon_3\cdot k_{12})-(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)))\\&-\epsilon_1\cdot f_5\cdot\epsilon_6\,(-(\epsilon_2\cdot\epsilon_4)\,(\epsilon_3\cdot k_{12})+(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)-(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3))-(\epsilon_2\cdot\epsilon_4)(\epsilon_1\cdot f_3\cdot f_5\cdot\epsilon_6))\\=&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)\,\{(\epsilon_5\cdot k_{1234})\,[(\epsilon_1\cdot f_3\cdot\epsilon_6)-(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot k_{12})]+(\epsilon_1\cdot f_4\cdot\epsilon_6)(\epsilon_3\cdot k_{12})-\epsilon_1\cdot f_3\cdot f_5\cdot\epsilon_6\}\\&+(k_1\cdot k_2)\,\{(\epsilon_1\cdot f_5\cdot\epsilon_6)-(\epsilon_1\cdot\epsilon_6)(\epsilon_5\cdot k_{1234})\}\,[(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)]\\=&-(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)(\epsilon_5\cdot k_{1234})(\epsilon_3\cdot k_{12})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\right]\cdot\epsilon_6\right\}\\&-(k_1\cdot k_2)(\epsilon_5\cdot k_{1234})\left\{\epsilon_1\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\right]\cdot\epsilon_6\right\}[(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)]\end{aligned}$$

$$\begin{aligned}N_{\{4,2,5,3\}}-N_{\{4,2,3,5\}} =&(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_5)\,((\epsilon_4\cdot k_{12})\,(\epsilon_1\cdot f_2\cdot\epsilon_6-(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot k_1))+(\epsilon_1\cdot f_4\cdot\epsilon_6)(\epsilon_2\cdot k_1)-\epsilon_1\cdot f_2\cdot f_4\cdot\epsilon_6\\&+(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_4))\\=&-(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_1)(\epsilon_4\cdot k_{12})\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{12}}\right]\cdot\epsilon_6\right\}\\&+(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_5)\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_4)\end{aligned}$$

$$\begin{aligned}N_{\{4,3,2,5\}}-N_{\{4,2,3,5\}} =&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)\,((\epsilon_5\cdot k_{1234})\,(\epsilon_1\cdot f_4\cdot\epsilon_6-(\epsilon_1\cdot\epsilon_6)(\epsilon_4\cdot k_1))+(\epsilon_1\cdot f_5\cdot\epsilon_6)(\epsilon_4\cdot k_1)-\epsilon_1\cdot f_4\cdot f_5\cdot\epsilon_6)\\=&-(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_1)\,(\epsilon_5\cdot k_{1234})\left\{\epsilon_1\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\right]\cdot\epsilon_6\right\}\end{aligned}$$

$$
\begin{aligned}
N_{\{4,3,5,2\}} - N_{\{4,3,2,5\}} =&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)\left((\epsilon_4\cdot k_{13})\left(\epsilon_1\cdot f_3\cdot\epsilon_6-(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot k_1)\right)+(\epsilon_1\cdot f_4\cdot\epsilon_6)(\epsilon_3\cdot k_1)-\epsilon_1\cdot f_3\cdot f_4\cdot\epsilon_6\right.\\
&\left.+\,(k_1\cdot k_3)(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot\epsilon_4)\right)\\
=&-(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot k_1)\,(\epsilon_4\cdot k_{13})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{13}}\right]\cdot\epsilon_6\right\}\\
&+(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)(k_1\cdot k_3)(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot\epsilon_4)
\end{aligned}
$$

$$
\begin{aligned}
N_{\{4,5,2,3\}} - N_{\{4,2,5,3\}} =&(k_1\cdot k_2)\left((\epsilon_2\cdot\epsilon_5)(\epsilon_1\cdot f_3\cdot\epsilon_6)(\epsilon_4\cdot k_1)+(\epsilon_2\cdot\epsilon_5)(\epsilon_1\cdot f_3\cdot\epsilon_6)(\epsilon_4\cdot k_3)\right.\\
&+(\epsilon_1\cdot f_4\cdot\epsilon_6)\left(\epsilon_2\cdot\epsilon_5\,(\epsilon_3\cdot k_{12})-(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_3)+(\epsilon_2\cdot\epsilon_3)(\epsilon_5\cdot k_3)\right)\\
&-(\epsilon_2\cdot\epsilon_5)(\epsilon_1\cdot f_3\cdot f_4\cdot\epsilon_6)+(k_1\cdot k_3)(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)-(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot k_1)(\epsilon_4\cdot k_1)\\
&-(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot k_2)(\epsilon_4\cdot k_1)+(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_3)(\epsilon_4\cdot k_1)-(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot k_1)(\epsilon_4\cdot k_3)\\
&\left.-\,(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_1)(\epsilon_5\cdot k_3)\right)\\
=&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)\left\{(\epsilon_4\cdot k_{13})(\epsilon_1\cdot f_3\cdot\epsilon_6)+(\epsilon_3\cdot k_{12})(\epsilon_1\cdot f_4\cdot\epsilon_6)-\epsilon_1\cdot f_3\cdot f_4\cdot\epsilon_6\right.\\
&\left.-(\epsilon_1\cdot\epsilon_6)\left[(\epsilon_3\cdot k_1)(\epsilon_4\cdot k_1)+(\epsilon_3\cdot k_2)(\epsilon_4\cdot k_3)+(\epsilon_3\cdot k_1)(\epsilon_4\cdot k_3)\right]\right\}\\
&+(k_1\cdot k_2)\left\{(\epsilon_1\cdot f_4\cdot\epsilon_6)-(\epsilon_1\cdot\epsilon_6)(\epsilon_4\cdot k_1)\right\}\left[(\epsilon_2\cdot\epsilon_3)(\epsilon_5\cdot k_3)-(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_3)\right]\\
&+(k_1\cdot k_3)(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)\\
=&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)\left\{(\epsilon_4\cdot k_{13})(\epsilon_1\cdot f_3\cdot\epsilon_6)+(\epsilon_3\cdot k_{12})(\epsilon_1\cdot f_4\cdot\epsilon_6)-\epsilon_1\cdot f_3\cdot f_4\cdot\epsilon_6\right.\\
&\left.-(\epsilon_1\cdot\epsilon_6)\left[(\epsilon_3\cdot k_{12})(\epsilon_4\cdot k_{13})-(\epsilon_3\cdot k_2)(\epsilon_4\cdot k_3)\right]\right\}\\
&+(k_1\cdot k_2)\left\{(\epsilon_1\cdot f_4\cdot\epsilon_6)-(\epsilon_1\cdot\epsilon_6)(\epsilon_4\cdot k_1)\right\}\left[(\epsilon_2\cdot\epsilon_3)(\epsilon_5\cdot k_3)-(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_3)\right]\\
&+(k_1\cdot k_2)(k_1\cdot k_3)(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)\\
=&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)\left\{(\epsilon_4\cdot k_{13})\left[(\epsilon_1\cdot f_3\cdot\epsilon_6)-(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot k_{12})\right]+(\epsilon_3\cdot k_{12})(\epsilon_1\cdot f_4\cdot\epsilon_6)-\epsilon_1\cdot f_3\cdot f_4\cdot\epsilon_6\right\}\\
&+(k_1\cdot k_2)\left\{(\epsilon_1\cdot f_4\cdot\epsilon_6)-(\epsilon_1\cdot\epsilon_6)(\epsilon_4\cdot k_1)\right\}\left[(\epsilon_2\cdot\epsilon_3)(\epsilon_5\cdot k_3)-(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_3)\right]\\
&+(k_1\cdot k_2)(k_1\cdot k_3)(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)+(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot k_2)(\epsilon_4\cdot k_3)\\
=&-(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot k_{12})(\epsilon_4\cdot k_{13})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{13}}\right]\cdot\epsilon_6\right\}\\
&-(k_1\cdot k_2)(\epsilon_4\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_1}\right]\cdot\epsilon_6\right\}\left[(\epsilon_2\cdot\epsilon_3)(\epsilon_5\cdot k_3)-(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_3)\right]\\
&+(k_1\cdot k_2)(k_1\cdot k_3)(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)+(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot k_2)(\epsilon_4\cdot k_3)
\end{aligned}
$$

$$
\begin{aligned}
N_{\{5,3,4,2\}} - N_{\{5,3,2,4\}} =&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)\left\{(\epsilon_5\cdot k_{13})\left(\epsilon_1\cdot f_3\cdot\epsilon_6-(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot k_1)\right)+(\epsilon_1\cdot f_5\cdot\epsilon_6)(\epsilon_3\cdot k_1)-\epsilon_1\cdot f_3\cdot f_5\cdot\epsilon_6\right\}\\
&+(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)(k_1\cdot k_3)(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot\epsilon_5)\\
=&-(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)(\epsilon_3\cdot k_1)(\epsilon_5\cdot k_{13})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{13}}\right]\cdot\epsilon_6\right\}\\
&+(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)(k_1\cdot k_3)(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot\epsilon_5)
\end{aligned}
$$

$$
\begin{aligned}
N_{\{4,5,3,2\}} - N_{\{4,5,2,3\}} =&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)\left((\epsilon_5\cdot k_{14})\left(\epsilon_1\cdot f_4\cdot\epsilon_6-(\epsilon_1\cdot\epsilon_6)(\epsilon_4\cdot k_1)\right)+(\epsilon_1\cdot f_5\cdot\epsilon_6)(\epsilon_4\cdot k_1)-\epsilon_1\cdot f_4\cdot f_5\cdot\epsilon_6\right)\\
=&-(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)\,(\epsilon_5\cdot k_{14})\,(\epsilon_4\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{14}}\right]\cdot\epsilon_6\right\}
\end{aligned}
$$

$$
\begin{aligned}
N_{\{5,3,2,4\}} - N_{\{5,2,3,4\}} =&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)\left\{(\epsilon_1\cdot f_4\cdot\epsilon_6)\,(\epsilon_5\cdot k_{14})+(\epsilon_1\cdot f_5\cdot\epsilon_6)\,(\epsilon_4\cdot k_{123})-\epsilon_1\cdot f_4\cdot f_5\cdot\epsilon_6\right.\\
&\left.+(\epsilon_1\cdot\epsilon_6)\left((k_1\cdot k_4)\,(\epsilon_4\cdot\epsilon_5)-(\epsilon_4\cdot k_{123})\,(\epsilon_5\cdot k_1)-(\epsilon_4\cdot k_1)(\epsilon_5\cdot k_4)\right)\right\}\\
=&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)\left\{(\epsilon_1\cdot f_4\cdot\epsilon_6)\,(\epsilon_5\cdot k_{14})+(\epsilon_1\cdot f_5\cdot\epsilon_6)\,(\epsilon_4\cdot k_{123})-\epsilon_1\cdot f_4\cdot f_5\cdot\epsilon_6\right.\\
&\left.+(\epsilon_1\cdot\epsilon_6)\left((k_1\cdot k_4)\,(\epsilon_4\cdot\epsilon_5)-(\epsilon_4\cdot k_{123})\,(\epsilon_5\cdot k_{14})+(\epsilon_4\cdot k_{23})(\epsilon_5\cdot k_4)\right)\right\}\\
=&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)\left\{(\epsilon_5\cdot k_{14})\left[(\epsilon_1\cdot f_4\cdot\epsilon_6)-(\epsilon_1\cdot\epsilon_6)(\epsilon_4\cdot k_{123})\right]+(\epsilon_1\cdot f_5\cdot\epsilon_6)\,(\epsilon_4\cdot k_{123})-\epsilon_1\cdot f_4\cdot f_5\cdot\epsilon_6\right\}\\
&+(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)\,(\epsilon_1\cdot\epsilon_6)\left((k_1\cdot k_4)\,(\epsilon_4\cdot\epsilon_5)+(\epsilon_4\cdot k_{23})(\epsilon_5\cdot k_4)\right)\\
=&-(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_{123})(\epsilon_5\cdot k_{14})\left\{\epsilon_1\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{123}}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{14}}\right]\cdot\epsilon_6\right\}\\
&+(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)\,(\epsilon_1\cdot\epsilon_6)\left((k_1\cdot k_4)\,(\epsilon_4\cdot\epsilon_5)+(\epsilon_4\cdot k_{23})(\epsilon_5\cdot k_4)\right)
\end{aligned}
$$

$$
\begin{aligned}
N_{\{5,2,3,4\}} - N_{\{2,5,3,4\}} =&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)\left\{(\epsilon_4\cdot k_{123})\,(\epsilon_1\cdot f_3\cdot\epsilon_6)+(\epsilon_3\cdot k_{12})\,(\epsilon_1\cdot f_4\cdot\epsilon_6)-\epsilon_1\cdot f_3\cdot f_4\cdot\epsilon_6\right\}\\
&+(k_1\cdot k_2)\left\{(\epsilon_2\cdot\epsilon_4)(\epsilon_1\cdot f_3\cdot\epsilon_6)(\epsilon_5\cdot k_4)-(\epsilon_4\cdot\epsilon_5)(\epsilon_1\cdot f_3\cdot\epsilon_6)\,(\epsilon_2\cdot k_4)-(\epsilon_3\cdot\epsilon_5)(\epsilon_1\cdot f_4\cdot\epsilon_6)(\epsilon_2\cdot k_3)\right.\\
&\left.+(\epsilon_2\cdot\epsilon_3)(\epsilon_1\cdot f_4\cdot\epsilon_6)(\epsilon_5\cdot k_3)\right\}\\
&-(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\left\{(\epsilon_4\cdot\epsilon_5)\left[(\epsilon_2\cdot k_3)(\epsilon_3\cdot k_4)-(k_3\cdot k_4)(\epsilon_2\cdot\epsilon_3)-(\epsilon_2\cdot k_4)\,(\epsilon_3\cdot k_{12})\right]\right.\\
&-(\epsilon_4\cdot k_{123})\left[(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_3)-(\epsilon_2\cdot\epsilon_5)\,(\epsilon_3\cdot k_{12})-(\epsilon_2\cdot\epsilon_3)(\epsilon_5\cdot k_3)\right]\\
&\left.+(\epsilon_5\cdot k_4)\left[(\epsilon_2\cdot\epsilon_4)\,(\epsilon_3\cdot k_{12})-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)+(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)\right]\right\}\\
=&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)\left\{(\epsilon_4\cdot k_{123})\left[(\epsilon_1\cdot f_3\cdot\epsilon_6)-(\epsilon_2\cdot\epsilon_5)\,(\epsilon_3\cdot k_{12})\right]+(\epsilon_3\cdot k_{12})\,(\epsilon_1\cdot f_4\cdot\epsilon_6)-\epsilon_1\cdot f_3\cdot f_4\cdot\epsilon_6\right\}\\
&+(k_1\cdot k_2)(\epsilon_1\cdot f_3\cdot\epsilon_6)\left[(\epsilon_2\cdot\epsilon_4)(\epsilon_5\cdot k_4)-(\epsilon_4\cdot\epsilon_5)\,(\epsilon_2\cdot k_4)\right]\\
&+(k_1\cdot k_2)(\epsilon_1\cdot f_4\cdot\epsilon_6)\left[(\epsilon_2\cdot\epsilon_3)(\epsilon_5\cdot k_3)-(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_3)\right]\\
&-(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\left\{(\epsilon_4\cdot\epsilon_5)\left[(\epsilon_2\cdot k_3)(\epsilon_3\cdot k_4)-(k_3\cdot k_4)(\epsilon_2\cdot\epsilon_3)-(\epsilon_2\cdot k_4)\,(\epsilon_3\cdot k_{12})\right]\right.\\
&-(\epsilon_4\cdot k_{123})\left[(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_3)-(\epsilon_2\cdot\epsilon_3)(\epsilon_5\cdot k_3)\right]\\
&\left.+(\epsilon_5\cdot k_4)\left[(\epsilon_2\cdot\epsilon_4)\,(\epsilon_3\cdot k_{12})-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)+(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)\right]\right\}\\
=&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)\left\{(\epsilon_4\cdot k_{123})\left[(\epsilon_1\cdot f_3\cdot\epsilon_6)-(\epsilon_2\cdot\epsilon_5)\,(\epsilon_3\cdot k_{12})\right]+(\epsilon_3\cdot k_{12})\,(\epsilon_1\cdot f_4\cdot\epsilon_6)-\epsilon_1\cdot f_3\cdot f_4\cdot\epsilon_6\right\}\\
&+(k_1\cdot k_2)\left\{(\epsilon_1\cdot f_3\cdot\epsilon_6)-(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot k_{12})\right\}\left[(\epsilon_2\cdot\epsilon_4)(\epsilon_5\cdot k_4)-(\epsilon_4\cdot\epsilon_5)\,(\epsilon_2\cdot k_4)\right]\\
&+(k_1\cdot k_2)\left\{(\epsilon_1\cdot f_4\cdot\epsilon_6)-(\epsilon_1\cdot\epsilon_6)(\epsilon_4\cdot k_{123})\right\}\left[(\epsilon_2\cdot\epsilon_3)(\epsilon_5\cdot k_3)-(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_3)\right]\\
&-(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)(\epsilon_4\cdot\epsilon_5)\left[(\epsilon_2\cdot k_3)(\epsilon_3\cdot k_4)-(k_3\cdot k_4)(\epsilon_2\cdot\epsilon_3)\right]\\
&-(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)(\epsilon_5\cdot k_4)\left[(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)\right]\\
=&-(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot k_{12})\,(\epsilon_4\cdot k_{123})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{123}}\right]\cdot\epsilon_6\right\}\\
&-(k_1\cdot k_2)(\epsilon_3\cdot k_{12})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\epsilon_6\right\}\left[(\epsilon_2\cdot\epsilon_4)(\epsilon_5\cdot k_4)-(\epsilon_4\cdot\epsilon_5)\,(\epsilon_2\cdot k_4)\right]\\
&-(k_1\cdot k_2)(\epsilon_4\cdot k_{123})\left\{\epsilon_1\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{123}}\right]\cdot\epsilon_6\right\}\left[(\epsilon_2\cdot\epsilon_3)(\epsilon_5\cdot k_3)-(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_3)\right]\\
&-(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)(\epsilon_4\cdot\epsilon_5)\left[(\epsilon_2\cdot k_3)(\epsilon_3\cdot k_4)-(k_3\cdot k_4)(\epsilon_2\cdot\epsilon_3)\right]\\
&-(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)(\epsilon_5\cdot k_4)\left[(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)\right]
\end{aligned}
$$

$$
\begin{aligned}
N_{\{5,2,4,3\}} - N_{\{5,2,3,4\}} =&(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_4)\left\{(\epsilon_5\cdot k_{12})\,(\epsilon_1\cdot f_2\cdot\epsilon_6-(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot k_1))+(\epsilon_1\cdot f_5\cdot\epsilon_6)(\epsilon_2\cdot k_1)-\epsilon_1\cdot f_2\cdot f_5\cdot\epsilon_6\right\}\\
&+(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_4)\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_5)\\
=&-(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_1)(\epsilon_5\cdot k_{12})\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{12}}\right]\cdot\epsilon_6\right\}\\
&+(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_4)\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_5)
\end{aligned}
$$

$$
\begin{aligned}
N_{\{5,4,2,3\}} - N_{\{5,2,4,3\}} =&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)\left\{(\epsilon_1\cdot f_3\cdot\epsilon_6)(\epsilon_5\cdot k_{13})+(\epsilon_1\cdot f_5\cdot\epsilon_6)\,(\epsilon_3\cdot k_{12})-\epsilon_1\cdot f_3\cdot f_5\cdot\epsilon_6\right\}\\
&+(k_1\cdot k_2)(\epsilon_1\cdot f_5\cdot\epsilon_6)\left[(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)\right]\\
&+(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_5\cdot k_1)\left[(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)-(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)\right]\\
&+(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_4)\left\{(k_1\cdot k_3)\,(\epsilon_3\cdot\epsilon_5)-(\epsilon_3\cdot k_{12})\,(\epsilon_5\cdot k_1)-(\epsilon_3\cdot k_1)(\epsilon_5\cdot k_3)\right\}\\
=&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)\left\{(\epsilon_1\cdot f_3\cdot\epsilon_6)(\epsilon_5\cdot k_{13})+(\epsilon_1\cdot f_5\cdot\epsilon_6)\,(\epsilon_3\cdot k_{12})-\epsilon_1\cdot f_3\cdot f_5\cdot\epsilon_6\right\}\\
&+(k_1\cdot k_2)\left\{(\epsilon_1\cdot f_5\cdot\epsilon_6)-(\epsilon_1\cdot\epsilon_6)(\epsilon_5\cdot k_1)\right\}\left[(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)\right]\\
&+(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_4)\left\{(k_1\cdot k_3)\,(\epsilon_3\cdot\epsilon_5)-(\epsilon_3\cdot k_{12})\,(\epsilon_5\cdot k_{13})+(\epsilon_3\cdot k_2)(\epsilon_5\cdot k_3)\right\}\\
=&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)\left\{(\epsilon_5\cdot k_{13})\left[(\epsilon_1\cdot f_3\cdot\epsilon_6)-(\epsilon_3\cdot k_{12})\,(\epsilon_1\cdot\epsilon_6)\right]+(\epsilon_1\cdot f_5\cdot\epsilon_6)\,(\epsilon_3\cdot k_{12})-\epsilon_1\cdot f_3\cdot f_5\cdot\epsilon_6\right\}\\
&+(k_1\cdot k_2)\left\{(\epsilon_1\cdot f_5\cdot\epsilon_6)-(\epsilon_1\cdot\epsilon_6)(\epsilon_5\cdot k_1)\right\}\left[(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)\right]\\
&+(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_4)\left\{(k_1\cdot k_3)\,(\epsilon_3\cdot\epsilon_5)+(\epsilon_3\cdot k_2)(\epsilon_5\cdot k_3)\right\}\\
=&-(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)(\epsilon_3\cdot k_{12})(\epsilon_5\cdot k_{13})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{13}}\right]\cdot\epsilon_6\right\}\\
&-(k_1\cdot k_2)(\epsilon_5\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_1}\right]\cdot\epsilon_6\right\}\left[(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)\right]\\
&+(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_4)\left\{(k_1\cdot k_3)\,(\epsilon_3\cdot\epsilon_5)+(\epsilon_3\cdot k_2)(\epsilon_5\cdot k_3)\right\}
\end{aligned}
$$

$$
\begin{aligned}
N_{\{5,4,3,2\}} - N_{\{5,4,2,3\}} =&(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)\left\{(\epsilon_5\cdot k_{14})\,(\epsilon_1\cdot f_4\cdot\epsilon_6-(\epsilon_1\cdot\epsilon_6)(\epsilon_4\cdot k_1))+(\epsilon_1\cdot f_5\cdot\epsilon_6)(\epsilon_4\cdot k_1)-\epsilon_1\cdot f_4\cdot f_5\cdot\epsilon_6\right\}\\
&+(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)(k_1\cdot k_4)(\epsilon_1\cdot\epsilon_6)(\epsilon_4\cdot\epsilon_5)\\
=&-(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_1)(\epsilon_5\cdot k_{14})\left\{\epsilon_1\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{14}}\right]\cdot\epsilon_6\right\}\\
&+(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)(k_1\cdot k_4)(\epsilon_1\cdot\epsilon_6)(\epsilon_4\cdot\epsilon_5)
\end{aligned}
$$

The reference order averaged numerator is then easily obtained, and after grouping similar terms we have

$$
\begin{aligned}
24N_6^{YM} =& 24(\epsilon_2\cdot k_1)(\epsilon_3\cdot k_{12})(\epsilon_4\cdot k_{123})(\epsilon_5\cdot k_{1234})\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{123}}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\right]\cdot\epsilon_6\right\}\\
&-12\,(k_4\cdot k_{123})\,(\epsilon_4\cdot\epsilon_5)(\epsilon_2\cdot k_1)\,(\epsilon_3\cdot k_{12})\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\epsilon_6\right\}\\
&-8\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_1)(\epsilon_4\cdot k_{123})\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{123}}\right]\cdot\epsilon_6\right\}\\
&-4\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_1)(\epsilon_4\cdot k_{12})\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{12}}\right]\cdot\epsilon_6\right\}\\
&-8\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_1)\,(\epsilon_5\cdot k_{1234})\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\right]\cdot\epsilon_6\right\}\\
&-4\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_1)(\epsilon_5\cdot k_{12})\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{12}}\right]\cdot\epsilon_6\right\}\\
&-6(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot k_{12})\,(\epsilon_4\cdot k_{123})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{123}}\right]\cdot\epsilon_6\right\}\\
&-2(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot k_1)(\epsilon_4\cdot k_{123})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{123}}\right]\cdot\epsilon_6\right\}\\
&-2(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot k_{12})(\epsilon_4\cdot k_{13})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{13}}\right]\cdot\epsilon_6\right\}\\
&-2(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot k_1)\,(\epsilon_4\cdot k_{13})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{13}}\right]\cdot\epsilon_6\right\}\\
&-6(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)(\epsilon_5\cdot k_{1234})(\epsilon_3\cdot k_{12})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\right]\cdot\epsilon_6\right\}\\
&-2(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)\,(\epsilon_5\cdot k_{1234})\,(\epsilon_3\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\right]\cdot\epsilon_6\right\}\\
&-2(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)(\epsilon_3\cdot k_{12})(\epsilon_5\cdot k_{13})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{13}}\right]\cdot\epsilon_6\right\}\\
&-2(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)\,(\epsilon_5\cdot k_{13})\,(\epsilon_3\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{13}}\right]\cdot\epsilon_6\right\}\\
&-6\,(k_2\cdot k_1)\,(\epsilon_2\cdot\epsilon_3)\,(\epsilon_4\cdot k_{123})\,(\epsilon_5\cdot k_{1234})\left\{\epsilon_1\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{123}}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\right]\cdot\epsilon_6\right\}\\
&-2(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_1)\,(\epsilon_5\cdot k_{1234})\left\{\epsilon_1\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\right]\cdot\epsilon_6\right\}\\
&-2(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_{123})(\epsilon_5\cdot k_{14})\left\{\epsilon_1\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{123}}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{14}}\right]\cdot\epsilon_6\right\}\\
&-2(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_1)\,(\epsilon_5\cdot k_{14})\left\{\epsilon_1\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{14}}\right]\cdot\epsilon_6\right\}\\
&-8(k_3\cdot k_{12})(\epsilon_2\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\epsilon_6\right\}[(\epsilon_3\cdot\epsilon_4)(\epsilon_5\cdot k_4)-(\epsilon_4\cdot\epsilon_5)(\epsilon_3\cdot k_4)]\\
&-6(k_1\cdot k_2)(\epsilon_3\cdot k_{12})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\epsilon_6\right\}[(\epsilon_2\cdot\epsilon_4)(\epsilon_5\cdot k_4)-(\epsilon_4\cdot\epsilon_5)\,(\epsilon_2\cdot k_4)]\\
&-2(k_1\cdot k_2)(\epsilon_3\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\epsilon_6\right\}[(\epsilon_2\cdot\epsilon_4)(\epsilon_5\cdot k_4)-(\epsilon_4\cdot\epsilon_5)(\epsilon_2\cdot k_4)]\\
&-6(k_1\cdot k_2)(\epsilon_4\cdot k_{123})\left\{\epsilon_1\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{123}}\right]\cdot\epsilon_6\right\}[(\epsilon_2\cdot\epsilon_3)(\epsilon_5\cdot k_3)-(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_3)]\\
&-2(k_1\cdot k_2)(\epsilon_4\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_1}\right]\cdot\epsilon_6\right\}[(\epsilon_2\cdot\epsilon_3)(\epsilon_5\cdot k_3)-(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_3)]\\
&-6(k_1\cdot k_2)(\epsilon_5\cdot k_{1234})\left\{\epsilon_1\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\right]\cdot\epsilon_6\right\}[(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)]\\
&-2(k_1\cdot k_2)(\epsilon_5\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_1}\right]\cdot\epsilon_6\right\}[(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)]\\
&-6\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\,\{(\epsilon_4\cdot\epsilon_5)\,[(\epsilon_2\cdot k_3)(\epsilon_3\cdot k_4)-(k_3\cdot k_4)(\epsilon_2\cdot\epsilon_3)]+(\epsilon_5\cdot k_4)\,[(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)]\}\\
&+3\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\,[(k_4\cdot k_{123})\,(\epsilon_4\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_3)+(\epsilon_2\cdot\epsilon_4)\,(k_1\cdot k_3)\,(\epsilon_3\cdot\epsilon_5)]\\
&+3\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\,(k_3\cdot k_{12})\,[(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_4)+(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_5)]\\
&+2(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)\,[(k_1\cdot k_3)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)+(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot k_2)(\epsilon_4\cdot k_3)+(\epsilon_2\cdot\epsilon_4)(\epsilon_3\cdot k_2)(\epsilon_5\cdot k_3)]\\
&+2(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\,(\epsilon_2\cdot\epsilon_3)\,[(k_1\cdot k_4)\,(\epsilon_4\cdot\epsilon_5)+(\epsilon_4\cdot k_{23})(\epsilon_5\cdot k_4)]\\
&+(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)\,[(k_1\cdot k_3)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)+(\epsilon_2\cdot\epsilon_3)(k_1\cdot k_4)(\epsilon_4\cdot\epsilon_5)]
\end{aligned}
$$

We will simplify this using the following identities

$$\begin{aligned}
&(\epsilon_i \cdot k_1)\left(\epsilon_j \cdot k_{1,abc\cdots z}\right)\left\{\epsilon_1 \cdot \exp\left[-\frac{f_i}{\epsilon_i \cdot k_1}\right] \cdot \exp\left[-\frac{f_j}{\epsilon_j \cdot k_{1,abc\cdots z}}\right] \cdot \epsilon_6\right\} \\
=&(\epsilon_i \cdot k_1)(\epsilon_j \cdot k_{1a})\left\{\epsilon_1 \cdot \exp\left[-\frac{f_i}{\epsilon_i \cdot k_1}\right] \cdot \exp\left[-\frac{f_j}{\epsilon_j \cdot k_{1a}}\right] \cdot \epsilon_6\right\} \\
&+(\epsilon_i \cdot k_1)(\epsilon_j \cdot k_{bc\cdots z})\left\{\epsilon_1 \cdot \exp\left[-\frac{f_i}{\epsilon_i \cdot k_1}\right] \cdot \epsilon_6\right\}
\end{aligned} \tag{B.0.1}$$

$$(\epsilon_i \cdot k_{1,abc\cdots z})\left\{\epsilon_1 \cdot \exp\left[-\frac{f_i}{\epsilon_i \cdot k_{1,abc\cdots z}}\right] \cdot \epsilon_6\right\} = (\epsilon_i \cdot k_1)\left\{\epsilon_1 \cdot \exp\left[-\frac{f_i}{\epsilon_i \cdot k_1}\right] \cdot \epsilon_6\right\} + (\epsilon_i \cdot k_{abc\cdots z})(\epsilon_1 \cdot \epsilon_6) \tag{B.0.2}$$

$$\begin{aligned}
&(\epsilon_i \cdot k_{1\gamma})\left(\epsilon_j \cdot k_{1,abc\cdots z}\right)\left\{\epsilon_1 \cdot \exp\left[-\frac{f_i}{\epsilon_i \cdot k_{1\gamma}}\right] \cdot \exp\left[-\frac{f_j}{\epsilon_j \cdot k_{1,abc\cdots z}}\right] \cdot \epsilon_6\right\} \\
=&(\epsilon_i \cdot k_1)(\epsilon_j \cdot k_{1a})\left\{\epsilon_1 \cdot \exp\left[-\frac{f_i}{\epsilon_i \cdot k_1}\right] \cdot \exp\left[-\frac{f_j}{\epsilon_j \cdot k_{1a}}\right] \cdot \epsilon_6\right\} \\
&+(\epsilon_i \cdot k_\gamma)(\epsilon_j \cdot k_1)\left\{\epsilon_1 \cdot \exp\left[-\frac{f_j}{\epsilon_j \cdot k_1}\right] \cdot \epsilon_6\right\} + (\epsilon_i \cdot k_1)(\epsilon_j \cdot k_{bc\cdots z})\left\{\epsilon_1 \cdot \exp\left[-\frac{f_i}{\epsilon_i \cdot k_1}\right] \cdot \epsilon_6\right\} \\
&+(\epsilon_1 \cdot \epsilon_6)(\epsilon_i \cdot k_\gamma)(\epsilon_j \cdot k_{abc\cdots z})
\end{aligned} \tag{B.0.3}$$

Using the identities we have the following simplifications

$$\begin{aligned}
&\Big[-6(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_5)(\epsilon_3 \cdot k_{12})(\epsilon_4 \cdot k_{123})\left\{\epsilon_1 \cdot \exp\left[-\frac{f_3}{\epsilon_3 \cdot k_{12}}\right] \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_{123}}\right] \cdot \epsilon_6\right\} \\
&-2(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_5)(\epsilon_3 \cdot k_1)(\epsilon_4 \cdot k_{123})\left\{\epsilon_1 \cdot \exp\left[-\frac{f_3}{\epsilon_3 \cdot k_1}\right] \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_{123}}\right] \cdot \epsilon_6\right\} \\
&-2(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_5)(\epsilon_3 \cdot k_{12})(\epsilon_4 \cdot k_{13})\left\{\epsilon_1 \cdot \exp\left[-\frac{f_3}{\epsilon_3 \cdot k_{12}}\right] \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_{13}}\right] \cdot \epsilon_6\right\} \\
&-2(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_5)(\epsilon_3 \cdot k_1)(\epsilon_4 \cdot k_{13})\left\{\epsilon_1 \cdot \exp\left[-\frac{f_3}{\epsilon_3 \cdot k_1}\right] \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_{13}}\right] \cdot \epsilon_6\right\} \\
&-6(k_1 \cdot k_2)(\epsilon_3 \cdot k_{12})\left\{\epsilon_1 \cdot \exp\left[-\frac{f_3}{\epsilon_3 \cdot k_{12}}\right] \cdot \epsilon_6\right\}[(\epsilon_2 \cdot \epsilon_4)(\epsilon_5 \cdot k_4) - (\epsilon_4 \cdot \epsilon_5)(\epsilon_2 \cdot k_4)] \\
&-2(k_1 \cdot k_2)(\epsilon_3 \cdot k_1)\left\{\epsilon_1 \cdot \exp\left[-\frac{f_3}{\epsilon_3 \cdot k_1}\right] \cdot \epsilon_6\right\}[(\epsilon_2 \cdot \epsilon_4)(\epsilon_5 \cdot k_4) - (\epsilon_4 \cdot \epsilon_5)(\epsilon_2 \cdot k_4)] \\
&-6(k_1 \cdot k_2)(\epsilon_4 \cdot k_{123})\left\{\epsilon_1 \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_{123}}\right] \cdot \epsilon_6\right\}[(\epsilon_2 \cdot \epsilon_3)(\epsilon_5 \cdot k_3) - (\epsilon_3 \cdot \epsilon_5)(\epsilon_2 \cdot k_3)] \\
&-2(k_1 \cdot k_2)(\epsilon_4 \cdot k_1)\left\{\epsilon_1 \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_1}\right] \cdot \epsilon_6\right\}[(\epsilon_2 \cdot \epsilon_3)(\epsilon_5 \cdot k_3) - (\epsilon_3 \cdot \epsilon_5)(\epsilon_2 \cdot k_3)]\Big] \\
\Big[=&-12(k_1 \cdot k_2)(\epsilon_2 \cdot \epsilon_5)(\epsilon_3 \cdot k_1)(\epsilon_4 \cdot k_{13})\left\{\epsilon_1 \cdot \exp\left[-\frac{f_3}{\epsilon_3 \cdot k_1}\right] \cdot \exp\left[-\frac{f_4}{\epsilon_4 \cdot k_{13}}\right] \cdot \epsilon_6\right\} \\
&+8(k_1 \cdot k_2)(\epsilon_3 \cdot k_1)\left\{\epsilon_1 \cdot \exp\left[-\frac{f_3}{\epsilon_3 \cdot k_1}\right] \cdot \epsilon_6\right\} N(2^1, 4^1, 5^1) \\
&+8(k_1 \cdot k_2)(\epsilon_4 \cdot k_1)\left\{\epsilon_1 \cdot \exp\left[-\frac{f_4}{\epsilon_3 \cdot k_1}\right] \cdot \epsilon_6\right\} N(2^1, 3^1, 5^1) \\
&-2(k_1 \cdot k_2)(\epsilon_1 \cdot \epsilon_6)(\epsilon_2 \cdot \epsilon_5)(\epsilon_3 \cdot k_2)(\epsilon_4 \cdot k_3) \\
&-6(k_1 \cdot k_2)(\epsilon_1 \cdot \epsilon_6)(\epsilon_3 \cdot k_2)[(\epsilon_2 \cdot \epsilon_4)(\epsilon_5 \cdot k_4) - (\epsilon_4 \cdot \epsilon_5)(\epsilon_2 \cdot k_4)] \\
&+6(k_1 \cdot k_2)(\epsilon_1 \cdot \epsilon_6)(\epsilon_4 \cdot k_{23}) N(2^1, 3^1, 5^1)\Big]
\end{aligned}$$

$$
\begin{aligned}
&\Big[-8\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_1)(\epsilon_4\cdot k_{123})\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_2}{\epsilon_2\cdot k_1}\Big]\cdot\exp\Big[-\frac{f_4}{\epsilon_4\cdot k_{123}}\Big]\cdot\epsilon_6\Big\} \\
&-4\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_1)(\epsilon_4\cdot k_{12})\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_2}{\epsilon_2\cdot k_1}\Big]\cdot\exp\Big[-\frac{f_4}{\epsilon_4\cdot k_{12}}\Big]\cdot\epsilon_6\Big\} \\
&-8(k_3\cdot k_{12})(\epsilon_2\cdot k_1)\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_2}{\epsilon_2\cdot k_1}\Big]\cdot\epsilon_6\Big\}\,[(\epsilon_3\cdot\epsilon_4)(\epsilon_5\cdot k_4)-(\epsilon_4\cdot\epsilon_5)(\epsilon_3\cdot k_4)] \\
&-8\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_1)\,(\epsilon_5\cdot k_{1234})\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_2}{\epsilon_2\cdot k_1}\Big]\cdot\exp\Big[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\Big]\cdot\epsilon_6\Big\} \\
&-4\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_1)(\epsilon_5\cdot k_{12})\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_2}{\epsilon_2\cdot k_1}\Big]\cdot\exp\Big[-\frac{f_5}{\epsilon_5\cdot k_{12}}\Big]\cdot\epsilon_6\Big\}\Big] \\
=&\Big[-12\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_1)(\epsilon_4\cdot k_{12})\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_2}{\epsilon_2\cdot k_1}\Big]\cdot\exp\Big[-\frac{f_4}{\epsilon_4\cdot k_{12}}\Big]\cdot\epsilon_6\Big\} \\
&-12\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_1)\,(\epsilon_5\cdot k_{12})\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_2}{\epsilon_2\cdot k_1}\Big]\cdot\exp\Big[-\frac{f_5}{\epsilon_5\cdot k_{12}}\Big]\cdot\epsilon_6\Big\} \\
&+8(k_3\cdot k_{12})(\epsilon_2\cdot k_1)\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_2}{\epsilon_2\cdot k_1}\Big]\cdot\epsilon_6\Big\}\,\big[N(3^1,4^1,5^1)-(\epsilon_3\cdot\epsilon_4)(\epsilon_5\cdot k_{34})\big]\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\Big[-6(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)(\epsilon_5\cdot k_{1234})(\epsilon_3\cdot k_{12})\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_3}{\epsilon_3\cdot k_{12}}\Big]\cdot\exp\Big[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\Big]\cdot\epsilon_6\Big\} \\
&-2(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)\,(\epsilon_5\cdot k_{1234})\,(\epsilon_3\cdot k_1)\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_3}{\epsilon_3\cdot k_1}\Big]\cdot\exp\Big[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\Big]\cdot\epsilon_6\Big\} \\
&-2(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)(\epsilon_3\cdot k_{12})(\epsilon_5\cdot k_{13})\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_3}{\epsilon_3\cdot k_{12}}\Big]\cdot\exp\Big[-\frac{f_5}{\epsilon_5\cdot k_{13}}\Big]\cdot\epsilon_6\Big\} \\
&-2(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)\,(\epsilon_5\cdot k_{13})\,(\epsilon_3\cdot k_1)\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_3}{\epsilon_3\cdot k_1}\Big]\cdot\exp\Big[-\frac{f_5}{\epsilon_5\cdot k_{13}}\Big]\cdot\epsilon_6\Big\}\Big] \\
&\Big[= -12(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)(\epsilon_3\cdot k_1)(\epsilon_5\cdot k_{13})\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_3}{\epsilon_3\cdot k_1}\Big]\cdot\exp\Big[-\frac{f_5}{\epsilon_5\cdot k_{13}}\Big]\cdot\epsilon_6\Big\} \\
&-8(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)(\epsilon_3\cdot k_2)(\epsilon_5\cdot k_1)\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_5}{\epsilon_5\cdot k_1}\Big]\cdot\epsilon_6\Big\} \\
&-8(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)(\epsilon_3\cdot k_1)(\epsilon_5\cdot k_{24})\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_3}{\epsilon_3\cdot k_1}\Big]\cdot\epsilon_6\Big\} \\
&-6(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_4)(\epsilon_3\cdot k_2)(\epsilon_5\cdot k_{234}) \\
&-2(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_4)(\epsilon_3\cdot k_2)(\epsilon_5\cdot k_3)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\Big[-6\,(k_2\cdot k_1)\,(\epsilon_2\cdot\epsilon_3)\,(\epsilon_4\cdot k_{123})\,(\epsilon_5\cdot k_{1234})\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_4}{\epsilon_4\cdot k_{123}}\Big]\cdot\exp\Big[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\Big]\cdot\epsilon_6\Big\} \\
&-2(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_1)\,(\epsilon_5\cdot k_{1234})\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_4}{\epsilon_4\cdot k_1}\Big]\cdot\exp\Big[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\Big]\cdot\epsilon_6\Big\} \\
&-2(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_{123})(\epsilon_5\cdot k_{14})\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_4}{\epsilon_4\cdot k_{123}}\Big]\cdot\exp\Big[-\frac{f_5}{\epsilon_5\cdot k_{14}}\Big]\cdot\epsilon_6\Big\} \\
&-2(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_1)\,(\epsilon_5\cdot k_{14})\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_4}{\epsilon_4\cdot k_1}\Big]\cdot\exp\Big[-\frac{f_5}{\epsilon_5\cdot k_{14}}\Big]\cdot\epsilon_6\Big\} \\
&-6(k_1\cdot k_2)(\epsilon_5\cdot k_{1234})\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\Big]\cdot\epsilon_6\Big\}\,[(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)] \\
&-2(k_1\cdot k_2)(\epsilon_5\cdot k_1)\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_5}{\epsilon_5\cdot k_1}\Big]\cdot\epsilon_6\Big\}\,[(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)]\Big] \\
&\Big[= -12(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_1)\,(\epsilon_5\cdot k_{14})\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_4}{\epsilon_4\cdot k_1}\Big]\cdot\exp\Big[-\frac{f_5}{\epsilon_5\cdot k_{14}}\Big]\cdot\epsilon_6\Big\} \\
&-8(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_1)\,(\epsilon_5\cdot k_{23})\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_4}{\epsilon_4\cdot k_1}\Big]\cdot\epsilon_6\Big\} \\
&-8(k_1\cdot k_2)(\epsilon_5\cdot k_1)\,\Big\{\epsilon_1\cdot\exp\Big[-\frac{f_5}{\epsilon_5\cdot k_1}\Big]\cdot\epsilon_6\Big\}\,[(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)+(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_{23})] \\
&-6\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_{23})(\epsilon_5\cdot k_{234}) \\
&-6(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_5\cdot k_{234})\,[(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)] \\
&-2(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_{23})\,(\epsilon_5\cdot k_4)\Big]
\end{aligned}
$$

All in all the simplifications add up to

$$
\begin{aligned}
24N_6^{YM} =&24(\epsilon_2\cdot k_1)(\epsilon_3\cdot k_{12})(\epsilon_4\cdot k_{123})(\epsilon_5\cdot k_{1234})\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{123}}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\right]\cdot\epsilon_6\right\}\\
&-12\,(k_4\cdot k_{123})\,(\epsilon_4\cdot\epsilon_5)(\epsilon_2\cdot k_1)\,(\epsilon_3\cdot k_{12})\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\epsilon_6\right\}\\
&-12\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_1)(\epsilon_4\cdot k_{12})\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{12}}\right]\cdot\epsilon_6\right\}\\
&-12\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_1)\,(\epsilon_5\cdot k_{12})\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{12}}\right]\cdot\epsilon_6\right\}\\
&-12(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot k_1)\,(\epsilon_4\cdot k_{13})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{13}}\right]\cdot\epsilon_6\right\}\\
&-12(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)(\epsilon_3\cdot k_1)(\epsilon_5\cdot k_{13})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{13}}\right]\cdot\epsilon_6\right\}\\
&-12(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_1)\,(\epsilon_5\cdot k_{14})\left\{\epsilon_1\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{14}}\right]\cdot\epsilon_6\right\}\\
&+8(k_3\cdot k_{12})(\epsilon_2\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\epsilon_6\right\}\left[N(3^1,4^1,5^1)-(\epsilon_3\cdot\epsilon_4)(\epsilon_5\cdot k_{34})\right]\\
&+8(k_1\cdot k_2)(\epsilon_3\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\epsilon_6\right\}\left[N(2^1,4^1,5^1)-(\epsilon_2\cdot\epsilon_4)(\epsilon_5\cdot k_{24})\right]\\
&+8(k_1\cdot k_2)(\epsilon_4\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_1}\right]\cdot\epsilon_6\right\}\left[N(2^1,3^1,5^1)-(\epsilon_2\cdot\epsilon_3)\,(\epsilon_5\cdot k_{23})\right]\\
&+8(k_1\cdot k_2)(\epsilon_5\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_1}\right]\cdot\epsilon_6\right\}\left[N(2^1,3^1,4^1)-(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_{23})\right]\\
&-6(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_5\cdot k_{234})\left[(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)+(\epsilon_2\cdot\epsilon_4)(\epsilon_3\cdot k_2)+(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_{23}\right]\\
&-6(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot k_2)\left[(\epsilon_2\cdot\epsilon_4)(\epsilon_5\cdot k_4)-(\epsilon_4\cdot\epsilon_5)(\epsilon_2\cdot k_4)\right]\\
&-6\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)(\epsilon_4\cdot\epsilon_5)\left[(\epsilon_2\cdot k_3)(\epsilon_3\cdot k_4)-(k_3\cdot k_4)(\epsilon_2\cdot\epsilon_3)\right]\\
&-6\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)(\epsilon_5\cdot k_4)\left[(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)-(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)\right]\\
&+6(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_4\cdot k_{23})N(2^1,3^1,5^1)\\
&+3\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\,(k_4\cdot k_{123})\,(\epsilon_4\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_3)\\
&+3\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_4)\\
&+3\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_5)\\
&+3(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_4)\,(k_1\cdot k_3)\,(\epsilon_3\cdot\epsilon_5)\\
&+3(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(k_1\cdot k_3)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)\\
&+3(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\,(\epsilon_2\cdot\epsilon_3)\,(k_1\cdot k_4)\,(\epsilon_4\cdot\epsilon_5)
\end{aligned}
$$

Now notice that the 4 point pure gluon numerator can be written as

$$
\begin{aligned}
6N(2^1,3^1,4^1,5^1)=&-3(k_3\cdot k_2)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)-6(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)(\epsilon_5\cdot k_4)+6(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot k_2)(\epsilon_4\cdot k_{23})\\
&-6(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_3)\,(\epsilon_4\cdot k_{23})-6(\epsilon_4\cdot\epsilon_5)\left[(\epsilon_2\cdot k_4)(\epsilon_3\cdot k_2)-(\epsilon_2\cdot k_3)(\epsilon_3\cdot k_4)+(k_3\cdot k_4)(\epsilon_3\cdot\epsilon_2)\right]\\
&+6(\epsilon_3\cdot\epsilon_2)(\epsilon_4\cdot k_{23})(\epsilon_5\cdot k_3)+6(\epsilon_3\cdot\epsilon_2)(\epsilon_4\cdot k_3)(\epsilon_5\cdot k_4)+6(\epsilon_4\cdot\epsilon_2)(\epsilon_3\cdot k_2)(\epsilon_5\cdot k_4)
\end{aligned}
$$

So we can put a part of the expression in a more convenient form

$$
\begin{aligned}
&+3\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\,(k_3\cdot k_2)\,(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)+6\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_3)(\epsilon_5\cdot k_4)\\
&-6(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_4\cdot k_{23})(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot k_2)+6(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_4\cdot k_{23})(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_3)\\
&+6\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)(\epsilon_4\cdot\epsilon_5)\left[(\epsilon_2\cdot k_4)(\epsilon_3\cdot k_2)-(\epsilon_2\cdot k_3)(\epsilon_3\cdot k_4)+(k_3\cdot k_4)(\epsilon_2\cdot\epsilon_3)\right]\\
&-6(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_{23})(\epsilon_5\cdot k_3)-6\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)(\epsilon_5\cdot k_4)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_3)\\
&-6(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_3\cdot k_2)\left[(\epsilon_2\cdot\epsilon_4)(\epsilon_5\cdot k_4)\right]+6(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_5\cdot k_{234})\left[N(2^1,3^1,4^1)-(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_{23}\right]\\
&+3\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\,(k_4\cdot k_{123})\,(\epsilon_4\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_3)+3\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_4)\\
&+3\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\,(k_3\cdot k_1)\,(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_5)+3(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_2\cdot\epsilon_4)\,(k_1\cdot k_3)\,(\epsilon_3\cdot\epsilon_5)\\
&+3(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(k_1\cdot k_3)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)+3(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\,(\epsilon_2\cdot\epsilon_3)\,(k_1\cdot k_4)\,(\epsilon_4\cdot\epsilon_5)\\
=&-6(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)N(2^1,3^1,4^1,5^1)\\
&+6(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)(\epsilon_5\cdot k_{234})\left[N(2^1,3^1,4^1)-(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_{23})\right]\\
&+6(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)\,(k_1\cdot k_3)\,(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)\\
&+3\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\,(k_4\cdot k_{123})\,(\epsilon_4\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_3)+3\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_4)\\
&+3(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)\,(k_1\cdot k_3)\,(\epsilon_2\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_5)+3(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\,(k_1\cdot k_4)\,(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot\epsilon_5)
\end{aligned}
$$

Putting it all together we have the final numerator

$$
\begin{aligned}
N_6^{\mathrm{YM}} =&(\epsilon_2\cdot k_1)(\epsilon_3\cdot k_{12})(\epsilon_4\cdot k_{123})(\epsilon_5\cdot k_{1234})\\
&\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{123}}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{1234}}\right]\cdot\epsilon_6\right\}\\
&-\frac{1}{2}\,(k_4\cdot k_{123})\,(\epsilon_4\cdot\epsilon_5)(\epsilon_2\cdot k_1)\,(\epsilon_3\cdot k_{12})\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_{12}}\right]\cdot\epsilon_6\right\}\\
&-\frac{1}{2}\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot k_1)(\epsilon_4\cdot k_{12})\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{12}}\right]\cdot\epsilon_6\right\}\\
&-\frac{1}{2}\,(k_3\cdot k_{12})\,(\epsilon_3\cdot\epsilon_4)(\epsilon_2\cdot k_1)\,(\epsilon_5\cdot k_{12})\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{12}}\right]\cdot\epsilon_6\right\}\\
&-\frac{1}{2}(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot k_1)\,(\epsilon_4\cdot k_{13})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_{13}}\right]\cdot\epsilon_6\right\}\\
&-\frac{1}{2}(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_4)(\epsilon_3\cdot k_1)(\epsilon_5\cdot k_{13})\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{13}}\right]\cdot\epsilon_6\right\}\\
&-\frac{1}{2}(k_1\cdot k_2)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_1)\,(\epsilon_5\cdot k_{14})\left\{\epsilon_1\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_1}\right]\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_{14}}\right]\cdot\epsilon_6\right\}\\
&+\frac{1}{3}(k_3\cdot k_{12})(\epsilon_2\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_2}{\epsilon_2\cdot k_1}\right]\cdot\epsilon_6\right\}\left[N(3^1,4^1,5^1)-(\epsilon_3\cdot\epsilon_4)(\epsilon_5\cdot k_{34})\right]\\
&+\frac{1}{3}(k_1\cdot k_2)(\epsilon_3\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_3}{\epsilon_3\cdot k_1}\right]\cdot\epsilon_6\right\}\left[N(2^1,4^1,5^1)-(\epsilon_2\cdot\epsilon_4)(\epsilon_5\cdot k_{24})\right]\\
&+\frac{1}{3}(k_1\cdot k_2)(\epsilon_4\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_4}{\epsilon_4\cdot k_1}\right]\cdot\epsilon_6\right\}\left[N(2^1,3^1,5^1)-(\epsilon_2\cdot\epsilon_3)\,(\epsilon_5\cdot k_{23})\right]\\
&+\frac{1}{3}(k_1\cdot k_2)(\epsilon_5\cdot k_1)\left\{\epsilon_1\cdot\exp\left[-\frac{f_5}{\epsilon_5\cdot k_1}\right]\cdot\epsilon_6\right\}\left[N(2^1,3^1,4^1)-(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_{23})\right]\\
&+\frac{1}{4}(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)\left\{(\epsilon_5\cdot k_{234})N(2^1,3^1,4^1))-N(2^1,3^1,4^1,5^1)-(\epsilon_5\cdot k_4)(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_{23})\right\}\\
&+\frac{1}{4}(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)\left\{(k_1\cdot k_4)\,(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot\epsilon_5)-(\epsilon_5\cdot k_{23})(\epsilon_2\cdot\epsilon_3)(\epsilon_4\cdot k_{23})\right\}\\
&+\frac{1}{4}(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_6)\,(k_1\cdot k_3)\left[(\epsilon_2\cdot\epsilon_5)(\epsilon_3\cdot\epsilon_4)+(\epsilon_2\cdot\epsilon_4)(\epsilon_3\cdot\epsilon_5)\right]\\
&+\frac{1}{8}\,(k_1\cdot k_2)\,(\epsilon_1\cdot\epsilon_6)\left\{(k_4\cdot k_{23})\,(\epsilon_4\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_3)+(k_3\cdot k_2)\,(\epsilon_3\cdot\epsilon_5)(\epsilon_2\cdot\epsilon_4)\right\}
\end{aligned}
\tag{B.0.4}
$$

Which is the same numerator obtained through the rules in chapter 8.

Bibliography

[1] S.J. Parke and T. Taylor, *An Amplitude for n Gluon Scattering*, *Phys. Rev. Lett.* **56** (1986) 2459.

[2] LIGO Scientific, Virgo collaboration, *Observation of Gravitational Waves from a Binary Black Hole Merger*, *Phys. Rev. Lett.* **116** (2016) 061102 [1602.03837].

[3] A. Guevara, A. Ochirov and J. Vines, *Scattering of Spinning Black Holes from Exponentiated Soft Factors*, *JHEP* **09** (2019) 056 [1812.06895].

[4] Y.F. Bautista and A. Guevara, *On the Double Copy for Spinning Matter*, 1908.11349.

[5] Y.F. Bautista and A. Guevara, *From Scattering Amplitudes to Classical Physics: Universality, Double Copy and Soft Theorems*, 1903.12419.

[6] H. Johansson and A. Ochirov, *Double copy for massive quantum particles with spin*, *JHEP* **09** (2019) 040 [1906.12292].

[7] A. Ochirov, *Helicity amplitudes for QCD with massive quarks*, *JHEP* **04** (2018) 089 [1802.06730].

[8] N. Arkani-Hamed, Y.-t. Huang and D. O'Connell, *Kerr black holes as elementary particles*, *JHEP* **01** (2020) 046 [1906.10100].

[9] M.-Z. Chung, Y.-T. Huang, J.-W. Kim and S. Lee, *The simplest massive S-matrix: from minimal coupling to Black Holes*, *JHEP* **04** (2019) 156 [1812.08752].

[10] F. Cachazo, S. He and E.Y. Yuan, *Scattering in Three Dimensions from Rational Maps*, *JHEP* **10** (2013) 141 [1306.2962].

[11] F. Cachazo, S. He and E.Y. Yuan, *Scattering of Massless Particles in Arbitrary Dimensions*, *Phys. Rev. Lett.* **113** (2014) 171601 [1307.2199].

[12] F. Cachazo, S. He and E.Y. Yuan, *Scattering of Massless Particles: Scalars, Gluons and Gravitons*, *JHEP* **07** (2014) 033 [1309.0885].

[13] F. Cachazo, S. He and E.Y. Yuan, *Scattering equations and Kawai-Lewellen-Tye orthogonality*, *Phys. Rev. D* **90** (2014) 065001 [1306.6575].

[14] F. Cachazo, S. He and E.Y. Yuan, *Einstein-Yang-Mills Scattering Amplitudes From Scattering Equations*, *JHEP* **01** (2015) 121 [1409.8256].

[15] F. Cachazo, S. He and E.Y. Yuan, *Scattering Equations and Matrices: From Einstein To Yang-Mills, DBI and NLSM*, *JHEP* **07** (2015) 149 [1412.3479].

[16] L. Dolan and P. Goddard, *The Polynomial Form of the Scattering Equations*, *JHEP* **07** (2014) 029 [1402.7374].

[17] L. Dolan and P. Goddard, *Proof of the Formula of Cachazo, He and Yuan for Yang-Mills Tree Amplitudes in Arbitrary Dimension*, *JHEP* **05** (2014) 010 [1311.5200].

[18] S.G. Naculich, *Scattering equations and BCJ relations for gauge and gravitational amplitudes with massive scalar particles*, *JHEP* **09** (2014) 029 [1407.7836].

[19] S.G. Naculich, *CHY representations for gauge theory and gravity amplitudes with up to three massive particles*, *JHEP* **05** (2015) 050 [1501.03500].

[20] S. He and E.Y. Yuan, *One-loop Scattering Equations and Amplitudes from Forward Limit*, *Phys. Rev. D* **92** (2015) 105004 [1508.06027].

[21] J. Agerskov, N. Bjerrum-Bohr, H. Gomez and C. Lopez-Arcos, *Yang-Mills Loop Amplitudes from Scattering Equations*, 1910.03602.

[22] B. Feng, *CHY-construction of Planar Loop Integrands of Cubic Scalar Theory*, *JHEP* **05** (2016) 061 [1601.05864].

[23] H. Kawai, D. Lewellen and S. Tye, *A Relation Between Tree Amplitudes of Closed and Open Strings*, *Nucl. Phys. B* **269** (1986) 1.

[24] N. Bjerrum-Bohr, P.H. Damgaard, B. Feng and T. Sondergaard, *Gravity and Yang-Mills Amplitude Relations*, *Phys. Rev. D* **82** (2010) 107702 [1005.4367].

[25] N. Bjerrum-Bohr, P.H. Damgaard, B. Feng and T. Sondergaard, *New Identities among Gauge Theory Amplitudes*, *Phys. Lett. B* **691** (2010) 268 [1006.3214].

[26] N. Bjerrum-Bohr, P.H. Damgaard, B. Feng and T. Sondergaard, *Proof of Gravity and Yang-Mills Amplitude Relations*, *JHEP* **09** (2010) 067 [1007.3111].

[27] N. Bjerrum-Bohr, P.H. Damgaard, T. Sondergaard and P. Vanhove, *The Momentum Kernel of Gauge and Gravity Theories*, *JHEP* **01** (2011) 001 [1010.3933].

[28] N. Bjerrum-Bohr, A. Cristofoli, P.H. Damgaard and H. Gomez, *Scalar-Graviton Amplitudes*, *JHEP* **11** (2019) 148 [1908.09755].

[29] N.E.J. Bjerrum-Bohr, P.H. Damgaard, P. Tourkine and P. Vanhove, *Scattering Equations and String Theory Amplitudes*, *Phys. Rev. D* **90** (2014) 106002 [1403.4553].

[30] A. Edison, S. He, O. Schlotterer and F. Teng, *One-loop Correlators and BCJ Numerators from Forward Limits*, 2005.03639.

[31] A. Edison and F. Teng, *Efficient Calculation of Crossing Symmetric BCJ Tree Numerators*, 2005.03638.

[32] Y.-J. Du and F. Teng, *BCJ numerators from reduced Pfaffian*, *JHEP* **04** (2017) 033 [1703.05717].

[33] F. Teng and B. Feng, *Expanding Einstein-Yang-Mills by Yang-Mills in CHY frame*, *JHEP* **05** (2017) 075 [1703.01269].

[34] V. Del Duca, L.J. Dixon and F. Maltoni, *New color decompositions for gauge amplitudes at tree and loop level*, *Nucl. Phys. B* **571** (2000) 51 [hep-ph/9910563].

[35] A. Guevara, A. Ochirov and J. Vines, *Black-hole scattering with general spin directions from minimal-coupling amplitudes*, *Phys. Rev. D* **100** (2019) 104024 [1906.10071].

[36] H. Elvang and Y. Huang, *Scattering Amplitudes*, 1308.1697.

[37] M. Srednicki, *Quantum field theory*, Cambridge University Press (1, 2007).

[38] L.J. Dixon, *A brief introduction to modern amplitude methods*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: Particle Physics: The Higgs Boson and Beyond*, pp. 31–67, 2014, DOI [1310.5353].

[39] M.D. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press (3, 2014).

[40] N. Bjerrum-Bohr, J.L. Bourjaily, P.H. Damgaard and B. Feng, *Manifesting Color-Kinematics Duality in the Scattering Equation Formalism*, *JHEP* **09** (2016) 094 [1608.00006].

[41] Z. Bern, J.J. Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, *The Duality Between Color and Kinematics and its Applications*, 1909.01358.

[42] N. Arkani-Hamed, T.-C. Huang and Y.-t. Huang, *Scattering Amplitudes For All Masses and Spins*, 1709.04891.

[43] H. Gomez and A. Helset, *Scattering equations and a new factorization for amplitudes. Part II. Effective field theories*, *JHEP* **05** (2019) 129 [1902.02633].

[44] H. Gomez, *Λ scattering equations*, *JHEP* **06** (2016) 101 [1604.05373].

[45] C. Cardona and H. Gomez, *Elliptic scattering equations*, *JHEP* **06** (2016) 094 [1605.01446].

[46] N. Bjerrum-Bohr, H. Gomez and A. Helset, *New factorization relations for nonlinear sigma model amplitudes*, *Phys. Rev. D* **99** (2019) 045009 [1811.06024].

[47] N. Bjerrum-Bohr, H. Gomez and A. Helset, *New factorization relations for nonlinear sigma model amplitudes*, *Phys. Rev. D* **99** (2019) 045009 [1811.06024].

[48] H. Gomez, *Scattering equations and a new factorization for amplitudes. Part I. Gauge theories*, *JHEP* **05** (2019) 128 [1810.05407].

[49] N. Bjerrum-Bohr, P.H. Damgaard and H. Gomez, *New Factorization Relations for Yang Mills Amplitudes*, *Phys. Rev. D* **99** (2019) 025014 [1810.05023].

[50] C. Baadsgaard, N. Bjerrum-Bohr, J.L. Bourjaily and P.H. Damgaard, *Integration Rules for Scattering Equations*, *JHEP* **09** (2015) 129 [1506.06137].

[51] N. Bjerrum-Bohr, J.L. Bourjaily, P.H. Damgaard and B. Feng, *Analytic representations of Yang–Mills amplitudes*, *Nucl. Phys. B* **913** (2016) 964 [1605.06501].

[52] K. Zhou and B. Feng, *Note on differential operators, chy integrands, and unifying relations for amplitudes*, *Journal of High Energy Physics* **2018** (2018) 160.

[53] Z. Bern, J. Carrasco and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, *Phys. Rev. D* **78** (2008) 085011 [0805.3993].

[54] Z. Bern, J.J.M. Carrasco and H. Johansson, *Perturbative Quantum Gravity as a Double Copy of Gauge Theory*, *Phys. Rev. Lett.* **105** (2010) 061602 [1004.0476].

[55] Z. Bern, T. Dennen, Y.-T. Huang and M. Kiermaier, *Gravity as the Square of Gauge Theory*, *Phys. Rev. D* **82** (2010) 065003 [1004.0693].

[56] H. Johansson and A. Ochirov, *Color-Kinematics Duality for QCD Amplitudes*, *JHEP* **01** (2016) 170 [1507.00332].

[57] C.-H. Fu, Y.-J. Du, R. Huang and B. Feng, *Expansion of Einstein-Yang-Mills Amplitude*, *JHEP* **09** (2017) 021 [1702.08158].

[58] N.E.J. Bjerrum-Bohr, T.V. Brown and H. Gomez, "In preperation.".