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High precision gravity observables: From Effective Field Theories (EFTs) to particle amplitudes

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Abstract

In this thesis we make progress within the effective field theory for gravitating spinning objects in the post-Newtonian (PN) approximation, which is used to describe the inspiral phase of a compact binary coalescence. Based on a newly computed interaction potential [1], we derive for the first time the complete dynamics of a compact binary system to the next-to-leading order with cubic-in-spin effects, which enter at the fourth and a half PN (4.5PN) order for maximally-rotating objects, beyond the current state of the art. After verifying the Feynman rules and the total evaluation of the Feynman diagrams, we compute the reduced potential, which no longer contains higher-order time derivatives, via lower-order variable redefinitions. Furthermore, we derive the corresponding correction to the equations of motion and to the general Hamiltonian, valid also for general compact objects, generic orbits, and with arbitrary spin orientations. Then, we also compute the complete gauge-invariant relations between the binding energy, angular momentum and orbital frequency of the binary for circular orbits and aligned spins. These results are of high interest for the community, as they can be used to develop the highest-in-spin gravitational corrections at next-to-leading order for the waveform templates of the emitted gravitational waves. On the other hand, this work helps to advance towards understanding the limits of the gravitational Compton scattering with massive particles of spin $s > 2$. Finally, we address the Poincaré invariance of the system, which constitutes the most stringent theoretical self-consistency check in PN gravity.

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Part I
Introduction

1 Background and motivation

We live in exciting times for theories of gravitation. In 1915, the general theory of relativity (GR) was formulated by Einstein [2], and shortly after, in 1916, he already predicted the existence of gravitational waves (GWs) in the theory [3]. Nevertheless, it was not until very recently, a century after the original prediction, that GWs were observed. The first direct detection of GWs, labeled “GW150914” and announced in 2016 [4], was accomplished by the Advanced LIGO detectors [5] in 2015, which launched the era of gravitational-wave astronomy. Since then, more than 50 detections have been announced in collaboration with the Advanced Virgo detector [6]. In fact, GWs have proven to be so promising that further ground detectors are planned, such as KAGRA [7] in Japan, which became operational last year, IndIGO [8] in India, the Einstein Telescope [9] in Europe, the Cosmic Explorer [10] in the United States, and even the space-based detector LISA [11].

What makes GWs so significant is that they stand among the few events that can probe classical gravity in a strong regime where Newtonian physics no longer holds, and could shed light onto previously unknown features of gravity. In particular, all of the GW detections made so far involve the coalescence of binaries of compact objects (CBCs), reporting mostly binaries of black holes (BHs) but even including neutron stars [12]. The CBC involves three stages [13]:

1. The inspiral, when the components of the binary still move at non-relativistic velocities and their orbital separation slowly decreases.

2. The merger, when the objects reach relativistic velocities and merge into a single object.

3. The ringdown (for BH binaries), when the merged object settles to a rotating Kerr BH via quasinormal oscillations.

Given that the detections are based on the matched-filtering technique, in which GW templates are superposed with the data in order to find agreement, it becomes thus crucial to provide high-precision theoretical models for the templates. With that objective, effective field theories (EFTs) precisely provide a framework designed for high-precision analytical perturbative descriptions of systems with distinct physical scales [14, 15]. In particular, the EFT of post-Newtonian (PN) gravity [13, 14, 16, 17] is employed to describe the orbital dynamics of the inspiral phase of the binary system, building on quantum field theory (QFT) techniques applied to classical gravity. With the analytical description provided by PN gravity and via the effective one-body formalism [18], which develops the Newtonian idea of mapping the 2-body problem to an effective one-body system, gravitational waveform templates can be modeled. However, it turns out that even then, high-order corrections such as the sixth PN (6PN) are at least required to obtain useful information

	(N^0)LO	$\mathrm{N}^{(1)}$LO	N^2LO	N^3LO	N^4LO	N^5LO
Non-spinning (S^0)	++	++	++	++	++	+
Spin-orbit (S^1)	++	++	++	+		
Quadratic-in-spin (S^2)	++	++	+	+		
Cubic-in-spin (S^3)	++	+				
Quartic-in-spin (S^4)	++	+				

Table 1.1: State of the art of PN gravity for the dynamics of the compact binary inspiral. The ++ and + entries denote sectors that have been fully completed and verified, or partially completed/not verified, respectively. The corresponding PN correction enters at the order $n+l+\text{Parity}(l)/2$, where n denotes the highest n-loop order at N^nLO and l the highest spin multipole S^l for each sector, where parity is 0 or 1 for l even or odd, respectively. The gray area corresponds to gravitational Compton scattering with spin $s \geq 3/2$, as elaborated below.

of the inner structure of the components of the binary from the GWs [19]. This propelled the community to push the frontier of PN gravity, pushing for the 5.5PN and 6PN orders in the non-spinning sectors very recently [20, 21].

Furthermore, astrophysical observations indicated that the components of such binary systems had large spins [22], meaning that they have an intrinsic rotation around an axis. Hence, the development of an EFT formulation for gravitating spinning objects for the binary system was also required [23, 24], with the state-of-the-art frontier, including the spinning sectors, summarized in Table 1.1. Within the EFT approach, the work in [23] approached the first rederivation of Tulczyjew's [25] and Barker and O'Connell's [26] results for the leading order (LO) corrections to the PN binary dynamics due to spin-orbit (SO) and quadratic-in-spin (S^2) effects. Then, based on [24], in [19] the LO corrections due to cubic- and quartic-in-spin (S^3 and S^4) effects were computed, which together with [27–29] completed the 4PN order for spinning objects. Since then, work has been done to approach both the high-in-spin sectors [1, 30] and to increase the loop order in the perturbation, with the next-to-leading order (NLO), the NNLO and the N^3LO dynamics up to quadratic-in-spin (S^2) effects derived in [24], [27–29] and [31, 32], respectively.

What is more, as first shown in [33], where the traditional spinor-helicity formalism for massless particles was extended to massive particles with arbitrary spin, the 4-particle gravitational Compton scattering amplitude cannot be uniquely fixed for spins $s > 2$. Hence, from that point onward spin effects cannot be treated from the scattering amplitudes point of view, and nowadays can only be approached using EFT methods. This is because classical effects with spin to the lth order correspond to scattering amplitudes with a quantum spin of $s = l/2$ [34, 35]. Thus, odd-in-spin effects in our EFT formalism correspond to scattering amplitudes with particles of half-integer spin, or fermions, making them more intricate than the even-in-spin effects, whose homologous scattering amplitudes describe bosons, of integer spin.

In this direction, the works carried out in [1, 30], where the interaction potentials to NLO S^3 and S^4 were respectively calculated, represent the state-of-the-art research. Their results stand close to the edge of the Compton ambiguity, with counterparts being 4-particle scattering amplitudes with $s = 3/2$ and $s = 2$, the first to be approached within the $s \geq 3/2$ condition represented by the gray area in Table 1.1.

Based on the pressing necessity to obtain high-precision theoretical predictions for the spinning binary inspiral problem, the aim of this thesis is to push even further the high-in-spin frontier. For that, starting from the basic formulation of the EFT of PN gravity, we compute the state-of-the-art dynamics at the NLO cubic-in-spin effects, corresponding to a 4.5PN correction for maximally-rotating objects, using a newly derived interaction potential [1]. These results have never been previously computed, and are of high interest for the GW community, as they provide the highest-in-spin corrections at 1-loop for the binary inspiral problem with general compact objects.

Throughout the thesis we use units with $c = \hbar = 1$, and choose the convention $\eta_{\mu\nu} \equiv \mathrm{Diag}[1, -1, -1, -1]$ for the Minkowski metric. Greek letters $(\mu, \nu, \rho, \dots)$ denote tensor indices in the global coordinate frame, running from 0 to 3, while lowercase Latin letters from the middle of the alphabet, $(i, j, k, \dots)$, denote spatial indices running from 1 to 3. Lowercase Latin letters from the beginning of the alphabet, $(a, b, c, \dots)$, denote indices in the local Lorentz frame, running from 0 to 3, whereas uppercase ones, $(A, B, C, \dots)$, denote indices in the body-fixed frame, also running from 0 to 3. Lastly, uppercase Latin letters from the middle of the alphabet, $(I, J, K, \dots)$, denote particle labels from 1 to 2.

2 Introduction to EFTs of compact binaries

In this section we review the formulation of EFTs and present its implementation to the binary inspiral problem with PN gravity, for which we build on [13, 14]. This way, it will also serve as the theoretical background needed for the methodology carried out in the thesis.

For formulations with a foundation in QFT, EFTs are effective theories that describe physics at a given energy (or length) scale ω, while neglecting all higher energy (or short distance) phenomena, characterized by a cut-off scale Λ. Thus, they are especially relevant for problems that involve several widely separated scales $\omega \ll \Lambda$, and are broadly used in many branches of physics. Intuitively, their motivation is yet very simple: We do not need to know about the high-energy behavior of the atoms that constitute a planet to make precise predictions about its orbit around a star, we only require information at the relevant scale.

To formulate an EFT, there are two distinct but equivalent approaches [14]. The first one is known as the top-down approach, in which the full theory action $S[\phi, \Phi]$ valid at the scale Λ is known. In the top-down approach we integrate out the high-energy (or heavy modes of mass $\gtrsim \Lambda$) degrees of freedom (DoFs) Φ by performing a functional integral. Then, the resulting effective action $S_{\text{eff}}[\phi]$ is relevant for the low-energy (or light modes of mass ω) DoFs ϕ,

$$e^{iS_{\text{eff}}[\phi]} = \int \mathcal{D}\Phi \, e^{iS[\phi,\Phi]}, \tag{2.1}$$

where $\mathcal{D}\Phi$ denotes integration over all modes Φ. Thus, the effects of the ultra-violet (UV) physics that we suppressed arise as a systematic expansion of the ratio $\omega/\Lambda \ll 1$. Diagrammatically, the heavy modes Φ appear as internal lines of the diagrams with ϕ particles, creating corrections to the low-energy result.

Alternatively, one can use the bottom-up approach, which is especially useful when the full theory is not known or is highly non-trivial. In this approach, the effective action $S_{\text{eff}}[\phi]$ is directly constructed as a functional of the fields $\phi(x)$,

$$S_{\text{eff}}[\phi] = \sum_i c_i \int d^4x \, \mathcal{O}_i(x), \tag{2.2}$$

so that it is given by an infinite set of the operators $\mathcal{O}_i(x)$, where the coefficients c_i are known as Wilson coefficients. In this approach, the effective action is built from scratch, by considering the most general set of operators $\mathcal{O}_i(x)$ that are allowed by the symmetries of the system. Moreover, this infinite series directly separates the physical scales, as the Wilson coefficients encapsulate all UV information, while the operators $\mathcal{O}_i(x)$ only depend on the low-energy scale, according to what is known as the decoupling theorem [13].

To fix the unknown Wilson coefficients, if the full theory is known and both approaches can be used and compared, then the coefficients can be directly matched to the full theory, or else they can also be determined from the data of experiments.

2.1 Post-Newtonian gravity and tower of EFTs

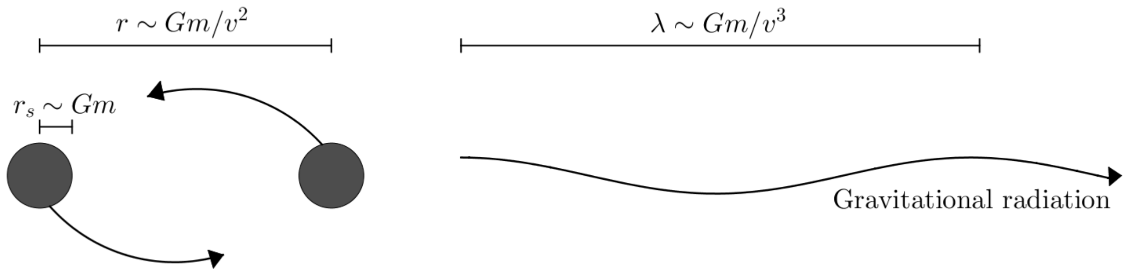


Figure 2.1: Representation of the binary inspiral setup with the relevant length scales for the EFT: the scale of the single compact object r_s, the scale of the orbital separation r, and the scale of the wavelength of the gravitational radiation λ.

As presented in §1, the inspiral phase of the CBC is characterized by non-relativistic velocities. Therefore, it is natural to assume the PN approximation $v \ll 1$, where the nPN order is defined as the v^{2n} correction from GR to Newtonian gravity (already entering at order v^2), as well as the weak field approximation [17]. Furthermore, in the binary inspiral problem we have a hierarchy of 3 widely separated scales, as depicted in Figure 2.1:

1. The scale of the internal structure of the single compact object, r_s. For BHs with Schwarzschild radius r_s, we can relate it to its mass m with Newton's constant G, $r_s \sim Gm$.

2. The scale of the orbital separation r between the components of the binary. For a bound binary system the virial theorem holds, relating $Gm/r \sim v^2$.

3. The scale of the wavelength of the gravitational radiation, λ. Since we measure on-shell gravitons, it holds that $\lambda^{-1} \sim k \sim \omega \sim v/r$, where k is the momentum and ω the frequency of the emitted gravitons in the GWs.

This way, the scales are related by

$$r_s \sim rv^2 \sim \lambda v^3, \tag{2.3}$$

which creates a hierarchy of 3 scales $r_s \ll r \ll \lambda$, controlled by the expansion parameter $v \ll 1$. Therefore, it is natural to address the binary inspiral problem via EFTs. In this case, in order to arrive at the EFT of orbital dynamics we will require a tower of EFTs. First, we will define a one-particle EFT in which we integrate out the small scale of the object r_s, next an EFT for the composite object, where the orbital field modes are integrated out, and finally we remove the radiation scale λ to obtain an EFT of dynamical multipoles. Moreover, due to the virial theorem we have $Gm/r \sim v^2$, which creates a combined perturbative expansion in both G and v, which can be interchanged[1].

[1]Recently, an EFT formulation for the post-Minkowskian approximation was put forward, in which the velocity is not small and only a perturbative expansion in terms of G is considered, i.e., only considering a weak field approximation. See [36] for the current state-of-the-art result for the binary inspiral problem at order $\mathcal{O}(G^4)$ for non-spinning objects.

2.1.1 One-particle EFT

Our first goal is to remove the scale of the single compact object r_s, obtaining an EFT that is valid far away from it. At this stage, the relevant DoFs for the system would be the low-energy modes of the gravitational field $\bar{g}_{\mu\nu}(x)$, the worldline coordinate $x^\mu(\lambda)$ parametrizing the location of the object, and the body-fixed orthonormal frame $e_A{}^\mu(\lambda)$ representing its rotation (spin). The latter is described by a tetrad field, which will be introduced in the next section.

First, we will have the gravitational field action in the bulk, which is the part of the effective action that does not contain matter components, and so generates pure gravity self-interactions in all points of the spacetime. From the full GR theory, we have that the action for the gravitational field, represented by a metric field $g_{\mu\nu}(x) \equiv \bar{g}_{\mu\nu} + g^s_{\mu\nu}$, where $g^s_{\mu\nu}$ stands for the strong modes, is given by the Einstein-Hilbert action $S_{\rm EH}$ [37]. However, here we just apply it to the low-energy gravitational field modes $\bar{g}_{\mu\nu}(x)$ that are not integrated out. Adding also a gauge-fixing term $S_{\rm GF}$, which we choose as the fully harmonic gauge, the bulk action reads

$$S_g[\bar{g}_{\mu\nu}] = S_{\rm EH} + S_{\rm GF} = -\frac{1}{16\pi G}\int d^4x\sqrt{\bar{g}}\, R + \frac{1}{32\pi G}\int d^4x\sqrt{\bar{g}}\,\bar{g}_{\mu\nu}\Gamma^\mu\Gamma^\nu, \tag{2.4}$$

as given in eq. (2.3) of [28], where $R = R(x)$ is the Ricci scalar for the low-energy modes $\bar{g}_{\mu\nu}$, and where we define $\bar{g} \equiv \det(\bar{g}_{\mu\nu})$ and $\Gamma^\mu \equiv \Gamma^\mu_{\rho\sigma}\bar{g}^{\rho\sigma}$, being $\Gamma^\mu_{\rho\sigma}$ the Christoffel symbols.

At this point, to define the effective action for the compact object we should adopt the bottom-up approach, so that we write

$$S_{\rm eff}[\bar{g}_{\mu\nu}, x^\mu, e_A{}^\mu] = S_g[\bar{g}_{\mu\nu}] + \sum_i C_i(r_s)\int d\lambda\, \mathcal{O}_i(\lambda), \tag{2.5}$$

where all the small scale dependence goes into the Wilson coefficients $C_i(r_s)$. To constrain the infinite series, we will consider the following symmetries [13, 14]:

- General coordinate invariance, including parity invariance;
- Worldline reparametrization invariance;
- Internal Lorentz invariance of the local frame field, $e_a{}^\mu$;
- SO(3) rotational invariance of the worldline spatial triad, $e_A{}^\mu$;
- Spin gauge invariance, which implies invariance under the completion of the worldline spatial triad to a tetrad.

With these symmetry considerations, we can infer some terms via the bottom-up approach. As seen from far away, the compact object can be approximated to a point-particle, with action

$$S_{\rm pp}[\bar{g}_{\mu\nu}, x^\mu] = -m\int d\tau = -m\int\sqrt{\bar{g}_{\mu\nu}dx^\mu dx^\nu} = -m\int d\lambda\,\sqrt{u^2}, \tag{2.6}$$

where τ is the proper time along the worldline and $u^\mu = \frac{dx^\mu}{d\lambda}$ is the coordinate velocity. Nevertheless, this is only a minimal coupling term, as we can also have couplings of the point-particle with the gravitational field arising from its finite-size effects. As explained in [13, 14], the non-minimal couplings would add the terms

$$S_{\rm pp}[\bar{g}_{\mu\nu}, x^\mu] \supset c_E \int d\lambda \, \frac{E^2_{\mu\nu}(x^\alpha(\lambda))}{\sqrt[3]{u^2}} + c_B \int d\lambda \, \frac{B^2_{\mu\nu}(x^\alpha(\lambda))}{\sqrt[3]{u^2}}, \tag{2.7}$$

where the Riemann tensor is decomposed into its electric and magnetic components,

$$E_{\mu\nu} \equiv R_{\mu\alpha\nu\beta} u^\alpha u^\beta, \qquad B_{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\alpha\beta\gamma\mu} R^{\alpha\beta}{}_{\delta\nu} u^\gamma u^\delta, \tag{2.8}$$

which have definite parity (even and odd, respectively), with $\varepsilon_{\alpha\beta\gamma\mu}$ being the Levi-Civita tensor density. Here, c_E and c_B are the Wilson coefficients, which also correspond to the Love numbers of the compact object, accounting for its finite-size effects that produce mass-induced tidal deformations. However, as pointed out in [13], they only start contributing as of the 5PN order for non-spinning objects. In addition, in [38] it was found that they vanish for Schwarzschild BHs in spacetime dimension $d = 4$.

However, if we consider that the point-particle is spinning, the following additions to the point-particle action take place [24, 39],

$$S_{\rm pp}[\bar{g}_{\mu\nu}, x^\mu, e_A{}^\mu] = \int d\lambda \left[-m\sqrt{u^2} - \frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu} + L_{\rm NMC}[\bar{g}_{\mu\nu}, u^\mu, S_{\mu\nu}] \right], \tag{2.9}$$

where the first two terms represent the point-particle minimal coupling terms in the spinning case, and where the spin tensor $S_{\mu\nu}$ is defined as the conjugate to the angular velocity tensor $\Omega^{\mu\nu} \equiv e_A{}^\mu \frac{De^{A\nu}}{D\lambda}$. $L_{\rm NMC}$ stands for the spin-induced non-minimal coupling Lagrangian, arising from finite-size spin-induced effects. It has a general expression given in eq. (4.16) of [24], where it was obtained via a direct product of SO(3) vectors S^μ. Up to cubic-in-spin order, it reads

$$L_{\rm ES^2} = -\frac{C_{\rm ES^2}}{2m} \frac{E_{\mu\nu}}{\sqrt{u^2}} S^\mu S^\nu, \tag{2.10}$$

$$L_{\rm BS^3} = -\frac{C_{\rm BS^3}}{6m^2} \frac{D_\lambda B_{\mu\nu}}{\sqrt{u^2}} S^\mu S^\nu S^\lambda, \tag{2.11}$$

where D_μ stands for the covariant derivative. Here, $C_{\rm ES^2}$ and $C_{\rm BS^3}$ are the Wilson coefficients that describe the quadrupolar and octupolar tidal deformations due to spin, respectively. For BHs, the coefficients are equal to 1, but they can be larger for neutron stars. Therefore, the effective action for a general spinning compact object becomes $S_{\rm eff} = S_g + S_{\rm pp}$, where S_g is given in eq. (2.4) and $S_{\rm pp}$ is given in eq. (2.9).

2.1.2 EFT of a composite object

To obtain an EFT that is valid for the composite object formed by the bound binary system, two steps are required using the top-down approach.

First, we need to define the EFT that is valid for two compact objects, seen from far away, that interact with the gravitational field. Based on the considerations of the previous section, the effective action would be the sum of the effective point-particle actions for the two spinning particles plus the bulk action,

$$S_{\text{eff}}[\bar{g}_{\mu\nu}, (x_1)^\mu, (x_2)^\mu, (e_1)_A{}^\mu, (e_2)_A{}^\mu] = S_g[\bar{g}_{\mu\nu}] + \sum_{I=1}^{2} S_{(I)\text{pp}}[\bar{g}_{\mu\nu}(x_I), (x_I)^\mu, (e_I)_A{}^\mu], \quad (2.12)$$

where S_g is given in eq. (2.4) and S_{pp} is given in eq. (2.9).

Second, to obtain an EFT for the binary system we should remove the orbital scale of the binary [24]. For that, we decompose the low-energy modes of the metric into

$$\bar{g}_{\mu\nu} \equiv \eta_{\mu\nu} + H_{\mu\nu} + \tilde{h}_{\mu\nu}, \quad (2.13)$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric, $H_{\mu\nu}$ are the orbital field modes, and $\tilde{h}_{\mu\nu}$ are the radiation modes. Then, we integrate out the hard-momentum orbital field modes by performing an explicit functional integral, which defines the EFT for the composite object,

$$e^{iS_{\text{eff}}[\tilde{g}_{\mu\nu}, (x_c)^\mu, (e_c)_A{}^\mu]} = \int \mathcal{D}H_{\mu\nu}\, e^{iS_{\text{eff}}[\bar{g}_{\mu\nu}, (x_1)^\mu, (x_2)^\mu, (e_1)_A{}^\mu, (e_2)_A{}^\mu]}, \quad (2.14)$$

where the subscript c denotes the worldline DoF of the composite object as a whole. This functional integral defines a diagrammatic expansion that consists of Feynman diagrams for the binary problem. However, we will take the classical limit and only consider tree-level diagrams in gravitons, without quantum graviton loops. This way, we can use the QFT methods for classical gravity.

For our purpose, we will only be interested in the conservative regime, where no dissipative effects are considered, hence there are no radiation modes present. So, our EFT formulation for the problem would be complete at this point.

As a last remark, if radiation modes are present, then a final EFT is defined by integrating out all field dependence. This creates an EFT of dynamical multipoles, where radiation-reaction and tail effects can be studied. For more details, see the review [13].

2.2 Tetrad fields and non-relativistic gravitational fields

As pointed out in the beginning of last section, an orthonormal frame $e_A{}^\mu(\lambda)$ must be used to represent the rotation of objects in GR. This is because up to now, the formulation is only valid for objects that behave like scalars, but rotation is closely related to spinor fields. To describe them, we require a new non-coordinate basis called tetrad or *vierbein* (*vielbein* in many dimensions), see §12.5 of [37] and §98 of [40], and especially §7.8 of [41] for a review.

The tetrad is defined by a set of 4 independent vector fields $e_a{}^\mu$, where $\mu = 0, \ldots, 3$ are the global coordinate indices and $a = 0, \ldots, 3$ is the index labeling the vectors in the tetrad. In particular, tetrads are useful because they satisfy the relation

$$g_{\mu\nu} e_a{}^\mu e_b{}^\nu = \eta_{ab}, \quad (2.15)$$

so that the label $a = 0, \ldots, 3$ becomes a label in the local tangent space[2]. This way, the tetrad internal indices are lowered/raised with the Minkowski metric,

$$e^{a\mu} \equiv \eta^{ab} e_b{}^\mu, \tag{2.16}$$

and so they define an orthonormal and complete basis,

$$g_{\mu\nu} e^{a\mu} e_b{}^\nu = \delta^a_b, \qquad e_{a\mu} e^{a\nu} = \delta^\nu_\mu. \tag{2.17}$$

Hence, we can write the metric and its inverse in terms of tetrads,

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu, \qquad g^{\mu\nu} = \eta^{ab} e_a{}^\mu e_b{}^\nu. \tag{2.18}$$

Similarly, they can be used to project any tensor to the locally flat tangent space,

$$V^a \equiv e^a{}_\mu V^\mu, \qquad V_a \equiv e_a{}^\mu V_\mu, \tag{2.19}$$

and viceversa,

$$V^\mu \equiv V^a\, e_a{}^\mu, \qquad V_\mu \equiv V_a\, e^a{}_\mu. \tag{2.20}$$

Recapitulating, the tetrads $e_a{}^\mu$ describe curved spacetime effects, but project them to the locally flat space. By contrast, to represent the rotation of the spinning object itself, a distinct body-fixed tetrad frame $e_A{}^\mu(\lambda)$ is used, which follows the worldline of the object. Since the tetrads satisfy the relation

$$e_A{}^\mu = \Lambda^a_A e_a{}^\mu, \tag{2.21}$$

they allow us to finally disentangle the point-particle DoFs, given in the worldline Lorentz matrices $\Lambda^a_A(\lambda)$, from the field DoFs, in $e_a{}^\mu(x)$. This way, tetrads are valid to capture the coupling of gravity to spin.

The last ingredient that we need in order to apply the EFT formalism to the binary inspiral problem is the metric. Although spacetime is 4-dimensional, from the non-relativistic point of view, where the gravitational interaction is instantaneous, time can be regarded as a compact dimension. Therefore, this motivates the use of a temporal Kaluza-Klein (KK) reduction of the metric, see §11 of [42]. As introduced in [43], it adopts the following form,

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = e^{2\phi}(dt - A_i dx^i)^2 - e^{-2\phi}\gamma_{ij} dx^i dx^j, \tag{2.22}$$

where we define the non-relativistic gravitational (NRG) fields: the Newtonian scalar ϕ, the gravito-magnetic vector A_i, which play the reminiscent role of the scalar and vector potentials of electrodynamics, and the tensor field $\gamma_{ij} \equiv \delta_{ij} + \sigma_{ij}$. Here, we just use the KK decomposition for the metric, the usefulness of which will become clear later on, but we do not proceed to the typical KK reduction of the action.

[2] Therefore, tetrads allow for 16 DoFs, accounting for the 10 DoFs of a symmetric metric $g_{\mu\nu}$ and 6 DoFs of Lorentz transformations (rotations + boosts), applying the Lorentz transformation matrices $\Lambda^{\mu\nu}$ of flat spacetime to the local frames in curved spacetime.

At this point, for the derivation of the Feynman rules it will be useful to express all variables in terms of the NRG fields. First, we can expand the metric in terms of them, given in eq. (2) of [44], where we also use the weak field approximation,

$$\begin{aligned} g_{\mu\nu} &= \begin{pmatrix} e^{2\phi} & -e^{2\phi}A_j \\ -e^{2\phi}A_i & -e^{-2\phi}\gamma_{ij} + e^{2\phi}A_iA_j \end{pmatrix} \\ &\simeq \begin{pmatrix} 1+2\phi+2\phi^2 & -A_j - 2A_j\phi \\ -A_i - 2A_i\phi & -\delta_{ij} + 2\phi\delta_{ij} - \sigma_{ij} - 2\phi^2\delta_{ij} + 2\phi\sigma_{ij} + A_iA_j \end{pmatrix} = \eta_{\mu\nu} + h_{\mu\nu}, \end{aligned} \tag{2.23}$$

as well as its inverse, given in eq. (3) of [44],

$$\begin{aligned} g^{\mu\nu} &= \begin{pmatrix} e^{-2\phi} - e^{2\phi}\gamma^{ij}A_iA_j & -e^{2\phi}A^j \\ -e^{2\phi}A^i & -e^{2\phi}\gamma^{ij} \end{pmatrix} \\ &\simeq \begin{pmatrix} 1-2\phi+2\phi^2 - A_kA_k & -A_j - 2\phi A_j + \sigma_{jk}A_k \\ -A_i - 2\phi A_i + \sigma_{ik}A_k & -\delta_{ij} - 2\phi\delta_{ij} + \sigma_{ij} - 2\phi^2\delta_{ij} + 2\phi\sigma_{ij} - \sigma_{il}\sigma_{lj} \end{pmatrix}, \end{aligned} \tag{2.24}$$

where γ^{ij} is the inverse of γ_{ij}, defined by $\gamma^{ij}\gamma_{jk} \equiv \delta^i_k$, and $A^i \equiv \gamma^{ij}A_j$. Using them, we can also calculate the determinant

$$\sqrt{-g} = \sqrt{1+\eta^{\mu\rho}h_{\mu\rho}} = 1 + \frac{1}{2}h_{00} - \frac{1}{2}\delta^{ij}h_{ij} = 1 - 2\phi + \ldots, \tag{2.25}$$

which appears in the measure of the Levi-Civita tensor density in curved spacetime,

$$\varepsilon_{\alpha\beta\gamma\mu} \equiv \sqrt{-g}\,\epsilon_{\alpha\beta\gamma\mu}, \tag{2.26}$$

where $\epsilon_{\alpha\beta\gamma\mu}$ is the totally antisymmetric Levi-Civita symbol with $\epsilon_{0123} = +1$, as well as the 4-velocity contraction

$$\sqrt{u^2} = \sqrt{g_{\mu\nu}u^\mu u^\nu} = \sqrt{1 - v^2 + 2\phi - 2A_iv^i + \ldots} = 1 - \frac{1}{2}v^2 + \phi - A_iv^i + \ldots. \tag{2.27}$$

Secondly, we require the tetrad expressed in terms of the NRG fields, as given in eq. (5.8) of [24],

$$e^a{}_\mu = \begin{pmatrix} e^\phi & -e^\phi A_i \\ 0 & e^{-\phi}\sqrt{\gamma_{ij}} \end{pmatrix}, \tag{2.28}$$

which obeys the Schwinger time gauge $\tilde{e}_{(i)}{}^0(x) = 0$, given in eq. (5.7) of [24], and from which we can also calculate:

$$\begin{aligned} e^{\mu a} &= g^{\mu\rho}\,e_\rho{}^a = g^{\mu\rho}\,(e^a{}_\rho)^\top \\ &\simeq \begin{pmatrix} 1-2\phi & -A_j \\ -A_i & -\delta_{ij} - 2\phi\delta_{ij} \end{pmatrix} \begin{pmatrix} 1+\phi & 0 \\ -A_i & \delta_{ij} - \phi\delta_{ij} \end{pmatrix} \simeq \begin{pmatrix} 1-\phi & -A_j \\ 0 & -\delta_{ij} - \phi\delta_{ij} \end{pmatrix}. \end{aligned} \tag{2.29}$$

This expressions will be used when projecting the spin variables onto the locally flat frame. With that, we are able to separate all field dependence, which goes with the tetrad, from the spin variable defined in flat space. Similarly, covariant derivatives are also projected onto the locally flat frame using the tetrad fields.

3 Outline

Addressing a new sector in PN gravity is not at all simple, since it directly builds on lower-order sectors, and it also grows in scale and complexity. Therefore, in order to attempt the computation at NLO S^3 we had to first master all previous sectors: from the LO Newtonian sector without spin to the LO S^3 and to the NLO S^2 corrections. For that, during the thesis, the exact reproduction of the results in the articles [1, 19, 24, 28, 29, 45] has been necessary, in order to later apply and expand their formalism for the first time to NLO S^3. Since the calculations are very complex and require a meticolous and thorough examination, even though some parts have also been computed by hand, the necessity of a computer program becomes imperative. With this objective, a code programmed in *Wolfram Mathematica* [46] has also been developed from scratch, including all calculations from the Newtonian to the NLO S^3 order present in the thesis. The code is completely analytic, and it builds on the *xTensor* package used for abstract computer tensor algebra, present in the *xAct* bundle [47], regularly used in GR. Additionally, the derivation of the Feynman rules and the evaluation of the respective diagrams is also available in the public EFTofPNG code [48], used in this thesis to reproduce the NLO S^3 Feynman diagrams first derived in [1].

The outline of the rest of this thesis is as follows. In Part II we present the methodology and formulation required to obtain the NLO S^3 dynamics. Since we need to be proficient in the lower-order sectors, which play an important role in the new results, we will exemplify the methodology deriving the relevant lower-order results, which include up to LO S^3 and to NLO S^2. In sections 4 and 5 we first derive the Feynman rules from the effective action and next evaluate the Feynman diagrams to obtain the interaction potentials for general compact objects. Since these potentials include higher-order time derivatives, in section 6 we perform a redefinition of the position and spin variables to eliminate them, obtaining the reduced potentials. In section 7 we similarly derive the physical equations of motion via proper variations of the reduced action. In section 8 we perform a Legendre transform on the reduced potentials to obtain the Hamiltonian, from which in section 9 the binding energy and its gauge-invariant relations with the angular momentum and with the orbital frequency are be derived. Lastly, in section 10 the Poincaré invariance of the system is addressed, which provides the most stringent self-consistency check in PN gravity.

Once the methodology of Part II has been exemplified and applied to all lower-order sectors, in Part III we present the novel results at NLO S^3. In section 11 we implement the methodology for the first time at NLO S^3, first addressing the calculation of the Feynman rules and diagrams, and then deriving the reduced potential, the equations of motion, the Hamiltonian, the gauge-invariant observables and tackling its Poincaré invariants. Finally, in Part IV we summarize our main conclusions and outlook.

Part II
Methodology

4 Derivation of Feynman rules

In this section we derive the Feynman rules from the effective action for the binary, given in eq. (2.12). We will address separately the different components of the action: the gravitational bulk action, the minimal couplings and the non-minimal couplings, as they will require different treatments.

4.1 Bulk action

Let us first describe the process required to extract the gravitational self-interaction vertices and propagators from the bulk action, given in eq. (2.4).

In order to extract from this action the Feynman rules for the NRG fields, we first need to express the Ricci scalar in terms of them. Nevertheless, the traditional method of calculating first the Christoffel symbols, next the Riemann tensor, then the Ricci tensor and finally the Ricci scalar turns out to be rather cumbersome, due to the non-linearity of the equations that origin from the metric in eq. (2.22). To this end, we can alternatively use Cartan's method of two-forms, see §7.8 of [41], which allows us to compute the curvature in terms of the NRG fields in an analytic and elegant manner. A review and a simple example of the use of Cartan's method of exterior forms is provided in Appendix A.

As first done in [49], or rewritten in our notation in eq. (5) of [44], after calculating the Ricci scalar via Cartan's method, the Einstein-Hilbert action becomes

$$S_{\rm EH} = -\frac{1}{16\pi G}\int dtd^3x\sqrt{\gamma}\left[-R[\gamma_{ij}] + 2\gamma^{ij}\partial_i\phi\partial_j\phi - \frac{1}{4}e^{4\phi}F_{ij}F_{kl}\gamma^{ik}\gamma^{jl}\right], \tag{4.1}$$

where $\gamma \equiv \det(\gamma_{ij})$, $F_{ij} \equiv \partial_i A_j - \partial_j A_i$, and $R[\gamma_{ij}]$ denotes the Ricci scalar for the spatial metric γ_{ij}. Similarly, the gauge-fixing term is expanded in terms of the NRG fields in eq. (4.3) of [49]. Expanding the actions in the weak field regime, we can obtain the propagators in the harmonic gauge, their relativistic time corrections, and the self-interaction vertices.

The propagators are obtained as usual in QFT, inverting the quadratic term in the action in Fourier space. Specifically, they are given in eqs. (2.4)-(2.6) of [28], and read

$$= \langle\phi(x_1)\,\phi(x_2)\rangle = 4\pi G\int_{\vec{k}}\frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{\vec{k}^2}\,\delta(t_1-t_2), \tag{4.2}$$

$$= \langle A_i(x_1)\,A_j(x_2)\rangle = -16\pi G\,\delta_{ij}\int_{\vec{k}}\frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{\vec{k}^2}\,\delta(t_1-t_2), \tag{4.3}$$

$$= \langle\sigma_{ij}(x_1)\,\sigma_{kl}(x_2)\rangle = 32\pi G\,P_{ij;kl}\int_{\vec{k}}\frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{\vec{k}^2}\,\delta(t_1-t_2), \tag{4.4}$$

where $P_{ij;kl} \equiv \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl})$, and where we introduce the abbreviated notation $\int_{\vec{k}} \equiv \int\frac{d^d\vec{k}}{(2\pi)^d}$. Note that a solid line will represent the propagation of the scalar field, a dashed line represents the vector field, and a double solid line that of the tensor field. Other mixed 2-point functions between different fields vanish, namely: $\langle\phi(x_1)\,A_i(x_2)\rangle = \langle\phi(x_1)\,\sigma_{jk}(x_2)\rangle = \langle A_i(x_1)\,\sigma_{jk}(x_2)\rangle = 0$.

The relativistic time corrections to the propagators arise because, as we will work in non-relativistic gravity, we want propagators to be instantaneous. Nonetheless, the propagators of the orbital field modes are in fact given by

$$\int \frac{d^4k}{(2\pi)^4} e^{-ikx}\frac{1}{k^2} = \int \frac{d^4k}{(2\pi)^4} e^{-ik_0t+i\vec{k}\cdot\vec{x}}\frac{1}{k_0^2-\vec{k}^2}. \tag{4.5}$$

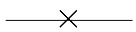
Expanding the denominator in the PN approximation, since $|\vec{k}| \sim \frac{1}{r}$ while $k_0 \sim \frac{v}{r}$,

$$\frac{1}{k_0^2-\vec{k}^2} = -\frac{1}{\vec{k}^2}\left(\frac{1}{1-\frac{k_0^2}{\vec{k}^2}}\right) = -\frac{1}{\vec{k}^2}\Big(1+\mathcal{O}(v^2)\Big), \tag{4.6}$$

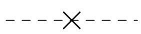
we have that to lowest order in the velocity, the propagator becomes

$$\int \frac{d^4k}{(2\pi)^4} e^{-ik_0t+i\vec{k}\cdot\vec{x}}\frac{1}{\vec{k}^2} = \int \frac{dk_0}{2\pi}e^{-ik_0t}\int\frac{d^3\vec{k}}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\frac{1}{\vec{k}^2} = \delta(t)\int\frac{d^3\vec{k}}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\frac{1}{\vec{k}^2}. \tag{4.7}$$

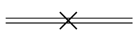
Therefore, we obtain the desired instantaneous propagator at LO, where the momenta is only in 3 dimensions. This detail will have to be taken into account later on, as we will have to set $d = 3$ in dimensional regularization. The velocity corrections to the instantaneous propagator, or relativistic time corrections, present in eq. (4.6), are then treated as self-gravitational quadratic vertices. They are expressed as follows, as given in eqs. (2.7)-(2.9) of [28],



$$= \frac{1}{8\pi G}\int d^4x\,(\partial_t\phi)^2, \tag{4.8}$$



$$= -\frac{1}{32\pi G}\int d^4x\,(\partial_t A_i)^2, \tag{4.9}$$

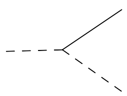


$$= \frac{1}{128\pi G}\int d^4x\,\Big[2(\partial_t\sigma_{ij})^2-(\partial_t\sigma_{ii})^2\Big], \tag{4.10}$$

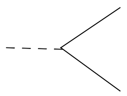
where the relativistic time correction is represented diagrammatically by a cross on a propagator.

The self-interactions, as well as the diagrams later on, will be given in position space rather than in momentum space. This is because it is more natural for the binary inspiral problem, in which the positions of the 2 bodies are specified, rather than the momenta of the particles as in usual QFT computations for collider physics.

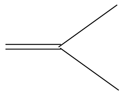
Finally, expanding the bulk actions of eq. (4.1) we also obtain n-graviton self-interaction vertices. In particular, some cubic self-interactions are given by



$$= \frac{1}{8\pi G}\int d^4x\,\phi\Big[\partial_iA_j(\partial_iA_j-\partial_jA_i)+(\partial_iA_i)^2\Big], \tag{4.11}$$



$$= -\frac{1}{4\pi G}\int d^4x\,A_i\partial_i\phi\partial_t\phi, \tag{4.12}$$



$$= \frac{1}{16\pi G}\int d^4x \left[2\sigma_{ij}\partial_i\phi\partial_j\phi - \sigma_{jj}\partial_i\phi\partial_i\phi\right], \tag{4.13}$$

but more interactions are gathered in eqs. (2.10)-(2.17) of [28]. Self-interactions are natural in our theory, as we are computing GR corrections to Newtonian gravity, and General Relativity is known for being a highly non-linear theory.

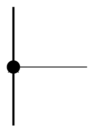
4.2 Minimal couplings

In order to obtain the Feynman rules for the graviton couplings to the worldline mass and spin dipole, we start from the minimal coupling part of the point-particle action, given in eq. (2.9). Choosing the parametrization $\lambda = x^0 = t$, so that $u^0 = 1$ and $u^i = \frac{dx^i}{dt} = v^i$, the non-spinning minimal coupling part reads

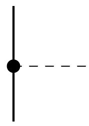
$$-m\int dt\sqrt{u^2} = -m\int dt\sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} = -m\int dt\left[e^{\phi}\sqrt{(1-A_iv^i)^2 - e^{-4\phi}\gamma_{ij}v^iv^j}\right], \tag{4.14}$$

which can be expanded in the velocities and in the NRG fields to obtain an infinite number of worldline couplings. Here, we use a classical source (the mass) to source a quantum field (the gravitational field), although at the end we will only consider its classical contributions.

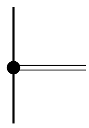
For instance, we obtain the following one-graviton couplings to the worldline mass, or monopole couplings, represented by a black dot, given to NLO in eqs. (2.19)-(2.21) of [28]:



$$= -m\int dt\,\phi\left[1 + \frac{3}{2}v^2 \dots\right], \tag{4.15}$$

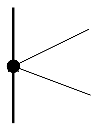


$$= m\int dt\,A_iv^i\left[1 + \frac{1}{2}v^2 + \dots\right], \tag{4.16}$$



$$= \frac{1}{2}m\int dt\,\sigma_{ij}v^iv^j\left[1 + \dots\right], \tag{4.17}$$

where the vertical line represents the worldline or trajectory of the classical source, which is one of the components of the binary, and where the ellipses indicate higher orders in v. For couplings with higher number of gravitons, see eqs. (2.22)-(2.26) of [28]. As an example, at NLO we have the 2-graviton scalar coupling



$$= -\frac{1}{2}m\int dt\,\phi^2\left[1 - \frac{9}{2}v^2 + \dots\right]. \tag{4.18}$$

Here we already start noting the benefits of the KK decomposition: There is a PN hierarchy in the coupling of the graviton fields to the mass, making the scalar ϕ dominant with respect to the vector A_i, and to the tensor field σ_{ij}. Therefore, depending on the order in velocity desired, there will be fields that will not contribute, simplifying the calculations. When spin is included, the hierarchy is not so explicit within the spin dipole, but there will also be dominant fields, as we will see.

From the non-spinning minimal coupling we also obtain kinetic terms that do not contain fields, such as

$$L_{\rm kin} = \frac{1}{2}mv^2 + \frac{1}{8}mv^4 + \frac{1}{16}mv^6 + \dots, \tag{4.19}$$

where we can identify the leading Newtonian contribution and its 1PN and 2PN corrections.

Similarly, from the minimal coupling part for spin we obtain Feynman rules. Nevertheless, here special care has to be taken with the spin variables, as explained in [24, 44]. First of all, we need to address the gauge freedom in the spin variable. In general, the spin tensor is commonly gauge-fixed using the covariant spin supplementary condition (SSC) $S_{\mu\nu}p^\nu = 0$, for $p^\mu = m\frac{u^\mu}{\sqrt{u^2}} + \mathcal{O}(R)$, as introduced in [50] and later extended to curvature-dependent higher-multipoles in [51]. Then, as explained in §3.2 of [24], we can relate it to the generic spin variable $\hat{S}^{\mu\nu}$ by the shift

$$S^{\mu\nu} = \hat{S}^{\mu\nu} - \hat{S}^{\mu\rho}\frac{u_\rho u^\nu}{u^2} + \hat{S}^{\nu\rho}\frac{u_\rho u^\mu}{u^2}. \tag{4.20}$$

Using the previous equation, we have that to leading order in the velocity,

$$S^{0k} = - \hat{S}^{kl}v^l, \tag{4.21}$$

$$S^{jk} = \hat{S}^{jk} + \frac{1}{2}\hat{S}^{jl}v^l v^k - \frac{1}{2}\hat{S}^{kl}v^l v^j. \tag{4.22}$$

Now, the spin vector is related to the spin tensor by the following expression, see eq. (2.13) of [19],

$$S_\mu = -\frac{1}{2}\epsilon_{\mu\nu\gamma\delta}S^{\nu\gamma}\frac{p^\delta}{\sqrt{p^2}}, \tag{4.23}$$

which acts as a classical version of the Pauli-Lubanski pseudovector $S^\mu \equiv -\frac{1}{2m}\epsilon^{\mu\nu\gamma\delta}J_{\nu\gamma}P_\delta$, with $J_{\nu\gamma}$ being the relativistic angular momentum tensor and P_δ the four-momentum. Thus, we find that to lowest order in the velocity,

$$S_0 = - \hat{S}_{[k]}v^k, \tag{4.24}$$

$$S_i = \hat{S}_{[i]} + \frac{1}{2}\hat{S}_{[l]}v^l v^i, \tag{4.25}$$

where we defined the Newtonian (or Euclidean) spin using square-bracketed indices, given in the local frame, $\hat{\vec{S}} = \{\hat{S}_{[i]}\}$, by $\hat{S}_{[i]} \equiv \frac{1}{2}\epsilon_{ijk}\hat{S}^{jk}$. As we will express at the end all Feynman rules in terms of the Newtonian spin, we will drop the hat and square indices notation, leaving $\vec{S} = \{S_i\}$. Moreover, we can also relate the spin tensor and the spin vector, both being Newtonian now, by

$$S^{ij} = \epsilon^{ijk}S_k, \qquad S^i = \frac{1}{2}\epsilon^{ijk}S_{jk}. \tag{4.26}$$

With the previous considerations, one can rewrite the minimal coupling part for spin $S_{{\rm pp}(S)}$ introducing the Ricci rotation coefficients, see eq. (50) of [44], defined by

$$\omega_\mu^{ab} = e^{b\nu}D_\mu e^a_\nu, \tag{4.27}$$

so that it reads

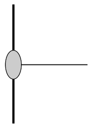
$$S_{\mathrm{pp(S)}} = \int d\lambda \left[-\frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu} \right] = \int d\lambda \left[-\frac{1}{2} \hat{S}_{ab} \Omega^{ab} - \frac{1}{2} \hat{S}_{ab} \omega_\mu^{ab} u^\mu - \frac{\hat{S}_{ab} p^b}{p^2} \frac{D p^a}{D\lambda} \right], \tag{4.28}$$

where the first term on the right-hand side is a flat spacetime rotation term. Expanding this expression, one can derive the following kinetic terms with one spin vector, given in eq. (2.27) of [28], which should be added to the potentials of the relevant spinning sectors,

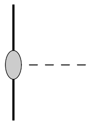
$$L_{\mathrm{kin}} = -\vec{S} \cdot \vec{\Omega} + \frac{1}{2} \vec{S} \cdot \vec{v} \times \vec{a} \left(1 + \frac{3}{4} v^2 + \frac{5}{8} v^4 + \dots \right), \tag{4.29}$$

where $a^i \equiv \dot{v}^i$ is the orbital acceleration.

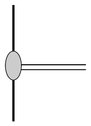
Moreover, we can extract the following one-graviton couplings with the worldline spin dipole, which again acts as a classical source of the field, given to NLO in eqs. (2.28)-(2.36) of [28]:



$$= \int dt \, \epsilon_{ijk} S_k v^i \left(2 \partial_j \phi + v^2 \partial_j \phi - 2 a^j \phi \right), \tag{4.30}$$



$$= \int dt \, \epsilon_{ijk} S_k \left(\frac{1}{2} \partial_i A_j + \frac{3}{4} v^i v^l (\partial_l A_j - \partial_j A_l) + v^i \partial_t A_j \right), \tag{4.31}$$



$$= \int dt \, \frac{1}{2} \epsilon_{ijk} S_k v^l \partial_i \sigma_{jl}, \tag{4.32}$$

where the gray oval represents the spin dipole on the worldline. Feynman rules with a higher number of graviton couplings can be found in §2 of [52].

4.3 Non-minimal couplings

Now let us consider the spin-induced non-minimal couplings, from which the higher-spin Feynman rules can be extracted, and which up to cubic-in-spin order are given in eqs. (2.10), (2.11). In order to obtain the Feynman rules, we just have to express all variables in terms of the NRG fields, and expand up to the desired order in the fields and in the velocities. For that, we can directly employ the expansions given in eqs. (2.23)-(2.29).

Using also the following expressions, valid in the weak gravitational field regime, see eq. (6.6.2) of [37], for the Riemann tensor,

$$\begin{aligned} R_{\alpha\beta\delta\nu} = \frac{1}{2} &\left[\partial_\beta \partial_\delta h_{\alpha\nu} + \partial_\alpha \partial_\nu h_{\beta\delta} - \partial_\beta \partial_\nu h_{\alpha\delta} - \partial_\alpha \partial_\delta h_{\beta\nu} \right] \\ &+ \eta_{\rho\sigma} \left[\Gamma^\rho_{\alpha\nu} \Gamma^\sigma_{\beta\delta} - \Gamma^\rho_{\alpha\delta} \Gamma^\sigma_{\beta\nu} \right] + \mathcal{O}(h^3), \end{aligned} \tag{4.33}$$

and (10.1.3) of [37] for the Christoffel symbol,

$$\Gamma^\rho_{\alpha\nu} = \frac{1}{2} \eta^{\rho\sigma} \left(\partial_\nu h_{\alpha\sigma} + \partial_\alpha h_{\nu\sigma} - \partial_\sigma h_{\alpha\nu} \right) + \mathcal{O}(h^2), \tag{4.34}$$

we have all the prescriptions necessary to evaluate the non-minimal higher-spin Feynman rules.

As a detailed example, we can derive the LO coupling of one scalar to the worldline octupole. For that, we start from eq. (2.11), which fully expanded and with the spin vectors projected onto the locally flat frame becomes

$$L_{\mathrm{BS}^3} = -\frac{C_{\mathrm{BS}^3}}{12m^2}\frac{1}{\sqrt{u^2}}D_\lambda\Big[\sqrt{-g}\,\epsilon_{\alpha\beta\gamma\mu}\,g^{\alpha\alpha'}g^{\beta\beta'}R_{\alpha'\beta'\delta\nu}\,u^\gamma u^\delta\Big]e^{\mu a}\,S_a\,e^{\nu b}\,S_b\,e^{\lambda c}\,S_c. \tag{4.35}$$

Now, to simplify the derivation we can realize that some quantities can be taken out of the covariant derivative. First of all, the 4-velocity u^μ is a worldline parameter, and so it can be pulled out. Similarly, the covariant derivative of the metric vanishes, see eqs. (4.6.16)-(4.6.17) of [37], i.e.,

$$D_\lambda\, g_{\mu\nu} = 0, \qquad\qquad D_\lambda\, g^{\mu\nu} = 0, \tag{4.36}$$

and the metric can also be taken out. Secondly, from eq. (4.4.2) of [37], we have the following transformation rule for the determinant of the metric under a coordinate transformation $x' \to x$:

$$\sqrt{-g'} = \left|\left|\frac{\partial x}{\partial x'}\right|\right|\sqrt{-g}, \tag{4.37}$$

which also means that $\sqrt{-g}$ is a scalar density of weight $w = 1$. Consequently, from eq. (4.4.9) of [37], we have the transformation rule for the Levi-Civita tensor density,

$$\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\frac{\partial x'^\eta}{\partial x^\lambda}\frac{\partial x'^\xi}{\partial x^\kappa}\varepsilon^{\mu\nu\lambda\kappa} = \left|\frac{\partial x'}{\partial x}\right|\varepsilon^{\rho\sigma\eta\xi}, \tag{4.38}$$

and so it is a tensor density of weight $w = -1$. Then, by virtue of the general form for the covariant derivative of a tensor density $\mathcal{J}^{\cdots}_{\cdots}$ with arbitrary indices and weight w, given in eq. (4.6.11) of [37],

$$D_\lambda\,\mathcal{J}^{\cdots}_{\cdots} \equiv g^{-w/2}D_\lambda\left(g^{w/2}\mathcal{J}^{\cdots}_{\cdots}\right), \tag{4.39}$$

we have that for the Levi-Civita tensor density $\varepsilon_{\alpha\beta\mu\nu} = \sqrt{-g}\,\epsilon_{\alpha\beta\mu\nu}$,

$$D_\lambda\Big(\sqrt{-g}\,\epsilon_{\alpha\beta\mu\nu}\Big) = (-g)^{1/2}D_\lambda\left((-g)^{-1/2}\sqrt{-g}\,\epsilon_{\alpha\beta\mu\nu}\right) = \sqrt{-g}\,D_\lambda\epsilon_{\alpha\beta\mu\nu} = 0, \tag{4.40}$$

which vanishes since $\epsilon_{\alpha\beta\mu\nu}$ is the Levi-Civita symbol, a constant. Hence, it can also be taken out of the covariant derivative. Therefore, we obtain that eq. (4.35) becomes

$$L_{\mathrm{BS}^3} = -\frac{C_{\mathrm{BS}^3}}{12m^2}\frac{1}{\sqrt{u^2}}\sqrt{-g}\,\epsilon_{\alpha\beta\gamma\mu}\,g^{\alpha\alpha'}g^{\beta\beta'}D_\lambda\Big(R_{\alpha'\beta'\delta\nu}\Big)u^\gamma u^\delta e^{\mu a}\,e^{\nu b}\,e^{\lambda c}\,S_a\,S_b\,S_c. \tag{4.41}$$

Now, since we want the LO Feynman rule, we can expand to leading order in the velocity. In addition, since we want the one-graviton coupling, we can neglect all quadratic (or higher) terms in the fields. For that, we can realize that for the Riemann tensor not to be zero, it has to contain derivatives of the gravitational fields, so it is the term containing the desired field dependence, and all other contributions with fields can be neglected. Thus, we can already substitute the covariant derivative $D_\lambda = \partial_\lambda + \mathcal{O}(\phi)$ by the partial derivative.

Then, using the equations (2.29), (4.24) and (4.25), we can realize that for $\lambda = 0$ we would obtain a velocity dependence $e^{0c}S_c = -\hat{S}_{[i]}v^i$ and $\partial_t \sim \frac{v}{r}$, whereas for $\lambda = i$ we would have $e^{ic}S_c = -\hat{S}_{[i]}$ and $\partial_i \sim \frac{1}{r}$. So, we can keep $\lambda = i$ to leading order in the velocities. With similar considerations and expanding all contractions, one can obtain that

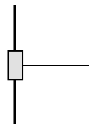
$$L_{\mathrm{BS}^3} = \frac{C_{\mathrm{BS}^3}}{12m^2}\epsilon_{kim}\Big[-\partial_n R_{kij0} + v^l\partial_n R_{kilj} + 2v^k\partial_n R_{i0j0}\Big] S_m S_j S_n, \tag{4.42}$$

where we already dropped the hat and bracketed indices in the Newtonian spin variables, as advertised. Finally, if we require the scalar vertex, we just have to use the metric given in eq. (2.23) in the definition (4.33) for the Riemann tensor, and take all other fields as zero. Specifically, we obtain that

$$R_{i0j0} = -\partial_i\partial_j\phi, \qquad R_{kij0} = \delta_{ij}\partial_k\partial_t\phi - \delta_{kj}\partial_i\partial_t\phi, \tag{4.43}$$

$$R_{kilj} = \delta_{kj}\partial_i\partial_l\phi + \delta_{il}\partial_k\partial_j\phi - \delta_{kl}\partial_i\partial_j\phi - \delta_{ij}\partial_k\partial_l\phi, \tag{4.44}$$

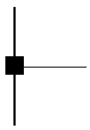
which upon substitution gives, to LO in the velocity, the Feynman rule



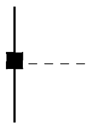
$$= \int dt\, \frac{C_{\mathrm{BS}^3}}{3m^2}\,\epsilon_{klm}\, S_i S_j S_m v^l\, \partial_i\partial_j\partial_k\phi, \tag{4.45}$$

in agreement with eq. (2.7) of [1].

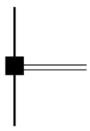
Proceeding analogously, for the quadrupole coupling, given in eq. (2.10), we have the following one-graviton couplings with the worldline spin-squared, given to NLO in eqs. (2.18)-(2.25) of [27]:



$$\begin{aligned} = \int dt\, \frac{C_{\mathrm{ES}^2}}{2m}\Bigg[& S^i S^j \Big(\partial_i\partial_j\phi\Big(1 + \frac{3}{2}v^2\Big) - 3v^i v^k \partial_j\partial_k\phi - 2v^j\partial_i\partial_t\phi\Big) \\ & - S^2\Big(\partial_i\partial_i\phi\Big(1 + \frac{3}{2}v^2\Big) - v^i v^j\partial_i\partial_j\phi + 2v^i\partial_i\partial_t\phi + 2\partial_t^2\phi\Big)\Bigg], \end{aligned} \tag{4.46}$$



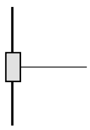
$$= \int dt\, \left(-\frac{C_{\mathrm{ES}^2}}{2m}\right)(S^i S^j - S^2\delta^{ij})\Big(v^k\partial_i\partial_j A_k - v^k\partial_i\partial_k A_j - \partial_i\partial_t A_j\Big), \tag{4.47}$$



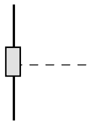
$$\begin{aligned} = \int dt\, \left(-\frac{C_{\mathrm{ES}^2}}{4m}\right)(S^i S^j - S^2\delta^{ij})\Big(& v^k v^l\partial_i\partial_j\sigma_{kl} + v^k v^l\partial_k\partial_l\sigma_{ij} \\ & - 2v^k v^l\partial_i\partial_k\sigma_{jl} - 2v^k\partial_t\partial_i\sigma_{jk} + 2v^k\partial_t\partial_k\sigma_{ij} + \partial_t^2\sigma_{ij}\Big), \end{aligned} \tag{4.48}$$

where the black square represents the electric quadrupolar spin coupling on the worldline.

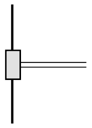
For the octupole coupling, expressed in eq. (2.11), we have the following cubic-in-spin one-graviton couplings, given to NLO in eqs. (2.6)-(2.8) of [1]:



$$= \int dt\, \frac{C_{\mathrm{BS}^3}}{3m^2}\,\epsilon_{klm}\, S_i S_j S_m v^l\Big(\partial_i\partial_j\partial_k\phi\Big(1 + \frac{1}{2}v^2\Big) + v^i\partial_t\partial_j\partial_k\phi\Big), \tag{4.49}$$



$$= \int dt \, \frac{C_{\mathrm{BS}^3}}{12m^2} S_i S_j \epsilon_{klm} \Big[\partial_i \partial_j \partial_l A_k \Big(S_m \Big(1 + \frac{1}{2} v^2 \Big) - \frac{1}{2} v^m S_n v^n \Big) + S_m \Big(v^i \partial_t \partial_j \partial_l A_k + v^l v^n (\partial_n \partial_j \partial_k A_i - \partial_i \partial_j \partial_k A_n) + v^l (\partial_t \partial_i \partial_j A_k + \partial_t \partial_j \partial_k A_i) \Big) \Big], \tag{4.50}$$



$$= \int dt \, \frac{C_{\mathrm{BS}^3}}{12m^2} S_i S_j \epsilon_{klm} S_m \partial_i \partial_l \Big(v^n (\partial_j \sigma_{kn} - \partial_n \sigma_{jk}) - \partial_t \sigma_{jk} \Big), \tag{4.51}$$

where the grey rectangle represents the magnetic octupolar spin coupling on the worldline.

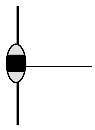
For Feynman rules with a higher number of graviton couplings, as well as higher contributions in the velocity, see §2 of [27] for quadrupole coulpings, eqs. (2.9)-(2.10) of [1] for octupole couplings, and eqs. (2.6)-(2.8) of [30] for higher orders in spin.

As noted for the mass monopole, for the spin multipoles there is also a hierarchy of fields depending on the parity of the multipole: The scalar field ϕ is dominant in the even-parity multipoles, e.g. the mass monopole and the spin quadrupole, while the vector field A_i is dominant for the odd-parity counterparts, like the spin dipole and octupole, and so on. This is a clear reminiscent of the scalar and vector potentials in electrodynamics, where the gravito-static component couples to the monopole and the gravito-magnetic vector component to the dipole. Hence, this hierarchy in the spin couplings also highlights the benefits of the KK decomposition of the gravitational field.

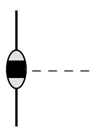
Finally, as first approached in [1], at NLO S^3 we can no longer consider the linear momentum to be given by its leading approximation, but we must add corrections arising from the non-minimal coupling in the action, expressed in eq. (2.10),

$$\begin{aligned} p_\mu = -\frac{\partial L}{\partial u^\mu} &= m \frac{u_\mu}{\sqrt{u^2}} + \mathcal{O}(R) \\ &= m \frac{u_\mu}{\sqrt{u^2}} + \frac{C_{\mathrm{ES}^2}}{2m} S^\rho S^\nu \left[\frac{2}{\sqrt{u^2}} R_{\rho\alpha\nu\mu} u^\alpha - \frac{1}{\sqrt[3]{u^2}} R_{\rho\alpha\nu\beta} u^\alpha u^\beta u_\mu \right]. \end{aligned} \tag{4.52}$$

When this difference is taken into account in eq. (4.28), it creates a new set of special Feynman rules, where we have the following one-graviton couplings, given in eqs. (4.6)-(4.7) of [1],



$$\begin{aligned} = \int dt \, \left(-\frac{C_{\mathrm{ES}^2}}{2m^2} \right) S_i S_j \epsilon_{klm} \Big[& 2 S_m a^k \Big(2 \Big(v^l \partial_i \partial_j \phi - v^j \partial_i \partial_l \phi \Big) \\ & + \delta_{ij} \Big(\partial_t \partial_l \phi + v^n \partial_l \partial_n \phi \Big) \Big) - \dot{S}_m v^k \Big(2 v^j \partial_i \partial_l \phi - \delta_{ij} \Big(\partial_t \partial_l \phi + v^n \partial_l \partial_n \phi \Big) \\ & + \delta_{il} \Big(\partial_t \partial_j \phi + v^n \partial_j \partial_n \phi \Big) \Big) \Big], \end{aligned} \tag{4.53}$$



$$= \int dt \, \frac{C_{\mathrm{ES}^2}}{4m^2} S_i S_j \epsilon_{klm} \Big[\Big(2 S_m a^k + \dot{S}_m v^k \Big) \Big(\partial_i \partial_j A_l - \partial_i \partial_l A_j \Big) \Big], \tag{4.54}$$

where the black square superposed on top of a gray oval represents the new cubic-in-spin coupling. Feynman rules with a higher number of graviton exchanges can be found also in eqs. (4.8)-(4.9) of [1].

5 Evaluation of Feynman diagrams

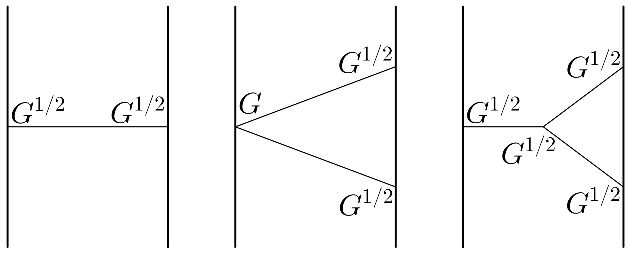
In this section we describe the method performed to evaluate the Feynman diagrams from the previously derived Feynman rules, in order to obtain the interaction potentials. As explained in §2.1.2, the diagrams arise in the second stage in the tower of EFTs for the conservative binary problem, when we integrate out the orbital field modes. First we will go over the bare topologies, which serve as the starting point to obtain the dressed diagrams. Once the prescription utilized to evaluate the non-spinning diagrams is presented, the generalization to spinning sectors will naturally arise. The evaluation of the diagrams is also automated in the public EFTofPNG code [48].

5.1 Bare topologies

To begin with, we remark that we will calculate graphs that contribute to two-particle interaction amplitudes, hence we have 2 worldlines for the 2 components of the binary, which are to be interchanged, as the gravitational interaction is symmetric under exchange. Moreover, some graph topologies are already excluded from the diagrammatic expansion of the theory [13]. Specifically, the graphs excluded consist in graphs with separate connectivity components, because the effective action $e^{iS_{\text{eff}}}$ is defined in the exponent; graphs with graviton loops, as we consider classical gravity with no internal purely quantum loops; and graphs that renormalize the Wilson coefficients. Then, graphs are drawn in position space with the direction of time flowing upwards, in accordance with spacetime representations in relativity, as opposed to Feynman diagrams for particle physics, in which time flows from left to right.

Next, in order to identify which topologies can potentially contribute to each PN order in the interaction, it is beneficial to address their power counting, as it will identify the relevant bare (without the specific vertices) topologies for each order. For that, we first have from the propagators in eqs. (4.2)-(4.4) that fields scale as $G^{\frac{1}{2}}$. Hence, each n-graviton coupling to the worldline scales as $G^{\frac{n}{2}}$. Due to an extra G^{-1} in the purely gravitational action, each n-graviton self-interaction vertex thus scales as $G^{\frac{n}{2}-1}$. Counting the powers of G in the bare graphs, we can identify the relevant topologies at each PN order, as it holds that for the nPN order we can have a weight up to G^{n+1} in the bare graphs. For the dressed graphs, which are the full graphs including the specific vertices, it is a matter of counting powers of velocity v, keeping in mind that from the virial theorem it holds that $Gm/r \sim v^2$. In particular, at nPN order we can have dressed graphs with orders G^{n+1}, $G^n v^2$, $\ldots$, Gv^{2n}. The order $G^0 v^{2(n+1)}$ is not present as it would be a purely kinetic term without coupling, not an interaction.

To make the prescription explicit, let us examine the bare topologies contributing to the first PN orders. At Newtonian order, or 0PN order ($\mathcal{O}(G)$), there is a single topology, consisting of a diagram with a one-graviton exchange, drawn in Figure 5.1(a). At 1PN order (up to $\mathcal{O}(G^2)$), we can have three topologies, represented in Figures 5.1(a)-(c): A one-graviton exchange, which is of order G but will contain additional factors of v^2 when dressed, a 2-graviton exchange, and a one-loop diagram with a cubic self-interaction. This loop is not quantum, as it comes from the inherent non-linearity of the theory, and contains



(a) **(b)** **(c)**

Figure 5.1: The single graph topology at $\mathcal{O}(G)$: (a) One-graviton exchange; and the graph topologies at $\mathcal{O}(G^2)$: (b) Two-graviton exchange, (c) Single cubic self-interaction. We also indicate the contribution to the power counting in Newton's constant G for all vertices.

an integral over the time and position of the self-interaction in the bulk. Nevertheless, when we dress the diagrams, since the cubic vertex composed of the scalar field only is of the form $\phi(\partial_t\phi)^2 \sim v^2$, it defers the diagram to the 2PN order, as it would be of order G^2v^2, beyond the allowed 1PN order. This signifies one of the advantages of the KK decomposition that was chosen: It delays higher loop topologies, more complicated to calculate, to higher PN orders.

For topologies of higher PN orders, we refer to §4.2 of the review [13], where the bare topologies are represented up to order G^5. At this order, there are four loops present in the graphs.

5.2 Dressed diagrams

In order to evaluate the diagrams, we dress the previously permitted bare topologies with the Feynman rules of §4, PN-weighted according to the allowed power in v, and then perform the usual quantum field theory techniques [53], which involve Wick contractions, symmetry factors and Feynman integrals. For the integrals, we recall that due to the KK decomposition of the metric, which makes the LO propagator instantaneous, we will take $d = 3$ in dimensional regularization.

To exemplify it, we can calculate the Newtonian interaction, or 0PN order. For that, we require $\mathcal{O}(Gv^0)$, and looking at the only bare topology at $\mathcal{O}(G)$ in Figure 5.1(a), we see that we cannot allow for more powers of v. To this end, and since we have no spin in the Newtonian interaction, we see that only the LO scalar graviton coupling to the mass monopole, given in eq. (4.15), can contribute. Thus, we only have one diagram, represented in Figure 5.2, which contracts as follows:

$$\text{Fig. 5.2} = (-m_1)\int dt_1\, \overline{\phi(x_1) \cdot (-m_2)\int dt_2\, \phi}(x_2), \tag{5.1}$$

where x_I for $I = 1, 2$ denotes the 4-vector specifying the position of each of the components of the binary, and where we have a trivial symmetry factor of 1, and the overline represents

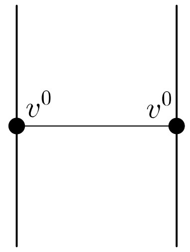


Figure 5.2: The one-graviton exchange diagram representing the Newtonian (0PN) interaction. We also indicate the contribution to the power counting in the velocity v for all vertices.

the Wick contraction between the fields. Using for the contraction the propagator given in eq. (4.2), we have:

$$\text{Fig. 5.2} = 4\pi G m_1 m_2 \int dt_1 dt_2\, \delta(t_1 - t_2) \int_{\vec{k}} \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2} = \int dt\, \frac{G m_1 m_2}{r}, \tag{5.2}$$

where we denote the orbital separation by $r \equiv |\vec{r}| \equiv |\vec{x}_1 - \vec{x}_2|$. Here we can see that time can be integrated straightforwardly, due to the delta function in time in the propagator. To evaluate the last Fourier integral, the following scalar master integral is used:

$$I(\alpha) \equiv \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{r}}}{(\vec{k}^2)^\alpha} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\frac{d}{2} - \alpha)}{\Gamma(\alpha)} \left(\frac{r^2}{4}\right)^{\alpha - \frac{d}{2}}, \tag{5.3}$$

which can be derived using Schwinger parametrization,

$$\frac{1}{A^n} = \frac{1}{\Gamma(n)} \int_0^\infty dx\, x^{n-1} e^{-xA}, \tag{5.4}$$

as given in eq. (4.4.1) of [54], and using the following integral form of the Gamma function:

$$\Gamma(z) \equiv \int_0^\infty dx\, x^{z-1} e^{-x}. \tag{5.5}$$

This way, we obtain from eq. (5.2) the expected classical Newtonian potential,

$$V_{\rm N} = -\frac{G m_1 m_2}{r}. \tag{5.6}$$

Adding the corresponding kinetic term given in eq. (4.19), we thus also obtain the Newtonian Lagrangian for the binary system,

$$L_{\rm N} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{G m_1 m_2}{r}. \tag{5.7}$$

To make the evaluation of higher-order graphs more transparent, we also exemplify the calculation of the 1PN order interaction potential in Appendix B, where we evaluate the

4 diagrams that contribute to both $\mathcal{O}(G^2 v^0)$ and $\mathcal{O}(G v^2)$. As a result, we obtain the first GR correction to Newtonian gravity, given by

$$V_{1\text{PN}} = \frac{G m_1 m_2}{2r}\Big[-3(v_1^2 + v_2^2) + 7\vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \cdot \vec{n}\ \vec{v}_2 \cdot \vec{n}\Big] + \frac{G^2 m_1 m_2}{2r^2}(m_1 + m_2), \tag{5.8}$$

which matches the well-known Einstein-Infeld-Hoffmann correction [55], where $\vec{n} \equiv \vec{r}/r$ is the unit vector in the separation direction. The 1PN Lagrangian then reads

$$L_{1\text{PN}} = \frac{1}{8}m_1 v_1^4 + \frac{1}{8}m_2 v_2^4 + \frac{G m_1 m_2}{2r}\Big[3(v_1^2 + v_2^2) - 7\vec{v}_1 \cdot \vec{v}_2 - \vec{v}_1 \cdot \vec{n}\ \vec{v}_2 \cdot \vec{n}\Big] - \frac{G^2 m_1 m_2}{2r^2}(m_1 + m_2). \tag{5.9}$$

Examples of non-spinning interaction potentials at higher PN orders can be found in [45], with [20, 21] being the highest-order calculations at 5.5PN and 6PN order.

5.3 Higher loops

As mentioned before, in general, at nPN order we first encounter n-loop topologies, although they are deferred to higher order by virtue of the KK decomposition in the non-spinning sectors. Nonetheless, in this section we shall see that there are further properties that can be used to calculate higher loop diagrams.

First of all, we saw that at tree-level (0-loop), the integral over the momentum of the propagator can be calculated using the scalar master integral in eq. (5.3). However, we also saw that there are higher-order topologies arising from the non-linearity of the theory that factorize onto a product of 2 tree integrals, as in the 1PN diagram of Figure B.1(c). For the genuine 1-loop topologies, as the graph in Figure 5.1(c), we require the following 1-loop scalar master integral:

$$\begin{aligned} J(\alpha, \beta) &\equiv \int \frac{d^d \vec{k}}{(2\pi)^d} \frac{1}{\left[\vec{k}^2\right]^\alpha \left[(\vec{k} - \vec{q})^2\right]^\beta} \\ &= \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\alpha + \beta - d/2)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(d/2 - \alpha)\Gamma(d/2 - \beta)}{\Gamma(d - \alpha - \beta)} \left(q^2\right)^{d/2 - \alpha - \beta}, \end{aligned} \tag{5.10}$$

which is obtained using Feynman parametrization,

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha - 1}(1 - x)^{\beta - 1}}{(xA + (1 - x)B)^{\alpha + \beta}}, \tag{5.11}$$

as given in eq. (4.4.10) of [54], and using the following integral form of the Beta function:

$$B(\alpha, \beta) \equiv \int_0^1 dx\, x^{\alpha - 1}(1 - x)^{\beta - 1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}. \tag{5.12}$$

Other extensions of these Feynman integrals, such as the vector and tensor master integrals that can be obtained by taking spatial derivatives, are gathered in appendix A of [52].

However, it is when one goes to 2-loop level that more methods are needed. At order G^3, with 9 different bare topologies shown in Figure 11 of [13], there is one "non-reducible" 2-loop diagram. All of the others factorize onto 3 tree-level integrals, or a tree and a 1-loop, or two 1-loops, or even onto two 1-loops where one is nested in the other.

Then, we can evaluate graphs using standard QFT multiloop techniques, such as Integration-by-parts (IBP) identities [56]. In particular, the "non-reducible" 2-loop topology can be reduced to a sum of factorized or nested 1-loop integrals using the IBP two-loop reduction relation

$$
\begin{aligned}
F(a_1,a_2,a_3,a_4,a_5) &\equiv \int_{\vec{k}_1,\vec{k}_2} \frac{1}{[\vec{k}_1^2]^{a_1}[(\vec{k}-\vec{k}_1)^2]^{a_2}[\vec{k}_2^2]^{a_3}[(\vec{k}-\vec{k}_2)^2]^{a_4}[(\vec{k}_1-\vec{k}_2)^2]^{a_5}} \\
&= \frac{a_1[F(a_1+,a_3-)-F(a_1+,a_5-)]+[1\leftrightarrow 2, 3\leftrightarrow 4]}{a_1+a_2+2a_5-d}, \qquad (5.13)
\end{aligned}
$$

where $F(a_1+,a_3-)\equiv F(a_1+1,a_2,a_3-1,a_4,a_5)$. Thus, at $\mathcal{O}(G^3)$ all topologies still involve only the evaluation of tree or 1-loop integrals. In fact, for the non-spinning sector even at $\mathcal{O}(G^4)$ this holds, and it is only at the next order that new master integrals are required [13].

5.4 Higher-in-spin diagrams

For diagrams that contain couplings with spin, we conduct an analogous procedure, but also taking into account the time dependence in the spin couplings, which upon action of the time derivatives can lead to derivatives of the spin vector, $\dot{\vec{S}}_I \equiv d\vec{S}_I/dt$, $\ddot{\vec{S}}_I \equiv d^2\vec{S}_I/dt^2$, as well as derivatives of the spin length $S^2=\vec{S}\cdot\vec{S}$, such as $(\dot{S}_I^2)$ and $(\ddot{S}_I^2)$.

In addition, we also have to maintain the desired order in the spin multipole coupling. Mainly, for the spin-orbit (denoted as SO) sectors we can only have 1 spin dipole, for the spin1-spin2 (bi-linear in spin, denoted as $\mathrm{S_1S_2}$) sectors we can have 1 spin dipole in each worldline, for the spin-squared[3] (denoted as SS) sectors we can have 1 spin quadrupole or 2 spin dipoles in one worldline, and so on.

Furthermore, odd-in-spin sectors carry an extra power of velocity due to their tensor structure, which will make them much more complex than the even-in-spin sectors in general, as we further elaborate at the end of the section. Moreover, as we are interested in the NLO S^3 sector, the quartic-in-spin and NNLO contributions will be irrelevant, as they have powers of G or spin beyond the relevant order. Hence, they will not be considered throughout the thesis.

Proceeding as described in the previous section, it can be calculated that the LO SO potential, originally computed in [25], with 2 graphs shown in Figure 1 of [44] and rewritten in our convention in eq. (72) there, is given by

$$
V_{\mathrm{SO}}^{\mathrm{LO}} = -2\frac{Gm_2}{r^2}\vec{S}_1\cdot(\vec{v}_1\times\vec{n}-\vec{v}_2\times\vec{n}) - \frac{1}{2}\vec{S}_1\cdot\vec{v}_1\times\vec{a}_1 + (1\leftrightarrow 2), \qquad (5.14)
$$

where we have already added the corresponding kinetic contribution, while the LO $\mathrm{S_1S_2}$ potential, first derived in [26] and reproduced in eq. (11) of [57], with a single graph also shown in Figure 1 there, is

$$
V_{\mathrm{S_1S_2}}^{\mathrm{LO}} = -\frac{G}{r^3}\left[\vec{S}_1\cdot\vec{S}_2 - 3\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right]. \qquad (5.15)
$$

[3] We remark that the quadratic-in-spin contributions are commonly separated in two sectors ($\mathrm{S_1S_2}$ and SS), as the spin-squared sector contains Wilson coefficients coming from the non-minimal coupling, i.e., finite-size effects.

It is significant to remark that under the exchange $1 \leftrightarrow 2$, the unit vector $\vec{n}$ changes sign, $\vec{n} \to -\vec{n}$, and that we can verify the correct PN weight of each expression counting powers of G, v and time derivatives $\partial_t \sim v$. The LO SS potential [26], given in eq. (2.17) of [19] with a single graph in Figure 1 there, becomes

$$V_{\rm SS}^{\rm LO} = -\frac{C_{1({\rm ES}^2)} G m_2}{2 m_1 r^3} \left[S_1^2 - 3(\vec{S}_1 \cdot \vec{n})^2\right] + (1 \leftrightarrow 2), \tag{5.16}$$

whereas the LO S^3 potential, expressed in eqs. (3.1)-(3.4) of [19] and obtained from 4 graphs in Figure 2 there, is

$$\begin{aligned}
V_{\rm S^3}^{\rm LO} = &-3\frac{C_{1({\rm ES}^2)} G}{m_1 r^4}\Big[\vec{S}_2 \cdot (\vec{v}_1 \times \vec{n} - \vec{v}_2 \times \vec{n})\, S_1^2 + 2\vec{S}_1 \cdot \left(\vec{S}_2 \times \vec{v}_1 - \vec{S}_2 \times \vec{v}_2\right) \vec{S}_1 \cdot \vec{n} \\
&- 5\vec{S}_2 \cdot (\vec{v}_1 \times \vec{n} - \vec{v}_2 \times \vec{n})\, (\vec{S}_1 \cdot \vec{n})^2\Big] \\
&+ 3\frac{C_{1({\rm ES}^2)} G}{m_1 r^3}\Big[\vec{S}_1 \cdot \vec{S}_2 \times \vec{n}\, \dot{\vec{S}}_1 \cdot \vec{n} + \dot{\vec{S}}_1 \cdot \vec{S}_2 \times \vec{n}\, \vec{S}_1 \cdot \vec{n}\Big] \\
&- \frac{C_{1({\rm BS}^3)} G m_2}{m_1^2 r^4} \vec{S}_1 \cdot (\vec{v}_1 \times \vec{n} - \vec{v}_2 \times \vec{n}) \Big[S_1^2 - 5(\vec{S}_1 \cdot \vec{n})^2\Big] + (1 \leftrightarrow 2).
\end{aligned} \tag{5.17}$$

Similarly, at the next-to-leading order in the PN approximation, we have that the NLO SO potential [44], with the kinetic term already included, rewritten in eq. (6.18) of [24] using our spin gauge and given by 15 graphs drawn in Figure 2 there, is

$$\begin{aligned}
V_{\rm SO}^{\rm NLO} = &-\frac{G m_2}{r^2}\Big[\vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\left(v_1^2 - 2\vec{v}_1 \cdot \vec{v}_2 + v_2^2 - 3\vec{v}_1 \cdot \vec{n}\, \vec{v}_2 \cdot \vec{n}\right) \\
&+ \vec{S}_1 \cdot \vec{v}_2 \times \vec{n}\left(\vec{v}_1 \cdot \vec{v}_2 - v_2^2 + 3\vec{v}_1 \cdot \vec{n}\, \vec{v}_2 \cdot \vec{n}\right) + \vec{S}_1 \cdot \vec{v}_1 \times \vec{v}_2\, \vec{v}_2 \cdot \vec{n}\Big] \\
&+ \frac{G^2 m_2^2}{2r^3}\Big[\vec{S}_1 \cdot \vec{v}_1 \times \vec{n} - \vec{S}_1 \cdot \vec{v}_2 \times \vec{n}\Big] + \frac{3}{8}\vec{S}_1 \cdot \vec{a}_1 \times \vec{v}_1\, v_1^2 \\
&+ \frac{G m_2}{r}\Big[2\vec{S}_1 \cdot \vec{a}_1 \times \vec{v}_1 - 3\vec{S}_1 \cdot \vec{a}_1 \times \vec{v}_2 + \vec{S}_1 \cdot \vec{a}_2 \times \vec{v}_1 - \vec{S}_1 \cdot \vec{a}_1 \times \vec{n}\, \vec{v}_2 \cdot \vec{n} \\
&- \vec{S}_1 \cdot \vec{a}_2 \times \vec{n}\, \vec{v}_1 \cdot \vec{n} - \dot{\vec{S}}_1 \cdot \vec{v}_1 \times \vec{n}\, \vec{v}_2 \cdot \vec{n} + \dot{\vec{S}}_1 \cdot \vec{v}_2 \times \vec{n}\, \vec{v}_2 \cdot \vec{n} - 3\dot{\vec{S}}_1 \cdot \vec{v}_1 \times \vec{v}_2\Big] \\
&+ G m_2 \dot{\vec{S}}_1 \cdot \vec{a}_2 \times \vec{n} + (1 \leftrightarrow 2).
\end{aligned} \tag{5.18}$$

We can verify that the PN order is now uniform with the presence of velocities and time derivatives, which act on both the position coordinates or the spin variables. The NLO $\mathrm{S}_1\mathrm{S}_2$ potential [57], which is derived from 6 graphs shown in Figure 3 of [24] and can be found in eq. (6.28) there, is given by

$$\begin{aligned}
V_{\rm S_1S_2}^{\rm NLO} = &-\frac{G}{r^3}\Big[\vec{S}_1 \cdot \vec{S}_2\Big(\frac{7}{2}v_1^2 - \frac{15}{2}\vec{v}_1 \cdot \vec{v}_2 + \frac{7}{2}v_2^2 - 6(\vec{v}_1 \cdot \vec{n})^2 + \frac{21}{2}\vec{v}_1 \cdot \vec{n}\, \vec{v}_2 \cdot \vec{n} - 6(\vec{v}_2 \cdot \vec{n})^2\Big) \\
&+ \vec{S}_1 \cdot \vec{v}_1\Big(-\frac{7}{2}\vec{S}_2 \cdot \vec{v}_1 + \frac{5}{2}\vec{S}_2 \cdot \vec{v}_2 + \frac{9}{2}\vec{S}_2 \cdot \vec{n}\, \vec{v}_1 \cdot \vec{n} - \frac{9}{2}\vec{S}_2 \cdot \vec{n}\, \vec{v}_2 \cdot \vec{n}\Big) \\
&+ \vec{S}_1 \cdot \vec{v}_2\Big(\frac{9}{2}\vec{S}_2 \cdot \vec{v}_1 - \frac{7}{2}\vec{S}_2 \cdot \vec{v}_2 - \frac{15}{2}\vec{S}_2 \cdot \vec{n}\, \vec{v}_1 \cdot \vec{n} + 6\vec{S}_2 \cdot \vec{n}\, \vec{v}_2 \cdot \vec{n}\Big) \\
&+ \vec{S}_1 \cdot \vec{n}\left(6\vec{S}_2 \cdot \vec{v}_1\, \vec{v}_1 \cdot \vec{n} - \frac{15}{2}\vec{S}_2 \cdot \vec{v}_1\, \vec{v}_2 \cdot \vec{n} - \frac{9}{2}\vec{S}_2 \cdot \vec{v}_2\, \vec{v}_1 \cdot \vec{n} + \frac{9}{2}\vec{S}_2 \cdot \vec{v}_2\, \vec{v}_2 \cdot \vec{n}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\left(-\frac{9}{2}v_1^2+\frac{21}{2}\vec{v}_1\cdot\vec{v}_2-\frac{9}{2}v_2^2+\frac{15}{2}\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right)\Big]\\
&+2\frac{G^2(m_1+m_2)}{r^4}\Big[\vec{S}_1\cdot\vec{S}_2-4\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\Big]+\frac{G}{2r}\Big[\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2+\dot{\vec{S}}_1\cdot\vec{n}\,\dot{\vec{S}}_2\cdot\vec{n}\Big]\\
&-\frac{G}{r^2}\Big[2\vec{S}_1\cdot\vec{S}_2\,\vec{a}_1\cdot\vec{n}-2\vec{S}_1\cdot\vec{S}_2\,\vec{a}_2\cdot\vec{n}+2\vec{S}_1\cdot\vec{a}_2\,\vec{S}_2\cdot\vec{n}-2\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{a}_1\\
&+2\dot{\vec{S}}_1\cdot\vec{S}_2\,\vec{v}_1\cdot\vec{n}-\frac{1}{2}\dot{\vec{S}}_1\cdot\vec{S}_2\,\vec{v}_2\cdot\vec{n}+\frac{1}{2}\vec{S}_1\cdot\dot{\vec{S}}_2\,\vec{v}_1\cdot\vec{n}-2\vec{S}_1\cdot\dot{\vec{S}}_2\,\vec{v}_2\cdot\vec{n}\\
&+\frac{1}{2}\dot{\vec{S}}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{n}-\frac{1}{2}\vec{S}_1\cdot\vec{n}\,\dot{\vec{S}}_2\cdot\vec{v}_1-2\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_1+\frac{1}{2}\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_2\\
&-\frac{1}{2}\vec{S}_1\cdot\vec{v}_1\,\dot{\vec{S}}_2\cdot\vec{n}+2\vec{S}_1\cdot\vec{v}_2\,\dot{\vec{S}}_2\cdot\vec{n}-\frac{3}{2}\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}+\frac{3}{2}\vec{S}_1\cdot\vec{n}\,\dot{\vec{S}}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\Big],
\end{aligned}
\tag{5.19}
$$

while the NLO SS potential, first approached in [58], and computed with 6 graphs shown in Figure 4 of [24], with a total result given in our convention in eq. (6.40) there, reads as follows:

$$
\begin{aligned}
V_{\rm SS}^{\rm NLO}=&-\frac{C_{1(\rm ES^2)}Gm_2}{2m_1r^3}\Big[S_1^2\left(\frac{5}{2}v_1^2-\frac{9}{2}\vec{v}_1\cdot\vec{v}_2+\frac{3}{2}v_2^2-3(\vec{v}_1\cdot\vec{n})^2+\frac{3}{2}\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right)\\
&+\vec{S}_1\cdot\vec{v}_1\Big(-\vec{S}_1\cdot\vec{v}_1+\vec{S}_1\cdot\vec{v}_2+3\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}-3\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\\
&-3\vec{S}_1\cdot\vec{v}_2\,\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}+(\vec{S}_1\cdot\vec{n})^2\left(-\frac{9}{2}v_1^2+\frac{21}{2}\vec{v}_1\cdot\vec{v}_2-\frac{9}{2}v_2^2+\frac{15}{2}\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right)\Big]\\
&+\frac{C_{1(\rm ES^2)}G^2m_2}{2r^4}\Big[S_1^2-3(\vec{S}_1\cdot\vec{n})^2\Big]+2\frac{C_{1(\rm ES^2)}G^2m_2^2}{m_1r^4}\Big[S_1^2-3(\vec{S}_1\cdot\vec{n})^2\Big]\\
&-\frac{G^2m_2}{r^4}(\vec{S}_1\cdot\vec{n})^2-\frac{C_{1(\rm ES^2)}Gm_2}{m_1r^2}\Big[\vec{S}_1\cdot\vec{a}_1\,\vec{S}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{n}-\frac{3}{2}\dot{\vec{S}}_1\cdot\vec{v}_2\,\vec{S}_1\cdot\vec{n}\\
&+\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1-\frac{3}{2}\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_2-\frac{3}{2}\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}-(\dot{S_1^2})\,\vec{v}_1\cdot\vec{n}+\frac{7}{4}(\dot{S_1^2})\,\vec{v}_2\cdot\vec{n}\Big]\\
&-\frac{C_{1(\rm ES^2)}Gm_2}{m_1r}(\ddot{S_1^2})+(1\leftrightarrow 2).
\end{aligned}
\tag{5.20}
$$

At NLO S^3 we have 53 diagrams, shown in Figures 2-5 of [1], including 4 diagrams containing the new Feynman rules coming from the dependence of the momentum on the spin, as presented at the end of §4.3. The corresponding interaction Lagrangian, which was recently first derived, is written in eqs. (5.1)-(5.22) of [1], and reads as follows:

$$
L_{\rm S^3}^{\rm NLO}=L_{\rm S_1^2S_2}^{\rm NLO}+L_{\rm S_1^3}^{\rm NLO}+(1\leftrightarrow 2),
\tag{5.21}
$$

where we have

$$
\begin{aligned}
L_{\rm S_1^2S_2}^{\rm NLO}=&\frac{G^2}{r^5}L_{(1)}+\frac{C_{1(\rm ES^2)}G}{r^4}\frac{1}{m_1}L_{(2)}+\frac{C_{1(\rm ES^2)}G^2}{r^5}L_{(3)}+\frac{C_{1(\rm ES^2)}G^2}{r^5}\frac{m_2}{m_1}L_{(4)}\\
&+\frac{G^2}{r^4}L_{(5)}+\frac{C_{1(\rm ES^2)}G}{r^3}\frac{1}{m_1}L_{(6)}+\frac{C_{1(\rm ES^2)}G^2}{r^4}L_{(7)}+\frac{C_{1(\rm ES^2)}G^2}{r^4}\frac{m_2}{m_1}L_{(8)}\\
&+\frac{C_{1(\rm ES^2)}G}{r^2}\frac{1}{m_1}L_{(9)}+\frac{C_{1(\rm ES^2)}G}{r}\frac{1}{m_1}L_{(10)},
\end{aligned}
\tag{5.22}
$$

with the following pieces, organized so that more higher-order time derivatives are present in the last terms:

$$\begin{aligned}L_{(1)} =& \frac{1}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\Big(-5\vec{S}_1\cdot\vec{v}_1+\vec{S}_1\cdot\vec{v}_2+9\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}-9\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\\ &+9\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}-\frac{5}{4}\vec{S}_2\cdot\vec{v}_2\times\vec{n}\,\Big(S_1^2+3\big(\vec{S}_1\cdot\vec{n}\big)^2\Big)\\ &+\vec{S}_1\cdot\vec{v}_2\times\vec{n}\,\Big(3\vec{S}_1\cdot\vec{S}_2-12\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\Big)-\frac{1}{4}\vec{S}_2\cdot\vec{v}_1\times\vec{n}\,\Big(7S_1^2-27\big(\vec{S}_1\cdot\vec{n}\big)^2\Big),\end{aligned}\tag{5.23}$$

$$\begin{aligned}L_{(2)} =& -3\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\Big(4\vec{S}_1\cdot\vec{v}_1\,\vec{v}_2\cdot\vec{n}+\vec{S}_1\cdot\vec{v}_2\,\vec{v}_2\cdot\vec{n}\Big)+\frac{9}{2}\vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\,S_1^2\,\vec{v}_1\cdot\vec{n}\\ &+3\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\Big(4\vec{S}_1\cdot\vec{v}_2\,\vec{v}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{n}\,\big(2\vec{v}_1\cdot\vec{v}_2-v_2^2\big)\Big)\\ &-3\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\,\vec{S}_1\cdot\vec{S}_2\,\vec{v}_2\cdot\vec{n}-\frac{15}{2}\vec{v}_1\cdot\vec{v}_2\times\vec{n}\,\vec{S}_2\cdot\vec{v}_1\,\big(\vec{S}_1\cdot\vec{n}\big)^2\\ &+\vec{S}_2\cdot\vec{v}_1\times\vec{n}\,\Big(\frac{15}{2}S_1^2\,\big(v_1^2-\vec{v}_1\cdot\vec{v}_2-2(\vec{v}_1\cdot\vec{n})^2-\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\big)\\ &+3\vec{S}_1\cdot\vec{v}_1\,\big(-2\vec{S}_1\cdot\vec{v}_1+2\vec{S}_1\cdot\vec{v}_2+5\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\big)-15\vec{S}_1\cdot\vec{v}_2\,\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\\ &+\frac{15}{2}\big(\vec{S}_1\cdot\vec{n}\big)^2\,\big(-v_1^2+7\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\big)\Big)+\vec{S}_2\cdot\vec{v}_2\times\vec{n}\,\Big(\frac{3}{2}S_1^2\big(-5v_1^2+4\vec{v}_1\cdot\vec{v}_2-v_2^2\\ &+10(\vec{v}_1\cdot\vec{n})^2+5\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\big)+\frac{15}{2}\big(\vec{S}_1\cdot\vec{n}\big)^2\big(2v_1^2-2\vec{v}_1\cdot\vec{v}_2+v_2^2-7\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\big)\\ &+3\vec{S}_1\cdot\vec{v}_1\,\big(2\vec{S}_1\cdot\vec{v}_1-\vec{S}_1\cdot\vec{v}_2-5\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}+5\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\big)\Big)\\ &-3\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\Big(\vec{S}_1\cdot\vec{S}_2\,\big(v_1^2-\vec{v}_1\cdot\vec{v}_2-5\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\big)+\vec{S}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{v}_1\\ &+\vec{S}_1\cdot\vec{v}_1\,\big(-\vec{S}_2\cdot\vec{v}_1+5\vec{S}_2\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\big)\Big)+3\vec{S}_1\cdot\vec{v}_2\times\vec{n}\,\Big(\vec{S}_1\cdot\vec{S}_2\,\big(v_1^2-5\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\big)\\ &-\vec{S}_1\cdot\vec{v}_1\,\vec{S}_2\cdot\vec{v}_1+15\vec{S}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\Big),\end{aligned}\tag{5.24}$$

$$\begin{aligned}L_{(3)} =& \Big(\vec{S}_1\cdot\vec{v}_1\times\vec{n}-\vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big)\,\vec{S}_1\cdot\vec{S}_2\\ &+\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\Big(-\frac{1}{2}\vec{S}_1\cdot\vec{v}_1+\frac{1}{2}\vec{S}_1\cdot\vec{v}_2-\frac{3}{2}\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}+\frac{3}{2}\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\\ &-\frac{1}{4}\Big(\vec{S}_2\cdot\vec{v}_1\times\vec{n}-\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big)\Big(7S_1^2-15(\vec{S}_1\cdot\vec{n})^2\Big),\end{aligned}\tag{5.25}$$

$$\begin{aligned}L_{(4)} =& 31\Big(\vec{S}_1\cdot\vec{v}_1\times\vec{n}-\vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big)\,\vec{S}_1\cdot\vec{S}_2\\ &-2\Big(\vec{S}_2\cdot\vec{v}_1\times\vec{n}-\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big)\Big(19S_1^2-21\big(\vec{S}_1\cdot\vec{n}\big)^2\Big)\\ &+\vec{S}_1\cdot\vec{S}_2\times\vec{n}\Big(-41\vec{S}_1\cdot\vec{v}_1+41\vec{S}_1\cdot\vec{v}_2+63\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}-66\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big),\end{aligned}\tag{5.26}$$

$$L_{(5)} = 3\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\dot{\vec{S}}_1\cdot\vec{n}-\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}-2\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}+2\vec{S}_1\cdot\dot{\vec{S}}_1\times\vec{S}_2,\tag{5.27}$$

$$
\begin{aligned}
L_{(6)} =& \frac{1}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\dot{\vec{S}}_1\cdot\vec{v}_2 + \frac{1}{2}\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{v}_2 \\
&+ \frac{1}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\left(\dot{\vec{S}}_1\cdot\vec{v}_1 - 2\dot{\vec{S}}_1\cdot\vec{v}_2 - 3\dot{\vec{S}}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n} - 3\dot{\vec{S}}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right) \\
&+ \vec{S}_1\cdot\vec{S}_2\times\vec{a}_1\,\vec{S}_1\cdot\vec{v}_2 + \left(\vec{S}_1\cdot\vec{S}_2\times\vec{a}_2 + \vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{v}_2\right)\left(-\vec{S}_1\cdot\vec{v}_1 + 3\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\right) \\
&- 2\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{v}_1 + 3\vec{S}_1\cdot\vec{v}_2\times\vec{a}_1\,\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n} \\
&+ \frac{1}{2}\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\left(\vec{S}_1\cdot\vec{v}_1 - \vec{S}_1\cdot\vec{v}_2 - 3\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n} - 3\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right) \\
&- \vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\left(\dot{\vec{S}}_1\cdot\vec{S}_1 - 3\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{n}\right) + \left(\vec{S}_2\cdot\vec{v}_2\times\vec{a}_2 + \frac{3}{2}\vec{S}_2\cdot\vec{v}_2\times\vec{a}_1\right) \\
&\left(S_1^2 - 3(\vec{S}_1\cdot\vec{n})^2\right) + \frac{3}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{n}\Big(2\vec{S}_1\cdot\vec{n}\,\vec{a}_1\cdot\vec{v}_2 + \dot{\vec{S}}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{v}_2 - \dot{\vec{S}}_1\cdot\vec{v}_2\,\vec{v}_1\cdot\vec{n} \\
&+ 5\dot{\vec{S}}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big) - \frac{1}{2}\left(\vec{S}_2\cdot\vec{v}_1\times\vec{a}_2 + \dot{\vec{S}}_2\cdot\vec{v}_1\times\vec{v}_2\right)\left(5S_1^2 - 9(\vec{S}_1\cdot\vec{n})^2\right) \\
&+ \frac{3}{2}\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\left(-\vec{S}_1\cdot\vec{v}_2\,\vec{v}_1\cdot\vec{n} + \vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{v}_2 + 5\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right) \\
&+ \frac{3}{2}\vec{S}_2\cdot\vec{v}_1\times\vec{n}\left(S_1^2\,\vec{a}_1\cdot\vec{n} + \dot{\vec{S}}_1\cdot\vec{S}_1\,(2\vec{v}_1\cdot\vec{n} + 4\vec{v}_2\cdot\vec{n}) - 10\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right) \\
&+ \frac{3}{2}\vec{S}_2\cdot\vec{a}_1\times\vec{n}\left(S_1^2\,\vec{v}_1\cdot\vec{n} + S_1^2\,\vec{v}_2\cdot\vec{n} - 5(\vec{S}_1\cdot\vec{n})^2\,\vec{v}_2\cdot\vec{n}\right) \\
&+ \frac{3}{2}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\Big(\vec{S}_1\cdot\vec{S}_2\,\vec{a}_1\cdot\vec{n} + 2\vec{S}_1\cdot\dot{\vec{S}}_2\,\vec{v}_1\cdot\vec{n} + \dot{\vec{S}}_1\cdot\vec{S}_2\,\vec{v}_1\cdot\vec{n} - \dot{\vec{S}}_1\cdot\vec{S}_2\,\vec{v}_2\cdot\vec{n} \\
&- 2\dot{\vec{S}}_2\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1 - \vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{a}_1 - \dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_1\Big) - 3\vec{S}_1\cdot\vec{v}_2\times\vec{n}\,\vec{S}_1\cdot\vec{S}_2\,\vec{a}_1\cdot\vec{n} \\
&+ \vec{S}_2\cdot\vec{v}_2\times\vec{n}\left(\frac{3}{2}(\dot{\vec{S}}_1\cdot\vec{v}_2\,\vec{S}_1\cdot\vec{n} + \vec{S}_1\cdot\vec{v}_2\,\dot{\vec{S}}_1\cdot\vec{n}) - 3\vec{v}_2\cdot\vec{n}\left(3\dot{\vec{S}}_1\cdot\vec{S}_1 - 5\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{n}\right)\right) \\
&- \frac{3}{2}\dot{\vec{S}}_2\cdot\vec{v}_1\times\vec{n}\left(S_1^2\,\vec{v}_1\cdot\vec{n} - 5(\vec{S}_1\cdot\vec{n})^2\,\vec{v}_1\cdot\vec{n}\right) \\
&+ \frac{3}{2}\dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{n}\left(\left(\vec{v}_1\cdot\vec{n} - \vec{v}_2\cdot\vec{n}\right)\vec{S}_1\cdot\vec{S}_2 - \vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_1\right) \\
&+ \frac{3}{2}\left(\vec{S}_2\cdot\vec{a}_2\times\vec{n} + \dot{\vec{S}}_2\cdot\vec{v}_2\times\vec{n}\right)\left(-S_1^2\,\vec{v}_1\cdot\vec{n} + 2\vec{S}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{n} - 5(\vec{S}_1\cdot\vec{n})^2\,\vec{v}_1\cdot\vec{n}\right) \\
&+ \frac{3}{2}\vec{S}_1\cdot\vec{a}_1\times\vec{n}\left(-\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_1 + \vec{S}_1\cdot\vec{S}_2\,\vec{v}_1\cdot\vec{n}\right),
\end{aligned} \tag{5.28}
$$

$$
L_{(7)} = \vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\dot{\vec{S}}_1\cdot\vec{n} + \dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}, \tag{5.29}
$$

$$
L_{(8)} = -4\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\dot{\vec{S}}_1\cdot\vec{n} - 13\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n} - 4\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}, \tag{5.30}
$$

$$
\begin{aligned}
L_{(9)} =& \vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{v}_2\,\dot{\vec{S}}_1\cdot\vec{n} + \dot{\vec{S}}_1\cdot\dot{\vec{S}}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n} + \vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{a}_1\,\vec{S}_1\cdot\vec{n} \\
&+ \vec{S}_1\cdot\vec{S}_2\times\vec{a}_2\,\dot{\vec{S}}_1\cdot\vec{n} + \dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{a}_2\,\vec{S}_1\cdot\vec{n} - \frac{1}{2}\vec{S}_1\cdot\vec{v}_2\times\vec{n}\,\ddot{\vec{S}}_1\cdot\vec{S}_2 \\
&- \frac{1}{2}\dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{n}\,\vec{S}_1\cdot\dot{\vec{S}}_2 - \frac{1}{2}\ddot{\vec{S}}_1\cdot\vec{v}_2\times\vec{n}\,\vec{S}_1\cdot\vec{S}_2 - \dot{\vec{S}}_1\cdot\vec{v}_2\times\vec{n}\,\dot{\vec{S}}_1\cdot\vec{S}_2 \\
&- \frac{1}{2}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2 - \vec{S}_2\cdot\vec{v}_2\times\vec{n}\left(\ddot{\vec{S}}_1\cdot\vec{S}_1 + \dot{\vec{S}}_1\cdot\dot{\vec{S}}_1\right)
\end{aligned}
$$

$$
\begin{aligned}
&- 3\Big(\vec{S}_2\cdot\vec{a}_2\times\vec{n} + \dot{\vec{S}}_2\cdot\vec{v}_2\times\vec{n}\Big)\left(\dot{\vec{S}}_1\cdot\vec{S}_1 - \dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{n}\right)\\
&- 3\dot{\vec{S}}_2\cdot\vec{v}_1\times\vec{n}\,\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{n} - \frac{1}{2}\dot{\vec{S}}_2\cdot\vec{a}_1\times\vec{n}\left(3\big(\vec{S}_1\cdot\vec{n}\big)^2 + S_1^2\right)\\
&- \frac{3}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\ddot{\vec{S}}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n} + \frac{3}{2}\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\,\dot{\vec{S}}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\\
&- 3\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\,\dot{\vec{S}}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n} - \frac{3}{2}\ddot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\\
&+ \frac{3}{2}\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n},
\end{aligned}
\tag{5.31}
$$

$$
L_{(10)} = -\frac{1}{2}\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\,\ddot{\vec{S}}_1\cdot\vec{n} - \dot{\vec{S}}_1\cdot\dot{\vec{S}}_2\times\vec{n}\,\dot{\vec{S}}_1\cdot\vec{n} - \frac{1}{2}\ddot{\vec{S}}_1\cdot\dot{\vec{S}}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n},
\tag{5.32}
$$

and also

$$
\begin{aligned}
L_{\mathrm{S}_1^3}^{\mathrm{NLO}} =& \frac{C_{1(\mathrm{ES}^2)}G^2}{r^5}\frac{m_2}{m_1}L_{[1]} + \frac{C_{1(\mathrm{ES}^2)}G^2}{r^5}\frac{m_2^2}{m_1^2}L_{[2]} + \frac{C_{1(\mathrm{BS}^3)}G}{r^4}\frac{m_2}{m_1^2}L_{[3]}\\
&+ \frac{C_{1(\mathrm{BS}^3)}G^2}{r^5}\frac{m_2}{m_1}L_{[4]} + \frac{C_{1(\mathrm{BS}^3)}G^2}{r^5}\frac{m_2^2}{m_1^2}L_{[5]} + \frac{C_{1(\mathrm{ES}^2)}G}{r^3}\frac{m_2}{m_1^2}L_{[6]}\\
&+ \frac{C_{1(\mathrm{ES}^2)}G^2}{r^4}\frac{m_2}{m_1}L_{[7]} + \frac{C_{1(\mathrm{BS}^3)}G}{r^3}\frac{m_2}{m_1^2}L_{[8]} + \frac{C_{1(\mathrm{BS}^3)}G}{r^2}\frac{m_2}{m_1^2}L_{[9]},
\end{aligned}
\tag{5.33}
$$

with the pieces:

$$
L_{[1]} = \frac{1}{2}\Big(-\vec{S}_1\cdot\vec{v}_1\times\vec{n} + \vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big)\Big(S_1^2 - 9\big(\vec{S}_1\cdot\vec{n}\big)^2\Big),
\tag{5.34}
$$

$$
L_{[2]} = 3\Big(\vec{S}_1\cdot\vec{v}_1\times\vec{n} - \vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big)\left(S_1^2 - 2(\vec{S}_1\cdot\vec{n})^2\right),
\tag{5.35}
$$

$$
\begin{aligned}
L_{[3]} =& \vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\,\vec{S}_1\cdot\vec{v}_2\,\vec{S}_1\cdot\vec{n} + \vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\Big(\frac{1}{2}S_1^2\left(v_1^2 - 2\vec{v}_1\cdot\vec{v}_2 + 2v_2^2 - 5\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right)\\
&+ \vec{S}_1\cdot\vec{v}_1\left(-\vec{S}_1\cdot\vec{v}_1 + \vec{S}_1\cdot\vec{v}_2 + \vec{S}_1\cdot\vec{n}\left(5\vec{v}_1\cdot\vec{n} - 6\vec{v}_2\cdot\vec{n}\right)\right) - 5\vec{S}_1\cdot\vec{v}_2\,\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\\
&- \frac{5}{2}(\vec{S}_1\cdot\vec{n})^2\left(v_1^2 - 2\vec{v}_1\cdot\vec{v}_2 + 2v_2^2 - 7\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right)\Big)\\
&+ \vec{S}_1\cdot\vec{v}_2\times\vec{n}\,\Big(-\frac{1}{2}S_1^2\left(v_2^2 - 5\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right) + \frac{5}{2}(\vec{S}_1\cdot\vec{n})^2\left(v_2^2 - 7\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right)\\
&+ \vec{S}_1\cdot\vec{v}_1\left(\vec{S}_1\cdot\vec{v}_1 - \vec{S}_1\cdot\vec{v}_2 - \vec{S}_1\cdot\vec{n}\,\left(4\vec{v}_1\cdot\vec{n} - 5\vec{v}_2\cdot\vec{n}\right)\right) + 5\vec{S}_1\cdot\vec{v}_2\,\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\Big)\\
&+ \vec{v}_1\cdot\vec{v}_2\times\vec{n}\,\Big(-\big(\vec{S}_1\cdot\vec{n}\big)^2\Big(\vec{S}_1\cdot\vec{v}_1 + \frac{5}{2}\vec{S}_1\cdot\vec{v}_2\Big) + \frac{1}{2}S_1^2\,\vec{S}_1\cdot\vec{v}_2\Big),
\end{aligned}
\tag{5.36}
$$

$$
L_{[4]} = \frac{1}{2}\Big(\vec{S}_1\cdot\vec{v}_1\times\vec{n} - \vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big)\Big(S_1^2 - 5\big(\vec{S}_1\cdot\vec{n}\big)^2\Big),
\tag{5.37}
$$

$$
L_{[5]} = -4\Big(\vec{S}_1\cdot\vec{v}_1\times\vec{n} - \vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big)\Big(S_1^2 - 5\big(\vec{S}_1\cdot\vec{n}\big)^2\Big),
\tag{5.38}
$$

$$
\begin{aligned}
L_{[6]} =3\Bigg[&\left(\vec{S}_1 \cdot \vec{v}_1 \times \vec{a}_1 - \vec{S}_1 \cdot \vec{v}_2 \times \vec{a}_1\right)\left(S_1^2 - 2\big(\vec{S}_1 \cdot \vec{n}\big)^2\right) \\
&+ \vec{S}_1 \cdot \vec{a}_1 \times \vec{n}\, \left(S_1^2 \left(\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}\right) - 2\vec{S}_1 \cdot \vec{n}\left(\vec{S}_1 \cdot \vec{v}_1 - \vec{S}_1 \cdot \vec{v}_2\right)\right)\Bigg] \\
&- \frac{3}{2}\Bigg[\dot{\vec{S}}_1 \cdot \vec{S}_1 \times \vec{v}_1\, \left(\vec{S}_1 \cdot \vec{v}_1 - \vec{S}_1 \cdot \vec{v}_2 - \vec{S}_1 \cdot \vec{n}\,\left(\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}\right)\right) \\
&- \dot{\vec{S}}_1 \cdot \vec{v}_1 \times \vec{n}\, \left(S_1^2 \left(\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}\right) - 2\vec{S}_1 \cdot \vec{n}\left(\vec{S}_1 \cdot \vec{v}_1 - \vec{S}_1 \cdot \vec{v}_2\right)\right) \\
&- \dot{\vec{S}}_1 \cdot \vec{v}_1 \times \vec{v}_2 \left(S_1^2 - 2(\vec{S}_1 \cdot \vec{n})^2\right)\Bigg],
\end{aligned} \tag{5.39}
$$

$$
L_{[7]} = -3\dot{\vec{S}}_1 \cdot \vec{S}_1 \times \vec{n}\, \vec{S}_1 \cdot \vec{n}, \tag{5.40}
$$

$$
\begin{aligned}
L_{[8]} =\frac{1}{6}\Bigg[&2\vec{S}_1 \cdot \vec{v}_1 \times \vec{v}_2 \left(\dot{\vec{S}}_1 \cdot \vec{S}_1 - 3\dot{\vec{S}}_1 \cdot \vec{n}\, \vec{S}_1 \cdot \vec{n}\right) + \dot{\vec{S}}_1 \cdot \vec{v}_1 \times \vec{v}_2 \left(S_1^2 - 3(\vec{S}_1 \cdot \vec{n})^2\right) \\
&+ \vec{S}_1 \cdot \vec{a}_1 \times \vec{v}_2 \left(S_1^2 - 3(\vec{S}_1 \cdot \vec{n})^2\right) + \vec{S}_1 \cdot \vec{v}_1 \times \vec{a}_2 \left(S_1^2 - 3(\vec{S}_1 \cdot \vec{n})^2\right) \\
&- 6\vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \Big(\vec{S}_1 \cdot \vec{v}_1\, \dot{\vec{S}}_1 \cdot \vec{n} + \dot{\vec{S}}_1 \cdot \vec{v}_1\, \vec{S}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{a}_1\, \vec{S}_1 \cdot \vec{n} \\
&- \vec{v}_2 \cdot \vec{n}\,\big(\dot{\vec{S}}_1 \cdot \vec{S}_1 - 5\dot{\vec{S}}_1 \cdot \vec{n}\, \vec{S}_1 \cdot \vec{n}\big)\Big) \\
&- 3\dot{\vec{S}}_1 \cdot \vec{v}_1 \times \vec{n} \left(2\vec{S}_1 \cdot \vec{v}_1\, \vec{S}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n} \left(S_1^2 - 5(\vec{S}_1 \cdot \vec{n})^2\right)\right) \\
&- 3\vec{S}_1 \cdot \vec{a}_1 \times \vec{n} \left(2\vec{S}_1 \cdot \vec{v}_1\, \vec{S}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n} \left(S_1^2 - 5(\vec{S}_1 \cdot \vec{n})^2\right)\right) \\
&+ 6\vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \Big(\vec{S}_1 \cdot \vec{v}_1\, \dot{\vec{S}}_1 \cdot \vec{n} + \dot{\vec{S}}_1 \cdot \vec{v}_1\, \vec{S}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{a}_1\, \vec{S}_1 \cdot \vec{n} \\
&- \vec{v}_2 \cdot \vec{n}\, \left(\dot{\vec{S}}_1 \cdot \vec{S}_1 - 5\dot{\vec{S}}_1 \cdot \vec{n}\, \vec{S}_1 \cdot \vec{n}\right)\Big) \\
&+ 3\dot{\vec{S}}_1 \cdot \vec{v}_2 \times \vec{n} \left(2\vec{S}_1 \cdot \vec{v}_1\, \vec{S}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n} \left(S_1^2 - 5(\vec{S}_1 \cdot \vec{n})^2\right)\right) \\
&+ 3\vec{S}_1 \cdot \vec{a}_2 \times \vec{n} \left(2\vec{S}_1 \cdot \vec{v}_1\, \vec{S}_1 \cdot \vec{n} + \vec{v}_1 \cdot \vec{n} \left(S_1^2 - 5(\vec{S}_1 \cdot \vec{n})^2\right)\right)\Bigg],
\end{aligned} \tag{5.41}
$$

$$
L_{[9]} = -\frac{1}{3}\vec{S}_1 \cdot \vec{a}_2 \times \vec{n} \left(\dot{\vec{S}}_1 \cdot \vec{S}_1 - 3\dot{\vec{S}}_1 \cdot \vec{n}\, \vec{S}_1 \cdot \vec{n}\right) - \frac{1}{6}\dot{\vec{S}}_1 \cdot \vec{a}_2 \times \vec{n} \left(S_1^2 - 3\big(\vec{S}_1 \cdot \vec{n}\big)^2\right). \tag{5.42}
$$

We highlight that the elements $L_{[2]}$ and $L_{[6]}$ exclusively receive contributions from the 4 graphs with new Feynman rules, which do not contribute to any other term.

Examples of higher PN corrections for other high-in-spin results can be found at tree and 1-loop level in [19, 30], where the LO and NLO quartic-in-spin potential were respectively first derived, at 2-loop in [27], where the NNLO spin-squared potential was calculated, and lastly at 3-loop in [31, 32], where the N^3LO spin-orbit and quadratic-in-spin interactions were recently addressed, respectively.

As can be seen, there exists an intrinsic difficulty within the odd-in-spin sectors, which are much more complicated than the even-in-spin counterparts, not only in the number of diagrams but also in the scale and complexity of their expressions, containing triple products that make calculations more cumbersome. This feature of PN theory reflects the fact that classical effects with spin to the lth order correspond to scattering amplitudes with a quantum spin of $s = l/2$ [34, 35], as presented in §1.

6 Elimination of higher-order time derivatives

In the previous section we obtained the interaction potentials for the relevant PN sectors. However, we should note that starting from the LO SO potential, given in eq. (5.14), there appear higher-order time derivatives of both the position, such as $\vec{a}_I = \dot{\vec{v}}_I$, and of the spin variables, as $\dot{\vec{S}}_I$. Hence, a common procedure in perturbative theories is to make a redefinition of the variables so as to eliminate the higher-order terms without altering the physical predictions. Such quantum operators are also known as redundant operators in effective field theories [14], because they vanish by the lower-order equations of motion (they vanish on-shell), making their effects non-physical.

In the context of the PN perturbative scheme, we will shift both the position and spin variables so that the variation of the action entails the PN equation of motion (EoM) of such variable, used to eliminate the higher-order terms at linear approximation only [59–61]. With this procedure, which represents the most subtle and laborious part of the thesis, we will obtain the so-called standard reduced potential, or reduced potential for short, which will no longer contain higher-order time derivatives.

6.1 Elimination of accelerations

More concretely, we will start describing the elimination of accelerations, as done in §5.1 of [45]. Let us consider a Lagrangian containing kinetic and potential terms, that depend on higher-order time derivatives of the position,

$$L(\vec{x}, \vec{v}, \dot{\vec{v}}, \dots) = \sum_{I=1}^{2} \frac{m_I}{2} v_I^2 - V(\vec{x}, \vec{v}, \dot{\vec{v}}, \dots). \tag{6.1}$$

Then, by the variation principle, an infinitesimal variation $\delta\vec{x}_1$ of the action yields the EoM,

$$\delta\left[\int dt L(\vec{x}, \vec{v}, \dot{\vec{v}}, \dots)\right] = 0 \quad \Longrightarrow \quad m_1\vec{a}_1 = -\frac{\partial V}{\partial \vec{x}_1} + \frac{d}{dt}\frac{\partial V}{\partial \vec{v}_1} - \frac{d^2}{dt^2}\frac{\partial V}{\partial \dot{\vec{v}}_1} + \dots, \tag{6.2}$$

and the same for particle 2. Similarly, if we consider a position redefinition $\vec{x}_1 \to \vec{x}_1 + \Delta\vec{x}_1$, it induces a change in the Lagrangian of the form

$$\begin{aligned} \Delta L &= \left[\frac{\partial L}{\partial \vec{x}_1} - \frac{d}{dt}\frac{\partial L}{\partial \vec{v}_1} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \dot{\vec{v}}_1} + \dots\right] \cdot \Delta\vec{x}_1 + \mathcal{O}[(\Delta\vec{x}_1)^2] \\ &= -\left[m_1\vec{a}_1 + \frac{\partial V}{\partial \vec{x}_1} - \frac{d}{dt}\frac{\partial V}{\partial \vec{v}_1} + \frac{d^2}{dt^2}\frac{\partial V}{\partial \dot{\vec{v}}_1} + \dots\right] \cdot \Delta\vec{x}_1 + \mathcal{O}[(\Delta\vec{x}_1)^2]. \end{aligned} \tag{6.3}$$

Therefore, if the acceleration terms in the original Lagrangian take the form

$$La_1 = \vec{A}_1 \cdot \vec{a}_1, \tag{6.4}$$

where $\vec{A}_1$ can itself depend on further accelerations, we can eliminate them by setting in eq. (6.3) the position shift

$$\Delta\vec{x}_1 = \frac{\vec{A}_1}{m_1}. \tag{6.5}$$

With this redefinition, we eliminate the accelerations[4] present in the Lagrangian at a certain PN order by pushing them to higher-orders, which can then be neglected to a certain approximation. At linear order in $\Delta\vec{x}_1$, one would obtain the same result substituting the lower-order EoM into the acceleration terms. Nonetheless, the $\mathcal{O}[(\Delta\vec{x}_1)^2]$ contributions also play an important role, as they would appear in higher-order PN sectors. Therefore, the full position redefinition has to be considered in general. Moreover, if the shift $\Delta\vec{x}_1$ still contains accelerations, then one has to iteratively eliminate them.

6.2 Elimination of time derivatives of spin

Let us now turn our attention to the elimination of time derivatives of spin, which was first addressed in §5.2 of [45]. In this case, we have a Lagrangian of the following form:

$$L(\vec{S},\dot{\vec{S}},\ddot{\vec{S}},\dots) = -\sum_{I=1}^{2}\frac{1}{2}S_I^{ij}\Omega_I^{ij} - V(\vec{S},\dot{\vec{S}},\ddot{\vec{S}},\dots). \tag{6.6}$$

Then, similarly as before, combining an independent infinitesimal variation of the Lorentz rotation matrix Λ_1^{ij} and of the spin S_1^{ij} yields the EoM for the spin, also known as the precession equation, given in eq. (5.9) of [45],

$$\dot{S}_1^{ij} = -4S_1^{k[i}\delta^{j]l}\left[\frac{\partial V}{\partial S_1^{kl}} - \frac{d}{dt}\frac{\partial V}{\partial \dot{S}_1^{kl}} + \dots\right], \tag{6.7}$$

and the same for the spin of particle 2. Analogously, if we consider a spin redefinition $\Delta S_1^{ij} = S_1^{ik}\omega^{kj} - S_1^{jk}\omega^{ki}$ so that the spin is rotated similarly to the rotation matrices, which change as $\Delta\Lambda_1^{ij} = \Lambda_1^{ik}\omega^{kj}$, it induces a contribution in the Lagrangian of the form

$$\Delta L = -\left[\frac{1}{2}\dot{S}_1^{ij} + 2S_1^{ki}\left(\frac{\partial V}{\partial S_1^{kj}} - \frac{d}{dt}\frac{\partial V}{\partial \dot{S}_1^{kj}} + \dots\right)\right]\omega_1^{ij} + \mathcal{O}[(\omega_1^{ij})^2]. \tag{6.8}$$

Therefore, if the terms in the Lagrangian containing $\dot{S}_1^{ij}$ take the form

$$L_{\dot{S}_1} = A_1^{ij}\dot{S}_1^{ij}, \tag{6.9}$$

where A_1^{ij} can itself depend on further time derivatives of spin, we can eliminate them by setting in eq. (6.8) the spin redefinition

$$\omega_1^{ij} = A_1^{ij} - A_1^{ji}. \tag{6.10}$$

With this redefinition, we eliminate the time derivatives[5] of spin present in the Lagrangian. Again, one would obtain the same result substituting the lower-order EoM of spin. Even

[4] To eliminate even higher-order time derivatives like $L\dot{a}_1 = \vec{A}_1\cdot\dot{\vec{a}}_1$, we use a position shift $\vec{x}_1 \to \vec{x}_1 + \tilde{\Delta}\vec{x}_1 = \vec{x}_1 + \frac{d}{dt}\Delta\vec{x}_1$ that itself is a total time derivative. This way, we can flip the total time derivative in eqs. (6.2) and (6.3), and eliminate the higher-order terms by setting $\tilde{\Delta}\vec{x}_1 = -\frac{d}{dt}\Delta\vec{x}_1 = -\frac{d}{dt}\frac{\vec{A}_1}{m_1}$.

[5] Again, to eliminate even higher-order time derivatives like $L_{\ddot{S}_1} = A_1^{ij}\ddot{S}_1^{ij}$, we set a spin redefinition $\omega_1^{ij} = -\frac{d}{dt}\left(A_1^{ij} - A_1^{ji}\right)$ that itself is a total time derivative.

though at first sight the procedure seems analogous to the elimination of accelerations, the necessity to work with the spin tensor makes the application of spin redefinitions technically more difficult.

Finally, once the higher-order time derivatives have been systematically eliminated, the spin variables satisfy the so(3) canonical Poisson bracket [45], given by

$$\{S^{ij}, S^{kl}\} = S^{ik}\delta^{jl} - S^{il}\delta^{jk} + S^{jl}\delta^{ik} - S^{jk}\delta^{il}, \tag{6.11}$$

which allows us to write the precession equation of eq. (6.7) in the form

$$\dot{S}_1^{ij} = \{S_1^{ij}, -V_s\}, \tag{6.12}$$

where V_s stands for the standard potential, which no longer contains higher-order time derivatives.

6.3 Reduced potentials at leading order

As pointed out in the beginning of the section, we can observe that accelerations start appearing as of the LO SO potential, given in eq. (5.14), which we rewrite here for convenience:

$$V_{\text{SO}}^{\text{LO}} = -2\frac{Gm_2}{r^2}\vec{S}_1 \cdot (\vec{v}_1 \times \vec{n} - \vec{v}_2 \times \vec{n}) - \frac{1}{2}\vec{S}_1 \cdot \vec{v}_1 \times \vec{a}_1 + (1 \leftrightarrow 2). \tag{6.13}$$

According to the previous prescription, and following §6.1.1 of [24], we can identify that the shift of the positions required to eliminate the LO SO acceleration terms is

$$\vec{x}_1 \to \vec{x}_1 + (\Delta\vec{x}_1)_{\text{SO}}^{\text{LO}} = \vec{x}_1 + \frac{1}{2m_1}\vec{S}_1 \times \vec{v}_1, \tag{6.14}$$

where we have taken an extra minus sign due to it being the potential, not the Lagrangian. Now, this position shift should be performed onto all sectors of the EFT Lagrangian, which includes lower and higher PN orders, not just the present LO SO potential. This will make different sectors start mixing among them, as a shift in one sector affects others. This interplay between sectors is depicted in Table 6.1, where it is shown the PN sectors at which different orders of the LO SO shift contribute, when applied on different potentials[6].

Hence, starting with the lowest PN order, by eq. (6.3) the change in the Newtonian potential $V_{\text{N}} = -\frac{Gm_1m_2}{r}$, at linear order in the shift (6.14), is given by

$$\Delta V_{\text{SO}}^{\text{LO}} = \frac{1}{2}\vec{S}_1 \cdot \vec{v}_1 \times \vec{a}_1 + \frac{Gm_2}{2r^2}\vec{S}_1 \cdot \vec{v}_1 \times \vec{n} + (1 \leftrightarrow 2). \tag{6.15}$$

We see that it contributes to LO SO and exactly cancels the acceleration terms that we had in the first place, as expected. Therefore, the so-called shifted LO SO potential reads:

$$(V_s)_{\text{SO}}^{\text{LO}} \equiv V_{\text{SO}}^{\text{LO}} + \Delta V_{\text{SO}}^{\text{LO}} = -\frac{3Gm_2}{2r^2}\vec{S}_1 \cdot \vec{v}_1 \times \vec{n} + \frac{2Gm_2}{r^2}\vec{S}_1 \cdot \vec{v}_2 \times \vec{n} + (1 \leftrightarrow 2), \tag{6.16}$$

[6] Instead of computing the contributions separately, all of the following calculations are done in practice just shifting everything at once, which guarantees that all the affected sectors will be taken into account. While our approach requires a careful examination and quite some work, it nevertheless reduces the running time of the code enormously.

	Linear order in $\Delta\vec{x}_{1,2}$	Quadratic order in $\Delta\vec{x}_{1,2}$	Cubic order in $\Delta\vec{x}_{1,2}$
V_{N}	LO SO	NLO S^2	NLO S^3
V_{1PN}	NLO SO		
$V_{\text{SO}}^{\text{LO}}$	NLO S^2	NLO S^3	
$V_{\text{SO}}^{\text{NLO}}$			
$V_{\text{S}^2}^{\text{LO}}$	LO S^3		
$V_{\text{S}^2}^{\text{NLO}}$	NLO S^3		
$V_{\text{S}^3}^{\text{LO}}$			

Table 6.1: PN sectors to which different orders in the LO SO shift $\Delta\vec{x}_{1,2}$ of eq. (6.14) contribute, when applied on the different sectors, up to the relevant NLO S^3 contributions, where any sector beyond NLO or S^3 is omitted here.

which is now free of accelerations.

In fact, we can argue that there are no other contributions to the LO SO sector coming from this shift, as can be seen in Table 6.1. First, if we were to consider the second order in the shift for the Newtonian potential, we would have two spin variables and an extra factor of v^2 coming from $(\Delta\vec{x}_1)^2$, thus contributing to NLO quadratic-in-spin sectors, not to LO SO. Similarly, applying the shift to linear order on the 1PN Lagrangian, in eq. (5.9), creates a contribution to NLO SO, beyond what we need at leading order. Finally, a shift in the LO SO potential itself would contribute to next-to-leading order, and would contain two spin variables.

Consequently, we can also see that the LO quadratic-in-spin potentials given in eqs. (5.15)-(5.16), which did not have accelerations, are neither modified due to the LO SO shift acting on lower-order sectors, nor do they need extra variable redefinitions.

By contrast, the LO S^3 potential of eq. (5.17) is modified by the LO SO shift. In this case, the LO SO shift of eq. (6.14) is applied to linear order on the LO quadratic-in-spin sectors of eqs. (5.15)-(5.16), as shown in Table 6.1, resulting in

$$\begin{aligned}\left[\Delta V_{\text{S}^3}^{\text{LO}}\right]_1 = \frac{3C_{1(\text{ES}^2)}Gm_2}{4m_1^2r^4}\Bigg[\left(\vec{S}_1\cdot\vec{v}_1\times\vec{n}-\frac{m_1}{m_2}\vec{S}_2\cdot\vec{v}_2\times\vec{n}\right)\left(S_1^2-5(\vec{S}_1\cdot\vec{n})^2\right) \\ -2\frac{m_1}{m_2}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}\Bigg]+\frac{3G}{2m_1r^4}\Bigg[\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(\vec{S}_1\cdot\vec{S}_2-5\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right) \\ -\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n}\Bigg]+(1\leftrightarrow2), \qquad (6.17)\end{aligned}$$

as given in eq. (3.8) of [19]. Note that the subscript 1 is a label indicating that this is the first contribution to LO S^3. Thus, adding this contribution we obtain the shifted LO S^3 potential. Nevertheless, we can observe that eq. (5.17) contains time derivatives of spin. Thus, to obtain the reduced potential we require a separate redefinition of the spin variables. Following the prescription given in §6.2, we calculate that we require a spin redefinition

$$(\omega_1^{ij})^{\rm LO}_{\rm S^3} = \frac{3C_{1({\rm ES}^2)}G}{m_1 r^3}\Big(- S_1^{ik}S_2^{jl}n^k n^l + S_1^{jl}S_2^{lk}n^i n^k\Big) - (i \leftrightarrow j). \tag{6.18}$$

To verify this result, we employ eq. (6.8) to obtain that the contribution to the LO $\rm S^3$ potential due to this spin redefinition is given by

$$[\Delta V^{\rm LO}_{\rm S^3}]_2 = -\frac{3C_{1({\rm ES}^2)}G}{m_1 r^3}\Big(\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\dot{\vec{S}}_1\cdot\vec{n} + \dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\Big) + (1\leftrightarrow 2). \tag{6.19}$$

Comparing with eq. (5.17), we see it exactly cancels the terms with time derivatives of spin. To the order we are interested in, this LO $\rm S^3$ spin redefinition will only affect our sector, the NLO $\rm S^3$, as will be addressed later on.

Consequently, the reduced LO $\rm S^3$ potential can also be obtained simply adding eqs. (5.17) and (6.17) and taking $\dot{\vec{S}}_I = 0$, which precisely corresponds to the insertion of the Newtonian EoM for the spin, as anticipated. More concretely, as given in eq. (3.10) of [19], it corresponds to

$$\begin{aligned}
(V_s)^{\rm LO}_{\rm S^3} =& -3\frac{C_{1({\rm ES}^2)}G}{m_1 r^4}\Big[\vec{S}_2\cdot(\vec{v}_1\times\vec{n}-\vec{v}_2\times\vec{n})\,S_1^2 + 2\vec{S}_1\cdot(\vec{S}_2\times\vec{v}_1-\vec{S}_2\times\vec{v}_2)\,\vec{S}_1\cdot\vec{n}\\
&-5\vec{S}_2\cdot(\vec{v}_1\times\vec{n}-\vec{v}_2\times\vec{n})\,(\vec{S}_1\cdot\vec{n})^2 - \frac{1}{4}\Big(-2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}\\
&+\Big(\frac{m_2}{m_1}\vec{S}_1\cdot\vec{v}_1\times\vec{n} - \vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big)\Big(S_1^2 - 5(\vec{S}_1\cdot\vec{n})^2\Big)\Big)\Big]\\
&-\frac{C_{1({\rm BS}^3)}Gm_2}{m_1^2 r^4}\vec{S}_1\cdot(\vec{v}_1\times\vec{n}-\vec{v}_2\times\vec{n})\Big[S_1^2 - 5(\vec{S}_1\cdot\vec{n})^2\Big]\\
&+\frac{3G}{2m_1 r^4}\Big[\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\Big(\vec{S}_1\cdot\vec{S}_2 - 5\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\Big) - \vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n}\Big] + (1\leftrightarrow 2).
\end{aligned} \tag{6.20}$$

6.4 Reduced potentials at next-to-leading order

Let us now proceed to the elimination of higher-order time derivatives at the next-to-leading order for spinning sectors. We will start with the NLO SO sector, with the interaction potential given in eq. (5.18). First of all, we have to add to it the contribution coming from the LO SO position shift of eq. (6.14) acting to linear order on the 1PN potential of eq. (5.8), as depicted in Table 6.1. Following eq. (6.3), it results in

$$\begin{aligned}
\Delta V^{\rm NLO}_{\rm SO} =& \frac{1}{4}\vec{S}_1\cdot\vec{v}_1\times\vec{a}_1\,v_1^2 + \frac{Gm_2}{r}\Big[\frac{3}{2}\vec{S}_1\cdot\vec{v}_1\times\vec{a}_1 - \frac{7}{4}\vec{S}_1\cdot\vec{v}_1\times\vec{a}_2\\
&-\frac{1}{4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\vec{a}_2\cdot\vec{n}\Big] + \frac{Gm_2}{r^2}\Big[\frac{3}{4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,v_1^2 - 2\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\vec{v}_1\cdot\vec{v}_2\\
&+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,v_2^2 + 2\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\,\vec{v}_1\cdot\vec{n} - \frac{3}{2}\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\,\vec{v}_2\cdot\vec{n}\\
&-\frac{3}{4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,(\vec{v}_2\cdot\vec{n})^2\Big] - \frac{G^2m_2(m_1+m_2)}{2r^3}\vec{S}_1\cdot\vec{v}_1\times\vec{n} + (1\leftrightarrow 2),
\end{aligned} \tag{6.21}$$

as given in eq. (6.19) of [24]. Now, since both the potential itself and the lower-order contribution contain higher-order time derivatives, we have to perform further position and spin redefinitions to eliminate them. In particular, the following redefinitions are required, as showed in eqs. (6.20)-(6.21) of [24],

$$(\Delta\vec{x}_1)^{\rm NLO}_{\rm SO} = \frac{v_1^2}{8m_1}\vec{S}_1\times\vec{v}_1 + \frac{Gm_2}{m_1 r}\left(\frac{1}{2}\vec{S}_1\times\vec{v}_1 - 3\vec{S}_1\times\vec{v}_2 - (\vec{v}_2\cdot\vec{n})\,\vec{S}_1\times\vec{n}\right)$$
$$+\frac{G}{r}\left(\frac{11}{4}\vec{S}_2\times\vec{v}_2 - (\vec{v}_2\cdot\vec{n})\,\vec{S}_2\times\vec{n} + \frac{1}{4}\Big(\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big)\,\vec{n}\right) - \dot{\vec{S}}_2\times\vec{n}, \tag{6.22}$$

$$(\omega_1^{ij})^{\rm NLO}_{\rm SO} = \frac{Gm_2}{r}\Big(3v_1^i v_2^j + v_1^i n^j\,(\vec{v}_2\cdot\vec{n}) - v_2^i n^j\,(\vec{v}_2\cdot\vec{n}) - (i\leftrightarrow j)\Big). \tag{6.23}$$

These redefinitions will not affect lower-order sectors, and will first affect the NLO S^3. Using them, the reduced NLO SO potential becomes

$$(V_s)^{\rm NLO}_{\rm SO} = \frac{Gm_2}{r^2}\Bigg[\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(-\frac{1}{8}v_1^2 + 3\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n} - \frac{3}{4}(\vec{v}_2\cdot\vec{n})^2\right)$$
$$+\vec{S}_1\cdot\vec{v}_2\times\vec{n}\left(v_2^2 - \vec{v}_1\cdot\vec{v}_2 - 3\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right) + \vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\Big(2\vec{v}_1\cdot\vec{n} - \frac{5}{2}\vec{v}_2\cdot\vec{n}\Big)\Bigg]$$
$$-\frac{7G^2m_1m_2}{2r^3}\vec{S}_1\cdot\vec{v}_1\times\vec{n} + \frac{G^2m_2^2}{r^3}\left[\vec{S}_1\cdot\vec{v}_1\times\vec{n} - \frac{7}{2}\vec{S}_1\cdot\vec{v}_2\times\vec{n}\right]. \tag{6.24}$$

Having finished with the spin-orbit sector, let us proceed to the quadratic-in-spin sectors: the NLO $\mathrm{S_1S_2}$ and NLO SS sectors, with potentials given in eqs. (5.19) and (5.20), respectively. According to Table 6.1, they receive contributions from the LO SO shift applied to second order onto the Newtonian potential, and applied to linear order onto the LO SO potential. The latter can be computed as usual, but the shift to second order requires more careful attention. In general, given a general function $f(x_1,x_2,\dot{x}_1,\dot{x}_2)$ that depends on the position and its time derivatives, a position redefinition $x_I\to x_I+\Delta x_I$ induces a change up to second order given by

$$f(x_1+\Delta x_1, x_2+\Delta x_2, \dot{x}_1+\Delta\dot{x}_1, \dot{x}_2+\Delta\dot{x}_2) =$$
$$= f(x_1,x_2,\dot{x}_1,\dot{x}_2) + \sum_{I=1}^{2}\frac{\partial f}{\partial x_I}\Delta x_I + \sum_{I=1}^{2}\frac{\partial f}{\partial \dot{x}_I}\Delta\dot{x}_I + \frac{1}{2}\sum_{I=1}^{2}\sum_{J=1}^{2}\frac{\partial^2 f}{\partial x_I\partial x_J}\Delta x_I\Delta x_J$$
$$+\frac{1}{2}\sum_{I=1}^{2}\sum_{J=1}^{2}\frac{\partial^2 f}{\partial \dot{x}_I\partial \dot{x}_J}\Delta\dot{x}_I\Delta\dot{x}_J + \frac{1}{2}\sum_{I=1}^{2}\sum_{J=1}^{2}\frac{\partial^2 f}{\partial x_I\partial \dot{x}_J}\Delta x_I\Delta\dot{x}_J + \mathcal{O}\Big((\Delta x_I)^3\Big), \tag{6.25}$$

where one usually assumes that $\Delta\dot{x}_I = \frac{d}{dt}\Delta x_I$. In the present case, we will consider the Newtonian Lagrangian,

$$L_{\rm N}(\vec{x}_1,\vec{x}_2,\vec{v}_1,\vec{v}_2) = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{Gm_1m_2}{r}. \tag{6.26}$$

Then, using the previous description, the NLO quadratic-in-spin contributions will be given by the terms

$$\frac{1}{2}\left(\frac{\partial^2 L_{\rm N}}{\partial x_1^i\partial x_1^j}\Delta x_1^i\Delta x_1^j + \frac{\partial^2 L_{\rm N}}{\partial x_1^i\partial x_2^j}\Delta x_1^i\Delta x_2^j + \frac{\partial^2 L_{\rm N}}{\partial x_2^i\partial x_1^j}\Delta x_2^i\Delta x_1^j + \frac{\partial^2 L_{\rm N}}{\partial x_2^i\partial x_2^j}\Delta x_2^i\Delta x_2^j\right)$$

$$+\frac{1}{2}\left(\frac{\partial^2 L_{\rm N}}{\partial v_1^i \partial v_1^j}\frac{d}{dt}\Delta x_1^i \frac{d}{dt}\Delta x_1^j + \frac{\partial^2 L_{\rm N}}{\partial v_2^i \partial v_2^j}\frac{d}{dt}\Delta x_2^i \frac{d}{dt}\Delta x_2^j\right), \tag{6.27}$$

where $\Delta\vec{x}_I$ is given by eq. (6.14), and where the crossed derivatives $\partial^2/\partial v_1^i \partial v_2^j$ and $\partial^2/\partial x^i \partial v^j$ vanish for the Newtonian Lagrangian. Separating the two sectors, we obtain that the complete contributions read

$$\begin{aligned}
\Delta V^{\rm NLO}_{\rm S_1S_2} =& \frac{G}{r^2}\left[\dot{\vec{S}}_2\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{n} - \dot{\vec{S}}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{n} - \vec{S}_1\cdot\dot{\vec{S}}_2\,\vec{v}_1\cdot\vec{n} + \dot{\vec{S}}_1\cdot\vec{S}_2\,\vec{v}_2\cdot\vec{n}\right]\\
&+\frac{G}{r^3}\Bigg[\vec{S}_1\cdot\vec{S}_2\left(v_1^2 - \frac{3}{2}\vec{v}_1\cdot\vec{v}_2 + v_2^2 - \frac{3}{4}\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right)\\
&+\vec{S}_1\cdot\vec{v}_1\left(-\vec{S}_2\cdot\vec{v}_1 + 3\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\right) + \vec{S}_1\cdot\vec{v}_2\left(\frac{3}{2}\vec{S}_2\cdot\vec{v}_1 - \vec{S}_2\cdot\vec{v}_2 - \frac{9}{4}\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\right)\\
&+\vec{S}_1\cdot\vec{n}\left(-\frac{9}{4}\vec{S}_2\cdot\vec{v}_1\,\vec{v}_2\cdot\vec{n} + 3\vec{S}_2\cdot\vec{v}_2\,\vec{v}_2\cdot\vec{n} + \vec{S}_2\cdot\vec{n}\left(-3v_1^2 + \frac{21}{4}\vec{v}_1\cdot\vec{v}_2 - 3v_2^2\right)\right)\Bigg],
\end{aligned} \tag{6.28}$$

$$\begin{aligned}
\Delta V^{\rm NLO}_{\rm SS} =& \frac{Gm_2}{m_1 r^3}\Bigg[\frac{3}{4}S_1^2 v_1^2 - S_1^2\,\vec{v}_1\cdot\vec{v}_2 - \frac{3}{4}(\vec{S}_1\cdot\vec{v}_1)^2 + \vec{S}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{v}_2 + \frac{9}{4}\vec{S}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\\
&-3\vec{S}_1\cdot\vec{v}_2\,\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n} - \frac{21}{8}(\vec{S}_1\cdot\vec{n})^2\,v_1^2 + 3(\vec{S}_1\cdot\vec{n})^2\,\vec{v}_1\cdot\vec{v}_2 + \frac{3}{8}S_1^2(\vec{v}_1\cdot\vec{n})^2\Bigg]\\
&+\frac{Gm_2}{m_1 r^2}\left[\dot{\vec{S}}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{n} - \dot{\vec{S}}_1\cdot\vec{S}_1\,\vec{v}_1\cdot\vec{n}\right]\\
&+\frac{1}{4m_1}\Bigg[-2S_1^2\,\dot{\vec{a}}_1\cdot\vec{v}_1 + 2\vec{S}_1\cdot\dot{\vec{a}}_1\,\vec{S}_1\cdot\vec{v}_1 - \frac{1}{2}S_1^2 a_1^2 + \frac{1}{2}(\vec{S}_1\cdot\vec{a}_1)^2\\
&+3\dot{\vec{S}}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{a}_1 + \ddot{\vec{S}}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{v}_1 - 4\dot{\vec{S}}_1\cdot\vec{S}_1\,\vec{v}_1\cdot\vec{a}_1 - \ddot{\vec{S}}_1\cdot\vec{S}_1\,v_1^2\\
&+\frac{1}{2}(\dot{\vec{S}}_1\cdot\vec{v}_1)^2 + \dot{\vec{S}}_1\cdot\vec{a}_1\,\vec{S}_1\cdot\vec{v}_1 - \frac{1}{2}\dot{\vec{S}}_1\cdot\dot{\vec{S}}_1\,v_1^2\Bigg] + (1\leftrightarrow 2),
\end{aligned} \tag{6.29}$$

in agreement with eqs. (6.29) and (6.41)-(6.42) of [24], but also including the omitted precession terms. Then, as before, to eliminate the remaining higher-order time derivatives, new position and spin redefinitions must be carried out. The spin-squared potential requires extra manipulations, as it contains terms that are quadratic in the accelerations. To eliminate them, two iterative position redefinitions must be carried out. Furthermore, it also contains time derivatives at even higher orders, like $\dot{\vec{a}}_I$ and $\ddot{\vec{S}}_I$, which should be treated first. With these considerations, we have

$$(\Delta\vec{x}_1)^{\rm NLO}_{\rm S_1S_2} = 2\frac{G}{m_1 r^2}\left[(\vec{S}_1\cdot\vec{S}_2)\,\vec{n} - (\vec{S}_1\cdot\vec{n})\,\vec{S}_2\right], \tag{6.30}$$

$$\begin{aligned}
(\omega_1^{ij})^{\rm NLO}_{\rm S_1S_2} =& \frac{G}{r^2}\Bigg[\left(2S_2^{ik}v_1^j n^k - \frac{1}{2}S_2^{ik}v_2^j n^k - \frac{3}{2}S_2^{ik}n^j v_2^k + \frac{3}{2}S_2^{ik}n^j n^k\,(\vec{v}_2\cdot\vec{n})\right) - (i\leftrightarrow j)\\
&- S_2^{ij}\,\vec{v}_2\cdot\vec{n}\Bigg] + \frac{G}{4r}\left[\dot{S}_2^{ik}n^j n^k - (i\leftrightarrow j) - 2\dot{S}_2^{ij}\right],
\end{aligned} \tag{6.31}$$

$$(\Delta\vec{x}_1)^{\rm NLO}_{\rm SS} = \frac{C_{1(\rm ES^2)}Gm_2}{m_1^2 r^2}(\vec{S}_1\cdot\vec{n})\vec{S}_1 + \frac{Gm_2}{8m_1^2 r^2}\left[(\vec{S}_1\cdot\vec{n})\vec{S}_1 - S_1^2\,\vec{n}\right]$$

$$
+\frac{1}{4m_1^2}\Big[\frac{3}{2}(\vec{S}_1\cdot\vec{a}_1)\vec{S}_1-\frac{3}{2}S_1^2\,\vec{a}_1+(\vec{S}_1\cdot\vec{v}_1)\dot{\vec{S}}_1-(\dot{\vec{S}}_1\cdot\vec{v}_1)\vec{S}_1\Big], \tag{6.32}
$$

$$
\begin{aligned}
(\omega_1^{ij})^{\rm NLO}_{\rm SS}=&\,\frac{C_{1(\rm ES^2)}Gm_2}{m_1r^2}\Big[-S_1^{ik}v_1^jn^k-S_1^{ik}n^jv_1^k+\frac{3}{2}S_1^{ik}v_2^jn^k+\frac{3}{2}S_1^{ik}n^jv_2^k\\
&+\frac{3}{2}S_1^{ik}n^jn^k(\vec{v}_2\cdot\vec{n})-(i\leftrightarrow j)+2S_1^{ij}(\vec{v}_1\cdot\vec{n})-3S_1^{ij}(\vec{v}_2\cdot\vec{n})\Big]\\
&-\frac{Gm_2}{4m_1r^2}\Big[S_1^{ik}v_1^jn^k-S_1^{ik}n^jv_1^k-(i\leftrightarrow j)\Big]\\
&-\frac{1}{4m_1}\Big[S_1^{ik}v_1^ja_1^k+S_1^{ik}a_1^jv_1^k+\frac{1}{2}\dot{S}_1^{ik}v_1^jv_1^k-(i\leftrightarrow j)\Big],
\end{aligned}\tag{6.33}
$$

as given in eqs. (6.30)-(6.31) and (6.43)-(6.44) of [24], including the omitted precession terms. This way, the reduced potentials become

$$
\begin{aligned}
(V_s)^{\rm NLO}_{\rm S_1S_2}=&\,\frac{G(m_1+m_2)}{r^4}\Big[4\vec{S}_1\cdot\vec{S}_2-10\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\Big]\\
&+\frac{G}{r^3}\Big[\vec{S}_1\cdot\vec{S}_2\Big(-\frac{5}{2}v_1^2+6\vec{v}_1\cdot\vec{v}_2-\frac{5}{2}v_2^2+6(\vec{v}_1\cdot\vec{n})^2-\frac{45}{4}\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}+6(\vec{v}_2\cdot\vec{n})^2\Big)\\
&+\vec{S}_1\cdot\vec{v}_1\Big(\frac{5}{2}\vec{S}_2\cdot\vec{v}_1-\frac{5}{2}\vec{S}_2\cdot\vec{v}_2-\frac{3}{2}\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}+\frac{9}{2}\vec{S}_2\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\\
&+\vec{S}_1\cdot\vec{v}_2\Big(-3\vec{S}_2\cdot\vec{v}_1+\frac{5}{2}\vec{S}_2\cdot\vec{v}_2+\frac{21}{4}\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}-6\vec{S}_2\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\\
&+\vec{S}_1\cdot\vec{n}\Big(-6\vec{S}_2\cdot\vec{v}_1\,\vec{v}_1\cdot\vec{n}+\frac{21}{4}\vec{S}_2\cdot\vec{v}_1\,\vec{v}_2\cdot\vec{n}+\frac{9}{2}\vec{S}_2\cdot\vec{v}_2\,\vec{v}_1\cdot\vec{n}-\frac{3}{2}\vec{S}_2\cdot\vec{v}_2\,\vec{v}_2\cdot\vec{n}\Big)\\
&+\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\Big(\frac{3}{2}v_1^2-\frac{21}{4}\vec{v}_1\cdot\vec{v}_2+\frac{3}{2}v_2^2-\frac{15}{2}\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\Big],
\end{aligned}\tag{6.34}
$$

$$
\begin{aligned}
(V_s)^{\rm NLO}_{\rm SS}=&\,\frac{C_{1(\rm ES^2)}G^2m_2^2}{m_1r^4}\Big[2S_1^2-5(\vec{S}_1\cdot\vec{n})^2\Big]+\frac{G^2m_2^2}{8m_1r^4}\Big[-S_1^2+(\vec{S}_1\cdot\vec{n})^2\Big]\\
&+\frac{C_{1(\rm ES^2)}G^2m_2}{2r^4}\Big[S_1^2-3(\vec{S}_1\cdot\vec{n})^2\Big]-\frac{Gm_2}{r^4}(\vec{S}_1\cdot\vec{n})^2\\
&+\frac{C_{1(\rm ES^2)}Gm_2}{m_1r^3}\Big[S_1^2\Big(-\frac{5}{4}v_1^2+\frac{9}{4}\vec{v}_1\cdot\vec{v}_2-\frac{3}{4}v_2^2+\frac{3}{2}(\vec{v}_1\cdot\vec{n})^2-\frac{3}{4}\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\\
&+\vec{S}_1\cdot\vec{v}_1\Big(\frac{1}{2}\vec{S}_1\cdot\vec{v}_1-\frac{1}{2}\vec{S}_1\cdot\vec{v}_2-\frac{3}{2}\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}+\frac{3}{2}\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\\
&+\frac{3}{2}\vec{S}_1\cdot\vec{v}_2\,\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}+(\vec{S}_1\cdot\vec{n})^2\Big(\frac{9}{4}v_1^2-\frac{21}{4}\vec{v}_1\cdot\vec{v}_2+\frac{9}{4}v_2^2-\frac{15}{4}\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\Big]\\
&+\frac{Gm_2}{m_1r^3}\Big[S_1^2\Big(\frac{5}{4}v_1^2-\frac{3}{2}\vec{v}_1\cdot\vec{v}_2-\frac{9}{8}(\vec{v}_1\cdot\vec{n})^2+\frac{3}{2}\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)-3\vec{S}_1\cdot\vec{v}_2\,\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\\
&+\vec{S}_1\cdot\vec{v}_1\Big(-\frac{5}{4}\vec{S}_1\cdot\vec{v}_1+\frac{3}{2}\vec{S}_1\cdot\vec{v}_2+\frac{15}{4}\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}-\frac{3}{2}\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\\
&-\frac{21}{8}(\vec{S}_1\cdot\vec{n})^2\,v_1^2+3(\vec{S}_1\cdot\vec{n})^2\,\vec{v}_1\cdot\vec{v}_2\Big]+(1\leftrightarrow2).
\end{aligned}\tag{6.35}
$$

At this point we have thus considered all relevant sectors at leading and next-to-leading orders below the NLO $\rm S^3$ sector, which will be addressed using similar considerations in §11.3 in the results section.

7 Equations of motion

In the previous section we obtained the reduced potentials by the elimination of higher-order time derivatives. In this section, we will use them to perform a proper variation of the action to obtain the physical equations of motion (EoMs) for the binary system, first for the non-spinning sectors and then generalizing it to spinning ones.

7.1 Equations of motion for non-spinning sectors

Starting from the already calculated standard reduced potentials V_s, in the non-spinning sectors we will have a Lagrangian that no longer depends on higher-order time derivatives of the position,

$$L(\vec{x},\vec{v}) = \sum_{I=1}^{2} \frac{m_I}{2} v_I^2 - V_s(\vec{x},\vec{v}), \tag{7.1}$$

where we explicitly take out the Newtonian kinetic term. Then, as we did in §6.1, by the variation principle, an infinitesimal variation $\delta\vec{x}_1$ of the action yields the EoM,

$$\delta\left[\int dt L(\vec{x},\vec{v})\right] = 0 \quad \Longrightarrow \quad m_1\vec{a}_1 = -\frac{\partial V_s}{\partial \vec{x}_1} + \frac{d}{dt}\frac{\partial V_s}{\partial \vec{v}_1}, \tag{7.2}$$

and the same for particle 2. Note that now we do not have to consider derivatives with respect to accelerations and higher-order derivatives of $\vec{x}_I$. Therefore, we can use eq. (7.2) to compute the physical equations of motion.

Starting with the Newtonian potential, given in eq. (5.6), which is not modified during the elimination of higher-order time derivatives, we compute

$$(\vec{a}_1)_{\rm N} = -\frac{Gm_2}{r^2}\vec{n}, \tag{7.3}$$

which is the usual acceleration derived from Newton's law of universal gravitation for a 2-body system undergoing gravitational interaction.

Going then to the 1PN sector, with a Lagrangian given in eq. (5.9) that is also not modified during the position and spin redefinitions, we obtain that

$$\begin{aligned}(\vec{a}_1)_{\rm 1PN} = &-\frac{3Gm_2}{r}\vec{a}_1 + \frac{7Gm_2}{2r}\vec{a}_2 - \frac{1}{2}v_1^2\vec{a}_1 - (\vec{v}_1\cdot\vec{a}_1)\,\vec{v}_1 + \frac{Gm_2}{2r}(\vec{a}_2\cdot\vec{n})\,\vec{n} \\ &+ \frac{Gm_2}{r^2}\left[\frac{Gm_1}{r} + \frac{Gm_2}{r} + \frac{3v_1^2}{2} + 4\vec{v}_1\cdot\vec{v}_2 + \frac{3(\vec{v}_2\cdot\vec{n})^2}{2} - 2v_2^2\right]\vec{n} \\ &+ \frac{3Gm_2}{r^2}\left[\vec{v}_1\cdot\vec{n} - \vec{v}_2\cdot\vec{n}\right]\vec{v}_1 + \frac{Gm_2}{r^2}\left[-4\vec{v}_1\cdot\vec{n} + 3\vec{v}_2\cdot\vec{n}\right]\vec{v}_2. \end{aligned} \tag{7.4}$$

As can be seen, due to the presence of a time derivative in eq. (7.2), intrinsic accelerations start appearing. Hence, at this point we have to use the lower-order EoMs to eliminate them iteratively. Therefore, this will create again an interplay between sectors, so that higher-order sectors receive contributions from combinations of lower-order sectors. In the present case, plugging the Newtonian EoM of eq. (7.3) into eq. (7.4), we obtain that the final 1PN correction to the physical EoMs is

$$(\vec{a}_1)_{\rm 1PN} = \frac{G^2m_2}{r^3}(5m_1+4m_2)\,\vec{n} + \frac{Gm_2}{r^2}\Bigg[-v_1^2\,\vec{n} + 4(\vec{v}_1\cdot\vec{v}_2)\,\vec{n} + \frac{3}{2}(\vec{v}_2\cdot\vec{n})^2\,\vec{n} - 2v_2^2\,\vec{n}$$

$$+4(\vec{v}_1\cdot\vec{n})(\vec{v}_1-\vec{v}_2)-3(\vec{v}_2\cdot\vec{n})(\vec{v}_1-\vec{v}_2)\Big], \tag{7.5}$$

in agreement with [17]. Analogously, higher-order PN corrections for the EoMs in the non-spinning sectors can be obtained. Nevertheless, for the NLO S^3 sector, we will only need the EoMs from the non-spinning sectors up to the 1PN order.

7.2 Equations of motion for spinning sectors

In the spinning sectors we will have now a Lagrangian that no longer depends on higher-order time derivatives of the position nor of the spin,

$$L(\vec{x},\vec{v},\vec{S})=\sum_{I=1}^{2}\frac{m_I}{2}v_I^2-\sum_{I=1}^{2}\vec{S}_I\cdot\vec{\Omega}-V_s(\vec{x},\vec{v},\vec{S}), \tag{7.6}$$

where we explicitly take out the leading kinetic terms. Then, as we did in §6.2, an infinitesimal variation of the action yields both the EoM for the position and for the spin variables,

$$m_1\vec{a}_1=-\frac{\partial V_s}{\partial\vec{x}_1}+\frac{d}{dt}\frac{\partial V_s}{\partial\vec{v}_1},\qquad \dot{S}_1^{ij}=-4S_1^{k[i}\,\delta^{j]l}\frac{\partial V_s}{\partial S_1^{kl}}=\{S_1^{ij},-V_s\}, \tag{7.7}$$

and the same for particle 2. Now we do not have to consider derivatives with respect to neither accelerations etc., nor time derivatives of spin. Nevertheless, we will have now a double interplay between sectors, as the EoMs for the position and for the spin will mix. To keep track of these contributions more conveniently, Table 7.1 has been created. There, we show the PN sectors to which the accelerations etc., and time derivatives of spin contribute, when we substitute different PN corrections to the EoMs. Note that since the EoM for spin cannot contain higher-order time derivatives, contributions to other sectors can only arise from the position EoMs.

Focusing first on the LO SO sector, with a reduced potential written in eq. (6.16), from eq. (7.7) we have that the EoM for the spin is

$$(\dot{S}_1^{ij})_{\rm SO}^{\rm LO}=\frac{Gm_2}{r^2}\Big[3n^{[i}S_1^{j]k}v_1^k-3n^kv_1^{[i}S_1^{j]k}-4n^{[i}S_1^{j]k}v_2^k+4n^kv_2^{[i}S_1^{j]k}\Big], \tag{7.8}$$

in agreement with eq. (5.19) of [45]. For the EoM of the position, we obtain

$$\begin{aligned}(a_1^i)_{\rm SO}^{\rm LO}=&-\frac{3Gm_2}{2m_1r^2}\dot{S}_1^{ij}n^j-\frac{2G}{r^2}\dot{S}_2^{ij}n^j+\frac{Gm_2}{m_1r^3}\Big[-\frac{9}{2}\Big(\vec{S}_1\cdot\vec{v}_1\times\vec{n}\Big)n^i+6\Big(\vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big)n^i\\&+\frac{9}{2}(\vec{v}_1\cdot\vec{n})S_1^{ij}n^j-\frac{9}{2}(\vec{v}_2\cdot\vec{n})S_1^{ij}n^j-3S_1^{ij}v_1^j+\frac{7}{2}S_1^{ij}v_2^j\Big]\\&+\frac{G}{r^3}\Big[-6\Big(\vec{S}_2\cdot\vec{v}_1\times\vec{n}\Big)n^i+\frac{9}{2}\Big(\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big)n^i+6(\vec{v}_1\cdot\vec{n})S_2^{ij}n^j-6(\vec{v}_2\cdot\vec{n})S_2^{ij}n^j\\&-4S_2^{ij}v_1^j+\frac{7}{2}S_2^{ij}v_2^j\Big].\end{aligned} \tag{7.9}$$

As we can observe, time derivatives of spin appear. Following Table 7.1, if we were to substitute the only spin EoM that we have so far, the LO SO correction that we just

	Newton	1PN	LO SO	LO S^2	LO S^3
$(\vec{a}_1)_{\rm N} = \dots$					
$(\vec{a}_1)_{\rm 1PN} = \dots + \#\vec{a}_I$	1PN		NLO SO	NLO S^2	NLO S^3
$(\vec{a}_1)^{\rm LO}_{\rm SO} = \dots + \#\dot{\vec{S}}_I$	LO SO		NLO SO	NLO S^2	NLO S^3
$(\vec{a}_1)^{\rm NLO}_{\rm SO} = \dots + \#\vec{a}_I$	NLO SO			NLO S^3	
$+\#\dot{\vec{S}}_I$	NLO SO				
$(\vec{a}_1)^{\rm LO}_{\rm S^2} = \dots$					
$(\vec{a}_1)^{\rm NLO}_{\rm S^2} = \dots + \#\vec{a}_I$	NLO S^2		NLO S^3		
$+\#\dot{\vec{S}}_I$	NLO S^2			NLO S^3	
$(\vec{a}_1)^{\rm LO}_{\rm S^3} = \dots + \#\dot{\vec{S}}_I$	LO S^3		NLO S^3		
$(\vec{a}_1)^{\rm NLO}_{\rm S^3} = \dots + \#\vec{a}_I$	NLO S^3				
$+\#\dot{\vec{S}}_I$	NLO S^3				

Table 7.1: PN sectors to which accelerations and time derivatives of spin contribute, when we substitute different PN corrections to the equations of motion, up to the relevant NLO S^3 contributions. The ellipses indicate terms with no higher-order time derivatives, while the symbol # indicates some coefficient multiplying the relevant higher-order time derivatives.

computed in eq. (7.8), it would create a contribution to NLO SO. Since we are computing the LO SO EoM, this contribution can be neglected. Hence, we can only substitute the Newtonian EoM for the spin, which is $\dot{S}_1^{ij} = 0$. Thus, we finally obtain that

$$\begin{aligned}(a_1^i)^{\rm LO}_{\rm SO} = \frac{Gm_2}{m_1 r^3}\Big[&-\frac{9}{2}\Big(\vec{S}_1\cdot\vec{v}_1\times\vec{n}\Big)n^i + 6\Big(\vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big)n^i + \frac{9}{2}(\vec{v}_1\cdot\vec{n})S_1^{ij}n^j - \frac{9}{2}(\vec{v}_2\cdot\vec{n})S_1^{ij}n^j \\ &- 3S_1^{ij}v_1^j + \frac{7}{2}S_1^{ij}v_2^j\Big] + \frac{G}{r^3}\Big[-6\Big(\vec{S}_2\cdot\vec{v}_1\times\vec{n}\Big)n^i + \frac{9}{2}\Big(\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big)n^i \\ &+ 6(\vec{v}_1\cdot\vec{n})S_2^{ij}n^j - 6(\vec{v}_2\cdot\vec{n})S_2^{ij}n^j - 4S_2^{ij}v_1^j + \frac{7}{2}S_2^{ij}v_2^j\Big], \end{aligned} \tag{7.10}$$

in agreement with eq. (5.18) of [45]. Similarly, if we were to substitute eq. (7.10) into the 1PN order EoM, in eq. (7.4), we would obtain a contribution to NLO SO.

The quadratic-in-spin sectors are much easier, since their LO reduced potentials do not contain velocities, as given in eqs. (5.15)-(5.16). Therefore, no higher-order time derivatives appear in their equations of motion, so they cannot receive contributions from lower-order sectors, as shown in Table 7.1, and become

$$(\dot{S}_1^{ij})^{\rm LO}_{\rm S_1S_2} = \frac{2G}{r^3}\Big[3(\vec{S}_2\cdot\vec{n})n^{[i}S_1^{j]} - S_1^{[i}S_2^{j]}\Big], \tag{7.11}$$

$$(\dot{S}_1^{ij})^{\rm LO}_{\rm SS} = \frac{6C_{1({\rm ES}^2)}Gm_2}{m_1 r^3}(\vec{S}_1\cdot\vec{n})n^{[i}S_1^{j]}, \tag{7.12}$$

$$(\vec{a}_1)^{\rm LO}_{\rm S_1S_2} = -\frac{3G}{m_1 r^4}\Big[(\vec{S}_1\cdot\vec{S}_2)\,\vec{n} - 5(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{n})\,\vec{n} + (\vec{S}_2\cdot\vec{n})\,\vec{S}_1 + (\vec{S}_1\cdot\vec{n})\,\vec{S}_2\Big], \tag{7.13}$$

$$(\vec{a}_1)^{\rm LO}_{\rm SS} = -\frac{3C_{1(\rm ES^2)}Gm_2}{m_1^2 r^4}\left[\frac{1}{2}S_1^2\,\vec{n} - \frac{5}{2}(\vec{S}_1\cdot\vec{n})^2\,\vec{n} + (\vec{S}_1\cdot\vec{n})\,\vec{S}_1\right]$$
$$-\frac{3C_{2(\rm ES^2)}G}{m_2 r^4}\left[\frac{1}{2}S_2^2\,\vec{n} - \frac{5}{2}(\vec{S}_2\cdot\vec{n})^2\,\vec{n} + (\vec{S}_2\cdot\vec{n})\,\vec{S}_2\right], \tag{7.14}$$

in agreement with eqs. (5.18)-(5.19) of [45], respectively. If we were to substitute these EoMs onto the 1PN order EoM of eq. (7.4) or onto the LO SO position EoM of eq. (7.9), we would obtain in both cases contributions to the NLO quadratic-in-spin sectors, as shown in Table 7.1.

Finally, for the LO $\rm S^3$ sector, with a reduced potential given in eq. (6.20), we have that

$$\begin{aligned}
(\dot{S}_1^{ij})^{\rm LO}_{\rm S^3} =& \frac{2C_{1(\rm BS^3)}Gm_2}{m_1^2 r^4}\Big[10\Big(\vec{S}_1\cdot\vec{v}_1\times\vec{n} - \vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big)(\vec{S}_1\cdot\vec{n})\,n^{[i}S_1^{j]}\\
&+\Big(5(\vec{S}_1\cdot\vec{n})^2 - S_1^2\Big)\,n^{[j}S_1^{i]k}(v_1^k - v_2^k) + \Big(5(\vec{S}_1\cdot\vec{n})^2 - S_1^2\Big)\,n^k S_1^{k[i}(v_1^{j]} - v_2^{j]})\Big]\\
&+\frac{2C_{1(\rm ES^2)}G}{m_1 r^4}\Big[\Big(30\vec{S}_2\cdot\vec{v}_1\times\vec{n} - \frac{45}{2}\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big)(\vec{S}_1\cdot\vec{n})\,n^{[i}S_1^{j]}\\
&+\Big(6\vec{S}_1\cdot\vec{v}_1\times\vec{n} - \frac{9}{2}\vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big)S_1^{[j}S_2^{i]} + \Big(6\vec{S}_1\cdot\vec{v}_1 - \frac{9}{2}\vec{S}_1\cdot\vec{v}_2\Big)S_1^{[j}S_2^{i]k}n^k\\
&-\Big(\vec{S}_1\cdot\vec{S}_2\times\vec{n}\Big)S_1^{[j}\Big(6v_1^{i]} - \frac{9}{2}v_2^{i]}\Big) + (\vec{S}_1\cdot\vec{S}_2)\,n^{[j}S_1^{i]k}\Big(6v_1^k - \frac{9}{2}v_2^k\Big)\\
&+(\vec{S}_1\cdot\vec{S}_2)\,n^k S_1^{k[i}\Big(6v_1^{j]} - \frac{9}{2}v_2^{j]}\Big) + \frac{m_2}{m_1}\Big(-\frac{15}{2}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,(\vec{S}_1\cdot\vec{n})\,n^{[i}S_1^{j]}\\
&+\frac{3}{4}\Big(-5(\vec{S}_1\cdot\vec{n})^2 + S_1^2\Big)\,n^{[j}S_1^{i]k}v_1^k + \frac{3}{4}\Big(-5(\vec{S}_1\cdot\vec{n})^2 + S_1^2\Big)\,n^k S_1^{k[i}v_1^{j]}\Big)\Big]\\
&+\frac{2C_{2(\rm ES^2)}G}{m_2 r^4}\Big[\vec{S}_1\cdot\vec{n}\Big(-9\vec{S}_2\cdot\vec{v}_1 + 12\vec{S}_2\cdot\vec{v}_2\Big)S_2^{ij} + \vec{S}_1\cdot\vec{S}_2\Big(9\vec{v}_1\cdot\vec{n} - 12\vec{v}_2\cdot\vec{n}\Big)S_2^{ij}\\
&-\Big(\frac{9}{2}\vec{S}_2\cdot\vec{v}_1 - 6\vec{S}_2\cdot\vec{v}_2\Big)\,n^{[i}S_2^{j]k}S_1^k + \Big(\frac{9}{2}\vec{v}_1\cdot\vec{n} - 6\vec{v}_2\cdot\vec{n}\Big)S_2^{[i}S_2^{j]k}S_1^k\\
&+\frac{9}{4}\Big(5(\vec{S}_2\cdot\vec{n})^2 - 3S_2^2\Big)\,n^{[j}S_1^{i]k}v_1^k + \frac{9}{4}\Big(5(\vec{S}_2\cdot\vec{n})^2 - 3S_2^2\Big)\,n^k S_1^{k[i}v_1^{j]}\\
&+(\vec{S}_1\cdot\vec{S}_2)\,n^{[i}S_2^{j]k}\Big(\frac{9}{2}v_1^k - 6v_2^k\Big) + (\vec{S}_1\cdot\vec{S}_2)\,n^k S_2^{k[i}\Big(\frac{9}{2}v_1^{j]} - 6v_2^{j]}\Big)\\
&-(\vec{S}_1\cdot\vec{n})S_2^{[i}S_2^{j]k}\Big(\frac{9}{2}v_1^k - 6v_2^k\Big) - \Big(\vec{S}_1\cdot\vec{S}_2\times\vec{n}\Big)S_2^{[i}\Big(\frac{9}{2}v_1^{j]} - 6v_2^{j]}\Big)\\
&+\Big(-15(\vec{S}_2\cdot\vec{n})^2 + 9S_2^2\Big)\,n^{[j}S_1^{i]k}v_2^k + \Big(-15(\vec{S}_2\cdot\vec{n})^2 + 9S_2^2\Big)\,n^k S_1^{k[i}v_2^{j]}\Big]\\
&+\frac{2G}{r^4}\Big[\frac{15}{2}\Big(-\frac{\vec{S}_1\cdot\vec{v}_1\times\vec{n}}{m_1} + \frac{\vec{S}_2\cdot\vec{v}_2\times\vec{n}}{m_2}\Big)(\vec{S}_2\cdot\vec{n})\,n^{[i}S_1^{j]}\\
&+3\Big(\frac{\vec{S}_1\cdot\vec{v}_1\times\vec{n}}{m_1} - \frac{\vec{S}_2\cdot\vec{v}_2\times\vec{n}}{2m_2}\Big)S_1^{[j}S_2^{i]} - \frac{3}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{n}\Big(\frac{S_1^{[j}v_1^{i]}}{m_1} + \frac{S_2^{[j}v_2^{i]}}{m_1}\Big)\\
&+3(\vec{S}_1\cdot\vec{S}_2)\,n^{[j}\Big(\frac{S_1^{i]k}v_1^k}{m_1} + \frac{S_2^{i]k}v_2^k}{2m_2}\Big) + 3(\vec{S}_1\cdot\vec{S}_2)\,n^k\Big(\frac{S_1^{k[i}v_1^{j]}}{m_1} - \frac{S_2^{k[i}v_2^{j]}}{2m_2}\Big)\\
&+\frac{3}{2m_1}\Big((\vec{S}_1\cdot\vec{v}_1)S_1^{[j}S_2^{i]k}n^k - 5(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{n})\,n^{[j}S_1^{i]k}v_1^k
\end{aligned}$$

$$
\begin{aligned}
&- 5(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{n})\, n^k S_1^{k[i} v_1^{j]}\Big) \\
&+ \frac{3}{2m_2}\bigg(2(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{v}_2)S_2^{ij} - 2(\vec{S}_1\cdot\vec{S}_2)(\vec{v}_2\cdot\vec{n})S_2^{ij} + (\vec{S}_2\cdot\vec{v}_2)n^{[i}S_2^{j]k}S_1^k \\
&- (\vec{v}_2\cdot\vec{n})S_2^{[i}S_2^{j]k}S_1^k + S_2^2 n^{[j}S_1^{i]k}v_2^k + (\vec{S}_1\cdot\vec{n})S_2^{[i}S_2^{j]k}v_2^k + S_2^2 n^k S_1^{k[i}v_2^{j]}\bigg)\bigg], \qquad (7.15)
\end{aligned}
$$

$$
\begin{aligned}
(a_1^i)^{\rm LO}_{\rm S^3} = &\frac{C_{1(\rm BS^3)}Gm_2}{m_1^3 r^4}\Big[5(\vec{S}_1\cdot\vec{n})^2\dot{S}_1^{ij}n^j - S_1^2\dot{S}_1^{ij}n^j + 10(\dot{\vec{S}}_1\cdot\vec{n})(\vec{S}_1\cdot\vec{n})S_1^{ij}n^j \\
&- 2(\dot{\vec{S}}_1\cdot\vec{S}_1)S_1^{ij}n^j\Big] + \frac{C_{1(\rm BS^3)}Gm_2}{m_1^3 r^5}\Big[35\Big(\vec{S}_1\cdot\vec{v}_1\times\vec{n} - \vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big)(\vec{S}_1\cdot\vec{n})^2\, n^i \\
&- 5\Big(\vec{S}_1\cdot\vec{v}_1\times\vec{n} - \vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big)S_1^2 n^i - 10\Big(\vec{S}_1\cdot\vec{v}_1\times\vec{n} - \vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big)(\vec{S}_1\cdot\vec{n})S_1^i \\
&+ 10\vec{S}_1\cdot\vec{n}\,\Big(\vec{S}_1\cdot\vec{v}_1 - \vec{S}_1\cdot\vec{v}_2\Big)S_1^{ij}n^j - 35(\vec{S}_1\cdot\vec{n})^2\,(\vec{v}_1\cdot\vec{n} - \vec{v}_2\cdot\vec{n})S_1^{ij}n^j \\
&+ 5S_1^2\,(\vec{v}_1\cdot\vec{n} - \vec{v}_2\cdot\vec{n})S_1^{ij}n^j + 10(\vec{S}_1\cdot\vec{n})^2\,S_1^{ij}(v_1^j - v_2^j) - 2S_1^2\,S_1^{ij}(v_1^j - v_2^j)\Big] \\
&+ \frac{C_{2(\rm BS^3)}G}{m_2^2 r^4}\Big[5(\vec{S}_2\cdot\vec{n})^2\,\dot{S}_2^{ij}n^j - S_2^2\,\dot{S}_2^{ij}n^j + 10(\dot{\vec{S}}_2\cdot\vec{n})(\vec{S}_2\cdot\vec{n})S_2^{ij}n^j \\
&- 2(\dot{\vec{S}}_2\cdot\vec{S}_2)S_2^{ij}n^j\Big] + \frac{C_{2(\rm BS^3)}G}{m_2^2 r^5}\Big[35\Big(\vec{S}_2\cdot\vec{v}_1\times\vec{n} - \vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big)(\vec{S}_2\cdot\vec{n})^2\, n^i \\
&- 5\Big(\vec{S}_2\cdot\vec{v}_1\times\vec{n} - \vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big)S_2^2 n^i - 10\Big(\vec{S}_2\cdot\vec{v}_1\times\vec{n} - \vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big)(\vec{S}_2\cdot\vec{n})S_2^i \\
&+ 10\vec{S}_2\cdot\vec{n}\,\Big(\vec{S}_2\cdot\vec{v}_1 - \vec{S}_2\cdot\vec{v}_2\Big)S_2^{ij}n^j - 35(\vec{S}_2\cdot\vec{n})^2\,(\vec{v}_1\cdot\vec{n} - \vec{v}_2\cdot\vec{n})S_2^{ij}n^j \\
&+ 5S_2^2\,(\vec{v}_1\cdot\vec{n} - \vec{v}_2\cdot\vec{n})S_2^{ij}n^j + 10(\vec{S}_2\cdot\vec{n})^2\,S_2^{ij}(v_1^j - v_2^j) - 2S_2^2\,S_2^{ij}(v_1^j - v_2^j)\Big] \\
&+ \frac{C_{1(\rm ES^2)}G}{m_1^2 r^4}\bigg[\frac{m_2}{m_1}\bigg(- \frac{15}{4}(\vec{S}_1\cdot\vec{n})^2\dot{S}_1^{ij}n^j + \frac{3}{4}S_1^2\dot{S}_1^{ij}n^j + \frac{3}{2}(\dot{\vec{S}}_1\cdot\vec{S}_1)S_1^{ij}n^j \\
&- \frac{15}{2}(\dot{\vec{S}}_1\cdot\vec{n})(\vec{S}_1\cdot\vec{n})S_1^{ij}n^j\bigg) + 15(\vec{S}_1\cdot\vec{n})^2\,\dot{S}_2^{ij}n^j - 3S_1^2\,\dot{S}_2^{ij}n^j + 6(\vec{S}_1\cdot\vec{n})\dot{S}_1^{ij}S_2^j \\
&+ 6(\vec{S}_1\cdot\vec{n})S_1^{ij}\dot{S}_2^j + 6(\dot{\vec{S}}_1\cdot\vec{n})S_1^{ij}S_2^j - 6(\dot{\vec{S}}_1\cdot\vec{S}_1)S_2^{ij}n^j + 30(\dot{\vec{S}}_1\cdot\vec{n})(\vec{S}_1\cdot\vec{n})S_2^{ij}n^j\bigg] \\
&+ \frac{C_{1(\rm ES^2)}G}{m_1^2 r^5}\bigg[\frac{m_2}{m_1}\bigg(- \frac{105}{4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}(\vec{S}_1\cdot\vec{n})^2\, n^i + \frac{15}{4}\Big(\vec{S}_1\cdot\vec{v}_1\times\vec{n}\Big)S_1^2 n^i \\
&+ \frac{15}{2}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,(\vec{S}_1\cdot\vec{n})S_1^i - \frac{15}{2}\vec{S}_1\cdot\vec{n}\,\Big(\vec{S}_1\cdot\vec{v}_1 - \vec{S}_1\cdot\vec{v}_2\Big)S_1^{ij}n^j \\
&+ \frac{105}{4}(\vec{S}_1\cdot\vec{n})^2\,(\vec{v}_1\cdot\vec{n} - \vec{v}_2\cdot\vec{n})S_1^{ij}n^j - \frac{15}{4}S_1^2\,(\vec{v}_1\cdot\vec{n} - \vec{v}_2\cdot\vec{n})S_1^{ij}n^j \\
&- \frac{15}{2}(\vec{S}_1\cdot\vec{n})^2\,S_1^{ij}\Big(v_1^j - \frac{1}{2}v_2^j\Big) + \frac{3}{2}S_1^2\,S_1^{ij}\Big(v_1^j - \frac{1}{2}v_2^j\Big)\bigg) \\
&- 30\vec{S}_1\cdot\vec{n}\,\Big(\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1 - \frac{3}{4}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\Big)\, n^i \\
&+ 105(\vec{S}_1\cdot\vec{n})^2\,\Big(\vec{S}_2\cdot\vec{v}_1\times\vec{n} - \frac{3}{4}\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big)\, n^i
\end{aligned}
$$

$$
\begin{aligned}
&- 15S_1^2 \left(\vec{S}_2 \cdot \vec{v}_1 \times \vec{n} - \frac{3}{4}\vec{S}_2 \cdot \vec{v}_2 \times \vec{n}\right) n^i + 6\Big(\vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 - \frac{3}{4}\vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2\Big) S_1^i \\
&- 30\vec{S}_1 \cdot \vec{n} \left(\vec{S}_2 \cdot \vec{v}_1 \times \vec{n} - \frac{3}{4}\vec{S}_2 \cdot \vec{v}_2 \times \vec{n}\right) S_1^i + 6\Big(\vec{S}_1 \cdot \vec{v}_1 - \frac{3}{4}\vec{S}_1 \cdot \vec{v}_2\Big) S_1^{ij} S_2^j \\
&- 30\vec{S}_1 \cdot \vec{n}\, (\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) S_1^{ij} S_2^j + 30\vec{S}_1 \cdot \vec{n} \left(\vec{S}_1 \cdot \vec{v}_1 - \vec{S}_1 \cdot \vec{v}_2\right) S_2^{ij} n^j \\
&- 105(\vec{S}_1 \cdot \vec{n})^2\, (\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) S_2^{ij} n^j + 15S_1^2\, (\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) S_2^{ij} n^j \\
&+ 30(\vec{S}_1 \cdot \vec{n})^2\, S_2^{ij} \left(v_1^j - \frac{7}{8} v_2^j\right) - 6S_1^2\, S_2^{ij} \left(v_1^j - \frac{7}{8} v_2^j\right)\Bigg] \\
&+ \frac{C_{2(\mathrm{ES}^2)} G}{m_1 m_2 r^4} \Bigg[\frac{45}{4}(\vec{S}_2 \cdot \vec{n})^2\, \dot{S}_1^{ij} n^j - \frac{9}{4} S_2^2\, \dot{S}_1^{ij} n^j - \frac{9}{2}(\vec{S}_2 \cdot \vec{n}) \dot{S}_1^{ij} S_2^j - \frac{9}{2}(\vec{S}_2 \cdot \vec{n}) S_1^{ij} \dot{S}_2^j \\
&- \frac{9}{2}(\dot{\vec{S}}_2 \cdot \vec{S}_2) S_1^{ij} n^j + \frac{45}{2}(\dot{\vec{S}}_2 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n}) S_1^{ij} n^j - \frac{9}{2}(\dot{\vec{S}}_2 \cdot \vec{n}) S_1^{ij} S_2^j\Bigg] \\
&+ \frac{C_{2(\mathrm{ES}^2)} G}{m_2 r^5} \Bigg[\frac{1}{m_1}\Bigg(- 30\vec{S}_2 \cdot \vec{n} \left(- \frac{3}{4}\vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 + \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2\right) n^i \\
&- 105(\vec{S}_2 \cdot \vec{n})^2 \left(- \frac{3}{4}\vec{S}_1 \cdot \vec{v}_1 \times \vec{n} + \vec{S}_1 \cdot \vec{v}_2 \times \vec{n}\right) n^i \\
&+ 15S_2^2\Big(- \frac{3}{4}\vec{S}_1 \cdot \vec{v}_1 \times \vec{n} + \vec{S}_1 \cdot \vec{v}_2 \times \vec{n}\Big) n^i + 6\Big(- \frac{3}{4}\vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 + \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2\Big) S_2^i \\
&+ 30\vec{S}_2 \cdot \vec{n} \left(- \frac{3}{4}\vec{S}_1 \cdot \vec{v}_1 \times \vec{n} + \vec{S}_1 \cdot \vec{v}_2 \times \vec{n}\right) S_2^i + \frac{45}{2}\vec{S}_2 \cdot \vec{n} \left(\vec{S}_2 \cdot \vec{v}_1 - \vec{S}_2 \cdot \vec{v}_2\right) S_1^{ij} n^j \\
&- \frac{315}{4}(\vec{S}_2 \cdot \vec{n})^2\, (\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) S_1^{ij} n^j + \frac{45}{4} S_2^2\, (\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) S_1^{ij} n^j \\
&- \frac{9}{2}\Big(\vec{S}_2 \cdot \vec{v}_1 - \vec{S}_2 \cdot \vec{v}_2\Big) S_1^{ij} S_2^j + \frac{45}{2}\vec{S}_2 \cdot \vec{n}\, (\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) S_1^{ij} S_2^j \\
&+ \frac{45}{2}(\vec{S}_2 \cdot \vec{n})^2\, S_1^{ij} \left(v_1^j - \frac{7}{6} v_2^j\right) - \frac{9}{2} S_2^2\, S_1^{ij} \left(v_1^j - \frac{7}{6} v_2^j\right)\Bigg) \\
&+ \frac{1}{m_2}\Bigg(\frac{105}{4}\vec{S}_2 \cdot \vec{v}_2 \times \vec{n}\, (\vec{S}_2 \cdot \vec{n})^2\, n^i - \frac{15}{4}\Big(\vec{S}_2 \cdot \vec{v}_2 \times \vec{n}\Big) S_2^2 n^i \\
&- \frac{15}{2}\vec{S}_2 \cdot \vec{v}_2 \times \vec{n}\, (\vec{S}_2 \cdot \vec{n}) S_2^i + \frac{15}{4}(\vec{S}_2 \cdot \vec{n})^2 S_2^{ij} v_2^j - \frac{3}{4} S_2^2\, S_2^{ij} v_2^j\Bigg)\Bigg] \\
&+ \frac{G}{m_1^2 r^4} \Bigg[\frac{3}{2}(\vec{S}_1 \cdot \vec{S}_2) \dot{S}_1^{ij} n^j - \frac{15}{2}(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n}) \dot{S}_1^{ij} n^j + \frac{3}{2}(\vec{S}_1 \cdot \vec{n}) \dot{S}_1^{ij} S_2^j \\
&+ \frac{3}{2}(\vec{S}_1 \cdot \vec{n}) S_1^{ij} \dot{S}_2^j + \frac{3}{2}(\dot{\vec{S}}_1 \cdot \vec{S}_2) S_1^{ij} n^j + \frac{3}{2}(\vec{S}_1 \cdot \dot{\vec{S}}_2) S_1^{ij} n^j - \frac{15}{2}(\dot{\vec{S}}_2 \cdot \vec{n})(\vec{S}_1 \cdot \vec{n}) S_1^{ij} n^j \\
&- \frac{15}{2}(\dot{\vec{S}}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n}) S_1^{ij} n^j + \frac{3}{2}(\dot{\vec{S}}_1 \cdot \vec{n}) S_1^{ij} S_2^j\Bigg] \\
&+ \frac{G}{m_1 r^5} \Bigg[\frac{1}{m_1}\Bigg(- \frac{15}{2}\vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 (\vec{S}_1 \cdot \vec{n})\, n^i + \frac{15}{2}\vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\, (\vec{S}_1 \cdot \vec{S}_2)\, n^i \\
&- \frac{105}{2}\vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\, (\vec{S}_1 \cdot \vec{n})(\vec{S}_1 \cdot \vec{n})\, n^i + \frac{3}{2}\Big(\vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1\Big) S_1^i \\
&+ \frac{15}{2}\vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\, (\vec{S}_2 \cdot \vec{n}) S_1^i + \frac{15}{2}\vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\, (\vec{S}_1 \cdot \vec{n}) S_2^i \\
&- \frac{15}{2}\vec{S}_2 \cdot \vec{n} \left(\vec{S}_1 \cdot \vec{v}_1 - \vec{S}_1 \cdot \vec{v}_2\right) S_1^{ij} n^j - \frac{15}{2}\vec{S}_1 \cdot \vec{n} \left(\vec{S}_2 \cdot \vec{v}_1 - \vec{S}_2 \cdot \vec{v}_2\right) S_1^{ij} n^j
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{15}{2} \vec{S}_1 \cdot \vec{S}_2 (\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) S_1^{ij} n^j + \frac{105}{2} (\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n})(\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) S_1^{ij} n^j \\
&+ \frac{3}{2} \Big(\vec{S}_1 \cdot \vec{v}_1 - \vec{S}_1 \cdot \vec{v}_2 \Big) S_1^{ij} S_2^j - \frac{15}{2} \vec{S}_1 \cdot \vec{n} \, (\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) S_1^{ij} S_2^j \\
&+ 3 (\vec{S}_1 \cdot \vec{S}_2) S_1^{ij} \Big(v_1^j - \frac{1}{2} v_2^j \Big) - 15 (\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n}) S_1^{ij} \Big(v_1^j - \frac{1}{2} v_2^j \Big) \Big) \\
&+ \frac{1}{m_2} \Big(- \frac{15}{2} \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} \, (\vec{S}_1 \cdot \vec{S}_2) \, n^i - \frac{15}{2} \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 (\vec{S}_2 \cdot \vec{n}) \, n^i \\
&+ \frac{105}{2} \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} \, (\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n}) \, n^i - \frac{15}{2} \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} \, (\vec{S}_2 \cdot \vec{n}) S_1^i \\
&+ \frac{3}{2} \Big(\vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \Big) S_2^i - \frac{15}{2} \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} \, (\vec{S}_1 \cdot \vec{n}) S_2^i - \frac{3}{2} (\vec{S}_1 \cdot \vec{S}_2) S_2^{ij} v_2^j \\
&+ \frac{15}{2} (\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n}) S_2^{ij} v_2^j \Big) \Big],
\end{aligned} \tag{7.16}
$$

so that the LO S^3 correction to the EoMs is obtained by substituting the Newtonian EoM for the spin, $\dot{S}_1^{ij} = 0$. If we were to substitute the LO SO EoM for the spin, given in eq. (7.8), we would obtain a contribution to NLO S^3, as will be addressed in our results.

At this point, we can continue to the next-to-leading order spinning sectors. Nonetheless, as can be seen in Table 7.1, only the EoMs of the position are relevant for the NLO S^3 sector, so we will only focus on them.

Starting with the NLO SO sector, following Table 7.1 its EoM receives contributions from iterations of the LO SO on the 1PN sector, and evaluating eq. (7.7) it reads

$$
\begin{aligned}
(a_1^i)_{\rm SO}^{\rm NLO} = \frac{G^2 m_2}{r^4} \Big[& \frac{9}{4} \Big(\vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \Big) n^i - 22 \Big(\vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \Big) n^i + 18 \Big(\vec{S}_2 \cdot \vec{v}_1 \times \vec{n} \Big) n^i \\
&+ \frac{1}{2} \Big(\vec{S}_2 \cdot \vec{v}_2 \times \vec{n} \Big) n^i - 7 (\vec{v}_1 \cdot \vec{n}) S_1^{ij} n^j + 7 (\vec{v}_2 \cdot \vec{n}) S_1^{ij} n^j - 18 (\vec{v}_1 \cdot \vec{n}) S_2^{ij} n^j \\
&+ 18 (\vec{v}_2 \cdot \vec{n}) S_2^{ij} n^j + \frac{21}{4} S_1^{ij} v_1^j + 12 S_2^{ij} v_1^j - \frac{21}{2} S_1^{ij} v_2^j - 7 S_2^{ij} v_2^j \\
&+ \frac{m_1}{m_2} \Big(\frac{161}{4} \Big(\vec{S}_2 \cdot \vec{v}_1 \times \vec{n} \Big) n^i - \frac{43}{2} \Big(\vec{S}_2 \cdot \vec{v}_2 \times \vec{n} \Big) n^i - \frac{135}{4} (\vec{v}_1 \cdot \vec{n}) S_2^{ij} n^j \\
&+ \frac{131}{4} (\vec{v}_2 \cdot \vec{n}) S_2^{ij} n^j + \frac{93}{4} S_2^{ij} v_1^j - \frac{35}{2} S_2^{ij} v_2^j \Big) \\
&+ \frac{m_2}{m_1} \Big(\frac{71}{4} \Big(\vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \Big) n^i - 35 \Big(\vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \Big) n^i - \frac{71}{4} (\vec{v}_1 \cdot \vec{n}) S_1^{ij} n^j \\
&+ \frac{37}{2} (\vec{v}_2 \cdot \vec{n}) S_1^{ij} n^j + \frac{49}{4} S_1^{ij} v_1^j - \frac{35}{2} S_1^{ij} v_2^j \Big) \Big] \\
&+ \frac{G}{r^3} \Big[- 15 (\vec{v}_1 \cdot \vec{n})^2 (\vec{v}_2 \cdot \vec{n}) S_2^{ij} n^j + 15 (\vec{v}_1 \cdot \vec{n}) (\vec{v}_2 \cdot \vec{n})^2 S_2^{ij} n^j - 3 (\vec{v}_1 \cdot \vec{n}) v_2^2 \, S_2^{ij} n^j \\
&+ 3 (\vec{v}_2 \cdot \vec{n}) v_1^2 \, S_2^{ij} n^j + 6 (\vec{v}_1 \cdot \vec{n}) (\vec{v}_2 \cdot \vec{n}) S_2^{ij} v_1^j + 2 (\vec{v}_1 \cdot \vec{v}_2) S_2^{ij} v_1^j \\
&- \frac{1}{2} \vec{S}_2 \cdot \vec{v}_1 \times \vec{v}_2 \Big(3 (\vec{v}_1 \cdot \vec{n}) \, n^i + 3 (\vec{v}_2 \cdot \vec{n}) \, n^i + 3 v_1^i + v_2^i \Big) \\
&+ \vec{S}_2 \cdot \vec{v}_1 \times \vec{n} \, \Big(6 (\vec{v}_1 \cdot \vec{v}_2) \, n^i + 15 (\vec{v}_2 \cdot \vec{n})^2 \, n^i - 3 v_2^2 \, n^i + 6 (\vec{v}_1 \cdot \vec{n}) \, v_1^i - 6 (\vec{v}_1 \cdot \vec{n}) \, v_2^i \Big) \\
&+ \frac{3}{2} \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} \, \Big(- \frac{5}{2} (\vec{v}_1 \cdot \vec{n})^2 \, n^i - \frac{1}{2} v_1^2 \, n^i - 3 (\vec{v}_1 \cdot \vec{v}_2) \, n^i + 5 (\vec{v}_1 \cdot \vec{n}) (\vec{v}_2 \cdot \vec{n}) \, n^i
\end{aligned}
$$

$$
\begin{aligned}
& - 10(\vec{v}_2\cdot\vec{n})^2\, n^i + \frac{9}{4} v_2^2\, n^i - 3(\vec{v}_1\cdot\vec{n})\, v_1^i + (\vec{v}_1\cdot\vec{n})\, v_2^i + 2(\vec{v}_2\cdot\vec{n})\, v_2^i\Big) \\
& + \frac{33}{4}(\vec{v}_1\cdot\vec{n})^2 S_2^{ij} v_2^j - \frac{39}{2}(\vec{v}_1\cdot\vec{n})(\vec{v}_2\cdot\vec{n}) S_2^{ij} v_2^j + 6(\vec{v}_2\cdot\vec{n})^2 S_2^{ij} v_2^j - \frac{13}{4} v_1^2\, S_2^{ij} v_2^j \\
& + \frac{7}{2}(\vec{v}_1\cdot\vec{v}_2) S_2^{ij} v_2^j - \frac{15}{8} v_2^2\, S_2^{ij} v_2^j\Big] \\
& + \frac{G m_2}{m_1 r^3}\Big[- 15(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n}) S_1^{ij} n^j + \frac{75}{4}(\vec{v}_1\cdot\vec{n})(\vec{v}_2\cdot\vec{n})^2 S_1^{ij} n^j - \frac{15}{4}(\vec{v}_2\cdot\vec{n})^3 S_1^{ij} n^j \\
& - \frac{15}{8}(\vec{v}_1\cdot\vec{n})\, v_1^2\, S_1^{ij} n^j + 3(\vec{v}_1\cdot\vec{n})(\vec{v}_1\cdot\vec{v}_2) S_1^{ij} n^j - 3(\vec{v}_1\cdot\vec{n})\, v_2^2 S_1^{ij} n^j \\
& + \frac{39}{8} v_1^2\, (\vec{v}_2\cdot\vec{n}) S_1^{ij} n^j - \frac{9}{2}(\vec{v}_1\cdot\vec{v}_2)(\vec{v}_2\cdot\vec{n}) S_1^{ij} n^j + \frac{3}{2}(\vec{v}_2\cdot\vec{n})\, v_2^2\, S_1^{ij} n^j \\
& + 6(\vec{v}_1\cdot\vec{n})(\vec{v}_2\cdot\vec{n}) S_1^{ij} v_1^j - \frac{3}{2}(\vec{v}_2\cdot\vec{n})^2 S_1^{ij} v_1^j + \frac{5}{4} v_1^2\, S_1^{ij} v_1^j \\
& + \vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\Big(- \frac{3}{2}(\vec{v}_2\cdot\vec{n})\, n^i - \frac{13}{4} v_1^i + \frac{3}{2} v_2^i\Big) \\
& + \vec{S}_1\cdot\vec{v}_2\times\vec{n}\,\Big(- 3 v_1^2\, n^i - 6(\vec{v}_1\cdot\vec{v}_2)\, n^i - 15(\vec{v}_2\cdot\vec{n})^2\, n^i + 6 v_2^2\, n^i - 6(\vec{v}_1\cdot\vec{n})\, v_1^i \\
& + 6(\vec{v}_1\cdot\vec{n})\, v_2^i\Big) + \vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\Big(\frac{15}{8} v_1^2\, n^i + 3(\vec{v}_1\cdot\vec{v}_2)\, n^i + \frac{45}{4}(\vec{v}_2\cdot\vec{n})^2\, n^i - 3 v_2^2\, n^i \\
& + \frac{3}{4}(\vec{v}_1\cdot\vec{n})\, v_1^i + \frac{15}{4}(\vec{v}_2\cdot\vec{n})\, v_1^i - 3(\vec{v}_1\cdot\vec{n})\, v_2^i - \frac{3}{2}(\vec{v}_2\cdot\vec{n})\, v_2^i\Big) \\
& - 6(\vec{v}_1\cdot\vec{n})^2 S_1^{ij} v_2^j + \frac{15}{2}(\vec{v}_1\cdot\vec{n})(\vec{v}_2\cdot\vec{n}) S_1^{ij} v_2^j - \frac{27}{4}(\vec{v}_2\cdot\vec{n})^2 S_1^{ij} v_2^j + \frac{3}{8} v_1^2\, S_1^{ij} v_2^j \\
& - \frac{11}{2}(\vec{v}_1\cdot\vec{v}_2) S_1^{ij} v_2^j + \frac{7}{2} v_2^2\, S_1^{ij} v_2^j\Big] \\
& + \frac{G}{r^2}\Big[- \frac{5}{2}\Big(\vec{S}_2\cdot\vec{v}_1\times\vec{a}_2\Big)\, n^i + \frac{5}{2}\Big(\vec{S}_2\cdot\vec{v}_2\times\vec{a}_1\Big)\, n^i + \frac{3}{2}\vec{S}_2\cdot\vec{a}_2\times\vec{n}\,(\vec{v}_1\cdot\vec{n})\, n^i \\
& + 3\vec{S}_2\cdot\vec{a}_1\times\vec{n}\,(\vec{v}_2\cdot\vec{n})\, n^i - 3\vec{S}_2\cdot\vec{a}_2\times\vec{n}\,(\vec{v}_2\cdot\vec{n})\, n^i \\
& + \vec{S}_2\cdot\vec{v}_2\times\vec{n}\,\Big(\frac{3}{2}(\vec{a}_1\cdot\vec{n})\, n^i - 3(\vec{a}_2\cdot\vec{n})\, n^i\Big) \\
& + \vec{S}_2\cdot\vec{v}_1\times\vec{n}\,\Big(- 2 a_1^i + a_2^i + 3(\vec{a}_2\cdot\vec{n})\, n^i\Big) - \frac{5}{2}(\vec{v}_1\cdot\vec{n})\, S_2^{ij} a_2^j + 2(\vec{v}_2\cdot\vec{n}) S_2^{ij} a_2^j \\
& + 3(\vec{v}_1\cdot\vec{n})(\vec{a}_2\cdot\vec{n}) S_2^{ij} n^j + 3(\vec{a}_1\cdot\vec{n})(\vec{v}_2\cdot\vec{n}) S_2^{ij} n^j - 2(\vec{v}_1\cdot\vec{a}_1) S_2^{ij} n^j \\
& + (\vec{v}_1\cdot\vec{a}_2) S_2^{ij} n^j + (\vec{v}_2\cdot\vec{a}_1) S_2^{ij} n^j - 2\Big(\vec{S}_2\cdot\vec{a}_1\times\vec{n}\Big)\, v_1^i + \Big(\vec{S}_2\cdot\vec{a}_1\times\vec{n}\Big)\, v_2^i \\
& - \frac{5}{2}(\vec{a}_1\cdot\vec{n}) S_2^{ij} v_2^j + 2(\vec{a}_2\cdot\vec{n}) S_2^{ij} v_2^j\Big] \\
& + \frac{G m_2}{m_1 r^2}\Big[2\Big(\vec{S}_1\cdot\vec{v}_1\times\vec{a}_2\Big)\, n^i - 2\Big(\vec{S}_1\cdot\vec{v}_2\times\vec{a}_1\Big)\, n^i + 3\vec{S}_1\cdot\vec{a}_1\times\vec{n}\,(\vec{v}_2\cdot\vec{n})\, n^i \\
& - 3\vec{S}_1\cdot\vec{a}_2\times\vec{n}\,(\vec{v}_2\cdot\vec{n})\, n^i + \vec{S}_1\cdot\vec{v}_2\times\vec{n}\,\Big(- a_2^i - 3(\vec{a}_2\cdot\vec{n})\, n^i\Big) \\
& + \vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\Big(- \frac{1}{4} a_1^i + 3(\vec{a}_2\cdot\vec{n})\, n^i\Big) + 2(\vec{v}_1\cdot\vec{n}) S_1^{ij} a_2^j - \frac{5}{2}(\vec{v}_2\cdot\vec{n}) S_1^{ij} a_2^j \\
& + 3(\vec{v}_1\cdot\vec{n})(\vec{a}_2\cdot\vec{n}) S_1^{ij} n^j + 3(\vec{a}_1\cdot\vec{n})(\vec{v}_2\cdot\vec{n}) S_1^{ij} n^j - \frac{3}{2}(\vec{a}_2\cdot\vec{n})(\vec{v}_2\cdot\vec{n}) S_1^{ij} n^j
\end{aligned}
$$

$$
- \frac{1}{4}(\vec{v}_1 \cdot \vec{a}_1) S_1^{ij} n^j - \frac{1}{4}\left(\vec{S}_1 \cdot \vec{a}_1 \times \vec{n}\right) v_1^i - \left(\vec{S}_1 \cdot \vec{a}_2 \times \vec{n}\right) v_2^i
+ 2(\vec{a}_1 \cdot \vec{n}) S_1^{ij} v_2^j - \frac{5}{2}(\vec{a}_2 \cdot \vec{n}) S_1^{ij} v_2^j \Big], \tag{7.17}
$$

where we have already taken $\dot{\vec{S}}_I = 0$ since the precession terms will not contribute to NLO S^3. For the exact NLO SO correction to the EoM, we would have to substitute the Newtonian EoM of eq. (7.3) into the accelerations. Yet, the substitution of the LO quadratic-in-spin EoM in those acceleration terms precisely generates NLO S^3 contributions.

As for the NLO $\mathrm{S_1S_2}$ sector, based on Table 7.1, it receives contributions from iterations of the LO $\mathrm{S_1S_2}$ sector with the 1PN and LO SO EoMs. Calculating the contributions, it results in

$$
\begin{aligned}
(a_1^i)^{\rm NLO}_{\rm S_1S_2} = \frac{G^2}{r^5}\Big[& 28(\vec{S}_1 \cdot \vec{S}_2)\, n^i + \frac{45}{2}(\vec{S}_2 \cdot \vec{n}) S_1^i + \vec{S}_1 \cdot \vec{n}\Big(- 123(\vec{S}_2 \cdot \vec{n})\, n^i + \frac{49}{2} S_2^i\Big) \\
&+ \frac{m_2}{m_1}\Big(25(\vec{S}_1 \cdot \vec{S}_2)\, n^i + 22(\vec{S}_2 \cdot \vec{n}) S_1^i + \vec{S}_1 \cdot \vec{n}\Big(- \frac{219}{2}(\vec{S}_2 \cdot \vec{n})\, n^i + \frac{42}{2} S_2^i\Big)\Big)\Big] \\
&+ \frac{G}{m_1 r^4}\Big[\frac{1}{2}\vec{S}_2 \cdot \vec{v}_2\Big(9(\vec{v}_1 \cdot \vec{n}) S_1^i - 21(\vec{v}_2 \cdot \vec{n}) S_1^i\Big) \\
&+ \vec{S}_2 \cdot \vec{v}_1\Big(- 3(\vec{v}_1 \cdot \vec{n}) S_1^i + \frac{27}{4}(\vec{v}_2 \cdot \vec{n}) S_1^i\Big) + \frac{1}{4}\vec{S}_1 \cdot \vec{v}_2\Big(9(\vec{S}_2 \cdot \vec{v}_1)\, n^i - 9(\vec{S}_2 \cdot \vec{v}_2)\, n^i \\
&+ 39(\vec{v}_1 \cdot \vec{n}) S_2^i - 33(\vec{v}_2 \cdot \vec{n}) S_2^i\Big) + \vec{S}_1 \cdot \vec{v}_1\Big(- \frac{3}{2}(\vec{S}_2 \cdot \vec{v}_2)\, n^i - 12(\vec{v}_1 \cdot \vec{n}) S_2^i \\
&+ \frac{33}{4}(\vec{v}_2 \cdot \vec{n}) S_2^i\Big) + \vec{S}_2 \cdot \vec{n}\,\Big(\frac{15}{2}(\vec{v}_1 \cdot \vec{n})^2 S_1^i - \frac{3}{2} v_1^2\, S_1^i + \frac{45}{4}(\vec{v}_1 \cdot \vec{v}_2) S_1^i \\
&- \frac{45}{2}(\vec{v}_1 \cdot \vec{n})(\vec{v}_2 \cdot \vec{n}) S_1^i + \frac{45}{2}(\vec{v}_2 \cdot \vec{n})^2 S_1^i - 6 v_2^2 S_1^i \\
&+ \vec{S}_1 \cdot \vec{v}_1\Big(\frac{15}{2}(\vec{v}_2 \cdot \vec{n})\, n^i + 6 v_1^i - \frac{33}{4} v_2^i\Big) + \vec{S}_1 \cdot \vec{v}_2\Big(\frac{15}{4}(\vec{v}_2 \cdot \vec{n})\, n^i - 3 v_1^i + 6 v_2^i\Big)\Big) \\
&+ \vec{S}_1 \cdot \vec{S}_2\Big(- 30(\vec{v}_1 \cdot \vec{n})^2\, n^i + 6 v_1^2\, n^i - \frac{21}{4}(\vec{v}_1 \cdot \vec{v}_2)\, n^i - \frac{105}{4}(\vec{v}_2 \cdot \vec{n})^2\, n^i + \frac{15}{4} v_2^2\, n^i \\
&+ \vec{v}_1 \cdot \vec{n}\,\Big(60(\vec{v}_2 \cdot \vec{n})\, n^i + 18 v_1^i - \frac{75}{4} v_2^i\Big) + \vec{v}_2 \cdot \vec{n}\,\Big(- 15 v_1^i + \frac{69}{4} v_2^i\Big)\Big) \\
&+ \vec{S}_1 \cdot \vec{n}\,\Big(30(\vec{v}_1 \cdot \vec{n})^2 S_2^i - 6 v_1^2\, S_2^i + \frac{33}{2}(\vec{v}_1 \cdot \vec{v}_2) S_2^i - \frac{195}{4}(\vec{v}_1 \cdot \vec{n})(\vec{v}_2 \cdot \vec{n}) S_2^i \\
&+ \frac{105}{4}(\vec{v}_2 \cdot \vec{n})^2 S_2^i - \frac{27}{4} v_2^2\, S_2^i + \vec{S}_2 \cdot \vec{v}_1\Big(- \frac{45}{4}(\vec{v}_2 \cdot \vec{n})\, n^i + 6 v_1^i - \frac{9}{2} v_2^i\Big) \\
&+ \vec{S}_2 \cdot \vec{v}_2\Big(\frac{45}{2}(\vec{v}_2 \cdot \vec{n})\, n^i - 3 v_1^i + \frac{9}{4} v_2^i\Big) \\
&+ \vec{S}_2 \cdot \vec{n}\,\Big(- \frac{135}{4}(\vec{v}_1 \cdot \vec{v}_2)\, n^i - \frac{105}{2}(\vec{v}_2 \cdot \vec{n})^2\, n^i + 15 v_2^2\, n^i + \frac{1}{4}\vec{v}_2 \cdot \vec{n}\,(60 v_1^i - 75 v_2^i) \\
&+ \frac{1}{4}\vec{v}_1 \cdot \vec{n}\,(-120 v_1^i + 135 v_2^i)\Big)\Big)\Big] \\
&+ \frac{G}{m_1 r^3}\Big[\frac{5}{2}(\vec{S}_2 \cdot \vec{v}_1)\dot{S}_1^i - \frac{5}{2}(\vec{S}_2 \cdot \vec{v}_2)\dot{S}_1^i + \frac{5}{2}(\vec{S}_1 \cdot \vec{v}_1)\dot{S}_2^i - 3(\vec{S}_1 \cdot \vec{v}_2)\dot{S}_2^i
\end{aligned}
$$

$$
\begin{aligned}
&+\vec{S}_1\cdot\vec{S}_2\Big(-5a_1^i+6a_2^i+12(\vec{a}_1\cdot\vec{n})\,n^i-\frac{45}{4}(\vec{a}_2\cdot\vec{n})\,n^i\Big)\\
&+\frac{1}{2}\dot{\vec{S}}_2\cdot\vec{v}_2\Big(9(\vec{S}_1\cdot\vec{n})\,n^i-5S_1^i\Big)+\frac{5}{2}(\vec{S}_2\cdot\vec{a}_1)S_1^i-\frac{5}{2}(\vec{S}_2\cdot\vec{a}_2)S_1^i\\
&+\dot{\vec{S}}_2\cdot\vec{v}_1\Big(-6(\vec{S}_1\cdot\vec{n})\,n^i+\frac{5}{2}S_1^i\Big)+\frac{3}{2}\vec{S}_2\cdot\vec{n}\,\Big(-(\vec{v}_1\cdot\vec{n})\dot{S}_1^i+3(\vec{v}_2\cdot\vec{n})\dot{S}_1^i\\
&-(\vec{S}_1\cdot\vec{a}_1)\,n^i+\frac{7}{2}(\vec{S}_1\cdot\vec{a}_2)\,n^i-(\vec{a}_1\cdot\vec{n})S_1^i+3(\vec{a}_2\cdot\vec{n})S_1^i\Big)\\
&+\dot{\vec{S}}_1\cdot\vec{v}_2\Big(\frac{21}{4}(\vec{S}_2\cdot\vec{n})\,n^i-3S_2^i\Big)+\frac{5}{2}(\vec{S}_1\cdot\vec{a}_1)S_2^i-3(\vec{S}_1\cdot\vec{a}_2)S_2^i\\
&+\frac{1}{2}\dot{\vec{S}}_1\cdot\vec{v}_1\Big(-3(\vec{S}_2\cdot\vec{n})\,n^i+5S_2^i\Big)+\vec{S}_1\cdot\vec{n}\,\Big(-6(\vec{v}_1\cdot\vec{n})\dot{S}_2^i+\frac{21}{4}(\vec{v}_2\cdot\vec{n})\dot{S}_2^i\\
&-6(\vec{S}_2\cdot\vec{a}_1)\,n^i+\frac{9}{2}(\vec{S}_2\cdot\vec{a}_2)\,n^i+\vec{S}_2\cdot\vec{n}\,\Big(3a_1^i-\frac{21}{4}a_2^i-\frac{15}{2}(\vec{a}_2\cdot\vec{n})\,n^i\Big)\\
&-6(\vec{a}_1\cdot\vec{n})S_2^i+\frac{21}{4}(\vec{a}_2\cdot\vec{n})S_2^i\Big)+\frac{3}{2}\dot{\vec{S}}_2\cdot\vec{n}\,\Big(-(\vec{S}_1\cdot\vec{v}_1)\,n^i+\frac{7}{2}(\vec{S}_1\cdot\vec{v}_2)\,n^i\\
&-(\vec{v}_1\cdot\vec{n})S_1^i+3(\vec{v}_2\cdot\vec{n})S_1^i+\vec{S}_1\cdot\vec{n}\,\Big(-5(\vec{v}_2\cdot\vec{n})\,n^i+2v_1^i-\frac{7}{2}v_2^i\Big)\Big)\\
&+\dot{\vec{S}}_1\cdot\vec{n}\,\Big(-6(\vec{S}_2\cdot\vec{v}_1)\,n^i+\frac{9}{2}(\vec{S}_2\cdot\vec{v}_2)\,n^i-6(\vec{v}_1\cdot\vec{n})S_2^i+\frac{21}{4}(\vec{v}_2\cdot\vec{n})S_2^i\\
&+\vec{S}_2\cdot\vec{n}\,\Big(-\frac{15}{2}(\vec{v}_2\cdot\vec{n})\,n^i+3v_1^i-\frac{21}{4}v_2^i\Big)\Big)\\
&+\Big(\dot{\vec{S}}_1\cdot\vec{S}_2+\vec{S}_1\cdot\dot{\vec{S}}_2\Big)\Big(12(\vec{v}_1\cdot\vec{n})\,n^i-\frac{45}{4}(\vec{v}_2\cdot\vec{n})\,n^i-5v_1^i+6v_2^i\Big)\Big],
\end{aligned}
\tag{7.18}
$$

where the NLO S_1S_2 correction to the physical EoM is obtained by further substituting the Newtonian EoM of eq. (7.3) in the accelerations and $\dot{\vec{S}}_I=0$.

Analogously, following Table 7.1, the NLO SS EoM receives contributions from the LO SS EoM substituted in the 1PN and in the LO SO EoMs. Explicitly, it reads

$$
\begin{aligned}
(a_1^i)_{\rm SS}^{\rm NLO}=&\frac{G^2}{r^5}\Big[-6(\vec{S}_2\cdot\vec{n})^2\,n^i+2(\vec{S}_2\cdot\vec{n})S_2^i+\frac{m_1}{4m_2}\Big(3(\vec{S}_2\cdot\vec{n})^2\,n^i-2S_2^2\,n^i-(\vec{S}_2\cdot\vec{n})S_2^i\Big)\\
&+\frac{m_2}{m_1}\Big(-6(\vec{S}_1\cdot\vec{n})^2\,n^i+2(\vec{S}_1\cdot\vec{n})S_1^i\Big)+\frac{m_2^2}{4m_1^2}\Big(3(\vec{S}_1\cdot\vec{n})^2\,n^i-2S_1^2\,n^i\\
&-(\vec{S}_1\cdot\vec{n})S_1^i\Big)\Big]+\frac{C_{1(\rm ES^2)}G^2m_2}{m_1r^5}\Big[-\frac{75}{2}(\vec{S}_1\cdot\vec{n})^2\,n^i+8S_1^2\,n^i+\frac{27}{2}(\vec{S}_1\cdot\vec{n})S_1^i\\
&+\frac{m_2}{m_1}\Big(-57(\vec{S}_1\cdot\vec{n})^2\,n^i+\frac{25}{2}S_1^2\,n^i+\frac{47}{2}(\vec{S}_1\cdot\vec{n})S_1^i\Big)\Big]\\
&+\frac{C_{2(\rm ES^2)}G^2}{r^5}\Big[-\frac{63}{2}(\vec{S}_2\cdot\vec{n})^2\,n^i+\frac{13}{2}S_2^2\,n^i+12(\vec{S}_2\cdot\vec{n})S_2^i\\
&+\frac{m_1}{m_2}\Big(-\frac{129}{2}(\vec{S}_2\cdot\vec{n})^2\,n^i+14S_2^2\,n^i+\frac{53}{2}(\vec{S}_2\cdot\vec{n})S_2^i\Big)\Big]\\
&+\frac{G}{m_2r^4}\Big[3(\vec{S}_2\cdot\vec{v}_1)(\vec{S}_2\cdot\vec{v}_2)\,n^i-\frac{9}{4}(\vec{S}_2\cdot\vec{v}_2)^2\,n^i\\
&+\vec{S}_2\cdot\vec{v}_2\Big(-3(\vec{v}_1\cdot\vec{n})S_2^i+\frac{15}{4}(\vec{v}_2\cdot\vec{n})S_2^i\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ S_2^2 \Big(- 3(\vec{v}_1\cdot\vec{v}_2)\, n^i + \frac{15}{8}(\vec{v}_2\cdot\vec{n})^2\, n^i + \frac{9}{4} v_2^2\, n^i + 3(\vec{v}_1\cdot\vec{n}) v_2^i - \frac{15}{4}(\vec{v}_2\cdot\vec{n}) v_2^i\Big)\\
&+ (\vec{S}_2\cdot\vec{n})^2 \Big(15(\vec{v}_1\cdot\vec{v}_2)\, n^i - \frac{105}{8} v_2^2\, n^i - 15(\vec{v}_1\cdot\vec{n}) v_2^i + 15(\vec{v}_2\cdot\vec{n}) v_2^i\Big)\\
&+ \vec{S}_2\cdot\vec{n}\,\Bigg(- 9(\vec{v}_1\cdot\vec{v}_2) S_2^i + 15(\vec{v}_1\cdot\vec{n})(\vec{v}_2\cdot\vec{n}) S_2^i - 15(\vec{v}_2\cdot\vec{n})^2 S_2^i + \frac{33}{4} v_2^2\, S_2^i\\
&+ \frac{1}{4}\vec{S}_2\cdot\vec{v}_2 \Big(45(\vec{v}_2\cdot\vec{n})\, n^i - 33 v_2^i\Big) + \vec{S}_2\cdot\vec{v}_1 \Big(- 15(\vec{v}_2\cdot\vec{n})\, n^i + 9 v_2^i\Big)\Bigg)\Bigg]\\
&+ \frac{G m_2}{m_1^2 r^4}\Bigg[3(\vec{S}_1\cdot\vec{v}_2)^2\, n^i + \vec{S}_1\cdot\vec{v}_1 \Big(- \frac{9}{4}(\vec{S}_1\cdot\vec{v}_2)\, n^i + \frac{15}{2}(\vec{v}_1\cdot\vec{n}) S_1^i - \frac{15}{2}(\vec{v}_2\cdot\vec{n}) S_1^i\Big)\\
&+ \vec{S}_1\cdot\vec{v}_2 \Big(- \frac{21}{4}(\vec{v}_1\cdot\vec{n}) S_1^i + 6(\vec{v}_2\cdot\vec{n}) S_1^i\Big) + \frac{1}{2}\vec{S}_1\cdot\vec{n}\,\Bigg(- \frac{75}{2}(\vec{v}_1\cdot\vec{n})^2 S_1^i + 18 v_1^2\, S_1^i\\
&- \frac{45}{2}(\vec{v}_1\cdot\vec{v}_2) S_1^i + \frac{105}{2}(\vec{v}_1\cdot\vec{n})(\vec{v}_2\cdot\vec{n}) S_1^i - 15(\vec{v}_2\cdot\vec{n})^2 S_1^i + 3 v_2^2\, S_1^i\\
&+ \vec{S}_1\cdot\vec{v}_2 \Big(- 30(\vec{v}_2\cdot\vec{n})\, n^i + 21 v_1^i - 6 v_2^i\Big) + \vec{S}_1\cdot\vec{v}_1 \Big(\frac{45}{2}(\vec{v}_2\cdot\vec{n})\, n^i - 21 v_1^i + \frac{15}{2} v_2^i\Big)\Bigg)\\
&+ \frac{1}{2} S_1^2\,\Bigg(\frac{45}{4}(\vec{v}_1\cdot\vec{n})^2\, n^i + 3 v_1^2\, n^i - \frac{3}{2}(\vec{v}_1\cdot\vec{v}_2)\, n^i + 15(\vec{v}_2\cdot\vec{n})^2\, n^i - 3 v_2^2\, n^i\\
&+ \vec{v}_2\cdot\vec{n}\,(15 v_1^i - 12 v_2^i) + \vec{v}_1\cdot\vec{n}\,\Big(- \frac{45}{2}(\vec{v}_2\cdot\vec{n})\, n^i - 15 v_1^i + \frac{21}{2} v_2^i\Big)\Bigg)\\
&+ (\vec{S}_1\cdot\vec{n})^2 \Bigg(- \frac{105}{8} v_1^2\, n^i + 15(\vec{v}_1\cdot\vec{v}_2)\, n^i + \Big(\vec{v}_1\cdot\vec{n} - \vec{v}_2\cdot\vec{n}\Big)\Big(\frac{105}{4} v_1^i - 15 v_2^i\Big)\Bigg)\Bigg]\\
&+ \frac{C_{1(\mathrm{ES}^2)} G m_2}{m_1^2 r^4}\Bigg[- \frac{3}{2}(\vec{S}_1\cdot\vec{v}_2)^2\, n^i + \vec{S}_1\cdot\vec{v}_2 \Big(\frac{3}{2}(\vec{v}_1\cdot\vec{n}) S_1^i - 3(\vec{v}_2\cdot\vec{n}) S_1^i\Big)\\
&+ \vec{S}_1\cdot\vec{v}_1 \Big(\frac{3}{2}(\vec{S}_1\cdot\vec{v}_2)\, n^i - 3(\vec{v}_1\cdot\vec{n}) S_1^i + 3(\vec{v}_2\cdot\vec{n}) S_1^i\Big)\\
&+ S_1^2\,\Bigg(- \frac{15}{2}(\vec{v}_1\cdot\vec{n})^2\, n^i + 3(\vec{v}_1\cdot\vec{v}_2)\, n^i - \frac{15}{4}(\vec{v}_2\cdot\vec{n})^2\, n^i - \frac{3}{2} v_2^2\, n^i\\
&+ \vec{v}_1\cdot\vec{n}\,\Big(15(\vec{v}_2\cdot\vec{n})\, n^i + 9 v_1^i - 9 v_2^i\Big) + \frac{15}{2}\vec{v}_2\cdot\vec{n}\,(- v_1^i + v_2^i)\Bigg)\\
&+ \vec{S}_1\cdot\vec{n}\,\Bigg(\frac{15}{2}(\vec{v}_1\cdot\vec{n})^2 S_1^i - \frac{9}{2} v_1^2\, S_1^i + \frac{27}{2}(\vec{v}_1\cdot\vec{v}_2) S_1^i - \frac{15}{2}(\vec{v}_1\cdot\vec{n})(\vec{v}_2\cdot\vec{n}) S_1^i\\
&+ \frac{15}{2}(\vec{v}_2\cdot\vec{n})^2 S_1^i - 6 v_2^2\, S_1^i + \vec{S}_1\cdot\vec{v}_1 \Big(- \frac{15}{2}(\vec{v}_2\cdot\vec{n})\, n^i + 12 v_1^i - \frac{21}{2} v_2^i\Big)\\
&+ \vec{S}_1\cdot\vec{v}_2 \Big(15(\vec{v}_2\cdot\vec{n})\, n^i - 9 v_1^i + 9 v_2^i\Big)\Bigg) + (\vec{S}_1\cdot\vec{n})^2 \Big(\frac{15}{2} v_1^2\, n^i - 30(\vec{v}_1\cdot\vec{v}_2)\, n^i\\
&- \frac{105}{4}(\vec{v}_2\cdot\vec{n})^2\, n^i + 15 v_2^2\, n^i + \frac{45}{2}\vec{v}_2\cdot\vec{n}\,(v_1^i - v_2^i) + 30\vec{v}_1\cdot\vec{n}\,(- v_1^i + v_2^i)\Big)\Bigg]\\
&+ \frac{C_{2(\mathrm{ES}^2)} G}{m_2 r^4}\Bigg[- \frac{3}{2}(\vec{S}_2\cdot\vec{v}_2)(\vec{v}_2\cdot\vec{n}) S_2^i + S_2^2\,\Bigg(- \frac{3}{2} v_1^2\, n^i + 6(\vec{v}_1\cdot\vec{v}_2)\, n^i\\
&+ \frac{15}{4}(\vec{v}_2\cdot\vec{n})^2\, n^i - 3 v_2^2\, n^i + 6\vec{v}_1\cdot\vec{n}\,(v_1^i - v_2^i) + \frac{9}{2}\vec{v}_2\cdot\vec{n}\,(- v_1^i + v_2^i)\Bigg)\\
&+ \vec{S}_2\cdot\vec{n}\,\Bigg(- 3 v_1^2\, S_2^i + 12(\vec{v}_1\cdot\vec{v}_2) S_2^i + \frac{15}{2}(\vec{v}_2\cdot\vec{n})^2 S_2^i - 6 v_2^2\, S_2^i
\end{aligned}
$$

$$
\begin{aligned}
&+12\vec{S}_2\cdot\vec{v}_1(v_1^i-v_2^i)+\vec{S}_2\cdot\vec{v}_2\Big(\frac{15}{2}(\vec{v}_2\cdot\vec{n})\,n^i-9v_1^i+\frac{21}{2}v_2^i\Big)\Big)\\
&+(\vec{S}_2\cdot\vec{n})^2\Big(\frac{15}{2}v_1^2\,n^i-30(\vec{v}_1\cdot\vec{v}_2)\,n^i-\frac{105}{4}(\vec{v}_2\cdot\vec{n})^2\,n^i+15v_2^2\,n^i\\
&+30\vec{v}_1\cdot\vec{n}\,(-v_1^i+v_2^i)+\frac{45}{2}\vec{v}_2\cdot\vec{n}\,(v_1^i-v_2^i)\Big)\Big]\\
&+\frac{G}{m_2r^3}\Big[3(\vec{S}_2\cdot\vec{n})^2a_2^i+\frac{3}{2}(\vec{S}_2\cdot\vec{v}_2)\dot{S}_2^i+\frac{3}{2}S_2^2\,\Big(-a_2^i+(\vec{a}_2\cdot\vec{n})\,n^i\Big)\\
&+\frac{3}{2}(\vec{S}_2\cdot\vec{a}_2)S_2^i+\frac{3}{2}\dot{\vec{S}}_2\cdot\vec{v}_2\Big(-(\vec{S}_2\cdot\vec{n})\,n^i+S_2^i\Big)\\
&+3\vec{S}_2\cdot\vec{n}\,\Big(-(\vec{v}_2\cdot\vec{n})\dot{S}_2^i-\frac{1}{2}(\vec{S}_2\cdot\vec{a}_2)\,n^i-(\vec{a}_2\cdot\vec{n})S_2^i\Big)+3\dot{\vec{S}}_2\cdot\vec{S}_2\Big((\vec{v}_2\cdot\vec{n})\,n^i-v_2^i\Big)\\
&+3\dot{\vec{S}}_2\cdot\vec{n}\,\Big(-\frac{1}{2}(\vec{S}_2\cdot\vec{v}_2)\,n^i-(\vec{v}_2\cdot\vec{n})S_2^i+2(\vec{S}_2\cdot\vec{n})v_2^i\Big)\Big]\\
&+\frac{Gm_2}{m_1^2r^3}\Big[(\vec{S}_1\cdot\vec{n})^2\Big(-\frac{21}{4}a_1^i+3a_2^i\Big)-\frac{5}{2}(\vec{S}_1\cdot\vec{v}_1)\dot{S}_1^i+\frac{3}{2}(\vec{S}_1\cdot\vec{v}_2)\dot{S}_1^i\\
&+\frac{1}{2}S_1^2\,\Big(5a_1^i-3a_2^i-\frac{9}{2}(\vec{a}_1\cdot\vec{n})\,n^i+3(\vec{a}_2\cdot\vec{n})\,n^i\Big)+\dot{\vec{S}}_1\cdot\vec{v}_1\Big(\frac{15}{4}(\vec{S}_1\cdot\vec{n})\,n^i-\frac{5}{2}S_1^i\Big)\\
&-\frac{5}{2}(\vec{S}_1\cdot\vec{a}_1)S_1^i+\frac{3}{2}(\vec{S}_1\cdot\vec{a}_2)S_1^i+\dot{\vec{S}}_1\cdot\vec{v}_2\Big(-3(\vec{S}_1\cdot\vec{n})\,n^i+\frac{3}{2}S_1^i\Big)\\
&+\frac{3}{2}\vec{S}_1\cdot\vec{n}\,\Big(\frac{5}{2}(\vec{v}_1\cdot\vec{n})\dot{S}_1^i-(\vec{v}_2\cdot\vec{n})\dot{S}_1^i+\frac{5}{2}(\vec{S}_1\cdot\vec{a}_1)\,n^i-2(\vec{S}_1\cdot\vec{a}_2)\,n^i+\frac{5}{2}(\vec{a}_1\cdot\vec{n})S_1^i\\
&-(\vec{a}_2\cdot\vec{n})S_1^i\Big)+\dot{\vec{S}}_1\cdot\vec{S}_1\Big(-\frac{9}{2}(\vec{v}_1\cdot\vec{n})\,n^i+3(\vec{v}_2\cdot\vec{n})\,n^i+5v_1^i-3v_2^i\Big)\\
&+\dot{\vec{S}}_1\cdot\vec{n}\,\Big(\frac{15}{4}(\vec{S}_1\cdot\vec{v}_1)\,n^i-3(\vec{S}_1\cdot\vec{v}_2)\,n^i+\frac{15}{4}(\vec{v}_1\cdot\vec{n})S_1^i-\frac{3}{2}(\vec{v}_2\cdot\vec{n})S_1^i\\
&+\vec{S}_1\cdot\vec{n}\,\Big(-\frac{21}{2}v_1^i+6v_2^i\Big)\Big)\Big]+\frac{C_{1(\mathrm{ES}^2)}Gm_2}{m_1^2r^3}\Big[(\vec{S}_1\cdot\vec{v}_1)\dot{S}_1^i-\frac{1}{2}(\vec{S}_1\cdot\vec{v}_2)\dot{S}_1^i\\
&+\frac{1}{4}(\vec{S}_1\cdot\vec{n})^2\Big(18a_1^i-21a_2^i-15(\vec{a}_2\cdot\vec{n})\,n^i\Big)\\
&+S_1^2\,\Big(-\frac{5}{2}a_1^i+\frac{9}{4}a_2^i+3(\vec{a}_1\cdot\vec{n})\,n^i-\frac{3}{4}(\vec{a}_2\cdot\vec{n})\,n^i\Big)+\frac{1}{2}\dot{\vec{S}}_1\cdot\vec{v}_2\Big(3(\vec{S}_1\cdot\vec{n})\,n^i-S_1^i\Big)\\
&+(\vec{S}_1\cdot\vec{a}_1)S_1^i-\frac{1}{2}(\vec{S}_1\cdot\vec{a}_2)S_1^i+\dot{\vec{S}}_1\cdot\vec{v}_1\Big(-\frac{3}{2}(\vec{S}_1\cdot\vec{n})\,n^i+S_1^i\Big)\\
&+\frac{3}{2}\vec{S}_1\cdot\vec{n}\,\Big(-(\vec{v}_1\cdot\vec{n})\dot{S}_1^i+(\vec{v}_2\cdot\vec{n})\dot{S}_1^i-(\vec{S}_1\cdot\vec{a}_1)\,n^i+(\vec{S}_1\cdot\vec{a}_2)\,n^i-(\vec{a}_1\cdot\vec{n})S_1^i\\
&+(\vec{a}_2\cdot\vec{n})S_1^i\Big)+\frac{3}{2}\dot{\vec{S}}_1\cdot\vec{n}\,\Big(-(\vec{S}_1\cdot\vec{v}_1)\,n^i+(\vec{S}_1\cdot\vec{v}_2)\,n^i-(\vec{v}_1\cdot\vec{n})S_1^i+(\vec{v}_2\cdot\vec{n})S_1^i\\
&+\vec{S}_1\cdot\vec{n}\,\Big(-5(\vec{v}_2\cdot\vec{n})\,n^i+6v_1^i-7v_2^i\Big)\Big)\\
&+\dot{\vec{S}}_1\cdot\vec{S}_1\Big(6(\vec{v}_1\cdot\vec{n})\,n^i-\frac{3}{2}(\vec{v}_2\cdot\vec{n})\,n^i-5v_1^i+\frac{9}{2}v_2^i\Big)\Big]\\
&+\frac{C_{2(\mathrm{ES}^2)}G}{m_2r^3}\Big[-\frac{1}{2}(\vec{S}_2\cdot\vec{v}_2)\dot{S}_2^i+\frac{3}{4}(\vec{S}_2\cdot\vec{n})^2\Big(6a_1^i-7a_2^i-5(\vec{a}_2\cdot\vec{n})\,n^i\Big)\\
&+\frac{3}{4}S_2^2\,\Big(-2a_1^i+3a_2^i-(\vec{a}_2\cdot\vec{n})\,n^i\Big)+\frac{1}{2}\dot{\vec{S}}_2\cdot\vec{v}_2\Big(3(\vec{S}_2\cdot\vec{n})\,n^i-S_2^i\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2}(\vec{S}_2 \cdot \vec{a}_2) S_2^i + \frac{3}{2}\vec{S}_2 \cdot \vec{n} \left((\vec{v}_2 \cdot \vec{n}) \dot{S}_2^i + (\vec{S}_2 \cdot \vec{a}_2)\, n^i + (\vec{a}_2 \cdot \vec{n}) S_2^i \right) \\
&+ \frac{3}{2}\dot{\vec{S}}_2 \cdot \vec{n} \left((\vec{S}_2 \cdot \vec{v}_2)\, n^i + (\vec{v}_2 \cdot \vec{n}) S_2^i + \vec{S}_2 \cdot \vec{n} \left(-5 (\vec{v}_2 \cdot \vec{n})\, n^i + 6 v_1^i - 21 v_2^i \right) \right) \\
&+ \frac{3}{2}\dot{\vec{S}}_2 \cdot \vec{S}_2 \left(-3 (\vec{v}_2 \cdot \vec{n})\, n^i - 2 v_1^i + 3 v_2^i \right) \Bigg],
\end{aligned}
\tag{7.19}
$$

where the NLO SS correction to the physical EoM is again obtained by further substituting the Newtonian EoM of eq. (7.3) in the accelerations and $\dot{\vec{S}}_I = 0$.

With this last result, we have all lower-order EoMs necessary for the evaluation of the NLO S^3 correction to the EoMs, which will be done in §11.4 in the results section.

8 Hamiltonian

In §6 we derived the reduced potentials for the relevant PN sectors. Now, we will use them to compute the Hamiltonian for the binary system. Even though having a Hamiltonian formulation of the problem is theoretically interesting on its own, the Hamiltonian is the central object used in the effective-one-body formalism to model gravitational waveform templates [18]. It will be calculated in the general coordinate frame, and will be valid for general compact objects, regardless if they are black holes or neutron stars, since it will contain general Wilson coefficient. In addition, it will be valid for generic orbits, not only circular, allowing for further studies based on the eccentricity. Finally, it will also hold for any orientations of the spins.

8.1 Legendre transformation and Hamiltonian

The computation of the Hamiltonian is much simpler than obtaining the reduced potential, and to derive it we will closely follow the description given in §6 of [45], see also §7.2 of [17]. The derivation starts as usual in analytical mechanics, by defining a canonical momentum, conjugate to the position variable,

$$\vec{p}_1 = \frac{\partial L}{\partial \vec{v}_1} = \frac{\partial}{\partial \vec{v}_1}\Big(\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - V_{\rm N} + L_{n\rm PN} - V_s\Big) = m_1\vec{v}_1 + \frac{\partial L_{n\rm PN}}{\partial \vec{v}_1} - \frac{\partial V_s}{\partial \vec{v}_1}, \quad (8.1)$$

and the same for particle 2. We expanded the Lagrangian into the leading Newtonian contribution and its higher PN corrections, which may include higher non-spinning Lagrangians $L_{n\rm PN}$ as well as spinning sectors, contained in a general standard reduced potential V_s. In our case, it will be sufficient to consider the contributions coming from the 1PN Lagrangian of eq. (5.9), from the LO reduced potentials up to cubic-in-spin, given in eqs. (6.16), (5.15)-(5.16), (6.20), and from the NLO quadratic-in-spin reduced potentials[7] of eqs. (6.34)-(6.35),

$$\vec{p}_1 = m_1\vec{v}_1 + \frac{\partial L_{1\rm PN}}{\partial \vec{v}_1} - \frac{\partial (V_s)^{\rm LO}_{\rm SO}}{\partial \vec{v}_1} - \frac{\partial (V_s)^{\rm LO}_{\rm S^2}}{\partial \vec{v}_1} - \frac{\partial (V_s)^{\rm NLO}_{\rm S^2}}{\partial \vec{v}_1} - \frac{\partial (V_s)^{\rm LO}_{\rm S^3}}{\partial \vec{v}_1}, \quad (8.2)$$

where the derivatives of the LO quadratic-in-spin potentials vanish as they do not contain velocities. One has then to invert this expression in a PN expansion to obtain the relation $\vec{v}_I(\vec{p})$. Expanded explicitly to LO SO it becomes

$$v_1^i = \frac{p_1^i}{m_1} - \frac{p_1^i\, p_1^2}{2m_1^3} + \frac{G}{r}\Big[-\frac{3m_2 p_1^i}{m_1} + \frac{7}{2}p_2^i + \frac{1}{2}(\vec{p}_2\cdot\vec{n})\, n^i\Big] - \frac{3Gm_2 S_1^{ij} n^j}{2m_1 r^2} - \frac{2GS_2^{ij} n^j}{r^2}, \quad (8.3)$$

as given in eq. (6.3) of [45]. The full expression expanded to LO $\rm S^3$ and to NLO quadratic-in-spin order is provided in §11.5, which is only used at the NLO $\rm S^3$.

Then, the total Hamiltonian H is obtained with an ordinary Legendre transformation, where we substitute the relation $\vec{v}_I(\vec{p})$ in the velocities,

$$H = \vec{v}_1\cdot\vec{p}_1 + \vec{v}_2\cdot\vec{p}_2 - L = \vec{v}_1\cdot\vec{p}_1 + \vec{v}_2\cdot\vec{p}_2 - \frac{1}{2}m_1 v_1^2 - \frac{1}{2}m_2 v_2^2 + V_{\rm N} - L_{n\rm PN} + V_s$$

[7] We do not need to consider the NLO SO sector in the canonical momentum as it would only create NLO $\rm S^3$ contributions when substituted in the LO quadratic-in-spin sectors, which do not have velocities.

$$
\begin{aligned}
&= \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{1}{2m_1}(\vec{p}_1 - m_1\vec{v}_1)^2 - \frac{1}{2m_2}(\vec{p}_2 - m_2\vec{v}_2)^2 + V_{\rm N} - L_{n\rm PN} + V_s \\
&= H_{\rm N} - \frac{1}{2m_1}\Big(\frac{\partial L_{n\rm PN}}{\partial \vec{v}_1} - \frac{\partial V_s}{\partial \vec{v}_1}\Big)^2 - \frac{1}{2m_2}\Big(\frac{\partial L_{n\rm PN}}{\partial \vec{v}_2} - \frac{\partial V_s}{\partial \vec{v}_2}\Big)^2 - L_{n\rm PN} + V_s, \qquad (8.4)
\end{aligned}
$$

and where we already identified the Newtonian Hamiltonian,

$$
H_{\rm N} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V_{\rm N} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{Gm_1m_2}{r}. \qquad (8.5)
$$

Identifying the relevant PN orders in the extra terms of eq. (8.4) with $\vec{p}_I \sim \vec{v}_I$, we can obtain all PN corrections to the Hamiltonian. More concretely, the 1PN correction is obtained by a direct Legendre transform of the 1PN Lagrangian in eq. (5.9), and is given by

$$
\begin{aligned}
H_{\rm 1PN} =& - L_{\rm 1PN}\Big(\vec{v}_I \to \frac{\vec{p}_I}{m_I}\Big) \\
=& - \frac{p_1^4}{8m_1^3} - \frac{p_2^4}{8m_2^3} + \frac{G}{2r}\Big[-3\Big(\frac{m_2}{m_1}p_1^2 + \frac{m_1}{m_2}p_2^2\Big) + 7\vec{p}_1\cdot\vec{p}_2 + \vec{p}_1\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}\Big] \\
&+ \frac{G^2m_1m_2}{2r^2}(m_1+m_2), \qquad (8.6)
\end{aligned}
$$

as in eq. (6.5) of [45]. Analogously, the LO SO Hamiltonian, derived from the reduced potential of eq. (6.16), is

$$
H_{\rm SO}^{\rm LO} = (V_s)_{\rm SO}^{\rm LO}\Big(\vec{v}_I \to \frac{\vec{p}_I}{m_I}\Big) = -\frac{3Gm_2}{2m_1r^2}\vec{S}_1\cdot\vec{p}_1\times\vec{n} + \frac{2G}{r^2}\vec{S}_1\cdot\vec{p}_2\times\vec{n} + (1\leftrightarrow 2), \qquad (8.7)
$$

as given also in eq. (6.7) of [45]. For the LO quadratic-in-spin sectors, given in eqs. (5.15)-(5.16), since their reduced potentials do not contain velocities, they will be equivalent to the corresponding Hamiltonians [24],

$$
\begin{aligned}
H_{\rm S_1S_2}^{\rm LO} &= (V_s)_{\rm S_1S_2}^{\rm LO} = -\frac{G}{r^3}\left[\vec{S}_1\cdot\vec{S}_2 - 3\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right], \qquad (8.8)\\
H_{\rm SS}^{\rm LO} &= (V_s)_{\rm SS}^{\rm LO} = -\frac{C_{1({\rm ES}^2)}Gm_2}{2m_1r^3}\left[S_1^2 - 3(\vec{S}_1\cdot\vec{n})^2\right] + (1\leftrightarrow 2). \qquad (8.9)
\end{aligned}
$$

The LO $\rm S^3$ Hamiltonian is again obtained from a straightforward Legendre transform of its reduced potential in eq. (6.20),

$$
\begin{aligned}
H_{\rm S^3}^{\rm LO} =& (V_s)_{\rm S^3}^{\rm LO}\Big(\vec{v}_I \to \frac{\vec{p}_I}{m_I}\Big) \\
=& -3\frac{C_{1({\rm ES}^2)}G}{m_1r^4}\Big[\vec{S}_2\cdot\Big(\frac{\vec{p}_1\times\vec{n}}{m_1} - \frac{\vec{p}_2\times\vec{n}}{m_2}\Big)S_1^2 + 2\vec{S}_1\cdot\Big(\frac{\vec{S}_2\times\vec{p}_1}{m_1} - \frac{\vec{S}_2\times\vec{p}_2}{m_2}\Big)\vec{S}_1\cdot\vec{n} \\
&- 5\vec{S}_2\cdot\Big(\frac{\vec{p}_1\times\vec{n}}{m_1} - \frac{\vec{p}_2\times\vec{n}}{m_2}\Big)(\vec{S}_1\cdot\vec{n})^2 - \frac{1}{4}\Big(-\frac{2}{m_2}\vec{S}_1\cdot\vec{S}_2\times\vec{p}_2\,\vec{S}_1\cdot\vec{n} \\
&+ \Big(\frac{m_2}{m_1^2}\vec{S}_1\cdot\vec{p}_1\times\vec{n} - \frac{1}{m_2}\vec{S}_2\cdot\vec{p}_2\times\vec{n}\Big)\Big(S_1^2 - 5(\vec{S}_1\cdot\vec{n})^2\Big)\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{C_{1(\mathrm{BS}^3)}Gm_2}{m_1^2r^4}\vec{S}_1\cdot\left(\frac{\vec{p}_1\times\vec{n}}{m_1}-\frac{\vec{p}_2\times\vec{n}}{m_2}\right)\left(S_1^2-5(\vec{S}_1\cdot\vec{n})^2\right)\\
&+\frac{3G}{2m_1^2r^4}\left[\vec{S}_1\cdot\vec{p}_1\times\vec{n}\left(\vec{S}_1\cdot\vec{S}_2-5\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right)-\vec{S}_1\cdot\vec{S}_2\times\vec{p}_1\,\vec{S}_1\cdot\vec{n}\right]+(1\leftrightarrow 2),
\end{aligned}
\tag{8.10}
$$

in agreement with eq. (3.12) of [19]. At next-to-leading order the procedure is the same, but now we also have to take into account contributions coming from lower-order sectors. Thus, the NLO SO Hamiltonian is obtained from the Legendre transform of its reduced potential, expressed in eq. (6.24), and also from combinations in eq. (8.4) of the 1PN and the LO SO sectors,

$$
\begin{aligned}
H_{\mathrm{SO}}^{\mathrm{NLO}}=&-L_{1\mathrm{PN}}\left(\vec{v}_I\rightarrow\frac{\vec{p}_I}{m_I}+\frac{1}{m_I}\frac{\partial(V_s)_{\mathrm{SO}}^{\mathrm{LO}}}{\partial\vec{v}_I}\right)+(V_s)_{\mathrm{SO}}^{\mathrm{LO}}\left(\vec{v}_I\rightarrow\frac{\vec{p}_I}{m_I}-\frac{1}{m_I}\frac{\partial L_{1\mathrm{PN}}}{\partial\vec{v}_I}\right)\\
&+\frac{1}{m_1}\frac{\partial L_{1\mathrm{PN}}}{\partial\vec{v}_1}\cdot\frac{\partial(V_s)_{\mathrm{SO}}^{\mathrm{LO}}}{\partial\vec{v}_1}+\frac{1}{m_2}\frac{\partial L_{1\mathrm{PN}}}{\partial\vec{v}_2}\cdot\frac{\partial(V_s)_{\mathrm{SO}}^{\mathrm{LO}}}{\partial\vec{v}_2}+(V_s)_{\mathrm{SO}}^{\mathrm{NLO}}\left(\vec{v}_I\rightarrow\frac{\vec{p}_I}{m_I}\right)\\
=&\frac{Gm_2}{r^2}\left[\frac{\vec{S}_1\cdot\vec{p}_1\times\vec{n}}{m_1}\left(\frac{5}{8}\frac{p_1^2}{m_1^2}+3\frac{\vec{p}_1\cdot\vec{n}}{m_1}\frac{\vec{p}_2\cdot\vec{n}}{m_2}-\frac{3}{4}\left(\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right)^2\right)\right.\\
&-\frac{\vec{S}_1\cdot\vec{p}_2\times\vec{n}}{m_2}\left(\frac{\vec{p}_1\cdot\vec{p}_2}{m_1m_2}+3\frac{\vec{p}_1\cdot\vec{n}}{m_1}\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right)+2\frac{\vec{S}_1\cdot\vec{p}_1\times\vec{p}_2}{m_1m_2}\frac{\vec{p}_1\cdot\vec{n}}{m_1}\\
&\left.-\frac{5}{2}\frac{\vec{S}_1\cdot\vec{p}_1\times\vec{p}_2}{m_1m_2}\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right]+\frac{G^2m_1m_2}{r^3}\left[\frac{7}{2}\frac{\vec{S}_1\cdot\vec{p}_1\times\vec{n}}{m_1}-6\frac{\vec{S}_1\cdot\vec{p}_2\times\vec{n}}{m_2}\right]\\
&+\frac{G^2m_2^2}{r^3}\left[5\frac{\vec{S}_1\cdot\vec{p}_1\times\vec{n}}{m_1}-\frac{35}{4}\frac{\vec{S}_1\cdot\vec{p}_2\times\vec{n}}{m_2}\right]+(1\leftrightarrow 2),
\end{aligned}
\tag{8.11}
$$

as obtained in eq. (6.22) of [24]. Since the LO quadratic-in-spin sectors did not contain velocities, the Legendre transform for the next-to-leading order case is much simpler, and only contains the reduced potentials of eqs. (6.34)-(6.35) and self-iterations of the LO SO sector,

$$
\begin{aligned}
H_{\mathrm{S_1S_2}}^{\mathrm{NLO}}=&(V_s)_{\mathrm{SO}}^{\mathrm{LO}}\left(\vec{v}_I\rightarrow\frac{\vec{p}_I}{m_I}+\frac{1}{m_I}\frac{\partial(V_s)_{\mathrm{SO}}^{\mathrm{LO}}}{\partial\vec{v}_I}\right)+(V_s)_{\mathrm{S_1S_2}}^{\mathrm{NLO}}\left(\vec{v}_I\rightarrow\frac{\vec{p}_I}{m_I}\right)\\
&-\frac{1}{2m_1}\left(\frac{\partial(V_s)_{\mathrm{SO}}^{\mathrm{LO}}}{\partial\vec{v}_1}\right)^2-\frac{1}{2m_2}\left(\frac{\partial(V_s)_{\mathrm{SO}}^{\mathrm{LO}}}{\partial\vec{v}_2}\right)^2\\
=&-\frac{G}{r^3}\left[\vec{S}_1\cdot\vec{S}_2\left(\frac{5}{2}\frac{p_1^2}{m_1^2}-6\frac{\vec{p}_1\cdot\vec{p}_2}{m_1m_2}+\frac{5}{2}\frac{p_2^2}{m_2^2}-6\left(\frac{\vec{p}_1\cdot\vec{n}}{m_1}\right)^2+\frac{45}{4}\frac{\vec{p}_1\cdot\vec{n}}{m_1}\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right.\right.\\
&\left.-6\left(\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right)^2\right)+\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\left(-\frac{3}{2}\frac{p_1^2}{m_1^2}+\frac{21}{4}\frac{\vec{p}_1\cdot\vec{p}_2}{m_1m_2}-\frac{3}{2}\frac{p_2^2}{m_2^2}+\frac{15}{2}\frac{\vec{p}_1\cdot\vec{n}}{m_1}\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right)\\
&+\frac{\vec{S}_1\cdot\vec{p}_1}{m_1}\left(-\frac{5}{2}\frac{\vec{S}_2\cdot\vec{p}_1}{m_1}+\frac{5}{2}\frac{\vec{S}_2\cdot\vec{p}_2}{m_2}+\frac{3}{2}\vec{S}_2\cdot\vec{n}\frac{\vec{p}_1\cdot\vec{n}}{m_1}-\frac{9}{2}\vec{S}_2\cdot\vec{n}\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right)\\
&+\frac{\vec{S}_1\cdot\vec{p}_2}{m_2}\left(3\frac{\vec{S}_2\cdot\vec{p}_1}{m_1}-\frac{5}{2}\frac{\vec{S}_2\cdot\vec{p}_2}{m_2}-\frac{21}{4}\vec{S}_2\cdot\vec{n}\frac{\vec{p}_1\cdot\vec{n}}{m_1}+6\vec{S}_2\cdot\vec{n}\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right)\\
&\left.+\vec{S}_1\cdot\vec{n}\left(6\frac{\vec{S}_2\cdot\vec{p}_1}{m_1}\frac{\vec{p}_1\cdot\vec{n}}{m_1}-\frac{21}{4}\frac{\vec{S}_2\cdot\vec{p}_1}{m_1}\frac{\vec{p}_2\cdot\vec{n}}{m_2}-\frac{9}{2}\frac{\vec{S}_2\cdot\vec{p}_2}{m_2}\frac{\vec{p}_1\cdot\vec{n}}{m_1}+\frac{3}{2}\frac{\vec{S}_2\cdot\vec{p}_2}{m_2}\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right)\right]
\end{aligned}
$$

$$+\frac{G^2\,(m_1+m_2)}{r^4}\Big[7\vec{S}_1\cdot\vec{S}_2-13\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\Big], \tag{8.12}$$

$$\begin{aligned}
H_{\rm SS}^{\rm NLO} &= (V_s)_{\rm SO}^{\rm LO}\Big(\vec{v}_I\rightarrow\frac{\vec{p}_I}{m_I}+\frac{1}{m_I}\frac{\partial(V_s)_{\rm SO}^{\rm LO}}{\partial\vec{v}_I}\Big)+(V_s)_{\rm SS}^{\rm NLO}\Big(\vec{v}_I\rightarrow\frac{\vec{p}_I}{m_I}\Big)\\
&\quad-\frac{1}{2m_1}\left(\frac{\partial(V_s)_{\rm SO}^{\rm LO}}{\partial\vec{v}_1}\right)^2-\frac{1}{2m_2}\left(\frac{\partial(V_s)_{\rm SO}^{\rm LO}}{\partial\vec{v}_2}\right)^2\\
&=-\frac{C_{1({\rm ES}^2)}Gm_2}{2m_1r^3}\Bigg[S_1^2\left(\frac{5}{2}\frac{p_1^2}{m_1^2}-\frac{9}{2}\frac{\vec{p}_1\cdot\vec{p}_2}{m_1m_2}+\frac{3}{2}\frac{p_2^2}{m_2^2}-3\left(\frac{\vec{p}_1\cdot\vec{n}}{m_1}\right)^2+\frac{3}{2}\frac{\vec{p}_1\cdot\vec{n}}{m_1}\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right)\\
&\quad+\frac{\vec{S}_1\cdot\vec{p}_1}{m_1}\left(-\frac{\vec{S}_1\cdot\vec{p}_1}{m_1}+\frac{\vec{S}_1\cdot\vec{p}_2}{m_2}+3\vec{S}_1\cdot\vec{n}\,\frac{\vec{p}_1\cdot\vec{n}}{m_1}-3\vec{S}_1\cdot\vec{n}\,\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right)\\
&\quad-3\frac{\vec{S}_1\cdot\vec{p}_2}{m_2}\,\vec{S}_1\cdot\vec{n}\,\frac{\vec{p}_1\cdot\vec{n}}{m_1}+(\vec{S}_1\cdot\vec{n})^2\left(-\frac{9}{2}\frac{p_1^2}{m_1^2}+\frac{21}{2}\frac{\vec{p}_1\cdot\vec{p}_2}{m_1m_2}-\frac{9}{2}\frac{p_2^2}{m_2^2}\right.\\
&\quad\left.+\frac{15}{2}\frac{\vec{p}_1\cdot\vec{n}}{m_1}\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right)\Bigg]+\frac{Gm_2}{m_1r^3}\Bigg[S_1^2\left(\frac{5}{4}\frac{p_1^2}{m_1^2}-\frac{3}{2}\frac{\vec{p}_1\cdot\vec{p}_2}{m_1m_2}+\frac{3}{2}\frac{\vec{p}_1\cdot\vec{n}}{m_1}\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right.\\
&\quad\left.-\frac{9}{8}\left(\frac{\vec{p}_1\cdot\vec{n}}{m_1}\right)^2\right)-3\frac{\vec{S}_1\cdot\vec{p}_2}{m_2}\,\vec{S}_1\cdot\vec{n}\,\frac{\vec{p}_1\cdot\vec{n}}{m_1}-\frac{21}{8}(\vec{S}_1\cdot\vec{n})^2\frac{p_1^2}{m_1^2}+3(\vec{S}_1\cdot\vec{n})^2\frac{\vec{p}_1\cdot\vec{p}_2}{m_1m_2}\\
&\quad+\frac{\vec{S}_1\cdot\vec{p}_1}{m_1}\left(-\frac{5}{4}\frac{\vec{S}_1\cdot\vec{p}_1}{m_1}+\frac{3}{2}\frac{\vec{S}_1\cdot\vec{p}_2}{m_2}+\frac{15}{4}\vec{S}_1\cdot\vec{n}\,\frac{\vec{p}_1\cdot\vec{n}}{m_1}-\frac{3}{2}\vec{S}_1\cdot\vec{n}\,\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right)\Bigg]\\
&\quad+\frac{C_{1({\rm ES}^2)}G^2m_2}{2r^4}\left[S_1^2-3\left(\vec{S}_1\cdot\vec{n}\right)^2\right]+\frac{C_{1({\rm ES}^2)}G^2m_2^2}{m_1r^4}\left[2S_1^2-5\left(\vec{S}_1\cdot\vec{n}\right)^2\right]\\
&\quad+\frac{G^2m_2}{r^4}\left[2S_1^2-3\left(\vec{S}_1\cdot\vec{n}\right)^2\right]+\frac{G^2m_2^2}{m_1r^4}\left[S_1^2-\left(\vec{S}_1\cdot\vec{n}\right)^2\right]+(1\leftrightarrow2),
\end{aligned} \tag{8.13}$$

as given in eqs. (6.32) and (6.45) of [24]. With these Hamiltonians, we will be able to compute the new NLO S^3 correction, which is done in §11.5 of the results.

8.2 Simplified Hamiltonian

Up to this point, all of the potentials and Hamiltonians that we already computed are gauge or coordinate dependent. By contrast, we would be interested in obtaining gauge-invariant quantities. For a binary system, the most natural observable that we can think of is the total energy of the system, or binding energy, which is global and gauge-invariant. Similarly, the total angular momentum of the system is also gauge-invariant, and so we would like to express the binding energy in terms of it. In fact, this relation is a very useful tool used in different descriptions of non-spinning binary systems with circular orbits [45], such as the effective-one-body formalism and even numerical simulations [62, 63], where they are used to refine the gravitational waveform templates.

To calculate the binding energy, for which we will follow the description given in §8 of [45], we can consider the center-of-mass frame, where we have vanishing total linear momentum,

$$\vec{p}\equiv\vec{p}_1=-\vec{p}_2, \tag{8.14}$$

and we also specify to the condition $r = \text{const.}$ for circular orbits, which implies a vanishing radial momentum p_r and thus the relations

$$p_r \equiv \vec{p}\cdot\vec{n} = 0, \qquad p^2 = \frac{L^2}{r^2}. \tag{8.15}$$

Then, to obtain the gauge-invariant relations, we rescale all variables so that they become dimensionless. With that purpose, we define the total mass $m = m_1 + m_2$ and the reduced mass $\mu = m_1 m_2/m^2$ of the system, so that we rescale

$$\tilde{H} \equiv \frac{H}{\mu}, \quad \tilde{r} \equiv \frac{r}{Gm}, \quad \tilde{L} \equiv \frac{L}{Gm\mu}, \quad \tilde{S}_I \equiv \frac{S_I}{Gm\mu}, \quad \text{for } I = 1,2, \tag{8.16}$$

where the orbital angular momentum is defined by $\vec{L} \equiv r\,\vec{n}\times\vec{p}$, and where $\tilde{H}$ is known as the simplified Hamiltonian. Here, H stands for the sum of all Hamiltonians except the rest-mass contribution, so that the total simplified Hamiltonian equals the dimensionless binding energy, $\tilde{H} \equiv e$.

For the general spinning case, the energy also depends on the spins of the components and their orientations. Therefore, to define a gauge-invariant relation, it is customary to consider the aligned-spins case, where the spins are aligned with the orbital angular momentum,

$$\vec{S}_I\cdot\vec{n} = \vec{S}_I\cdot\vec{p} = 0, \quad \text{for } I = 1,2. \tag{8.17}$$

Furthermore, this configuration of circular orbits with aligned spins corresponds to the case in which gravitational-wave detectors have the highest signal-to-noise ratio [63], the so-called "orbital configuration".

Finally, it is also useful to define the mass ratio q and the symmetric mass ratio ν, given by

$$q \equiv \frac{m_1}{m_2}, \qquad \nu \equiv \frac{m_1 m_2}{m^2} = \frac{\mu}{m} = \frac{q}{(1+q)^2}, \tag{8.18}$$

out of which all results can be expressed. Using all conditions (8.17)-(8.18), as well as the Hamiltonians that we derived in (8.5)-(8.13), we can obtain all of the corresponding simplified Hamiltonians, which are much simpler than in the general case. To 1PN order in the non-spinning case, they are

$$\tilde{H}_{\mathrm{N}} = \frac{1}{\tilde{r}}\left[-1 + \frac{\tilde{L}^2}{2\tilde{r}}\right], \tag{8.19}$$

$$\tilde{H}_{1\mathrm{PN}} = \frac{1}{2\tilde{r}^2}\left[1 - \frac{\tilde{L}^2}{\tilde{r}}(3+\nu) + \frac{\tilde{L}^4}{4\tilde{r}^2}(-1+3\nu)\right], \tag{8.20}$$

as given in eqs. (4.23)-(4.24) of [28]. For the spinning sectors, to leading order they become

$$\tilde{H}^{\mathrm{LO}}_{\mathrm{SO}} = \frac{\nu\tilde{L}\tilde{S}_1}{\tilde{r}^3}\left[2 + \frac{3}{2q}\right] + (1\leftrightarrow 2), \tag{8.21}$$

$$\tilde{H}^{\mathrm{LO}}_{\mathrm{S_1S_2}} = -\frac{\nu\tilde{S}_1\tilde{S}_2}{\tilde{r}^3}, \tag{8.22}$$

$$\tilde{H}^{\mathrm{LO}}_{\mathrm{SS}} = -\frac{\nu\tilde{S}_1^2}{2q\tilde{r}^3}C_{1(\mathrm{ES}^2)} + (1\leftrightarrow 2), \tag{8.23}$$

$$
\begin{aligned}
\tilde{H}^{\mathrm{LO}}_{\mathrm{S}^3} &= \frac{\nu \tilde{L}\tilde{S}_1^3}{\tilde{r}^5}\left[\frac{3\nu}{4}C_{1(\mathrm{ES}^2)} - \nu\, C_{1(\mathrm{BS}^3)} + \frac{1}{q}\Big(C_{1(\mathrm{ES}^2)}\Big(-\frac{3}{4}+\frac{3\nu}{2}\Big) + C_{1(\mathrm{BS}^3)}(1-\nu)\Big)\right] \\
&\quad + \frac{\nu^2\tilde{L}}{\tilde{r}^5}\tilde{S}_1^2\tilde{S}_2\left[\frac{9}{4}C_{1(\mathrm{ES}^2)} + \frac{1}{q}\Big(-\frac{3}{2}+3C_{1(\mathrm{ES}^2)}\Big)\right] + (1\leftrightarrow 2),
\end{aligned} \tag{8.24}
$$

whereas to next-to-leading order they read

$$
\tilde{H}^{\mathrm{NLO}}_{\mathrm{SO}} = \frac{\nu\tilde{L}\tilde{S}_1}{\tilde{r}^4}\left[-6-\frac{5\nu}{4}+\frac{13\nu\tilde{L}^2}{8\tilde{r}}+\frac{1}{q}\Big(-5-\frac{5\nu}{4}+\frac{\tilde{L}^2}{8\tilde{r}}(-5+10\nu)\Big)\right] + (1\leftrightarrow 2), \tag{8.25}
$$

$$
\tilde{H}^{\mathrm{NLO}}_{\mathrm{S}_1\mathrm{S}_2} = \frac{\nu\tilde{S}_1\tilde{S}_2}{\tilde{r}^4}\left[7-\frac{\tilde{L}^2}{2\tilde{r}}(5+2\nu)\right], \tag{8.26}
$$

$$
\begin{aligned}
\tilde{H}^{\mathrm{NLO}}_{\mathrm{SS}} &= \frac{\nu\tilde{S}_1^2}{2\tilde{r}^4}\bigg[2\nu - 3\nu C_{1(\mathrm{ES}^2)} + \frac{\tilde{L}^2}{2\tilde{r}}\Big(-5\nu+2\nu C_{1(\mathrm{ES}^2)}\Big) + \frac{1}{q}\Big(2+2\nu+C_{1(\mathrm{ES}^2)}(4-3\nu) \\
&\quad + \frac{\tilde{L}^2}{2\tilde{r}}\Big(5-4\nu+C_{1(\mathrm{ES}^2)}(-5+\nu)\Big)\Big)\bigg] + (1\leftrightarrow 2),
\end{aligned} \tag{8.27}
$$

in agreement with (4.26)-(4.33) of [28]. Note that under exchange of particles $(1 \leftrightarrow 2)$, the mass ratio q goes to $1/q$, but the symmetric mass ratio ν is invariant. For examples of simplified Hamiltonians at LO quartic-in-spin and up to NNLO quadratic-in-spin, see [29] and [28, 45], respectively.

9 Observables and gauge-invariant relations

In this section we will use the simplified Hamiltonian derived in §8 to compute gauge-invariant observables for circular orbits, namely the relation between the binding energy and the angular momentum or with the orbital frequency of the binary.

9.1 Binding energy and angular momentum

The simplified Hamiltonians are still gauge-dependent, as they are a function of the rescaled radial coordinate $\tilde{r}$. In order to eliminate it we will use the condition for circular orbits in eq. (8.15), which is preserved in time and reads, by Hamilton's equations,

$$\dot{\tilde{p}}_r = -\frac{\partial \tilde{H}(\tilde{r}, \tilde{L})}{\partial \tilde{r}} = 0, \tag{9.1}$$

to obtain a relation $\tilde{r}(\tilde{L})$. Then, we will substitute this relation in the binding energy $\tilde{H}(\tilde{r}, \tilde{L}) \equiv e(\tilde{r}, \tilde{L})$ to obtain the gauge-invariant relation $e(\tilde{L})$.

The simplest way to obtain $\tilde{r}(\tilde{L})$ is to define the following ansatz for the solution:

$$\frac{1}{\tilde{r}} = \sum_{n=2}^{11} \frac{C_n}{\tilde{L}^n}, \tag{9.2}$$

where the summation starts at $n = 2$ so that $(\tilde{r}(\tilde{L}))^{-1} \propto \tilde{L}^{-2} \propto v^2$ is of order 0PN, and corresponds to the Newtonian case, and ends at $n = 11$ so that $(\tilde{r}(\tilde{L}))^{-1} \supseteq \tilde{L}^{-11} \propto v^2 \cdot v^9$ is of order 4.5PN, as our desired NLO S^3 correction. Hence, the substitution of the ansatz of eq. (9.2) into the equation of motion for circular orbits of eq. (9.1) allows for all sectors to intertwine perturbatively. Solving order by order in the orbital angular momentum, we obtain the solution

$$\begin{aligned}
\frac{1}{\tilde{r}} = {} & \frac{1}{\tilde{L}^2} + \frac{4}{\tilde{L}^4} + \frac{\nu \tilde{S}_1}{\tilde{L}^5}\left[-6 + \frac{1}{\tilde{L}^2}\left(-81 + \frac{47\nu}{8}\right) + \frac{1}{q}\left(-\frac{9}{2} + \frac{1}{\tilde{L}^2}\left(-\frac{445}{8} + \frac{11\nu}{2}\right)\right)\right] \\
& + \frac{\nu \tilde{S}_1 \tilde{S}_2}{\tilde{L}^6}\left[3 + \frac{1}{\tilde{L}^2}\left(145 + \frac{19\nu}{2}\right)\right] + \frac{\nu \tilde{S}_1^2}{\tilde{L}^6}\left[\frac{\nu}{\tilde{L}^2}\left(\frac{135}{4} + \frac{7}{2} C_{1(\mathrm{ES}^2)}\right)\right. \\
& \left. + \frac{1}{q}\left(\frac{3}{2} C_{1(\mathrm{ES}^2)} + \frac{1}{4\tilde{L}^2}\left(121 + 112\nu + C_{1(\mathrm{ES}^2)}(98 + 10\nu)\right)\right)\right] \\
& + \frac{\nu \tilde{S}_1^3}{\tilde{L}^9}\left[\frac{93\nu}{4} C_{1(\mathrm{ES}^2)} + 5\nu\, C_{1(\mathrm{BS}^3)} + \frac{1}{q}\left(C_{1(\mathrm{ES}^2)}\left(-\frac{93}{4} + \frac{21\nu}{2}\right) + C_{1(\mathrm{BS}^3)}(-5 + 5\nu)\right)\right] \\
& + \frac{\nu^2 \tilde{S}_1^2 \tilde{S}_2}{\tilde{L}^9}\left[-72 - \frac{153}{4} C_{1(\mathrm{ES}^2)} + \frac{1}{q}\left(-\frac{93}{2} - 51 C_{1(\mathrm{ES}^2)}\right)\right] + (1 \leftrightarrow 2),
\end{aligned} \tag{9.3}$$

in agreement with eq. (5.1) of [28].

Given the solution for $\tilde{r}(\tilde{L})$, we substitute it in the simplified Hamiltonians $\tilde{H}(\tilde{r}, \tilde{L})$ given in eqs. (8.19)-(8.27) to obtain the gauge-invariant relation for the binding energy as a function of the angular momentum, which reads

$$e(\tilde{L}) = -\frac{1}{2\tilde{L}^2} - \frac{1}{8\tilde{L}^4}(9 + \nu) + \frac{\nu \tilde{S}_1}{\tilde{L}^5}\left[2 + \frac{1}{\tilde{L}^2}\left(18 + \frac{3\nu}{8}\right) + \frac{1}{q}\left(\frac{3}{2} + \frac{99}{8\tilde{L}^2}\right)\right]$$

$$
\begin{aligned}
&-\frac{\nu\tilde{S}_1\tilde{S}_2}{\tilde{L}^6}\left[1+\frac{1}{\tilde{L}^2}\left(\frac{69}{2}+\frac{13\nu}{4}\right)\right]-\frac{\nu\tilde{S}_1^2}{\tilde{L}^6}\left[\frac{\nu}{\tilde{L}^2}\left(\frac{65}{8}+C_{1(\mathrm{ES}^2)}\right)\right.\\
&\left.+\frac{1}{q}\left(\frac{1}{2}C_{1(\mathrm{ES}^2)}+\frac{1}{8\tilde{L}^2}(63+54\nu)+\frac{1}{4\tilde{L}^2}C_{1(\mathrm{ES}^2)}(21+5\nu)\right)\right]\\
&+\frac{\nu\tilde{S}_1^3}{\tilde{L}^9}\left[-6\nu\, C_{1(\mathrm{ES}^2)}-\nu\, C_{1(\mathrm{BS}^3)}+\frac{1}{q}\left(C_{1(\mathrm{ES}^2)}(6-3\nu)+C_{1(\mathrm{BS}^3)}(1-\nu)\right)\right]\\
&+\frac{\nu^2\tilde{S}_1^2\tilde{S}_2}{\tilde{L}^9}\left[18+9C_{1(\mathrm{ES}^2)}+\frac{1}{q}\left(12+12C_{1(\mathrm{ES}^2)}\right)\right]+(1\leftrightarrow 2),
\end{aligned}
\tag{9.4}
$$

in complete agreement with [17] for the non-spinning sectors, and with [28] for the spinning case.

9.2 Binding energy and orbital frequency

As a second application, we can also derive the binding energy as a function of the orbital frequency of the binary, also gauge-invariant as measured by an asymptotic observer. This relation is very useful, e.g. it can be used to obtain the change of the frequency of the emitted gravitational wave over time, or in other words, the phasing of the wave. To do so, we must obtain first a relation between the angular momentum and the orbital frequency of the system, to be substituted in eq. (9.4).

With that goal in mind, we use the equation of motion for the orbital phase ϕ, which by Hamilton's equations is related as canonical conjugate to the angular momentum $\tilde{L}$ by

$$
\tilde{\omega}\equiv\frac{d\phi}{d\tilde{t}}=\frac{\partial\tilde{H}(\tilde{r},\tilde{L})}{\partial\tilde{L}},
\tag{9.5}
$$

where we also define the orbital frequency $\tilde{\omega}$. After evaluating the derivative, we substitute the solution for $\tilde{r}(\tilde{L})$ of eq. (9.3), and we obtain a relation for $\tilde{\omega}(\tilde{L})$, which schematically reads, up to NLO S^3,

$$
\tilde{\omega}=\frac{1}{\tilde{L}^3}+\frac{9+\nu}{2\tilde{L}^5}+\frac{\tilde{S}_1\alpha_1}{\tilde{L}^6}+\frac{\tilde{S}_1\tilde{S}_2\alpha_2+\tilde{S}_1^2\alpha_3}{\tilde{L}^7}+\cdots+\frac{\tilde{S}_1^3\alpha_n+\tilde{S}_1^2\tilde{S}_2\alpha_{n+1}}{\tilde{L}^{12}}+(1\leftrightarrow 2),
\tag{9.6}
$$

for α_i being general coefficients. Then, it is useful to define the gauge-invariant parameter x, given by

$$
x\equiv\tilde{\omega}^{2/3},
\tag{9.7}
$$

which by Kepler's third law, $\omega^2=Gm/r^3$, acts as a measure of the inverse of the orbital separation of the binary. Schematically, we obtain

$$
\begin{aligned}
x&=\frac{1}{\tilde{L}^2}\left[1+\frac{9+\nu}{2\tilde{L}^2}+\frac{\tilde{S}_1\alpha_1}{\tilde{L}^3}+\frac{\tilde{S}_1\tilde{S}_2\alpha_2+\tilde{S}_1^2\alpha_3}{\tilde{L}^4}+\cdots+\frac{\tilde{S}_1^3\alpha_n+\tilde{S}_1^2\tilde{S}_2\alpha_{n+1}}{\tilde{L}^9}+(1\leftrightarrow 2)\right]^{2/3}\\
&=\frac{1}{\tilde{L}^2}+\frac{9+\nu}{3\tilde{L}^4}+\frac{\tilde{S}_1\beta_1}{\tilde{L}^5}+\frac{\tilde{S}_1\tilde{S}_2\beta_2+\tilde{S}_1^2\beta_3}{\tilde{L}^6}+\cdots+\frac{\tilde{S}_1^3\beta_n+\tilde{S}_1^2\tilde{S}_2\beta_{n+1}}{\tilde{L}^{11}}+(1\leftrightarrow 2),
\end{aligned}
\tag{9.8}
$$

for β_i again representing general coefficients, where we took out a factor of $1/\tilde{L}^2$ to use the Taylor expansion of $(1+z)^{2/3}=1+\frac{2}{3}z+\dots$ to order $\mathcal{O}(z^5)$. Now, we would like to invert this expression to obtain $\tilde{L}(x)$. To do so, we take out the factor of $1/\tilde{L}^2$ and make the following manipulation:

$$
\begin{aligned}
\frac{1}{\tilde{L}^2} &= x\left[1+\frac{9+\nu}{3\tilde{L}^2}+\frac{\tilde{S}_1\beta_1}{\tilde{L}^3}+\frac{\tilde{S}_1\tilde{S}_2\beta_2+\tilde{S}_1^2\beta_3}{\tilde{L}^4}+\dots+\frac{\tilde{S}_1^3\beta_n+\tilde{S}_1^2\tilde{S}_2\beta_{n+1}}{\tilde{L}^9}+(1\leftrightarrow 2)\right]^{-1}\\
&= x\left(1-\frac{9+\nu}{3\tilde{L}^2}+\frac{\tilde{S}_1\delta_1}{\tilde{L}^3}+\frac{\tilde{S}_1\tilde{S}_2\delta_2+\tilde{S}_1^2\delta_3}{\tilde{L}^4}+\dots+\frac{\tilde{S}_1^3\delta_n+\tilde{S}_1^2\tilde{S}_2\delta_{n+1}}{\tilde{L}^9}+(1\leftrightarrow 2)\right),
\end{aligned}
\tag{9.9}
$$

for δ_i again being general, and where we use the Taylor expansion for $(1+z)^{-1}=1-z+\dots$ to order $\mathcal{O}(z^6)$. Then, to obtain $\tilde{L}(x)$ we substitute iteratively the equation into itself, for which we use, for instance, $1/\tilde{L}^3=(1/\tilde{L}^2)^{3/2}$ and its respective Taylor expansion. This way, it results in the relation

$$
\begin{aligned}
\frac{1}{\tilde{L}^2} &= x-x^2\left(3+\frac{\nu}{3}\right)+\nu x^{5/2}\tilde{S}_1\left[\frac{20}{3}-x\left(16+\frac{337\nu}{36}\right)+\frac{1}{q}\left(5-x\left(\frac{69}{4}+\frac{25\nu}{3}\right)\right)\right]\\
&+\nu x^3\tilde{S}_1\tilde{S}_2\left[-4+x\left(\frac{196}{3}+\frac{95\nu}{18}\right)\right]+\nu x^3\tilde{S}_1^2\left[\nu x\left(\frac{365}{36}-\frac{16}{3}C_{1(\mathrm{ES}^2)}\right)\right.\\
&\left.+\frac{1}{q}\left(-2C_{1(\mathrm{ES}^2)}+x\left(\frac{107}{4}+\frac{59\nu}{6}\right)-xC_{1(\mathrm{ES}^2)}(-5+3\nu)\right)\right]\\
&+\nu x^{9/2}\tilde{S}_1^3\left[24\nu\,C_{1(\mathrm{ES}^2)}-6\nu\,C_{1(\mathrm{BS}^3)}+\frac{1}{q}\left(C_{1(\mathrm{ES}^2)}(-24+22\nu)+C_{1(\mathrm{BS}^3)}(6-6\nu)\right)\right]\\
&+\nu^2x^{9/2}\tilde{S}_1^2\tilde{S}_2\left[-52-6C_{1(\mathrm{ES}^2)}+\frac{1}{q}\left(-48-8C_{1(\mathrm{ES}^2)}\right)\right]+(1\leftrightarrow 2),
\end{aligned}
\tag{9.10}
$$

as given in eq. (5.2) of [28]. Finally, we can insert this relation into the expression for $e(\tilde{L})$, given in eq. (9.4), to obtain the gauge-invariant relation between binding energy and orbital frequency,

$$
\begin{aligned}
e(x) &= -\frac{x}{2}+\left(\frac{3}{8}+\frac{\nu}{24}\right)x^2+\nu x^{5/2}\tilde{S}_1\left[-\frac{4}{3}+x\left(-4+\frac{31\nu}{18}\right)+\frac{1}{q}\left(-1+x\left(-\frac{3}{2}+\frac{5\nu}{3}\right)\right)\right]\\
&+\nu x^3\tilde{S}_1\tilde{S}_2\left[1+x\left(\frac{5}{6}+\frac{5\nu}{18}\right)\right]+\nu x^3\tilde{S}_1^2\left[\nu x\left(\frac{25}{18}+\frac{5}{3}C_{1(\mathrm{ES}^2)}\right)\right.\\
&\left.+\frac{1}{q}\left(\frac{1}{2}C_{1(\mathrm{ES}^2)}+\frac{5x}{6}(-3+\nu)+\frac{5x}{4}C_{1(\mathrm{ES}^2)}(1+\nu)\right)\right]\\
&+\nu x^{9/2}\tilde{S}_1^3\left[-3\nu\,C_{1(\mathrm{ES}^2)}+2\nu\,C_{1(\mathrm{BS}^3)}+\frac{1}{q}\left(C_{1(\mathrm{ES}^2)}(3-4\nu)+C_{1(\mathrm{BS}^3)}(-2+2\nu)\right)\right]\\
&+\nu^2x^{9/2}\tilde{S}_1^2\tilde{S}_2\left[4-3C_{1(\mathrm{ES}^2)}+\frac{1}{q}\left(6-4C_{1(\mathrm{ES}^2)}\right)\right]+(1\leftrightarrow 2).
\end{aligned}
\tag{9.11}
$$

in agreement with [17] for the non-spinning sectors, and with [28] for the spinning case.

The new NLO S^3 corrections for the previous gauge-invariant observables will be presented in §11.6 in the results section.

10 Conserved integrals of motion: Poincaré algebra

In the previous section we derived the general Hamiltonian for different PN sectors. The next step would naturally be to verify its validity. That could be addressed in two ways: comparing with independent derivations or via a self-consistency check. Unfortunately, as the NLO S^3 sector represents the cutting-edge result obtained exclusively with the EFT of spinning objects, there are no other independent sources available for a comparison of our comprehensive result. Therefore, to validate our Hamiltonian, we will resort to a very strong self-consistency check coming from the global Poincaré symmetry.

For isolated N-body systems in GR, the full Poincaré group acts as a global symmetry. As first shown in [64], the conservative PN Lagrangian is also Poincaré invariant. Thus, from Noether's theorem, this global symmetry implies the existence of conserved integrals of motion, also constructed in [64]. In phase space, this integrals of motion form a representation of the Poincaré algebra [65]. Therefore, the usual self-consistency check performed in PN theory is to verify whether there exist generators that realize the Poincaré algebra, as it would mean that the Hamiltonian obeys the global Poincaré symmetry. Consequently, in this section we verify that the lower-order PN corrections to the Hamiltonian admit global Poincaré symmetry by explicitly finding the corresponding PN corrections to the generators of the Poincaré algebra.

10.1 The Poincaré algebra

The Poincaré transformations, which include translations in space and time, rotations and boosts, are generated by the total linear momentum $\vec{P}$, the Hamiltonian H, the total angular momentum $\vec{J}$, and the boost generator $\vec{K}$, respectively. Furthermore, we can decompose the boost generator as $\vec{K} = \vec{G} - t\vec{P}$, for $\vec{G}$ being the center-of-mass generator[8]. Then, these generators realize the Poincaré algebra [65], which is explicitly given by

$$\{P_i, P_j\} = \{P_i, H\} = \{J_i, H\} = 0, \qquad \{J_i, P_j\} = \epsilon_{ijk} P_k, \qquad \{J_i, J_j\} = \epsilon_{ijk} J_k, \tag{10.1}$$

$$\{J_i, G_j\} = \epsilon_{ijk} G_k, \quad \{G_i, H\} = P_i, \quad \{G_i, P_j\} = \delta_{ij} H, \quad \{G_i, G_j\} = -\epsilon_{ijk} J_k. \tag{10.2}$$

In the non-spinning two-body phase space $(\vec{x}_1, \vec{x}_2, \vec{p}_1, \vec{p}_2)$, the Poincaré algebra is expressed via the Poisson brackets

$$\{f(\vec{x}, \vec{p}), g(\vec{x}, \vec{p})\}_{(x,p)} \equiv \sum_{I=1}^{2} \left(\frac{\partial f}{\partial x_I^i} \frac{\partial g}{\partial p_I^i} - \frac{\partial f}{\partial p_I^i} \frac{\partial g}{\partial x_I^i} \right), \tag{10.3}$$

where the position and momentum variables are conjugates,

$$\{x_I^i, p_J^j\} = \delta^{ij} \delta_{IJ}, \quad \text{for } I, J = 1, 2. \tag{10.4}$$

In the present case, we also have to include the spin variables, which satisfy the so(3) canonical Poisson bracket of eq. (6.11). In terms of the spin vector, it implies the relation

$$\{S_I^i, S_J^j\} = \epsilon^{ijk} S_I^k \delta_{IJ}, \quad \text{for } I, J = 1, 2, \tag{10.5}$$

[8] To avoid confusion, in this section we will use G_{N} to denote Newton's constant.

as given in eq. (6.8) of [24]. Thus, we extend the Poisson brackets $\{,\}_{(x,p)}$ expressed in eq. (10.3) to its spinning generalization, see §3.A of [39] and [66], given by

$$\begin{aligned}\{f(\vec{x},\vec{p},\vec{S}),g(\vec{x},\vec{p},\vec{S})\} &\equiv \{f,g\}_{(x,p)}+\{f,g\}_{\rm spin}\\ &=\{f,g\}_{(x,p)}+\sum_{I=1}^{2}\epsilon_{klm}S_I^k\frac{\partial f}{\partial S_I^l}\frac{\partial g}{\partial S_I^m}.\end{aligned}\tag{10.6}$$

Having introduced the Poisson brackets used to express the Poincaré algebra, we can then address the calculation of its generators. To do so, we will follow the description and conventions used in [65] for the non-spinning sectors, and in [29, 67–70] for the spinning case. First of all, since our gauge choices do not affect the symmetries of translation and rotation invariance, the Hamiltonian by construction satisfies the Euclidean group symmetry, with algebra written in eq. (10.1). Thus, these Euclidean generators are still the total linear momentum and total angular momentum of the binary system:

$$\vec{P}=\vec{p}_1+\vec{p}_2,\qquad \vec{J}=\vec{x}_1\times\vec{p}_1+\vec{x}_2\times\vec{p}_2+\vec{S}_1+\vec{S}_2.\tag{10.7}$$

Hence, given the Hamiltonian, the construction and verification of the full Poincaré algebra of eqs. (10.1)-(10.2) reduces to the existence of a vector $\vec{G}(\vec{x}_I,\vec{p}_I)$ satisfying such conditions. Starting with the condition $\{G_i,P_j\}=\delta_{ij}H$, we can see that it is solved by the following ansatz:

$$\vec{G}=h_1\vec{x}_1+h_2\vec{x}_2+\vec{Y},\qquad h_1+h_2=H,\tag{10.8}$$

where h_I and $\vec{Y}$ are translation invariant: $\{h_I,P^j\}=\{Y^i,P^j\}=0$. Then, separating the point-mass (PM) and the spinning contributions, we can write

$$h_I=\frac{H}{2}+h_I^{\rm PM}+h_I^{\rm SO}+h_I^{\rm S_1S_2}+h_I^{\rm SS}+h_I^{\rm S^3},\tag{10.9}$$

so that the condition $h_1+h_2=H$ implies antisymmetry under particle exchange in the extra terms $h_I^{\rm PM},\ldots,h_I^{\rm S^3}$. By contrast, we require

$$\vec{Y}=\vec{Y}^{\rm PM}+\vec{Y}^{\rm SO}+\vec{Y}^{\rm S_1S_2}+\vec{Y}^{\rm SS}+\vec{Y}^{\rm S^3},\tag{10.10}$$

to be symmetric under particle exchange. Then, it turns out that the ansatz for $\vec{G}$ can be uniquely fixed using only the condition $\{G_i,H\}=P_i$, while the other Poisson brackets are automatically fulfilled [65].

Recapitulating, this means that the verification of the Poincaré algebra boils down to the existence of a generator $\vec{G}=h_1\vec{x}_1+h_2\vec{x}_2+\vec{Y}$, which is unique, and can be fixed just using one relation of the algebra. To obtain it, we will resort to the method of undetermined coefficients [65], writing the most general form that the generator can have, and constraining the coefficients.

There are several considerations to be taken into account when constructing the ansatz for the generator. First of all, as will be justified in the examples later, the generator is always of one-PN order less than the sector that we want to verify. For instance, the NLO SO Poincaré algebra requires a correction to the generator $\vec{G}$ of the order of the LO

SO. Second, the ansatz for the generator must preserve the tensor structure of the sector according to parity, i.e., it should include triple products for odd-in-spin sectors and scalar products for even-in-spin sectors.

Third, the form of the ansatz is heavily constrained by dimensional analysis. From eq. (10.9) we have that $[h_I] = m$ has mass dimensions, while $[\vec{Y}] = 1$ is dimensionless. Therefore, their mass dimensions have to be built from a combination of powers of $[\frac{1}{r}] = m$, $[G_{\mathrm{N}}] = m^{-2}$, $[S] = 1$ and $[C_{\mathrm{ES}^n}] = [C_{\mathrm{BS}^n}] = 1$, compensated with powers of the mass of the components.

Finally, while in principle one defines an ansatz for all powers in the gravitational constant G_{N} in the generator, it turns out that all $\mathcal{O}(G_{\mathrm{N}}^0)$ terms can be fixed from the special-relativistic limit [71], where no curved spacetime effects are considered. In flat spacetime, a closed form for the generator is given in eq. (4.27) of [71], and reads

$$\vec{G} = \sum_{I=1}^{2} \left(\gamma_I \, m_I \vec{x}_I - \frac{1}{m_I} (1 + \gamma_I)^{-1} \vec{S}_I \times \vec{p}_I \right), \tag{10.11}$$

where $\gamma_I = (1 - v_I^2)^{-1/2} = (1 + p_I^2/m_I^2)^{1/2}$ is the relativistic Lorentz factor. Expanding in the PN scheme, it becomes

$$\vec{G} = \left[m_1 + \frac{p_1^2}{2m_1} - \frac{p_1^4}{8m_1^3} + \dots \right] \vec{x}_1 - \frac{1}{2m_1} \vec{S}_1 \times \vec{p}_1 + \frac{p_1^2}{8m_1^3} \vec{S}_1 \times \vec{p}_1 - \frac{p_1^4}{16m_1^5} \vec{S}_1 \times \vec{p}_1 + \dots + (1 \leftrightarrow 2). \tag{10.12}$$

Therefore, we can already take from this expression the $\mathcal{O}(G_{\mathrm{N}}^0)$ terms for the different PN sectors. As can be seen, for quadratic-in-spin order and beyond there are no contributions. This is because before the accelerations are removed there are no kinetic terms beyond the spin-orbit sectors, as can be observed in the potentials (5.14)-(5.20).

To illustrate the procedure of verification of the Poincaré algebra and the construction of the center-of-mass generator, we will exemplify first some non-spinning sectors, and then we will consider the spinning ones.

10.2 Poincaré algebra for non-spinning sectors

In the non-spinning case, the Poisson brackets are simply given by the usual definition in eq. (10.3). Then, let us start with the lowest PN sector, the Newtonian sector. In this case, the generator $\vec{G}$ is known to be the Newtonian center-of-mass position [65]

$$\vec{G}_{0\mathrm{PN}} = m_1 \vec{x}_1 + m_2 \vec{x}_2. \tag{10.13}$$

Computing the Poincaré algebra, we observe that

$$\{(G_{0\mathrm{PN}})_i, P_j\} = \delta_{ij}(m_1 + m_2). \tag{10.14}$$

From this, we can extract that the Hamiltonian appearing in the Poisson bracket $\{G_i, P_j\} = \delta_{ij} H$ and in eqs. (10.8)-(10.9) is the rest-mass energy $E = m_1(c^2) + m_2(c^2)$. Therefore, we also have that

$$h_1^{\mathrm{PM}}\Big|_{\mathrm{Newton}} = \frac{m_1}{2} - (1 \leftrightarrow 2), \qquad \vec{Y}^{\mathrm{PM}}\Big|_{\mathrm{Newton}} = 0. \tag{10.15}$$

By contrast, the relation $\{G_i, H\} = P_i$ is satisfied only if we use here the Newtonian Hamiltonian, given in eq. (8.5). Consequently, we encounter here the PN order offset announced earlier: To compute the relation $\{G_i, H\} = P_i$ at Newtonian order, we require a generator of one-PN order less, i.e., of order of the rest-mass energy.

Let us then proceed to the next non-spinning PN order, the 1PN sector. Based on the previous offset, at 1PN we require an ansatz for $\vec{G}$, and hence for $h_I^{\rm PM}$ and $\vec{Y}^{\rm PM}$, of 0PN order, so at most proportional to $G_{\rm N}m/r \sim v^2 \sim p^2$. Taking also into account the considerations given in §10 that constrain the form of the ansatz, it reads

$$h_1^{\rm PM}\Big|_{\rm 1PN} = \alpha_1 \frac{p_1^2}{m_1} + \alpha_2 \frac{\vec{p}_1\cdot\vec{p}_2}{m_1} + \alpha_3 \frac{(\vec{p}_1\cdot\vec{n})^2}{m_1} + \alpha_4 \frac{(\vec{p}_1\cdot\vec{n})(\vec{p}_2\cdot\vec{n})}{m_1} - (1\leftrightarrow 2), \tag{10.16}$$

$$\vec{Y}^{\rm PM}\Big|_{\rm 1PN} = \left(\beta_1 \frac{p_1^2}{m_1^2} + \beta_2 \frac{(\vec{p}_1\cdot\vec{n})^2}{m_1^2}\right)\vec{n} + \left(\beta_3 \frac{\vec{p}_1\cdot\vec{n}}{m_1} + \beta_4 \frac{\vec{p}_2\cdot\vec{n}}{m_2}\right)\frac{\vec{p}_1}{m_1} + (1\leftrightarrow 2), \tag{10.17}$$

with α_i and β_i being unknown numerical coefficients. In order to determine them and uniquely fix the ansatz, we must use the condition $\{G_i, H\} = P_i$. Expanding both the generator and the Hamiltonian into its corresponding PN corrections, we obtain the following:

$$\begin{aligned}\vec{P} &= \{\vec{G}, H\}\\ &= \{\vec{G}_{\rm 0PN} + \vec{G}_{\rm 1PN}, H_{\rm N} + H_{\rm 1PN}\}\\ &= \{\vec{G}_{\rm 0PN}, H_{\rm N}\} + \{\vec{G}_{\rm 0PN}, H_{\rm 1PN}\} + \{\vec{G}_{\rm 1PN}, H_{\rm N}\} + \underbrace{\{\vec{G}_{\rm 1PN}, H_{\rm 1PN}\}}_{\mathcal{O}(2PN)},\end{aligned} \tag{10.18}$$

where the last term can be neglected at 1PN order, as it would be of order 2PN. Then, since at Newtonian order the Poisson bracket already resulted in $\{\vec{G}_{\rm 0PN}, H_{\rm N}\} = \vec{P}$, at 1PN we have that

$$\{\vec{G}_{\rm 0PN}, H_{\rm 1PN}\} + \{\vec{G}_{\rm 1PN}, H_{\rm N}\} = 0. \tag{10.19}$$

Therefore, one cannot just naively compute $\{\vec{G}_{n\rm PN}, H\}$ at nPN order, but instead we have to take into account all combinations of generators and Hamiltonians contributing to that certain PN order. To make the combinations contributing to each PN sector clearer, we have included the multiplication Table 10.1, where we show the combinations of generator and Hamiltonian used in the Poisson bracket of eq. (10.3). Since they act differently, we separate the contributions coming from the usual Poisson brackets $\{,\}_{(x,p)}$, given by eq. (10.3), in Table 10.1, and the contributions coming from the spinning addition to the Poisson bracket $\{,\}_{\rm spin}$, defined in eq. (10.6), in Table 10.2. For the latter, which will only be used in the following section, non-spinning sectors are not necessary, as they do not include spin variables and vanish upon substitution in $\{,\}_{\rm spin}$.

In the present case, using the Newtonian generator given in eq. (10.13) and the 1PN ansatz of eqs. (10.16)-(10.17), the condition of eq. (10.19) fixes the undetermined coefficients of the ansatz to be

$$\alpha_1 = \frac{1}{4}, \qquad \alpha_2 = \cdots = \beta_4 = 0, \tag{10.20}$$

	H_{N}	$H_{1\mathrm{PN}}$	$H_{\mathrm{SO}}^{\mathrm{LO}}$	$H_{\mathrm{SO}}^{\mathrm{NLO}}$	$H_{\mathrm{S}^2}^{\mathrm{LO}}$	$H_{\mathrm{S}^2}^{\mathrm{NLO}}$	$H_{\mathrm{S}^3}^{\mathrm{LO}}$	$H_{\mathrm{S}^3}^{\mathrm{NLO}}$
$\vec{G}_{0\mathrm{PN}}$	Newton	1PN	LO SO	NLO SO	LO S^2	NLO S^2	LO S^3	NLO S^3
$\vec{G}_{1\mathrm{PN}}$	1PN		NLO SO		NLO S^2		NLO S^3	
$\vec{G}_{\mathrm{SO}}^{\mathrm{LO}}$	LO SO	NLO SO	NLO S^2		LO S^3	NLO S^3		
$\vec{G}_{\mathrm{SO}}^{\mathrm{NLO}}$	NLO SO				NLO S^3			
$\vec{G}_{\mathrm{S}^2}^{\mathrm{LO}}$	LO S^2	NLO S^2	LO S^3	NLO S^3				
$\vec{G}_{\mathrm{S}^2}^{\mathrm{NLO}}$	NLO S^2		NLO S^3					
$\vec{G}_{\mathrm{S}^3}^{\mathrm{LO}}$	LO S^3	NLO S^3						
$\vec{G}_{\mathrm{S}^3}^{\mathrm{NLO}}$	NLO S^3							

Table 10.1: PN sectors to which different combinations of the center-of-mass generator $\vec{G}$ and Hamiltonian H contribute, when substituted in the Poisson bracket $\{G_i, H\}_{(x,p)}$ of eq. (10.3), up to the relevant NLO S^3 contributions.

	$H_{\mathrm{SO}}^{\mathrm{LO}}$	$H_{\mathrm{SO}}^{\mathrm{NLO}}$	$H_{\mathrm{S}^2}^{\mathrm{LO}}$	$H_{\mathrm{S}^2}^{\mathrm{NLO}}$	$H_{\mathrm{S}^3}^{\mathrm{LO}}$	$H_{\mathrm{S}^3}^{\mathrm{NLO}}$
$\vec{G}_{\mathrm{SO}}^{\mathrm{LO}}$	NLO SO		NLO S^2		NLO S^3	
$\vec{G}_{\mathrm{SO}}^{\mathrm{NLO}}$						
$\vec{G}_{\mathrm{S}^2}^{\mathrm{LO}}$	NLO S^2		LO S^3	NLO S^3		
$\vec{G}_{\mathrm{S}^2}^{\mathrm{NLO}}$			NLO S^3			
$\vec{G}_{\mathrm{S}^3}^{\mathrm{LO}}$	NLO S^3					
$\vec{G}_{\mathrm{S}^3}^{\mathrm{NLO}}$						

Table 10.2: PN sectors to which different combinations of the center-of-mass generator $\vec{G}$ and Hamiltonian H contribute, when substituted in the spinning addition to the Poisson bracket $\{G_i, H\}_{\mathrm{spin}}$ of eq. (10.6), up to the relevant NLO S^3 contributions.

in agreement with eqs. (18a)-(18b) of [65], and with the expected value predicted by the special-relativistic limit of eq. (10.12). To solve for the coefficients, we make the expression vanish order by order in G_{N} and in the momenta and masses. Then, we obtain a set of equations that uniquely fixes all coefficients.

Even though we will not need it at NLO S^3, let us continue to 2PN order to exemplify the course of action once more. At 2PN we require an ansatz of 1PN order, so at most proportional to $G_{\mathrm{N}}^2 m^2/r^2 \sim (G_{\mathrm{N}} m/r)p^2 \sim p^4$. In particular, it is given by

$$h_1^{\mathrm{PM}}\Big|_{2\mathrm{PN}} = \frac{G_{\mathrm{N}}^2}{r^2}\alpha_1 m_1^2 m_2 + \frac{G_{\mathrm{N}}}{m_1 m_2 r}\Big(\alpha_2 m_2^2 p_1^2 + \alpha_3 m_2^2 (\vec{p}_1\cdot\vec{n})^2\Big) - \frac{p_1^4}{16 m_1^3} - (1\leftrightarrow 2), \tag{10.21}$$

$$\begin{aligned}\vec{Y}^{\mathrm{PM}}\Big|_{2\mathrm{PN}} = {}& \frac{G_{\mathrm{N}}^2}{r}\beta_1 m_1^2 m_2 \vec{n} + \frac{G_{\mathrm{N}}}{m_1 m_2}\Big[\Big(\beta_2 m_2^2 p_1^2 + \beta_3 m_2^2 (\vec{p}_1\cdot\vec{n})^2\Big)\vec{n} + \Big(\beta_4 m_2 (\vec{p}_1\cdot\vec{n}) \\ & + \beta_5 m_1 (\vec{p}_2\cdot\vec{n})\Big) m_2 \vec{p}_1\Big] + (1\leftrightarrow 2),\end{aligned} \tag{10.22}$$

where we have already fixed the $\mathcal{O}(G_{\mathrm{N}}^0)$ term with eq. (10.12). To fix the undetermined coefficients, we then use the 2PN order condition

$$\{\vec{G}_{0\mathrm{PN}}, H_{2\mathrm{PN}}\} + \{\vec{G}_{1\mathrm{PN}}, H_{1\mathrm{PN}}\} + \{\vec{G}_{2\mathrm{PN}}, H_{\mathrm{N}}\} = 0, \tag{10.23}$$

where the 2PN Hamiltonian can be found in eq. (6.6) of [45]. Then, from the condition resulting from eq. (10.23), we obtain that the 2PN correction to the generator becomes

$$h_1^{\mathrm{PM}}\Big|_{2\mathrm{PN}} = \frac{G_{\mathrm{N}}^2 m_1^2 m_2}{4r^2} - \frac{p_1^4}{16 m_1^3} - (1\leftrightarrow 2), \tag{10.24}$$

$$\vec{Y}^{\mathrm{PM}}\Big|_{2\mathrm{PN}} = -\frac{G_{\mathrm{N}}}{4}(\vec{p}_2\cdot\vec{n})\vec{p}_1 + (1\leftrightarrow 2), \tag{10.25}$$

in agreement with eqs. (4.9) and (4.13) of [29].

10.3 Poincaré algebra for spinning sectors

In order to address the sectors including spin variables, we have to take into account the spinning generalization of the Poisson bracket, given in eq. (10.6). Then, as always, we have to work our way upwards in the PN orders, notwithstanding that the spinning addition to the Poisson bracket $\{,\}_{\mathrm{spin}}$ has a different power counting, as it mixes higher-in-spin sectors without modifying their PN order. Hence, additional different combinations of generator and Hamiltonian are to be used here, and they are represented in the multiplication Table 10.2.

Starting with the LO SO sector, we would require an ansatz of one-PN order less, so at most proportional to $G_{\mathrm{N}}^0 S^1 p$. Since it does not contain powers of G_{N}, we can completely read it from the special-relativistic limit in eq. (10.12), resulting in

$$h_1^{\mathrm{SO}}\Big|_{\mathrm{LO}} = 0, \tag{10.26}$$

$$\vec{Y}^{\mathrm{SO}}\Big|_{\mathrm{LO}} = -\frac{1}{2m_1}\vec{S}_1\times\vec{p}_1 + (1\leftrightarrow 2), \tag{10.27}$$

in agreement with eqs. (4.10) and (4.14) of [29]. Using it in the LO SO Poincaré algebra condition

$$\{\vec{G}_{0\mathrm{PN}}, H_{\mathrm{SO}}^{\mathrm{LO}}\}_{(x,p)} + \{\vec{G}_{\mathrm{SO}}^{\mathrm{LO}}, H_{\mathrm{N}}\}_{(x,p)} = 0, \tag{10.28}$$

we find that it is indeed satisfied.

The following LO spinning sectors do not contain kinetic terms, and so their corrections to the center-of-mass generator vanish by the special-relativistic limit,

$$\vec{G}_{\mathrm{S_1S_2}}^{\mathrm{LO}} = \vec{G}_{\mathrm{SS}}^{\mathrm{LO}} = \vec{G}_{\mathrm{S^3}}^{\mathrm{LO}} = \cdots = 0, \tag{10.29}$$

as given also in eqs. (4.11)-(4.12) and (4.15)-(4.16) of [29]. This is because they can be at most proportional to $G_{\mathrm{N}}^0 S_1 S_2$, $G_{\mathrm{N}}^0 S^2$, $G_{\mathrm{N}}^0 S^3 p$, ..., respectively, fixed to zero by eq. (10.12). Even if their corrections to the generator are zero, their Poincaré algebra conditions are non-trivially satisfied, and up to LO S^3 they are

$$\begin{aligned}
0 &= \{\vec{G}_{0\mathrm{PN}}, H_{\mathrm{S_1S_2}}^{\mathrm{LO}}\}_{(x,p)} + \{\vec{G}_{\mathrm{S_1S_2}}^{\mathrm{LO}}, H_{\mathrm{N}}\}_{(x,p)}, &(10.30)\\
0 &= \{\vec{G}_{0\mathrm{PN}}, H_{\mathrm{SS}}^{\mathrm{LO}}\}_{(x,p)} + \{\vec{G}_{\mathrm{SS}}^{\mathrm{LO}}, H_{\mathrm{N}}\}_{(x,p)}, &(10.31)\\
0 &= \{\vec{G}_{0\mathrm{PN}}, H_{\mathrm{S^3}}^{\mathrm{LO}}\}_{(x,p)} + \{\vec{G}_{\mathrm{SO}}^{\mathrm{LO}}, H_{\mathrm{S_1S_2}}^{\mathrm{LO}}\}_{(x,p)} + \{\vec{G}_{\mathrm{SO}}^{\mathrm{LO}}, H_{\mathrm{SS}}^{\mathrm{LO}}\}_{(x,p)} + \{\vec{G}_{\mathrm{S_1S_2}}^{\mathrm{LO}}, H_{\mathrm{SO}}^{\mathrm{LO}}\}_{(x,p)}\\
&\quad + \{\vec{G}_{\mathrm{SS}}^{\mathrm{LO}}, H_{\mathrm{SO}}^{\mathrm{LO}}\}_{(x,p)} + \{\vec{G}_{\mathrm{S^3}}^{\mathrm{LO}}, H_{\mathrm{N}}\}_{(x,p)} + \{\vec{G}_{\mathrm{S_1S_2}}^{\mathrm{LO}}, H_{\mathrm{S_1S_2}}^{\mathrm{LO}}\}_{\mathrm{spin}}\\
&\quad + \{\vec{G}_{\mathrm{S_1S_2}}^{\mathrm{LO}}, H_{\mathrm{SS}}^{\mathrm{LO}}\}_{\mathrm{spin}} + \{\vec{G}_{\mathrm{SS}}^{\mathrm{LO}}, H_{\mathrm{S_1S_2}}^{\mathrm{LO}}\}_{\mathrm{spin}} + \{\vec{G}_{\mathrm{SS}}^{\mathrm{LO}}, H_{\mathrm{SS}}^{\mathrm{LO}}\}_{\mathrm{spin}}. &(10.32)
\end{aligned}$$

We can observe that starting at LO S^3, there start appearing contributions from the spinning Poisson bracket.

Having attained all relevant leading order sectors, we can progress to the next-to-leading orders, starting with the NLO SO. There, we require an ansatz for the generator of the order of the LO SO, so at most proportional to $(G_{\mathrm{N}} m/r) S^1 p \sim S^1 p^3$. Fixing the $\mathcal{O}(G_{\mathrm{N}}^0)$ terms from the special-relativistic limit in eq. (10.12), the ansatz reads

$$\begin{aligned}
h_1^{\mathrm{SO}}\Big|_{\mathrm{NLO}} &= \frac{G_{\mathrm{N}}}{r^2}\left[\alpha_1 m_2 \frac{\vec{S}_1\cdot\vec{p}_1\times\vec{n}}{m_1} + \alpha_2 m_2 \frac{\vec{S}_1\cdot\vec{p}_2\times\vec{n}}{m_2}\right] - (1\leftrightarrow 2), &(10.33)\\
\vec{Y}^{\mathrm{SO}}\Big|_{\mathrm{NLO}} &= \frac{G_{\mathrm{N}}}{r}\left(\left[\beta_1 m_2 \frac{\vec{S}_1\cdot\vec{p}_1\times\vec{n}}{m_1} + \beta_2 m_2 \frac{\vec{S}_1\cdot\vec{p}_2\times\vec{n}}{m_2}\right]\vec{n} + \left[\beta_3 \frac{\vec{p}_1\cdot\vec{n}}{m_1} + \beta_4 \frac{\vec{p}_2\cdot\vec{n}}{m_2}\right] m_2 \vec{S}_1\times\vec{n}\right.\\
&\quad \left. + \beta_5 m_2 \frac{\vec{S}_1\times\vec{p}_1}{m_1} + \beta_6 m_2 \frac{\vec{S}_1\times\vec{p}_2}{m_2}\right) + \frac{p_1^2}{8m_1^3}\vec{S}_1\times\vec{p}_1 + (1\leftrightarrow 2). &(10.34)
\end{aligned}$$

To fix the undetermined coefficients, the following NLO SO Poincaré algebra condition is used:

$$\begin{aligned}
0 &= \{\vec{G}_{0\mathrm{PN}}, H_{\mathrm{SO}}^{\mathrm{NLO}}\}_{(x,p)} + \{\vec{G}_{1\mathrm{PN}}, H_{\mathrm{SO}}^{\mathrm{LO}}\}_{(x,p)} + \{\vec{G}_{\mathrm{SO}}^{\mathrm{LO}}, H_{1\mathrm{PN}}\}_{(x,p)}\\
&\quad + \{\vec{G}_{\mathrm{SO}}^{\mathrm{NLO}}, H_{\mathrm{N}}\}_{(x,p)} + \{\vec{G}_{\mathrm{SO}}^{\mathrm{LO}}, H_{\mathrm{SO}}^{\mathrm{LO}}\}_{\mathrm{spin}}, &(10.35)
\end{aligned}$$

which uniquely fixes the ansatz to

$$h_1^{\mathrm{SO}}\Big|_{\mathrm{NLO}} = \frac{G_{\mathrm{N}}}{4r^2} m_2 \frac{\vec{S}_1\cdot\vec{p}_1\times\vec{n}}{m_1} - (1\leftrightarrow 2), \tag{10.36}$$

$$\vec{Y}^{\mathrm{SO}}\Big|_{\mathrm{NLO}} = \frac{G_{\mathrm{N}}}{r}\bigg(-\frac{\vec{p}_2\cdot\vec{n}}{4}\vec{S}_1\times\vec{n} + \frac{5m_2}{2m_1}\vec{S}_1\times\vec{p}_1 - \frac{11}{4}\vec{S}_1\times\vec{p}_2\bigg) + \frac{p_1^2}{8m_1^3}\vec{S}_1\times\vec{p}_1 + (1\leftrightarrow 2), \tag{10.37}$$

in agreement with eqs. (4.10) and (4.14) of [29].

At NLO $\mathrm{S_1S_2}$ we require an ansatz for the generator of the order of the LO $\mathrm{S_1S_2}$, so at most proportional to $(G_{\mathrm{N}}m/r)S_1S_2 \sim S_1S_2p^2$. In particular, it is given by

$$h_1^{\mathrm{S_1S_2}}\Big|_{\mathrm{NLO}} = 0, \tag{10.38}$$
$$\vec{Y}^{\mathrm{S_1S_2}}\Big|_{\mathrm{NLO}} = \frac{G_{\mathrm{N}}}{r^2}\beta_1(\vec{S}_2\cdot\vec{n})\vec{S}_1 + (1\leftrightarrow 2), \tag{10.39}$$

where the $\mathcal{O}(G_{\mathrm{N}}^0)$ terms vanish. To fix the only unknown coefficient, we use the NLO $\mathrm{S_1S_2}$ Poincaré algebra condition

$$\begin{aligned}0 = {}&\{\vec{G}_{\mathrm{0PN}}, H^{\mathrm{NLO}}_{\mathrm{S_1S_2}}\}_{(x,p)} + \{\vec{G}_{\mathrm{1PN}}, H^{\mathrm{LO}}_{\mathrm{S_1S_2}}\}_{(x,p)} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{SO}}, H^{\mathrm{LO}}_{\mathrm{SO}}\}_{(x,p)}\\ &+ \{\vec{G}^{\mathrm{LO}}_{\mathrm{S_1S_2}}, H_{\mathrm{1PN}}\}_{(x,p)} + \{\vec{G}^{\mathrm{NLO}}_{\mathrm{S_1S_2}}, H_{\mathrm{N}}\}_{(x,p)} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{SO}}, H^{\mathrm{LO}}_{\mathrm{S_1S_2}}\}_{\mathrm{spin}}\\ &+ \{\vec{G}^{\mathrm{LO}}_{\mathrm{SO}}, H^{\mathrm{LO}}_{\mathrm{SS}}\}_{\mathrm{spin}} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S_1S_2}}, H^{\mathrm{LO}}_{\mathrm{SO}}\}_{\mathrm{spin}} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{SS}}, H^{\mathrm{LO}}_{\mathrm{SO}}\}_{\mathrm{spin}},\end{aligned} \tag{10.40}$$

which implies a unique correction to the generator at NLO $\mathrm{S_1S_2}$ of the form

$$h_1^{\mathrm{S_1S_2}}\Big|_{\mathrm{NLO}} = 0, \tag{10.41}$$
$$\vec{Y}^{\mathrm{S_1S_2}}\Big|_{\mathrm{NLO}} = \frac{3G_{\mathrm{N}}}{2r^2}(\vec{S}_2\cdot\vec{n})\vec{S}_1 + (1\leftrightarrow 2), \tag{10.42}$$

in accordance with eqs. (4.11) and (4.15) of [29]. Finally, the last sector that we require to tackle the NLO S^3 is the NLO SS, where a correction of the order of the LO SS is required for the generator, so at most proportional to $(G_{\mathrm{N}}m/r)S^2 \sim S^2p^2$. However, in this case we have to take into account the possibility of having Wilson coefficients coming from the spin-squared non-minimal interaction, so more unknown coefficients will be used. More concretely, the ansatz is

$$h_1^{\mathrm{SS}}\Big|_{\mathrm{NLO}} = \frac{G_{\mathrm{N}}}{r^3}\bigg[\Big(\alpha_1 + \delta_1 C_{1(\mathrm{ES}^2)}\Big)m_2\frac{S_1^2}{m_1} + \Big(\alpha_2 + \delta_2 C_{1(\mathrm{ES}^2)}\Big)m_2\frac{(\vec{S}_1\cdot\vec{n})^2}{m_1}\bigg] - (1\leftrightarrow 2), \tag{10.43}$$

$$\begin{aligned}\vec{Y}^{\mathrm{SS}}\Big|_{\mathrm{NLO}} = \frac{G_{\mathrm{N}}}{r^2}\Bigg(&\bigg[\Big(\beta_1 + \omega_1 C_{1(\mathrm{ES}^2)}\Big)m_2\frac{S_1^2}{m_1} + \Big(\beta_2 + \omega_2 C_{1(\mathrm{ES}^2)}\Big)m_2\frac{(\vec{S}_1\cdot\vec{n})^2}{m_1}\bigg]\vec{n}\\ &+ \Big(\beta_3 + \omega_3 C_{1(\mathrm{ES}^2)}\Big)m_2\frac{\vec{S}_1\cdot\vec{n}}{m_1}\vec{S}_1\Bigg) + (1\leftrightarrow 2),\end{aligned} \tag{10.44}$$

where α_i, β_i, δ_i and ω_i are the undetermined numerical coefficients. They are fixed by the NLO SS condition

$$\begin{aligned}0 = {}&\{\vec{G}_{\mathrm{0PN}}, H^{\mathrm{NLO}}_{\mathrm{SS}}\}_{(x,p)} + \{\vec{G}_{\mathrm{1PN}}, H^{\mathrm{LO}}_{\mathrm{SS}}\}_{(x,p)} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{SO}}, H^{\mathrm{LO}}_{\mathrm{SO}}\}_{(x,p)}\\ &+ \{\vec{G}^{\mathrm{LO}}_{\mathrm{SS}}, H_{\mathrm{1PN}}\}_{(x,p)} + \{\vec{G}^{\mathrm{NLO}}_{\mathrm{SS}}, H_{\mathrm{N}}\}_{(x,p)} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{SO}}, H^{\mathrm{LO}}_{\mathrm{S_1S_2}}\}_{\mathrm{spin}}\end{aligned}$$

$$+\{\vec{G}^{\text{LO}}_{\text{SO}}, H^{\text{LO}}_{\text{SS}}\}_{\text{spin}} + \{\vec{G}^{\text{LO}}_{\text{S}_1\text{S}_2}, H^{\text{LO}}_{\text{SO}}\}_{\text{spin}} + \{\vec{G}^{\text{LO}}_{\text{SS}}, H^{\text{LO}}_{\text{SO}}\}_{\text{spin}}, \tag{10.45}$$

which forces the generator to have the following form:

$$h_1^{\text{SS}}\Big|_{\text{NLO}} = \frac{G_{\text{N}}}{2r^3}\Big(1 + C_{1(\text{ES}^2)}\Big) m_2 \frac{S_1^2}{m_1} - (1 \leftrightarrow 2), \tag{10.46}$$

$$\vec{Y}^{\text{SS}}\Big|_{\text{NLO}} = -\frac{G_{\text{N}}}{2r^2}\Big(1 + C_{1(\text{ES}^2)}\Big) m_2 \frac{\vec{S}_1 \cdot \vec{n}}{m_1} \vec{S}_1 + (1 \leftrightarrow 2), \tag{10.47}$$

equivalent to the solution written eqs. (4.12) and (4.16) of [29].

With this last contribution, we have exemplified the verification of the Poincaré algebra for all relevant lower-order PN sectors. Hence, the next step is to apply this strong self-consistency check precisely to the NLO S^3 Hamiltonian, which is addressed in §11.7 in our results.

Part III
Results

11 Next-to-leading cubic-in-spin sector

In this section we will apply the methodology described in Part II to calculate for the first time the dynamics at the NLO S^3 sector, which represents the highest-in-spin and state-of-the-art result for the compact binary inspiral. More concretely, we will first address the derivation of the corresponding Feynman rules and diagrams, and then calculate the standard reduced potential from the interaction potential, as well as the equations of motion. Then we will obtain the Hamiltonian, which is valid for general compact objects and holds in general coordinate frames, with general orbits, and with arbitrary orientations of the spins. From the Hamiltonian, we will define the binding energy, and proceed to obtain gauge-invariant relations for the binding energy as a function of the angular momentum and of the orbital frequency. Lastly, we address the Poincaré invariance of the system, which constitutes the most stringent self-consistency check in PN gravity.

11.1 Feynman rules

Before turning our attention to the new results at the NLO S^3 sector, let us first address a possible source of ambiguity in the interaction potentials, which has never been studied in detail before. In particular, the spin-induced non-minimal coupling Lagrangian L_{NMC} appearing in eq. (2.9) can in principle be freely defined using either a tensor product of *spin vectors*, e.g. $(S^\mu S^\nu)$, or a contraction of *spin tensors* $(S^{\mu\alpha} S_\alpha{}^\nu)$.

Consequently, we should evaluate whether the two formulations are equivalent, or if they lead to different results in the lower-order sectors, which include up to LO S^3 and to NLO SS. More concretely, the quadrupolar and octupolar non-minimal couplings can be defined with a product of *spin vectors*, as given in eqs. (2.10)-(2.11), and rewritten here for convenience,

$$L_{\mathrm{ES}^2} = -\frac{C_{\mathrm{ES}^2}}{2m} \frac{E_{\mu\nu}}{\sqrt{u^2}} S^\mu S^\nu, \qquad L_{\mathrm{BS}^3} = -\frac{C_{\mathrm{ES}^2}}{6m^2} \frac{D_\lambda B_{\mu\nu}}{\sqrt{u^2}} S^\mu S^\nu S^\lambda, \tag{11.1}$$

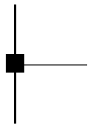
or with a contraction of *spin tensors*, where an extra minus sign appears with respect to the product of spin vectors S^μ, which are spacelike,

$$L_{\mathrm{ES}^2} = \frac{C_{\mathrm{ES}^2}}{2m} \frac{E_{\mu\nu}}{\sqrt{u^2}} S^{\mu\alpha} S_\alpha{}^\nu, \qquad L_{\mathrm{BS}^3} = \frac{C_{\mathrm{ES}^2}}{6m^2} \frac{D_\lambda B_{\mu\nu}}{\sqrt{u^2}} S^{\mu\gamma} S_\gamma{}^\nu S^\lambda, \tag{11.2}$$

$$L_{\mathrm{BS}^3} = \frac{C_{\mathrm{ES}^2}}{6m^2} \frac{D_\lambda B_{\mu\nu}}{\sqrt{u^2}} S^\mu S^{\nu\gamma} S_\gamma{}^\lambda, \qquad L_{\mathrm{BS}^3} = \frac{C_{\mathrm{ES}^2}}{6m^2} \frac{D_\lambda B_{\mu\nu}}{\sqrt{u^2}} S^{\mu\gamma} S^\nu S_\gamma{}^\lambda. \tag{11.3}$$

We see that in the octupolar coupling three different options for the contraction of the *spin tensors* are possible, so that the free spin vector can be contracted either with the covariant derivative, or with the first or second index of the magnetic component of the Riemann tensor.

Following the description presented in §4 and computing all different cases, it was found that the resulting Feynman rules corresponding to the *spin tensor* prescriptions can be obtained from the *spin vector* counterparts, via the explicit rule of thumb substitution $S^iS^j \to (S^iS^j - S^2\delta^{ij})$ for the pair of spin variables that are contracted. For instance, using the *spin vector* prescription of eq. (11.1), we obtain the following NLO one-scalar graviton coupling to the worldline spin-squared,



$$= \int dt \, \frac{C_{\mathrm{ES}^2}}{2m} \Big[S^iS^j \Big(\partial_i\partial_j\phi \Big(1 + \frac{3}{2}v^2\Big) - 3v^iv^k\partial_j\partial_k\phi - 2v^j\partial_i\partial_t\phi \Big) \\ - S^2 \Big(-v^iv^j\partial_i\partial_j\phi - 2v^i\partial_i\partial_t\phi - \partial_t^2\phi \Big) \Big], \tag{11.4}$$

so that its *spin tensor* analog, derived from eq. (11.2) and with a result given in eq. (4.46), can also be obtained realizing the substitution $S^iS^j \to (S^iS^j - S^2\delta^{ij})$, as well as $S^2 = S^iS^j\delta_{ij} \to (S^iS^j - S^2\delta^{ij})\delta_{ij} = -2S^2$ in eq. (11.4).

Now, it turns out that to leading order, the *spin vector* and the *spin tensor* prescriptions lead to identical results already at the level of the Feynman diagrams, calculated following §5. This is because, in all instances, during the evaluation of the diagrams we encounter a contraction like

$$(S^iS^j - S^2\delta^{ij})\partial_i\partial_j\frac{1}{r} = (S^iS^j - \underbrace{S^2\delta^{ij})\frac{1}{r^3}(-\delta_{ij} + 3n_in_j)}_{0}, \tag{11.5}$$

that makes the extra term between the two Feynman rules vanish. As a consequence, we find that the two formulations are equivalent, and we recover the same potential, Hamiltonian and binding energy at LO SS and at LO S^3.

By contrast, at NLO SS, the prescriptions yield different results in each Feynman diagram. Nonetheless, they were also found to be equivalent, but at the level of the interaction potential, when the sum of all graphs is taken into account. To calculate the NLO SS potential we require six diagrams, given in figure 4 of [24]. Using the *spin tensor* prescription of eq. (11.2), we have that all diagrams are in exact agreement with eqs. (6.34)-(6.39) of [24], so that we obtain the same potential $V_{\mathrm{SS}}^{\mathrm{NLO}}$ as in eq. (5.20).

On the other hand, using the *spin vector* prescription of eq. (11.1), we obtain the following differences for each diagram, where $\Delta\mathrm{Fig.}\ 4(\cdot) \equiv \mathrm{Fig.}\ 4(\cdot)|_{\mathrm{vector}} - \mathrm{Fig.}\ 4(\cdot)|_{\mathrm{tensor}}$ signifies the difference with respect to the results given in the paper:

$$\Delta\mathrm{Fig.}\ 4(\mathrm{a}) = \frac{C_{1(\mathrm{ES}^2)}Gm_2}{2m_1r^3}\Big[-S_1^2v_1^2 + 3S_1^2(\vec{v}_1\cdot\vec{n})^2 \Big] - \frac{3C_{1(\mathrm{ES}^2)}Gm_2}{2m_1r}\ddot{S}_1^2 \\ + \frac{C_{1(\mathrm{ES}^2)}Gm_2}{m_1r^2}\Big[-\frac{1}{2}S_1^2(\vec{a}_1\cdot\vec{n}) + \dot{S}_1^2(\vec{v}_1\cdot\vec{n}) \Big] + (1 \leftrightarrow 2), \tag{11.6}$$

$$\Delta\mathrm{Fig.}\ 4(\mathrm{b}) = -\frac{C_{1(\mathrm{ES}^2)}Gm_2}{4m_1r^3}\Big[-2S_1^2(\vec{v}_1\cdot\vec{v}_2) + 6S_1^2(\vec{v}_1\cdot\vec{n})(\vec{v}_2\cdot\vec{n}) \Big] \\ + \frac{C_{1(\mathrm{ES}^2)}Gm_2}{2m_1r^2}\dot{S}_1^2(\vec{v}_2\cdot\vec{n}) + (1 \leftrightarrow 2), \tag{11.7}$$

$$\Delta\text{Fig. 4(c)} = -\frac{2C_{1(\text{ES}^2)}Gm_2}{m_1 r^2}\dot{S_1^2}(\vec{v}_2\cdot\vec{n}) + (1\leftrightarrow 2), \tag{11.8}$$

$$\Delta\text{Fig. 4(d)} = 0, \tag{11.9}$$

$$\Delta\text{Fig. 4(e)} = 0, \tag{11.10}$$

$$\Delta\text{Fig. 4(f)} = 0. \tag{11.11}$$

As it turns out, the total difference between the two prescriptions is actually given by a total time derivative,

$$\sum_\kappa \Delta\text{Fig. 4}(\kappa) = -\frac{C_{1(\text{ES}^2)}Gm_2}{2m_1}\frac{d}{dt}\left[\frac{3}{r}\dot{S_1^2} + \frac{1}{r^2}S_1^2(\vec{v}_1\cdot\vec{n})\right] + (1\leftrightarrow 2), \tag{11.12}$$

which does not affect the equations of motion and thus can be dropped from the Lagrangian, making the two results equivalent also at next-to-leading order. Yet, we could go on to perform the relevant position and spin redefinitions to eliminate higher-order time derivatives, and then perform a Legendre transform to obtain the Hamiltonian for each prescription, following the descriptions provided in §6 and §8, obtaining a difference given by

$$\begin{aligned}\Delta H_{\text{SS}}^{\text{NLO}} &= H_{\text{SS}}^{\text{NLO}}|_{\text{vector}} - H_{\text{SS}}^{\text{NLO}}|_{\text{tensor}} \\ &= -\frac{C_{1(\text{ES}^2)}G^2m_2^2}{2m_1r^4}S_1^2 + \frac{C_{1(\text{ES}^2)}Gm_2}{2m_1r^3}S_1^2\Bigg[-3\left(\frac{\vec{p}_1\cdot\vec{n}}{m_1}\right)^2 + \frac{p_1^2}{m_1^2} \\ &\quad -\frac{\vec{p}_1\cdot\vec{p}_2}{m_1m_2} + 3\frac{\vec{p}_1\cdot\vec{n}}{m_1}\frac{\vec{p}_2\cdot\vec{n}}{m_2}\Bigg] + (1\leftrightarrow 2),\end{aligned} \tag{11.13}$$

where $H_{\text{SS}}^{\text{NLO}}|_{\text{tensor}}$ reproduces the result in eq. (8.13). In this case, the Hamiltonians are actually related by a canonical transformation. For that, we check whether there exists an infinitesimal generator g of a canonical transformation such that

$$\Delta H_{\text{SS}}^{\text{NLO}} = \{H, g\} = -\frac{dg}{dt}, \tag{11.14}$$

as detailed in Appendix B of [44]. In this case, it was found that the generator reads

$$g = -\frac{C_{1(\text{ES}^2)}Gm_2}{2m_1r^2}S_1^2\frac{\vec{p}_1\cdot\vec{n}}{m_1} + (1\leftrightarrow 2). \tag{11.15}$$

Therefore, also at the level of the Hamiltonians the two prescriptions are equivalent, being indeed related by a canonical transformation.

If we were to still proceed to compute the binding energy as in §9, we reproduce the NLO SS correction in eq. (9.4) for both the *spin tensor* and the *spin vector* prescriptions, without any additions. Therefore, even if we missed the total time derivative (or canonical transformation) in the potential (or in the Hamiltonian), the difference vanishes identically in the binding energy, which is a physical observable.

With this analysis, we verify that all calculations and predictions up to LO S^3 and to NLO S^2 are not modified when using either *spin vectors* or a contraction of *spin tensors* in the spin-induced non-minimal coupling. The analysis to NLO S^3, which involves the calculation of the Feynman rules entering the 53 Feynman diagrams using the different prescriptions of eqs. (11.2)-(11.3), is left for a possible future check.

The Feynman rules for the NLO S^3 were obtained using the *spin vector* prescription, combining both pen-and-paper calculations and using the publicly available EFTofPNG code [48], reproducing the Feynman rules provided in eqs. (4.49)-(4.51) and (4.53)-(4.54), as well as those with a higher number of gravitons given in [1].

11.2 Feynman diagrams

Using the Feynman rules that we verified, and employing again the EFTofPNG code, the corresponding 53 Feynman diagrams were also calculated, verifying the total result for the interaction Lagrangian provided in eqs. (5.21)-(5.42). Nevertheless, due to the presence of triple products containing both scalar and vector products, its total result has been further simplified using the following vectorial identity for 4 vectors in 3 dimensions, presented in eq. (3.14) of [19],

$$\vec{N}[\vec{A}_a] \equiv \vec{A}_1\,\vec{A}_2\cdot\vec{A}_3\times\vec{A}_4 - \vec{A}_2\,\vec{A}_3\cdot\vec{A}_4\times\vec{A}_1 + \vec{A}_3\,\vec{A}_4\cdot\vec{A}_1\times\vec{A}_2 - \vec{A}_4\,\vec{A}_1\cdot\vec{A}_2\times\vec{A}_3 \equiv \vec{0}. \quad (11.16)$$

This identity will play a central role in the calculation of all of the forthcoming NLO S^3 corrections, as it will be regularly used to simplify the results.

11.3 Reduced potential

Following §6, for the NLO S^3 sector we have to first take into consideration the 4 contributions coming from the position shift of eq. (6.14) at LO SO, as shown in Table 6.1. They consist of:

1. The LO SO shift of eq. (6.14) applied to linear order on the NLO quadratic-in-spin potentials of eqs. (5.19)-(5.20);

2. The LO SO shift of eq. (6.14) applied to quadratic order on the LO SO potential of eq. (5.14);

3. The LO SO shift of eq. (6.14) applied to cubic order on the Newtonian Lagrangian of eq. (5.7).

Nonetheless, the NLO S^3 sector also starts receiving contributions from further position shifts and spin redefinitions. In particular, they are:

4. The NLO SO position shift of eq. (6.22) applied to linear order on the LO quadratic-in-spin potentials of eqs. (5.15)-(5.16);

5. The NLO quadratic-in-spin position shifts of eqs. (6.30)-(6.32) applied to linear order on the shifted LO SO potential of eq. (6.16);

6. The NLO quadratic-in-spin spin redefinitions of eqs. (6.31)-(6.33) applied to linear order on the LO quadratic-in-spin potentials of eqs. (5.15)-(5.16);

7. The LO S^3 spin redefinition of eq. (6.18) applied to linear order on the LO SO potential of eq. (5.14).

These contributions from lower-order sectors amount to a total of 484 terms (plus crossed $1 \leftrightarrow 2$ terms) in the NLO S^3 potential. Since the redefinition of variables represents the most subtle step in the thesis, we provide here the exact values of these highly non-trivial NLO S^3 contributions for future reference.

Particularly, when the LO SO shift of eq. (6.14) is applied to linear order on the NLO S_1S_2 potential of eq. (5.19), the NLO S^3 sector receives the contribution

$$
\begin{aligned}
\left[\Delta V^{\rm NLO}_{\rm S^3}\right]_1 = \frac{G^2}{r^5}\Bigg[& \Big(1+\frac{m_2}{m_1}\Big) 4\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n} + \vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\Big(-4\vec{S}_1\cdot\vec{S}_2 \\
& + 24\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\Big)\Bigg] + \frac{G}{m_1 r^2}\Bigg[-\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{v}_1\,\dot{\vec{S}}_1\cdot\vec{n} - \frac{3}{4}\vec{S}_1\cdot\ddot{\vec{S}}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n} \\
& + \vec{S}_1\cdot\vec{v}_1\times\vec{n}\Big(-\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2 - \frac{3}{4}\vec{S}_1\cdot\ddot{\vec{S}}_2 + \frac{3}{4}\ddot{\vec{S}}_2\cdot\vec{n}\,\vec{S}_1\cdot\vec{n}\Big)\Bigg] \\
& + \frac{G}{m_1 r^3}\Bigg[\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\Big(\frac{3}{4}\dot{\vec{S}}_1\cdot\vec{v}_1 - \dot{\vec{S}}_1\cdot\vec{v}_2 + \frac{3}{4}\vec{S}_1\cdot\vec{a}_1 - \frac{1}{4}\vec{S}_1\cdot\vec{a}_2 \\
& + \frac{3}{4}\vec{S}_1\cdot\vec{n}\,\vec{a}_2\cdot\vec{n}\Big) + \vec{S}_1\cdot\vec{v}_1\times\vec{a}_1\Big(\frac{3}{2}\vec{S}_1\cdot\vec{S}_2 - \frac{9}{2}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\Big) \\
& + \vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\Big(-\frac{5}{2}\dot{\vec{S}}_1\cdot\vec{S}_2 - \vec{S}_1\cdot\dot{\vec{S}}_2 + \frac{9}{2}\dot{\vec{S}}_2\cdot\vec{n}\,\vec{S}_1\cdot\vec{n} + 6\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\Big) \\
& + \vec{S}_1\cdot\vec{v}_1\times\vec{a}_2\Big(-\frac{7}{4}\vec{S}_1\cdot\vec{S}_2 + \frac{21}{4}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\Big) \\
& + \vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{v}_1\Big(-\frac{1}{4}\vec{S}_1\cdot\vec{v}_1 + \frac{1}{2}\vec{S}_1\cdot\vec{v}_2 + 3\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n} - \frac{3}{2}\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big) \\
& + \vec{S}_1\cdot\vec{v}_1\times\vec{n}\Big(3\dot{\vec{S}}_2\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{n} - 3\dot{\vec{S}}_2\cdot\vec{v}_2\,\vec{S}_1\cdot\vec{n} - \frac{3}{4}\vec{S}_1\cdot\vec{S}_2\,\vec{a}_2\cdot\vec{n} + \frac{9}{4}\dot{\vec{S}}_2\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1 \\
& - \frac{3}{2}\dot{\vec{S}}_2\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_2 - \frac{9}{4}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{a}_2 + \frac{9}{4}\dot{\vec{S}}_1\cdot\vec{v}_1\,\vec{S}_2\cdot\vec{n} - 3\dot{\vec{S}}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{n} \\
& + \frac{9}{4}\vec{S}_1\cdot\vec{a}_1\,\vec{S}_2\cdot\vec{n} - \frac{3}{4}\vec{S}_1\cdot\vec{a}_2\,\vec{S}_2\cdot\vec{n} + \frac{15}{4}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\,\vec{a}_2\cdot\vec{n} - \frac{3}{2}\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_2\dot{\vec{v}}_2 \\
& + \frac{3}{2}\dot{\vec{S}}_1\cdot\vec{S}_2\,\vec{v}_2\cdot\vec{n} - 3\vec{S}_1\cdot\dot{\vec{S}}_2\,\vec{v}_2\cdot\vec{n} + \frac{15}{2}\dot{\vec{S}}_2\cdot\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big) \\
& + \dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{v}_1\Big(-\frac{3}{4}\vec{S}_2\cdot\vec{v}_1 + \vec{S}_2\cdot\vec{v}_2 + \frac{9}{4}\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{n} - \frac{3}{2}\vec{S}_2\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\Bigg] \\
& + \frac{G}{m_1 r^4}\Bigg[\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\Big(\frac{21}{4}\vec{S}_1\cdot\vec{v}_1\,\vec{S}_2\cdot\vec{n} - \frac{9}{2}\vec{S}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{n} + 6\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_1 \\
& - \frac{21}{4}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_2 + 6\vec{S}_1\cdot\vec{S}_2\,\vec{v}_1\cdot\vec{n} - 30\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{n} - \frac{9}{2}\vec{S}_1\cdot\vec{S}_2\,\vec{v}_2\cdot\vec{n} \\
& + \frac{45}{2}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big) + \vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\Big(-\frac{9}{4}\vec{S}_1\cdot\vec{n}\,v_1^2 + 6\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{v}_2 \\
& + \frac{3}{4}\vec{S}_1\cdot\vec{v}_1\,\vec{v}_2\cdot\vec{n} - \frac{3}{2}\vec{S}_1\cdot\vec{v}_2\,\vec{v}_2\cdot\vec{n} + \frac{15}{4}\vec{S}_1\cdot\vec{n}\,(\vec{v}_2\cdot\vec{n})^2 - 3\vec{S}_1\cdot\vec{n}\,v_2^2\Big) \\
& + \vec{S}_1\cdot\vec{v}_1\times\vec{n}\Big(-\frac{3}{4}\vec{S}_1\cdot\vec{v}_1\,\vec{S}_2\cdot\vec{v}_2 + \frac{3}{4}\vec{S}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{v}_2 + \frac{9}{4}\vec{S}_1\cdot\vec{S}_2\,v_1^2 \\
& - \frac{45}{4}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\,v_1^2 - 6\vec{S}_1\cdot\vec{S}_2\,\vec{v}_1\cdot\vec{v}_2 + 30\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{v}_2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{15}{4}\vec{S}_1\cdot\vec{v}_1\,\vec{S}_2\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}-\frac{15}{2}\vec{S}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}-\frac{15}{4}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_2\,\vec{v}_2\cdot\vec{n}\\
&-\frac{15}{4}\vec{S}_1\cdot\vec{S}_2\,(\vec{v}_2\,\vec{n})^2+\frac{105}{4}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\,(\vec{v}_2\cdot\vec{n})^2+3\vec{S}_1\cdot\vec{S}_2\,v_2^2\\
&-15\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\,v_2^2\Big)\Bigg]+(1\leftrightarrow 2).
\end{aligned}
\tag{11.17}
$$

Likewise, when applied to linear order on the NLO SS potential of eq. (5.20), the NLO S^3 sector receives the contribution

$$
\begin{aligned}
[\Delta V_{\mathrm{S}^3}^{\mathrm{NLO}}]_2 =&\frac{C_{1(\mathrm{ES}^2)}G^2m_2}{m_1r^5}\left[\left(1+\frac{4m_2}{m_1}\right)\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(\frac{9}{2}(\vec{S}_1\cdot\vec{n})^2-S_1^2\right)\right]\\
&+\frac{C_{2(\mathrm{ES}^2)}G^2}{r^5}\left[\left(1+\frac{4m_1}{m_2}\right)\left(\frac{3}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_2\cdot\vec{n}\right.\right.\\
&\left.\left.+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(\frac{9}{2}(\vec{S}_1\cdot\vec{n})^2-S_1^2\right)\right)\right]\\
&+\frac{G^2}{r^5}\left[\frac{3m_2}{m_1}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2+\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_2\cdot\vec{n}+3\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,(\vec{S}_2\cdot\vec{n})^2\right]\\
&+\frac{3C_{2(\mathrm{ES}^2)}G}{4m_2r^2}\left[-\frac{1}{2}\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{v}_1\,\dot{\vec{S}}_2\cdot\vec{n}-\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\ddot{\vec{S}}_2\cdot\vec{n}-\vec{S}_1\cdot\ddot{\vec{S}}_2\times\vec{v}_1\,\vec{S}_2\cdot\vec{n}\right.\\
&\left.+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(-\ddot{\vec{S}}_2\cdot\vec{S}_2-\dot{\vec{S}}_2\cdot\dot{\vec{S}}_2+(\dot{\vec{S}}_2\cdot\vec{n})^2+\ddot{\vec{S}}_2\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right)\right]\\
&+\frac{C_{1(\mathrm{ES}^2)}Gm_2}{m_1^2r^3}\left[\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\left(-3\dot{\vec{S}}_1\cdot\vec{S}_1+6\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{n}\right)\right.\\
&+\vec{S}_1\cdot\vec{v}_1\times\vec{a}_2\left(\frac{21}{8}(\vec{S}_1\cdot\vec{n})^2-\frac{9}{8}S_1^2\right)+\vec{S}_1\cdot\vec{v}_1\times\vec{a}_1\left(-\frac{9}{4}(\vec{S}_1\cdot\vec{n})^2+\frac{5}{4}S_1^2\right)\\
&+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(\frac{9}{4}\dot{\vec{S}}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{n}-3\dot{\vec{S}}_1\cdot\vec{v}_2\,\vec{S}_1\cdot\vec{n}+\frac{9}{4}\vec{S}_1\cdot\vec{a}_1\,\vec{S}_1\cdot\vec{n}\right.\\
&-\frac{3}{4}\vec{S}_1\cdot\vec{a}_2\vec{S}_1\cdot\vec{n}+\frac{15}{8}(\vec{S}_1\cdot\vec{n})^2\,\vec{a}_2\cdot\vec{n}-\frac{3}{2}S_1^2\,\vec{a}_1\cdot\vec{n}+\frac{3}{8}S_1^2\,\vec{a}_2\cdot\vec{n}\\
&\left.+\frac{9}{4}\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1-3\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_2-3\dot{\vec{S}}_1\cdot\vec{S}_1\,\vec{v}_1\cdot\vec{n}+3\dot{\vec{S}}_1\cdot\vec{S}_1\,\vec{v}_2\cdot\vec{n}\right)\\
&\left.+\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{v}_1\left(-\frac{3}{2}\vec{S}_1\cdot\vec{v}_1+\frac{3}{2}\vec{S}_1\cdot\vec{v}_2+\frac{9}{4}\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}-\frac{3}{2}\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right)\right]\\
&+\frac{C_{2(\mathrm{ES}^2)}G}{m_2r^3}\left[\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\left(-\frac{3}{2}\dot{\vec{S}}_2\cdot\vec{S}_2+\frac{9}{2}\dot{\vec{S}}_2\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right)\right.\\
&+\vec{S}_1\cdot\vec{v}_1\times\vec{a}_2\left(\frac{21}{8}(\vec{S}_2\cdot\vec{n})^2-\frac{9}{8}S_2^2\right)+\vec{S}_1\cdot\vec{v}_1\times\vec{a}_1\left(-\frac{9}{4}(\vec{S}_2\cdot\vec{n})^2+\frac{3}{4}S_2^2\right)\\
&+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(3\dot{\vec{S}}_2\cdot\vec{v}_1\,\vec{S}_2\cdot\vec{n}-3\dot{\vec{S}}_2\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{n}+\frac{9}{4}\vec{S}_2\cdot\vec{a}_2\,\vec{S}_2\cdot\vec{n}\right.\\
&+\frac{15}{8}(\vec{S}_2\cdot\vec{n})^2\,\vec{a}_2\cdot\vec{n}+\frac{3}{8}S_2^2\,\vec{a}_2\cdot\vec{n}+3\dot{\vec{S}}_2\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_1\\
&\left.-3\dot{\vec{S}}_2\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_2-\frac{3}{2}\dot{\vec{S}}_2\cdot\vec{S}_2\,\vec{v}_2\cdot\vec{n}+\frac{15}{2}\dot{\vec{S}}_2\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right)\\
&+\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{v}_1\left(3\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}-\frac{3}{2}\vec{S}_2\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\Big(-\frac{3}{4}\vec{S}_2\cdot\vec{a}_2+\frac{3}{4}\vec{S}_2\cdot\vec{n}\,\vec{a}_2\cdot\vec{n}+3\dot{\vec{S}}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}-\frac{3}{2}\dot{\vec{S}}_2\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\Big]\\
&+\frac{C_{1(\mathrm{ES}^2)}Gm_2}{m_1^2r^4}\Big[\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\Big(\frac{21}{4}\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1-\frac{9}{2}\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_2\\
&-15(\vec{S}_1\cdot\vec{n})^2\,\vec{v}_1\cdot\vec{n}+\frac{9}{2}S_1^2\,\vec{v}_1\cdot\vec{n}+\frac{45}{4}(\vec{S}_1\cdot\vec{n})^2\,\vec{v}_2\cdot\vec{n}-\frac{15}{4}S_1^2\,\vec{v}_2\cdot\vec{n}\Big)\\
&+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\Big(-\frac{3}{4}\vec{S}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{v}_2+\frac{3}{4}(\vec{S}_1\cdot\vec{v}_2)^2+\frac{15}{4}S_1^2\,(\vec{v}_1\cdot\vec{n})^2\\
&-\frac{45}{8}(\vec{S}_1\cdot\vec{n})^2\,v_1^2+\frac{3}{8}S_1^2\,v_1^2+15(\vec{S}_1\cdot\vec{n})^2\,\vec{v}_1\cdot\vec{v}_2-\frac{3}{2}S_1^2\,\vec{v}_1\cdot\vec{v}_2\\
&+\frac{15}{4}\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1\,\vec{v}_2\cdot\vec{n}-\frac{15}{2}\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_2\,\vec{v}_2\cdot\vec{n}-\frac{15}{2}S_1^2\,\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\\
&+\frac{105}{8}(\vec{S}_1\cdot\vec{n})^2\,(\vec{v}_2\cdot\vec{n})^2+\frac{15}{8}S_1^2\,(\vec{v}_2\cdot\vec{n})^2-\frac{15}{2}(\vec{S}_1\cdot\vec{n})^2\,v_2^2+\frac{3}{4}S_1^2\,v_2^2\Big)\Big]\\
&+\frac{C_{2(\mathrm{ES}^2)}G}{m_2r^4}\Big[\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\Big(-\frac{9}{4}\vec{S}_2\cdot\vec{n}\,v_1^2+6\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{v}_2-\frac{3}{4}\vec{S}_2\cdot\vec{v}_2\,\vec{v}_2\cdot\vec{n}\\
&+\frac{15}{4}\vec{S}_2\cdot\vec{n}\,(\vec{v}_2\cdot\vec{n})^2-3\vec{S}_2\cdot\vec{n}\,v_2^2\Big)+\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\Big(6\vec{S}_2\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_1\\
&-\frac{21}{4}\vec{S}_2\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_2-15(\vec{S}_2\cdot\vec{n})^2\,\vec{v}_1\cdot\vec{n}+3S_2^2\,\vec{v}_1\cdot\vec{n}+\frac{45}{4}(\vec{S}_2\cdot\vec{n})^2\,\vec{v}_2\cdot\vec{n}\\
&-\frac{9}{4}S_2^2\,\vec{v}_2\cdot\vec{n}\Big)+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\Big(-\frac{45}{8}(\vec{S}_2\cdot\vec{n})^2\,v_1^2+\frac{9}{8}S_2^2\,v_1^2+15(\vec{S}_2\cdot\vec{n})^2\,\vec{v}_1\cdot\vec{v}_2\\
&-3S_2^2\,\vec{v}_1\cdot\vec{v}_2-\frac{15}{4}\vec{S}_2\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_2\,\vec{v}_2\cdot\vec{n}+\frac{105}{8}(\vec{S}_2\cdot\vec{n})^2\,(\vec{v}_2\cdot\vec{n})^2\\
&-\frac{15}{8}S_2^2\,(\vec{v}_2\cdot\vec{n})^2-\frac{15}{2}(\vec{S}_2\cdot\vec{n})^2\,v_2^2+\frac{3}{2}S_2^2\,v_2^2\Big)\Big]+(1\leftrightarrow 2).
\end{aligned}
\tag{11.18}
$$

When the LO SO shift of eq. (6.14) is applied to quadratic order on the LO SO potential of eq. (5.14), we obtain the NLO S^3 contribution

$$
\begin{aligned}
[\Delta V^{\mathrm{NLO}}_{\mathrm{S}^3}]_3=\;&\frac{Gm_2}{m_1^2r^3}\Big[\frac{1}{2}\vec{S}_1\cdot\vec{v}_1\times\vec{a}_1\,S_1^2-\frac{1}{2}\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{v}_1\,\vec{S}_1\cdot\vec{v}_1\\
&+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\Big(\frac{3}{2}\vec{S}_1\cdot\vec{a}_1\,\vec{S}_1\cdot\vec{n}-\frac{3}{2}S_1^2\,\vec{a}_1\cdot\vec{n}+\frac{3}{2}\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1-\frac{3}{2}\dot{\vec{S}}_1\cdot\vec{S}_1\,\vec{v}_1\cdot\vec{n}\Big)\Big]\\
&+\frac{G}{4m_1r^3}\Big[\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\Big(\dot{\vec{S}}_1\cdot\vec{v}_2-\vec{S}_1\cdot\vec{a}_2\Big)+\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\Big(\dot{\vec{S}}_1\cdot\vec{S}_2-\vec{S}_1\cdot\dot{\vec{S}}_2\Big)\\
&+2\vec{S}_1\cdot\vec{v}_1\times\vec{a}_1\,\vec{S}_1\cdot\vec{S}_2-\vec{S}_1\cdot\vec{v}_1\times\vec{a}_2\,\vec{S}_1\cdot\vec{S}_2-\vec{S}_1\cdot\vec{v}_2\times\vec{a}_1\,\vec{S}_1\cdot\vec{S}_2\\
&-\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{v}_2+\vec{S}_1\cdot\vec{S}_2\times\vec{a}_1\,\vec{S}_1\cdot\vec{v}_2\\
&-2\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{v}_1\,\vec{S}_2\cdot\vec{v}_1+\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{v}_2\,\vec{S}_2\cdot\vec{v}_1\\
&+6\vec{S}_2\cdot\vec{v}_1\times\vec{n}\,\Big(\vec{S}_1\cdot\vec{a}_1\,\vec{S}_1\cdot\vec{n}-S_1^2\,\vec{a}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1-\dot{\vec{S}}_1\cdot\vec{S}_1\,\vec{v}_1\cdot\vec{n}\Big)\\
&+3\vec{S}_1\cdot\vec{v}_2\times\vec{n}\,\Big(\vec{S}_1\cdot\vec{S}_2\,\vec{a}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{a}_1-\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_1+\dot{\vec{S}}_1\cdot\vec{S}_2\,\vec{v}_1\cdot\vec{n}\Big)\\
&-\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{S}_2\,\vec{v}_1\cdot\vec{v}_2+3\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\Big(-2\vec{S}_1\cdot\vec{v}_1\,\vec{a}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{v}_2\,\vec{a}_1\cdot\vec{n}\\
&+2\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{a}_1-2\dot{\vec{S}}_1\cdot\vec{v}_1\,\vec{v}_1\cdot\vec{n}-\dot{\vec{S}}_1\cdot\vec{v}_2\,\vec{v}_1\cdot\vec{n}+2\dot{\vec{S}}_1\cdot\vec{n}\,v_1^2
\end{aligned}
$$

$$
\begin{aligned}
&+\dot{\vec{S}}_1\,\vec{n}\,\vec{v}_1\cdot\vec{v}_2+\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{a}_1\Big)+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\Big(\vec{S}_1\cdot\vec{S}_2\,\vec{a}_2\cdot\vec{n}-\vec{S}_1\cdot\vec{v}_2\,\dot{\vec{S}}_2\cdot\vec{n}\\
&-\vec{S}_1\cdot\vec{a}_2\,\vec{S}_2\cdot\vec{n}+\vec{S}_1\cdot\dot{\vec{S}}_2\,\vec{v}_2\cdot\vec{n}\Big)\Big]+\frac{G}{4m_2r^3}\Big[\dot{\vec{S}}_2\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{v}_2\\
&-\vec{S}_2\cdot\vec{v}_1\times\vec{a}_2\,\vec{S}_1\cdot\vec{S}_2+\vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\Big(\dot{\vec{S}}_1\cdot\vec{S}_2-\vec{S}_1\cdot\dot{\vec{S}}_2\Big)-\vec{S}_2\cdot\vec{v}_2\times\vec{a}_1\,\vec{S}_1\cdot\vec{S}_2\\
&+\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\Big(-\dot{\vec{S}}_2\cdot\vec{v}_1+\vec{S}_2\cdot\vec{a}_1\Big)+\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_2\cdot\vec{v}_1-\vec{S}_1\cdot\vec{S}_2\times\vec{a}_2\,\vec{S}_2\cdot\vec{v}_1\\
&+\vec{S}_2\cdot\vec{v}_2\times\vec{n}\,\Big(\vec{S}_1\cdot\vec{S}_2\,\vec{a}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{a}_1-\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_1+\dot{\vec{S}}_1\,\vec{S}_2\,\vec{v}_1\cdot\vec{n}\Big)\\
&-\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{S}_2\,\vec{v}_1\cdot\vec{v}_2+\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\Big(\vec{S}_2\cdot\vec{v}_1\,\vec{a}_2\cdot\vec{n}-\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{a}_2\\
&-\dot{\vec{S}}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{v}_2+\dot{\vec{S}}_2\,\vec{v}_1\,\vec{v}_2\cdot\vec{n}\Big)+\vec{S}_2\cdot\vec{v}_1\times\vec{n}\,\Big(\vec{S}_1\cdot\vec{S}_2\,\vec{a}_2\cdot\vec{n}-\dot{\vec{S}}_2\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_2\\
&-\vec{S}_1\cdot\vec{a}_2\,\vec{S}_2\cdot\vec{n}+\vec{S}_1\cdot\dot{\vec{S}}_2\,\vec{v}_2\cdot\vec{n}\Big)\Big]\\
&+\frac{3Gm_2}{2m_1^2r^4}\Big[\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\Big(\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1-S_1^2\,\vec{v}_1\cdot\vec{n}\Big)\\
&+\Big(\vec{S}_1\cdot\vec{v}_1\times\vec{n}-\vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big)\Big((\vec{S}_1\cdot\vec{v}_1)^2-5\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1\,\vec{v}_1\cdot\vec{n}+\frac{5}{2}S_1^2\,(\vec{v}_1\cdot\vec{n})^2\\
&+\frac{5}{2}(\vec{S}_1\cdot\vec{n})^2\,v_1^2-S_1^2\,v_1^2\Big)\Big]+\frac{3G}{4m_1r^4}\Big[-\vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\,S_1^2\,\vec{v}_1\cdot\vec{n}\\
&+\Big(\vec{S}_2\cdot\vec{v}_1\times\vec{n}-\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big)\Big(2(\vec{S}_1\cdot\vec{v}_1)^2-10\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1\,\vec{v}_1\cdot\vec{n}\\
&+5S_1^2\,(\vec{v}_1\cdot\vec{n})^2+5(\vec{S}_1\cdot\vec{n})^2\,v_1^2-2S_1^2\,v_1^2\Big)+\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\Big(-\vec{S}_1\cdot\vec{v}_1\,\vec{v}_1\cdot\vec{n}\\
&-\vec{S}_1\cdot\vec{v}_2\,\vec{v}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{v}_2\Big)+\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\Big(2\vec{S}_1\cdot\vec{v}_1\,\vec{v}_1\cdot\vec{n}-2\vec{S}_1\cdot\vec{n}\,v_1^2\\
&+\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{v}_2\Big)+\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\Big(-\vec{S}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{n}+\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_1+\vec{S}_1\cdot\vec{S}_2\,\vec{v}_2\cdot\vec{n}\Big)\\
&+\Big(\vec{S}_1\cdot\vec{v}_1\times\vec{n}-\vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big)\Big(-2\vec{S}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{v}_1+5\vec{S}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\\
&+2\vec{S}_1\cdot\vec{S}_2\,\vec{v}_1\cdot\vec{v}_2-5\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{v}_2+5\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_1\,\vec{v}_2\cdot\vec{n}\\
&-5\vec{S}_1\cdot\vec{S}_2\,\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\Big]+\frac{3G}{4m_2r^4}\Big[\vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\Big(-\vec{S}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{n}\\
&-\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_1+\vec{S}_1\cdot\vec{S}_2\,\vec{v}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{S}_2\,\vec{v}_2\cdot\vec{n}\Big)\\
&+\Big(\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1-\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\Big)\Big(\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{v}_2-\vec{S}_2\,\vec{v}_1\,\vec{v}_2\cdot\vec{n}\Big)\\
&+\Big(\vec{S}_2\cdot\vec{v}_1\times\vec{n}-\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big)\Big(-2\vec{S}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{v}_1+5\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_1\,\vec{v}_2\cdot\vec{n}\\
&+2\vec{S}_1\cdot\vec{S}_2\,\vec{v}_1\cdot\vec{v}_2-5\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{v}_2+5\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_1\,\vec{v}_2\cdot\vec{n}\\
&-5\vec{S}_1\cdot\vec{S}_2\,\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\Big]+\frac{1}{8m_1^2}\Big[-2\dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{a}_1\,\dot{\vec{S}}_1\cdot\vec{S}_1+\ddot{\vec{S}}_1\cdot\vec{S}_1\times\vec{v}_1\,\vec{S}_1\cdot\vec{a}_1\\
&+2\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{a}_1\,\vec{S}_1\cdot\vec{a}_1-\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{v}_1\,\vec{S}_1\cdot\dot{\vec{a}}_1+\ddot{\vec{S}}_1\cdot\vec{v}_1\times\vec{a}_1\,S_1^2-\dot{\vec{S}}_1\cdot\vec{v}_1\times\dot{\vec{a}}_1\,S_1^2\\
&+\vec{S}_1\cdot\dot{\vec{a}}_1\times\vec{a}_1\,S_1^2+\ddot{\vec{S}}_1\cdot\dot{\vec{S}}_1\times\vec{v}_1\,\vec{S}_1\cdot\vec{v}_1\Big]+(1\leftrightarrow2). \qquad (11.19)
\end{aligned}
$$

When the LO SO shift of eq. (6.14) is applied to cubic order on the Newtonian Lagrangian of eq. (5.7), we obtain the NLO S^3 contribution

$$\begin{aligned}
[\Delta V_{\mathrm{S}^3}^{\mathrm{NLO}}]_4 = \frac{G}{8m_1r^4}\Bigg[&\left(\vec{S}_2\cdot\vec{v}_2\times\vec{n}-\frac{m_2}{m_1}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\right)\Big((\vec{S}_1\cdot\vec{v}_1)^2-5\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1\,\vec{v}_1\cdot\vec{n}\\
&+\frac{5}{2}S_1^2\,(\vec{v}_1\cdot\vec{n})^2+\frac{5}{2}(\vec{S}_1\cdot\vec{n})^2\,v_1^2-S_1^2\,v_1^2\Big)+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\Big(2\vec{S}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{v}_1\\
&-5\vec{S}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}-2\vec{S}_1\cdot\vec{S}_2\,\vec{v}_1\cdot\vec{v}_2+5\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{v}_2\\
&-5\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{v}_1\,\vec{v}_2\cdot\vec{n}+5\vec{S}_1\cdot\vec{S}_2\,\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\Bigg]+(1\leftrightarrow2).
\end{aligned}\tag{11.20}$$

When the NLO SO position shift of eq. (6.22) is applied to linear order on the LO $\mathrm{S}_1\mathrm{S}_2$ potential of eq. (5.15), the NLO S^3 sector obtains the contribution

$$\begin{aligned}
[\Delta V_{\mathrm{S}^3}^{\mathrm{NLO}}]_5 = \frac{G^2}{r^5}\Bigg[&\frac{33}{4}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_2\cdot\vec{n}+\vec{S}_2\cdot\vec{v}_2\times\vec{n}\left(9\vec{S}_1\cdot\vec{S}_2-\frac{87}{2}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right)\\
&-3\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_2\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}+\frac{m_2}{m_1}\Bigg(-\frac{3}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n}+9\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}\\
&+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(\frac{3}{2}\vec{S}_1\cdot\vec{S}_2-\frac{15}{2}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right)\\
&+\vec{S}_1\cdot\vec{v}_2\times\vec{n}\left(-9\vec{S}_1\cdot\vec{S}_2+45\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right)+3\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Bigg)\Bigg]\\
&+\frac{3G^2}{r^4}\left[\dot{\vec{S}}_2\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}-\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\,\vec{S}_2\cdot\vec{n}\right]\\
&+\frac{G}{m_1r^4}\Bigg[-\frac{3}{8}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n}\,v_1^2\\
&+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(\frac{3}{8}\vec{S}_1\cdot\vec{S}_2\,v_1^2-\frac{15}{8}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\,v_1^2\right)\Bigg]+(1\leftrightarrow2).
\end{aligned}\tag{11.21}$$

Similarly, when it is applied to linear order on the LO SS potential of eq. (5.16), the NLO S^3 sector obtains the contribution

$$\begin{aligned}
[\Delta V_{\mathrm{S}^3}^{\mathrm{NLO}}]_6 = \frac{C_{1(\mathrm{ES}^2)}G^2m_2}{m_1r^5}\Bigg[&\frac{33}{4}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}+\vec{S}_2\cdot\vec{v}_2\times\vec{n}\left(-\frac{87}{4}(\vec{S}_1\cdot\vec{n})^2+\frac{9}{2}S_1^2\right)\\
&-3\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}+\frac{m_2}{m_1}\Bigg(\vec{S}_1\cdot\vec{v}_2\times\vec{n}\left(\frac{45}{2}(\vec{S}_1\cdot\vec{n})^2-\frac{9}{2}S_1^2\right)\\
&+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(-\frac{15}{4}(\vec{S}_1\cdot\vec{n})^2+\frac{3}{4}S_1^2\right)\Bigg)\Bigg]+\frac{C_{2(\mathrm{ES}^2)}G^2}{r^5}\Bigg[-\frac{3}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_2\cdot\vec{n}\\
&+9\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_2\cdot\vec{n}+\vec{S}_1\cdot\vec{v}_2\times\vec{n}\left(\frac{45}{2}(\vec{S}_2\cdot\vec{n})^2-\frac{9}{2}S_2^2\right)\\
&+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(-\frac{15}{4}(\vec{S}_2\cdot\vec{n})^2+\frac{3}{4}S_2^2\right)+3\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_2\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\\
&+\frac{m_1}{m_2}\vec{S}_2\cdot\vec{v}_2\times\vec{n}\left(-\frac{87}{4}(\vec{S}_2\cdot\vec{n})^2+\frac{9}{2}S_2^2\right)\Bigg]\\
&-\frac{3C_{1(\mathrm{ES}^2)}G^2m_2}{m_1r^4}\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}+3\frac{C_{2(\mathrm{ES}^2)}G^2m_1}{m_2r^4}\dot{\vec{S}}_2\cdot\vec{S}_2\times\vec{n}\,\vec{S}_2\cdot\vec{n}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{3C_{1(\mathrm{ES}^2)}Gm_2}{16m_1^2r^4}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\Big(-5(\vec{S}_1\cdot\vec{n})^2\,v_1^2+S_1^2\,v_1^2\Big)\\
&+\frac{3C_{2(\mathrm{ES}^2)}G}{8m_2r^4}\Big[-\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_2\cdot\vec{n}\,v_1^2\\
&+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\Big(-\frac{5}{2}(\vec{S}_2\cdot\vec{n})^2\,v_1^2+\frac{1}{2}S_2^2\,v_1^2\Big)\Big]+(1\leftrightarrow 2).
\end{aligned}
\tag{11.22}
$$

When the NLO $\mathrm{S_1S_2}$ position shift of eq. (6.30) is applied to linear order on the shifted LO SO potential of eq. (6.16), the NLO S^3 sector receives the contribution

$$
\begin{aligned}
[\Delta V^{\mathrm{NLO}}_{\mathrm{S}^3}]_7=&\frac{G^2}{r^4}\Big[-4\dot{\vec{S}}_2\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}-3\frac{m_2}{m_1}\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\Big]\\
&+\frac{G^2}{r^5}\Big[\vec{S}_2\cdot\vec{v}_1\times\vec{n}\Big(4\vec{S}_1\cdot\vec{S}_2-12\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\Big)\\
&+\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big(-2\vec{S}_1\cdot\vec{S}_2+9\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\Big)\\
&+\frac{m_2}{m_1}\Big(3\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n}+7\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}+3\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\vec{S}_1\cdot\vec{S}_2\\
&-9\vec{S}_2\cdot\vec{v}_1\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2+\vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big(-5\vec{S}_1\cdot\vec{S}_2+12\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\Big)\\
&-9\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\Big]+(1\leftrightarrow 2).
\end{aligned}
\tag{11.23}
$$

Likewise, when the NLO SS position shift of eq. (6.32) is applied to linear order on the shifted LO SO potential of eq. (6.16), the NLO S^3 sector receives the contribution

$$
\begin{aligned}
[\Delta V^{\mathrm{NLO}}_{\mathrm{S}^3}]_8=&\frac{C_{1(\mathrm{ES}^2)}G^2m_2}{m_1r^5}\Big[2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n}+\frac{7}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}\\
&-\frac{9}{2}\vec{S}_2\cdot\vec{v}_2\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2+6\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}-6\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\\
&+\frac{m_2}{m_1}\Big(\frac{9}{2}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2-6\vec{S}_1\cdot\vec{v}_2\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2\Big)\Big]\\
&+\frac{G^2m_2}{4m_1r^5}\Big[\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n}+\frac{7}{4}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}-\vec{S}_2\cdot\vec{v}_1\times\vec{n}\,S_1^2\\
&+\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big(-\frac{9}{4}(\vec{S}_1\cdot\vec{n})^2+\frac{1}{2}S_1^2\Big)+3\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\\
&-3\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}+\frac{m_2}{m_1}\Big(\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\Big(\frac{9}{4}(\vec{S}_1\cdot\vec{n})^2-\frac{3}{4}S_1^2\Big)\\
&+\vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big(-3(\vec{S}_1\cdot\vec{n})^2+\frac{5}{4}S_1^2\Big)\Big)\Big]\\
&+\Big(1+8C_{1(\mathrm{ES}^2)}\Big)\frac{G^2m_2}{4m_1r^4}\Big[-\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}+\frac{3m_2}{4m_1}\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{n}\,\vec{S}_1\cdot\vec{n}\Big]\\
&+\frac{G}{4m_1r^3}\Big[\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\Big(-2\dot{\vec{S}}_1\cdot\vec{v}_1+3\vec{S}_1\cdot\vec{a}_1\Big)\\
&+\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\Big(-\frac{7}{2}\dot{\vec{S}}_1\cdot\vec{v}_1+\frac{21}{4}\vec{S}_1\cdot\vec{a}_1\Big)+2\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{v}_1
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{7}{2}\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{v}_1-3\vec{S}_2\cdot\vec{v}_1\times\vec{a}_1\,S_1^2-\frac{21}{4}\vec{S}_2\cdot\vec{v}_2\times\vec{a}_1\,S_1^2\\
&+\vec{S}_2\cdot\vec{v}_2\times\vec{n}\left(\frac{9}{2}\dot{\vec{S}}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{n}-\frac{27}{4}\vec{S}_1\cdot\vec{a}_1\,\vec{S}_1\cdot\vec{n}+\frac{27}{4}S_1^2\,\vec{a}_1\cdot\vec{n}-\frac{9}{2}\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1\right)\\
&+6\dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{n}\,\vec{S}_1\cdot\vec{v}_1\,\vec{S}_2\cdot\vec{n}+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(-6\dot{\vec{S}}_1\cdot\vec{v}_1\,\vec{S}_2\cdot\vec{n}+9\vec{S}_1\cdot\vec{a}_1\,\vec{S}_2\cdot\vec{n}\right)\\
&+9\vec{v}_1\cdot\vec{a}_1\times\vec{n}\,S_1^2\,\vec{S}_2\cdot\vec{n}-6\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{v}_1\,\vec{v}_2\cdot\vec{n}-9\vec{S}_2\cdot\vec{a}_1\times\vec{n}\,S_1^2\,\vec{v}_2\cdot\vec{n}\\
&+\vec{S}_1\cdot\vec{S}_2\times\vec{n}\left(6\dot{\vec{S}}_1\cdot\vec{v}_1\,\vec{v}_2\cdot\vec{n}-9\vec{S}_1\cdot\vec{a}_1\,\vec{v}_2\cdot\vec{n}\right)\\
&+\frac{m_2}{m_1}\Bigg(\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(-\frac{9}{2}\dot{\vec{S}}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{n}+\frac{27}{4}\vec{S}_1\cdot\vec{a}_1\,\vec{S}_1\cdot\vec{n}\right)-\frac{9}{4}\vec{S}_1\cdot\vec{v}_1\times\vec{a}_1\,S_1^2\\
&-\frac{21}{4}\vec{S}_1\cdot\vec{v}_2\times\vec{a}_1\,S_1^2+\frac{3}{2}\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{v}_1\,\vec{S}_1\cdot\vec{v}_1+\frac{7}{2}\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{v}_2\,\vec{S}_1\cdot\vec{v}_1\\
&+\frac{9}{2}\dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1+\vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big(6\dot{\vec{S}}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{n}-9\vec{S}_1\cdot\vec{a}_1\,\vec{S}_1\cdot\vec{n}\\
&+9S_1^2\,\vec{a}_1\cdot\vec{n}-6\dot{\vec{S}}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1\Big)+\frac{27}{4}\vec{v}_1\cdot\vec{a}_1\times\vec{n}\,S_1^2\,\vec{S}_1\cdot\vec{n}\\
&-\frac{27}{4}\vec{S}_1\cdot\vec{a}_1\times\vec{n}\,S_1^2\,\vec{v}_2\cdot\vec{n}-\frac{9}{2}\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{n}\,\vec{S}_1\cdot\vec{v}_1\,\vec{v}_2\cdot\vec{n}\Bigg)\Bigg]\\
&+\frac{G}{4m_1r^2}\Bigg[\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\left(2\dot{\vec{S}}_1\cdot\vec{v}_1-3\vec{S}_1\cdot\vec{a}_1\right)-3\dot{\vec{S}}_2\,\vec{a}_1\times\vec{n}\,S_1^2\\
&-2\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2\times\vec{n}\,\vec{S}_1\cdot\vec{v}_1+\frac{m_2}{m_1}\Bigg(\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{n}\left(-\frac{3}{2}\dot{\vec{S}}_1\cdot\vec{v}_1+\frac{9}{4}\vec{S}_1\cdot\vec{a}_1\right)\\
&-\frac{9}{4}\dot{\vec{S}}_1\cdot\vec{a}_1\times\vec{n}\,S_1^2\Bigg)\Bigg]+(1\leftrightarrow 2).
\end{aligned}
\tag{11.24}
$$

When the NLO S_1S_2 spin redefinition of eq. (6.31) is applied to linear order on the LO quadratic-in-spin potentials of eqs. (5.15)-(5.16), the NLO S^3 sector receives the respective contributions

$$
\begin{aligned}
[\Delta V_{S^3}^{\rm NLO}]_9=&\frac{G^2}{2r^4}\Big[\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{S}_2-\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\dot{\vec{S}}_2\cdot\vec{n}-3\dot{\vec{S}}_2\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\Big]\\
&+\frac{G^2}{r^5}\Big[-6\vec{S}_2\cdot\vec{v}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}-3\vec{S}_1\cdot\vec{v}_2\times\vec{n}\,(\vec{S}_2\cdot\vec{n})^2\\
&+\vec{S}_1\cdot\vec{S}_2\times\vec{n}\left(-4\vec{S}_2\cdot\vec{v}_1+\vec{S}_2\cdot\vec{v}_2\right)\Big]+(1\leftrightarrow 2),
\end{aligned}
\tag{11.25}
$$

$$
\begin{aligned}
[\Delta V_{S^3}^{\rm NLO}]_{10}=&\frac{3C_{1({\rm ES}^2)}G^2m_2}{2m_1r^4}\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}+\frac{C_{1({\rm ES}^2)}G^2m_2}{m_1r^5}\Big[9\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}\\
&-9\vec{S}_2\cdot\vec{v}_2\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2-12\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\Big]+(1\leftrightarrow 2).
\end{aligned}
\tag{11.26}
$$

Analogously, when the NLO SS spin redefinition of eq. (6.33) is applied to linear order on the LO quadratic-in-spin potentials of eqs. (5.15)-(5.16), the NLO S^3 sector receives the respective contributions

$$
[\Delta V_{S^3}^{\rm NLO}]_{11}=\frac{C_{1({\rm ES}^2)}G^2m_2}{m_1r^5}\Big[2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n}-3\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}
$$

$$
\begin{aligned}
&- 6\vec{S}_2\cdot\vec{v}_1\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2 + 9\vec{S}_2\cdot\vec{v}_2\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2 + \vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\Big(2\vec{S}_1\cdot\vec{v}_1\\
&- 3\vec{S}_1\cdot\vec{v}_2 + 6\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\Big] + \frac{G^2m_2}{2m_1r^5}\Big[- \vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n}\\
&+ \vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{v}_1 + 3\vec{S}_2\cdot\vec{v}_1\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2\Big]\\
&+ \frac{G}{4m_1r^3}\Big[2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{a}_1 + \dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{v}_1 + 2\vec{S}_1\cdot\vec{S}_2\times\vec{a}_1\,\vec{S}_1\cdot\vec{v}_1\\
&- 6\vec{S}_2\cdot\vec{a}_1\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1 + \vec{S}_2\cdot\vec{v}_1\times\vec{n}\,\Big(- 3\dot{\vec{S}}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{n} - 3\vec{S}_1\cdot\vec{a}_1\,\vec{S}_1\cdot\vec{n}\Big)\\
&- \dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{v}_1\,\vec{S}_2\cdot\vec{v}_1 - 12\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{a}_1\\
&- 3\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,v_1^2\Big] + (1\leftrightarrow 2),
\end{aligned}
\tag{11.27}
$$

$$
\begin{aligned}
[\Delta V^{\text{NLO}}_{\text{S}^3}]_{12} = {}& \frac{C_{1(\text{ES}^2)}Gm_2}{4m_1^2r^3}\Big[- 6\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\vec{S}_1\cdot\vec{a}_1\,\vec{S}_1\cdot\vec{n} - 3\dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1\\
&- 6\vec{S}_1\cdot\vec{a}_1\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1 - 3\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{v}_1\,\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\Big]\\
&+ \frac{C_{1(\text{ES}^2)}^2G^2m_2^2}{m_1^2r^5}\Big[- 6\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2 + 9\vec{S}_1\cdot\vec{v}_2\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2\Big]\\
&+ \frac{3C_{1(\text{ES}^2)}G^2m_2^2}{2m_1^2r^5}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2 + (1\leftrightarrow 2).
\end{aligned}
\tag{11.28}
$$

Lastly, when the LO S^3 spin redefinition of eq. (6.18) is applied to linear order on the LO SO potential of eq. (5.14), the NLO S^3 sector receives the contribution

$$
\begin{aligned}
[\Delta V^{\text{NLO}}_{\text{S}^3}]_{13} = {}& \frac{12C_{1(\text{ES}^2)}G^2m_2}{m_1r^5}\Big[- \vec{S}_2\cdot\vec{v}_1\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2 + \vec{S}_2\cdot\vec{v}_2\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2\\
&+ \vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\Big(- \vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n} + \vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\Big]\\
&+ \frac{C_{1(\text{ES}^2)}G}{m_1r^3}\Big[- 3\vec{S}_2\cdot\vec{v}_1\times\vec{n}\,\vec{S}_1\cdot\vec{a}_1\,\vec{S}_1\cdot\vec{n}\\
&+ \vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\Big(- 6\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{a}_1 + 3\vec{S}_1\cdot\vec{a}_1\,\vec{v}_1\cdot\vec{n}\Big)\Big] + (1\leftrightarrow 2).
\end{aligned}
\tag{11.29}
$$

Note that the subscripts $1,\ldots,13$, denote the different contributions to NLO S^3. Hence, at this first stage we can already note the enormous increase in complexity and in scale that going to the next odd-in-spin sector implies.

When combined with the Lagrangian in eqs. (5.21)-(5.42), which came from the evaluation of the 53 Feynman graphs, it results in a potential that requires the following position and spin redefinitions to eliminate the higher-order time derivatives still there:

$$
\begin{aligned}
(\Delta\vec{x}_1)^{\text{NLO}}_{\text{S}^3} = {}& -\frac{1}{8m_1^3}\Big[\Big(\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{v}_1\Big)\,\dot{\vec{S}}_1 + \Big(\ddot{\vec{S}}_1\cdot\vec{S}_1\times\vec{v}_1\Big)\,\vec{S}_1 + 2\Big(\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{a}_1\Big)\,\vec{S}_1\\
&+ 2(\dot{\vec{S}}_1\cdot\vec{S}_1)\,\vec{S}_1\times\vec{a}_1 + (\vec{S}_1\cdot\vec{a}_1)\,\dot{\vec{S}}_1\times\vec{S}_1 + S_1^2\,\ddot{\vec{S}}_1\times\vec{v}_1 + 2S_1^2\,\dot{\vec{S}}_1\times\vec{a}_1
\end{aligned}
$$

$$
\begin{aligned}
&+ S_1^2\,\vec{S}_1\times\dot{\vec{a}}_1\Big] + \frac{G}{4m_1^2r^2}\Big[3\Big(\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\Big)\,\vec{S}_1 - 3S_1^2\,\dot{\vec{S}}_2\times\vec{n}\\
&+ \frac{m_2}{m_1}\Big(-\frac{7}{4}\Big(\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{n}\Big)\,\vec{S}_1 + \frac{1}{2}(\vec{S}_1\cdot\vec{n})\,\dot{\vec{S}}_1\times\vec{S}_1 - \frac{9}{4}S_1^2\,\dot{\vec{S}}_1\times\vec{n}\Big)\Big]\\
&+ \frac{C_{1(\mathrm{ES}^2)}G}{m_1^2r^2}\Big[(\vec{S}_1\cdot\vec{n})\,\vec{S}_1\times\dot{\vec{S}}_2 + \frac{3}{2}(\vec{S}_1\cdot\vec{n})^2\,\dot{\vec{S}}_2\times\vec{n} + \frac{1}{2}S_1^2\,\dot{\vec{S}}_2\times\vec{n}\Big]\\
&+ \frac{C_{2(\mathrm{ES}^2)}G}{m_1m_2r^2}\Big[(\dot{\vec{S}}_2\cdot\vec{n})\,\vec{S}_1\times\vec{S}_2 - 3(\dot{\vec{S}}_2\cdot\vec{S}_2)\,\vec{S}_1\times\vec{n} + (\vec{S}_2\cdot\vec{n})\,\vec{S}_1\times\dot{\vec{S}}_2\\
&+ 3(\dot{\vec{S}}_2\cdot\vec{n})(\vec{S}_2\cdot\vec{n})\,\vec{S}_1\times\vec{n}\Big] + \frac{C_{2(\mathrm{BS}^3)}G}{m_2^2r^2}\Big[-\frac{1}{3}(\dot{\vec{S}}_2\cdot\vec{S}_2)\,\vec{S}_2\times\vec{n}\\
&+ (\dot{\vec{S}}_2\cdot\vec{n})(\vec{S}_2\cdot\vec{n})\,\vec{S}_2\times\vec{n} + \frac{1}{2}(\vec{S}_2\cdot\vec{n})^2\,\dot{\vec{S}}_2\times\vec{n} - \frac{1}{6}S_2^2\,\dot{\vec{S}}_2\times\vec{n}\Big]\\
&+ \frac{G}{m_1^2r^3}\Big[\frac{3}{2}\vec{S}_2\cdot\vec{v}_1\times\vec{n}\,S_1^2\,\vec{n} - 2\Big(\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\Big)\,\vec{S}_1 - \frac{21}{16}\Big(\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\Big)\,\vec{S}_1\\
&- \frac{9}{2}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,(\vec{S}_2\cdot\vec{n})\,\vec{S}_1 + \frac{27}{16}\vec{S}_2\cdot\vec{v}_2\times\vec{n}\,\Big(-S_1^2\,\vec{n} + (\vec{S}_1\cdot\vec{n})\,\vec{S}_1\Big)\\
&+ \frac{3}{2}\vec{S}_1\cdot\vec{v}_2\times\vec{n}\,\Big(-(\vec{S}_1\cdot\vec{S}_2)\,\vec{n} + (\vec{S}_1\cdot\vec{n})\,\vec{S}_2\Big) + \frac{3}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\Big((\vec{S}_1\cdot\vec{v}_1)\,\vec{n}\\
&+ (\vec{S}_1\cdot\vec{v}_2)\,\vec{n} + \frac{3}{2}(\vec{v}_2\cdot\vec{n})\,\vec{S}_1 + (\vec{S}_1\cdot\vec{n})\,\vec{v}_1 - (\vec{S}_1\cdot\vec{n})\,\vec{v}_2\Big)\\
&- \frac{1}{2}\Big(\vec{S}_1\cdot\vec{v}_1 + \vec{S}_1\cdot\vec{v}_2\Big)\,\vec{S}_1\times\vec{S}_2 - \frac{3}{2}\Big(\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1 - \frac{3}{2}S_1^2\,\vec{v}_2\cdot\vec{n}\Big)\,\vec{S}_2\times\vec{n}\\
&+ \Big(-2\vec{S}_1\cdot\vec{S}_2 + \frac{9}{2}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\Big)\vec{S}_1\times\vec{v}_1 + \frac{1}{2}(\vec{S}_1\cdot\vec{S}_2)\,\vec{S}_1\times\vec{v}_2\\
&+ \frac{3}{4}S_1^2\,\vec{S}_2\times\vec{v}_1 + \frac{21}{16}S_1^2\,\vec{S}_2\times\vec{v}_2 + \frac{9}{4}S_1^2\,(\vec{S}_2\cdot\vec{n})\,\vec{v}_1\times\vec{n}\\
&+ \frac{m_2}{m_1}\Big(\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\Big(\frac{3}{2}S_1^2\,\vec{n} - \frac{51}{16}(\vec{S}_1\cdot\vec{n})\,\vec{S}_1\Big)\\
&+ \frac{9}{4}\vec{S}_1\cdot\vec{v}_2\times\vec{n}\,\Big(-S_1^2\,\vec{n} + (\vec{S}_1\cdot\vec{n})\,\vec{S}_1\Big) - \Big(\frac{3}{8}S_1^2\,\vec{v}_1\cdot\vec{n} + \frac{21}{16}S_1^2\,\vec{v}_2\cdot\vec{n}\Big)\,\vec{S}_1\times\vec{n}\\
&+ \frac{3}{16}S_1^2\,\vec{S}_1\times\vec{v}_1 + \frac{19}{16}S_1^2\,\vec{S}_1\times\vec{v}_2 + \frac{27}{16}S_1^2(\vec{S}_1\cdot\vec{n})\,\vec{v}_1\times\vec{n}\Big)\\
&+ \frac{m_1}{m_2}\Big(\frac{3}{4}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\Big((\vec{S}_2\cdot\vec{n})\,\vec{n} - \vec{S}_2\Big) + \frac{3}{4}\vec{S}_2\cdot\vec{v}_2\times\vec{n}\,\Big(-(\vec{S}_1\cdot\vec{S}_2)\,\vec{n}\\
&- 5(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{n})\,\vec{n} + 3(\vec{S}_2\cdot\vec{n})\,\vec{S}_1 + 3(\vec{S}_1\cdot\vec{n})\,\vec{S}_2\Big)\\
&+ \frac{9}{4}(\vec{S}_1\cdot\vec{S}_2)\,\vec{S}_2\times\vec{v}_2 - \frac{21}{4}(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{n})\,\vec{S}_2\times\vec{v}_2\Big)\Big]\\
&+ \frac{C_{1(\mathrm{ES}^2)}G}{m_1^2r^3}\Big[-3\vec{S}_1\cdot\vec{v}_2\times\vec{n}\,(\vec{S}_1\cdot\vec{S}_2)\,\vec{n} + \frac{3}{4}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\Big((\vec{S}_1\cdot\vec{n})\,\vec{n} - \vec{S}_1\Big)\\
&+ \vec{S}_2\cdot\vec{v}_2\times\vec{n}\,\Big(-\frac{15}{8}(\vec{S}_1\cdot\vec{n})^2\,\vec{n} - \frac{3}{8}S_1^2\,\vec{n} + \frac{9}{4}(\vec{S}_1\cdot\vec{n})\,\vec{S}_1\Big)\\
&+ \vec{S}_2\cdot\vec{v}_1\times\vec{n}\Big(\frac{3}{2}S_1^2\,\vec{n} + 3(\vec{S}_1\cdot\vec{n})\,\vec{S}_1\Big) + \frac{3}{2}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\Big((\vec{S}_1\cdot\vec{S}_2)\,\vec{n} - (\vec{S}_1\cdot\vec{n})\,\vec{S}_2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \Big(- 3(\vec{v}_1 \cdot \vec{n})\, \vec{S}_1 + 6(\vec{S}_1 \cdot \vec{n})\, \vec{v}_1 + 3(\vec{S}_1 \cdot \vec{n})\, \vec{v}_2 \Big) + (\vec{S}_1 \cdot \vec{v}_2)\, \vec{S}_1 \times \vec{S}_2 \\
&+ \frac{3}{2}\Big(\vec{S}_1 \cdot \vec{n}\, \vec{S}_2 \cdot \vec{v}_1 - \vec{S}_1 \cdot \vec{S}_2\, \vec{v}_1 \cdot \vec{n} \Big)\, \vec{S}_1 \times \vec{n} + \frac{3}{2}\Big(- S_1^2\, \vec{v}_1 \cdot \vec{n} + 5(\vec{S}_1 \cdot \vec{n})^2\, \vec{v}_2 \cdot \vec{n} \\
&- S_1^2\, \vec{v}_2 \cdot \vec{n} \Big)\, \vec{S}_2 \times \vec{n} + 3(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n})\, \vec{S}_1 \times \vec{v}_2 \\
&+ \frac{1}{8}\Big(- 57(\vec{S}_1 \cdot \vec{n})^2 + 21 S_1^2 \Big)\, \vec{S}_2 \times \vec{v}_2 + \frac{m_2}{m_1}\Bigg(\vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\, \Big(\frac{3}{2} S_1^2\, \vec{n} - \frac{3}{4}(\vec{S}_1 \cdot \vec{n})\, \vec{S}_1 \Big) \\
&+ \Big(\frac{9}{2} \vec{S}_1 \cdot \vec{n}\, \vec{S}_1 \cdot \vec{v}_1 - 6 \vec{S}_1 \cdot \vec{n}\, \vec{S}_1 \cdot \vec{v}_2 - 3 S_1^2\, \vec{v}_1 \cdot \vec{n} + 3 S_1^2\, \vec{v}_2 \cdot \vec{n} \Big)\, \vec{S}_1 \times \vec{n} \\
&+ \frac{1}{4}\Big(- 15(\vec{S}_1 \cdot \vec{n})^2 + 7 S_1^2 \Big)\, \vec{S}_1 \times \vec{v}_1 + \Big(6(\vec{S}_1 \cdot \vec{n})^2 - 3 S_1^2 \Big)\, \vec{S}_1 \times \vec{v}_2 \Bigg) \Bigg] \\
&+ \frac{C_{2(\mathrm{ES}^2)} G}{m_2^2 r^3}\Bigg[\frac{3}{4} \vec{S}_2 \cdot \vec{v}_2 \times \vec{n}\, \Big(- \frac{5}{2}(\vec{S}_2 \cdot \vec{n})^2\, \vec{n} - \frac{1}{2} S_2^2\, \vec{n} + (\vec{S}_2 \cdot \vec{n})\, \vec{S}_2 \Big) \\
&+ \frac{1}{8}\Big(- 21(\vec{S}_2 \cdot \vec{n})^2 + 9 S_2^2 \Big)\, \vec{S}_2 \times \vec{v}_2 + \frac{m_2}{m_1}\Bigg(\Big(\vec{S}_2 \cdot \vec{v}_2 - 3 \vec{S}_2 \cdot \vec{n}\, \vec{v}_2 \cdot \vec{n} \Big)\, \vec{S}_1 \times \vec{S}_2 \\
&+ 3\Big(- \vec{S}_2 \cdot \vec{n}\, \vec{S}_2 \cdot \vec{v}_2 + \frac{5}{2}(\vec{S}_2 \cdot \vec{n})^2\, \vec{v}_2 \cdot \vec{n} + \frac{1}{2} S_2^2\, \vec{v}_2 \cdot \vec{n} \Big)\, \vec{S}_1 \times \vec{n} \\
&+ \frac{1}{4}\Big(- 3(\vec{S}_2 \cdot \vec{n})^2 + S_2^2 \Big)\, \vec{S}_1 \times \vec{v}_1 + \frac{1}{2}\Big(9(\vec{S}_2 \cdot \vec{n})^2 - 5 S_2^2 \Big)\, \vec{S}_1 \times \vec{v}_2 \Bigg) \Bigg] \\
&+ \frac{C_{1(\mathrm{BS}^3)} G m_2}{m_1^3 r^3}\Bigg[- \vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\, (\vec{S}_1 \cdot \vec{n})\, \vec{S}_1 + \vec{S}_1 \cdot \vec{v}_2 \times \vec{n}\, (\vec{S}_1 \cdot \vec{n})\, \vec{S}_1 \\
&+ \Big(\vec{S}_1 \cdot \vec{n}\, \vec{S}_1 \cdot \vec{v}_1 + \frac{5}{2}(\vec{S}_1 \cdot \vec{n})^2\, \vec{v}_2 \cdot \vec{n} - \frac{1}{2} S_1^2\, \vec{v}_2 \cdot \vec{n} \Big)\, \vec{S}_1 \times \vec{n} \\
&+ \frac{1}{2}\Big((\vec{S}_1 \cdot \vec{n})^2 - \frac{1}{3} S_1^2 \Big)\, \vec{S}_1 \times \vec{v}_2 \Bigg] + \frac{C_{2(\mathrm{BS}^3)} G}{2 m_2^2 r^3}\Bigg[\Big(- 2 \vec{S}_2 \cdot \vec{n}\, \vec{S}_2 \cdot \vec{v}_2 \\
&+ 5(\vec{S}_2 \cdot \vec{n})^2\, \vec{v}_2 \cdot \vec{n} - S_2^2\, \vec{v}_2 \cdot \vec{n} \Big)\, \vec{S}_2 \times \vec{n} - \Big((\vec{S}_2 \cdot \vec{n})^2 - \frac{1}{3} S_2^2 \Big) \vec{S}_2 \times \vec{v}_2 \Bigg], \qquad (11.30)
\end{aligned}
$$

$$
\begin{aligned}
(\omega_1^{ij})^{\mathrm{NLO}}_{\mathrm{S}^3} = \frac{1}{8 m_1^2}\Bigg[& - 2(\dot{\vec{S}}_1 \cdot \vec{S}_1)\, a_1^i v_1^j - S_1^2\, \dot{a}_1^i v_1^j + (\dot{\vec{S}}_1 \cdot \vec{v}_1)\, \dot{S}_1^i v_1^j + \vec{S}_1 \cdot \vec{v}_1 \Big(\dot{S}_1^i a_1^j + \ddot{S}_1^i v_1^j \Big) \\
&+ \vec{S}_1 \cdot \vec{a}_1 \Big(S_1^i a_1^j + 2 \dot{S}_1^i v_1^j \Big) + (\dot{\vec{S}}_1 \cdot \vec{a}_1)\, S_1^i v_1^j + (\vec{S}_1 \cdot \dot{\vec{a}}_1)\, S_1^i v_1^j \Bigg] \\
&+ \frac{G^2}{r^4}\Bigg[\frac{m_1}{2 m_2}(\vec{S}_2 \cdot \vec{n})\, S_2^i n^j + \frac{m_2^2}{8 m_1^2}(\vec{S}_1 \cdot \vec{n})\, S_1^i n^j + \frac{3 m_2}{m_1}(\vec{S}_1 \cdot \vec{n})\, S_2^i n^j \\
&- \frac{3}{2}(\vec{S}_1 \cdot \vec{n})\, S_2^i n^j + 2(\vec{S}_2 \cdot \vec{n})\, S_2^i n^j \Bigg] + \frac{21 C_{2(\mathrm{ES}^2)} G^2 m_1}{2 m_2 r^4}(\vec{S}_2 \cdot \vec{n})\, S_2^i n^j \\
&+ \frac{C_{1(\mathrm{ES}^2)} G^2 m_2}{m_1 r^4}\Bigg[4(\vec{S}_2 \cdot \vec{n})\, S_1^i n^j + \vec{S}_1 \cdot \vec{n}\, \Big(- 6 S_1^i n^j - 8 S_2^i n^j \Big) - 4 S_1^i S_2^j \\
&- \frac{3 m_2}{2 m_1}(\vec{S}_1 \cdot \vec{n})\, S_1^i n^j + \frac{m_1}{m_2}\Big(- 2(\vec{S}_2 \cdot \vec{n})\, S_1^i n^j + 4(\vec{S}_1 \cdot \vec{n})\, S_2^i n^j + 2 S_1^i S_2^j \Big) \Bigg] \\
&+ \frac{C_{1(\mathrm{ES}^2)} G}{3 m_1 r}\Bigg[- \dot{S}_1^i \dot{S}_2^j + (\dot{\vec{S}}_2 \cdot \vec{n})\, \dot{S}_1^i n^j - 2(\dot{\vec{S}}_1 \cdot \vec{n})\, \dot{S}_2^i n^j \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{C_{2(\mathrm{ES}^2)}G}{m_2 r}\Big[-\frac{1}{2}\dot{S}_1^i\dot{S}_2^j+(\vec{S}_2\cdot\vec{n})\,\ddot{S}_1^i n^j-\frac{1}{2}(\dot{\vec{S}}_1\cdot\vec{n})\,\dot{S}_2^i n^j\\
&+\dot{\vec{S}}_2\cdot\vec{n}\,\Big(\dot{S}_1^i n^j+\frac{1}{3}\dot{S}_2^i n^j\Big)-\frac{1}{2}(\ddot{\vec{S}}_1\cdot\vec{n})\,S_2^i n^j-\frac{1}{2}\ddot{S}_1^i S_2^j\Big]\\
&+\frac{G}{4m_1 r^2}\Big[(\vec{v}_1\cdot\vec{n})\,S_1^i\dot{S}_2^j-(\dot{\vec{S}}_2\cdot\vec{v}_1)\,S_1^i n^j+2(\vec{S}_1\cdot\dot{\vec{S}}_2)\,v_1^i n^j+2(\vec{S}_1\cdot\vec{n})\,\dot{S}_2^i v_1^j\\
&+\frac{m_2}{m_1}\Big(-\frac{1}{4}(\vec{S}_1\cdot\vec{v}_1)\,\dot{S}_1^i n^j+\frac{3}{2}(\dot{\vec{S}}_1\cdot\vec{v}_1)\,S_1^i n^j+\frac{3}{4}(\vec{v}_1\cdot\vec{n})\,\dot{S}_1^i S_1^j-(\dot{\vec{S}}_1\cdot\vec{S}_1)\,v_1^i n^j\\
&-\frac{1}{2}(\vec{S}_1\cdot\vec{n})\,\dot{S}_1^i v_1^j\Big)+\frac{m_1}{m_2}\Big(-3(\vec{S}_2\cdot\vec{n})^2\,a_2^i n^j+3S_2^2\,a_2^i n^j-2(\vec{S}_2\cdot\vec{v}_2)\,\dot{S}_2^i n^j\\
&-2(\dot{\vec{S}}_2\cdot\vec{v}_2)\,S_2^i n^j-3(\vec{S}_2\cdot\vec{a}_2)\,S_2^i n^j+4(\dot{\vec{S}}_2\cdot\vec{S}_2)\,v_2^i n^j\\
&+3\vec{S}_2\cdot\vec{n}\,\Big((\vec{v}_2\cdot\vec{n})\,\dot{S}_2^i n^j-S_2^i a_2^j+(\vec{a}_2\cdot\vec{n})\,S_2^i n^j-\dot{S}_2^i v_2^j-2(\dot{\vec{S}}_2\cdot\vec{n})\,v_2^i n^j\Big)\\
&+3\dot{\vec{S}}_2\cdot\vec{n}\,\Big((\vec{v}_2\cdot\vec{n})\,S_2^i n^j-S_2^i v_2^j\Big)\Big)\Big]\\
&+\frac{C_{1(\mathrm{ES}^2)}G}{4m_1 r^2}\Big[11(\vec{S}_1\cdot\vec{S}_2)\,a_2^i n^j-(\vec{S}_1\cdot\vec{v}_2)\,\dot{S}_2^i n^j+3(\vec{v}_1\cdot\vec{n})\,S_1^i\dot{S}_2^j+5(\vec{v}_2\cdot\vec{n})\,S_1^i\dot{S}_2^j\\
&+(\dot{\vec{S}}_2\cdot\vec{v}_1)\,S_1^i n^j-2(\dot{\vec{S}}_2\cdot\vec{v}_2)\,S_1^i n^j-2(\vec{S}_2\cdot\vec{a}_2)\,S_1^i n^j\\
&+\vec{S}_2\cdot\vec{n}\,\Big(5S_1^i a_2^j-3(\vec{a}_2\cdot\vec{n})\,S_1^i n^j\Big)-7(\vec{S}_1\cdot\vec{a}_2)\,S_2^i n^j+3(\vec{a}_2\cdot\vec{n})\,S_1^i S_2^j\\
&+\vec{S}_1\cdot\dot{\vec{S}}_2\,\Big(-2v_1^i n^j+5v_2^i n^j\Big)+\dot{\vec{S}}_2\cdot\vec{n}\,\Big(-3(\vec{v}_1\cdot\vec{n})\,S_1^i n^j-6(\vec{v}_2\cdot\vec{n})\,S_1^i n^j\\
&-S_1^i v_1^j+3S_1^i v_2^j\Big)+\vec{S}_1\cdot\vec{n}\,\Big(-3(\vec{S}_2\cdot\vec{n})\,a_2^i n^j+3(\vec{v}_1\cdot\vec{n})\,\dot{S}_2^i n^j+9(\vec{v}_2\cdot\vec{n})\,\dot{S}_2^i n^j\\
&-2S_2^i a_2^j+15(\vec{a}_2\cdot\vec{n})\,S_2^i n^j-6\dot{S}_2^i v_1^j+2\dot{S}_2^i v_2^j+3\dot{\vec{S}}_2\cdot\vec{n}\,\Big(-2v_1^i n^j+v_2^i n^j\Big)\Big)\Big]\\
&+\frac{C_{2(\mathrm{ES}^2)}G}{m_2 r^2}\Big[(\vec{S}_2\cdot\vec{v}_1)\,\dot{S}_1^i n^j+\vec{S}_2\cdot\vec{v}_2\Big(-\dot{S}_1^i n^j-\frac{1}{4}\dot{S}_2^i n^j\Big)+\frac{1}{2}(\vec{v}_1\cdot\vec{n})\,\dot{S}_1^i S_2^j\\
&-\frac{1}{2}(\vec{v}_2\cdot\vec{n})\,\dot{S}_1^i S_2^j-\frac{1}{2}(\dot{\vec{S}}_1\cdot\vec{v}_1)\,S_2^i n^j+\frac{1}{2}(\dot{\vec{S}}_1\cdot\vec{v}_2)\,S_2^i n^j-\frac{1}{4}(\dot{\vec{S}}_2\cdot\vec{v}_2)\,S_2^i n^j\\
&+\frac{1}{2}\dot{\vec{S}}_2\cdot\vec{S}_2\Big(3v_1^i n^j+v_2^i n^j\Big)+\frac{1}{4}\dot{\vec{S}}_2\cdot\vec{n}\,\Big(3(\vec{v}_2\cdot\vec{n})\,S_2^i n^j+2S_2^i v_1^j+S_2^i v_2^j\Big)\\
&+\frac{1}{2}\dot{\vec{S}}_1\cdot\vec{n}\,\Big(3(\vec{v}_1\cdot\vec{n})\,S_2^i n^j-3(\vec{v}_2\cdot\vec{n})\,S_2^i n^j-S_2^i v_1^j+S_2^i v_2^j\Big)\\
&+\vec{S}_2\cdot\vec{n}\,\Big(-3(\vec{v}_1\cdot\vec{n})\,\dot{S}_1^i n^j+\vec{v}_2\cdot\vec{n}\,\Big(3\dot{S}_1^i n^j+\frac{3}{4}\dot{S}_2^i n^j\Big)+\dot{S}_1^i v_1^j+\frac{1}{2}\dot{S}_2^i v_1^j\\
&-\dot{S}_1^i v_2^j+\frac{1}{4}\dot{S}_2^i v_2^j+\frac{3}{2}\dot{\vec{S}}_2\cdot\vec{n}\,\Big(-v_1^i n^j+v_2^i n^j\Big)\Big)\Big]\\
&+\frac{G}{2m_1 r^3}\Big[3(\vec{v}_1\cdot\vec{n})^2\,S_1^i S_2^j-\frac{7}{2}v_1^2\,S_1^i S_2^j+\frac{3}{4}(\vec{v}_1\cdot\vec{v}_2)\,S_1^i S_2^j\\
&+\vec{S}_2\cdot\vec{v}_2\Big(9(\vec{v}_1\cdot\vec{n})\,S_1^i n^j-8S_1^i v_1^j\Big)+\frac{1}{4}\vec{S}_1\cdot\vec{v}_2\Big(3(\vec{v}_1\cdot\vec{n})\,S_2^i n^j+13S_2^i v_1^j\Big)\\
&+\vec{S}_1\cdot\vec{v}_1\Big(3(\vec{v}_1\cdot\vec{n})\,S_2^i n^j-5S_2^i v_1^j\Big)+\vec{S}_2\cdot\vec{v}_1\Big(-3(\vec{v}_1\cdot\vec{n})\,S_1^i n^j-9(\vec{v}_2\cdot\vec{n})\,S_1^i n^j
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{7}{2}S_1^i v_1^j+\frac{33}{4}S_1^i v_2^j\Big)+\vec{S}_1\cdot\vec{n}\,\Big(-3v_1^2\,S_2^i n^j-\frac{3}{4}(\vec{v}_1\cdot\vec{v}_2)\,S_2^i n^j-9(\vec{S}_2\cdot\vec{v}_2)\,v_1^i n^j\\
&+\frac{9}{2}(\vec{v}_1\cdot\vec{n})\,S_2^i v_1^j-\frac{9}{4}(\vec{v}_2\cdot\vec{n})\,S_2^i v_1^j-\frac{9}{4}(\vec{S}_2\cdot\vec{n})\,v_1^i v_2^j+\vec{S}_2\cdot\vec{v}_1\Big(\frac{3}{2}v_1^i n^j+\frac{51}{4}v_2^i n^j\Big)\Big)\\
&+\vec{S}_1\cdot\vec{S}_2\Big(\frac{27}{4}(\vec{v}_2\cdot\vec{n})\,v_1^i n^j-\frac{23}{4}v_1^i v_2^j+\vec{v}_1\cdot\vec{n}\,\Big(3v_1^i n^j-\frac{51}{4}v_2^i n^j\Big)\Big)\\
&+\vec{S}_2\cdot\vec{n}\,\Big(\frac{9}{2}v_1^2\,S_1^i n^j-3(\vec{v}_1\cdot\vec{v}_2)\,S_1^i n^j-\frac{15}{2}(\vec{S}_1\cdot\vec{v}_1)\,v_1^i n^j+\frac{33}{4}(\vec{S}_1\cdot\vec{v}_2)\,v_1^i n^j\\
&+6(\vec{v}_2\cdot\vec{n})\,S_1^i v_1^j+\vec{v}_1\cdot\vec{n}\,\Big(-6S_1^i v_1^j-3S_1^i v_2^j\Big)\Big)\Big]\\
&+\frac{G}{4m_2r^3}\Big[\vec{S}_2\cdot\vec{v}_2\Big(-27(\vec{v}_1\cdot\vec{n})\,S_2^i n^j-3(\vec{v}_2\cdot\vec{n})\,S_2^i n^j+21S_2^i v_1^j-4S_2^i v_2^j\Big)\\
&+\vec{S}_2\cdot\vec{v}_1\Big(21(\vec{v}_2\cdot\vec{n})\,S_2^i n^j-15S_2^i v_2^j\Big)+(\vec{S}_2\cdot\vec{n})^2\,\Big(-15(\vec{v}_1\cdot\vec{n})\,v_2^i n^j\\
&-15(\vec{v}_2\cdot\vec{n})\,v_2^i n^j-9v_1^i v_2^j\Big)+S_2^2\,\Big(27(\vec{v}_1\cdot\vec{n})\,v_2^i n^j+21v_1^i v_2^j\\
&+\vec{v}_2\cdot\vec{n}\,\Big(-30v_1^i n^j+9v_2^i n^j\Big)\Big)+\vec{S}_2\cdot\vec{n}\,\Big(-9(\vec{v}_1\cdot\vec{v}_2)\,S_2^i n^j+15(\vec{v}_2\cdot\vec{n})^2\,S_2^i n^j\\
&+9v_2^2\,S_2^i n^j-3(\vec{S}_2\cdot\vec{v}_1)\,v_2^i n^j+\vec{S}_2\cdot\vec{v}_2\Big(30v_1^i n^j-15v_2^i n^j\Big)\\
&+\vec{v}_1\cdot\vec{n}\,\Big(15(\vec{v}_2\cdot\vec{n})\,S_2^i n^j+3S_2^i v_2^j\Big)+\vec{v}_2\cdot\vec{n}\,\Big(-9S_2^i v_1^j-3S_2^i v_2^j\Big)\Big)\Big]\\
&+\frac{Gm_2}{2m_1^2r^3}\Big[-3(\vec{S}_1\cdot\vec{n})^2\,v_1^i v_2^j+\vec{S}_1\cdot\vec{v}_2\Big(-3(\vec{v}_1\cdot\vec{n})\,S_1^i n^j+\frac{13}{4}S_1^i v_1^j\Big)\\
&+\vec{S}_1\cdot\vec{v}_1\Big(\frac{9}{4}(\vec{v}_2\cdot\vec{n})\,S_1^i n^j-3S_1^i v_1^j-7S_1^i v_2^j\Big)\\
&+\vec{S}_1\cdot\vec{n}\,\Big(-\frac{9}{4}v_1^2\,S_1^i n^j+3(\vec{v}_1\cdot\vec{v}_2)\,S_1^i n^j+3(\vec{S}_1\cdot\vec{v}_2)\,v_1^i n^j+6(\vec{v}_1\cdot\vec{n})\,S_1^i v_1^j\\
&-\frac{15}{4}(\vec{v}_2\cdot\vec{n})\,S_1^i v_1^j-3\vec{S}_1\cdot\vec{v}_1\Big(v_1^i n^j+v_2^i n^j\Big)\Big)+S_1^2\,\Big(-\frac{15}{4}(\vec{v}_2\cdot\vec{n})\,v_1^i n^j\\
&+\frac{13}{4}v_1^i v_2^j+\vec{v}_1\cdot\vec{n}\,\Big(\frac{15}{4}v_1^i n^j+3v_2^i n^j\Big)\Big)\Big]\\
&+\frac{C_{1(\mathrm{ES}^2)}G}{4m_1r^3}\Big[\Big(13(\vec{v}_1\cdot\vec{v}_2)+3(\vec{v}_1\cdot\vec{n})(\vec{v}_2\cdot\vec{n})+15(\vec{v}_2\cdot\vec{n})^2-7v_2^2\Big)\,S_1^i S_2^j\\
&+\vec{S}_2\cdot\vec{v}_2\Big(12(\vec{v}_1\cdot\vec{n})\,S_1^i n^j-4S_1^i v_1^j+S_1^i v_2^j\Big)+\vec{S}_2\cdot\vec{v}_1\Big(-6S_1^i v_1^j+3S_1^i v_2^j\Big)\\
&+\vec{S}_1\cdot\vec{v}_1\Big(-12(\vec{v}_1\cdot\vec{n})\,S_2^i n^j+21(\vec{v}_2\cdot\vec{n})\,S_2^i n^j-18S_2^i v_2^j\Big)\\
&+\vec{S}_1\cdot\vec{v}_2\Big(-39(\vec{v}_1\cdot\vec{n})\,S_2^i n^j+6(\vec{v}_2\cdot\vec{n})\,S_2^i n^j+29S_2^i v_1^j-5S_2^i v_2^j\Big)\\
&+\vec{S}_1\cdot\vec{S}_2\Big(21v_1^i v_2^j+\vec{v}_1\cdot\vec{n}\,\Big(24v_1^i n^j+15v_2^i n^j\Big)+\vec{v}_2\cdot\vec{n}\,\Big(-30v_1^i n^j-3v_2^i n^j\Big)\Big)\\
&+\vec{S}_2\cdot\vec{n}\,\Big(-12(\vec{v}_1\cdot\vec{v}_2)\,S_1^i n^j-30(\vec{v}_2\cdot\vec{n})^2\,S_1^i n^j+6v_2^2\,S_1^i n^j-33(\vec{S}_1\cdot\vec{v}_1)\,v_2^i n^j\\
&+\vec{S}_1\cdot\vec{v}_2\Big(42v_1^i n^j-3v_2^i n^j\Big)+\vec{v}_1\cdot\vec{n}\,\Big(6S_1^i v_1^j-3S_1^i v_2^j\Big)\\
&+\vec{v}_2\cdot\vec{n}\,\Big(-12S_1^i v_1^j+15S_1^i v_2^j\Big)\Big)+\vec{S}_1\cdot\vec{n}\,\Big(15(\vec{v}_1\cdot\vec{v}_2)\,S_2^i n^j+15(\vec{v}_2\cdot\vec{n})^2\,S_2^i n^j
\end{aligned}
$$

$$
\begin{aligned}
&- 9v_2^2\, S_2^i n^j - 3(\vec{S}_2\cdot\vec{v}_2)\, v_2^i n^j + \vec{S}_2\cdot\vec{v}_1\Big(- 12v_1^i n^j + 9v_2^i n^j\Big)\\
&+ \vec{v}_1\cdot\vec{n}\,\Big(75(\vec{v}_2\cdot\vec{n})\, S_2^i n^j + 12S_2^i v_1^j - 18S_2^i v_2^j\Big) + \vec{v}_2\cdot\vec{n}\,\Big(- 33S_2^i v_1^j + 9S_2^i v_2^j\Big)\\
&+ \vec{S}_2\cdot\vec{n}\,\Big(- 15(\vec{v}_1\cdot\vec{n})\, v_2^i n^j - 9v_1^i v_2^j + \vec{v}_2\cdot\vec{n}\,\big(- 60v_1^i n^j + 45v_2^i n^j\big)\Big)\Big)\Bigg]\\
&+ \frac{C_{1(\mathrm{ES}^2)}Gm_2}{m_1^2 r^3}\Bigg[\vec{S}_1\cdot\vec{v}_2\Big(6(\vec{v}_1\cdot\vec{n})\, S_1^i n^j - 3S_1^i v_1^j\Big) + \vec{S}_1\cdot\vec{v}_1\Big(- \frac{3}{4}(\vec{v}_1\cdot\vec{n})\, S_1^i n^j\\
&- 6(\vec{v}_2\cdot\vec{n})\, S_1^i n^j - \frac{9}{4}S_1^i v_1^j + 6S_1^i v_2^j\Big) + 9S_1^2\,\Big(\frac{1}{2}(\vec{v}_2\cdot\vec{n})\, v_1^i n^j - \frac{1}{2}v_1^i v_2^j\\
&+ \vec{v}_1\cdot\vec{n}\,\Big(\frac{1}{2}v_1^i n^j - v_2^i n^j\Big)\Big) + \vec{S}_1\cdot\vec{n}\,\Big(\frac{9}{4}v_1^2\, S_1^i n^j - 3(\vec{v}_1\cdot\vec{v}_2)\, S_1^i n^j + 3(\vec{v}_2\cdot\vec{n})\, S_1^i v_1^j\\
&+ \vec{S}_1\cdot\vec{v}_1\Big(- \frac{27}{4}v_1^i n^j + 9v_2^i n^j\Big) + \vec{v}_1\cdot\vec{n}\,\Big(\frac{3}{4}S_1^i v_1^j - 3S_1^i v_2^j\Big)\Big)\Bigg]\\
&+ \frac{C_{2(\mathrm{ES}^2)}G}{m_2 r^3}\Bigg[\vec{S}_2\cdot\vec{v}_2\Big(S_2^i v_1^j - S_2^i v_2^j\Big) + \frac{1}{2}(\vec{S}_2\cdot\vec{n})^2\,\Big(- 9v_1^i v_2^j\\
&+ \vec{v}_2\cdot\vec{n}\,\Big(- 15v_1^i n^j + 15v_2^i n^j\Big)\Big) + \frac{1}{2}S_2^2\,\Big(5v_1^i v_2^j + \vec{v}_2\cdot\vec{n}\,\Big(- 3v_1^i n^j + 3v_2^i n^j\Big)\Big)\\
&+ \vec{S}_2\cdot\vec{n}\,\Big(\vec{S}_2\cdot\vec{v}_2\Big(3v_1^i n^j - 3v_2^i n^j\Big) + \vec{v}_2\cdot\vec{n}\,\Big(- 3S_2^i v_1^j + 3S_2^i v_2^j\Big)\Big)\Bigg]\\
&+ \frac{C_{1(\mathrm{BS}^3)}Gm_2}{m_1^2 r^3}\Bigg[\vec{S}_1\cdot\vec{v}_2\Big((\vec{v}_2\cdot\vec{n})\, S_1^i n^j + \frac{1}{3}S_1^i v_1^j\Big) + \vec{S}_1\cdot\vec{v}_1\Big((\vec{v}_1\cdot\vec{n})\, S_1^i n^j\\
&- 2(\vec{v}_2\cdot\vec{n})\, S_1^i n^j - S_1^i v_1^j + \frac{2}{3}S_1^i v_2^j\Big) + \frac{1}{2}(\vec{S}_1\cdot\vec{n})^2\,\Big(- 3v_1^i v_2^j\\
&+ \vec{v}_2\cdot\vec{n}\,\Big(- 15v_1^i n^j + 15v_2^i n^j\Big)\Big) + \frac{1}{2}S_1^2\,\Big(v_1^i v_2^j + \vec{v}_2\cdot\vec{n}\,\Big(3v_1^i n^j - 3v_2^i n^j\Big)\Big)\\
&+ \vec{S}_1\cdot\vec{n}\,\Big(v_1^2\, S_1^i n^j - (\vec{v}_1\cdot\vec{v}_2)\, S_1^i n^j - 5(\vec{v}_2\cdot\vec{n})^2\, S_1^i n^j\\
&+ \vec{S}_1\cdot\vec{v}_1\Big(- 3v_1^i n^j + 3v_2^i n^j\Big) + \vec{v}_1\cdot\vec{n}\,\Big(5(\vec{v}_2\cdot\vec{n})\, S_1^i n^j - S_1^i v_1^j + 2S_1^i v_2^j\Big)\\
&+ \vec{v}_2\cdot\vec{n}\,\Big(- 5S_1^i v_1^j + 4S_1^i v_2^j\Big)\Big)\Bigg] - (i\leftrightarrow j). \qquad (11.31)
\end{aligned}
$$

As a check of the variable redefinitions, it has been verified that the resulting reduced potential corresponds also to the insertion of the lower-order equations of motion in the higher-order time derivatives, as it should at this order. This way, we obtain that the NLO S^3 reduced potential reads as follows:

$$
(V_s)^{\mathrm{NLO}}_{\mathrm{S}^3} = (V_s)^{\mathrm{NLO}}_{\mathrm{S}_1^2\mathrm{S}_2} + (V_s)^{\mathrm{NLO}}_{\mathrm{S}_1^3} + (1\leftrightarrow 2), \qquad (11.32)
$$

where we have

$$
\begin{aligned}
(V_s)^{\mathrm{NLO}}_{\mathrm{S}_1^2\mathrm{S}_2} =& \frac{G^2}{r^5}V_{(1)} + \frac{G^2}{r^5}\frac{m_2}{m_1}V_{(2)} + \frac{C_{1(\mathrm{ES}^2)}G^2}{r^5}V_{(3)} + \frac{C_{1(\mathrm{ES}^2)}G^2}{r^5}\frac{m_2}{m_1}V_{(4)}\\
&+ \frac{G}{r^4}\frac{1}{m_1}V_{(5)} + \frac{C_{1(\mathrm{ES}^2)}G}{r^4}\frac{1}{m_1}V_{(6)}, \qquad (11.33)
\end{aligned}
$$

with the following pieces:

$$
V_{(1)} = \frac{49}{4}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n} + \vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n} + \frac{1}{4}\vec{S}_2\cdot\vec{v}_1\times\vec{n}\,\Big(- 15(\vec{S}_1\cdot\vec{n})^2 + 7S_1^2\Big)
$$

$$
\begin{aligned}
&+\frac{1}{4}\vec{S}_2\cdot\vec{v}_2\times\vec{n}\left(3(\vec{S}_1\cdot\vec{n})^2+5S_1^2\right)+\vec{S}_1\cdot\vec{v}_2\times\vec{n}\left(-7\vec{S}_1\cdot\vec{S}_2+24\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right)\\
&+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(-\frac{25}{2}\vec{S}_1\cdot\vec{S}_2+60\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right)\\
&+\frac{1}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{n}\left(7\vec{S}_1\cdot\vec{v}_1-9\vec{S}_1\cdot\vec{v}_2+\vec{S}_1\cdot\vec{n}\left(-15\vec{v}_1\cdot\vec{n}+9\vec{v}_2\cdot\vec{n}\right)\right),
\end{aligned}
\tag{11.34}
$$

$$
\begin{aligned}
V_{(2)}=&\frac{13}{4}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n}+\frac{121}{8}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}+\vec{S}_2\cdot\vec{v}_1\times\vec{n}\left(-\frac{15}{2}(\vec{S}_1\cdot\vec{n})^2+2S_1^2\right)\\
&+\frac{1}{4}\vec{S}_2\cdot\vec{v}_2\times\vec{n}\left(\frac{9}{2}(\vec{S}_1\cdot\vec{n})^2-S_1^2\right)+\frac{1}{2}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(-3\vec{S}_1\cdot\vec{S}_2+\frac{69}{2}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right)\\
&+\vec{S}_1\cdot\vec{v}_2\times\vec{n}\left(-15\vec{S}_1\cdot\vec{S}_2+\frac{117}{2}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right)\\
&+\vec{S}_1\cdot\vec{S}_2\times\vec{n}\left(\frac{3}{2}\vec{S}_1\cdot\vec{v}_1+\vec{S}_1\cdot\vec{v}_2+\vec{S}_1\cdot\vec{n}\left(\frac{3}{2}\vec{v}_1\cdot\vec{n}-6\vec{v}_2\cdot\vec{n}\right)\right),
\end{aligned}
\tag{11.35}
$$

$$
\begin{aligned}
V_{(3)}=&9\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{v}_1\times\vec{n}\,\vec{S}_1\cdot\vec{S}_2+\vec{S}_1\cdot\vec{v}_2\times\vec{n}\,\vec{S}_1\cdot\vec{S}_2\\
&+\frac{1}{4}\vec{S}_2\cdot\vec{v}_1\times\vec{n}\left(-123(\vec{S}_1\cdot\vec{n})^2+35S_1^2\right)+\frac{1}{4}\vec{S}_2\cdot\vec{v}_2\times\vec{n}\left(15(\vec{S}_1\cdot\vec{n})^2-7S_1^2\right)\\
&+\frac{1}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{n}\left(3\vec{S}_1\cdot\vec{v}_1-\vec{S}_1\cdot\vec{v}_2+\vec{S}_1\cdot\vec{n}\,(3\vec{v}_1\cdot\vec{n}-3\vec{v}_2\cdot\vec{n})\right),
\end{aligned}
\tag{11.36}
$$

$$
\begin{aligned}
V_{(4)}=&4\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n}+\frac{95}{4}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}+\vec{S}_2\cdot\vec{v}_1\times\vec{n}\left(-57(\vec{S}_1\cdot\vec{n})^2+\frac{79}{2}S_1^2\right)\\
&+\vec{S}_2\cdot\vec{v}_2\times\vec{n}\left(3(\vec{S}_1\cdot\vec{n})^2-\frac{109}{4}S_1^2\right)+\vec{S}_1\cdot\vec{v}_2\times\vec{n}\left(28\vec{S}_1\cdot\vec{S}_2+3\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right)\\
&+\frac{1}{2}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(-59\vec{S}_1\cdot\vec{S}_2+9\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right)\\
&+\vec{S}_1\cdot\vec{S}_2\times\vec{n}\left(43\vec{S}_1\cdot\vec{v}_1-43\vec{S}_1\cdot\vec{v}_2+\vec{S}_1\cdot\vec{n}\,(-84\vec{v}_1\cdot\vec{n}+78\vec{v}_2\cdot\vec{n})\right),
\end{aligned}
\tag{11.37}
$$

$$
\begin{aligned}
V_{(5)}=&-\frac{3}{4}\vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\,S_1^2\,\vec{v}_1\cdot\vec{n}+\frac{1}{2}\vec{S}_2\cdot\vec{v}_1\times\vec{n}\left(3(\vec{S}_1\cdot\vec{v}_1)^2-15\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1\,\vec{v}_1\cdot\vec{n}\right.\\
&\left.+S_1^2\left(\frac{15}{2}(\vec{v}_1\cdot\vec{n})^2-3v_1^2\right)+\frac{15}{2}(\vec{S}_1\cdot\vec{n})^2\,v_1^2\right)+\frac{1}{8}\vec{S}_2\cdot\vec{v}_2\times\vec{n}\left(-11(\vec{S}_1\cdot\vec{v}_1)^2\right.\\
&\left.+55\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1\,\vec{v}_1\cdot\vec{n}-\frac{55}{2}(\vec{S}_1\cdot\vec{n})^2\,v_1^2+S_1^2\left(-\frac{55}{2}(\vec{v}_1\cdot\vec{n})^2+11v_1^2\right)\right)\\
&+\frac{1}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\left(-\frac{3}{2}\vec{S}_1\cdot\vec{v}_1\,\vec{v}_1\cdot\vec{n}-3\vec{S}_1\cdot\vec{v}_2\,\vec{v}_1\cdot\vec{n}+3\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{v}_2\right)\\
&+\vec{S}_1\cdot\vec{v}_2\times\vec{n}\left(3\vec{S}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{v}_1-\frac{15}{2}\vec{S}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\right.\\
&\left.+\frac{15}{2}\vec{S}_1\cdot\vec{n}\left(\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{v}_2-\vec{S}_2\cdot\vec{v}_1\,\vec{v}_2\cdot\vec{n}\right)+\vec{S}_1\cdot\vec{S}_2\left(-3\vec{v}_1\cdot\vec{v}_2+\frac{15}{2}\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right)\right)\\
&+\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\left(\vec{S}_2\cdot\vec{n}\left(\frac{21}{4}\vec{S}_1\cdot\vec{v}_1-6\vec{S}_1\cdot\vec{v}_2\right)+\vec{S}_1\cdot\vec{S}_2\left(\frac{27}{4}\vec{v}_1\cdot\vec{n}-3\vec{v}_2\cdot\vec{n}\right)\right.\\
&\left.+\vec{S}_1\cdot\vec{n}\left(6\vec{S}_2\cdot\vec{v}_1-\frac{21}{4}\vec{S}_2\cdot\vec{v}_2+\frac{1}{2}\vec{S}_2\cdot\vec{n}(-60\vec{v}_1\cdot\vec{n}+45\vec{v}_2\cdot\vec{n})\right)\right)\\
&+\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\left(\frac{3}{2}\vec{S}_1\cdot\vec{v}_2\left(\frac{1}{2}\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\right)+\frac{3}{2}\vec{S}_1\cdot\vec{v}_1\left(\vec{v}_1\cdot\vec{n}+\frac{1}{2}\vec{v}_2\cdot\vec{n}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\vec{S}_1\cdot\vec{n}\left(-\frac{33}{8}v_1^2+6\vec{v}_1\cdot\vec{v}_2+\frac{15}{4}(\vec{v}_2\cdot\vec{n})^2-3v_2^2\right)\Big)\\
&+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(\frac{1}{4}\vec{S}_1\cdot\vec{v}_2\Big(-11\vec{S}_2\cdot\vec{v}_1+3\vec{S}_2\cdot\vec{v}_2\Big)-\frac{3}{4}\vec{S}_1\cdot\vec{v}_1\,\vec{S}_2\cdot\vec{v}_2\right.\\
&+\vec{S}_2\cdot\vec{n}\left(\frac{5}{8}\vec{S}_1\cdot\vec{v}_2(11\vec{v}_1\cdot\vec{n}-12\vec{v}_2\cdot\vec{n})+\frac{15}{4}\vec{S}_1\cdot\vec{v}_1\,\vec{v}_2\cdot\vec{n}\right)\\
&+\vec{S}_1\cdot\vec{n}\left(\frac{55}{8}\vec{S}_2\cdot\vec{v}_1\,\vec{v}_2\cdot\vec{n}-\frac{15}{4}\vec{S}_2\cdot\vec{v}_2\,\vec{v}_2\cdot\vec{n}\right.\\
&+\left.\frac{5}{8}\vec{S}_2\cdot\vec{n}\,(-21v_1^2+37\vec{v}_1\cdot\vec{v}_2+42(\vec{v}_2\cdot\vec{n})^2-120v_2^2)\right)\\
&+\vec{S}_1\cdot\vec{S}_2\Big(\frac{21}{8}v_1^2-\frac{13}{4}\vec{v}_1\cdot\vec{v}_2-\frac{55}{8}\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}-\frac{15}{4}(\vec{v}_2\cdot\vec{n})^2+3v_2^2\Big)\Big),
\end{aligned}
\tag{11.38}
$$

$$
\begin{aligned}
V_{(6)}=\;&\frac{15}{2}\vec{v}_1\cdot\vec{v}_2\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2\,\vec{S}_2\cdot\vec{v}_1+3\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\,\vec{S}_1\cdot\vec{S}_2\,\vec{v}_2\cdot\vec{n}\\
&+\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\Big(12\vec{S}_1\cdot\vec{v}_1\,\vec{v}_2\cdot\vec{n}+3\vec{S}_1\cdot\vec{v}_2\,\vec{v}_2\cdot\vec{n}\Big)\\
&+\vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\Big(\vec{S}_1\cdot\vec{n}\left(-\frac{21}{4}\vec{S}_1\cdot\vec{v}_1+6\vec{S}_1\cdot\vec{v}_2\right)+(\vec{S}_1\cdot\vec{n})^2\left(\frac{45}{4}\vec{v}_1\cdot\vec{n}-15\vec{v}_2\cdot\vec{n}\right)\\
&+S_1^2\left(-\frac{27}{4}\vec{v}_1\cdot\vec{n}+3\vec{v}_2\cdot\vec{n}\right)\Big)+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(-3\vec{S}_1\cdot\vec{v}_1\,\vec{S}_2\cdot\vec{v}_1+3\vec{S}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{v}_1\right.\\
&+\left.15\vec{S}_1\cdot\vec{v}_1\,\vec{S}_2\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}+\vec{S}_1\cdot\vec{S}_2\Big(3v_1^2-3\vec{v}_1\cdot\vec{v}_2-15\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\right)\\
&+\vec{S}_2\cdot\vec{v}_1\times\vec{n}\left(6(\vec{S}_1\cdot\vec{v}_1)^2-6\vec{S}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{v}_2\right.\\
&+\vec{S}_1\cdot\vec{n}\left(-15\vec{S}_1\cdot\vec{v}_1\,\vec{v}_1\cdot\vec{n}+15\vec{S}_1\cdot\vec{v}_2\,\vec{v}_1\cdot\vec{n}\right)\\
&+(\vec{S}_1\cdot\vec{n})^2\left(\frac{15}{2}v_1^2-\frac{105}{2}\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right)+S_1^2\left(15(\vec{v}_1\cdot\vec{n})^2-\frac{15}{2}v_1^2+\frac{15}{2}\vec{v}_1\cdot\vec{v}_2\right.\\
&+\left.\left.\frac{15}{2}\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right)\right)+\vec{S}_1\cdot\vec{v}_2\times\vec{n}\left(3\vec{S}_1\cdot\vec{v}_1\,\vec{S}_2\cdot\vec{v}_1-15\vec{S}_1\cdot\vec{v}_2\,\vec{S}_2\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\right.\\
&+\left.\vec{S}_1\cdot\vec{S}_2\Big(-3v_1^2+15\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\Big)\right)+\vec{S}_2\cdot\vec{v}_2\times\vec{n}\left(-6(\vec{S}_1\cdot\vec{v}_1)^2+3\vec{S}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{v}_2\right.\\
&+\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{v}_1\Big(\frac{75}{4}\vec{v}_1\cdot\vec{n}-15\vec{v}_2\cdot\vec{n}\Big)+(\vec{S}_1\cdot\vec{n})^2\left(-\frac{105}{8}(\vec{v}_1\cdot\vec{n})^2-\frac{15}{2}v_1^2\right.\\
&+\left.\frac{105}{2}\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}-\frac{15}{16}v_2^2\right)+S_1^2\left(-\frac{105}{8}(\vec{v}_1\cdot\vec{n})^2+6v_1^2-3\vec{v}_1\cdot\vec{v}_2\right.\\
&-\left.\left.\frac{15}{2}\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}+\frac{3}{16}v_2^2\right)\right)+\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\Big(-\frac{3}{4}\vec{S}_1\cdot\vec{v}_1\,\vec{v}_1\cdot\vec{n}-12\vec{S}_1\cdot\vec{v}_2\,\vec{v}_1\cdot\vec{n}\\
&+\vec{S}_1\cdot\vec{n}\left(\frac{15}{4}(\vec{v}_1\cdot\vec{n})^2-3v_1^2+\frac{3}{8}v_2^2\right)\Big),
\end{aligned}
\tag{11.39}
$$

and also

$$
\begin{aligned}
(V_s)^{\rm NLO}_{\rm S_1^3}=\;&\frac{G^2}{r^5}\frac{m_2}{m_1}V_{[1]}+\frac{G^2}{r^5}\frac{m_2^2}{m_1^2}V_{[2]}+\frac{C_{1(\rm ES^2)}G^2}{r^5}\frac{m_2}{m_1}V_{[3]}+\frac{C_{1(\rm ES^2)}G^2}{r^5}\frac{m_2^2}{m_1^2}V_{[4]}\\
&+\frac{C^2_{1(\rm ES^2)}G^2}{r^5}\frac{m_2^2}{m_1^2}V_{[5]}+\frac{C_{1(\rm BS^3)}G^2}{r^5}\frac{m_2}{m_1}V_{[6]}+\frac{C_{1(\rm BS^3)}G^2}{r^5}\frac{m_2^2}{m_1^2}V_{[7]}\\
&+\frac{G}{r^4}\frac{m_2}{m_1^2}V_{[8]}+\frac{C_{1(\rm ES^2)}G}{r^4}\frac{m_2}{m_1^2}V_{[9]}+\frac{C_{1(\rm BS^3)}G}{r^4}\frac{m_2}{m_1^2}V_{[10]},
\end{aligned}
\tag{11.40}
$$

with the pieces:

$$V_{[1]} = 3\vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\,(\vec{S}_1 \cdot \vec{n})^2, \tag{11.41}$$

$$V_{[2]} = \vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\,\Big(- \frac{21}{8}(\vec{S}_1 \cdot \vec{n})^2 + \frac{3}{2}S_1^2\Big) + \vec{S}_1 \cdot \vec{v}_2 \times \vec{n}\,\Big(\frac{3}{2}(\vec{S}_1 \cdot \vec{n})^2 - \frac{3}{4}S_1^2\Big), \tag{11.42}$$

$$V_{[3]} = \vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\,\Big(\frac{51}{2}(\vec{S}_1 \cdot \vec{n})^2 - \frac{23}{4}S_1^2\Big) + \vec{S}_1 \cdot \vec{v}_2 \times \vec{n}\,\Big(\frac{9}{2}(\vec{S}_1 \cdot \vec{n})^2 - \frac{1}{2}S_1^2\Big), \tag{11.43}$$

$$V_{[4]} = \vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\,\Big(\frac{87}{4}(\vec{S}_1 \cdot \vec{n})^2 - 3S_1^2\Big) + \vec{S}_1 \cdot \vec{v}_2 \times \vec{n}\,\Big(\frac{33}{2}(\vec{S}_1 \cdot \vec{n})^2 - \frac{9}{2}S_1^2\Big), \tag{11.44}$$

$$V_{[5]} = -6\vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\,(\vec{S}_1 \cdot \vec{n})^2 + 9\vec{S}_1 \cdot \vec{v}_2 \times \vec{n}\,(\vec{S}_1 \cdot \vec{n})^2, \tag{11.45}$$

$$V_{[6]} = \vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\,\Big(3(\vec{S}_1 \cdot \vec{n})^2 - \frac{2}{3}S_1^2\Big) + \vec{S}_1 \cdot \vec{v}_2 \times \vec{n}\,\Big(- \frac{5}{2}(\vec{S}_1 \cdot \vec{n})^2 + \frac{1}{2}S_1^2\Big), \tag{11.46}$$

$$V_{[7]} = \vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\,\Big(- 21(\vec{S}_1 \cdot \vec{n})^2 + 4S_1^2\Big) + \vec{S}_1 \cdot \vec{v}_2 \times \vec{n}\,\Big(\frac{43}{2}(\vec{S}_1 \cdot \vec{n})^2 - \frac{25}{6}S_1^2\Big), \tag{11.47}$$

$$\begin{aligned} V_{[8]} = {} & \frac{3}{2}\vec{S}_1 \cdot \vec{v}_1 \times \vec{v}_2\Big(\vec{S}_1 \cdot \vec{n}\,\vec{S}_1 \cdot \vec{v}_1 - S_1^2\,\vec{v}_1 \cdot \vec{n}\Big) + \Big(\frac{11}{8}\vec{S}_1 \cdot \vec{v}_1 \times \vec{n} - \frac{3}{2}\vec{S}_1 \cdot \vec{v}_2 \times \vec{n}\Big) \\ & \Big((\vec{S}_1 \cdot \vec{v}_1)^2 - 5\vec{S}_1 \cdot \vec{n}\,\vec{S}_1 \cdot \vec{v}_1\,\vec{v}_1 \cdot \vec{n} + S_1^2\,\Big(\frac{5}{2}(\vec{v}_1 \cdot \vec{n})^2 - v_1^2\Big) + \frac{5}{2}(\vec{S}_1 \cdot \vec{n})^2\,v_1^2\Big), \end{aligned} \tag{11.48}$$

$$\begin{aligned} V_{[9]} = {} & \vec{S}_1 \cdot \vec{v}_1 \times \vec{v}_2\Big(\vec{S}_1 \cdot \vec{n}\,\Big(\frac{21}{4}\vec{S}_1 \cdot \vec{v}_1 - \frac{9}{2}\vec{S}_1 \cdot \vec{v}_2\Big) + S_1^2\,\Big(\frac{9}{2}\vec{v}_1 \cdot \vec{n} - \frac{15}{4}\vec{v}_2 \cdot \vec{n}\Big) \\ & + (\vec{S}_1 \cdot \vec{n})^2\,\Big(- 15\vec{v}_1 \cdot \vec{n} + \frac{45}{4}\vec{v}_2 \cdot \vec{n}\Big)\Big) + \frac{3}{4}\vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\,\Big(- \vec{S}_1 \cdot \vec{v}_1\,\vec{S}_1 \cdot \vec{v}_2 + (\vec{S}_1 \cdot \vec{v}_2)^2 \\ & + \vec{S}_1 \cdot \vec{n}\,\Big(5\vec{S}_1 \cdot \vec{v}_1\,\vec{v}_2 \cdot \vec{n} - 10\vec{S}_1 \cdot \vec{v}_2\,\vec{v}_2 \cdot \vec{n}\Big) \\ & + (\vec{S}_1 \cdot \vec{n})^2\,\Big(- \frac{35}{4}v_1^2 + 45\vec{v}_1 \cdot \vec{v}_2 + \frac{35}{2}(\vec{v}_2 \cdot \vec{n})^2 - 10v_2^2\Big) \\ & + S_1^2\,\Big(5(\vec{v}_1 \cdot \vec{n})^2 + \frac{3}{4}v_1^2 - 2\vec{v}_1 \cdot \vec{v}_2 - 10\vec{v}_1 \cdot \vec{n}\,\vec{v}_2 \cdot \vec{n} + \frac{5}{2}(\vec{v}_2 \cdot \vec{n})^2 + v_2^2\Big)\Big), \end{aligned} \tag{11.49}$$

$$\begin{aligned} V_{[10]} = {} & \vec{v}_1 \cdot \vec{v}_2 \times \vec{n}\,\Big(- \frac{1}{2}S_1^2\,\vec{S}_1 \cdot \vec{v}_2 + (\vec{S}_1 \cdot \vec{n})^2\,\vec{S}_1 \cdot \vec{v}_1 + \frac{5}{2}(\vec{S}_1 \cdot \vec{n})^2\,\vec{S}_1 \cdot \vec{v}_2\Big) \\ & - \vec{S}_1 \cdot \vec{v}_1 \times \vec{v}_2\,\vec{S}_1 \cdot \vec{n}\,\vec{S}_1 \cdot \vec{v}_2 + \vec{S}_1 \cdot \vec{v}_2 \times \vec{n}\,\Big(- (\vec{S}_1 \cdot \vec{v}_1)^2 + \vec{S}_1 \cdot \vec{v}_1\,\vec{S}_1 \cdot \vec{v}_2 \\ & + \vec{S}_1 \cdot \vec{n}\,\Big(- 5\vec{S}_1 \cdot \vec{v}_2\,\vec{v}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{v}_1\,(4\vec{v}_1 \cdot \vec{n} - 5\vec{v}_1 \cdot \vec{n})\Big) \\ & + \frac{5}{2}(\vec{S}_1 \cdot \vec{n})^2\,(7\vec{v}_1 \cdot \vec{n}\,\vec{v}_2 \cdot \vec{n} - v_2^2) + \frac{1}{2}S_1^2\,(-5\vec{v}_1 \cdot \vec{n}\,\vec{v}_2 \cdot \vec{n} + v_2^2)\Big) \\ & + \vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\,\Big((\vec{S}_1 \cdot \vec{v}_1)^2 - \vec{S}_1 \cdot \vec{v}_1\,\vec{S}_1 \cdot \vec{v}_2 + \vec{S}_1 \cdot \vec{n}\,\Big(5\vec{S}_1 \cdot \vec{v}_2\,\vec{v}_1 \cdot \vec{n} \end{aligned}$$

$$+ \vec{S}_1 \cdot \vec{v}_1 \left(-5\vec{v}_1 \cdot \vec{n} + 6\vec{v}_2 \cdot \vec{n}\right)\Big) + S_1^2 \left(-\frac{1}{2} v_1^2 + \vec{v}_1 \cdot \vec{v}_2 + \frac{5}{2} \vec{v}_1 \cdot \vec{n}\, \vec{v}_2 \cdot \vec{n} - v_2^2 \right)$$
$$+ (\vec{S}_1 \cdot \vec{n})^2 \left(\frac{5}{2} v_1^2 - 5\vec{v}_1 \cdot \vec{v}_2 - \frac{35}{2} \vec{v}_1 \cdot \vec{n}\, \vec{v}_2 \cdot \vec{n} + 5 v_2^2 \right)\Big). \tag{11.50}$$

Comparing with the original Lagrangian in eqs. (5.21)-(5.42), we can see that the reduced potential is considerably smaller in scale. Note that there is a peculiar $V_{[5]}$ term that is multiplied by a Wilson coefficient squared. This is unusual since Wilson coefficients capture the UV physics of an extended object that is modeled as a point-like particle. Thus, a product of them could signal the breakdown of the point-particle picture at higher spins, as it could indicate the emergence of composite effects. In particular, looking at it from the EoM perspective instead of from the variable redefinitions, we have that at LO SS the precession equation is

$$(\dot{S}_1^{ij})_{\rm SS}^{\rm LO} = \frac{6 C_{1({\rm ES}^2)} G m_2}{m_1 r^3} (\vec{S}_1 \cdot \vec{n})\, n^{[i} S_1^{j]}, \tag{11.51}$$

as given in eq. (7.12), while the NLO SS potential of eq. (5.20) contains the terms

$$V_{\rm SS}^{\rm NLO} \supset -\frac{C_{1({\rm ES}^2)} G m_2}{m_1 r^2} (\vec{S}_1 \cdot \vec{n}) \left[\dot{\vec{S}}_1 \cdot \vec{v}_1 - \frac{3}{2} \dot{\vec{S}}_1 \cdot \vec{v}_2 \right]. \tag{11.52}$$

Substituting the lower-order EoM, we exactly obtain the term $V_{[5]}$, thus the outcome of the precession effects at LO SS on the NLO SS potential is the presence of Wilson coefficients squared at NLO S^3. For black holes these Wilson coefficients are expected to be 1, so numerically this feature would not be exceptional, yet for neutron stars where this Wilson coefficient may be considerably larger than 1, then this feature could become notably large and dominant. Therefore, it would be interesting to investigate the physical interpretation of this term in the future.

Moreover, the terms $L_{[2]}$ and $L_{[6]}$ of the original Lagrangian in eqs. (5.21)-(5.42), which exclusively received contributions from the 4 graphs with new spin-dependent Feynman rules, conspire to exactly cancel each other in the reduced potential. Consequently, the NLO S^3 reduced action does not depend on the gauge used in the new rules, which is thus equivalent to the original gauge used in lower-order sectors, and nor will the resulting Hamiltonian and physical observables.

11.4 Equations of motion

Since we have obtained the reduced potential, we can also use it to derive the NLO S^3 correction to the physical EoMs, for which we follow the description provided in §7. Based on Table 7.1, we can see that the NLO S^3 EoMs receive 11 contributions from lower-order sectors. In particular, they come from:

1. The substitution of the LO S^3 EoM for the position, in eq. (7.16), into the 1PN EoM, in eq. (7.4);

2. The substitution of the LO S^3 EoM for the spin, in eq. (7.15), into the LO SO EoM for the position, in eq. (7.9);

3. The substitution of the LO SO EoM for the spin, in eq. (7.8), into the LO S^3 EoM for the position, in eq. (7.16);

4. The substitution of the LO quadratic-in-spin EoMs for the position, in eqs. (7.13) and (7.14), into the NLO SO EoM for the position, in eq. (7.17);

5. The substitution of the LO SO EoM for the position, in eq. (7.10), into the NLO quadratic-in-spin EoMs for the position, in eqs. (7.18) and (7.19);

6. The substitution of the LO quadratic-in-spin EoMs for the spin, in eqs. (7.11) and (7.12), into the NLO quadratic-in-spin EoMs for the position, in eqs. (7.18) and (7.19).

Taking their results into consideration, and adding the contributions to the EoMs coming precisely from the NLO S^3 reduced potential, given in eqs. (11.32)-(11.50), we obtain that the NLO S^3 corrections to the physical equations of motion of the position and the spin amount to a total of 2216 and 3809 terms, respectively. Since the specific form of the equations of motion is gauge-dependent, although their solution is not, and they are too lengthy to be included in the thesis, the complete result for the EoMs can be provided in a *Mathematica* notebook upon request.

11.5 Hamiltonian

Continuing in the spirit of §8, at the NLO S^3 sector we will receive numerous contributions from lower-order sectors, and its Hamiltonian will come from the Legendre transform of the reduced potential in eqs. (11.32)-(11.50), as well as from iterations of the 1PN sector with the LO S^3, and of the LO SO with both of the NLO quadratic-in-spin sectors. More specifically, we use the following Legendre transform for the relation $\vec{v}_I(\vec{p})$, which extends that used in eq. (8.3) up to LO S^3 and to NLO quadratic-in-spin,

$$
\begin{aligned}
v_1^i &= \frac{p_1^i}{m_1} - \frac{1}{m_1}\frac{\partial L_{1\mathrm{PN}}}{\partial \vec{v}_1} + \frac{1}{m_1}\frac{\partial (V_s)^{\mathrm{LO}}_{\mathrm{SO}}}{\partial \vec{v}_1} + \frac{1}{m_1}\frac{\partial (V_s)^{\mathrm{LO}}_{\mathrm{S}^2}}{\partial \vec{v}_1} + \frac{1}{m_1}\frac{\partial (V_s)^{\mathrm{NLO}}_{\mathrm{S}^2}}{\partial \vec{v}_1} + \frac{1}{m_1}\frac{\partial (V_s)^{\mathrm{LO}}_{\mathrm{S}^3}}{\partial \vec{v}_1} \\
&= \frac{p_1^i}{m_1} - \frac{p_1^i\, p_1^2}{2m_1^3} + \frac{G}{r}\Big[- \frac{3m_2 p_1^i}{m_1} + \frac{7}{2}p_2^i + \frac{1}{2}(\vec{p}_2\cdot\vec{n})\, n^i \Big] - \frac{3Gm_2 S_1^{ij} n^j}{2m_1 r^2} - \frac{2G S_2^{ij} n^j}{r^2} \\
&\quad + \frac{G}{m_1^2 r^3}\Big[\vec{S}_1\cdot\vec{S}_2\Big(12(\vec{p}_1\cdot\vec{n})\, n^i - 5p_1^i\Big) - \frac{3}{2}\vec{S}_2\cdot\vec{n}\,\Big((\vec{S}_1\cdot\vec{p}_1)\, n^i + (\vec{p}_1\cdot\vec{n})\, S_1^i\Big) \\
&\quad + \frac{5}{2}(\vec{S}_2\cdot\vec{p}_1)\, S_1^i + \frac{5}{2}(\vec{S}_1\cdot\vec{p}_1)\, S_2^i + \vec{S}_1\cdot\vec{n}\,\Big(- 6(\vec{S}_2\cdot\vec{p}_1)\, n^i + 3(\vec{S}_2\cdot\vec{n})\, p_1^i - 6(\vec{p}_1\cdot\vec{n})\, S_2^i\Big)\Big] \\
&\quad + \frac{G}{m_1 m_2 r^3}\Big[\vec{S}_1\cdot\vec{S}_2\Big(- \frac{45}{4}(\vec{p}_2\cdot\vec{n})\, n^i + 6p_2^i\Big) - \frac{5}{2}(\vec{S}_2\cdot\vec{p}_2)\, S_1^i \\
&\quad + \vec{S}_2\cdot\vec{n}\,\Big(\frac{21}{4}(\vec{S}_1\cdot\vec{p}_2)\, n^i + \frac{9}{2}(\vec{p}_2\cdot\vec{n})\, S_1^i\Big) - 3(\vec{S}_1\cdot\vec{p}_2)\, S_2^i \\
&\quad + \vec{S}_1\cdot\vec{n}\,\Big(\frac{9}{2}(\vec{S}_2\cdot\vec{p}_2)\, n^i - \frac{15}{2}(\vec{S}_2\cdot\vec{n})\,(\vec{p}_2\cdot\vec{n})\, n^i - \frac{21}{4}(\vec{S}_2\cdot\vec{n})\, p_2^i + \frac{21}{4}(\vec{p}_2\cdot\vec{n})\, S_2^i\Big)\Big] \\
&\quad + \frac{G}{m_1^2 r^3}\Big[\frac{3}{2}S_1^2\,\Big((\vec{p}_2\cdot\vec{n})\, n^i - p_2^i\Big) + 3(\vec{S}_1\cdot\vec{n})^2\, p_2^i + \frac{3}{2}(\vec{S}_1\cdot\vec{p}_2)\, S_1^i
\end{aligned}
$$

$$
\begin{aligned}
&+ \vec{S}_1\cdot\vec{n}\,\Big(-3(\vec{S}_1\cdot\vec{p}_2)\,n^i-\frac{3}{2}(\vec{p}_2\cdot\vec{n})\,S_1^i\Big)\Big]\\
&+\frac{G}{m_2^2r^3}\Big[\frac{3}{2}S_2^2\,\Big((\vec{p}_2\cdot\vec{n})\,n^i-p_2^i\Big)+3(\vec{S}_2\cdot\vec{n})^2\,p_2^i+\frac{3}{2}(\vec{S}_2\cdot\vec{p}_2)\,S_2^i\\
&+\vec{S}_2\cdot\vec{n}\,\Big(-\frac{3}{2}(\vec{S}_2\cdot\vec{p}_2)\,n^i-3(\vec{p}_2\cdot\vec{n})\,S_2^i\Big)\Big]\\
&+\frac{Gm_2}{m_1^3r^3}\Big[-\frac{21}{4}(\vec{S}_1\cdot\vec{n})^2\,p_1^i+S_1^2\,\Big(-\frac{9}{4}(\vec{p}_1\cdot\vec{n})\,n^i+\frac{5}{2}p_1^i\Big)-\frac{5}{2}(\vec{S}_1\cdot\vec{p}_1)\,S_1^i\\
&+\frac{15}{4}\vec{S}_1\cdot\vec{n}\,\Big((\vec{S}_1\cdot\vec{p}_1)\,n^i+(\vec{p}_1\cdot\vec{n})\,S_1^i\Big)\Big]\\
&+\frac{C_{1(\mathrm{ES}^2)}G}{m_1^2r^3}\Big[(\vec{S}_1\cdot\vec{n})^2\,\Big(-\frac{15}{4}(\vec{p}_2\cdot\vec{n})\,n^i-\frac{21}{4}p_2^i\Big)+S_1^2\,\Big(-\frac{3}{4}(\vec{p}_2\cdot\vec{n})\,n^i+\frac{9}{4}p_2^i\Big)\\
&-\frac{1}{2}(\vec{S}_1\cdot\vec{p}_2)\,S_1^i+\frac{3}{2}\vec{S}_1\cdot\vec{n}\,\Big((\vec{S}_1\cdot\vec{p}_2)\,n^i+(\vec{p}_2\cdot\vec{n})\,S_1^i\Big)\Big]\\
&+\frac{C_{1(\mathrm{ES}^2)}Gm_2}{m_1^3r^3}\Big[S_1^2\,\Big(3(\vec{p}_1\cdot\vec{n})\,n^i-\frac{5}{2}p_1^i\Big)+\frac{9}{2}(\vec{S}_1\cdot\vec{n})^2\,p_1^i+(\vec{S}_1\cdot\vec{p}_1)\,S_1^i\\
&-\frac{3}{2}\vec{S}_1\cdot\vec{n}\,\Big((\vec{S}_1\cdot\vec{p}_1)\,n^i+(\vec{p}_1\cdot\vec{n})\,S_1^i\Big)\Big]\\
&+\frac{C_{2(\mathrm{ES}^2)}G}{m_2^2r^3}\Big[(\vec{S}_2\cdot\vec{n})^2\,\Big(-\frac{15}{4}(\vec{p}_2\cdot\vec{n})\,n^i-\frac{21}{4}p_2^i\Big)+S_2^2\,\Big(-\frac{3}{4}(\vec{p}_2\cdot\vec{n})\,n^i+\frac{9}{4}p_2^i\Big)\\
&-\frac{1}{2}(\vec{S}_2\cdot\vec{p}_2)\,S_2^i+\frac{3}{2}\vec{S}_2\cdot\vec{n}\,\Big((\vec{S}_2\cdot\vec{p}_2)\,n^i+(\vec{p}_2\cdot\vec{n})\,S_2^i\Big)\Big]\\
&+\frac{3C_{2(\mathrm{ES}^2)}G}{2m_1m_2r^3}\Big[3(\vec{S}_2\cdot\vec{n})^2\,p_1^i-S_2^2\,p_1^i\Big]\\
&+\frac{G}{m_1^2r^4}\Big[\frac{3}{2}(\vec{S}_1\cdot\vec{n})\,S_1^{ij}S_2^j-\frac{15}{2}(\vec{S}_1\cdot\vec{n})\,(\vec{S}_2\cdot\vec{n})\,S_1^{ij}n^j+\frac{3}{2}(\vec{S}_1\cdot\vec{S}_2)\,S_1^{ij}n^j\Big]\\
&+\frac{C_{1(\mathrm{ES}^2)}G}{m_1^2r^4}\Big[6(\vec{S}_1\cdot\vec{n})\,S_1^{ij}S_2^j+15(\vec{S}_1\cdot\vec{n})^2\,S_2^{ij}n^j-3S_1^2\,S_2^{ij}n^j\\
&+\frac{m_2}{m_1}\Big(-\frac{15}{4}(\vec{S}_1\cdot\vec{n})^2\,S_1^{ij}n^j+\frac{3}{4}S_1^2\,S_1^{ij}n^j\Big)\Big]\\
&+\frac{C_{2(\mathrm{ES}^2)}G}{m_1m_2r^4}\Big[-\frac{9}{2}(\vec{S}_2\cdot\vec{n})\,S_1^{ij}S_2^j+\frac{45}{4}(\vec{S}_2\cdot\vec{n})^2\,S_1^{ij}n^j-\frac{9}{4}S_2^2\,S_1^{ij}n^j\Big]\\
&+\frac{C_{1(\mathrm{BS}^3)}Gm_2}{m_1^3r^4}\Big[5(\vec{S}_1\cdot\vec{n})^2S_1^{ij}n^j-S_1^2\,S_1^{ij}n^j\Big]+\frac{C_{2(\mathrm{BS}^3)}G}{m_2^2r^4}\Big[5(\vec{S}_2\cdot\vec{n})^2S_2^{ij}n^j-S_2^2\,S_2^{ij}n^j\Big].
\end{aligned}
\tag{11.53}
$$

Then, the NLO S^3 correction to the Hamiltonian is obtained from

$$
\begin{aligned}
H^{\mathrm{NLO}}_{\mathrm{S}^3}=&-L_{1\mathrm{PN}}\Big(\vec{v}_I\to\frac{\vec{p}_I}{m_I}+\frac{1}{m_I}\frac{\partial(V_s)^{\mathrm{LO}}_{\mathrm{S}^3}}{\partial\vec{v}_I}\Big)+(V_s)^{\mathrm{LO}}_{\mathrm{S}^3}\Big(\vec{v}_I\to\frac{\vec{p}_I}{m_I}-\frac{1}{m_I}\frac{\partial L_{1\mathrm{PN}}}{\partial\vec{v}_I}\Big)\\
&+\frac{1}{m_1}\frac{\partial L_{1\mathrm{PN}}}{\partial\vec{v}_1}\cdot\frac{\partial(V_s)^{\mathrm{LO}}_{\mathrm{S}^3}}{\partial\vec{v}_1}+\frac{1}{m_2}\frac{\partial L_{1\mathrm{PN}}}{\partial\vec{v}_2}\cdot\frac{\partial(V_s)^{\mathrm{LO}}_{\mathrm{S}^3}}{\partial\vec{v}_2}
\end{aligned}
$$

$$
\begin{aligned}
&+ (V_s)^{\rm NLO}_{\rm S_1S_2}\Big(\vec{v}_I \to \frac{\vec{p}_I}{m_I} + \frac{1}{m_I}\frac{\partial (V_s)^{\rm LO}_{\rm SO}}{\partial \vec{v}_I}\Big) + (V_s)^{\rm NLO}_{\rm SS}\Big(\vec{v}_I \to \frac{\vec{p}_I}{m_I} + \frac{1}{m_I}\frac{\partial (V_s)^{\rm LO}_{\rm SO}}{\partial \vec{v}_I}\Big) \\
&- \frac{1}{m_1}\frac{\partial (V_s)^{\rm NLO}_{\rm S_1S_2}}{\partial \vec{v}_1}\cdot\frac{\partial (V_s)^{\rm LO}_{\rm SO}}{\partial \vec{v}_1} - \frac{1}{m_2}\frac{\partial (V_s)^{\rm NLO}_{\rm S_1S_2}}{\partial \vec{v}_2}\cdot\frac{\partial (V_s)^{\rm LO}_{\rm SO}}{\partial \vec{v}_2} \\
&- \frac{1}{m_1}\frac{\partial (V_s)^{\rm NLO}_{\rm SS}}{\partial \vec{v}_1}\cdot\frac{\partial (V_s)^{\rm LO}_{\rm SO}}{\partial \vec{v}_1} - \frac{1}{m_2}\frac{\partial (V_s)^{\rm NLO}_{\rm SS}}{\partial \vec{v}_2}\cdot\frac{\partial (V_s)^{\rm LO}_{\rm SO}}{\partial \vec{v}_2} \\
&+ (V_s)^{\rm LO}_{\rm SO}\Big(\vec{v}_I \to \frac{\vec{p}_I}{m_I} + \frac{1}{m_I}\frac{\partial (V_s)^{\rm NLO}_{\rm S_1S_2}}{\partial \vec{v}_I}\Big) + (V_s)^{\rm LO}_{\rm SO}\Big(\vec{v}_I \to \frac{\vec{p}_I}{m_I} + \frac{1}{m_I}\frac{\partial (V_s)^{\rm NLO}_{\rm SS}}{\partial \vec{v}_I}\Big) \\
&+ (V_s)^{\rm NLO}_{\rm S^3}\Big(\vec{v}_I \to \frac{\vec{p}_I}{m_I}\Big). \qquad (11.54)
\end{aligned}
$$

Computing all contributions, the NLO S^3 Hamiltonian reads

$$
H^{\rm NLO}_{\rm S^3} = H^{\rm NLO}_{\rm S_1^2S_2} + H^{\rm NLO}_{\rm S_1^3} + (1 \leftrightarrow 2), \qquad (11.55)
$$

where we have

$$
\begin{aligned}
H^{\rm NLO}_{\rm S_1^2S_2} =& \frac{G^2}{r^5}\frac{1}{m_1}H_{(1)} + \frac{G^2}{r^5}\frac{1}{m_2}H_{(2)} + \frac{G^2}{r^5}\frac{m_2}{m_1^2}H_{(3)} + \frac{C_{1(\rm ES^2)}G^2}{r^5}\frac{1}{m_1}H_{(4)} \\
&+ \frac{C_{1(\rm ES^2)}G^2}{r^5}\frac{1}{m_2}H_{(5)} + \frac{C_{1(\rm ES^2)}G^2}{r^5}\frac{m_2}{m_1^2}H_{(6)} + \frac{G}{r^4}\frac{1}{m_1^4}H_{(7)} + \frac{G}{r^4}\frac{1}{m_1^3m_2}H_{(8)} \\
&+ \frac{G}{r^4}\frac{1}{m_1^2m_2^2}H_{(9)} + \frac{C_{1(\rm ES^2)}G}{r^4}\frac{1}{m_1^4}H_{(10)} + \frac{C_{1(\rm ES^2)}G}{r^4}\frac{1}{m_1^3m_2}H_{(11)} \\
&+ \frac{C_{1(\rm ES^2)}G}{r^4}\frac{1}{m_1^2m_2^2}H_{(12)} + \frac{C_{1(\rm ES^2)}G}{r^4}\frac{1}{m_1m_2^3}H_{(13)}, \qquad (11.56)
\end{aligned}
$$

with the following expressions:

$$
\begin{aligned}
H_{(1)} =& \frac{49}{4}\vec{S}_1\cdot\vec{S}_2\times\vec{p}_1\,\vec{S}_1\cdot\vec{n} + \frac{79}{8}\vec{S}_1\cdot\vec{S}_2\times\vec{p}_2\,\vec{S}_1\cdot\vec{n} \\
&+ \vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\Big(\vec{S}_1\cdot\vec{n}\,\Big(6\vec{p}_1\cdot\vec{n} - \frac{141}{8}\vec{p}_2\cdot\vec{n}\Big) - \frac{15}{4}\vec{S}_1\cdot\vec{p}_1 + \frac{17}{2}\vec{S}_1\cdot\vec{p}_2\Big) \\
&+ \vec{S}_2\cdot\vec{p}_1\times\vec{n}\,\Big(\frac{3}{4}(\vec{S}_1\cdot\vec{n})^2 - \frac{1}{2}S_1^2\Big) + \vec{S}_2\cdot\vec{p}_2\times\vec{n}\,\Big(-\frac{39}{8}(\vec{S}_1\cdot\vec{n})^2 + \frac{11}{4}S_1^2\Big) \\
&+ \vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\Big(-\frac{1}{2}\vec{S}_1\cdot\vec{S}_2 + \frac{99}{2}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\Big) \\
&+ \vec{S}_1\cdot\vec{p}_2\times\vec{n}\,\Big(-\frac{75}{4}\vec{S}_1\cdot\vec{S}_2 + \frac{321}{8}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\Big), \qquad (11.57)
\end{aligned}
$$

$$
\begin{aligned}
H_{(2)} =& \vec{S}_1\cdot\vec{S}_2\times\vec{p}_2\,\vec{S}_1\cdot\vec{n} + \frac{1}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\Big(3\vec{S}_1\cdot\vec{n}\,\vec{p}_2\cdot\vec{n} + \vec{S}_1\cdot\vec{p}_2\Big) \\
&+ \vec{S}_2\cdot\vec{p}_2\times\vec{n}\,\Big(\frac{3}{4}(\vec{S}_1\cdot\vec{n})^2 + \frac{5}{4}S_1^2\Big) + \vec{S}_1\cdot\vec{p}_2\times\vec{n}\,\Big(-17\vec{S}_1\cdot\vec{S}_2 + 30\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\Big), \qquad (11.58)
\end{aligned}
$$

$$
H_{(3)} = \frac{31}{4}\vec{S}_1\cdot\vec{S}_2\times\vec{p}_1\,\vec{S}_1\cdot\vec{n} + \vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\Big(18\vec{S}_1\cdot\vec{n}\,\vec{p}_1\cdot\vec{n} - \frac{29}{4}\vec{S}_1\cdot\vec{p}_1\Big)
$$

$$+\vec{S}_2\cdot\vec{p}_1\times\vec{n}\left(3(\vec{S}_1\cdot\vec{n})^2-3S_1^2\right)+\vec{S}_1\cdot\vec{p}_1\times\vec{n}\left(\frac{3}{2}\vec{S}_1\cdot\vec{S}_2+\frac{141}{4}\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right), \tag{11.59}$$

$$\begin{aligned}H_{(4)}=&\frac{11}{4}\left(9\vec{S}_1\cdot\vec{S}_2\times\vec{p}_1+\vec{S}_1\cdot\vec{S}_2\times\vec{p}_2\right)\vec{S}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{S}_2\times\vec{n}\left(\vec{S}_1\cdot\vec{n}\left(\frac{3}{2}\vec{p}_1\cdot\vec{n}+78\vec{p}_2\cdot\vec{n}\right)\right.\\&\left.+\frac{9}{4}\vec{S}_1\cdot\vec{p}_1-44\vec{S}_1\cdot\vec{p}_2\right)-\vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\vec{S}_1\cdot\vec{S}_2\\&+\vec{S}_1\cdot\vec{p}_2\times\vec{n}\left(28\vec{S}_1\cdot\vec{S}_2+3\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right)+\vec{S}_2\cdot\vec{p}_1\times\vec{n}\left(-78(\vec{S}_1\cdot\vec{n})^2+20S_1^2\right)\\&+\vec{S}_2\cdot\vec{p}_2\times\vec{n}\left(66(\vec{S}_1\cdot\vec{n})^2-\frac{169}{4}S_1^2\right),\end{aligned} \tag{11.60}$$

$$\begin{aligned}H_{(5)}=&-\frac{27}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{p}_2\,\vec{S}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{S}_2\times\vec{n}\left(-\frac{3}{2}\vec{S}_1\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}-\frac{1}{2}\vec{S}_1\cdot\vec{p}_2\right)\\&+\vec{S}_1\cdot\vec{p}_2\times\vec{n}\,\vec{S}_1\cdot\vec{S}_2+\vec{S}_2\cdot\vec{p}_2\times\vec{n}\left(\frac{177}{4}(\vec{S}_1\cdot\vec{n})^2-\frac{43}{4}S_1^2\right),\end{aligned} \tag{11.61}$$

$$\begin{aligned}H_{(6)}=&22\vec{S}_1\cdot\vec{S}_2\times\vec{p}_1\,\vec{S}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{S}_2\times\vec{n}\left(-87\vec{S}_1\cdot\vec{n}\,\vec{p}_1\cdot\vec{n}+45\vec{S}_1\cdot\vec{p}_1\right)\\&+\vec{S}_2\cdot\vec{p}_1\times\vec{n}\left(-111(\vec{S}_1\cdot\vec{n})^2+\frac{107}{2}S_1^2\right)\\&+\frac{1}{2}\vec{S}_1\cdot\vec{p}_1\times\vec{n}\left(-59\vec{S}_1\cdot\vec{S}_2+9\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right),\end{aligned} \tag{11.62}$$

$$\begin{aligned}H_{(7)}=&\vec{S}_1\cdot\vec{S}_2\times\vec{p}_1\left(-\frac{27}{8}\vec{S}_1\cdot\vec{n}\,p_1^2+\frac{3}{2}\vec{S}_1\cdot\vec{p}_1\,\vec{p}_1\cdot\vec{n}\right)\\&+\frac{1}{2}\vec{S}_2\cdot\vec{p}_1\times\vec{n}\left(\frac{15}{2}(\vec{S}_1\cdot\vec{n})^2\,p_1^2-15\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_1\,\vec{p}_1\cdot\vec{n}+3(\vec{S}_1\cdot\vec{p}_1)^2\right.\\&\left.+S_1^2\left(\frac{15}{2}(\vec{p}_1\cdot\vec{n})^2-3p_1^2\right)\right)+\frac{15}{8}\vec{S}_1\cdot\vec{p}_1\times\vec{n}\left(\vec{S}_1\cdot\vec{S}_2\,p_1^2-5\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\,p_1^2\right),\end{aligned} \tag{11.63}$$

$$\begin{aligned}H_{(8)}=&-\frac{3}{4}\vec{S}_1\cdot\vec{S}_2\times\vec{p}_2\,\vec{S}_1\cdot\vec{p}_1\,\vec{p}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{S}_2\times\vec{p}_1\left(6\vec{S}_1\cdot\vec{n}\,\vec{p}_1\cdot\vec{p}_2+\frac{3}{4}\vec{S}_1\cdot\vec{p}_1\,\vec{p}_2\cdot\vec{n}\right.\\&\left.+\frac{3}{4}\vec{S}_1\cdot\vec{p}_2\,\vec{p}_1\cdot\vec{n}\right)-\frac{3}{4}\vec{S}_2\cdot\vec{p}_1\times\vec{p}_2\,S_1^2\,\vec{p}_1\cdot\vec{n}\\&+\frac{1}{8}\vec{S}_2\cdot\vec{p}_2\times\vec{n}\left(-\frac{55}{2}(\vec{S}_1\cdot\vec{n})^2\,p_1^2+55\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_1\,\vec{p}_1\cdot\vec{n}-11(\vec{S}_1\cdot\vec{p}_1)^2\right.\\&\left.+S_1^2\left(-\frac{55}{2}(\vec{p}_1\cdot\vec{n})^2+11p_1^2\right)\right)+\vec{S}_1\cdot\vec{p}_1\times\vec{p}_2\left(\frac{27}{4}\vec{S}_1\cdot\vec{S}_2\,\vec{p}_1\cdot\vec{n}\right.\\&\left.+\frac{21}{4}\vec{S}_1\cdot\vec{p}_1\,\vec{S}_2\cdot\vec{n}+\vec{S}_1\cdot\vec{n}\left(-30\vec{S}_2\cdot\vec{n}\,\vec{p}_1\cdot\vec{n}+6\vec{S}_2\cdot\vec{p}_1\right)\right)\\&+\frac{1}{8}\vec{S}_1\cdot\vec{p}_1\times\vec{n}\left(\vec{S}_1\cdot\vec{S}_2\left(-26\vec{p}_1\cdot\vec{p}_2-55\vec{p}_1\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}\right)\right.\\&+\vec{S}_2\cdot\vec{n}\left(30\vec{S}_1\cdot\vec{p}_1\,\vec{p}_2\cdot\vec{n}+55\vec{S}_1\cdot\vec{p}_2\,\vec{p}_1\cdot\vec{n}\right)-22\vec{S}_1\cdot\vec{p}_2\,\vec{S}_2\cdot\vec{p}_1\\&\left.+\vec{S}_1\cdot\vec{n}\left(185\vec{S}_2\cdot\vec{n}\,\vec{p}_1\cdot\vec{p}_2+55\vec{S}_2\cdot\vec{p}_1\,\vec{p}_2\cdot\vec{n}\right)-6\vec{S}_1\cdot\vec{p}_1\,\vec{S}_2\cdot\vec{p}_2\right),\end{aligned} \tag{11.64}$$

$$
\begin{aligned}
H_{(9)} =& \frac{3}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{p}_2\Big(\vec{S}_1\cdot\vec{n}\,\vec{p}_1\cdot\vec{p}_2-\vec{S}_1\cdot\vec{p}_2\,\vec{p}_1\cdot\vec{n}\Big)\\
&+\vec{S}_1\cdot\vec{S}_2\times\vec{p}_1\Big(\vec{S}_1\cdot\vec{n}\,\Big(\frac{15}{4}(\vec{p}_2\cdot\vec{n})^2-3p_2^2\Big)-\frac{3}{2}\vec{S}_1\cdot\vec{p}_2\,\vec{p}_2\cdot\vec{n}\Big)\\
&+\vec{S}_1\cdot\vec{p}_2\times\vec{n}\,\Big(\vec{S}_1\cdot\vec{S}_2\Big(-3\vec{p}_1\cdot\vec{p}_2+\frac{15}{2}\vec{p}_1\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}\Big)-\frac{15}{2}\vec{S}_1\cdot\vec{p}_2\,\vec{S}_2\cdot\vec{n}\,\vec{p}_1\cdot\vec{n}\\
&+3\vec{S}_1\cdot\vec{p}_2\,\vec{S}_2\cdot\vec{p}_1+\frac{15}{2}\vec{S}_1\cdot\vec{n}\,\Big(\vec{S}_2\cdot\vec{n}\,\vec{p}_1\cdot\vec{p}_2-\vec{S}_2\cdot\vec{p}_1\,\vec{p}_2\cdot\vec{n}\Big)\Big)\\
&+\vec{S}_1\cdot\vec{p}_1\times\vec{p}_2\Big(-3\vec{S}_1\cdot\vec{S}_2\,\vec{p}_2\cdot\vec{n}-6\vec{S}_1\cdot\vec{p}_2\,\vec{S}_2\cdot\vec{n}\\
&+\vec{S}_1\cdot\vec{n}\,\Big(\frac{45}{2}\vec{S}_2\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}-\frac{21}{4}\vec{S}_2\cdot\vec{p}_2\Big)\Big)+\frac{3}{4}\vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\Big(\vec{S}_1\cdot\vec{S}_2\Big(-5(\vec{p}_2\cdot\vec{n})^2+4p_2^2\Big)\\
&-10\vec{S}_1\cdot\vec{p}_2\,\vec{S}_2\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}+\vec{S}_1\cdot\vec{p}_2\,\vec{S}_2\cdot\vec{p}_2\\
&+\vec{S}_1\cdot\vec{n}\,\Big(\vec{S}_2\cdot\vec{n}\,(35(\vec{p}_2\cdot\vec{n})^2-20p_2^2)-5\vec{S}_2\cdot\vec{p}_2\,\vec{p}_2\cdot\vec{n}\Big)\Big),
\end{aligned}
\tag{11.65}
$$

$$
\begin{aligned}
H_{(10)} =& 3\vec{S}_1\cdot\vec{S}_2\times\vec{p}_1\,\vec{S}_1\cdot\vec{n}\,p_1^2+\vec{S}_2\cdot\vec{p}_1\times\vec{n}\,\Big(-15\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_1\,\vec{p}_1\cdot\vec{n}+6(\vec{S}_1\cdot\vec{p}_1)^2\\
&+S_1^2\,\Big(15(\vec{p}_1\cdot\vec{n})^2-6p_1^2\Big)\Big)+\vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\Big(3\vec{S}_1\cdot\vec{S}_2\,p_1^2-3\vec{S}_1\cdot\vec{p}_1\,\vec{S}_2\cdot\vec{p}_1\Big),
\end{aligned}
\tag{11.66}
$$

$$
\begin{aligned}
H_{(11)} =& 12\vec{S}_1\cdot\vec{S}_2\times\vec{p}_1\,\vec{S}_1\cdot\vec{p}_1\,\vec{p}_2\cdot\vec{n}+\vec{S}_1\cdot\vec{S}_2\times\vec{p}_2\Big(\vec{S}_1\cdot\vec{n}\,\Big(\frac{15}{4}(\vec{p}_1\cdot\vec{n})^2-3p_1^2\Big)\\
&-\frac{3}{4}\vec{S}_1\cdot\vec{p}_1\,\vec{p}_1\cdot\vec{n}\Big)+\vec{S}_2\cdot\vec{p}_1\times\vec{n}\,\Big(-\frac{105}{2}(\vec{S}_1\cdot\vec{n})^2\,\vec{p}_1\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}\\
&+15\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_2\,\vec{p}_1\cdot\vec{n}-6\vec{S}_1\cdot\vec{p}_1\,\vec{S}_1\cdot\vec{p}_2+\frac{15}{2}S_1^2\,\Big(\vec{p}_1\cdot\vec{p}_2+\vec{p}_1\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}\Big)\Big)\\
&+\frac{1}{4}\vec{S}_2\cdot\vec{p}_1\times\vec{p}_2\Big(45(\vec{S}_1\cdot\vec{n})^2\,\vec{p}_1\cdot\vec{n}-21\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_1-27S_1^2\,\vec{p}_1\cdot\vec{n}\Big)\\
&+\vec{S}_2\cdot\vec{p}_2\times\vec{n}\,\Big((\vec{S}_1\cdot\vec{n})^2\,\Big(-\frac{105}{8}(\vec{p}_1\cdot\vec{n})^2-\frac{15}{2}p_1^2\Big)+\frac{75}{4}\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_1\,\vec{p}_1\cdot\vec{n}\\
&-6(\vec{S}_1\cdot\vec{p}_1)^2+S_1^2\,\Big(-\frac{105}{8}(\vec{p}_1\cdot\vec{n})^2+6p_1^2\Big)\Big)+\frac{15}{2}\vec{p}_1\cdot\vec{p}_2\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2\,\vec{S}_2\cdot\vec{p}_1\\
&+\vec{S}_1\cdot\vec{p}_2\times\vec{n}\,\Big(-3\vec{S}_1\cdot\vec{S}_2\,p_1^2+3\vec{S}_1\cdot\vec{p}_1\,\vec{S}_2\cdot\vec{p}_1\Big)\\
&+\vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\Big(\vec{S}_1\cdot\vec{S}_2\Big(-3\vec{p}_1\cdot\vec{p}_2-15\vec{p}_1\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}\Big)\\
&+15\vec{S}_1\cdot\vec{p}_1\,\vec{S}_2\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}+3\vec{S}_1\cdot\vec{p}_2\,\vec{S}_2\cdot\vec{p}_1\Big),
\end{aligned}
\tag{11.67}
$$

$$
\begin{aligned}
H_{(12)} =& 3\vec{S}_1\cdot\vec{S}_2\times\vec{p}_1\,\vec{S}_1\cdot\vec{p}_2\,\vec{p}_2\cdot\vec{n}-12\vec{S}_1\cdot\vec{S}_2\times\vec{p}_2\,\vec{S}_1\cdot\vec{p}_2\,\vec{p}_1\cdot\vec{n}+3\vec{S}_1\cdot\vec{p}_1\times\vec{p}_2\,\vec{S}_1\cdot\vec{S}_2\,\vec{p}_2\cdot\vec{n}\\
&+\vec{S}_2\cdot\vec{p}_1\times\vec{p}_2\Big(-15(\vec{S}_1\cdot\vec{n})^2\,\vec{p}_2\cdot\vec{n}+6\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_2+3S_1^2\,\vec{p}_2\cdot\vec{n}\Big)\\
&+\vec{S}_2\cdot\vec{p}_2\times\vec{n}\,\Big(\frac{105}{2}(\vec{S}_1\cdot\vec{n})^2\,\vec{p}_1\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}-15\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_1\,\vec{p}_2\cdot\vec{n}+3\vec{S}_1\cdot\vec{p}_1\,\vec{S}_1\cdot\vec{p}_2\\
&+S_1^2\,\Big(-3\vec{p}_1\cdot\vec{p}_2-\frac{15}{2}\vec{p}_1\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}\Big)\Big)+\vec{S}_1\cdot\vec{p}_2\times\vec{n}\,\Big(15\vec{S}_1\cdot\vec{S}_2\,\vec{p}_1\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}\\
&-15\vec{S}_1\cdot\vec{p}_2\,\vec{S}_2\cdot\vec{n}\,\vec{p}_1\cdot\vec{n}\Big),
\end{aligned}
\tag{11.68}
$$

$$H_{(13)} = -\frac{15}{8}\vec{S}_1\cdot\vec{S}_2\times\vec{p}_2\,\vec{S}_1\cdot\vec{n}\,p_2^2 + \vec{S}_2\cdot\vec{p}_2\times\vec{n}\,\Big(\frac{75}{16}(\vec{S}_1\cdot\vec{n})^2\,p_2^2 - \frac{15}{16}S_1^2\,p_2^2\Big), \tag{11.69}$$

and also

$$\begin{aligned}
H^{\rm NLO}_{\rm S_1^3} =& \frac{G^2}{r^5}\frac{m_2}{m_1^2}H_{[1]} + \frac{G^2}{r^5}\frac{m_2^2}{m_1^3}H_{[2]} + \frac{C_{1(\rm ES^2)}G^2}{r^5}\frac{1}{m_1}H_{[3]} + \frac{C_{1(\rm ES^2)}G^2}{r^5}\frac{m_2}{m_1^2}H_{[4]}\\
&+ \frac{C_{1(\rm ES^2)}G^2}{r^5}\frac{m_2^2}{m_1^3}H_{[5]} + \frac{C^2_{1(\rm ES^2)}G^2}{r^5}\frac{m_2}{m_1^2}H_{[6]} + \frac{C^2_{1(\rm ES^2)}G^2}{r^5}\frac{m_2^2}{m_1^3}H_{[7]}\\
&+ \frac{C_{1(\rm BS^3)}G^2}{r^5}\frac{1}{m_1}H_{[8]} + \frac{C_{1(\rm BS^3)}G^2}{r^5}\frac{m_2}{m_1^2}H_{[9]} + \frac{C_{1(\rm BS^3)}G^2}{r^5}\frac{m_2^2}{m_1^3}H_{[10]} + \frac{G}{r^4}\frac{1}{m_1^4}H_{[11]}\\
&+ \frac{G}{r^4}\frac{m_2}{m_1^5}H_{[12]} + \frac{C_{1(\rm ES^2)}G}{r^4}\frac{1}{m_1^4}H_{[13]} + \frac{C_{1(\rm ES^2)}G}{r^4}\frac{1}{m_1^3m_2}H_{[14]} + \frac{C_{1(\rm ES^2)}G}{r^4}\frac{m_2}{m_1^5}H_{[15]}\\
&+ \frac{C_{1(\rm BS^3)}G}{r^4}\frac{1}{m_1^4}H_{[16]} + \frac{C_{1(\rm BS^3)}G}{r^4}\frac{1}{m_1^3m_2}H_{[17]} + \frac{C_{1(\rm BS^3)}G}{r^4}\frac{m_2}{m_1^5}H_{[18]},
\end{aligned} \tag{11.70}$$

with the pieces:

$$H_{[1]} = \vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\Big(9(\vec{S}_1\cdot\vec{n})^2 - 3S_1^2\Big) + \vec{S}_1\cdot\vec{p}_2\times\vec{n}\,\Big(-3(\vec{S}_1\cdot\vec{n})^2 + \frac{3}{2}S_1^2\Big), \tag{11.71}$$

$$H_{[2]} = \vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\Big(\frac{21}{4}(\vec{S}_1\cdot\vec{n})^2 - \frac{9}{4}S_1^2\Big),\quad H_{[3]} = \vec{S}_1\cdot\vec{p}_2\times\vec{n}\,\Big(\frac{27}{2}(\vec{S}_1\cdot\vec{n})^2 - \frac{7}{2}S_1^2\Big), \tag{11.72}$$

$$H_{[4]} = \vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\Big(15(\vec{S}_1\cdot\vec{n})^2 - \frac{5}{4}S_1^2\Big) + \vec{S}_1\cdot\vec{p}_2\times\vec{n}\,\Big(\frac{45}{4}(\vec{S}_1\cdot\vec{n})^2 - \frac{21}{4}S_1^2\Big), \tag{11.73}$$

$$H_{[5]} = \vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\Big(\frac{105}{4}(\vec{S}_1\cdot\vec{n})^2 - \frac{3}{4}S_1^2\Big),\quad H_{[6]} = 9\vec{S}_1\cdot\vec{p}_2\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2, \tag{11.74}$$

$$H_{[7]} = -6\vec{S}_1\cdot\vec{p}_1\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2,\quad H_{[8]} = \vec{S}_1\cdot\vec{p}_2\times\vec{n}\,\Big(\frac{25}{2}(\vec{S}_1\cdot\vec{n})^2 - \frac{5}{2}S_1^2\Big), \tag{11.75}$$

$$H_{[9]} = \vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\Big(-\frac{29}{2}(\vec{S}_1\cdot\vec{n})^2 + \frac{17}{6}S_1^2\Big) + \vec{S}_1\cdot\vec{p}_2\times\vec{n}\,\Big(39(\vec{S}_1\cdot\vec{n})^2 - \frac{23}{3}S_1^2\Big), \tag{11.76}$$

$$H_{[10]} = \vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\Big(-36(\vec{S}_1\cdot\vec{n})^2 + 7S_1^2\Big), \tag{11.77}$$

$$\begin{aligned}
H_{[11]} =& \frac{3}{2}\vec{S}_1\cdot\vec{p}_1\times\vec{p}_2\Big(\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_1 - S_1^2\,\vec{p}_1\cdot\vec{n}\Big) + \frac{3}{2}\vec{S}_1\cdot\vec{p}_2\times\vec{n}\,\Big(-\frac{5}{2}(\vec{S}_1\cdot\vec{n})^2\,p_1^2\\
&+ 5\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_1\,\vec{p}_1\cdot\vec{n} - (\vec{S}_1\cdot\vec{p}_1)^2 + S_1^2\,\Big(-\frac{5}{2}(\vec{p}_1\cdot\vec{n})^2 + p_1^2\Big)\Big),
\end{aligned} \tag{11.78}$$

$$\begin{aligned}
H_{[12]} =& \frac{11}{16}\vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\Big(5(\vec{S}_1\cdot\vec{n})^2\,p_1^2 - 10\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_1\,\vec{p}_1\cdot\vec{n} + 2(\vec{S}_1\cdot\vec{p}_1)^2\\
&+ S_1^2\,\Big(5(\vec{p}_1\cdot\vec{n})^2 - 2p_1^2\Big)\Big),
\end{aligned} \tag{11.79}$$

$$
\begin{aligned}
H_{[13]} =& \vec{S}_1\cdot\vec{p}_1\times\vec{p}_2\Big(-15(\vec{S}_1\cdot\vec{n})^2\,\vec{p}_1\cdot\vec{n}+\frac{21}{4}\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_1+\frac{9}{2}S_1^2\,\vec{p}_1\cdot\vec{n}\Big)\\
&+\vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\Big(15(\vec{S}_1\cdot\vec{n})^2\,\vec{p}_1\cdot\vec{p}_2+\frac{15}{4}\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_1\,\vec{p}_2\cdot\vec{n}-\frac{3}{4}\vec{S}_1\cdot\vec{p}_1\,\vec{S}_1\cdot\vec{p}_2\\
&+S_1^2\,\Big(-\frac{3}{2}\vec{p}_1\cdot\vec{p}_2-\frac{15}{2}\vec{p}_1\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}\Big)\Big),
\end{aligned}
\tag{11.80}
$$

$$
\begin{aligned}
H_{[14]} =& \vec{S}_1\cdot\vec{p}_1\times\vec{p}_2\Big(\frac{45}{4}(\vec{S}_1\cdot\vec{n})^2\,\vec{p}_2\cdot\vec{n}-\frac{9}{2}\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_2-\frac{15}{4}S_1^2\,\vec{p}_2\cdot\vec{n}\Big)\\
&+\vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\Big((\vec{S}_1\cdot\vec{n})^2\,\Big(\frac{105}{8}(\vec{p}_2\cdot\vec{n})^2-\frac{15}{2}p_2^2\Big)-\frac{15}{2}\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_2\,\vec{p}_2\cdot\vec{n}\\
&+\frac{3}{4}(\vec{S}_1\cdot\vec{p}_2)^2+S_1^2\,\Big(\frac{15}{8}(\vec{p}_2\cdot\vec{n})^2+\frac{3}{4}p_2^2\Big)\Big),
\end{aligned}
\tag{11.81}
$$

$$
H_{[15]} = \frac{3}{16}\vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\Big(-25(\vec{S}_1\cdot\vec{n})^2\,p_1^2+S_1^2\,\Big(20(\vec{p}_1\cdot\vec{n})^2+p_1^2\Big)\Big),
\tag{11.82}
$$

$$
\begin{aligned}
H_{[16]} =& \vec{p}_1\cdot\vec{p}_2\times\vec{n}\,(\vec{S}_1\cdot\vec{n})^2\,\vec{S}_1\cdot\vec{p}_1+\vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\Big((\vec{S}_1\cdot\vec{n})^2\,\Big(-5\vec{p}_1\cdot\vec{p}_2-\frac{35}{2}\vec{p}_1\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}\Big)\\
&-\vec{S}_1\cdot\vec{p}_1\,\vec{S}_1\cdot\vec{p}_2+\vec{S}_1\cdot\vec{n}\,\Big(6\vec{S}_1\cdot\vec{p}_1\,\vec{p}_2\cdot\vec{n}+5\vec{S}_1\cdot\vec{p}_2\,\vec{p}_1\cdot\vec{n}\Big)+S_1^2\,\Big(\vec{p}_1\cdot\vec{p}_2\\
&+\frac{5}{2}\vec{p}_1\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}\Big)\Big)+\vec{S}_1\cdot\vec{p}_2\times\vec{n}\,\Big(4\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_1\,\vec{p}_1\cdot\vec{n}-(\vec{S}_1\cdot\vec{p}_1)^2\Big),
\end{aligned}
\tag{11.83}
$$

$$
\begin{aligned}
H_{[17]} =& -\vec{S}_1\cdot\vec{p}_1\times\vec{p}_2\,\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_2+\frac{1}{2}\vec{p}_1\cdot\vec{p}_2\times\vec{n}\,\Big(5(\vec{S}_1\cdot\vec{n})^2\,\vec{S}_1\cdot\vec{p}_2-S_1^2\,\vec{S}_1\cdot\vec{p}_2\Big)\\
&+\vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\Big(5(\vec{S}_1\cdot\vec{n})^2\,p_2^2-S_1^2\,p_2^2\Big)+\vec{S}_1\cdot\vec{p}_2\times\vec{n}\,\Big(\frac{35}{2}(\vec{S}_1\cdot\vec{n})^2\,\vec{p}_1\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}\\
&+\vec{S}_1\cdot\vec{p}_1\,\vec{S}_1\cdot\vec{p}_2-5\vec{S}_1\cdot\vec{n}\,\Big(\vec{S}_1\cdot\vec{p}_1\,\vec{p}_2\cdot\vec{n}+\vec{S}_1\cdot\vec{p}_2\,\vec{p}_1\cdot\vec{n}\Big)-\frac{5}{2}S_1^2\,\vec{p}_1\cdot\vec{n}\,\vec{p}_2\cdot\vec{n}\Big),
\end{aligned}
\tag{11.84}
$$

$$
H_{[18]} = \vec{S}_1\cdot\vec{p}_1\times\vec{n}\,\Big(-5\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{p}_1\,\vec{p}_1\cdot\vec{n}+(\vec{S}_1\cdot\vec{p}_1)^2\Big).
\tag{11.85}
$$

Note that, still at the level of the Hamiltonian, there are the two peculiar terms $H_{[6]}$ and $H_{[7]}$ that are multiplied by a Wilson coefficient squared, which arose from the aforementioned corresponding contributions to the reduced potential in §11.3.

With this general Hamiltonian, we follow the procedure described in §8.2 to compute the simplified Hamiltonian in the center-of-mass frame for circular orbits and in the aligned-spins case, which to NLO S^3 becomes

$$
\begin{aligned}
\tilde{H}^{\mathrm{NLO}}_{\mathrm{S}^3} =& \frac{\nu\tilde{L}\tilde{S}_1^3}{\tilde{r}^6}\Big[-\frac{9}{4}\nu-\frac{9\nu^2}{4}+C_{1(\mathrm{ES}^2)}\Big(-\frac{3\nu}{2}+2\nu^2\Big)+C_{1(\mathrm{BS}^3)}(7\nu+\nu^2)\\
&+\frac{\tilde{L}^2}{\tilde{r}}\Big(-\frac{11\nu}{8}+\frac{5\nu^2}{4}+C_{1(\mathrm{ES}^2)}\Big(\frac{3\nu}{16}+\frac{9\nu^2}{8}\Big)-\nu^2\,C_{1(\mathrm{BS}^3)}\Big)\\
&+\frac{1}{q}\Big(\frac{9}{4}-\frac{9\nu}{4}-\frac{9\nu^2}{4}+C_{1(\mathrm{ES}^2)}\Big(\frac{3}{2}-\frac{17\nu}{2}+2\nu^2\Big)+C_{1(\mathrm{BS}^3)}\Big(-7+\frac{21\nu}{2}+\nu^2\Big)
\end{aligned}
$$

$$+\frac{\tilde{L}^2}{\tilde{r}}\bigg(\frac{11}{8}-4\nu+\frac{9\nu^2}{8}+C_{1(\mathrm{ES}^2)}\Big(-\frac{3}{16}-\frac{3\nu}{4}+\frac{27\nu^2}{16}\Big)+C_{1(\mathrm{BS}^3)}\big(\nu-\nu^2\big)\bigg)\bigg)\bigg]$$
$$+\frac{\nu^2\tilde{L}\tilde{S}_1^2\tilde{S}_2}{\tilde{r}^6}\bigg[-\frac{63}{4}-\frac{3\nu}{4}+C_{1(\mathrm{ES}^2)}\Big(-\frac{39}{4}+\frac{\nu}{2}\Big)$$
$$+\frac{\tilde{L}^2}{\tilde{r}}\bigg(\frac{3\nu}{8}+C_{1(\mathrm{ES}^2)}\Big(-\frac{15}{16}+\frac{15\nu}{8}\Big)\bigg)+\frac{1}{q}\bigg(\frac{3}{2}-\frac{3\nu}{4}+C_{1(\mathrm{ES}^2)}\Big(-24+\frac{\nu}{2}\Big)$$
$$+\frac{\tilde{L}^2}{\tilde{r}}\bigg(-\frac{3}{8}-\frac{9\nu}{8}+C_{1(\mathrm{ES}^2)}\Big(3+\frac{39\nu}{16}\Big)\bigg)\bigg)\bigg]+(1\leftrightarrow 2). \tag{11.86}$$

Notice how, since $\vec{S}_I\cdot\vec{n}=0$, the peculiar terms $H_{[6]}$ and $H_{[7]}$ that were multiplied by a Wilson coefficient squared in the general Hamiltonian, given in eqs. (11.55)-(11.85), now vanish in the restricted aligned-spins case. Therefore, other partial results in the literature that may have been computed only for circular orbits and in the aligned-spins case do not capture these possible composite effects. It would be interesting to note if this would have observable effects for the unaligned-spins case.

11.6 Gauge-invariant observables and relations

Using the previous simplified Hamiltonian, we can derive the gauge-invariant observables and relations for the binding energy of the binary system, as presented in §9. First of all, the circular orbit relation $\tilde{r}(\tilde{L})$ of eq. (9.3) has an addition of the form

$$\frac{1}{\tilde{r}}=\cdots+\frac{\nu\tilde{S}_1^3}{\tilde{L}^{11}}\bigg[220\nu-224\nu^2+C_{1(\mathrm{ES}^2)}\Big(\frac{10251\nu}{16}-\frac{423\nu^2}{4}\Big)+C_{1(\mathrm{BS}^3)}\Big(\frac{171\nu}{2}-\frac{13\nu^2}{2}\Big)$$
$$+\frac{1}{q}\bigg(-220-176\nu-195\nu^2+C_{1(\mathrm{ES}^2)}\Big(-\frac{10251}{16}+\frac{2313\nu}{8}-\frac{1467\nu^2}{16}\Big)$$
$$+C_{1(\mathrm{BS}^3)}\Big(-\frac{171}{2}+65\nu-\frac{13\nu^2}{2}\Big)\bigg)\bigg]$$
$$+\frac{\nu^2\tilde{S}_1^2\tilde{S}_2}{\tilde{L}^{11}}\bigg[-\frac{7707}{2}-\frac{963\nu}{8}+C_{1(\mathrm{ES}^2)}\Big(-\frac{16287}{16}+\frac{189\nu}{4}\Big)$$
$$+\frac{1}{q}\bigg(-\frac{19839}{8}-\frac{123\nu}{2}+C_{1(\mathrm{ES}^2)}\Big(-\frac{2577}{2}+\frac{981\nu}{16}\Big)\bigg)\bigg]+(1\leftrightarrow 2), \tag{11.87}$$

which implies that to NLO S^3, the gauge-invariant relation between the binding energy e and the orbital angular momentum $\tilde{L}$ reads

$$e(\tilde{L})=\cdots+\frac{\nu\tilde{S}_1^3}{\tilde{L}^{11}}\bigg[-\frac{389\nu}{8}+\frac{167\nu^2}{4}+C_{1(\mathrm{ES}^2)}\Big(-\frac{927\nu}{8}+11\nu^2\Big)-13\nu\, C_{1(\mathrm{BS}^3)}$$
$$+\frac{1}{q}\bigg(\frac{389}{8}+29\nu+\frac{291\nu^2}{8}+C_{1(\mathrm{ES}^2)}\Big(\frac{927}{8}-\frac{169\nu}{4}+\frac{103\nu^2}{4}\Big)$$
$$+C_{1(\mathrm{BS}^3)}\Big(13-\frac{17\nu}{2}\Big)\bigg)\bigg]+\frac{\nu^2\tilde{S}_1^2\tilde{S}_2}{\tilde{L}^{11}}\bigg[\frac{2913}{4}+\frac{201\nu}{4}+C_{1(\mathrm{ES}^2)}\Big(\frac{1437}{8}+5\nu\Big)$$
$$+\frac{1}{q}\bigg(\frac{1917}{4}+\frac{243\nu}{8}+C_{1(\mathrm{ES}^2)}\Big(222+\frac{55\nu}{8}\Big)\bigg)\bigg]+(1\leftrightarrow 2), \tag{11.88}$$

where the lower orders are given in eq. (9.4).

Secondly, the relation for the angular momentum as a function of the orbital frequency, i.e., to the variable x, written in eq. (9.10), receives the NLO S^3 correction

$$\begin{aligned}\frac{1}{\tilde{L}^2} &= \cdots + \nu x^{11/2} \tilde{S}_1^3 \bigg[-\frac{757\nu}{6} + \frac{247\nu^2}{81} + C_{1(\mathrm{ES}^2)} \Big(-\frac{361\nu}{4} - \frac{443\nu^2}{9} \Big) \\ &+ C_{1(\mathrm{BS}^3)} \Big(\frac{92\nu}{3} + 14\nu^2 \Big) + \frac{1}{q} \bigg(\frac{757}{6} - \frac{202\nu}{3} + \frac{59\nu^2}{18} + C_{1(\mathrm{ES}^2)} \Big(\frac{361}{4} - \frac{293\nu}{2} - \frac{617\nu^2}{12} \Big) \\ &+ C_{1(\mathrm{BS}^3)} \Big(-\frac{92}{3} + \frac{149\nu}{3} + 14\nu^2 \Big) \bigg) \bigg] + \nu^2 x^{11/2} \tilde{S}_1^2 \tilde{S}_2 \bigg[\frac{9329}{18} + \frac{4835\nu}{54} \\ &+ C_{1(\mathrm{ES}^2)} \Big(-\frac{83}{12} + \frac{\nu}{9} \Big) + \frac{1}{q} \bigg(\frac{3083}{6} + \frac{1703\nu}{18} + C_{1(\mathrm{ES}^2)} \Big(\frac{40}{3} - \frac{25\nu}{12} \Big) \bigg) \bigg] + (1 \leftrightarrow 2),\end{aligned} \tag{11.89}$$

which results in the following NLO S^3 addition to the binding energy as a function of the orbital frequency, given to lower orders in eq. (9.11),

$$\begin{aligned}e(x) &= \cdots + \nu x^{11/2} \tilde{S}_1^3 \bigg[\frac{4\nu}{3} - \frac{128\nu^2}{81} + C_{1(\mathrm{ES}^2)} \Big(2\nu - \frac{20\nu^2}{9} \Big) + C_{1(\mathrm{BS}^3)} \Big(-\frac{4\nu}{3} - 4\nu^2 \Big) \\ &+ \frac{1}{q} \bigg(-\frac{4}{3} + \frac{8\nu}{3} - \frac{8\nu^2}{9} + C_{1(\mathrm{ES}^2)} \Big(-2 + 16\nu + \frac{2\nu^2}{3} \Big) \\ &+ C_{1(\mathrm{BS}^3)} \Big(\frac{4}{3} - \frac{28\nu}{3} - 4\nu^2 \Big) \bigg) \bigg] + \nu^2 x^{11/2} \tilde{S}_1^2 \tilde{S}_2 \bigg[\frac{82}{9} + \frac{4\nu}{27} + C_{1(\mathrm{ES}^2)} \Big(-\frac{32}{3} + \frac{64\nu}{9} \Big) \\ &+ \frac{1}{q} \bigg(\frac{28}{3} - \frac{32\nu}{9} + C_{1(\mathrm{ES}^2)} \Big(-\frac{32}{3} + 10\nu \Big) \bigg) \bigg] + (1 \leftrightarrow 2).\end{aligned} \tag{11.90}$$

From these results, the state-of-the-art next-to-leading order with cubic-in-spin effects gravitational waveform template can be obtained, beyond the current state-of-the-art result. Moreover, it will still be valid for generic compact objects, as we include generic Wilson coefficients.

11.7 Poincaré algebra

In §11.5 we derived the new NLO S^3 Hamiltonian. Hence, to verify its validity, in this section we address whether it admits global Poincaré symmetry by explicitly finding the corresponding NLO S^3 correction to the generators for the Poincaré algebra to hold at this order. This would result in the most stringent self-consistency check that we can perform on the Hamiltonian.

Proceeding as described in §10, at NLO S^3 we require a generator of the order of the LO S^3, so at most proportional[9] to $(G_{\mathrm{N}} m/r) S^3 p \sim S^3 p^3$. But, as seen first in the NLO SS sector, we also have to account for possible Wilson coefficients, which can now appear in the form of both spin-squared and cubic-in-spin coefficients. In particular, now the correction to the generator is given by the ansatz

$$h_1^{\mathrm{S}_1^3}\Big|_{\mathrm{NLO}} = \frac{G_{\mathrm{N}}}{r^4} \bigg[\frac{\vec{S}_1 \cdot \vec{p}_1 \times \vec{n}}{m_1} \bigg(\Big(\alpha_1 + \delta_1 C_{1(\mathrm{ES}^2)} + \zeta_1 C_{1(\mathrm{BS}^3)} \Big) \frac{S_1^2 m_2}{m_1^2}$$

[9] In this section we again adopt the convention of G_{N} to denote Newton's constant.

$$
\begin{aligned}
&+\Big(\alpha_2+\delta_2 C_{1(\mathrm{ES}^2)}+\zeta_2 C_{1(\mathrm{BS}^3)}\Big)\frac{S_1^2}{m_1}\\
&+\Big(\alpha_3+\delta_3 C_{1(\mathrm{ES}^2)}+\zeta_3 C_{1(\mathrm{BS}^3)}\Big)\frac{(\vec{S}_1\cdot\vec{n})^2 m_2}{m_1^2}\\
&+\Big(\alpha_4+\delta_4 C_{1(\mathrm{ES}^2)}+\zeta_4 C_{1(\mathrm{BS}^3)}\Big)\frac{(\vec{S}_1\cdot\vec{n})^2}{m_1}\Bigg)\\
&+\frac{\vec{S}_1\cdot\vec{p}_2\times\vec{n}}{m_2}\Bigg(\Big(\alpha_5+\delta_5 C_{1(\mathrm{ES}^2)}+\zeta_5 C_{1(\mathrm{BS}^3)}\Big)\frac{S_1^2 m_2}{m_1^2}\\
&+\Big(\alpha_6+\delta_6 C_{1(\mathrm{ES}^2)}+\zeta_6 C_{1(\mathrm{BS}^3)}\Big)\frac{S_1^2}{m_1}\\
&+\Big(\alpha_7+\delta_7 C_{1(\mathrm{ES}^2)}+\zeta_7 C_{1(\mathrm{BS}^3)}\Big)\frac{(\vec{S}_1\cdot\vec{n})^2 m_2}{m_1^2}\\
&+\Big(\alpha_8+\delta_8 C_{1(\mathrm{ES}^2)}+\zeta_8 C_{1(\mathrm{BS}^3)}\Big)\frac{(\vec{S}_1\cdot\vec{n})^2}{m_1}\Bigg)\Bigg], \qquad (11.91)
\end{aligned}
$$

$$
\begin{aligned}
h_1^{\mathrm{S}_1^2\mathrm{S}_2}\Big|_{\mathrm{NLO}} = \frac{G_{\mathrm{N}}}{r^4}\Bigg[&\frac{\vec{S}_1\cdot\vec{S}_2\times\vec{n}}{m_1}\Bigg(\Big(\alpha_9+\delta_9 C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{p}_1}{m_1}+\Big(\alpha_{10}+\delta_{10} C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{p}_2}{m_2}\\
&+\Big(\alpha_{11}+\delta_{11} C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})(\vec{p}_1\cdot\vec{n})}{m_1}+\Big(\alpha_{12}+\delta_{12} C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})(\vec{p}_2\cdot\vec{n})}{m_2}\Bigg)\\
&+\frac{\vec{S}_1\cdot\vec{S}_2\times\vec{n}}{m_2}\Bigg(\Big(\alpha_{13}+\delta_{13} C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{p}_1}{m_1}+\Big(\alpha_{14}+\delta_{14} C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{p}_2}{m_2}\\
&+\Big(\alpha_{15}+\delta_{15} C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})(\vec{p}_1\cdot\vec{n})}{m_1}+\Big(\alpha_{16}+\delta_{16} C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})(\vec{p}_2\cdot\vec{n})}{m_2}\Bigg)\\
&+\frac{\vec{S}_1\cdot\vec{S}_2\times\vec{p}_1}{m_1}\Bigg(\Big(\alpha_{17}+\delta_{17} C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{n}}{m_1}+\Big(\alpha_{18}+\delta_{18} C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{n}}{m_2}\Bigg)\\
&+\frac{\vec{S}_1\cdot\vec{S}_2\times\vec{p}_2}{m_2}\Bigg(\Big(\alpha_{19}+\delta_{19} C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{n}}{m_1}+\Big(\alpha_{20}+\delta_{20} C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{n}}{m_2}\Bigg)\\
&+\frac{\vec{S}_1\cdot\vec{p}_1\times\vec{n}}{m_1}\Bigg(\Big(\alpha_{21}+\delta_{21} C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{S}_2}{m_1}+\Big(\alpha_{22}+\delta_{22} C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{S}_2}{m_2}\\
&+\Big(\alpha_{23}+\delta_{23} C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{n})}{m_1}+\Big(\alpha_{24}+\delta_{24} C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{n})}{m_2}\Bigg)\\
&+\frac{\vec{S}_1\cdot\vec{p}_2\times\vec{n}}{m_2}\Bigg(\Big(\alpha_{25}+\delta_{25} C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{S}_2}{m_1}+\Big(\alpha_{26}+\delta_{26} C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{S}_2}{m_2}\\
&+\Big(\alpha_{27}+\delta_{27} C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{n})}{m_1}+\Big(\alpha_{28}+\delta_{28} C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{n})}{m_2}\Bigg)\\
&+\frac{\vec{S}_2\cdot\vec{p}_1\times\vec{n}}{m_1}\Bigg(\Big(\alpha_{29}+\delta_{29} C_{1(\mathrm{ES}^2)}\Big)\frac{S_1^2}{m_1}+\Big(\alpha_{30}+\delta_{30} C_{1(\mathrm{ES}^2)}\Big)\frac{S_1^2}{m_2}\\
&+\Big(\alpha_{31}+\delta_{31} C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})^2}{m_1}+\Big(\alpha_{32}+\delta_{32} C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})^2}{m_2}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\vec{S}_2\cdot\vec{p}_2\times\vec{n}}{m_2}\Bigg(\Big(\alpha_{33}+\delta_{33}C_{1(\mathrm{ES}^2)}\Big)\frac{S_1^2}{m_1}+\Big(\alpha_{34}+\delta_{34}C_{1(\mathrm{ES}^2)}\Big)\frac{S_1^2}{m_2}\\
&+\Big(\alpha_{35}+\delta_{35}C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})^2}{m_1}+\Big(\alpha_{36}+\delta_{36}C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})^2}{m_2}\Bigg)\Bigg], \qquad (11.92)\\
\vec{Y}^{\mathrm{S}_1^3}\Big|_{\mathrm{NLO}} &= \frac{G_{\mathrm{N}}}{r^3}\Bigg\{\Bigg[\frac{\vec{S}_1\cdot\vec{p}_1\times\vec{n}}{m_1}\Bigg(\Big(\beta_1+\omega_1C_{1(\mathrm{ES}^2)}+\lambda_1C_{1(\mathrm{BS}^3)}\Big)\frac{S_1^2m_2}{m_1^2}\\
&+\Big(\beta_2+\omega_2C_{1(\mathrm{ES}^2)}+\lambda_2C_{1(\mathrm{BS}^3)}\Big)\frac{S_1^2}{m_1}\\
&+\Big(\beta_3+\omega_3C_{1(\mathrm{ES}^2)}+\lambda_3C_{1(\mathrm{BS}^3)}\Big)\frac{(\vec{S}_1\cdot\vec{n})^2m_2}{m_1^2}\\
&+\Big(\beta_4+\omega_4C_{1(\mathrm{ES}^2)}+\lambda_4C_{1(\mathrm{BS}^3)}\Big)\frac{(\vec{S}_1\cdot\vec{n})^2}{m_1}\Bigg)\\
&+\frac{\vec{S}_1\cdot\vec{p}_2\times\vec{n}}{m_2}\Bigg(\Big(\beta_5+\omega_5C_{1(\mathrm{ES}^2)}+\lambda_5C_{1(\mathrm{BS}^3)}\Big)\frac{S_1^2m_2}{m_1^2}\\
&+\Big(\beta_6+\omega_6C_{1(\mathrm{ES}^2)}+\lambda_6C_{1(\mathrm{BS}^3)}\Big)\frac{S_1^2}{m_1}\\
&+\Big(\beta_7+\omega_7C_{1(\mathrm{ES}^2)}+\lambda_7C_{1(\mathrm{BS}^3)}\Big)\frac{(\vec{S}_1\cdot\vec{n})^2m_2}{m_1^2}\\
&+\Big(\beta_8+\omega_8C_{1(\mathrm{ES}^2)}+\lambda_8C_{1(\mathrm{BS}^3)}\Big)\frac{(\vec{S}_1\cdot\vec{n})^2}{m_1}\Bigg)\Bigg]\vec{n}\\
&+\Bigg[\frac{\vec{S}_1\cdot\vec{p}_1\times\vec{n}}{m_1}\Big(\beta_9+\omega_9C_{1(\mathrm{ES}^2)}+\lambda_9C_{1(\mathrm{BS}^3)}\Big)\frac{(\vec{S}_1\cdot\vec{n})m_2}{m_1}\\
&+\frac{\vec{S}_1\cdot\vec{p}_2\times\vec{n}}{m_2}\Big(\beta_{10}+\omega_{10}C_{1(\mathrm{ES}^2)}+\lambda_{10}C_{1(\mathrm{BS}^3)}\Big)\frac{(\vec{S}_1\cdot\vec{n})m_2}{m_1}\Bigg]\frac{\vec{S}_1}{m_1}\\
&+\Bigg[\frac{\vec{S}_1\cdot\vec{p}_1\times\vec{n}}{m_1}\Big(\beta_{11}+\omega_{11}C_{1(\mathrm{ES}^2)}+\lambda_{11}C_{1(\mathrm{BS}^3)}\Big)\frac{(\vec{S}_1\cdot\vec{n})m_2}{m_1}\\
&+\frac{\vec{S}_1\cdot\vec{p}_2\times\vec{n}}{m_2}\Big(\beta_{12}+\omega_{12}C_{1(\mathrm{ES}^2)}+\lambda_{12}C_{1(\mathrm{BS}^3)}\Big)\frac{(\vec{S}_1\cdot\vec{n})m_2}{m_1}\Bigg]\frac{\vec{S}_1}{m_2}\\
&+\Bigg[\frac{S_1^2}{m_1}\Bigg(\Big(\beta_{13}+\omega_{13}C_{1(\mathrm{ES}^2)}+\lambda_{13}C_{1(\mathrm{BS}^3)}\Big)\frac{(\vec{p}_1\cdot\vec{n})m_2}{m_1^2}\\
&+\Big(\beta_{14}+\omega_{14}C_{1(\mathrm{ES}^2)}+\lambda_{14}C_{1(\mathrm{BS}^3)}\Big)\frac{\vec{p}_2\cdot\vec{n}}{m_1}\Bigg)\\
&+\frac{S_1^2}{m_2}\Bigg(\Big(\beta_{15}+\omega_{15}C_{1(\mathrm{ES}^2)}+\lambda_{15}C_{1(\mathrm{BS}^3)}\Big)\frac{(\vec{p}_1\cdot\vec{n})m_2}{m_1^2}\\
&+\Big(\beta_{16}+\omega_{16}C_{1(\mathrm{ES}^2)}+\lambda_{16}C_{1(\mathrm{BS}^3)}\Big)\frac{\vec{p}_2\cdot\vec{n}}{m_1}\Bigg)\\
&+\frac{(\vec{S}_1\cdot\vec{n})^2}{m_1}\Bigg(\Big(\beta_{17}+\omega_{17}C_{1(\mathrm{ES}^2)}+\lambda_{17}C_{1(\mathrm{BS}^3)}\Big)\frac{(\vec{p}_1\cdot\vec{n})m_2}{m_1^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\beta_{18}+\omega_{18}C_{1(\mathrm{ES}^2)}+\lambda_{18}C_{1(\mathrm{BS}^3)}\right)\frac{\vec{p}_2\cdot\vec{n}}{m_1}\Bigg)\\
&+\frac{(\vec{S}_1\cdot\vec{n})^2}{m_2}\Bigg(\left(\beta_{19}+\omega_{19}C_{1(\mathrm{ES}^2)}+\lambda_{19}C_{1(\mathrm{BS}^3)}\right)\frac{(\vec{p}_1\cdot\vec{n})m_2}{m_1^2}\\
&+\left(\beta_{20}+\omega_{20}C_{1(\mathrm{ES}^2)}+\lambda_{20}C_{1(\mathrm{BS}^3)}\right)\frac{\vec{p}_2\cdot\vec{n}}{m_1}\Bigg)\\
&+\frac{\vec{S}_1\cdot\vec{p}_1}{m_1}\Bigg(\left(\beta_{21}+\omega_{21}C_{1(\mathrm{ES}^2)}+\lambda_{21}C_{1(\mathrm{BS}^3)}\right)\frac{(\vec{S}_1\cdot\vec{n})m_2}{m_1^2}\\
&+\left(\beta_{22}+\omega_{22}C_{1(\mathrm{ES}^2)}+\lambda_{22}C_{1(\mathrm{BS}^3)}\right)\frac{\vec{S}_1\cdot\vec{n}}{m_1}\Bigg)\\
&+\frac{\vec{S}_1\cdot\vec{p}_2}{m_2}\Bigg(\left(\beta_{23}+\omega_{23}C_{1(\mathrm{ES}^2)}+\lambda_{23}C_{1(\mathrm{BS}^3)}\right)\frac{(\vec{S}_1\cdot\vec{n})m_2}{m_1^2}\\
&+\left(\beta_{24}+\omega_{24}C_{1(\mathrm{ES}^2)}+\lambda_{24}C_{1(\mathrm{BS}^3)}\right)\frac{\vec{S}_1\cdot\vec{n}}{m_1}\Bigg)\Bigg]\vec{S}_1\times\vec{n}\\
&+\Bigg[\left(\beta_{25}+\omega_{25}C_{1(\mathrm{ES}^2)}+\lambda_{25}C_{1(\mathrm{BS}^3)}\right)\frac{S_1^2m_2}{m_1^2}\\
&+\left(\beta_{26}+\omega_{26}C_{1(\mathrm{ES}^2)}+\lambda_{26}C_{1(\mathrm{BS}^3)}\right)\frac{S_1^2}{m_1}\\
&+\left(\beta_{27}+\omega_{27}C_{1(\mathrm{ES}^2)}+\lambda_{27}C_{1(\mathrm{BS}^3)}\right)\frac{(\vec{S}_1\cdot\vec{n})^2m_2}{m_1^2}\\
&+\left(\beta_{28}+\omega_{28}C_{1(\mathrm{ES}^2)}+\lambda_{28}C_{1(\mathrm{BS}^3)}\right)\frac{(\vec{S}_1\cdot\vec{n})^2}{m_1}\Bigg]\frac{\vec{S}_1\times\vec{p}_1}{m_1}\\
&+\Bigg[\left(\beta_{29}+\omega_{29}C_{1(\mathrm{ES}^2)}+\lambda_{29}C_{1(\mathrm{BS}^3)}\right)\frac{S_1^2m_2}{m_1^2}\\
&+\left(\beta_{30}+\omega_{30}C_{1(\mathrm{ES}^2)}+\lambda_{30}C_{1(\mathrm{BS}^3)}\right)\frac{S_1^2}{m_1}\\
&+\left(\beta_{31}+\omega_{31}C_{1(\mathrm{ES}^2)}+\lambda_{31}C_{1(\mathrm{BS}^3)}\right)\frac{(\vec{S}_1\cdot\vec{n})^2m_2}{m_1^2}\\
&+\left(\beta_{32}+\omega_{32}C_{1(\mathrm{ES}^2)}+\lambda_{32}C_{1(\mathrm{BS}^3)}\right)\frac{(\vec{S}_1\cdot\vec{n})^2}{m_1}\Bigg]\frac{\vec{S}_1\times\vec{p}_2}{m_2}\Bigg\}, \qquad (11.93)
\end{aligned}
$$

$$
\begin{aligned}
\vec{Y}^{\mathrm{S}_1^2\mathrm{S}_2}\Big|_{\mathrm{NLO}}=\frac{G_{\mathrm{N}}}{r^3}\Bigg\{\Bigg[&\frac{\vec{S}_1\cdot\vec{S}_2\times\vec{n}}{m_1}\Bigg(\left(\beta_{33}+\omega_{33}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_1\cdot\vec{p}_1}{m_1}+\left(\beta_{34}+\omega_{34}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_1\cdot\vec{p}_2}{m_2}\\
&+\left(\beta_{35}+\omega_{35}C_{1(\mathrm{ES}^2)}\right)\frac{(\vec{S}_1\cdot\vec{n})(\vec{p}_1\cdot\vec{n})}{m_1}+\left(\beta_{36}+\omega_{36}C_{1(\mathrm{ES}^2)}\right)\frac{(\vec{S}_1\cdot\vec{n})(\vec{p}_2\cdot\vec{n})}{m_2}\Bigg)\\
&+\frac{\vec{S}_1\cdot\vec{S}_2\times\vec{n}}{m_2}\Bigg(\left(\beta_{37}+\omega_{37}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_1\cdot\vec{p}_1}{m_1}+\left(\beta_{38}+\omega_{38}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_1\cdot\vec{p}_2}{m_2}\\
&+\left(\beta_{39}+\omega_{39}C_{1(\mathrm{ES}^2)}\right)\frac{(\vec{S}_1\cdot\vec{n})(\vec{p}_1\cdot\vec{n})}{m_1}+\left(\beta_{40}+\omega_{40}C_{1(\mathrm{ES}^2)}\right)\frac{(\vec{S}_1\cdot\vec{n})(\vec{p}_2\cdot\vec{n})}{m_2}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\vec{S}_1\cdot\vec{S}_2\times\vec{p}_1}{m_1}\Bigg(\Big(\beta_{41}+\omega_{41}C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{n}}{m_1}+\Big(\beta_{42}+\omega_{42}C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{n}}{m_2}\Bigg)\\
&+\frac{\vec{S}_1\cdot\vec{S}_2\times\vec{p}_2}{m_2}\Bigg(\Big(\beta_{43}+\omega_{43}C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{n}}{m_1}+\Big(\beta_{44}+\omega_{44}C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{n}}{m_2}\Bigg)\\
&+\frac{\vec{S}_1\cdot\vec{p}_1\times\vec{n}}{m_1}\Bigg(\Big(\beta_{45}+\omega_{45}C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{S}_2}{m_1}+\Big(\beta_{46}+\omega_{46}C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{S}_2}{m_2}\\
&+\Big(\beta_{47}+\omega_{47}C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{n})}{m_1}+\Big(\beta_{48}+\omega_{48}C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{n})}{m_2}\Bigg)\\
&+\frac{\vec{S}_1\cdot\vec{p}_2\times\vec{n}}{m_2}\Bigg(\Big(\beta_{49}+\omega_{49}C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{S}_2}{m_1}+\Big(\beta_{50}+\omega_{50}C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{S}_2}{m_2}\\
&+\Big(\beta_{51}+\omega_{51}C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{n})}{m_1}+\Big(\beta_{52}+\omega_{52}C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{n})}{m_2}\Bigg)\\
&+\frac{\vec{S}_2\cdot\vec{p}_1\times\vec{n}}{m_1}\Bigg(\Big(\beta_{53}+\omega_{53}C_{1(\mathrm{ES}^2)}\Big)\frac{S_1^2}{m_1}+\Big(\beta_{54}+\omega_{54}C_{1(\mathrm{ES}^2)}\Big)\frac{S_1^2}{m_2}\\
&+\Big(\beta_{55}+\omega_{55}C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})^2}{m_1}+\Big(\beta_{56}+\omega_{56}C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})^2}{m_2}\Bigg)\\
&+\frac{\vec{S}_2\cdot\vec{p}_2\times\vec{n}}{m_2}\Bigg(\Big(\beta_{57}+\omega_{57}C_{1(\mathrm{ES}^2)}\Big)\frac{S_1^2}{m_1}+\Big(\beta_{58}+\omega_{58}C_{1(\mathrm{ES}^2)}\Big)\frac{S_1^2}{m_2}\\
&+\Big(\beta_{59}+\omega_{59}C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})^2}{m_1}+\Big(\beta_{60}+\omega_{60}C_{1(\mathrm{ES}^2)}\Big)\frac{(\vec{S}_1\cdot\vec{n})^2}{m_2}\Bigg)\Bigg]\vec{n}\\
&+\Bigg[\frac{\vec{S}_1\cdot\vec{S}_2\times\vec{n}}{m_1}\Big(\beta_{61}+\omega_{61}C_{1(\mathrm{ES}^2)}\Big)\vec{S}_1\cdot\vec{n}\\
&+\frac{\vec{S}_1\cdot\vec{S}_2\times\vec{n}}{m_2}\Big(\beta_{62}+\omega_{62}C_{1(\mathrm{ES}^2)}\Big)\vec{S}_1\cdot\vec{n}\Bigg]\frac{\vec{p}_1}{m_1}\\
&+\Bigg[\frac{\vec{S}_1\cdot\vec{S}_2\times\vec{n}}{m_1}\Big(\beta_{63}+\omega_{63}C_{1(\mathrm{ES}^2)}\Big)\vec{S}_1\cdot\vec{n}\\
&+\frac{\vec{S}_1\cdot\vec{S}_2\times\vec{n}}{m_2}\Big(\beta_{64}+\omega_{64}C_{1(\mathrm{ES}^2)}\Big)\vec{S}_1\cdot\vec{n}\Bigg]\frac{\vec{p}_2}{m_2}\\
&+\Bigg[\vec{S}_1\cdot\vec{S}_2\times\vec{n}\Bigg(\Big(\beta_{65}+\omega_{65}C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{p}_1\cdot\vec{n}}{m_1}+\Big(\beta_{66}+\omega_{66}C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{p}_2\cdot\vec{n}}{m_2}\Bigg)\\
&+\Big(\beta_{67}+\omega_{67}C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{S}_2\times\vec{p}_1}{m_1}+\Big(\beta_{68}+\omega_{68}C_{1(\mathrm{ES}^2)}\Big)\frac{\vec{S}_1\cdot\vec{S}_2\times\vec{p}_2}{m_2}\\
&+\frac{\vec{S}_1\cdot\vec{p}_1\times\vec{n}}{m_1}\Big(\beta_{69}+\omega_{69}C_{1(\mathrm{ES}^2)}\Big)\vec{S}_2\cdot\vec{n}+\frac{\vec{S}_1\cdot\vec{p}_2\times\vec{n}}{m_2}\Big(\beta_{70}+\omega_{70}C_{1(\mathrm{ES}^2)}\Big)\vec{S}_2\cdot\vec{n}\\
&+\frac{\vec{S}_2\cdot\vec{p}_1\times\vec{n}}{m_1}\Big(\beta_{71}+\omega_{71}C_{1(\mathrm{ES}^2)}\Big)\vec{S}_1\cdot\vec{n}\\
&+\frac{\vec{S}_2\cdot\vec{p}_2\times\vec{n}}{m_2}\Big(\beta_{72}+\omega_{72}C_{1(\mathrm{ES}^2)}\Big)\vec{S}_1\cdot\vec{n}\Bigg]\frac{\vec{S}_1}{m_1}
\end{aligned}
$$

$$
\begin{aligned}
&+ \Bigg[\vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \bigg(\Big(\beta_{73} + \omega_{73} C_{1(\mathrm{ES}^2)}\Big) \frac{\vec{p}_1 \cdot \vec{n}}{m_1} + \Big(\beta_{74} + \omega_{74} C_{1(\mathrm{ES}^2)}\Big) \frac{\vec{p}_2 \cdot \vec{n}}{m_2} \bigg) \\
&+ \Big(\beta_{75} + \omega_{75} C_{1(\mathrm{ES}^2)}\Big) \frac{\vec{S}_1 \cdot \vec{S}_2 \times \vec{p}_1}{m_1} + \Big(\beta_{76} + \omega_{76} C_{1(\mathrm{ES}^2)}\Big) \frac{\vec{S}_1 \cdot \vec{S}_2 \times \vec{p}_2}{m_2} \\
&+ \frac{\vec{S}_1 \cdot \vec{p}_1 \times \vec{n}}{m_1} \Big(\beta_{77} + \omega_{77} C_{1(\mathrm{ES}^2)}\Big) \vec{S}_2 \cdot \vec{n} + \frac{\vec{S}_1 \cdot \vec{p}_2 \times \vec{n}}{m_2} \Big(\beta_{78} + \omega_{78} C_{1(\mathrm{ES}^2)}\Big) \vec{S}_2 \cdot \vec{n} \\
&+ \frac{\vec{S}_2 \cdot \vec{p}_1 \times \vec{n}}{m_1} \Big(\beta_{79} + \omega_{79} C_{1(\mathrm{ES}^2)}\Big) \vec{S}_1 \cdot \vec{n} \\
&+ \frac{\vec{S}_2 \cdot \vec{p}_2 \times \vec{n}}{m_2} \Big(\beta_{80} + \omega_{80} C_{1(\mathrm{ES}^2)}\Big) \vec{S}_1 \cdot \vec{n} \Bigg] \frac{\vec{S}_1}{m_2} \\
&+ \Bigg[\frac{\vec{S}_1 \cdot \vec{p}_1 \times \vec{n}}{m_1} \Big(\beta_{81} + \omega_{81} C_{1(\mathrm{ES}^2)}\Big) \vec{S}_1 \cdot \vec{n} \\
&+ \frac{\vec{S}_1 \cdot \vec{p}_2 \times \vec{n}}{m_2} \Big(\beta_{82} + \omega_{82} C_{1(\mathrm{ES}^2)}\Big) \vec{S}_1 \cdot \vec{n} \Bigg] \frac{\vec{S}_2}{m_1} \\
&+ \Bigg[\frac{\vec{S}_1 \cdot \vec{p}_1 \times \vec{n}}{m_1} \Big(\beta_{83} + \omega_{83} C_{1(\mathrm{ES}^2)}\Big) \vec{S}_1 \cdot \vec{n} \\
&+ \frac{\vec{S}_1 \cdot \vec{p}_2 \times \vec{n}}{m_2} \Big(\beta_{84} + \omega_{84} C_{1(\mathrm{ES}^2)}\Big) \vec{S}_1 \cdot \vec{n} \Bigg] \frac{\vec{S}_2}{m_2} \\
&+ \Bigg[\Big(\beta_{85} + \omega_{85} C_{1(\mathrm{ES}^2)}\Big) \frac{\vec{S}_1 \cdot \vec{p}_1}{m_1} + \Big(\beta_{86} + \omega_{86} C_{1(\mathrm{ES}^2)}\Big) \frac{\vec{S}_1 \cdot \vec{p}_2}{m_2} \\
&+ \Big(\beta_{87} + \omega_{87} C_{1(\mathrm{ES}^2)}\Big) \frac{(\vec{S}_1 \cdot \vec{n})(\vec{p}_1 \cdot \vec{n})}{m_1} \\
&+ \Big(\beta_{88} + \omega_{88} C_{1(\mathrm{ES}^2)}\Big) \frac{(\vec{S}_1 \cdot \vec{n})(\vec{p}_2 \cdot \vec{n})}{m_2} \Bigg] \frac{\vec{S}_1 \times \vec{S}_2}{m_1} \\
&+ \Bigg[\Big(\beta_{89} + \omega_{89} C_{1(\mathrm{ES}^2)}\Big) \frac{\vec{S}_1 \cdot \vec{p}_1}{m_1} + \Big(\beta_{90} + \omega_{90} C_{1(\mathrm{ES}^2)}\Big) \frac{\vec{S}_1 \cdot \vec{p}_2}{m_2} \\
&+ \Big(\beta_{91} + \omega_{91} C_{1(\mathrm{ES}^2)}\Big) \frac{(\vec{S}_1 \cdot \vec{n})(\vec{p}_1 \cdot \vec{n})}{m_1} \\
&+ \Big(\beta_{92} + \omega_{92} C_{1(\mathrm{ES}^2)}\Big) \frac{(\vec{S}_1 \cdot \vec{n})(\vec{p}_2 \cdot \vec{n})}{m_2} \Bigg] \frac{\vec{S}_1 \times \vec{S}_2}{m_2} \\
&+ \Bigg[\frac{\vec{S}_1 \cdot \vec{S}_2}{m_1} \bigg(\Big(\beta_{93} + \omega_{93} C_{1(\mathrm{ES}^2)}\Big) \frac{\vec{p}_1 \cdot \vec{n}}{m_1} + \Big(\beta_{94} + \omega_{94} C_{1(\mathrm{ES}^2)}\Big) \frac{\vec{p}_2 \cdot \vec{n}}{m_2} \bigg) \\
&+ \frac{\vec{S}_1 \cdot \vec{S}_2}{m_2} \bigg(\Big(\beta_{95} + \omega_{95} C_{1(\mathrm{ES}^2)}\Big) \frac{\vec{p}_1 \cdot \vec{n}}{m_1} + \Big(\beta_{96} + \omega_{96} C_{1(\mathrm{ES}^2)}\Big) \frac{\vec{p}_2 \cdot \vec{n}}{m_2} \bigg) \\
&+ \frac{(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n})}{m_1} \bigg(\Big(\beta_{97} + \omega_{97} C_{1(\mathrm{ES}^2)}\Big) \frac{\vec{p}_1 \cdot \vec{n}}{m_1} + \Big(\beta_{98} + \omega_{98} C_{1(\mathrm{ES}^2)}\Big) \frac{\vec{p}_2 \cdot \vec{n}}{m_2} \bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{n})}{m_2}\left(\left(\beta_{99}+\omega_{99}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{p}_1\cdot\vec{n}}{m_1}+\left(\beta_{100}+\omega_{100}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right)\\
&+\frac{\vec{S}_1\cdot\vec{p}_1}{m_1}\left(\left(\beta_{101}+\omega_{101}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_2\cdot\vec{n}}{m_1}+\left(\beta_{102}+\omega_{102}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_2\cdot\vec{n}}{m_2}\right)\\
&+\frac{\vec{S}_1\cdot\vec{p}_2}{m_2}\left(\left(\beta_{103}+\omega_{103}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_2\cdot\vec{n}}{m_1}+\left(\beta_{104}+\omega_{104}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_2\cdot\vec{n}}{m_2}\right)\\
&+\frac{\vec{S}_2\cdot\vec{p}_1}{m_1}\left(\left(\beta_{105}+\omega_{105}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_1\cdot\vec{n}}{m_1}+\left(\beta_{106}+\omega_{106}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_1\cdot\vec{n}}{m_2}\right)\\
&+\frac{\vec{S}_2\cdot\vec{p}_2}{m_2}\left(\left(\beta_{107}+\omega_{107}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_1\cdot\vec{n}}{m_1}+\left(\beta_{108}+\omega_{108}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_1\cdot\vec{n}}{m_2}\right)\Bigg]\vec{S}_1\times\vec{n}\\
&+\Bigg[\frac{S_1^2}{m_1}\left(\left(\beta_{109}+\omega_{109}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{p}_1\cdot\vec{n}}{m_1}+\left(\beta_{110}+\omega_{110}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right)\\
&+\frac{S_1^2}{m_2}\left(\left(\beta_{111}+\omega_{111}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{p}_1\cdot\vec{n}}{m_1}+\left(\beta_{112}+\omega_{112}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right)\\
&+\frac{(\vec{S}_1\cdot\vec{n})^2}{m_1}\left(\left(\beta_{113}+\omega_{113}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{p}_1\cdot\vec{n}}{m_1}+\left(\beta_{114}+\omega_{114}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right)\\
&+\frac{(\vec{S}_1\cdot\vec{n})^2}{m_2}\left(\left(\beta_{115}+\omega_{115}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{p}_1\cdot\vec{n}}{m_1}+\left(\beta_{116}+\omega_{116}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{p}_2\cdot\vec{n}}{m_2}\right)\\
&+\frac{\vec{S}_1\cdot\vec{p}_1}{m_1}\left(\left(\beta_{117}+\omega_{117}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_1\cdot\vec{n}}{m_1}+\left(\beta_{118}+\omega_{118}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_1\cdot\vec{n}}{m_2}\right)\\
&+\frac{\vec{S}_1\cdot\vec{p}_2}{m_2}\left(\left(\beta_{119}+\omega_{119}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_1\cdot\vec{n}}{m_1}+\left(\beta_{120}+\omega_{120}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_1\cdot\vec{n}}{m_2}\right)\Bigg]\vec{S}_2\times\vec{n}\\
&+\Bigg[\left(\beta_{121}+\omega_{121}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_1\cdot\vec{S}_2}{m_1}+\left(\beta_{122}+\omega_{122}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_1\cdot\vec{S}_2}{m_2}\\
&+\left(\beta_{123}+\omega_{123}C_{1(\mathrm{ES}^2)}\right)\frac{(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{n})}{m_1}\\
&+\left(\beta_{124}+\omega_{124}C_{1(\mathrm{ES}^2)}\right)\frac{(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{n})}{m_2}\Bigg]\frac{\vec{S}_1\times\vec{p}_1}{m_1}\\
&+\Bigg[\left(\beta_{125}+\omega_{125}C_{1(\mathrm{ES}^2)}\right)\frac{S_1^2}{m_1}+\left(\beta_{126}+\omega_{126}C_{1(\mathrm{ES}^2)}\right)\frac{S_1^2}{m_2}\\
&+\left(\beta_{127}+\omega_{127}C_{1(\mathrm{ES}^2)}\right)\frac{(\vec{S}_1\cdot\vec{n})^2}{m_1}+\left(\beta_{128}+\omega_{128}C_{1(\mathrm{ES}^2)}\right)\frac{(\vec{S}_1\cdot\vec{n})^2}{m_2}\Bigg]\frac{\vec{S}_2\times\vec{p}_1}{m_1}\\
&+\Bigg[\left(\beta_{129}+\omega_{129}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_1\cdot\vec{S}_2}{m_1}+\left(\beta_{130}+\omega_{130}C_{1(\mathrm{ES}^2)}\right)\frac{\vec{S}_1\cdot\vec{S}_2}{m_2}\\
&+\left(\beta_{131}+\omega_{131}C_{1(\mathrm{ES}^2)}\right)\frac{(\vec{S}_1\cdot\vec{n})(\vec{S}_2\cdot\vec{n})}{m_1}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\beta_{132} + \omega_{132} C_{1(\mathrm{ES}^2)}\right) \frac{(\vec{S}_1 \cdot \vec{n})(\vec{S}_2 \cdot \vec{n})}{m_2} \Bigg] \frac{\vec{S}_1 \times \vec{p}_2}{m_2} \\
&+ \Bigg[\left(\beta_{133} + \omega_{133} C_{1(\mathrm{ES}^2)}\right) \frac{S_1^2}{m_1} + \left(\beta_{134} + \omega_{134} C_{1(\mathrm{ES}^2)}\right) \frac{S_1^2}{m_2} \\
&+ \left(\beta_{135} + \omega_{135} C_{1(\mathrm{ES}^2)}\right) \frac{(\vec{S}_1 \cdot \vec{n})^2}{m_1} \\
&+ \left(\beta_{136} + \omega_{136} C_{1(\mathrm{ES}^2)}\right) \frac{(\vec{S}_1 \cdot \vec{n})^2}{m_2} \Bigg] \frac{\vec{S}_2 \times \vec{p}_2}{m_2} \Bigg\},
\end{aligned}
\tag{11.94}
$$

where α_i, β_i, δ_i, ω_i, ζ_i and λ_i are the undetermined numerical coefficients, and where we have also used the special-relativistic limit of eq. (10.12) to fix to zero all $\mathcal{O}(G_{\mathrm{N}}^0)$ terms. Moreover, we have split the generator into the different spin contributions, $\vec{G}^{\mathrm{NLO}}_{\mathrm{S}^3} = \vec{G}^{\mathrm{NLO}}_{\mathrm{S}_1^3} + \vec{G}^{\mathrm{NLO}}_{\mathrm{S}_1^2\mathrm{S}_2} + (1 \leftrightarrow 2)$.

Then, for the Poincaré algebra the NLO S^3 sector receives contributions from 22 combinations of sectors, as shown in the multiplication Tables 10.1, 10.2. Splitting again into the different spin contributions, $\mathrm{S}_1^3/\mathrm{S}_2^3$ and $\mathrm{S}_1^2\mathrm{S}_2/\mathrm{S}_1\mathrm{S}_2^2$, the Poincaré algebra conditions read

$$
\begin{aligned}
0 = {} & \{\vec{G}_{0\mathrm{PN}}, H^{\mathrm{NLO}}_{\mathrm{S}_1^3}\}_{(x,p)} + \{\vec{G}_{1\mathrm{PN}}, H^{\mathrm{LO}}_{\mathrm{S}_1^3}\}_{(x,p)} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{O}}, H^{\mathrm{NLO}}_{\mathrm{S}_1^2}\}_{(x,p)} + \{\vec{G}^{\mathrm{NLO}}_{\mathrm{S}_1\mathrm{O}}, H^{\mathrm{LO}}_{\mathrm{S}_1^2}\}_{(x,p)} \\
& + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_1^2}, H^{\mathrm{NLO}}_{\mathrm{S}_1\mathrm{O}}\}_{(x,p)} + \{\vec{G}^{\mathrm{NLO}}_{\mathrm{S}_1^2}, H^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{O}}\}_{(x,p)} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_1^3}, H_{1\mathrm{PN}}\}_{(x,p)} + \{\vec{G}^{\mathrm{NLO}}_{\mathrm{S}_1^3}, H_{\mathrm{N}}\}_{(x,p)} \\
& + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{O}}, H^{\mathrm{LO}}_{\mathrm{S}_1^3}\}_{\mathrm{spin}} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_1^2}, H^{\mathrm{NLO}}_{\mathrm{S}_1^2}\}_{\mathrm{spin}} + \{\vec{G}^{\mathrm{NLO}}_{\mathrm{S}_1^2}, H^{\mathrm{LO}}_{\mathrm{S}_1^2}\}_{\mathrm{spin}} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_1^3}, H^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{O}}\}_{\mathrm{spin}} \\
& + (1 \leftrightarrow 2),
\end{aligned}
\tag{11.95}
$$

$$
\begin{aligned}
0 = {} & \{\vec{G}_{0\mathrm{PN}}, H^{\mathrm{NLO}}_{\mathrm{S}_1^2\mathrm{S}_2}\}_{(x,p)} + \{\vec{G}_{1\mathrm{PN}}, H^{\mathrm{LO}}_{\mathrm{S}_1^2\mathrm{S}_2}\}_{(x,p)} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{O}}, H^{\mathrm{NLO}}_{\mathrm{S}_1\mathrm{S}_2}\}_{(x,p)} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_2\mathrm{O}}, H^{\mathrm{NLO}}_{\mathrm{S}_1^2}\}_{(x,p)} \\
& + \{\vec{G}^{\mathrm{NLO}}_{\mathrm{S}_1\mathrm{O}}, H^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{S}_2}\}_{(x,p)} + \{\vec{G}^{\mathrm{NLO}}_{\mathrm{S}_2\mathrm{O}}, H^{\mathrm{LO}}_{\mathrm{S}_1^2}\}_{(x,p)} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{S}_2}, H^{\mathrm{NLO}}_{\mathrm{S}_1\mathrm{O}}\}_{(x,p)} + \{\vec{G}^{\mathrm{NLO}}_{\mathrm{S}_1\mathrm{S}_2}, H^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{O}}\}_{(x,p)} \\
& + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_1^2}, H^{\mathrm{NLO}}_{\mathrm{S}_2\mathrm{O}}\}_{(x,p)} + \{\vec{G}^{\mathrm{NLO}}_{\mathrm{S}_1^2}, H^{\mathrm{LO}}_{\mathrm{S}_2\mathrm{O}}\}_{(x,p)} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_1^2\mathrm{S}_2}, H_{1\mathrm{PN}}\}_{(x,p)} + \{\vec{G}^{\mathrm{NLO}}_{\mathrm{S}_1^2\mathrm{S}_2}, H_{\mathrm{N}}\}_{(x,p)} \\
& + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{O}}, H^{\mathrm{LO}}_{\mathrm{S}_1^2\mathrm{S}_2}\}_{\mathrm{spin}} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_2\mathrm{O}}, H^{\mathrm{LO}}_{\mathrm{S}_1^2\mathrm{S}_2}\}_{\mathrm{spin}} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{S}_2}, H^{\mathrm{NLO}}_{\mathrm{S}_1\mathrm{S}_2}\}_{\mathrm{spin}} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{S}_2}, H^{\mathrm{NLO}}_{\mathrm{S}_1^2}\}_{\mathrm{spin}} \\
& + \{\vec{G}^{\mathrm{NLO}}_{\mathrm{S}_1\mathrm{S}_2}, H^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{S}_2}\}_{\mathrm{spin}} + \{\vec{G}^{\mathrm{NLO}}_{\mathrm{S}_1\mathrm{S}_2}, H^{\mathrm{LO}}_{\mathrm{S}_1^2}\}_{\mathrm{spin}} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_1^2}, H^{\mathrm{NLO}}_{\mathrm{S}_1\mathrm{S}_2}\}_{\mathrm{spin}} + \{\vec{G}^{\mathrm{NLO}}_{\mathrm{S}_1^2}, H^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{S}_2}\}_{\mathrm{spin}} \\
& + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_1^2\mathrm{S}_2}, H^{\mathrm{LO}}_{\mathrm{S}_1\mathrm{O}}\}_{\mathrm{spin}} + \{\vec{G}^{\mathrm{LO}}_{\mathrm{S}_1^2\mathrm{S}_2}, H^{\mathrm{LO}}_{\mathrm{S}_2\mathrm{O}}\}_{\mathrm{spin}} + (1 \leftrightarrow 2),
\end{aligned}
\tag{11.96}
$$

where we denote by $\mathrm{S}_I\mathrm{O}$ the terms in the spin-orbit sector containing one spin S_I. These equations result in a total of 5692 terms, and at the present moment we have not yet found the solution for the undetermined coefficients. The difficulty is mainly due to the vectorial identity for 4 vectors in 3 dimensions of eq. (11.16), which mixes terms that contain the same powers of momenta, spins and masses, and thus makes the resolution of the Poincaré algebra more complicated. Work towards the solution is still in progress.

Part IV
Conclusions and Outlook

12 Conclusions

The recent first observation of gravitational waves reinforced the urgency of the development of high-precision theoretical models for their waveform templates, for which the analytical description provided by the EFT of PN gravity plays a central role. In this thesis, we used this formalism to push the high-in-spin frontier, implementing it for the first time to derive the dynamics of a generic compact binary system at the next-to-leading order with cubic-in-spin effects. The most pressing application of our results would be to implement the Hamiltonian and the gauge-invariant relations within the effective-one-body formalism to compute the 4.5PN correction to the waveform templates for the emitted GWs.

For that purpose, we have presented the methodology required to obtain the physical observables, starting from the basic formulation of the effective action. Since a deep understanding of the lower-order sectors is necessary to approach any new PN sector, we have exemplified the methodology to all relevant lower-order PN sectors, from the non-spinning Newtonian to LO S^3 and to NLO S^2.

We then implemented the methodology for the first time at NLO S^3, deriving the standard reduced potential, where higher-order time derivatives have been eliminated via lower-order variable redefinitions, the equations of motion, and the Hamiltonian. All of them have been derived in a general coordinate frame, and are valid for generic compact objects, generic orbits, and with arbitrary spin orientations. Considering the aligned-spins case in a circular orbit in the center-of-mass frame, we also computed physical observables, such as the binding energy, and two gauge-invariant relations between the binding energy, angular momentum and orbital frequency of the binary. These results, which correspond to a 4.5PN correction, go beyond the current complete state of the art at 4PN order. Moreover, as could be observed, the scale and complexity of the calculations increased for each odd-in-spin sector, being enormously enhanced at the present NLO S^3 sector, which prompted the development of a specialized *Mathematica* code within this master's project.

As an additional inquiry, we have proven the equivalence of the dynamics for the lower-order sectors when defining the spin-induced non-minimal coupling either with spin vectors or with a contraction of spin tensors. While at leading order the difference vanished at the level of the Feynman rules and diagrams, at next-to-leading order both descriptions were related by a total time derivative or by a canonical transformation in the interaction potential and in the Hamiltonian, respectively, which did not modify the physical predictions. The check at the NLO S^3 sector, which involves the calculation of the 53 Feynman diagrams using these different prescriptions, is left for a possible future work.

Furthermore, we have proven that the special diagrams that contained the new Feynman rules in this sector, arising from the subleading dependence of the linear momentum in the curvature and the spin, cancel out altogether in the reduced action. Therefore, the rotational gauge used for these new rules turns out to be equivalent to the simple one used in lower-order sectors, and so neither the resulting Hamiltonian nor the physical observables are modified by this change of gauge.

Besides, it is left for future research to reach a better physical understanding of the presence of Wilson coefficients squared at NLO S^3. They arose at the level of the reduced potential, and could represent composite effects, signaling the breakdown of the original point-particle picture for higher spins. Approaching it from the EoMs perspective, we saw that the Wilson coefficients squared at NLO S^3 originated as an outcome of the precession effects at LO SS on the NLO SS potential. However, we found that their effects vanish in the Hamiltonian in the center-of-mass frame in the aligned-spins case. Thus, it would be interesting to study whether this would have observable effects for the unaligned-spins case. Although for black holes these Wilson coefficients are expected to be equal to 1, in which case this feature would not be numerically exceptional, for neutron stars these Wilson coefficients may be considerably larger than 1, where it could induce a notably large and dominant effect.

Finally, it remains to complete the verification of the Poincaré invariance of the system at NLO S^3, which would provide the most stringent self-consistency check for the results presented in this thesis.

13 Prospective work

Based on the work presented in this thesis, the possibilities of future research directions are two-fold: Pursue higher-in-spin corrections, or further apply the results reported in this thesis, allowing for comparisons with the literature, where restricted partial results have been computed at this order. For the first direction, the next natural follow-up work would be to apply the formalism and technology developed in this thesis also in the newly derived interaction potential at NLO S^4 [30], to derive the 5PN correction to the dynamics of a generic compact binary system. This derivation would depend on the results at NLO S^3 reported in this thesis, even though it would be computationally simpler, as it corresponds to an even-in-spin sector. Nevertheless, at this order the effective action for the spinning particle has to be extended with operators that are quadratic in the curvature, entailing new Wilson coefficients, as well as the ones encapsulating hexadecapolar finite-size effects. In the long term, this line of research also attempts to pave the way towards a classical point of view for understanding the non-uniqueness of the 4-particle gravitational Compton scattering amplitude for spins $s > 2$, whose counterpart are classical effects with spins at the order $l = 2s$.

For the second future research direction, it remains to compare the results presented in this thesis with other independent calculations, in the restricted limit where there is overlap. Particularly, a comparison of the reduced potential should be made with that of [72], where it was computed in momentum space from the Post-Minkowskian approximation at $\mathcal{O}(G^2)$, using the gravitational Compton amplitude to evaluate the diagrams. Similarly, work has been done to compute the scattering angle in the case of BHs in the aligned-spins configuration [73]. Therefore, it would also be interesting to compute the correction to the scattering angle from our results.

Work is expected to progress in all of these directions in the short-term future.

Appendices

A Review of Cartan's method of exterior forms

In this Appendix we describe Cartan's method of exterior forms, see §7.8 of [41], used in §4.1 to derive an expression for the Ricci scalar in terms of the NRG fields.

The first step in Cartan's method of exterior forms is to define a new non-coordinate basis $\{\hat{\theta}^\alpha\}$, which is a dual basis to the vierbein basis, and is defined as

$$\hat{\theta}^\alpha = e^\alpha{}_\mu dx^\mu. \tag{A.1}$$

In terms of this basis, the metric becomes

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = g_{\mu\nu}\, dx^\mu \otimes dx^\nu = \eta_{\alpha\beta}\, \hat{\theta}^\alpha \otimes \hat{\theta}^\beta, \tag{A.2}$$

where $\otimes$ denotes a direct product, or tensor product. Thus, we see that the elements of the dual basis are orthonormal, as they reduce the general metric $g_{\mu\nu}$ to the Minkowski metric $\eta_{\alpha\beta}$. For a non-Lorentzian metric without time component, it would reduce to the Euclidean metric $\delta_{\alpha\beta}$.

Now, due to the presence of the differential dx^μ in the definition of the dual basis, in eq. (A.1), its elements actually are 1-forms. In general, a differential form of order r, or r-form, is a totally antisymmetric covariant tensor of rank r. Defining the wedge product (or exterior product) $\wedge$ of r 1-forms by the totally antisymmetric tensor product

$$dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_r} = \sum_{P \in S_r} sign(P)\, dx^{\mu_{P(1)}} \otimes dx^{\mu_{P(2)}} \otimes \cdots \otimes dx^{\mu_{P(r)}}, \tag{A.3}$$

where S_r is the set of non-cyclic permutations of r elements, an r-form ω is defined by

$$\omega = \frac{1}{r!} \omega_{[\mu_1 \ldots \mu_r]} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}, \tag{A.4}$$

where $\omega_{[\mu_1 \ldots \mu_r]}$ denotes the antisymmetrization of a tensor $\omega_{\mu_1 \ldots \mu_r}$. Moreover, we can define the exterior derivative d of an r-form, given by

$$d\omega \equiv \frac{1}{r!} \left(\frac{\partial}{\partial x^\nu} \omega_{\mu_1 \ldots \mu_r} \right) dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}, \tag{A.5}$$

which creates an $(r+1)$-form, adding a wedge product with a partial derivative with respect to the variable we differentiate.

With the previous definitions, we can introduce a matrix-valued 1-form $\{\omega^\alpha{}_\beta\}$, called the connection one-form, given by

$$\omega^\alpha{}_\beta \equiv \Gamma^\alpha_{\gamma\beta}\, \hat{\theta}^\gamma, \tag{A.6}$$

where $\Gamma^\alpha_{\gamma\beta} = e^\alpha{}_\nu e_\gamma{}^\mu \left(\partial_\mu e_\beta{}^\nu + e_\beta{}^\lambda \Gamma^\nu_{\mu\lambda} \right) = e^\alpha{}_\nu e_\gamma{}^\mu \nabla_\mu\, e_\beta{}^\nu$ are the Levi-Civita connection coefficients, projected onto the local frame, which also satisfies the antisymmetry $\omega_{\mu\nu} =$

$-\omega_{\nu\mu}$ due to the condition $\nabla_\mu g = 0$ for the metric (see eq. (7.155) of [41]). Then, connection one-forms satisfy Cartan's structure equations [41], given by

$$\begin{cases} d\hat{\theta}^\alpha + \omega^\alpha{}_\beta \wedge \hat{\theta}^\beta = T^\alpha, \\ d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta = R^\alpha{}_\beta, \end{cases} \tag{A.7}$$

where $T^\alpha \equiv \frac{1}{2} T^\alpha{}_{\beta\gamma}\, \hat{\theta}^\beta \wedge \hat{\theta}^\gamma$ is the torsion two-form, which we will take as zero since we will consider a torsion-free theory, and where $R^\alpha{}_\beta \equiv \frac{1}{2} R^\alpha{}_{\beta\gamma\delta}\, \hat{\theta}^\gamma \wedge \hat{\theta}^\delta$ is the curvature two-form, out of which the Riemann tensor and thus the Ricci tensor can be finally extracted.

To make the usefulness of Cartan's method manifest, we can consider the following simple example: a unit sphere $\mathcal{S}^2$ in 3 dimensions, with line-element in spherical coordinates given by

$$ds^2 = d\theta \otimes d\theta + \sin^2\theta\, d\phi \otimes d\phi. \tag{A.8}$$

In order to obtain $ds^2 = \delta_{\alpha\beta}\, \hat{\theta}^\alpha \otimes \hat{\theta}^\beta = \hat{\theta}^1 \otimes \hat{\theta}^1 + \hat{\theta}^2 \otimes \hat{\theta}^2$, we can define the dual basis as $\hat{\theta}^1 = d\theta$ and $\hat{\theta}^2 = \sin\theta\, d\phi$, so that by eq. (A.1), the vierbeins are $e^1{}_\theta = 1$, $e^2{}_\theta = 0$, $e^1{}_\phi = 0$ and $e^2{}_\phi = \sin\theta$. Then, the torsion-free equations for the two elements of the basis are

$$\begin{cases} d(d\theta) + \omega^1{}_2 \wedge (\sin\theta\, d\phi) = 0, \\ d(\sin\theta\, d\phi) + \omega^2{}_1 \wedge (d\theta) = 0, \end{cases} \tag{A.9}$$

which are solved by $\omega^2{}_1 = -\omega^1{}_2 = \cos\theta\, d\phi$, since by eq. (A.5), $d(d\theta) = 0$ and $d(\sin\theta\, d\phi) = \cos\theta\, d\theta \wedge d\phi$. Then, from the second Cartan structure equation for $\alpha = 1$ and $\beta = 2$, we have

$$d\omega^1{}_2 = \sin\theta\, d\theta \wedge d\phi = \frac{1}{2} R^1{}_{212} d\theta \wedge (\sin\theta\, d\phi) + \frac{1}{2} R^1{}_{221} (\sin\theta\, d\phi) \wedge d\theta, \tag{A.10}$$

so that $R^1{}_{212} = 1$, or, in coordinate basis, $R^\theta{}_{\phi\theta\phi} = e_\alpha{}^\theta e^\beta{}_\phi e^\gamma{}_\theta e^\delta{}_\phi R^\alpha{}_{\beta\gamma\delta} = \sin^2\theta\, R^1{}_{212} = \sin^2\theta$, in agreement with the traditional derivation via Christoffel symbols. In a similar manner, all components of the Riemann tensor can be calculated, and thereafter the Ricci tensor and Ricci scalar.

In our present case, we wish to compute the Ricci scalar for the metric

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = e^{2\phi}(dt - A_i dx^i)^2 - e^{-2\phi}\gamma_{ij} dx^i dx^j, \tag{A.11}$$

given in terms of the NRG fields. However, solving the torsion-free equations to obtain the connection 1-forms is still highly non-trivial. Alternatively, one can compute some coefficients $c_{\mu\nu}{}^\alpha$ via

$$d\hat{\theta}^\alpha = -c_{|\mu\nu|}{}^\alpha\, \hat{\theta}^\mu \wedge \hat{\theta}^\nu, \tag{A.12}$$

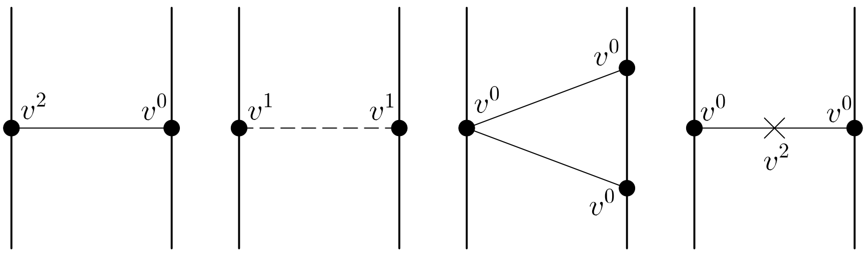
where $|\mu\nu|$ is restricted to $\mu < \nu$, and then evaluate the connection one-forms as

$$\omega_{\mu\nu} = \frac{1}{2}(c_{\mu\nu\alpha} + c_{\mu\alpha\nu} - c_{\nu\alpha\mu})\, \hat{\theta}^\alpha, \tag{A.13}$$

as detailed in §14.6 of [74]. Proceeding this way, as first done in [49], or rewritten in our notation in eq. (5) of [44], the Einstein-Hilbert action becomes

$$S_{\rm EH} = -\frac{1}{16\pi G}\int dt d^3x \sqrt{\gamma}\left[-R[\gamma_{ij}] + 2\gamma^{ij}\partial_i\phi\partial_j\phi - \frac{1}{4}e^{4\phi}F_{ij}F_{kl}\gamma^{ik}\gamma^{jl}\right], \tag{A.14}$$

where $\gamma \equiv \det(\gamma_{ij})$, $F_{ij} \equiv \partial_i A_j - \partial_j A_i$, and $R[\gamma_{ij}]$ denotes the Ricci scalar for γ_{ij}.



(a) (b) (c) (d)

Figure B.1: The diagrams representing the 1PN interaction: (a) One scalar-graviton exchange, (b) One vector-graviton exchange, (c) Two scalar-graviton exchange, (d) One scalar-graviton exchange with a relativistic time correction. We also indicate the contribution to the power counting in the velocity v for all vertices.

B Evaluation of the 1PN interaction potential

In this Appendix we provide the details required to calculate the Feynman diagrams that make up the 1PN interaction potential, used in §5.2.

For that, at 1PN we can have both orders $\mathcal{O}(G^2v^0)$ and $\mathcal{O}(Gv^2)$. Dressing the bare topologies represented in Figure 5.1, we obtain 4 possible graphs that contribute to this order, as shown in Figure B.1.

The first diagram is calculated in an analogous way as the Newtonian case, but now taking the next-to-leading order in velocity of the scalar graviton Feynman rule, given in eq. (4.15). This will introduce a velocity in the calculation, which depends on time but not on the position, as it is dependent on the worldline parameter. Moreover, as opposed to the Newtonian case where the interaction was symmetric, since here we take the LO rule for one worldline and the NLO for the other, we have to add the same diagram with the worldlines interchanged, to explicitly make the interaction symmetric under exchange. It then evaluates as follows:

$$\begin{aligned}\text{Fig. } B.1(a) &= (-m_1)\int dt_1\, \phi(x_1)\,\frac{3}{2}\,v_1^2(t_1)\cdot(-m_2)\int dt_2\, \phi(x_2) + (1\leftrightarrow 2)\\ &= \int dt\,\frac{3}{2}(v_1^2+v_2^2)\,\frac{Gm_1m_2}{r},\end{aligned} \tag{B.1}$$

where the dependence of the velocity on time does not change the calculation, as we have again a trivial delta function in time.

The second diagram is calculated very similarly, but now using the rules for the gravitomagnetic vector. Hence, we use eq. (4.16) for the vector coupling to the mass and eq. (4.3) for the propagator of the vector field:

$$\begin{aligned}\text{Fig. } B.1(b) &= m_1\int dt_1\, A_i(x_1)\,v_1^i(t_1)\cdot m_2\int dt_2\, A_j(x_2)\,v_2^j(t_2)\\ &= -\int dt\,\frac{4Gm_1m_2}{r}\,\vec{v}_1\cdot\vec{v}_2.\end{aligned} \tag{B.2}$$

Again, the propagator results in a trivial delta function that can be integrated directly.

The third diagram already becomes more interesting, as we have the 2-graviton exchange given in eq. (4.18), which modifies the symmetry factor, trivial up to now. Schematically, writing the Feynman rules, we will have 2 possible identical contractions:

$$m_1\phi_1\phi_1 \cdot m_2\phi_2 \cdot m_2\phi_2' + m_1\phi_1\phi_1 \cdot m_2\phi_2 \cdot m_2\phi_2'. \tag{B.3}$$

To calculate the symmetry factor, we have to multiply the number of identical contractions with the symmetry coefficient of the contraction. For the latter, we have to expand the exponential of the action, from which we collect the graphs:

$$\begin{aligned} e^{S_{\rm eff}} &\sim e^{S_g+S_{(1)\rm pp}+S_{(2)\rm pp}} \\ &\sim \left[1+S_g+\frac{S_g^2}{2!}+\dots\right]\left[1+S_{(1)\rm pp}+\frac{S_{(1)\rm pp}^2}{2!}+\dots\right]\left[1+S_{(2)\rm pp}+\frac{S_{(2)\rm pp}^2}{2!}+\dots\right], \end{aligned} \tag{B.4}$$

where we remind that S_g represents the bulk action and $S_{(I)\rm pp}$ the point-particle action for each of the components of the binary, from §2.1.1. Then, we have to select the order that is specific for our diagram. In our case, we have no bulk self-interaction, we have one vertex in the worldline 1, and two vertices in the worldline 2. Since the two vertices in worldline 2 are identical[10], we have that the symmetry coefficient of the contraction in our diagram is

$$1\cdot S_{(1)\rm pp}\cdot\frac{S_{(2)\rm pp}^2}{2!} \quad\Rightarrow\quad \text{sym. coeff. of contraction} = 1\cdot 1\cdot\frac{1}{2!}=\frac{1}{2}, \tag{B.5}$$

and so the symmetry factor of the diagram is

$$\text{sym. factor} = 2\cdot\frac{1}{2}=1. \tag{B.6}$$

Then, it evaluates as follows, where now we have to include 2 different times for the different vertices in worldline 2, which we distinguish using a prime:

$$\begin{aligned} \text{Fig. } B.1(c) =& -\frac{1}{2}m_1\int dt_1\,\phi(x_1)\phi(x_1)\cdot(-m_2)\int dt_2\,\phi(x_2)\cdot(-m_2)\int dt_2'\,\phi(x_2') + (1\leftrightarrow 2) \\ =& -8\pi^2G^2m_1m_2^2\int dt\int_{\vec k}\frac{e^{i\vec k\cdot\vec r}}{\vec k^2}\int_{\vec k'}\frac{e^{i\vec k'\cdot\vec r}}{\vec k'^2}+(1\leftrightarrow 2) \\ =& -\int dt\,\frac{G^2m_1m_2}{2r^2}(m_1+m_2). \end{aligned} \tag{B.7}$$

Again, the delta functions of the two propagators are integrated directly, leading to an overall single time integral. Furthermore, instead of a 1-loop integral, we can observe that the two integrals decouple into 2 exchanges of one-graviton Fourier integrals. This factorization behavior is addressed in §5.3, as it will be a general feature of the theory.

[10] If the vertices were to be different, we would have that $S_{(2)\rm pp}^2 = (\text{vertex}_1+\text{vertex}_2)^2 \to 2\,\text{vertex}_1\cdot\text{vertex}_2$, so it would include an extra factor of 2.

For the last diagram, we have an additional complication, it being the time derivative in the relativistic time correction to the propagator of the scalar graviton, as given in eq. (4.8). This derivative will dramatically change the evaluation of the diagram, and we will have to leave all time dependence explicit. In particular, the following generic identity for the time derivative of a delta function, from eq. (4.36) of [13], will be useful:

$$\int dt_1 dt_2 \partial_{t_1} \delta(t_1 - t_2) f(t_1) g(t_2) = - \int dt_1 dt_2 \partial_{t_2} \delta(t_1 - t_2) f(t_1) g(t_2). \tag{B.8}$$

Following the previous discussion, in this case the symmetry factor is equal to 2, and the diagram evaluates as follows, where we denote the coordinates in the bulk using a prime:

$$\begin{aligned} \text{Fig. } B.1(d) =& \, 2\,(-m_1) \int dt_1\, \phi(x_1) \cdot \frac{1}{8\pi G} \int d^4x'\, (\partial_{t'}\phi(x'))(\partial_{t'}\phi(x')) \cdot (-m_2) \int dt_2\, \phi(x_2) \\ =& \, 4\pi G m_1 m_2 \int dt_1 \int d^4x' \int dt_2\, \partial_{t'}\delta(t_1 - t')\, \partial_{t'}\delta(t' - t_2) \\ & \times \int_{\vec{k}_1} \frac{e^{i\vec{k}_1 \cdot (\vec{x}_1(t_1) - \vec{x}')}}{\vec{k}_1^2} \int_{\vec{k}_2} \frac{e^{i\vec{k}_2 \cdot (\vec{x}' - \vec{x}_2(t_2))}}{\vec{k}_2^2}. \end{aligned} \tag{B.9}$$

Here it is important to remark that the position $\vec{x}'$ of the self-interaction in the bulk does not depend on t', because it does not follow a worldline trajectory. Moreover, one could wonder why does the derivative only act on the delta functions. This is because, going back to eq. (4.7), we see that in the full propagator the explicit time dependence goes with the delta function. Then, to evaluate the time derivatives we use eq. (B.8) to swap the derivative with respect to the bulk time t' to a derivative with respect to the worldline times t_i, and next we integrate by parts to remove the derivative from the delta function. Doing so, we obtain

$$\text{Fig. } B.1(d) = 4\pi G m_1 m_2 \int dt \int d^3\vec{x}'\, v_{1i}\, v_{2j} \int_{\vec{k}_1} k_1^i \frac{e^{i\vec{k}_1 \cdot (\vec{x}_1 - \vec{x}')}}{\vec{k}_1^2} \int_{\vec{k}_2} k_2^j \frac{e^{i\vec{k}_2 \cdot (\vec{x}' - \vec{x}_2)}}{\vec{k}_2^2}, \tag{B.10}$$

where we used that $\dot{\vec{x}}_I(t) = \vec{v}_I$. As we see, we can integrate over the position in the bulk, which leads to momentum conservation in the vertex:

$$\int d^3\vec{x}' e^{i(\vec{k}_2 - \vec{k}_1) \cdot \vec{x}'} = (2\pi)^3 \delta^3(\vec{k}_1 - \vec{k}_2), \tag{B.11}$$

to obtain that

$$\begin{aligned} \text{Fig. } B.1(d) =& \, 4\pi G m_1 m_2 \int dt\, v_{1i}\, v_{2j} \int_{\vec{k}} k^i\, k^j\, \frac{e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)}}{\vec{k}^4} \\ =& \int dt\, \frac{G m_1 m_2}{2r} (\vec{v}_1 \cdot \vec{v}_2 - \vec{v}_1 \cdot \vec{n}\; \vec{v}_2 \cdot \vec{n}), \end{aligned} \tag{B.12}$$

where $\vec{n} \equiv \vec{r}/r$ is the unit vector in the separation direction. The last integral is just related to the scalar master integral in eq. (5.3) by 2 spatial derivatives.

All in all, we obtain that the 1PN correction potential, which conforms with the first GR correction to Newtonian gravity, is given by

$$V_{1\text{PN}} = \frac{G m_1 m_2}{2r} \Big[-3(v_1^2 + v_2^2) + 7\vec{v}_1 \cdot \vec{v}_2 + \vec{v}_1 \cdot \vec{n}\; \vec{v}_2 \cdot \vec{n} \Big] + \frac{G^2 m_1 m_2}{2r^2}(m_1 + m_2), \tag{B.13}$$

which matches the well-known Einstein-Infeld-Hoffmann correction [55].

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