UNIVERSITY OF COPENHAGEN
FACULTY OF SCIENCE

Master's thesis

Roberto Forbicia León

On non-Lorentzian geometry and the weak field limit of non-relativistic gravity

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Supervisor: Niels A. Obers

Abstract

In this work we study the weak field limit of non-relativistic gravity (NRG). To this end, we review the covariant formulation of the latter as obtained from an appropriate large speed of light expansion of GR, as well as modern perspectives on non-Lorentzian geometry. We explore the two possible paths in the description of the weak field limit of NRG. The first one corresponds to a non-relativistic expansion of the well-known theory of linearised GR. We derive the resulting theories at LO and NLO, as well as the corresponding EOM. The second one amounts to a linearisation of the geometric fields of NRG around a flat NC background. We show explicitly that the two paths yield the same theory at LO, which suggests that our formulation renders the two approaches compatible. We argue that the weak field limit of NRG is already richer than Newtonian gravity in two senses: by allowing for small perturbations on the closedness of the clock-form and by allowing time-dependence. Finally, building on the knowledge provided by the recently discovered covariant formulation of Carroll gravity as obtained from an ultra-local expansion of GR, we propose an interpretation of a truncated sector of the NLO theory in the non-relativistic expansion of GR as the non-relativistic magnetic limit of the latter.

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Contents

Chapter 1

Introduction

1.1 From Galilei to Cartan

In 1632, G. Galilei became the first scientist to propose a universal principle of relativity. In his attempt of arguing against a stationary Earth in a geocentric model of the Universe and in favour of an Earth in motion around the Sun, he designed a thought experiment consisting of two observers 'in the main cabin below decks on some large ship' and able to carry out physical experiments when the ship is standing still. Then, he argued:

> *(...) When you have observed all these things carefully (though doubtless when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still.* [28]

As discussed in [14], Galilei did not mention what sort of transformations should relate different inertial reference frames to each other, but he did suggest that the laws of physics should be the same in all of them (at least those known by him at the time). In particular, he did not introduce what we now know as Galilean transformations.

Some years later, in 1687, I. Newton provided the foundations of classical mechanics in the famous *Principia*. In this work, Newton implemented Galilei's relativity principle in his laws of motion and of universal gravitation, together with an assumption on the absoluteness of time. From a modern perspective, it is now generally understood that by Galilean transformations we mean the transformations under which Newtonian mechanics is invariant. And it is in this sense that we say that the latter obeys a Galilean relativity principle.

Newtonian mechanics and its Galilean symmetry provided an appropriate description of the laws of nature until its incompatibility with Maxwell's theory of electrodynamics was noted by H. Lorentz in 1892 [64]. This realisation eventually led to the introduction of Lorentz transformations, which rendered electrodynamics compatible with the principle of relativity. In 1905, A. Einstein published his theory of Special Relativity (SR), where he elevated the principle of relativity to a postulate and further postulated the constancy of the speed of light in any inertial reference frame. He then derived Lorentz transformations as the only possible set of transformations relating two different inertial reference frames consistent with the two postulates. Subsequently, H. Minkowski introduced his eponymous spacetime in 1908, which provided a natural mathematical framework for SR. In this context, Lorentz transformations are simply the homogeneous transformations of the group of isometries of Minkowski's spacetime, i. e., the Poincaré group. Following the success of SR in the description of physical phenomena in the absence of gravity, Newtonian physics was eventually labelled as 'non-relativistic'. Of course, this is an abuse of language since the latter is still based on a Galilean relativity principle, as we mentioned. However, the interpretation of a theory being 'non-relativistic' in the sense of 'being invariant under Galilean transformations (or generalisations thereof)' is widely extended and ubiquitous in the literature, so we shall stick to it.

The question of how gravity could fit in this new landscape of physics was answered by Einstein himself in the theory of General Relativity (GR), published in 1915. The latter stands on two brilliant ideas: the equivalence principle and the statement that "gravity is geometry". The former implements SR locally and forces the geometry of spacetime to be Lorentzian, thus turning the previous statement into "gravity is Lorentzian geometry". It may not be obvious that these two ideas, yet closely related, are completely independent. Indeed, following Einsten's celebrated ideas, É. Cartan tried to formulate a geometric description of Newtonian gravity [16, 17]. His idea was that, in the same way that Einstein's equivalence principle leads to the notion of gravity as dynamical Lorentzian geometry, an equivalence principle based on local Galilean symmetry instead of local Lorentzian symmetry should lead to Newtonian gravity. The result of his work was a covariant formulation of the Poisson equation[1] and the introduction of Newton-Cartan (NC) geometry as the geometry arising from local Galilean symmetry, thus providing the appropriate geometrical framework for the study of non-relativistic physics. In order to make contact with the Newtonian notion of absolute time, Cartan restricted the geometry to be torsionfree. The geometric description of Newtonian gravity was subsequently developed following this approach in, e.g., [36, 20, 72, 26, 29].

1.2 Background and motivation

GR is written in terms of two fundamental constants of nature: Newton's gravitational constant G_N and the speed of light c. In particular, the local Lorentzian symmetry of spacetime according to GR is explicitly dependent on c, which enters as a parameter in the definition of Lorentz boosts. It is therefore interesting on its own right to study the two natural limits $c \to \infty$ and $c \to 0$. Being c a dimensionful quantity, a comment about what is meant by such two limits is in order. In the $c \to \infty$ case, what is meant is that we set $c = \hat{c}/\sqrt{\sigma}$ with σ a dimensionless parameter, and choose units in which $\hat{c} = 1$, so that $\sigma = 1/c^2$ and we can consider instead the $\sigma \to 0$ limit. Such a limit results in the vanishing of the slopes of the light cones, which are completely flattened out and define a causal structure that allows action at a distance. Similarly, Lorentz boosts become Galilei boosts and local Galilean symmetry is restored when $c \to \infty$, which is why we refer to the latter as the non-relativistic limit. In the $c \to 0$ case, what is meant is that we set $c = \hat{c}\epsilon$ with ϵ dimensionless, choose units such that $\hat{c} = 1$ and consider instead the $\epsilon \to 0$ limit. This is known as an ultra-local limit[2], because the slope of light cones becomes arbitrarily large and they collapse into a line, effectively removing interactions from the picture. The ultra-local

[1]We refer the reader to [2] for a brief and interesting review on this.

[2]The ultra-local limit should not be confused with the ultra-relativistic limit. The latter is taken by letting $c/v_c \to 1$, where v_c is a characteristic velocity of the problem in question. In the ultra-local limit, one takes instead $c/v_c \to 0$.

limit of a Lorentz boosts yields a Carroll boost, and the underlying local symmetry is then determined by the Carroll group[3], first introduced in 1965 by J.M. Lévy-Leblond [62]. The geometry resulting from this local symmetry is then called Carrollian or Carroll geometry.

In recent years, the interest in non-relativistic physics and NC geometry has resurged[4] following the discovery of two torsionful generalisations of the latter, leading to the notion of torsional Newton-Cartan (TNC) geometry. The first one, now referred to as type I TNC [43], was first found to arise as the boundary geometry in the context of Lifshitz holography [18, 19], and has been subsequently studied in, e.g., [51, 52]. This discovery, as well as earlier work on the covariant formulation of Post-Newtonian approximations of GR [21, 70], spurred the interest in exploring a non-relativistic expansion, i.e. an expansion around $\sigma = 0$, of the theory. Following the earlier work [25], an action principle for Newtonian gravity based on a new non-relativistic local symmetry was derived in [43]. The underlying geometry was proven to arise from a non-relativistic expansion of the Poincaré algebra and dubbed type II TNC geometry. The resulting non-relativistic theory of gravity goes beyond the Newtonian version by allowing for time dilation. In particular, it was shown in [44] that such non-relativistic theory passes the three classical tests of GR: perihelion precession, deflection of light and gravitational redshift, thus elucidating the difference between strong gravitational field effects and relativistic effects. All this knowledge crystallised in the work [46], where a covariant formulation of non-relativistic gravity (NRG) is obtained from a non-relativistic expansion of GR. The methods presented there have been subsequently used in [48] to obtain a covariant formulation of Carroll gravity from an ultra-local expansion of GR. Interestingly, the latter work also provides an interpretation of the Carroll electric and magnetic limits of GR (previously considered in [55, 69]) as, respectively, the leading order and a truncated sector of the next-to-leading order theories in the ultra-local expansion.

The motivations for the study of non-Lorentzian geometries such as NC or Carrollian geometry are manifold. From a mathematical perspective, non-Lorentzian geometries have a natural interpretation as G-structures ($G < \mathrm{GL}(D, \mathbb{R})$) on a manifold [33, 34]. This more general geometrical setting also provides meaningful insight on Lorentzian geometry through a better understanding of notions like torsion or non-metricity. Similarly, the study of dynamical non-Lorentzian geometries such as non-relativistic or Carroll gravity is interesting in its own right as they both emerge from relevant limits of a well-tested theory like GR. By focusing on the corresponding restricted settings, one can gain insight on the full theory. Moreover, there are many situations in which nature effectively behaves as non-relativistic. The study of NC geometry and non-relativistic field theory is also relevant in these contexts, which include condensed matter and biophysics (see, e.g., [57, 3]). In parallel, applications of Carroll symmetries have been found in black hole physics [68, 27] and, more recently, Carroll symmetry has been suggested to be relevant for the study of dark energy and inflation [24]. Finally, both type I and type II TNC geometry have been studied in the context of non-relativistic string theory in, e. g., [49, 9, 12].

The study of NRG is also motivated by its role in the landscape of $G_N c^{-1}\hbar$-physics. Indeed, in light of the recent developments, it seems that non-relativistic gravity has overcome Newtonian gravity as the correct theoretical framework to describe the $(G_N, 0, 0)$ corner of the Bronstein cube, shown in Fig. 1.1. It follows than a better understanding of this theory, as well as its eventual quantum version, opens up an unexplored path towards quantum gravity opposed to the usual attempts from the GR or QFT corners. Investigat-

[3]Named by Lévy-Leblond himself after L. Carroll because, in his owns words: 'the behaviour of a universe that would be governed by this group of invariance is reminiscent of that of *Alice in Wonderland*'.

[4]We refer the reader to the future work [67], for an overview of the historical development of NC geometry.

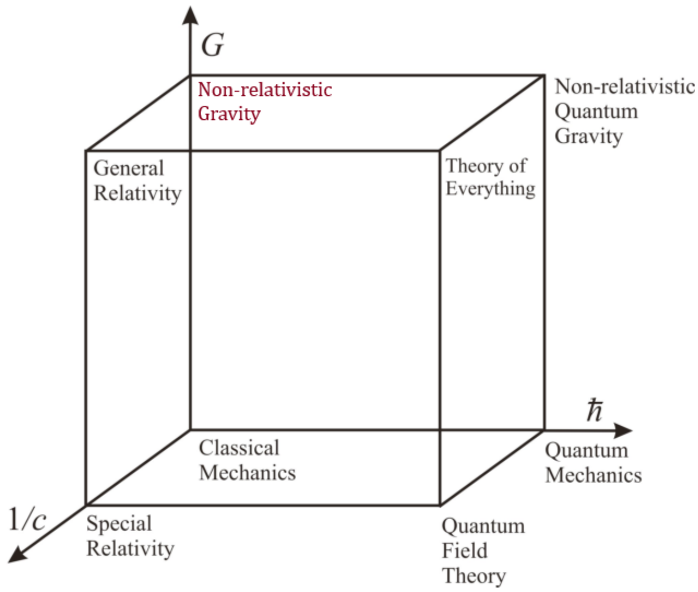

Figure 1.1: The Bronstein cube of physical theories.

ing the weak field limit of NRG is a natural step towards its better understanding, and therefore a (humble) contribution to this bigger picture. However, we believe the weak field limit of NRG to be of theoretical interest in its own right. Indeed, having established that strong gravitational field effects and relativistic effects are completely independent, it seems reasonable to consider a theory that describes gravity under the assumptions that

1. relativistic effects are small,
2. the gravitational field is weak.

This might sound just like Newtonian gravity at first sight, but the richness of NRG seems to be enough to go beyond the latter even in its weak field limit. This can happen in two directions: one is by allowing for small perturbations around the closedness of the clock-form, and the other by allowing such perturbations to be time-dependent. Finally, being the result of taking two limiting cases of GR (non-relativistic expansion and weak field limit), there are two natural routes towards its description. That is, depending on which of the two is considered first. Although expected to be equivalent, to show that explicitly in a formulation that renders the two descriptions compatible is also of theoretical interest. In particular, this involves carrying out a large speed of light expansion of the well known relativistic theory of linearised GR.

1.3 Outline

This thesis is structured as follows. In Chapter 2, we offer a review on Lorentzian geometry. This involves a discussion on the notions of torsion and non-metricity, a description of the vielbein formulation of Lorentzian geometry and GR and its obtention via gauging the Poincaré algebra. The vielbeine formulation is presented in two different ways: first through the introduction of non-coordinate bases in the tangent spaces of a Lorentzian manifold, and later through the more abstract notion of Lorentzian frame bundles. The latter is especially relevant for the study of non-Lorentzian geometry in Chapter 3, where we present a description of NC geometry and generalisations thereof, and of Carrollian geometry. Chapter 4 is devoted to the non-relativistic expansion of GR, which includes the PNR parametrisation and subsequent expansion of the underlying Lorentzian geometry as well as the expansion of the EH Lagrangian. More precisely, we review in detail how NC geometry arises as the underlying geometry of the LO theory, and how putting the theory on shell restricts the geometry to be twistless TNC geometry. We also show how the geometry of the NNLO theory corresponds to type II TNC, and how the latter can be obtained by gauging the level one expansion of the Poincaré algebra. For completeness, we also comment briefly on the NNLO theory. The chapter ends with the first original contribution of this work: the interpretation of a truncated sector of the NLO theory as the non-relativistic magnetic limit of GR, obtained in complete analogy with the Carroll magnetic limit of GR described in [48].

The rest of original work is presented in Chapter 5, devoted to the study of the weak field limit of NRG. The chapter starts with a discussion on the two natural routes towards its description, namely the non-relativistic expansion of linearised GR and the linearisation of NRG around a flat NC background. The first one requires what we have called a perturbative pre-non-relativistic (PPNR) parametrisation of the Fierz-Pauli Lagrangian describing linearised GR. The so-called PPNR fields provide an adequate starting point for the non-relativistic expansion of linearised GR, just as the PNR fields of [46] do for the full theory. The expansion gives rise to the notion of non-relativistic linearised GR. We obtain its Lagrangian and EOM at leading and next-to-leading orders. We also discuss how the gauge symmetry of linearised GR can be used to simplify the EOM of its non-relativistic version. In particular, we explore a PPNR version of the harmonic gauge condition wich upon expanding yields a considerable simplification of the EOM of the LO theory. Finally, we study the linearisation around a flat NC background of the LO theory resulting from the non-relativistic expansion of GR. We show that this theory is exactly the LO theory of non-relativistic linearised GR. We consider this to be a central result, and conjecture that the equivalence holds beyond LO. In particular, this result suggests that our formulation of non-relativistic linearised GR provides an adequate framework for the study of the weak field limit of NRG.

The last chapter is devoted to a discussion of the results and the potential interesting research directions that they open up.

Chapter 2

Lorentzian geometry revisited

In this chapter, we review some standard notions on differential and Lorentzian geometry. We aim to do so from a slightly more general perspective than the one provided by its usual study in the context of GR. This includes a discussion on the notions of torsion and non-metricity, which are usually overlooked due to the natural choice of the Levi-Civita connection, as well as the description of two alternative and closely related formulations of Lorentzian geometry. The first one corresponds to the vielbein formulation of GR, or in more general terms, the description of Lorentzian frame bundles. The second one is its formulation as a gauge theory of the Poincaré algebra. Both will prove to be essential for our subsequent study of non-Lorentzian geometries.

2.1 Curvature, torsion and non-metricity

Let M be a $(d+1)$-dimensional smooth manifold together with an affine connection $\tilde{\Gamma}$ with associated covariant derivative $\tilde{\nabla}$. If (U,φ) is a chart on M with coordinate functions $\{x^\mu\}_{\mu=0,\ldots,d}$, then $\tilde{\Gamma}$ is locally determined by the coefficients $\tilde{\Gamma}^\rho_{\mu\nu}$ through

$$\tilde{\nabla}_{\partial_\mu}\partial_\nu = \tilde{\Gamma}^\rho_{\mu\nu}\partial_\rho, \tag{2.1}$$

where $\{\partial_\mu\}_{\mu=0,\ldots,d}$ is the local coordinate basis for vector fields on U. We then define the *torsion tensor* of $\tilde{\Gamma}$ by[1]

$$\tilde{T}^\rho{}_{\mu\nu} := 2\tilde{\Gamma}^\rho_{[\mu\nu]}. \tag{2.2}$$

A connection is said to be *torsionful* (resp. *torsionless* or *torsionfree*) if it has non-vanishing (resp. vanishing) torsion. The Riemann curvature tensor is then defined by the action of the commutator of two covariant derivatives on any vector field X^ρ,

$$[\tilde{\nabla}_\mu,\tilde{\nabla}_\nu]X^\rho = -\tilde{R}_{\mu\nu\sigma}{}^\rho X^\sigma - \tilde{T}^\sigma{}_{\mu\nu}\tilde{\nabla}_\sigma X^\rho, \tag{2.3}$$

so that

$$\tilde{R}_{\mu\nu\sigma}{}^\rho := -\partial_\mu\tilde{\Gamma}^\rho_{\nu\sigma} + \partial_\nu\tilde{\Gamma}^\rho_{\mu\sigma} - \tilde{\Gamma}^\rho_{\mu\lambda}\tilde{\Gamma}^\lambda_{\nu\sigma} + \tilde{\Gamma}^\rho_{\nu\lambda}\tilde{\Gamma}^\lambda_{\mu\sigma}. \tag{2.4}$$

The expression (2.3) realises the intuitive notion that curvature manifests itself as the rotation experienced by a vector when parallel transported around a closed loop. But it

[1] Of course, these are actually just the components of the torsion tensor with respect to the same basis in which the coefficients $\tilde{\Gamma}^\rho_{\mu\nu}$ are defined. However, we shall throughout this work stick to the common practice of blurring the difference between objects and their components.

also provides insight on the geometric interpretation of torsion as the failure to close of the parallelogram formed after the parallel transport of two vectors along each other [58].

Let us now assume M to have a Lorentzian structure characterised by a Lorentzian metric tensor $g_{\mu\nu}$. We can then define contractions of the curvature tensor. In particular, the Ricci tensor is defined by

$$\tilde{R}_{\mu\nu} := \tilde{R}_{\mu\rho\nu}{}^{\rho}. \tag{2.5}$$

If the connection coefficients satisfy $\tilde{\Gamma}^{\rho}_{\rho\nu} = \partial_\nu f$ for some $f \in \mathcal{C}^\infty(M)$, then the antisymmetric part of the Ricci tensor is given by

$$2\tilde{R}_{[\mu\nu]} = -2\tilde{T}^{\lambda}{}_{\rho[\mu}\tilde{T}^{\rho}{}_{\nu]\lambda} + \tilde{T}^{\lambda}{}_{\mu\nu}\tilde{T}^{\rho}{}_{\lambda\rho} + \tilde{\nabla}_\mu \tilde{T}^{\rho}{}_{\nu\rho} - \tilde{\nabla}_\nu \tilde{T}^{\rho}{}_{\mu\rho} + \tilde{\nabla}_\rho \tilde{T}^{\rho}{}_{\mu\nu}. \tag{2.6}$$

It follows that the Ricci tensor is symmetric if the connection is torsionless.

With the introduction of the metric structure we can now consider how adapted to it the connection is. This is measured by the so-called *non-metricity tensor*, defined by

$$\tilde{Q}_{\rho\mu\nu} := \tilde{\nabla}_\rho g_{\mu\nu}. \tag{2.7}$$

Like curvature and torsion, non-metricity also has a geometric interpretation: it measures the variation in the norm of a vector when the latter is parallel transported. Indeed, assuming that X^μ is parallel transported along the coordinate directions, i. e. $\tilde{\nabla}_\rho X^\mu = 0$, we have

$$\tilde{\nabla}_\rho (g_{\mu\nu} X^\mu X^\nu) = \tilde{Q}_{\rho\mu\nu} X^\mu X^\nu. \tag{2.8}$$

Any connection with vanishing non-metricity is said to be *metric compatible*. The fundamental theory of semi-Riemannian geometry then states that there is a unique connection Γ that is both torsionless and metric compatible, known as the Levi-Civita connection. If we denote its associated covariant derivative by ∇, then we can write

$$T^{\rho}{}_{\mu\nu} := 2\Gamma^{\rho}_{[\mu\nu]} = 0, \tag{2.9a}$$

$$Q_{\rho\mu\nu} := \nabla_\rho g_{\mu\nu} = 0. \tag{2.9b}$$

The coefficients of Γ are the Christoffel symbols and are given in terms of the metric by

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \tag{2.10}$$

Because of its special features, the Levi-Civita is usually the preferred connection for the study of Lorentzian geometry. In particular, it is natural to take it as the origin of the affine vector space[2] of all possible affine connections on M, meaning that we can decompose any arbitrary connection $\tilde{\Gamma}$ according to [58]

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + \tilde{K}^{\rho}{}_{\mu\nu} + \tilde{L}^{\rho}{}_{\mu\nu}, \tag{2.11}$$

where

$$\tilde{K}^{\rho}{}_{\mu\nu} := \frac{1}{2}\tilde{T}^{\rho}{}_{\mu\nu} + \tilde{T}_{(\mu}{}^{\rho}{}_{\nu)}, \tag{2.12a}$$

$$\tilde{L}^{\rho}{}_{\mu\nu} := \frac{1}{2}\tilde{Q}^{\rho}{}_{\mu\nu} - \tilde{Q}_{(\mu}{}^{\rho}{}_{\nu)}, \tag{2.12b}$$

[2] As argued in [6], this is just the formalisation of the well-known fact that the difference of two connections is a tensor.

are the *contortion tensor* and the *disformation tensor*, respectively. Therefore, one is free to define any connection in order to introduce a notion of covariant differentiation in a Lorentzian manifold, but the choice of the Levi-Civita connection seems like a sensible one. As a dynamic theory of Lorentzian geometry, for example, GR is formulated in terms of the Levi-Civita connection. Ultimately, this is a consequence of the fact that GR geometrises gravity by identifying it with curvature, while removing torsion and non-metricity from the picture. As recently determined in [58], however, it is possible to formulate GR by removing curvature from the picture and attributing the gravitational effects to the torsion and/or non-metricity of flat spacetimes, thus giving rise to the so-called *geometrical trinity of gravity*.

2.2 Vielbein formulation

In the standard formulation of GR, one takes the Lorentzian metric tensor $g_{\mu\nu}$ to be the dynamical variable. However, this is not the most adequate framework to carry out the non-relativistic and ultra-local expansions of the theory. Instead, we shall make use of the vielbein (or frame) formulation of Lorentzian geometry and take the relativistic vielbeine to be the dynamical variables of GR. Such a formulation has been extensively studied in the literature and can be found in most textbooks on the subject, e.g. [15, 66], which are the main references for this section. Our goal is to review its most important aspects, for the sake of completeness. In passing, we shall also review some standard definitions on differential geometry and set some notation.

Let M be a $(d+1)$-dimensional smooth manifold and $p \in M$. We denote the tangent and cotangent bundles by TM and T^*M, respectively. Given a local chart (U, φ) on M with coordinate functions $\{x^\mu\}_{\mu=0,\dots,d}$ such that $p \in U$, each fibre T_pM of TM has a natural basis

$$\{\partial_\mu|_p\}_{\mu=0,\dots,d}, \tag{2.13}$$

given by the vectors tangent to M at p in the x^μ coordinate direction. Similarly, each fibre T_p^*M of T^*M has a natural basis

$$\{dx^\mu|_p\}_{\mu=0,\dots,d}, \tag{2.14}$$

given by the gradients of the coordinate functions. In particular, this is the dual basis of (2.13), in the sense that

$$dx^\mu|_p(\partial_\nu|_p) = \frac{\partial x^\mu}{\partial x^\nu}\Big|_p = \delta^\mu_\nu. \tag{2.15}$$

For $r, s \in \mathbb{N}$, we can also consider the more general tensor bundle $T^{r,s}(M)$, whose sections are tensor fields on M of type (r,s). In this case, each fibre $\bigotimes^r T_pM \otimes \bigotimes^s T_p^*M$ has a natural basis given by

$$\left\{\partial_{\mu_1}\big|_p \otimes \cdots \otimes \partial_{\mu_r}\big|_p \otimes dx^{\nu_1}\big|_p \otimes \cdots \otimes dx^{\nu_s}\big|_p\right\}_{\mu_i,\nu_j=0,\dots,d},$$

with $i = 1, \dots, r$ and $j = 1, \dots, s$.

It follows now that if X is a vector field on U, then we can write it as

$$X = X^\mu \partial_\mu, \tag{2.16}$$

where ∂_μ is the vector field on U assigning to every point $q \in U$ the tangent vector $\partial_\mu|_q \in T_qM$. Likewise, given a one-form ω on U, we can write it as

$$\omega = \omega_\mu dx^\mu, \tag{2.17}$$

where dx^μ is the one-form assigning to every point $q \in U$ the linear form $dx^\mu|_q \in T_q^* M$. More generally, any tensor field T on U of type (r, s) can be written as

$$T = T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_r} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_s}. \tag{2.18}$$

As mentioned before, we will usually refer to a vector field X as X^μ, to a one-form ω as ω_μ, and so on for tensors of any type.

The choice of the bases (2.13) and (2.14) is very natural and convenient for most calculations. In order to simplify the notation, let us rename the corresponding basis vectors by

$$\vec{e}_{(\mu)} := \partial_\mu|_p, \qquad \vec{\theta}^{(\mu)} := dx^\mu|_p. \tag{2.19}$$

The indices in parenthesis here are meant to stress that these label the different basis vectors, instead of denoting its components. Nothing prevents us, however, from choosing different bases $\{\vec{e}_{(A)}\}_{A=0,\dots,d}$ and $\{\vec{\theta}^{(A)}\}_{A=0,\dots,d}$ for T_pM and T_p^*M, respectively, where the use of Latin indices stresses the fact that these new bases are not related to any coordinate system. For this reason, they are usually referred to as *non-coordinate bases.* We shall still require them to be dual to each other by

$$\vec{\theta}^{(A)}(\vec{e}_{(B)}) = \delta^A_B. \tag{2.20}$$

In both cases, the change from coordinate to non-coordinate bases and vice versa will be realised by general linear transformations and their inverses. In particular, we have

$$\vec{e}_{(\mu)} = e_\mu{}^A \vec{e}_{(A)}, \qquad \vec{e}_{(A)} = e^\mu{}_A \vec{e}_{(\mu)}, \tag{2.21}$$

where the components $e_\mu{}^A$ form a matrix of $\mathrm{GL}(d+1, \mathbb{R})$ whose inverse has components $e^\mu{}_A$. It follows that

$$e^\mu{}_A e_\nu{}^A = \delta^\mu_\nu, \qquad e_\mu{}^A e^\mu{}_B = \delta^A_B. \tag{2.22}$$

Regarding the bases for the cotangent space, we have as a consequence of (2.20):

$$\vec{\theta}^{(\mu)} = e^\mu{}_A \vec{\theta}^{(A)}, \qquad \vec{\theta}^{(A)} = e_\mu{}^A \vec{\theta}^{(\mu)}. \tag{2.23}$$

Using (2.21), (2.23) and linearity we can easily relate the components in the non-coordinate and the coordinate basis of any vector or form. The same applies for tensors of mixed indices and/or higher rank. For instance, for a $(1,1)$-tensor

$$T = T^\mu{}_\nu \vec{e}_{(\mu)} \otimes \vec{\theta}^{(\nu)} = T^A{}_B \vec{e}_{(A)} \otimes \vec{\theta}^{(B)},$$

we have

$$T^A{}_B = e_\mu{}^A T^\mu{}_B = e^\nu{}_B T^A{}_\nu = e_\mu{}^A e^\nu{}_B T^\mu{}_\nu. \tag{2.24}$$

Up to this point, we have assumed M to be just a smooth manifold with no metric structure, and there seemed to be no reason as to introduce non-coordinate bases. Let us now consider M to be endowed with a Lorentzian metric structure given by the metric tensor $g_{\mu\nu}$, that turns every tangent space into an inner product space. It is then clear that orthonormal bases offer a preferable choice of non-coordinate bases. The requirement that $\{\vec{e}_{(A)}\}_{A=0,\dots,d}$ be orthonormal reads

$$g_{\mu\nu} e^\mu{}_A e^\nu{}_B = \eta_{AB}, \tag{2.25}$$

or equivalently

$$g_{\mu\nu} = e_\mu{}^A e_\nu{}^B \eta_{AB}, \tag{2.26}$$

where $\eta_{AB} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric on T_pM. The latter can be used to raise and lower non-coordinate indices at will, while coordinate indices are raised and lowered with $g_{\mu\nu}$ as usual. Notice that the associated dual basis in T_p^*M defined by (2.20) is then also orthonormal. The orthonormal basis $\{\vec{e}_{(A)}\}_{A=0,\ldots,d}$ is then called a *vielbein* (from the German for "many legs"). In four dimensions, it is sometimes called a *tetrad* or a *vierbein* (from the German for "four legs"). For simplicity and following the literature, we shall stretch this definition and refer to the components $e_\mu{}^A$ as the vielbein or vielbeine (in plural), and to $e^\mu{}_A$ as the inverse vielbein(e)[3]. In this way, we can interpret the vielbeine as components of a $(1,1)$-tensor

$$e = e_\mu{}^A dx^\mu \otimes \vec{e}_{(A)}. \tag{2.27}$$

What we have done so far applies only for the tangent and cotangent spaces at $p \in M$, even though we dropped it from the notation in (2.19). However, the usual pointwise assignation (the same we discussed before for the coordinate vector field ∂_μ and one-form dx^μ) allows to define *vielbeine fields* and *inverse vielbeine fields* in a neighbourhood U of p, serving as a local basis for tensor fields on U. Accordingly, in what follows we shall pay little or no attention to the difference between tensors and vielbeine at T_pM and tensor fields and vielbeine fields at U.

With the introduction of new bases for vector fields and one-forms comes the study of the transformation properties of different objects under a change of such bases. We are familiar with the tensor transformation law following a general coordinate transformation, but now we are dealing with non-coordinate bases that we can change independently of the coordinates. Since we are interested in the new basis being orthonormal as well, we shall require the orthonormality condition (2.25) to be preserved. Therefore, we consider changes of basis of the form

$$\vec{e}_{(A)} \to \vec{e}_{(A')} = \vec{e}_{(A)} (\Lambda^{-1})^A{}_{A'}(x), \tag{2.28a}$$
$$\vec{\theta}^{(A)} \to \vec{\theta}^{(A')} = \Lambda^{A'}{}_A(x) \vec{\theta}^{(A)}. \tag{2.28b}$$

where $\Lambda^A{}_{A'}(x)$ is a spacetime dependent Lorentz transformation, thus preserving the metric η_{AB} at each point. The vielbeine and their inverses then transform as

$$e_\mu{}^A \to e_\mu{}^{A'} = \Lambda^{A'}{}_A e_\mu{}^A, \qquad e^\mu{}_A \to e^\mu{}_{A'} = (\Lambda^{-1})^A{}_{A'} e^\mu{}_A \tag{2.29}$$

We can perform these transformations at each point in space, hence we call them *local Lorentz transformations* (LLTs). Besides these, we still have the freedom to change coordinates by means of *general coordinate transformations* (GCTs). Therefore, if we consider a tensor $T^{A\mu}{}_{B\nu}$ with both coordinate and non-coordinate indices, its transformation law when performing simultaneously a LLT and a GCT is given by

$$T^{A'\mu'}{}_{B'\nu'} = \Lambda^{A'}{}_A \frac{\partial x^{\mu'}}{\partial x^\mu} (\Lambda^{-1})^B{}_{B'} \frac{\partial x^\nu}{\partial x^{\nu'}} T^{A\mu}{}_{B\nu}, \tag{2.30}$$

[3]There is no common agreement in the literature regarding the terminology for the vielbeine and their inverses. In this regard, we follow [15] but also [46, 48] and most works in the context of non-relativistic and ultra-local expansions of GR. However, some references such as [66] refer to what we here call the vielbeine as the inverse vielbeine, and vice versa.

which generalises to any general tensor in a straightforward way.

Let us now see how the use of a non-coordinate basis affects the usual notion of covariant differentiation with respect to a coordinate basis. In this case, one introduces a covariant derivative operator $\mathcal{D}$ associated to an affine connection with coefficients $\tilde{\Gamma}^\rho_{\mu\nu}$ on the coordinate basis $\{\partial_\mu\}_{\mu=0,\dots,d}$ such that

$$\mathcal{D}_{\partial_\mu}\partial_\nu \equiv \tilde{\Gamma}^\rho_{\mu\nu}\partial_\rho. \tag{2.31}$$

One can then evaluate the same covariant derivative in the non-coordinate basis to obtain the coefficients $\tilde{\Gamma}^C_{AB}$ of $\tilde{\Gamma}$ with respect to such basis, given by

$$\mathcal{D}_{\vec{e}_{(A)}}\vec{e}_{(B)} \equiv \tilde{\Gamma}^C_{AB}\vec{e}_{(C)}. \tag{2.32}$$

We now introduce the *spin connection* as the spacetime one-form $\omega^A{}_B$ defined by

$$\omega^A{}_B := \tilde{\Gamma}^A_{CB}\vec{\theta}^{(C)} = \tilde{\Gamma}^A_{CB}e_\mu{}^C dX^\mu, \tag{2.33}$$

so that

$$\omega_\mu{}^A{}_B = \bar{\Gamma}^A_{CB}e_\mu{}^C. \tag{2.34}$$

It transforms inhomogeneously under LLTs according to

$$\omega^A{}_B \to \omega^{A'}{}_{B'} = \Lambda^{A'}{}_A\omega^A{}_B(\Lambda^{-1})^B{}_{B'} + \Lambda^{A'}{}_C(\mathrm{d}\Lambda^{-1})^C{}_{B'} \tag{2.35}$$

or equivalently,

$$\omega_\mu{}^A{}_B \to \omega_\mu{}^{A\prime}{}_{B'} = \Lambda^{A'}{}_A\omega_\mu{}^A{}_B(\Lambda^{-1})^B{}_{B'} + \Lambda^{A'}{}_C\partial_\mu(\Lambda^{-1})^C{}_{B'}. \tag{2.36}$$

Under an infinitesimal LLT

$$\Lambda^A{}_B = \delta^A_B + \lambda^A{}_B + \mathcal{O}(\lambda^2), \tag{2.37}$$

the transformation law then reads

$$\delta_{\mathrm{LLT}}\omega^A{}_B = \lambda^A{}_C\omega^C{}_B - \lambda^C{}_B\omega^A{}_C - \mathrm{d}\lambda^A{}_B. \tag{2.38}$$

We can now take covariant derivatives of tensors with mixed indices. For example, for a tensor $T^{A\mu}{}_{B\nu}$ we can write

$$\mathcal{D}_\rho T^{A\mu}{}_{B\nu} = \partial_\rho T^{A\mu}{}_{B\nu} + \omega_\rho{}^A{}_C T^{C\mu}{}_{B\nu} + \tilde{\Gamma}^\mu_{\lambda\rho}T^{A\lambda}{}_{B\nu} - \omega_\rho{}^C{}_B T^{A\mu}{}_{C\nu} - \tilde{\Gamma}^\lambda_{\nu\rho}T^{A\mu}{}_{B\lambda}. \tag{2.39}$$

The transformation law (2.35) results in the covariant derivative $\mathcal{D}$ transforming homogeneously under both GCTs and LLTs. The spin connection thus takes care of correcting the extra non-tensorial terms that appear due to the partial differentiation of objects with non-coordinate indices, in the same way that the affine connection does so for objects with coordinate indices. Of course, any tensor must be independent of the basis we use to express it. For instance, if $X = X^\mu\partial_\mu = X^A\vec{e}_{(A)}$ is a vector field, then we require that

$$\mathcal{D}X = (\mathcal{D}_\rho X^\mu)\, dx^\rho \otimes \partial_\mu = \left(\mathcal{D}_\mu X^A\right) dx^\rho \otimes \vec{e}_{(A)}. \tag{2.40}$$

This requirement establishes a relation between the spin connection, the affine connection and the vielbeine, that can be written down as

$$\mathcal{D}_\mu e_\nu{}^A = \partial_\mu e_\nu{}^A - \tilde{\Gamma}^\rho_{\mu\nu}e_\rho{}^A + \omega_\mu{}^A{}_B e_\nu{}^B = 0, \tag{2.41}$$

and is known as the *vielbein postulate.* A bit of manipulation allows to show that this implies

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = e^{\rho}{}_{A}\partial_{\mu}e_{\nu}{}^{A} + e^{\rho}{}_{A}e_{\nu}{}^{B}\omega_{\mu}{}^{A}{}_{B}, \tag{2.42}$$

or equivalently,

$$\omega_{\mu}{}^{A}{}_{B} = e_{\rho}{}^{A}e^{\nu}{}_{B}\tilde{\Gamma}^{\rho}_{\mu\nu} - e^{\nu}{}_{B}\partial_{\mu}e_{\nu}{}^{A}. \tag{2.43}$$

An advantage of the vielbein formulation is that we can think of some tensors as tensor-valued differential forms. For instance, a tensor $F_{\mu\nu}{}^{AB}{}_{C}$ that is antisymmetric in μ and ν can be interpreted as a $(2,1)$-tensor-valued two-form and a tensor $V_{\mu}{}^{A}$ can be thought of as a vector-valued one-form. It is customary then to suppress indices on differential forms when it is understood from the context, and write simply $F^{AB}{}_{C}$ or V^{A}. For instance, for the vielbeine and the spin connection we may write

$$e^{A} = e_{\mu}{}^{A}dx^{\mu}, \tag{2.44}$$

Notice that e^{A} is what we previously called $\vec{\theta}^{(A)}$. The vielbeine are in this sense vector valued one-forms, and $\omega^{A}{}_{B}$ is a one-form but it is not tensor valued because it transforms inhomogeneously under LLTs. This interpretation is useful because we can now take exterior derivatives and products of tensor-valued forms. Doing this will yield new differential forms under GCTs but having in general a non-tensorial character under LLTs. For example, the object

$$(\mathrm{d}V)_{\mu\nu}{}^{A} = 2\partial_{[\mu}V_{\nu]}{}^{A} \tag{2.45}$$

transforms as a two-form under GCTs but not as a vector under LLTs. There is however a natural way to remedy this by an appropriate use of the spin connection and its non-tensorial character. Indeed, one can check that the object

$$(dV)_{\mu\nu}{}^{A} + (\omega \wedge V)_{\mu\nu}{}^{A} = 2\partial_{[\mu}V_{\nu]}{}^{A} + 2\omega_{[\mu}{}^{A}{}_{B}V_{\nu]}{}^{B}, \tag{2.46}$$

does transform covariantly under both GCTs and LLTs.

This formulation is in particular very adequate for the description of the torsion and curvature tensors associated to $\tilde{\Gamma}$. Indeed, using their antisymmetry properties, the former can be thought of as a vector-valued two-form $T_{\mu\nu}{}^{A}$ and the latter as a $(1,1)$-tensor-valued two-form $R^{A}{}_{B\mu\nu}$. Their definitions in (2.2) and (2.4) can be proven to be equivalent to

$$T^{A} = \mathrm{d}e^{A} + \omega^{A}{}_{B} \wedge e^{B}, \tag{2.47a}$$

$$R^{A}{}_{B} = \mathrm{d}\omega^{A}{}_{B} + \omega^{A}{}_{C} \wedge \omega^{C}{}_{B}, \tag{2.47b}$$

known as the *Cartan structure equations.*

The results above hold for an arbitrary affine connection, and in particular for torsionful and not metric compatible ones. Let us now restrict our attention to the Levi-Civita connection Γ, associated to which there is a covariant derivative ∇. If we express the metric compatibility condition (2.9b) in the non-coordinate basis we find

$$0 = \nabla_{\mu}\eta_{AB} = \partial_{\mu}\eta_{AB} - \omega_{\mu}{}^{C}{}_{A}\eta_{CB} - \omega_{\mu}{}^{C}{}_{B}\eta_{AC} = -\omega_{\mu AB} - \omega_{\mu BA}. \tag{2.48}$$

Therefore, metric compatibility is equivalent to the antisymmetry of the spin connection in its non-coordinate indices:

$$\omega_{\mu AB} = -\omega_{\mu BA}. \tag{2.49}$$

The torsionlessness of Γ amounts to setting (2.47a) to zero. The spin connection is then determined completely by the vielbeine through

$$\omega^A{}_B \wedge e^B = -\mathrm{d}e^A, \tag{2.50}$$

which can be solved using (2.49).

At this point, it is clear that the vielbein formulation of Lorentzian geometry can be used to describe GR by using the relation (2.26) between the metric tensor and the vielbeine and considering the latter to be the dynamical fields of the theory. A comment regarding the counting of degrees of freedom in each approach is in place [66]. Indeed, putting $D = d+1$, the metric tensor $g_{\mu\nu}$ has $D(D+1)/2$ degrees of freedom, while the vielbeine $e_\mu{}^A$ have D^2 degrees of freedom. Of course, there is a redundancy in their definition, as all non-coordinate bases related by LLTs yield the same metric tensor. But the dimension of the Lorentz group $\mathrm{SO}(1,d)$ is $D(D-1)/2 = D^2 - D(D+1)/2$, which is precisely the difference between the degrees of freedom of $e_\mu{}^A$ and $g_{\mu\nu}$.

2.3 Lorentzian frame bundles

The discussion carried out above in order to present the vielbein formulation of Lorentzian geometry and GR hides some deep mathematical meaning. We thus wish to address the same issue from a more abstract perspective, following two main motivations. On one hand, we expect this to provide some valuable insight. On the other hand, this more general approach will later serve our purpose of exploring geometries beyond the Lorentzian (and semi-Riemannian) case. In particular, we shall come back to it in Section 3.2. The main references for this section are [40, 61]

2.3.1 Tangent bundle

Let M be a D-dimensional smooth manifold and

$$TM \xrightarrow{\pi} M,$$

its tangent bundle, whose fibres at each point $p \in M$ are given by

$$\pi^{-1}(\{p\}) \cong T_pM. \tag{2.51}$$

Being a vector bundle, TM admits a covering set $\{(U_k, \varphi_k)\}_{k\in K}$ of local trivialisations such that for any U_i, U_j $(i, j \in K)$ with $U_i \cap U_j \neq \emptyset$, we have the following commutative diagram

$$\begin{tikzcd}
(U_i \cap U_j) \times \mathbb{R}^D \arrow[r, "\varphi_{ij}"] \arrow[d, "\varphi_j"] & (U_i \cap U_j) \times \mathbb{R}^D \arrow[dl, "\varphi_i"] \\
\pi^{-1}(U_i \cap U_j), &
\end{tikzcd}$$

where

$$\varphi_{ij} := \varphi_i{}^{-1} \circ \varphi_j : (U_i \cap U_j) \times \mathbb{R}^D \longrightarrow (U_i \cap U_j) \times \mathbb{R}^D \tag{2.52}$$

is a smooth map given by

$$\varphi_{ij}(p, x) = (p, t_{ij}(p)x)\,. \tag{2.53}$$

The smooth map

$$t_{ij} : U_i \cap U_j \longrightarrow \mathrm{GL}(D, \mathbb{R}).$$

is called a *transition function* and $\mathrm{GL}(D,\mathbb{R})$ is said to be the *structure group* of TM[4], which weaves the fibres together to form the bundle structure. Another way of looking at this is the following. Let (U,ϕ) and (V,ψ) be overlapping charts for M with respective coordinate functions $\{x^\mu\}_{\mu=0,\ldots,d}$ and $\{y^\mu\}_{\mu=0,\ldots,d}$ such that $p \in U \cap V$. Then a vector $v \in T_pM$ has two coordinate representations

$$v = v^\mu \frac{\partial}{\partial x^\mu}\bigg|_p = \tilde{v}^\mu \frac{\partial}{\partial y^\mu}\bigg|_p, \tag{2.54}$$

that are related by

$$\tilde{v}^\nu = G^\nu{}_\mu v^\mu, \tag{2.55}$$

where $G^\nu{}_\mu = \frac{\partial y^\nu}{\partial x^\mu}\big|_p$ are the components of a non-singular matrix, so that

$$G \in \mathrm{GL}(D, \mathbb{R}).$$

Therefore, fibre coordinates are rotated by an element of the structure group $\mathrm{GL}(D,\mathbb{R})$ every time we change the coordinates on the manifold.

2.3.2 Frame bundle

For each $p \in M$ we define a *frame at* p as an ordered basis for the vector space T_pM, and we denote by F_pM the set of all frames at p. The basis vectors of a frame $u_p = \{e_{(A)}(p)\}_{A=0,\ldots,d}$ at p can be expressed in terms of the usual coordinate basis by

$$e_{(A)}(p) = e^\mu{}_A \partial_\mu\big|_p, \qquad (e^\mu{}_A) \in \mathrm{GL}(D,\mathbb{R}). \tag{2.56}$$

In some cases, however, it is more convenient to think of a frame u at p as a linear isomorphism

$$u_p : \mathbb{R}^D \longrightarrow T_pM, \qquad v = (v^0, \ldots, v^d) \longmapsto u_p(v) := v^A e_{(A)}(p). \tag{2.57}$$

The general linear group $\mathrm{GL}(D,\mathbb{R})$ acts transitively on F_pM by change of basis

$$F_pM \times \mathrm{GL}(D,\mathbb{R}) \longrightarrow F_pM, \qquad (u, g) \longmapsto u \circ g. \tag{2.58}$$

In particular,

$$F_pM \cong \mathrm{GL}(D,\mathbb{R})$$

as topological spaces.

We can now introduce the *(tangent) frame bundle* FM of M. As a set, it is defined as the disjoint union

$$FM := \bigsqcup_{p\in M} F_pM. \tag{2.59}$$

There is a natural projection $\pi_F : FM \to M$ such that

$$\pi_F^{-1}(\{p\}) \cong F_pM, \tag{2.60}$$

[4]More, generally, $\mathrm{GL}(D,\mathbb{R})$ is the structure group of any vector bundle up to reductions thereof, as we shall see later.

and a fibre-preserving $\mathrm{GL}(D,\mathbb{R})$-action given by (2.58), whose orbits are just the fibres themselves. Endowing FM with the appropriate smooth bundle structure, all the above implies that FM becomes a principal $\mathrm{GL}(D,\mathbb{R})$-bundle associated to TM. In particular, FM and TM have the same structure group. Finally, let us stress that the frame bundle construction is completely general to any vector bundle.

The (local) sections of FM are called *(local) frame fields* (or just frames when there is no need to distinguish them from frames at a point). Given $V \subset M$ open, for instance, a local frame field u at V is a smooth map

$$u : V \longrightarrow FM, \qquad p \longmapsto (p, u_p),$$

where u_p is the frame at p defined in (2.57). Notice that local frames define a local trivialisations of the tangent bundle. Indeed, with the notation above, the smooth map

$$\varphi_u : V \times T_pM \longrightarrow \pi^{-1}(V), \qquad (p, v) \longmapsto (p, u_p(v)).$$

is a local trivialisation for the neighbourhood V.

2.3.3 Reduction of the structure group

Let us now consider how the situation above changes when we endow the manifold M with a metric structure. In particular, we consider a Lorentzian metric tensor g on M, but we stress that the following discussion generalises naturally to any semi-Riemannian metric structure, as we shall comment on later in Section 3.2. For every $p \in M$, the metric tensor g induces a Lorentzian scalar product g_p in the tangent space T_pM, which turns the latter into an inner product space with a well-defined notion of orthonormality. This allows to define an *orthonormal frame at p* as an ordered g_p-orthonormal basis for the inner product space T_pM. We can then reproduce the frame bundle construction that we discussed above but for orthonormal frames only, with almost no changes. In particular, we can write an orthonormal frame $u_p = \{e_{(A)}(p)\}_{A=0,\ldots,d}$ at p as in (2.56) and (2.57), and consider the set $\hat{F}_pM$ of orthonormal frames at p. The main difference comes when considering changes of bases, since in this case orthonormality must be preserved. This means that we still have a natural transitive right action on $\hat{F}_pM$, but by $\mathrm{O}(1,d)$ instead of the whole general linear group:

$$\hat{F}_pM \times \mathrm{O}(1,d) \longrightarrow \hat{F}_pM, \qquad (u, g) \longmapsto u \circ g, \tag{2.61}$$

so that

$$\hat{F}_pM \cong \mathrm{O}(1,d),$$

as topological spaces. We shall also introduce the *orthogonal frame bundle* $\hat{F}M$ of M in analogy with (2.59) and (2.60), whose fibres will be given at every $p \in M$ by $\hat{F}_pM$ and whose (local) sections will be *(local) orthonormal frame fields.* With the appropriate smooth bundle structure, $\hat{F}M$ then becomes a principal $\mathrm{O}(1,d)$-bundle associated to TM. Such a structure coincides with the natural one inherited from FM as a subspace, turning $\hat{F}M$ into a principal $\mathrm{O}(1,d)$-subbundle of FM. In this case, we say that $\hat{F}M$ is an $\mathrm{O}(1,d)$*-reduction of* FM, and an $\mathrm{O}(1,d)$*-structure on* M.

As a consequence of the existence of the Lorentzian metric structure, the structure group of TM is reduced too:

$$\mathrm{GL}(D,\mathbb{R}) \longrightarrow \mathrm{O}(1,d). \tag{2.62}$$

Indeed, if we now choose a covering set $\{(U_k, \varphi_k)\}_{k\in K}$ of local trivialisations given by local frame fields, then the transition functions t_{ij} $(i, j \in K)$ take values in the the Lorentz group in $(d+1)$-dimensions,

$$t_{ij} : U_i \cap U_j \longrightarrow \mathrm{O}(1, d).$$

To see this, consider local trivialisations (U_i, φ_i) and (U_j, φ_j) such that $p \in U_i \cap U_j$, where φ_i, φ_j are associated, respectively, to the local frames $\tilde{u}$ and u pointwise defined by

$$\tilde{u}_p : \mathbb{R}^D \longrightarrow T_pM, \qquad v \longmapsto \tilde{u}_p(v) = v^A \tilde{e}_{(A)}(p), \tag{2.63}$$

and

$$u_p : \mathbb{R}^D \longrightarrow T_pM, \qquad v \longmapsto u_p(v) = v^A e_{(A)}(p). \tag{2.64}$$

The two frames are related by a Lorentz transformation $\Lambda \in \mathrm{O}(1, d)$ as

$$e_{(A)}(p) = \Lambda^B{}_A \tilde{e}_{(B)}(p), \qquad \tilde{e}_{(A)}(p) = (\Lambda^{-1})^B{}_A e_{(B)}(p). \tag{2.65}$$

It follows that given a certain vector $w = w^A e_{(A)}(p) = \tilde{w}^A \tilde{e}_{(A)}(p) \in T_pM$, its components in the two frames are related by

$$w^A = (\Lambda^{-1})^A{}_B \tilde{w}^B, \qquad \tilde{w}^A = \Lambda^A{}_B w^B. \tag{2.66}$$

Note how the equality in the right is completely analogous to (2.55), but with $\Lambda \in \mathrm{O}(1, d)$ in this case. This is of course related to the reduction of the structure group of TM. Indeed, taking all this into account the smooth map φ_{ij} in (2.52) is in this case given by

$$\varphi_{ij}(p, v) \equiv {\varphi_i}^{-1}(\varphi_j(p, x)) = {\varphi_i}^{-1}\left(p, v^A e_{(A)}(p)\right) = {\varphi_i}^{-1}\left(p, \tilde{v}^A \tilde{e}_{(A)}(p)\right) = (p, \tilde{v}), \tag{2.67}$$

with

$$\tilde{v} = \Lambda v, \qquad \Lambda \in \mathrm{O}(1, d), \tag{2.68}$$

as follows from (2.66). Comparing to (2.53), we see that the covering set $\{(U_k, \varphi_k)\}_{k\in K}$ of TM by local orthonormal frames has $\mathrm{O}1, d)$-valued transition functions.

Notice that we could have also assumed M to be oriented, in which case the frame bundle FM admits an $\mathrm{SO}(1, d)$-reduction given by the bundle of positively-oriented orthogonal frames. Indeed, all the arguments above still apply simply by replacing $\mathrm{O}(1, d)$ with $\mathrm{SO}(1, d)$, and in particular the structure groups is then reduced by

$$\mathrm{GL}(D, \mathbb{R}) \longrightarrow \mathrm{SO}(1, d).$$

We can at this point make contact with the vielbein formulation of GR studied in Section 2.2. What we then introduced as orthonormal non-coordinate bases are (positively-oriented) orthonormal frames in this context. Moreover, nothing prevents us from writing Lorentzian frames $u = \{e_{(A)}\}_{A=0,\ldots,d}$ as in (2.56), and identify

$$({e_\mu}^A) \in \mathrm{GL}(D, \mathbb{R})$$

as the vielbeine of the previous section.

2.4 Lorentzian geometry from gauging the Poincaré algebra

Gauging procedures are ubiquitous in physics, especially as a way of obtaining Lagrangians with a desired local symmetry in the context of gauge theories. In this case, one usually starts with a theory invariant under the global transformations of a Lie symmmetry group G with Lie algebra $\mathfrak{g}$. Requiring this symmetry to be local then demands the introduction of a gauge connection taking values in $\mathfrak{g}$, in order to compensate the extra non-covariant factors appearing when making the transformations spacetime dependent. The components of this connection as an element of the symmetry algebra are the gauge fields, so that we have one for each of the generators of $\mathfrak{g}$. Their transformation properties under the gauge symmetry transformations and their associated gauge covariant field strengths are then determined by the structure constants of $\mathfrak{g}$.

It turns out that the vielbeine formulation of GR can be understood as a gauge theory of the Poincaré algebra. This was already considered in [53] and has been also reviewed and studied in, e. g., [35, 2, 52, 63]. In particular, a slightly different approach based on the redefinition of the gauge transformations instead of the imposition of constraints on the curvature was considered in [52]. We shall review such an approach here, following also [74]. The systematics of the gauging procedure of the Poincaré algebra generalise easily to other symmetry algebras, as we shall see when we study non-Lorentzian geometries.

2.4.1 The Poincaré group

The Poincaré group is the Lie group of isometries of Minkowski spacetime and consists of spacetime translations and Lorentz transformations. In the case of a D-dimensional ($D = d+1$) spacetime, the Poincaré group is $\frac{1}{2}D(D+1)$-dimensional and has the following semidirect product structure

$$\mathrm{Poin}(1,d) \cong \mathrm{SO}(1,d) \ltimes \mathbb{R}^{1+d}, \tag{2.69}$$

where $\mathrm{SO}(1,d)$ is the proper Lorentz group consisting of orientation-preserving space rotations and Lorentz boosts. Its Lie algebra has generators P_A and M_{AB} ($A, B = 0, \ldots, d$) satisfying the following non-zero commutation relations

$$\begin{aligned} [M_{AB}.P_C] &= \eta_{AC} P_B - \eta_{BC} P_A, \\ [M_{AB}, M_{CD}] &= \eta_{AC} M_{BD} - \eta_{BC} M_{AD} - \eta_{AD} M_{BC} + \eta_{BD} M_{AC}. \end{aligned} \tag{2.70}$$

It will be useful for our purposes to split these generators according to $A = (0, a)$ into their space and time components. To this end, we define

$$H := P_0, \quad K_a := M_{0a}, \quad J_{ab} := M_{ab}. \tag{2.71}$$

and consider the new set $\{H, P_a, K_a, J_{ab}\}$ of generators of the Poincaré algebra, corresponding to time translations, space translations, Lorentz boosts and orientation-preserving space rotations, respectively. They satisfy the commutation relations

$$\begin{aligned} [H, K_a] &= P_a, & [J_{ab}, P_c] &= \delta_{ac} P_b - \delta_{bc} P_a, \\ [P_a, K_b] &= \delta_{ab} H, & [J_{ab}, K_c] &= \delta_{ac} K_b - \delta_{bc} K_a, \\ [K_a, K_b] &= -J_{ab}, & [J_{ab}, J_{cd}] &= \delta_{ac} J_{bd} - \delta_{bc} J_{ad} - \delta_{ad} J_{bc} + \delta_{bd} J_{ac}. \end{aligned} \tag{2.72}$$

Here, and in what follows, it will be understood that any commutation relation that is omitted in the description of an algebra is zero.

2.4.2 Gauging procedure

As we said, the starting point is to associate a gauge field to every generator of the Poincaré algebra given by (2.70). To this end, we introduce a gauge connection $\mathcal{A}_\mu$ taking values in the Poincaré algebra, that we write as

$$\mathcal{A}_\mu = P_A E_\mu{}^A + \frac{1}{2} M_{AB} \omega_\mu{}^{AB}. \tag{2.73}$$

Therefore, $E_\mu{}^A$ and $\omega_\mu{}^{AB}$ here are the gauge fields associated to spacetime translations and rotations, respectively. In particular $M_{AB} = -M_{BA}$ and we can take $\omega_\mu{}^{AB}$ to be antisymmetric in A and B. As we shall see, these gauge fields will later be identified with the usual vielbeine and spin connection, respectively, of the vielbeine formulation of GR.

The connection $\mathcal{A}_\mu$, and hence the gauge fields, transforms as a one-form under diffeomorphisms,

$$\delta_\xi \mathcal{A}_\mu = \mathcal{L}_\xi \mathcal{A}_\mu, \tag{2.74}$$

where ξ^μ is a diffeomorphism generating vector field. Similarly, the transformation of the gauge fields under the action of the Poincaré (gauge) group follows from the transformation law for $\mathcal{A}_\mu$,

$$\delta_\Lambda \mathcal{A}_\mu = \partial_\mu \Lambda + [\mathcal{A}_\mu, \Lambda], \tag{2.75}$$

where Λ generates an infinitesimal Poincaré transformation and is given by

$$\Lambda = P_A \zeta^A + \frac{1}{2} M_{AB} \sigma^{AB}, \tag{2.76}$$

and that we can now interpret as a gauge parameter. Indeed, plugging (2.73) in both sides of (2.75) yields

$$\delta_\Lambda E_\mu{}^A = \partial_\mu \zeta^A - \omega_\mu{}^A{}_B \zeta^B + \sigma^A{}_B E_\mu{}^B, \tag{2.77a}$$

$$\delta_\Lambda \omega_\mu{}^{AB} = \partial_\mu \sigma^{AB} + 2\sigma^{[A}{}_C \omega_\mu{}^{CB]}. \tag{2.77b}$$

We can also define a gauge covariant field strength curvature $\mathcal{F}_{\mu\nu}$ associated to $\mathcal{A}_\mu$ by

$$\begin{aligned} \mathcal{F}_{\mu\nu} &:= \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \\ &= P_A R_{\mu\nu}{}^A(P) + \frac{1}{2} M_{AB} R_{\mu\nu}{}^{AB}(M), \end{aligned} \tag{2.78}$$

where $R_{\mu\nu}{}^A(P)$ and $R_{\mu\nu}{}^{AB}(M)$ are the curvatures associated to the gauge fields $E_\mu{}^A$ and $\omega_\mu{}^{AB}$, respectively. They are given by

$$R_{\mu\nu}{}^A(P) = 2\partial_{[\mu} E_{\nu]}{}^A - 2\omega_{[\mu}{}^A{}_B E_{\nu]}{}^B, \tag{2.79a}$$

$$R_{\mu\nu}{}^{AB}(M) = 2\partial_{[\mu} \omega_{\nu]}{}^{AB} - 2\omega_{[\mu}{}^{CA} \omega_{\nu]}{}^B{}_C. \tag{2.79b}$$

So far we have simply written down a gauge theory for the Poincaré algebra, consisting of two independent gauge fields that transform under local spacetime translations and rotations according to (2.77). If we want to make contact with Lorentzian geometry, we would like to interpret the gauge fields as the vielbeine and the spin connection of Section 2.2. To this end, we want to use (2.75) to define a new set of transformations for $\mathcal{A}_\mu$ that replaces local spacetime translations by diffeomorphisms, therefore including (2.74). In order to achieve this, we start by replacing the parameter ζ^A in (4.86) corresponding to spacetime translations by the spacetime vector ξ^μ via $\zeta^A = \xi^\mu E_\mu{}^A$. With this identification, we can now write Λ as

$$\Lambda = \xi^\mu \mathcal{A}_\mu + \Sigma, \tag{2.80}$$

where

$$\Sigma = \frac{1}{2} M_{AB} \lambda^{AB}, \tag{2.81}$$

with $\lambda^{AB} := \sigma^{AB} - \xi^\mu \omega_\mu{}^{AB}$. Notice that now Σ generates an infinitesimal local *Lorentz* transformation. We can now define a new set of local transformations (that we will refer to from now on simply as gauge transformations) by

$$\delta \mathcal{A}_\mu := \delta_\Lambda \mathcal{A}_\mu - \xi^\nu \mathcal{F}_{\mu\nu} = \mathcal{L}_\xi \mathcal{A}_\mu + \partial_\mu \Sigma + [\mathcal{A}_\mu, \Sigma]. \tag{2.82}$$

The gauge connection $\mathcal{A}_\mu$ then transforms covariantly under diffeomorphisms while keeping the transformation law (2.77) under local spacetime rotations. One can then derive the following transformation laws for the gauge fields:

$$\delta E_\mu{}^A = \mathcal{L}_\xi E_\mu{}^A + \lambda^A{}_B E_\mu{}^B, \tag{2.83a}$$

$$\delta \omega_\mu{}^{AB} = \mathcal{L}_\xi \omega_\mu{}^{AB} + \partial_\mu \lambda^{AB} + 2\lambda^{[A}{}_C \omega_\mu{}^{CB]}. \tag{2.83b}$$

The first one is equivalent to the infinitesimal version of (2.29) together with the term arising from the diffeomorphism. The second one is equivalent to (2.38)[5], after using that we are free to antisymmetrise in A and B because such a term always appears contracted with M_{AB}, that is antisymmetric, together with the corresponding term due to diffeomorphisms. Therefore, the gauge fields transform like the vielbeine and spin connection introduced in Section 2.2 under diffeomorphisms and LLTs.

The next logical step is to introduce a derivative operator $\mathcal{D}$ that behaves covariantly under these gauge transformations. We do so by its action on the vielbeine

$$\mathcal{D}_\mu E_\nu{}^A := \partial_\mu E_\nu{}^A - \tilde{\Gamma}^\rho_{\mu\nu} E_\rho{}^A - \omega_\mu{}^A{}_B E_\nu{}^B. \tag{2.84}$$

Demanding this to be gauge-covariant, namely that

$$\delta(\mathcal{D}_\mu E_\nu{}^A) = \mathcal{L}_\xi(\mathcal{D}_\mu E_\nu{}^A) + \lambda^A{}_B(\mathcal{D}_\mu E_\nu{}^B), \tag{2.85}$$

and using (2.83) we obtain the following tranformation law for $\tilde{\Gamma}^\rho_{\mu\nu}$:

$$\delta \tilde{\Gamma}^\rho_{\mu\nu} = \partial_\mu \partial_\nu \xi^\rho + \xi^\sigma \partial_\sigma \tilde{\Gamma}^\rho_{\mu\nu} + \tilde{\Gamma}^\rho_{\sigma\nu} \partial_\mu \xi^\sigma + \tilde{\Gamma}^\rho_{\mu\sigma} \partial_\nu \xi^\sigma - \tilde{\Gamma}^\sigma_{\mu\nu} \partial_\sigma \xi^\rho, \tag{2.86}$$

which is precisely the expected transformation law for an affine connection under an infinitesimal diffeomorphism. In particular, it is unaffected by the LLTs. Again, the relation between the affine connection $\tilde{\Gamma}^\rho_{\mu\nu}$ and the spin connection $\omega_\mu{}^{AB}$ is given by the vielbein postulate

$$\mathcal{D}_\mu E_\nu{}^A = 0, \tag{2.87}$$

from which follows that

$$\tilde{\Gamma}^\rho_{\mu\nu} = e^\rho{}_A \partial_\mu e_\nu{}^A - e^\rho{}_A e_\nu{}^B \omega_\mu{}^A{}_B, \tag{2.88}$$

and

$$\omega_\mu{}^A{}_B = -e_\rho{}^A e^\nu{}_B \tilde{\Gamma}^\rho_{\mu\nu} + e^\nu{}_B \partial_\mu e_\nu{}^A. \tag{2.89}$$

[5]It is actually equivalent up to a sign, due to the choice of different conventions in (2.75) and (2.38) for the transformation rule of a connection. The first one is more customary in the context of gauging procedures, so we have chosen to follow it here.

These two expressions will allow us to relate the curvatures $R_{\mu\nu}{}^{A}(P)$ and $R_{\mu\nu}{}^{AB}(M)$ to $\tilde{\Gamma}^{\rho}_{\mu\nu}$. For example, taking the antisymmetric part of (2.87) yields

$$R_{\mu\nu}{}^{A}(P) = 2\partial_{[\mu}E_{\nu]}{}^{A} - 2\omega_{[\mu}{}^{A}{}_{B}E_{\nu]}{}^{B} = 2\tilde{\Gamma}^{\rho}_{[\mu\nu]}E_{\rho}{}^{A} \equiv \tilde{T}^{\rho}{}_{\mu\nu}E_{\rho}{}^{A}, \tag{2.90}$$

where $\tilde{T}^{\rho}{}_{\mu\nu}$ is the torsion. Therefore, we conclude that the curvature $R_{\mu\nu}{}^{A}(P)$ is to be identified with the torsion two-form $\tilde{T}^{A}{}_{\mu\nu}$. Let us now consider the covariant derivative $\tilde{\nabla}$ associated to the affine connection (that is, not containing the spin conection). We know that the action of its commutator implicitly defines the Riemann tensor $R_{\mu\nu\rho}{}^{\sigma}$ according to (2.3). Using the vielbein postulate through (2.88), one can show that the definition (2.4) of the Riemann tensor is equivalent to

$$R_{\mu\nu\sigma}{}^{\rho} = -E_{\sigma A}E^{\rho}{}_{B}R_{\mu\nu}{}^{AB}(M), \tag{2.91}$$

from which we conclude that the curvature $R_{\mu\nu}{}^{AB}(M)$ is to be identified with the Riemann curvature two-form.

We can now define a metric by

$$g_{\mu\nu} = \eta_{AB}E_{\mu}{}^{A}E_{\nu}{}^{B}, \tag{2.92}$$

which is the only rank-2 Lorentz invariant tensor that we can build out of the vielbeine. The affine connection $\tilde{\Gamma}^{\rho}_{\mu\nu}$ has so far been considered to be arbitrary. However, as discussed in Section 2.2, the antisymmetry of the spin connection that we assumed in the beginning is equivalent to the metric compatibility condition

$$\tilde{\nabla}_{\rho}g_{\mu\nu} = 0. \tag{2.93}$$

This is known to completely determine the symmetric part of the connection,

$$\tilde{\Gamma}^{\rho}_{(\mu\nu)} = \Gamma^{\rho}_{\mu\nu}, \tag{2.94}$$

where $\Gamma^{\rho}_{\mu\nu}$ is the Levi-Civita connection. Therefore, our affine connection is fixed to be equal to the Levi-Civita one plus torsion terms that are a priori left unfixed. More precisely, we have

$$\bar{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + \frac{1}{2}\tilde{T}^{\rho}{}_{\mu\nu}. \tag{2.95}$$

From the gauging perspective, the usual choice in GR to work with the Levi-Civita connection then amounts to imposing the curvature constraint

$$R_{\mu\nu}{}^{A}(P) = 0. \tag{2.96}$$

This makes the spin connection completely dependent on the vielbeine and their derivatives, in analogy with (2.50). Without fixing the torsion, however, the two remain independent.

Chapter 3

Non-Lorentzian geometry

In this chapter, we present the description of spacetime geometries beyond the Lorentzian case. These arise from imposing local symmetry principles different from Einstein's equivalence principle, and yield non-Lorentzian theories of gravity when made dynamical. Especially relevant for our purposes is Newton-Cartan geometry, which is roughly speaking the geometry obtained by imposing Galilei's relativity principle to hold locally. The latter is closely related (and in a sense dual, as we will see) to Carroll geometry. can be obtained from Lorentzian geometry by, respectively, a non-relativistic expansion and an ultra-local expansion of GR [46, 48].

In order to understand these geometries, we first describe the non-Lorentzian symmetry groups, and corresponding algebras, that realise their local symmetries. This description includes the Galilei group, its central extension the Bargmann group, and the Carroll group, and is intended to provide all the group theory background necessary for our work. The main reference for this section is [42], but we have also followed [38, 47].

3.1 Non-Lorentzian symmetry groups

3.1.1 The Galilei group

The Galilei group is the symmetry group of Newtonian mechanics. It consists of space and time translations, orientation-preserving space rotations and Galilean boosts, that act on spacetime coordinates by

$$\begin{aligned} t &\to t, \\ \vec{x} &\to \vec{x} - t \cdot \vec{v}. \end{aligned} \tag{3.1}$$

The Galilei algebra is generated by the set $\{H, P_a, G_a, J_{ab}\}$, where G_a is the generator of Galilean boosts, and is given by the commutation relations

$$\begin{aligned} &[H, G_a] = P_a, &&[J_{ab}, P_c] = \delta_{ac}P_b - \delta_{bc}P_a, \\ &[P_a, G_b] = 0, &&[J_{ab}, G_c] = \delta_{ac}G_b - \delta_{bc}G_a, \\ &[G_a, G_b] = 0, &&[J_{ab}, J_{cd}] = \delta_{ac}J_{bd} - \delta_{bc}J_{ad} - \delta_{ad}J_{bc} + \delta_{bd}J_{ac}. \end{aligned} \tag{3.2}$$

The fact that the boost (3.1) leaves the time coordinate invariant shows one of the defining characteristics of Newtonian mechanics: the absoluteness of time. Note that a sufficient (although not necessary) condition for this is the vanishing of the commutator $[P_a, G_b]$. The group has the following semidirect product structure

$$\mathrm{Gal}(1, d) \cong \left(\mathrm{SO}(d) \ltimes \mathbb{R}^d\right) \ltimes \mathbb{R}^{1,d}, \tag{3.3}$$

where the subgroup in parenthesis corresponds to the *homogeneous Galilei group*

$$\mathrm{HGal}(1,d) \cong \mathrm{SO}(d) \ltimes \mathbb{R}^d, \tag{3.4}$$

consisting of orientation-preserving space rotations and Galilean boosts. It is the equivalent with respect to the whole Galilei group of the (proper) Lorentz group with respect to the Poincaré group, in the sense that they both only consist of the transformations that leave the origin fixed. In the same way that the Minkowski metric η_{AB} is by definition invariant under the action of $\mathrm{SO}(1,d)$, $\mathrm{HomGal}(1,d)$ has two invariant tensors t_A and π^{AB} satisfying

$$t_A \pi^{AB} = 0, \tag{3.5}$$

from which follows that π^{AB} has a degeneracy in the direction spanned by t_A. In the (fundamental) $\mathbb{R}^{1+d}$-representation of $\mathrm{HomGal}(1,d)$ [38] and in an appropriate basis these tensors take the form

$$t_A = \delta^0_A, \quad \pi^{AB} = \delta_{ab}\delta^a_A\delta^b_B. \tag{3.6}$$

It is well known that the Galilei group can be obtained by an Inönü-Wigner contraction of the Poincaré group [56], which essentially carries out a non-relativistic limit $\sigma \to 0$ of the latter. At the level of the algebra, this can easily be seen by the following redefinition of the Poincaré generators

$$\tilde{H} := H, \quad \tilde{P}_a := \sqrt{\sigma} P_a, \quad \tilde{G}_a := \sqrt{\sigma} K_a, \quad \tilde{J}_{ab} := J_{ab}. \tag{3.7}$$

Substituting in (2.72) and taking the contraction limit $\sigma \to 0$ then trivialises some of the commutation relations and yields an algebra isomorphic to (3.2). In particular, the Galilei group in $D = d+1$ dimensions is $\frac{1}{2}D(D+1)$-dimensional like the Poincaré group.

3.1.2 The Bargmann group

The Galilei group is actually incomplete for the description of massive fields. When these are present, one needs to extend the Galilei algebra with a mass generator N. Indeed, under a Galilean boost by k_a the momentum p_a of a particle of mass m will be shifted by

$$p_a \to p_a + m k_a. \tag{3.8}$$

Moreover, since the mass is invariant under any Galilean transformation, the new generator N must be in the center of the Galilean algebra. It then makes sense to consider the algebra generated by $\{H, P_a, G_a, J_{ab}, N\}$ satisfying the commutation relations (3.2) together with the only additional non-zero commutator

$$[P_a, G_b] = \delta_{ab} N, \tag{3.9}$$

known as the *Bargmann algebra*. At the group level, the result is a central extension of the Galilei group by a $U(1)_N$ factor called the *Bargmann group* [5]. Its structure is given by

$$\mathrm{Barg}(1,d) \cong \left(\mathrm{SO}(d) \ltimes \mathbb{R}^d\right) \ltimes \left(\mathbb{R}^{1,d} \otimes U(1)_N\right). \tag{3.10}$$

The Bargmann group cannot be obtained by any contraction of the Poincaré group, since it has one generator more. However, it turns out that it can be obtained by an Inönü-Wigner contraction of the Poincaré group trivially extended with a $U(1)$ factor (see e.g. [39]).

It is instructive to look now at the action on Hilbert space of a translation $\vec{x} \to \vec{x} + \vec{a}$ followed by a boost $\vec{x} \to \vec{x} - v \cdot t$. This is given by

$$\exp(-i\vec{K} \cdot \vec{v}) \exp(-i\vec{P} \cdot \vec{a}) = \exp\left(-\frac{i}{2} N \vec{a} \cdot \vec{v}\right) \exp\left(-i(\vec{K} \cdot \vec{v} + \vec{P} \cdot \vec{a})\right). \tag{3.11}$$

This is only the same as the action of the composed transformation $\vec{x} \to \vec{x}+\vec{a}-v\cdot t$ up to the phase factor $e^{-\frac{1}{2}N\vec{a}\cdot\vec{v}}$, showing that particle states only provide projective representations of the Galilei group, but they provide ordinary representations of the Bargmann group [73]. These are labelled by the 3 Casimir invariants of the Bargmann algebra [42]. In particular, the Casimir invariant

$$N = m\mathrm{Id}, \tag{3.12}$$

where Id is the identity, labels the representations according to the particle's mass m.

3.1.3 The Carroll group

The Carroll group was first introduced by J.M. Lévy-Leblond in [62, 4] and can be understood as the $c \to 0$ limit of the Poincaré group. It consists of space and time translations, space rotations and Carroll boosts. The latter act on spacetime coordinates $x^\mu = (t, x)$ by

$$\begin{aligned} t &\to t - \vec{v} \cdot \vec{x}, \\ \vec{x} &\to \vec{x}. \end{aligned} \tag{3.13}$$

The Carroll algebra is generated by the set $\{H, P_a, C_a, J_{ab}\}$, where C_a is the generator of Carroll boosts, and is given by the commutation relations

$$\begin{aligned} &[H, C_a] = 0 && [J_{ab}, P_c] = \delta_{ac}P_b - \delta_{bc}P_a \\ &[P_a, C_b] = \delta_{ab}H && [J_{ab}, C_c] = \delta_{ac}C_b - \delta_{bc}C_a \\ &[C_a, C_b] = 0 && [J_{ab}, J_{cd}] = \delta_{ac}J_{bd} - \delta_{bc}J_{ad} - \delta_{ad}J_{bc} + \delta_{bd}J_{ac}. \end{aligned} \tag{3.14}$$

As hinted above, the Carroll group can be obtained by a different Inönü-Wigner contraction of the Poincaré group, that formally takes its $\sigma \to \infty$ limit. In this case, one redefines the Poincaré generators through the following rescaling

$$\tilde{H} := \frac{H}{\sqrt{\sigma}}, \quad \tilde{P_a} := P_a, \quad \tilde{C_a} := \frac{K_a}{\sqrt{\sigma}}, \quad \tilde{J_{ab}} := J_{ab}. \tag{3.15}$$

The contraction limit $\sigma \to \infty$ then trivialises some of the commutation relations in (2.72) and yields and algebra isomorphic to (3.14). In particular, the Carroll group has also the same dimension as the Poincaré.

Note how H enters the translation-boost commutator in (3.14) exactly as N does in the Bargmann algebra. In particular, H is a central charge of the Carroll algebra. In fact, both the Bargmann and Carroll algebras can be obtained as subalgebras of the Poincaré algebra in one dimension higher, via null reduction [37, 30, 50]. Moreover, the Carroll group has the same structure as the Galilei group

$$\mathrm{Car}(1, d) \cong \left(\mathrm{SO}(d) \ltimes \mathbb{R}^d\right) \ltimes \mathbb{R}^{1,d}, \tag{3.16}$$

although they of course yield different physics. Indeed, (3.13) essentially tells us that Carrollian symmetry is compatible with a notion of absolute space, dual to the notion of absolute time present in the Galilean case[1]. The *homogeneous Carroll group* consisting of space rotations and Carroll boosts also has two invariant tensors t^A and π_{AB} satisfying

$$t^A \pi_{AB} = 0. \tag{3.17}$$

Again in the fundamental $\mathbb{R}^{d+1}$-representation of $\mathrm{Car}(1,d)$ and an adequate basis, these take the form

$$t^A = -\delta^A_0, \qquad \pi_{AB} = \delta_{ab}\delta^a_A\delta^b_B. \tag{3.18}$$

3.2 Non-Lorentzian frame bundles

Before diving in the study of Newton-Cartan and Carrollian geometries, let us first address the frame bundle perspective introduced in Section 2.3 for Lorentzian geometry in full generality. In particular, let us go back to the discussion about the reduction of the structure group of the tangent and frame bundles, TM and FM respectively, when introducing a metric structure on M. Indeed, we considered there the introduction of a Lorentzian metric tensor and the subsequent reduction of the structure group of both FM and TM by

$$\mathrm{GL}(D,\mathbb{R}) \longrightarrow \mathrm{SO}(1,d),$$

as a consequence of the existence of an $\mathrm{SO}(1,d)$-reduction of FM given by the bundle of orientation-preserving orthonormal frames. We then said that a Lorentzian metric tensor, together with an orientation, defines an $\mathrm{SO}(1,d)$-structure on M.

It is easy to see that all the arguments presented in (2.3.3) generalise in a straightforward way for any semi-Riemannian metric structure. Indeed, for any $n \in \mathbb{N}$, a semi-Riemannian metric g of signature $\mathrm{O}(n, D-n)$ defines an $\mathrm{O}(n, D-n)$-structure and, together with an orientation, an $\mathrm{SO}(n, D-n)$-structure on M. In the latter case, the structure group is then reduced by

$$\mathrm{GL}(D,\mathbb{R}) \longrightarrow \mathrm{SO}(n, D-n),$$

after the reduction of the frame bundle to the bundle of orientation-preserving g-orthonormal frames.

The notion of reduction of the structure group of a principal bundle is even more general and goes beyond its application for frame bundles and semi-Riemannian metric structures. Given a Lie group G and a Lie subgroup $H < G$, for instance, one can study the existence of H-reductions of a principal G-bundle. We will not need any further applications beyond frame bundles, but we shall actually consider the slightly more general picture of H-reductions and H-structures for subgroups

$$H < \mathrm{GL}(D,\mathbb{R}),$$

other than the orthonormal group $\mathrm{O}(n, D-n)$. On a physical level, the structure group of a frame bundle is nothing but the group of transformations relating one reference frame to another. Therefore, the existence of an H-structure on M is tantamount to the existence on M of a local relativity principle defined by the transformations of H, namely one where two different frames at $p \in M$ are related by a transformation $\Lambda \in H$ as

$$e_{(A)}(p) = \Lambda^B{}_A \tilde{e}_{(B)}(p), \qquad \tilde{e}_{(A)}(p) = (\Lambda^{-1})^B{}_A e_{(B)}(p). \tag{3.19}$$

[1]We refer the reader to [30] for a detailed discussion about this.

This is very relevant to our purposes, as it formalises the idea of a spacetime having a certain non-Lorentzian local symmetry. In particular, we will see how the invariant tensors of $\mathrm{HGal}(1,d)$ and $\mathrm{HCar}(1,d)$ can be used to define degenerate metric structures giving rise to reductions of the structure group

$$\mathrm{GL}(D,\mathbb{R}) \longrightarrow \mathrm{HGal}(1,d), \qquad \mathrm{GL}(D,\mathbb{R}) \longrightarrow \mathrm{HCar}(1,d).$$

3.3 Newton-Cartan geometry and extensions

NC geometry in its original formulation is characterised by having the Galilei group as its local symmetry group. As argued in Section 3.1.1, the latter is obtained from an Inönü-Wigner contraction of the Poincaré group, which essentially takes its $c \to \infty$ limit. It follows that NC geometry is the appropriate geometrical framework for the description of non-relativistic physics.

In the following, we present a description of NC geometry and its torsionful extensions: type I and type II TNC geometry. We pay special attention to the underlying symmetry principles in each case. This has been vastly considered in the literature from the gauging perspective introduced in Section 2.4 (e.g., [2, 11, 52, 46]).

3.3.1 Newton-Cartan geometry

Newton-Cartan geometry on a smooth manifold M is defined in terms of a pair

$$(t_\mu, h^{\mu\nu}),$$

where t_μ is a nowhere-vanishing one-form and $h^{\mu\nu}$ is a degenerate symmetric tensor of type $(2,0)$ with degeneracy of degree one in the direction spanned by t_μ. The last statement translates into the condition

$$t_\mu h^{\mu\nu} = 0. \tag{3.20}$$

One usually refers to t_μ as the *clock one-form* and to $h^{\mu\nu}$ as the *inverse spatial metric*, even though it is not the inverse of anything. The pair $(t_\mu, h^{\mu\nu})$ can then be thought of as a degenerate metric structure on M. More precisely, one has two degenerate symmetric tensor fields $t_{\mu\nu} := t_\mu t_\nu$ with signature $(1,0,\ldots,0)$ and $h^{\mu\nu}$ with signature $(0,1,\ldots,1)$.

Crucially, the clock form and inverse spatial metric do not characterise NC geometry completely[2]. In order to implement local Galilean symmetry, which is the defining feature of NC geometry in its original formulation, the manifold M needs to serve as the base manifold of a Galilean frame bundle and its dual:

$$F_{\mathrm{HGal}}(M) \xrightarrow{\pi} M, \qquad F^*_{\mathrm{HGal}}(M) \xrightarrow{\pi^*} M.$$

Their sections then define Galilean vielbeine ${e_\mu}^A$ and inverse vielbeine ${e^\mu}_A$, respectively. In this way, the clock one-form and inverse spatial metric are related to the invariant tensors t_A and π^{AB} of the homogeneous Galilean group by

$$t_\mu = {e_\mu}^A t_A, \qquad h^{\mu\nu} = {e^\mu}_A {e^\nu}_B \pi^{AB}. \tag{3.21}$$

It is now obvious that (3.20) follows from the analogous relation for the invariant tensors t_A and π^{AB} in (3.5). It also follows that t_μ and $h^{\mu\nu}$ will be invariant under local Galilean transformations (LGTs).

[2]We will show in Section 4.2 how such a degenerate metric structure exists implicitly in any Lorentzian manifold.

The degenerate metric structure implies that one cannot raise and lower indices at will, since contravariant and covariant tensors of the same rank in M are not in one-to-one correspondence. One can however introduce a vector field v^μ and a degenerate symmetric tensor $h_{\mu\nu}$ on M satisfying

$$v^\mu h_{\mu\nu} = 0, \tag{3.22}$$

and acting like projective inverses of t_μ and $h^{\mu\nu}$, respectively, in the following sense:

$$t_\mu v^\mu = -1, \tag{3.23a}$$

$$h_{\mu\rho} h^{\rho\nu} = \delta^\nu_\mu + v^\nu t_\mu. \tag{3.23b}$$

It is easy to see that the pair $(v^\mu, h_{\mu\nu})$ is not unique. For instance, following a transformation

$$v^\mu \to v^\mu + k^\mu, \qquad \text{with } k^\mu t_\mu = 0, \tag{3.24}$$

usually known as a Milne boost, the relation (3.23a) will still hold. It can then be proven [65] that once v^μ is determined, the requirements (3.22) and (3.23b) completely determine $h_{\mu\nu}$.

In a frame where (3.6) holds, usually called an *adapted frame*, the Galilean vielbeine and their inverses can then be written as

$$e_\mu{}^A = (t_\mu, e_\mu{}^a), \qquad e^\mu{}_A = (-v^\mu, e^\mu{}_a), \tag{3.25}$$

and the spatial metrics then take the form

$$h^{\mu\nu} = \delta^{ab} e^\mu{}_a e^\nu{}_b, \qquad h_{\mu\nu} = \delta_{ab} e_\mu{}^a e_\nu{}^b \tag{3.26}$$

From their standard transformations under an infinitesimal diffeomorphisms and LGTs we can obtain the corresponding transformations for the clock one-form, the inverse spatial metric and their projective inverses,

$$\delta t_\mu = \mathcal{L}_\xi t_\mu, \tag{3.27a}$$

$$\delta h^{\mu\nu} = \mathcal{L}_\xi h^{\mu\nu}, \tag{3.27b}$$

$$\delta v^\mu = \mathcal{L}_\xi v^\mu + \lambda^a e^\mu{}_a, \tag{3.27c}$$

$$\delta h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu} + 2t_{(\mu}\lambda_{\nu)}, \tag{3.27d}$$

where ξ^μ is a diffeomorphism generating vector field, $\lambda^a{}_b$ and λ^a are the infinitesimal generators for space rotations and local Galilean boosts respectively, and we have defined $\lambda_\mu := \lambda_a e_\mu{}^a$. Here, we have used that the spatial vielbeine transform as

$$\delta e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a + \lambda^a{}_b e_\mu{}^b + \lambda^a t_\mu, \tag{3.28a}$$

$$\delta e^\mu{}_a = \mathcal{L}_\xi e^\mu{}_a + \lambda_a{}^b e^\mu{}_b. \tag{3.28b}$$

As usual, we now want to introduce an appropriate connection $\tilde{C}$ on M with respect to which we can define a covariant derivative operator $\tilde{\nabla}$ and eventually build diffeomorphic invariant actions supported in NC geometry. In the spirit of having a connection that is somehow adapted to the underlying NC structure, we require that the covariant derivative satisfy

$$\tilde{\nabla}_\mu t_\nu = 0, \qquad \tilde{\nabla}_\mu h^{\nu\rho} = 0. \tag{3.29}$$

Indeed, these conditions are the Newton-Cartan version of the usual metric compatibility condition in Lorentzian geometry and any connection that satisfies them is said to be *Newton-Cartan compatible*. Expanding the first equation in (3.29) we get

$$\partial_\mu t_\nu - \tilde{C}^\rho_{\mu\nu} t_\rho = 0, \tag{3.30}$$

and taking the antisymmetric part yields

$$(\mathrm{d}t)_{\mu\nu} \equiv 2\partial_{[\mu} t_{\nu]} = t_\rho \tilde{C}^\rho_{[\mu\nu]}. \tag{3.31}$$

Therefore, any NC compatible connection $\tilde{C}$ must have the same temporal projection of the torsion. Notice also that due to the degeneracy of the metric structure, the NC metric compatibility conditions in (3.29) do not imply that the projective inverses v^μ and $h_{\mu\nu}$ are covariantly constant. Now, it can be proven that any NC compatible connection $\tilde{C}$ is of the form [52]

$$\tilde{C}^\rho_{\mu\nu} = \check{C}^\rho_{\mu\nu} + \frac{1}{2} h^{\rho\sigma} \left(2\tau_{(\mu} A_{\sigma\nu)} + B_{\sigma\mu\nu} \right), \tag{3.32}$$

where $A_{\mu\nu}$ and $B_{\sigma\mu\nu}$ satisfy $A_{\mu\nu} = -A_{\nu\mu}$ and $B_{\sigma\mu\nu} = -B_{\nu\mu\sigma}$ and

$$\check{C}^\rho_{\mu\nu} = -v^\rho \partial_\mu t_\nu + \frac{1}{2} h^{\rho\sigma} (\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}). \tag{3.33}$$

It follows that the NC compatibility conditions do not determine completely the connection. Moreover, this is still the case after imposing that the connection be torsionless, in contrast with the analogous situation in semi-Riemannian geometry, where the Levi-Civita connection is, by the fundamental theorem of differential geometry, the unique metric compatible and torsionfree connection. But there is yet another fundamental difference between the two cases. Indeed, (3.31) tells us that imposing torsionlessness on a NC compatible connection necessarily puts a constraint on the clock-form t_μ, and hence the geometry. From these observations we see that torsion, as well as non-metricity, arise naturally in NC geometry.

Note that (3.33) is the NC compatible connection obtained when setting $A_{\mu\nu}$ and $B_{\sigma\mu\nu}$ to zero. We will sometimes refer to it as the $\check{C}$–connection, and it satisfies

$$\check{\nabla}_\mu v^\nu = \frac{1}{2} h^{\nu\rho} \mathcal{L}_v h_{\rho\mu}, \qquad \check{\nabla}_\mu h_{\nu\rho} = t_{(\nu} \mathcal{L}_v h_{\rho)\mu}, \tag{3.34}$$

where $\check{\nabla}$ is its associated covariant derivative. Its torsion is given by

$$\check{T}^\rho{}_{\mu\nu} := 2\check{C}^\rho_{[\mu\nu]} = -2v^\rho \partial_{[\mu} t_{\nu]}. \tag{3.35}$$

It is also useful in this case to define the *torsion vector* a_μ by

$$a_\mu := \check{T}^\rho{}_{\mu\rho} = \mathcal{L}_v t_\mu, \tag{3.36}$$

where the equality follows from (3.23a), and *extrinsic curvature* $K_{\mu\nu}$ by

$$K_{\mu\nu} := -\frac{1}{2} \mathcal{L}_v h_{\mu\nu}. \tag{3.37}$$

The $\check{C}$-connection has the advantage of being built solely out of the metric fields and their inverses. It is to be interpreted as some sort of analogue to the Levi-Civita connection in Lorentzian geometry, although with some notable differences. For example, it is not invariant under LGTs[3], in contrast with the LLT-invariance of the Levi-Civita connection. Moreover, the $\check{C}$-connection is torsionful, unlike the Levi-Civita connection, although the spatial projection of its torsion is zero

$$e_\rho{}^a \check{T}^\rho{}_{\mu\nu} = 0. \tag{3.38}$$

However, we know from (3.31) that we cannot completely remove the torsion without putting constrains on the geometry, which means that the $\check{C}$-connection can be thought of as the NC metric compatible connection with less torsion, just like the Levi-Civita in the Lorentzian case. Following this discussion, we say that NC geometry has *intrinsic*[4] torsion [42], which allows to classify NC geometry in terms of $\mathrm{d}t$:

1. **(Torsionless) Newton-Cartan geometry (TlessNC):** This is the type of NC geometry that Cartan originally formulated. It is characterised by the clock form being closed, i.e., by
$$\mathrm{d}t = 0. \tag{3.39}$$
This allows for a notion of absolute time T since we can write
$$t_\mu = \partial_\mu T, \tag{3.40}$$
which in turn implies
$$\oint t = 0. \tag{3.41}$$
Therefore, the time interval between any two events is independent of the worldline of the observers.

2. **Twistless Torsional Newton-Cartan (TTNC) geometry:** It is defined by
$$t \wedge \mathrm{d}t = 0 \tag{3.42}$$
In this case, t_μ is not closed and $\oint t \neq 0$, allowing for time dilation. By the dual formulation of Frobenius theorem [15], the condition (3.42) is equivalent to t_μ being orthogonal to a foliation of spacelike hypersurfaces of simultaneity. This is somehow equivalent to the global hyperbolicity condition in Lorentzian geometry, that ensures the well-posedness of an initial value problem on the manifold. A general solution to (3.42) is given by
$$t_\mu = N\partial_\mu T, \tag{3.43}$$
where $T = T(x)$ is a time function and $N = N(x)$ is the *lapse function* measuring local time dilation. One can always set T as the time coordinate through diffeomorphisms but N cannot be set to one [44].

3. **(General) Torsional Newton-Cartan (TNC) geometry:** It is defined by
$$t \wedge \mathrm{d}t \neq 0, \tag{3.44}$$
which results in the theory being acausal, since any two events can be connected by a spacelike curve [38]. It is however interesting to have a completely unconstrained clock form when studying the coupling of non-relativistic fields to the geometry.

[3] It is however invariant under the local $U(1)$ transformations of type I TNC geometry.
[4] The notion of intrinsic torsion of a spacetime is thoroughly addressed in [33].

TNC geometry from gauging the Galilei algebra

In the same way that one can obtain a description of (torsional) Lorentzian geometry by gauging the Poincaré algebra, it is reasonable to consider gauging the Galilei algebra as a method to obtain the relevant geometric fields of TNC geometry and their transformations. This was achieved in [52], following the earlier work [2], where TlessNC was obtained from gauging the Bargmann algebra. We shall review the main steps of the procedure, which is in complete analogy with the one described in Section 2.4 for the Poincaré algebra.

Consider the Galilean algebra (3.2) and a connection one-form $\mathcal{A}_\mu$ taking values in the latter,

$$\mathcal{A}_\mu = H t_\mu + P_a e_\mu{}^a + G_a \omega_\mu{}^a + \frac{1}{2} J_{ab} \omega_\mu{}^{ab}. \tag{3.45}$$

In this context, t_μ and $e_\mu{}^a$ are the gauge fields associated to time and space translations, respectively. They will later be identified with the temporal and spatial Galilean vielbeine, so we use the same notation for simplicity. The connection transforms as

$$\delta_\Lambda \mathcal{A}_\mu = \partial_\mu \Lambda + [\mathcal{A}_\mu, \Lambda] \tag{3.46}$$

under an infinitesimal Galilei transformation generated by

$$\Lambda = H\alpha + P_a \zeta^a + G_a \sigma^a + \frac{1}{2} J_{ab} \sigma^{ab}. \tag{3.47}$$

We can also definte a gauge-covariant curvature

$$\begin{aligned} \mathcal{F}_{\mu\nu} &:= \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \\ &= H R_{\mu\nu}(H) + P_a R_{\mu\nu}{}^a(P) + G_a R_{\mu\nu}{}^a(G) + \frac{1}{2} J_{ab} R_{\mu\nu}{}^{ab}(J), \end{aligned} \tag{3.48}$$

with components given by

$$R_{\mu\nu}(H) = 2\partial_{[\mu} t_{\nu]}, \tag{3.49a}$$
$$R_{\mu\nu}{}^a(P) = 2\partial_{[\mu} e_{\nu]}{}^a - 2\omega_{[\mu}{}^a t_{\nu]} - 2\omega_{[\mu}{}^a{}_b e_{\nu]}{}^b, \tag{3.49b}$$
$$R_{\mu\nu}{}^a(G) = 2\partial_{[\mu} \omega_{\nu]}{}^a - 2\omega_{[\mu}{}^a{}_b \omega_{\nu]}{}^b, \tag{3.49c}$$
$$R_{\mu\nu}{}^{ab}(J) = 2\partial_{[\mu} \omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{ca} \omega_{\nu]}{}^b{}_c. \tag{3.49d}$$

In order to relate local time and space translations with diffeomorphisms we identify the parameters α and ζ^a with the spacetime vector ξ through

$$\alpha = \xi^\mu t_\mu, \qquad \zeta^a = \xi^\mu e_\mu{}^a. \tag{3.50}$$

We can then write Λ as

$$\Lambda = \xi^\mu \mathcal{A}_\mu + \Sigma, \tag{3.51}$$

where

$$\Sigma = G_a \lambda^a + \frac{1}{2} J_{ab} \lambda^{ab}, \tag{3.52}$$

with

$$\lambda^a = \sigma^a - \xi^\mu \omega_\mu{}^a, \qquad \lambda^{ab} = \sigma^{ab} - \xi^\mu \omega_\mu{}^{ab}. \tag{3.53}$$

In particular, Σ now generates an infinitesimal local *homogeneous* Galilean transformation. We can now introduce a new set of local transformations defined by

$$\delta \mathcal{A}_\mu := \delta_\Lambda \mathcal{A}_\mu - \xi^\nu \mathcal{F}_{\mu\nu} = \mathcal{L}_\xi \mathcal{A}_\mu + \partial_\mu \Sigma + [\mathcal{A}_\mu, \Sigma], \tag{3.54}$$

which leads to

$$\delta t_\mu = \mathcal{L}_\xi t_\mu, \tag{3.55a}$$

$$\delta e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a + \lambda^a{}_b e_\mu{}^b + \lambda^a t_\mu, \tag{3.55b}$$

$$\delta \omega_\mu{}^a = \mathcal{L}_\xi \omega_\mu{}^a + \partial_\mu \lambda^a + \lambda^a{}_b \omega_\mu{}^b + \lambda^b \omega_{\mu b}{}^a, \tag{3.55c}$$

$$\delta \omega_\mu{}^{ab} = \mathcal{L}_\xi \omega_\mu{}^{ab} + \partial_\mu \lambda^{ab} + 2\lambda^{[a}{}_c \omega_\mu{}^{cb]}. \tag{3.55d}$$

The first equality reproduces the transformation law (3.27a) for the NC clock-form and the second one the transformation law (3.28a) for the spatial Galilean vielbeine. The last two equalities follow from replacing Lorentz boosts for Galilei boosts and splitting $A = (0, a)$ in (2.83b). Therefore, we can identify the gauge fields $(t_\mu, e_\mu{}^a)$ with the Galilean vielbeine and $(\omega_\mu{}^a, \omega_\mu{}^{ab})$ with Galilean spin connections.

As usual, we can define inverse Galilean vielbeine v^μ and $e^\mu{}_a$ satisfying

$$v^\mu t_\mu = -1, \qquad v^\mu e_\mu{}^a = 0, \qquad e^\mu{}_a t_\mu = 0, \qquad e^\mu{}_a e_\mu{}^b = \delta^b_a, \qquad e^\mu{}_a e_\nu{}^a = \delta^\mu_\nu - v^\mu t_\nu, \tag{3.56}$$

and spatial metrics $h_{\mu\nu} = \delta_{ab} e_\mu{}^a e_\nu{}^b$ and $h^{\mu\nu} = \delta^{ab} e^\mu{}_a e^\nu{}_b$.

We also introduce a gauge covariant derivative defined by its action on the Galilean vielbeine

$$\mathcal{D}_\mu t_\nu = \partial_\mu t_\nu - \bar{\Gamma}^\rho_{\mu\nu} t_\rho, \tag{3.57a}$$

$$\mathcal{D}_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \bar{\Gamma}^\rho_{\mu\nu} e_\rho{}^a - \omega_\mu{}^a t_\nu - \omega_\mu{}^a{}_b e_\nu{}^b, \tag{3.57b}$$

where $\bar{\Gamma}^\rho_{\mu\nu}$ is an affine connection transforming like

$$\delta \bar{\Gamma}^\rho_{\mu\nu} = \partial_\mu \partial_\nu \xi^\rho + \xi^\sigma \partial_\sigma \bar{\Gamma}^\rho_{\mu\nu} + \bar{\Gamma}^\rho_{\sigma\nu} \partial_\mu \xi^\sigma + \bar{\Gamma}^\rho_{\mu\sigma} \partial_\nu \xi^\sigma - \bar{\Gamma}^\sigma_{\mu\nu} \partial_\sigma \xi^\rho. \tag{3.58}$$

As in Section 2.4, we impose the vielbeine postulate(s)

$$\mathcal{D}_\mu t_\nu = 0, \tag{3.59a}$$

$$\mathcal{D}_\mu e_\nu{}^a = 0. \tag{3.59b}$$

Using the antisymmetry of $\omega_\mu{}^{ab}$ together with the vielbeine postulates we immediately obtain the NC metric compatibility conditions:

$$\bar{\nabla}_\rho t_\mu = 0, \qquad \bar{\nabla}_\rho h^{\mu\nu} = 0, \tag{3.60}$$

where $\bar{\nabla}$ is the covariant derivative containing only the affine connection. These conditions put constrains on $\bar{\Gamma}^\rho_{\mu\nu}$, which we can now express as

$$\Gamma^{\bar{\rho}}_{\mu\nu} = -v^\rho \partial_\mu t_\nu + e^\rho{}_a \left(\partial_\mu e_\nu{}^a - \omega_\mu{}^a t_\nu - \omega_\mu{}^a{}_b e_\nu{}^b \right), \tag{3.61}$$

using the vielbeine postulates. From them also follows that

$$R_{\mu\nu}(H) = 2\bar{\Gamma}^\rho_{[\mu\nu]} t_\rho, \qquad R_{\mu\nu}{}^a(P) = 2\bar{\Gamma}^\rho_{[\mu\nu]} e_\rho{}^a. \tag{3.62}$$

Therefore, we can identify the curvatures associated to time and space translations with the corresponding projections of the torsion tensor:

$$\bar{T}^\rho{}_{\mu\nu} \equiv 2\bar{\Gamma}^\rho_{[\mu\nu]} = -v^\rho R_{\mu\nu}(H) + e^\rho{}_a R_{\mu\nu}{}^a(P). \tag{3.63}$$

Similarly, the other curvatures are related to the Riemann tensor (defined as usual) by

$$R_{\mu\nu\sigma}{}^\rho = e^\rho{}_a t_\sigma R_{\mu\nu}{}^a(G) - e_{\sigma a} e^\rho{}_b R_{\mu\nu}{}^{ab}(J). \tag{3.64}$$

3.3.2 Type I TNC geometry

We have described above NC geometry in its original formulation as well as torsionful generalisations thereof. Now we study the extensions of NC geometry that arise when enhancing the local symmetry beyond the Galilean group. The first natural way is to consider the Bargmann group as the local symmetry group, which gives rise to what was dubbed type I TNC geometry in [43]. It is characterised by a triple

$$(t_\mu, h^{\mu\nu}, m_\mu),$$

where t_μ and $h^{\mu\nu}$ are the clock one-form and (inverse) spacial metric introduced before, transforming as (3.27a) and (3.27b) under local Bargmann transformations, and m_μ is a one-form transforming like

$$\delta m_\mu = \mathcal{L}_\xi m_\mu + e_\mu{}^a \lambda_a + \partial_\mu \Omega, \tag{3.65}$$

where Ω is the parameter of a local U(1) transformation. When making contact with Newtonian gravity, it is related to the Newtonian potential Φ through

$$\Phi \equiv -v^\mu m_\mu. \tag{3.66}$$

The role of the extra field m_μ is better understood from the gauging perspective, where it is identified with the gauge field corresponding to the extra central generator N entering the Bargmann algebra through the commutator (3.9). This was determined in [52], following the earlier work [11], where type I TNC was obtained from gauging the Schrödinger algebra, i.e., the conformal extension of the Bargmann algebra. The gauging of the Bargmann algebra to yield type I TNC is straightforward after having worked out the details of the gauging of the Galilei algebra. Indeed, in this case we consider a connection

$$\mathcal{A}_\mu = H t_\mu + P_a e_\mu{}^a + G_a \omega_\mu^a + \frac{1}{2} J_{ab} \omega_\mu{}^{ab} + N m_\mu, \tag{3.67}$$

taking values in the Bargmann algebra and with associated curvature given by

$$\begin{aligned} \mathcal{F}_{\mu\nu} &:= \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \\ &= H R_{\mu\nu}(H) + P_a R_{\mu\nu}{}^a(P) + G_a R_{\mu\nu}{}^a(G) + \frac{1}{2} J_{ab} R_{\mu\nu}{}^{ab}(J) + R_{\mu\nu}(N). \end{aligned} \tag{3.68}$$

We can then write an infinitesimal local Bargmann transformation Λ as in (3.51), but with

$$\Sigma = G_a \lambda^a + \frac{1}{2} J_{ab} \lambda^{ab} + N\varphi. \tag{3.69}$$

If we now introduce the new set of transformations as given by (3.54) all the previous results hold, because N is central. In particular, the curvature components for the generators common to the Galilei algebra are given by (3.49). For the curvature associated to the extra generator N, we find

$$R_{\mu\nu}(N) = 2\partial_{[\mu} m_{\nu]} + e_\mu{}^a \omega_{\nu a}. \tag{3.70}$$

And, most importantly, the gauge field m_μ associated to it transforms like (3.65) under the gauge transformations.

We argued in Section 3.3.1 that for NC geometry t_μ and $h^{\mu\nu}$ are the only local Galilei-boost invariant objects. In particular, the projective inverses v^μ and $h_{\mu\nu}$ are shifted under such transformations, as follows from (3.27c) and (3.27d). The introduction of m_μ allows to build three other local Galilei-boost invariant objects[5]:

$$\bar{v}^\mu := v^\mu - h^{\mu\nu} m_\nu, \tag{3.71}$$

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - 2t_{(\mu} m_{\nu)}, \tag{3.72}$$

$$\bar{\Phi} := \Phi + \frac{1}{2} h^{\mu\nu} m_\mu m_\nu. \tag{3.73}$$

It is easy to check that that taking these combinations the term $e_\mu{}^a \lambda_a$ appearing in δv^μ and $\delta h_{\mu\nu}$ cancels with the one appearing in δm_μ. The boost-invariant quantites $\bar{v}^\mu$ and $\bar{h}_{\mu\nu}$ are projective inverses of t_μ and $h^{\mu\nu}$, respectively, since they satisfy

$$h^{\mu\rho} \bar{h}_{\rho\nu} = \delta^\mu_\nu + \bar{v}^\mu t_\nu, \qquad \bar{v}^\mu t_\mu = -1, \tag{3.74}$$

although they are not orthogonal to each other:

$$\bar{v}^\mu \bar{h}_{\mu\nu} = h^{\mu\rho} m_\mu m_\rho t_\nu. \tag{3.75}$$

Interestingly, these boost-invariant projective inverses allow to define a torsionful NC compatible connection that, unlike (3.33), is invariant under local Galilean boosts but not under local $U(1)$ transformations:

$$\bar{C}^\rho_{\mu\nu} = -\bar{v}^\rho \partial_\mu t_\nu + \frac{1}{2} h^{\rho\sigma} (\partial_\mu \bar{h}_{\nu\sigma} + \partial_\nu \bar{h}_{\mu\sigma} + \partial_\sigma \bar{h}_{\mu\nu}). \tag{3.76}$$

Besides the insight provided by the gauging perspective, it is sometimes useful to think of type I TNC geometry in D dimensions as a null reduction of Lorentzian geometry in $D+1$ dimensions, something that has been extensively studied in the literature (see, e.g., [50, 46] and the earlier work [29, 59]). At the level of symmetries, this implies that the Bargmann algebra in D dimensions is a subalgebra of the Poincaré algebra in $D+1$ dimensions, as mentioned in Section 3.1. Moreover, it is precisely the null reduction perspective that allows to show that Newtonian gravity cannot arise from a Bargmann invariant theory. Indeed, in a NC geometric framework, the (sourceful) Poisson equation of Newtonian gravity amounts to the following EOM [72]

$$\bar{R}_{\mu\nu} = 8\pi G_N \frac{d-2}{d-1} \rho t_\mu t_\nu, \tag{3.77a}$$

$$\mathrm{d}t = 0, \tag{3.77b}$$

where $\bar{R}_{\mu\nu}$ is the Ricci tensor obtained from the connection (3.76) and ρ is the mass density of the matter distribution sourcing Newtonian gravity. As established in [43] and further argued in [46], it turns out that (3.77a) is not compatible with a Bargmann invariant coupling of NC geometry to matter, since ρ sources torsion (and in fact forces $t \wedge \mathrm{d}t \neq 0$), in contradiction with (3.77b) and hence the Newtonian notion of absolute time.

3.3.3 Type II TNC geometry

Newtonian gravity arises from a non-relativistic limit of GR, but the geometrical framework of this limit cannot be type I TNC. As shown in [43], the correct underlying symmetry

[5]The objects that we define here as $\bar{v}^\mu$ and $\bar{\Phi}$ are denoted $\hat{v}^\mu$ and $\hat{\Phi}$ in previous works like [46]. We choose to do this in order to avoid confusion with the notation in Section 5, where hatted variables are reserved to perturbations

allowing for an off-shell formulation of Newtonian gravity is type II TNC, which is obtained by the gauging of a novel $D(D+1)$-dimensional non-relativistic algebra generated by

$$\{H, P_a, G_a, J_{ab}, N, T_a, B_a, S_{ab}\},$$

where H, P_a, G_a and J_{ab} are just the generators of the Galilei algebra, and defined by the following non-zero commutation relations

$$\begin{aligned}
[H, G_a] &= P_a, & [J_{ab}, X_c] &= \delta_{ac}X_b - \delta_{bc}X_a, \\
[P_a, G_b] &= \delta_{ab}N, & [S_{ab}, P_c] &= \delta_{ac}T_b - \delta_{bc}T_a, \\
[G_a, G_b] &= -S_{ab}, & [S_{ab}, G_c] &= \delta_{ac}B_b - \delta_{bc}B_a, \\
[H, B_a] &= T_a & [J_{ab}, J_{cd}] &= \delta_{ac}J_{bd} - \delta_{bc}J_{ad} - \delta_{ad}J_{bc} + \delta_{bd}J_{ac}, \\
[N, G_a] &= T_a & [J_{ab}, S_{cd}] &= \delta_{ac}S_{bd} - \delta_{bc}S_{ad} - \delta_{ad}S_{bc} + \delta_{bd}S_{ac},
\end{aligned} \tag{3.78}$$

where $X_a \in \{P_a, T_a, G_a, B_a\}$. This algebra, as we will show in Section 4.3.3, arises from a large speed of light expansion of the Poincaré algebra and does not contain the Bargmann algebra as a subalgebra. Before that, in Section 4.3.2, we will show that type II TNC is precisely the geometry that arises from the large speed of light expansion of Lorentzian geometry at next-to-leading order. We postpone the details about its field content and transformation properties until then, but we anticipate that type II TNC can be realised by the NC pair

$$(t_\mu, h^{\mu\nu}),$$

with the standard transformation properties, together with the pair

$$(m_\mu, \Phi_{\mu\nu}),$$

where $\Phi_{\mu\nu}$ is symmetric and m_μ is analogous to its cousin in type I TNC but with a transformation law that is only equivalent to (3.65) when t_μ is closed. Another crucial difference appears when considering how matter couples to type II TNC geometry. In this case, m_μ couples to the energy current while in type I TNC it couples to the mass current [46].

3.4 Carrollian geometry

Carrollian geometry is characterised by having the Carroll group as its group of local symmetries. It was first described in this sense in [22], although it had already appeared in [54] as the geometry describing the zero signature limit of the Hamiltonian formulation of GR. Following the analogy with NC geometry, Carrollian geometry also has a natural description from the gauging perspective, as the geometry obtained by gauging the Carroll algebra [50]. Moreover, just as NC geometry can be obtained from a non-relativistic expansion of GR, it has recently been determined how Carrollian geometry arises from an ultra-local expansion of GR [48]. As mentioned before, Carrollian geometry also plays a role in the context of the strong coupling limit of GR [54, 1]. More recently, Carroll symmetry has been suggested to be relevant for the study of dark energy and inflation [24].

3.4.1 General formulation

Carrollian geometry on a D-dimensional smooth manifold M is defined in terms of degenerate metric structure given by a pair

$$(v^\mu, h_{\mu\nu}),$$

where v^μ is a nowhere-vanishing vector field on M and $h_{\mu\nu}$ is a degenerate symmetric (0,2)-tensor field on M with degeneracy of degree one in the direction spanned by v^μ:

$$\ker h = \mathrm{span}\{v\} \quad \Longrightarrow \quad v^\mu h_{\mu\nu} = 0. \tag{3.79}$$

The frame bundle FM of M admits a reduction of the structure group

$$\mathrm{GL}(D,\mathbb{R}) \longrightarrow \mathrm{HCar(1,d)},$$

so that M is the base manifold of Carrollian frame bundles

$$F_{\mathrm{HCar}}(M) \xrightarrow{\pi} M, \qquad F^*_{\mathrm{HCar}}(M) \xrightarrow{\pi^*} M,$$

the sections of which define Carrollian vielbeine $e_\mu{}^A$ and their inverses $e^\mu{}_A$. The fields v^μ and $h_{\mu\nu}$ are related to the invariant tensors t_A, π_{AB} of the homogeneous Carroll group by

$$v^\mu = t^A e^\mu{}_A, \qquad h_{\mu\nu} = \pi_{AB} e_\mu{}^A e_\nu{}^B. \tag{3.80}$$

In particular, they are invariant under local Carrollian transformations (LCTs).

We can introduce (non-unique) projective inverses

$$(t_\mu, h^{\mu\nu}),$$

where t_μ is a nowhere-vanishing one-form and $h^{\mu\nu}$ a symmetric degenerate tensor satisfying

$$t_\mu v^\mu = -1, \qquad t_\mu h^{\mu\nu} = 0, \qquad h_{\mu\rho} h^{\rho\nu} = \delta^\nu_\mu + v^\nu t_\mu, \tag{3.81}$$

just as in the NC case. In an adapted frame, where the invariant Carroll tensors take the form (3.18), the Carrollian vielbeine can be expressed as in (3.25) and the spatial metrics as in (3.26). Notice that we use the same symbols for the fields of Carrollian geometry as we did for the NC case, and that they share some of their properties. Of course, the key difference results from their inequivalent bundle structure, which implies different transformation laws for the geometric fields. In particular, the relevant fields of Carrollian geometry have the following transformation laws

$$\delta v^\mu = \mathcal{L}_\xi v^\mu, \tag{3.82a}$$
$$\delta h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu}, \tag{3.82b}$$
$$\delta t_\mu = \mathcal{L}_\xi t_\mu + \lambda_a e_\mu{}^a, \tag{3.82c}$$
$$\delta h^{\mu\nu} = \mathcal{L}_\xi h^{\mu\nu} + 2v^{(\mu}\lambda^{\nu)}, \tag{3.82d}$$

where ξ^μ is a diffeomorphism generating vector field, and the parameters λ_a and $\lambda^a{}_b$ correspond to Carroll boosts and spatial rotations, respectively. Finally, we have also defined $\lambda^\mu := e^\mu{}_a \lambda^a$. The transformations of the spatial metrics follow as usual from the transformations of the spatial Carrollian vielbeine and their inverses

$$\delta e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a + \lambda^a{}_b e_\mu{}^b, \tag{3.83a}$$
$$\delta e^\mu{}_a = \mathcal{L}_\xi e^\mu{}_a - \lambda^b{}_a e^\mu{}_b + \lambda_a v^\mu. \tag{3.83b}$$

We are again interested in defining a covariant derivative operator $\tilde{\nabla}$ associated to a connection $\tilde{C}^\rho_{\mu\nu}$ that is as suitable as possible to the underlying Carrollian structure. In this case, the analogue of Lorentzian metric compatibility and NC metric compatiblity is given by the natural requirement that the Carrollian pair $(v^\mu, h_{\mu\nu})$ be covariantly constant, namely

$$\tilde{\nabla}_\mu v^\nu = 0, \qquad \tilde{\nabla}_\mu h_{\nu\rho} = 0. \tag{3.84}$$

A connection satisfying the expressions above is said to be *Carroll compatible.* The most general Carroll compatible connection $\tilde{C}^\rho_{\mu\nu}$ can be written as [50]

$$\tilde{C}^\rho_{\mu\nu} = \check{C}^\rho_{\mu\nu} - h^{\rho\sigma}\tau_\nu K_{\mu\sigma} - v^\rho X_{\mu\nu} + h^{\rho\sigma} Y_{\sigma\mu\nu}, \tag{3.85}$$

where $\check{C}^\rho_{\mu\nu}$ and $K_{\mu\nu}$ are, respectively, the $\check{C}$-connection and extrinsic curvature defined in (3.33) and (3.37) for the NC case, and $X_{\mu\nu}$ and $Y_{\sigma\mu\nu}$ are tensors satisfying

$$v^\nu X_{\mu\nu} = 0, \tag{3.86a}$$
$$v^\sigma Y_{\sigma\mu\nu} = 0 = v^\mu Y_{\sigma\mu\nu}. \tag{3.86b}$$

It is then easy to see that the tensor $X_{\mu\nu}$ measures the non-metricity of $\tilde{C}^\rho_{\mu\nu}$ in the temporal direction, since we have

$$\tilde{\nabla}_\mu \tau_\nu = -X_{\mu\nu}. \tag{3.87}$$

In analogy with NC geometry, where we introduced a preferred NC compatible connection characterised by having zero spatial torsion, here we introduce a preferred Carroll compatible connection by requiring its temporal torsion to be zero

$$2t_\rho \tilde{C}^\rho_{[\mu\nu]} = 2\partial_{[\mu} t_{\nu]} + X_{[\mu\nu]} = 0. \tag{3.88}$$

This fixes the antisymmetric part $X_{\mu\nu}$ but leaves freedom to choose the symmetric part as long as (3.86a) holds. A particularly suitable choice comes from setting

$$X_{\mu\nu} = -2\partial_{[\mu} t_{\nu]} + t_{(\mu}\mathcal{L}_v t_{\nu)}, \tag{3.89}$$

which together with the choice $Y_{\sigma\mu\nu} = 0$ yields the following Carroll compatible connection

$$\hat{C}^\rho_{\mu\nu} := -v^\rho \partial_{(\mu} t_{\nu)} - v^\rho t_{(\mu}\mathcal{L}_v t_{\nu)} + \frac{1}{2} h^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) - h^{\rho\sigma} t_\nu K_{\mu\sigma}, \tag{3.90}$$

that has (purely spatial) torsion given by

$$\hat{T}^\rho{}_{\mu\nu} := 2\hat{C}^\rho_{\mu\nu} = h^{\rho\sigma} t_{[\mu} K_{\nu]\sigma}, \tag{3.91}$$

Following [48] based on the work [33], this results in a classification of Carrollian geometry in four classes

1. $K_{\mu\nu} = 0$,
2. $\mathcal{K} = 0$,
3. $K_{\mu\nu} = f h_{\mu\nu}$, for some non-zero $f \in \mathcal{C}^\infty(M)$,
4. none of the above.

Notice that this resembles the classification of NC geometry in terms of the intrinsic torsion measured by dt.

Chapter 4

Non-relativistic expansion of General Relativity

The goal of this chapter is to present the systematics of a non-relativistic expansion of GR, following its description in the original work [46]. We start from a broader perspective by presenting the general features of the expansion at the Lagrangian level of any parameter-dependent theory for small values of the parameter. This was first studied in [45] precisely in order to address the non-relativistic expansion of the EH Lagrangian, but the results are completely general and have been subsequently used to obtain the ultra-local expansion of GR [48]. As argued before, a non-relativistic expansion corresponds to a large speed of light expansion or, more precisely, to an expansion around $\sigma = 0$. In particular, this means that we will consider the most conventional[1] case of expanding in even powers of $1/c$, and work under the assumption that all relevant fields are analytic in σ. This will allow us to perform the non-relativistic expansion of the Lorentzian geometry of GR, after a suitable reparametrisation of the latter in terms of the so-called *pre-non-relativistic* (PNR) fields. We shall see how Newton-Cartan geometry naturally arises from such an expansion. Moreover, we will show that the underlying non-relativistic local symmetry of the geometry obtained at each order of the expansion can be obtained by gauging the Lie algebra expansion of the Poincaré algebra, truncated at the desired order. Then, we shall make these geometries dynamical by applying our knowledge on general Lagrangian expansions to the small σ expansion of the Einstein-Hilbert Lagrangian, leading to the notion of non-relativistic gravity. Finally, we propose an interpretation of a truncated sector of the NLO theory as the non-relativistic magnetic limit of GR.

4.1 Generalities on Lagrangian expansions

Consider a Lagrangian $\mathcal{L}$ that is a function of some fields ϕ^I, with I shorthand for all space-time and/or internal indices, and their derivatives. We also consider an explicit analytic dependence of the Lagrangian on a dimensionless parameter α (for example, through factors appearing in the kinetic or potential terms), and write $\mathcal{L} = \mathcal{L}(\alpha, \phi^I, \partial_\mu \phi^I)$. The fields themselves can also depend on α, that is $\phi^I = \phi^I(\alpha, x)$, which accounts for an implicit dependence in $\mathcal{L}$. We shall assume that every $\phi^I(\alpha, x)$ is analytic in α such that it admits a Taylor expansion around $\alpha = 0$:

$$\phi^I(\alpha, x) = \phi^I_0(x) + \alpha \phi^I_1(x) + \alpha^2 \phi^I_2(x) + \mathcal{O}(\alpha^3). \tag{4.1}$$

[1] A study including odd powers of $1/c$ can be found in [31].

Here, we take $\phi_0^I(x)$ to be non-zero, factoring out any overall power of α when necessary. Our goal is then to write our Lagrangian as a power series in α starting at some order $N \in \mathbb{Z}$ of the form

$$\mathcal{L}(\alpha, \phi^I, \partial_\mu \phi^I) = \alpha^N \mathcal{L}_{\text{LO}} + \alpha^{N+1}\mathcal{L}_{\text{NLO}} + \alpha^{N+2}\mathcal{L}_{\text{NNLO}} + \mathcal{O}(\alpha^{N+3}), \tag{4.2}$$

where all the α-dependence is in the prefactors.

In order to find the different coefficients in the expansion, we start by defining

$$\tilde{\mathcal{L}}(\alpha) := \alpha^{-N}\mathcal{L}(\alpha, \phi, \partial_\mu\phi), \tag{4.3}$$

which is by assumption analytic in α and such that its Taylor expansion around $\alpha = 0$ starts at order zero

$$\tilde{\mathcal{L}}(\alpha) = \tilde{\mathcal{L}}(0) + \alpha\tilde{\mathcal{L}}'(0) + \frac{1}{2}\tilde{\mathcal{L}}''(0) + \mathcal{O}(\alpha^3). \tag{4.4}$$

The prime here denotes differentiation with respect to α, and we have

$$\frac{d}{d\alpha} = \frac{\partial}{\partial\alpha} + \frac{\partial\phi^I}{\partial\alpha}\frac{\partial}{\partial\phi^I} + \frac{\partial\partial_\mu\phi^I}{\partial\alpha}\frac{\partial}{\partial\partial_\mu\phi^I}. \tag{4.5}$$

Therefore,

$$\begin{aligned}\tilde{\mathcal{L}}(\alpha) =& \tilde{\mathcal{L}}(0) + \alpha\left(\frac{\partial\tilde{\mathcal{L}}}{\partial\alpha} + \frac{\partial\phi^I}{\partial\alpha}\frac{\partial\tilde{\mathcal{L}}}{\partial\phi^I} + \frac{\partial\partial_\mu\phi^I}{\partial\alpha}\frac{\partial\tilde{\mathcal{L}}}{\partial\partial_\mu\phi^I}\right)\bigg|_{\alpha=0} + \dots \\ =& \tilde{\mathcal{L}}(0) + \alpha\left(\frac{\partial\tilde{\mathcal{L}}}{\partial\alpha}\Big|_{\alpha=0} + \phi_1^I\frac{\partial\tilde{\mathcal{L}}(0)}{\partial\phi_0^I} + \partial_\mu\phi_1^I\frac{\partial\tilde{\mathcal{L}}(0)}{\partial\partial_\mu\phi_0^I}\right) + \dots\end{aligned} \tag{4.6}$$

From this follows that the LO and NLO coefficients in the expansion (4.2) are given by

$$\mathcal{L}_{\text{LO}} = \mathcal{L}_{\text{LO}}(\phi_0^I, \partial_\mu\phi_0^I) = \tilde{\mathcal{L}}(0), \tag{4.7}$$

$$\begin{aligned}\mathcal{L}_{\text{NLO}} &= \mathcal{L}_{\text{NLO}}\left(\{\phi_i^I, \partial_\mu\phi_i^I\}_{i=0,1}\right) = \tilde{\mathcal{L}}'(0) = \frac{\partial\tilde{\mathcal{L}}}{\partial\alpha}\Big|_{\alpha=0} + \phi_1^I\frac{\partial\mathcal{L}_{\text{LO}}}{\partial\phi_0^I} + \partial_\mu\phi_1^I\frac{\partial\mathcal{L}_{\text{LO}}}{\partial\partial_\mu\phi_0^I} \\ &= \frac{\partial\tilde{\mathcal{L}}}{\partial\alpha}\Big|_{\alpha=0} + \phi_1^I\left[\frac{\partial\mathcal{L}_{\text{LO}}}{\partial\phi_0^I} - \partial_\mu\left(\frac{\partial\mathcal{L}_{\text{LO}}}{\partial(\partial_\mu\phi_0^I)}\right)\right] = \frac{\partial\tilde{\mathcal{L}}}{\partial\alpha}\Big|_{\alpha=0} + \phi_1^I \overset{(\text{LO})}{G}{}_I^{\phi_0},\end{aligned} \tag{4.8}$$

where the second-to-last equality holds up to a total derivative and in the last one we have introduced

$$\overset{(\text{LO})}{G}{}_I^{\phi_0} := \frac{\partial\mathcal{L}_{\text{LO}}}{\partial\phi_0^I} - \partial_\mu\left(\frac{\partial\mathcal{L}_{\text{LO}}}{\partial(\partial_\mu\phi_0^I)}\right). \tag{4.9}$$

This is nothing but the LHS of the EOM for the LO field ϕ_0^I in the LO Lagrangian $\mathcal{L}_{\text{LO}}$. It follows from (4.8) that such EOM are reproduced at NLO as the EOM of the subleading field ϕ_1^I in the subleading Lagrangian $\mathcal{L}_{\text{NLO}}$. More generally, we define for every $n \in \mathbb{N}$

$$\overset{(\text{N}^n\text{LO})}{G}{}_I^{\phi_n} := \frac{\partial\mathcal{L}_{\text{N}^n\text{LO}}}{\partial\phi_n^I} - \partial_\mu\left(\frac{\partial\mathcal{L}_{\text{N}^n\text{LO}}}{\partial(\partial_\mu\phi_n^I)}\right), \tag{4.10}$$

with the understanding that $\text{N}^0\text{LO} = \text{LO}$. In this notation, the previous statement reads

$$\overset{(\text{NLO})}{G}{}_I^{\phi_1} = \overset{(\text{LO})}{G}{}_I^{\phi_0}. \tag{4.11}$$

The NNLO Lagrangian can be obtained by an analogous calculation that yields, up to total derivative,

$$
\begin{aligned}
\mathcal{L}_{\mathrm{NNLO}} &= \mathcal{L}_{\mathrm{NNLO}}\left(\{\phi_i^I, \partial_\mu \phi_i^I\}_{i=0,1,2}\right) = \frac{1}{2}\tilde{\mathcal{L}}''(0) \\
&= \frac{1}{2}\frac{\partial^2 \tilde{\mathcal{L}}}{\partial \alpha^2}\Big|_{\alpha=0} + \phi_2^I \overset{(\mathrm{LO})}{G}{}_I^{\phi_0} + \phi_1^I \left\{ \frac{\partial}{\partial \phi_0^I}\left(\frac{\partial \tilde{\mathcal{L}}}{\partial \alpha}\Big|_{\alpha=0}\right) - \partial_\mu \left[\frac{\partial}{\partial(\partial_\mu \phi_0^I)}\left(\frac{\partial \tilde{\mathcal{L}}}{\partial \alpha}\Big|_{\alpha=0}\right)\right]\right\} \\
&\quad + \frac{1}{2}\left[\phi_1^I \phi_1^J \frac{\partial^2 \mathcal{L}_{\mathrm{LO}}}{\partial \phi_0^I \partial \phi_0^J} + 2\phi_1^I \partial_\mu \phi_1^J \frac{\partial^2 \mathcal{L}_{\mathrm{LO}}}{\partial \phi_0^I \partial(\partial_\mu \phi_0^J)} + \partial_\mu \phi_1^I \partial_\nu \phi_1^J \frac{\partial^2 \mathcal{L}_{\mathrm{LO}}}{\partial(\partial_\mu \phi_0^I)\partial(\partial_\nu \phi_0^J)}\right].
\end{aligned}
\tag{4.12}
$$

One can easily see how the variation of the NNLO Lagrangian with respect to ϕ_2^I yields the EOM of the LO Lagrangian. Moreover, it can be shown that

$$
\frac{\delta \mathcal{L}_{\mathrm{NNLO}}}{\delta \phi_1^I} = \frac{\delta \mathcal{L}_{\mathrm{NLO}}}{\delta \phi_0^I}, \tag{4.13}
$$

up to a total derivative. Therefore, the EOM of the NLO Lagrangian are all reproduced at NNLO and we have the following equalities

$$
\begin{array}{ccccc}
\overset{(\mathrm{LO})}{G}{}_I^{\phi_0} & & & & \\
 & \rotatebox{-45}{$=$} & & & \\
\overset{(\mathrm{NLO})}{G}{}_I^{\phi_0} & & \overset{(\mathrm{LO})}{G}{}_I^{\phi_1} & & \\
 & \rotatebox{-45}{$=$} & & \rotatebox{-45}{$=$} & \\
\overset{(\mathrm{NNLO})}{G}{}_I^{\phi_0} & & \overset{(\mathrm{NNLO})}{G}{}_I^{\phi_1} & & \overset{(\mathrm{NNLO})}{G}{}_I^{\phi_2}
\end{array}
$$

The expansion can be extended through analogous calculations to include higher powers in α. The recursive structure of the EOM holds beyond NNLO and is a general feature of Lagrangian expansions: lower-order EOM are reproduced at higher orders. More precisely, at each order in the expansion of the Lagrangian the only new set of EOM is the one corresponding to ϕ_0^I, while the variations with respect to subleading fields simply reproduce the EOM of lower-order Lagrangians.

4.2 Pre-non-relativistic parametrisation

As we have mentioned, the first step in order to carry out the large speed of light expansion of Lorentzian geometry is to parametrise it in terms of variables that are more suitable for the expansion. This is accomplished by, first of all, splitting the Lorentzian vielbeine in its temporal and spatial directions, in order to account for their different scaling with c. This is a very logical step, in anticipation of the natural separation between time and space that emerges in a non-relativistic limit. However, this PNR parametrisation also requires the introduction of an affine connection different from the Levi-Civita one, as well as expressing the relevant curvature objects in terms of it. As we will see, this PNR connection is particularly adapted to the expansion due to its relation to the "preferred" NC compatible connection (3.33) that we introduced in Section 3.3.1.

4.2.1 Pre-non-relativistic fields

Let us consider the relativistic vielbeine fields $E_\mu{}^A$ and their inverses $E^\mu{}_A$, characterising a $(d+1)$-dimensional Lorentzian manifold. Here, $\mu = 0, \ldots, d$ are spacetime coordinate indices and $A = 0, \ldots, d$ are frame indices. We then have the following orthonormality and completeness relations

$$E_\mu{}^A E^\mu{}_B = \delta^A_B, \qquad E_\mu{}^A E^\nu{}_A = \delta^\nu_\mu. \tag{4.14}$$

As we said, the time and space components of the vielbeine and their inverses should scale differently with c. Therefore, we need to carefully choose the overall factors of c in these, if we want every field expansion to start at order c^0. To this end, we introduce a 'timelike' one-form T_μ and vector V^μ defined through the relations

$$E_\mu{}^A = (cT_\mu, E_\mu{}^a), \qquad E^\mu{}_A = \left(-\frac{1}{c}V^\mu, E^\mu{}_a\right), \tag{4.15}$$

where $a = 1, \ldots, d$ is a spatial frame index. We can then rewrite the orhonormality and completeness relations (4.14) as

$$T_\mu V^\mu = -1, \quad T_\mu E^\mu{}_a = 0, \quad V^\mu E_\mu{}^a = 0, \quad E_\mu{}^a E^\mu{}_b = \delta^a_b, \quad E_\mu{}^a E^\nu{}_a = \delta^\nu_\mu + V^\nu T_\mu. \tag{4.16}$$

From the spatial vielbeine and their inverses we can build the following symmetric tensors,

$$\Pi_{\mu\nu} := \delta_{ab} E_\mu{}^a E_\nu{}^b, \qquad \Pi^{\mu\nu} := \delta^{ab} E^\mu{}_a E^\nu{}_b. \tag{4.17}$$

We shall refer to the fields T_μ, V^μ, $\Pi_{\mu\nu}$ and $\Pi^{\mu\nu}$ as the *pre-non-relativistic fields*. They satisfy the following orthogonality and completeness relations,

$$T_\mu V^\mu = -1, \qquad T_\mu \Pi^{\mu\nu} = 0, \qquad V^\mu \Pi_{\mu\nu} = 0, \qquad \Pi_{\mu\rho}\Pi^{\rho\nu} = \delta^\nu_\mu + V^\nu T_\mu, \tag{4.18}$$

as follows from (4.16). Note that $\Pi_{\mu\nu}$ and $\Pi^{\mu\nu}$ have a degeneracy spanned by the directions of V^μ and T_μ, respectively. In particular, they are not invertible, but they can be thought as projective inverses with projector defined by

$$P^\nu{}_\mu := \delta^\nu_\mu + V^\nu T_\mu, \tag{4.19}$$

so that $\Pi_{\mu\rho}\Pi^{\rho\nu} = P^\nu{}_\mu$. We can now write the Lorentzian metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ in terms of the PNR fields,

$$g_{\mu\nu} \equiv \eta_{AB} E_\mu{}^A E_\nu{}^B = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}, \tag{4.20a}$$

$$g^{\mu\nu} \equiv \eta^{AB} E^\mu{}_A E^\nu{}_B = -\frac{1}{c^2} V^\mu V^\nu + \Pi^{\mu\nu}. \tag{4.20b}$$

Similarly, for the factor $\sqrt{-g}$ appearing in the integral measure we write

$$\sqrt{-g} = cE, \qquad E := \sqrt{-\det(-T_\alpha T_\beta + \Pi_{\alpha\beta})}. \tag{4.21}$$

The vielbeine and their inverses transform under diffeomorphisms and LLTs that preserve η_{AB} on the frame bundle according to

$$\delta E_\mu{}^A = \mathcal{L}_\Xi E_\mu{}^A + \Lambda^A{}_B E_\mu{}^B, \tag{4.22a}$$

$$\delta E^\mu{}_A = \mathcal{L}_\Xi E^\mu{}_A - \Lambda^B{}_A E^\mu{}_B, \tag{4.22b}$$

where Ξ^μ is a diffeomorphism-generating vector field and $\Lambda^A{}_B$, with $\Lambda_{AB} = -\Lambda_{BA}$ is the generator of infinitesimal LLTs. Introducing the rescaled generators $\Lambda^a = c\Lambda^a{}_0$ and $\Lambda_a = c\Lambda^0{}_a$, and using (4.15), we find that the PNR fields transform as follows under diffeomorphisms and LLTs,

$$\delta T_\mu = \mathcal{L}_\Xi T_\mu + c^{-2}\Lambda_a E_\mu{}^a, \tag{4.23a}$$
$$\delta \Pi_{\mu\nu} = \mathcal{L}_\Xi \Pi_{\mu\nu} + 2\Lambda_a T_{(\mu} E_{\nu)}{}^a, \tag{4.23b}$$
$$\delta V^\mu = \mathcal{L}_\Xi V^\mu + \Lambda^a E^\mu{}_a, \tag{4.23c}$$
$$\delta \Pi^{\mu\nu} = \mathcal{L}_\Xi \Pi^{\mu\nu} + 2c^{-2}\Lambda^a V^{(\mu} E^{\nu)}{}_a, \tag{4.23d}$$

where we have used that the spatial vielbeine transform as

$$\delta E_\mu{}^a = \mathcal{L}_\Xi E_\mu{}^a + \Lambda^a T_\mu + \Lambda^a{}_b E_\mu{}^b, \tag{4.24a}$$
$$\delta E^\mu{}_a = \mathcal{L}_\Xi E^\mu{}_a - \Lambda^b{}_a E^\mu{}_b + c^{-2}\Lambda_a V^\mu. \tag{4.24b}$$

Notice that one could be tempted to conclude that the pair $(T_\mu, \Pi^{\mu\nu})$ is a NC pair in the sense of Section 3.3.1, but it is not because of the transformation laws in (4.23), which are nothing but LLTs. Of course, this comes as no surprise since the vielbeine that compose the PNR fields are Lorentzian.

4.2.2 Pre-non-relativistic connection and curvature

As we shall see later, the PNR parametrisation of the metric tensor in (4.20a) greatly simplifies the non-relativistic expansion of GR. Following our starting assumption (4.53), such expansion will require to rewrite GR in terms of fields the small σ expansion of which starts at order σ^0 with unconstrained leading order fields. However, the leading order term of the metric in the form (4.20a) is not unconstrained, as it is bound to be a product of two 1-forms. In order to proceed, it is therefore necessary to write the whole Einstein-Hilbert Lagrangian in terms of the PNR fields. To do so, we will first need to find an appropriate PNR parametrisation of the Levi-Civita connection and see how this translates into a PNR parametrisation of the curvature tensors. In each case, this will make the explicit c dependence due to (4.20) obvious.

For the purpose of what follows, it will also prove convenient to introduce a connection different than the Levi-Civita one, that we call the *pre-non-relativistic connection* and define by

$$C^\rho_{\mu\nu} := -V^\rho \partial_\mu T_\nu + \frac{1}{2}\Pi^{\rho\sigma}\left(\partial_\mu \Pi_{\nu\sigma} + \partial_\nu \Pi_{\mu\sigma} - \partial_\sigma \Pi_{\mu\nu}\right). \tag{4.25}$$

Note that this is a torsionful connection with torsion given by

$$\overset{(C)}{T}{}^\rho{}_{\mu\nu} := 2C^\rho_{[\mu\nu]} = -V^\rho T_{\mu\nu}, \tag{4.26}$$

where we have defined the two-form

$$T_{\mu\nu} := 2\partial_{[\mu} T_{\nu]}. \tag{4.27}$$

After these considerations, we can now write the Levi-Civita connection

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \tag{4.28}$$

as

$$\Gamma^{\rho}_{\mu\nu} = c^2 \overset{(-2)}{C^{\rho}_{\mu\nu}} + \overset{(0)}{C^{\rho}_{\mu\nu}} + c^{-2} \overset{(2)}{C^{\rho}_{\mu\nu}}, \tag{4.29}$$

where

$$\overset{(0)}{C^{\rho}_{\mu\nu}} = C^{\rho}_{\mu\nu} + S^{\rho}_{\mu\nu}, \tag{4.30}$$

and

$$\overset{(-2)}{C^{\rho}_{\mu\nu}} = T_{(\mu}\Pi^{\rho\sigma}T_{\sigma\nu)}, \tag{4.31a}$$

$$S^{\rho}_{\mu\nu} = \frac{1}{2}\left(T_{\mu\nu} - 2T_{(\mu}\mathcal{L}_V T_{\nu)}\right), \tag{4.31b}$$

$$\overset{(2)}{C^{\rho}_{\mu\nu}} = \frac{1}{2}V^{\rho}\mathcal{L}_V\Pi_{\mu\nu}. \tag{4.31c}$$

Note that we have split $\overset{(0)}{C^{\rho}_{\mu\nu}}$ in (4.30) in order to relate $C^{\rho}_{\mu\nu}$ to the PNR parametrisation of the Levi-Civita connection. The extra factor $S^{\rho}_{\mu\nu}$ accounts for the fact that the PNR connection is torsionful while the Levi-Civita one is not. Indeed, $2S^{\rho}_{[\mu\nu]} = V^{\rho}T_{\mu\nu}$ so that $\overset{(0)}{C^{\rho}_{\mu\nu}}$ is torsionless, as it should.

The PNR connection is particularly useful for calculating covariant derivatives, that we denote by $\overset{(C)}{\nabla}_{\mu}$. It is also useful to consider the Riemann tensor associated to the PNR connection, that we denote by $\overset{(C)}{R}_{\mu\nu\rho}{}^{\sigma}$, and its usual contractions. The covariant derivative with respect to the PNR connection satisfies

$$\overset{(C)}{\nabla}_{\mu}T_{\nu} = 0, \qquad \overset{(C)}{\nabla}_{\mu}\Pi^{\nu\rho} = 0, \qquad \overset{(C)}{\nabla}_{\mu}V^{\nu} = \frac{1}{2}\Pi^{\nu\rho}\mathcal{L}_V\Pi_{\rho\mu}, \qquad \overset{(C)}{\nabla}_{\mu}\Pi_{\nu\rho} = T_{(\nu}\mathcal{L}_V\Pi_{\rho)\nu}. \tag{4.32}$$

The vanishing of the coviariant derivatives of T_{μ} and $\Pi^{\mu\nu}$ is to be interpreted as a PNR version of the metric compatibility condition satisfied by the Levi-Civita connection.

The Ricci tensor $R_{\mu\nu}$ associated to the Levi-Civita connection takes the following PNR form

$$R_{\mu\nu} = c^4 \overset{(-4)}{R}_{\mu\nu} + c^2 \overset{(-2)}{R}_{\mu\nu} + \overset{(0)}{R}_{\mu\nu} + c^{-2} \overset{(2)}{R}_{\mu\nu} + c^{-4} \overset{(4)}{R}_{\mu\nu}, \tag{4.33}$$

where

$$\overset{(-4)}{R}_{\mu\nu} = \frac{1}{4}T_{\mu}T_{\nu}\Pi^{\rho\sigma}\Pi^{\lambda\kappa}T_{\rho\lambda}T_{\sigma\kappa}, \tag{4.34a}$$

$$\overset{(-2)}{R}_{\mu\nu} = \overset{(C)}{\nabla}_{\rho}\overset{(-2)}{C^{\rho}_{\mu\nu}} + \overset{(-2)}{C^{\lambda}_{\mu\nu}}S^{\rho}_{\rho\lambda} - \overset{(-2)}{C^{\rho}_{\mu\lambda}}S^{\lambda}_{\rho\nu} - \overset{(-2)}{C^{\lambda}_{\rho\nu}}S^{\rho}_{\mu\lambda} - 2C^{\lambda}_{[\mu\rho]}\overset{(-2)}{C^{\rho}_{\lambda\nu}}, \tag{4.34b}$$

$$\overset{(0)}{R}_{\mu\nu} = \overset{(C)}{R}_{\mu\nu} - \overset{(-2)}{C^{\rho}_{\mu\lambda}}\overset{(2)}{C^{\lambda}_{\rho\nu}} - \overset{(-2)}{C^{\rho}_{\nu\lambda}}\overset{(2)}{C^{\lambda}_{\rho\mu}} - \overset{(C)}{\nabla}_{\mu}S^{\rho}_{\rho\nu} + \overset{(C)}{\nabla}_{\rho}S^{\rho}_{\mu\nu} - 2C^{\lambda}_{[\mu\rho]}S^{\rho}_{\lambda\nu}, \tag{4.34c}$$

$$\overset{(2)}{R}_{\mu\nu} = \overset{(C)}{\nabla}_{\rho}\overset{(2)}{C^{\rho}_{\mu\nu}}, \tag{4.34d}$$

$$\overset{(4)}{R}_{\mu\nu} = 0. \tag{4.34e}$$

Let us stress again the fact that we have not yet performed any large speed of light expansion. In expressions (4.29) and (4.33), all we have done is to rewrite the Levi-Civita connection and its associated Ricci tensor in terms of the PNR fields, and then collect together the terms that scale equally in powers of c^{-2}. Before moving on to the PNR version of the Ricci scalar (which is what we are interested in since it enters the EH Lagrangian), some comments are in order.

Stokes' Theorem and PNR connection Consider a D-dimensional Lorentzian manifold M with boundary ∂M and Levi-Civita connection $\Gamma^{\rho}_{\mu\nu}$, to which we associate a covariant derivative ∇. The trace of the Levi-Civita connection is given by

$$\Gamma^{\mu}_{\mu\nu} = \partial_{\nu}\left(\log\sqrt{-g}\right), \tag{4.35}$$

from which follows that, for any vector field X^{μ} on M,

$$\nabla_{\mu}X^{\mu} = \frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}X^{\mu}\right). \tag{4.36}$$

This last expression followed by an integration by parts is enough to prove Stokes' Theorem,

$$\int_M d^D x\sqrt{-g}\nabla_{\mu}X^{\mu} = \int_{\partial M} d^{D-1}x\sqrt{-\gamma}n_{\mu}X^{\mu}, \tag{4.37}$$

where n^{μ} is the unit vector normal to the boundary and γ the determinant of the induced metric on the boundary. The RHS of (4.37) is a boundary term, which can usually be set to zero in an action integral. Since this whole reasoning is what allows us to neglect terms of the form $\sqrt{-g}\nabla_{\mu}X^{\mu}$ that appear in Lagrangian densities, it follows that this need not be true when working with a general (possibly torsionful) connection. For instance, the analogue version of (4.35) for the PNR connection reads

$$C^{\mu}_{\mu\nu} = \partial_{\nu}\left(\log E\right) - \mathcal{L}_V T_{\nu}. \tag{4.38}$$

This in turn modifies (4.36) to

$$\overset{(C)}{\nabla}_{\mu}X^{\mu} = \frac{1}{E}\partial_{\mu}(EX^{\mu}) - X^{\nu}\mathcal{L}_V T_{\nu}. \tag{4.39}$$

Using that $\mathcal{L}_V T_{\nu} = V^{\mu}T_{\mu\nu} = -\overset{(C)}{T}{}^{\rho}{}_{\mu\nu}$, we realise that the extra term is indeed due to the fact that $C^{\rho}_{\mu\nu}$ has non-zero torsion. Thus, the analogue of Stokes' Theorem in this case reads

$$\int_M d^D x E\left(\overset{(C)}{\nabla}_{\mu}X^{\mu} + T_{\mu\nu}V^{\mu}X^{\nu}\right) = \int_{\partial M} d^{D-1}x\sqrt{-\gamma}n_{\mu}X^{\mu}. \tag{4.40}$$

The PNR parametrisation of the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ can be obtained from (4.20a) and (4.33). In the process, it is convenient to use the following identities

$$\Pi^{\mu\nu}\overset{(-2)}{R}_{\mu\nu} = \frac{1}{2}\Pi^{\mu\nu}\Pi^{\rho\sigma}T_{\mu\rho}T_{\nu\sigma}, \tag{4.41a}$$

$$V^{\mu}V^{\nu}\overset{(-2)}{R}_{\mu\nu} = 0, \tag{4.41b}$$

$$\Pi^{\mu\nu}\overset{(0)}{R}_{\mu\nu} = \Pi^{\mu\nu}\overset{(C)}{R}_{\mu\nu}, \tag{4.41c}$$

$$V^{\mu}V^{\nu}\overset{(0)}{R}_{\mu\nu} = V^{\mu}V^{\nu}\overset{(C)}{R}_{\mu\nu}, \tag{4.41d}$$

$$\Pi^{\mu\nu}\overset{(2)}{R}_{\mu\nu} = 0, \tag{4.41e}$$

$$V^{\mu}V^{\nu}\overset{(2)}{R}_{\mu\nu} = 0. \tag{4.41f}$$

All of the identities above, except for the first one, only hold up to a total derivative, in the sense of

$$\frac{1}{E}\partial_{\mu}(EX^{\mu}) = \overset{(C)}{\nabla}_{\mu}X^{\mu} + T_{\mu\nu}V^{\mu}X^{\nu}, \tag{4.42}$$

explained in the previous remark. For this to work one first needs to realise that the objects in (4.31) are all tensors. Therefore, the Ricci scalar associated with the Levi-Civita connection has the following PNR parametrisation

$$R = \frac{c^2}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma}T_{\mu\rho}T_{\nu\sigma} + \Pi^{\mu\nu}\overset{(C)}{R}_{\mu\nu} - \frac{1}{c^2}V^\mu V^\nu \overset{(C)}{R}_{\mu\nu}. \tag{4.43}$$

This expression, as well as the PNR version of the measure (4.21), are all the ingredients we need to write down the PNR parametrisation of the EH Lagrangian.

4.2.3 Einstein-Hilbert Lagrangian

Let us start discussing the dimensionful normalisation of the EH Lagrangian. The dimension of the line element does not change when writing the metric in PNR form, so its dimension is still L^2 with L denoting length. This means that $T_\mu dx^\mu$ has dimensions of length (with our choice of units $\hat{c} = 1$) and $\Pi_{\mu\nu}dx^\mu dx^\nu$ has dimensions of length squared. Regarding the measure $Edtd^dx$, it has dimensions of TL^d, with T denoting time. We then take the EH action to be

$$S_{EH} = \frac{c^3}{16\pi G_N}\int dtd^dx\sqrt{-g}R, \tag{4.44}$$

where the unusual overall power of c^3 (opposed to the most common in the literature c^4) accounts for the fact that there is an implicit factor of c in $\sqrt{-g}$. We are now equipped with all the ingredients to write down the EH Lagrangian as a function of the PNR fields and connection. In particular, using (4.21) and (4.43), we find that the EH Lagrangian can be rewritten as

$$\mathcal{L}_{EH} = c^6\tilde{\mathcal{L}}_{EH}(\sigma, T, \Pi, \partial), \tag{4.45}$$

where

$$\tilde{\mathcal{L}}_{EH} = \frac{E}{16\pi G_N}\left(\frac{1}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma}T_{\mu\rho}T_{\nu\sigma} + \sigma\Pi^{\mu\nu}\overset{(C)}{R}_{\mu\nu} - \sigma^2 V^\mu V^\nu \overset{(C)}{R}_{\mu\nu}\right). \tag{4.46}$$

This is the form of the EH Lagrangian to which we will apply the results of Section 4.1. Note that in (4.46) we have factored out the overall power of c^6, coming from the fact that R is order c^2 and $\sqrt{-g}$ is order c, following (4.3). We emphasise one more time that no large c expansion has been performed yet, meaning that the Lagrangian (4.45) is completely equivalent to the usual EH Lagrangian and describes the same physics as GR. The PNR parametrisation, however, makes the c dependence of the theory explicit by accounting for the different way in which time and space components scale with c. Indeed, one can for example wonder how the Einstein field equations (EFEs) look in PNR form. Of course, the usual Einstein tensor $G^{\mu\nu}$, being defined as the variation of the EH Lagrangian with respect to the metric tensor, is no longer a "valid" object in the PNR parametrisation. Instead, we introduce the tensors E^μ_G and $E^{\mu\nu}_G$, defined implicitly by

$$\delta\mathcal{L}_{EH} := -\frac{c^6}{8\pi G_N}E\left(E^\mu_G\delta T_\mu + \frac{1}{2}E^{\mu\nu}_G\delta\Pi_{\mu\nu}\right). \tag{4.47}$$

Let us also consider the coupling to some matter fields ϕ described by a Lagrangian $\mathcal{L}_M = \mathcal{L}_M(\sigma, \phi, \partial_\mu\phi)$ starting at order c^N. We can now consider a variation of $\mathcal{L}_M$ with respect to the PNR fields and write

$$\delta\mathcal{L}_M := c^N E\left(E^\mu_M\delta T_\mu + \frac{1}{2}E^{\mu\nu}_M\delta\Pi_{\mu\nu}\right). \tag{4.48}$$

Then, we have the following PNR analogue of the EFEs

$$E^{\mu}_{G} = 8\pi G_N c^{N-6} E^{\mu}_{M}, \qquad E^{\mu\nu}_{G} = 8\pi G_N c^{N-6} E^{\mu\nu}_{M}. \tag{4.49}$$

One can also find an analogous version of the divergencelessness of the usual Einstein tensor due to diffeomorphism invariance of the EH action. Indeed, considering the variations

$$\delta_\Xi T_\mu = \mathcal{L}_\Xi T_\mu, \qquad \delta_\Xi \Pi_{\mu\nu} = \mathcal{L}_\Xi \Pi_{\mu\nu}, \tag{4.50}$$

of T_μ and $\Pi_{\mu\nu}$ under a diffeomorphism generated by Ξ^μ and requiring $\delta_\Xi \mathcal{L}_{EH} = 0$ (up to a total derivative) yields the following identity

$$T_\rho \left(\overset{(C)}{\nabla}_\mu + \mathcal{L}_V T_\mu \right) E^{\mu}_{G} + \Pi_{\nu\rho} \left(\overset{(C)}{\nabla}_\mu + \mathcal{L}_V T_\mu \right) E^{\mu\nu}_{G} + E^{\mu}_{G} T_{\mu\rho} + \frac{1}{2} T_\rho E^{\mu\nu}_{G} \mathcal{L}_V \Pi_{\mu\nu} = 0. \tag{4.51}$$

To obtain this result, we have used the following identity

$$\begin{aligned}
\mathcal{L}_\Xi X^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} = & \Xi^\alpha \nabla_\alpha X^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \\
& - \nabla_\alpha \Xi^{\mu_1} X^{\alpha \dots \mu_r}{}_{\nu_1 \dots \nu_s} - \nabla_\alpha \Xi^{\mu_2} X^{\mu_1 \alpha \dots \mu_r}{}_{\nu_1 \dots \nu_s} - \dots \\
& - \Xi^\alpha T^{\mu_1}{}_{\alpha\beta} X^{\beta \dots \mu_r}{}_{\nu_1 \dots \nu_s} - \Xi^\alpha T^{\mu_2}{}_{\alpha\beta} X^{\mu_1 \beta \dots \mu_r}{}_{\nu_1 \dots \nu_s} - \dots \\
& + \nabla_{\nu_1} \Xi^\alpha X^{\mu_1 \dots \mu_r}{}_{\alpha \dots \nu_s} + \nabla_{\nu_2} \Xi^\alpha X^{\mu_1 \dots \mu_r}{}_{\nu_1 \alpha \dots \nu_s} + \dots \\
& + \Xi^\alpha T^{\beta}{}_{\alpha\nu_1} X^{\mu_1 \dots \mu_r}{}_{\beta \dots \nu_s} + \Xi^\alpha T^{\beta}{}_{\alpha\nu_2} X^{\mu_1 \dots \mu_r}{}_{\nu_1 \beta \dots \nu_s} + \dots,
\end{aligned} \tag{4.52}$$

which holds for an arbitrary tensor $X^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}$ and any torsionful connection with associated covariant derivative ∇ and torsion $T^\rho{}_{\mu\nu}$.

4.3 Large speed of light expansion

In the following, we carry out the actual large c expansion of the Lorentzian geometry of GR. In analogy with (4.1), our starting assumption is that all the relevant fields (PNR fields, diffeomorphism-generating vector fields, LLTs...), denoted generally by $\phi^I(\sigma, x)$, are analytic in σ such that they admit a Taylor expansion around $\sigma = 0$,

$$\phi^I(\sigma, x) = \phi^I(\sigma, x) = \phi^I_0(x) + \sigma \phi^I_1(x) + \sigma^2 \phi^I_2(x) + \mathcal{O}(\sigma^3). \tag{4.53}$$

Expanding the PNR fields and their transformation laws will elucidate how NC geometry arises from Lorentzian geometry. In particular, at LO one finds that the expansion results in a NC metric structure, which will eventually result in TTNC geometry when we put the theory on shell in Section 4.4, while type II TNC geometry emerges at NLO. We shall also show how the latter can be obtained by gauging the level one expansion of the Poincaré algebra.

4.3.1 Vielbeine and metric

Assuming that the PNR fields enjoy a Taylor expansion around $\sigma = 0$ of the form (4.53), we can write them as

$$T_\mu = \tau_\mu + \sigma m_\mu + \sigma^2 B_\mu + \mathcal{O}(\sigma^3), \tag{4.54a}$$

$$\Pi_{\mu\nu} = h_{\mu\nu} + \sigma \Phi_{\mu\nu} + \sigma^2 \psi_{\mu\nu} + \mathcal{O}(\sigma^3), \tag{4.54b}$$

$$V^\mu = v^\mu + \sigma \left(v^\mu v^\rho m_\rho - e^\mu{}_b v^\rho \pi_\rho{}^b \right) + \mathcal{O}(\sigma^2), \tag{4.54c}$$

$$\Pi^{\mu\nu} = h^{\mu\nu} + \sigma \left(2h^{\rho(\mu} v^{\nu)} m_\rho - h^{\mu\rho} h^{\nu\sigma} \Phi_{\rho\sigma} \right) + \mathcal{O}(\sigma^2), \tag{4.54d}$$

where we have used the following expansion for the spatial components of the vielbeine and their inverses,

$$E_\mu{}^a = e_\mu{}^a + \sigma\pi_\mu{}^a + \sigma^2\kappa_\mu{}^a + \mathcal{O}(\sigma^3), \tag{4.55a}$$
$$E^\mu{}_a = e^\mu{}_a + \sigma\left(v^\mu e^\rho{}_a m_\rho - e^\mu{}_b e^\rho{}_a \pi_\rho{}^b\right) + \mathcal{O}(\sigma^2), \tag{4.55b}$$

and defined

$$h_{\mu\nu} := \delta_{ab} e_\mu{}^a e_\nu{}^b, \tag{4.56a}$$
$$h^{\mu\nu} := \delta^{ab} e^\mu{}_a e^\nu{}_b, \tag{4.56b}$$
$$\Phi_{\mu\nu} := 2\delta_{ab} e_{(\mu}{}^a \pi_{\nu)}{}^b, \tag{4.56c}$$
$$\psi_{\mu\nu} := \delta_{ab}\left(\pi_\mu{}^a \pi_\nu{}^b + 2e_{(\mu}{}^a \kappa_{\nu)}{}^b\right) \tag{4.56d}$$

Note that we have not introduced new variables for the subleading terms in the expansions of V^μ, $E^\mu{}_a$ or $\Pi^{\mu\nu}$, since these are determined through (4.18) by the leading order terms and the subleading ones in the expansion of T_μ, $E_\mu{}^a$ and $\Pi_{\mu\nu}$. For the metric tensor and its inverse (4.20), this means that they are large c expanded according to

$$g_{\mu\nu} = -c^2\tau_\mu\tau_\nu + \bar{h}_{\mu\nu} + \sigma\bar{\Phi}_{\mu\nu} + \mathcal{O}(\sigma^2), \tag{4.57a}$$
$$g^{\mu\nu} = h^{\mu\nu} - \sigma(\bar{v}^\mu\bar{v}^\nu + h^{\mu\rho}h^{\nu\sigma}\bar{\Phi}_{\rho\sigma}) + \mathcal{O}(\sigma^2), \tag{4.57b}$$

where we have introduced the tensors

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - 2\tau_{(\mu}m_{\nu)}, \tag{4.58a}$$
$$\bar{v}^\mu := v^\mu - h^{\mu\rho}m_\rho, \tag{4.58b}$$
$$\bar{\Phi}_{\mu\nu} := \Phi_{\mu\nu} - m_\mu m_\nu - 2B_{(\mu}\tau_{\nu)}. \tag{4.58c}$$

The LO fields in (4.54) satisfy the following orthogonality and completeness relations, obtained by expanding (4.18) and collecting terms at leading order,

$$\tau_\mu v^\mu = -1, \qquad \tau_\mu h^{\mu\nu} = 0, \qquad v^\mu h_{\mu\nu} = 0, \qquad h_{\mu\rho}h^{\rho\nu} = \delta_\mu^\nu + v^\nu\tau_\mu. \tag{4.59}$$

4.3.2 Newton-Cartan geometry from Lorentzian geometry

We see from (4.57) that the LO terms in the expansion of the metric and its inverse give rise to a degenerate metric structure given by $\tau_\mu\tau_\nu$ and $h^{\mu\nu}$, which has a degeneracy of degree one. It then follows from (4.59) that

$$\ker h = \mathrm{span}\{\tau\}. \tag{4.60}$$

As argued before, $(\tau_\mu, h^{\mu\nu})$ is not a NC pair until they realise the transformation laws resulting from local Galilean symmetry. To see how these arise from the diffeomorphisms and LLTs of GR, we expand in small σ the diffeomorphism generating vector field Ξ^μ and the generator of infinitesimal LLTs $\Lambda^A{}_B$ as

$$\Xi^\mu = \xi^\mu + \sigma\zeta^\mu + \mathcal{O}(\sigma^2), \tag{4.61a}$$
$$\Lambda^a = \lambda^a + \sigma\eta^a + \mathcal{O}(\sigma^2), \tag{4.61b}$$
$$\Lambda^a{}_b = \lambda^a{}_b + \sigma\rho^a{}_b + \mathcal{O}(\sigma^2), \tag{4.61c}$$

where recall that $\Lambda^a := c\Lambda^a{}_0$. We interpret this as ξ^μ being a vector field generating diffeomorphisms, while ζ^μ generates gauge transformations acting on the subleading fields m_μ and $\pi_\mu{}^a$. As for the LLTs, we interpret the expansion as giving rise to a local Galilean boost parameter λ^a and its subleading version η^a and to a local spatial rotation parameter $\lambda^a{}_b$ and its subleading version $\rho^a{}_b$. This follows from the fact that taking the non-relativistic limit of a Lorentz boost yields a Galilean boost.

We can now expand (4.23) and (4.24) according to (4.54) and (4.61). Collecting terms order by order in σ will then yield the transformation laws that implement the corresponding non-relativistic local symmetry of the resulting geometry at each order in the expansion. At leading order, for instance, we obtain a geometry defined in terms of the pair

$$(\tau_\mu, h^{\mu\nu}),$$

with projective inverses $(v^\mu, h_{\mu\nu})$. Moreover, we can also expand the PNR connection (4.25), so that its leading order term is given by

$$\check{C}^\rho_{\mu\nu} := C^\rho_{\mu\nu}\big|_{\sigma=0} = -v^\rho\partial_\mu\tau_\nu + \frac{1}{2}h^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}). \tag{4.62}$$

Its associated covariant derivative operator $\check{\nabla}$ then satisfies

$$\check{\nabla}_\mu\tau_\nu = 0, \qquad \check{\nabla}_\mu h^{\nu\rho} = 0, \tag{4.63}$$

as follows from the LO expansion of (4.32). Therefore, it satisfies the NC metric compatibility conditions (3.29).

The LO fields then transform under diffeomorphisms and local Galilean transformations according to

$$\begin{aligned} \delta\tau_\mu &= \mathcal{L}_\xi\tau_\mu, && (4.64a)\\ \delta h^{\mu\nu} &= \mathcal{L}_\xi h^{\mu\nu}, && (4.64b)\\ \delta v^\mu &= \mathcal{L}_\xi v^\mu + \lambda^a e^\mu{}_a, && (4.64c)\\ \delta h_{\mu\nu} &= \mathcal{L}_\xi h_{\mu\nu} + 2\tau_{(\mu}\lambda_{\nu)}, && (4.64d) \end{aligned}$$

where $\lambda_\mu := e_\mu{}^a\lambda_a$ and using that the spatial vielbeine transform as

$$\begin{aligned} \delta e_\mu{}^a &= \mathcal{L}_\xi e_\mu{}^a + \lambda^a\tau_\mu + \lambda^a{}_b e_\mu{}^b, && (4.65a)\\ \delta e^\mu{}_a &= \mathcal{L}_\xi + \lambda_a{}^b e^\mu{}_b, && (4.65b) \end{aligned}$$

as follows from truncating (4.61) at LO and substituting in (4.23) and (4.24). The transformations (4.64) reproduce the ones in (3.27), from which we conclude that the large speed of light expansion of the Lorentzian geometry of GR yields NC geometry at LO.

Let us now see what is the geometry that arises at NLO in the expansion. In this case, from the expansion of (4.23) one obtains that the LO fields transform as in (4.64) and (4.65). For the subleading fields in the expansions of the vielbeine we find

$$\begin{aligned} \delta m_\mu &= \mathcal{L}_\xi m_\mu + \mathcal{L}_\zeta\tau_\mu + \lambda_a e_\mu{}^a, && (4.66a)\\ \delta\pi_\mu{}^a &= \mathcal{L}_\xi\pi_\mu{}^a + \mathcal{L}_\zeta\pi_\mu{}^a + \lambda^a m_\mu + \eta^a\tau_\mu + \lambda^a{}_b\pi_\mu{}^b + \rho^a{}_b e_\mu{}^b. && (4.66b) \end{aligned}$$

In some contexts, it is preferable to work with the Milne boost invariant quantites $\bar{v}^\mu$, $\bar{h}_{\mu\nu}$, $\bar{\Phi}_{\mu\nu}$ and $\bar{\Phi}$ appearing in the expansion (4.57) of the metric and its inverse. For completeness, we also record here their transformations properties

$$\delta\bar{v}^\mu = \mathcal{L}_\xi \bar{v}^\mu - h^{\mu\rho}\mathcal{L}_\zeta \tau_\rho, \tag{4.67a}$$
$$\delta\bar{h}_{\mu\nu} = \mathcal{L}_\xi \bar{h}_{\mu\nu} - 2\tau_{(\mu}\mathcal{L}_\zeta \tau_{\nu)}, \tag{4.67b}$$
$$\delta\bar{\Phi}_{\mu\nu} = \mathcal{L}_\xi \bar{\Phi}_{\mu\nu} + \mathcal{L}_\zeta \bar{h}_{\mu\nu}, \tag{4.67c}$$
$$\delta\bar{\Phi} = \mathcal{L}_\xi \bar{\Phi} - \bar{v}^\mu \mathcal{L}_\zeta \tau_\mu. \tag{4.67d}$$

Notice that these fields are also invariant under local spatial rotations and subleading tansformations with parameters η^a and $\rho^a{}_b$.

If we now write, without loss of generality, the subleading diffeomorphism generating vector field ζ^μ as

$$\zeta^\mu = -\Omega v^\mu + h^{\mu\nu}\zeta_\nu, \tag{4.68}$$

we can then rewrite the transformation for m_μ and use (4.66b) to obtain the transformation for $\Phi_{\mu\nu}$. This yields

$$\delta m_\mu = \mathcal{L}_\xi m_\mu + \lambda_\mu + \partial_\mu \Omega - \Omega a_\mu + 2h^{\rho\sigma}\zeta_\sigma \partial_{[\rho}\tau_{\mu]}, \tag{4.69a}$$
$$\delta\Phi_{\mu\nu} = \mathcal{L}_\xi \Phi_{\mu\nu} + 2\lambda_a \left(\tau_{(\mu}\pi_{\nu)}{}^a + m_{(\mu}e_{\nu)}{}^a\right) + 2\eta_a e_{(\mu}{}^a\tau_{\nu)} + 2\Omega K_{\mu\nu} + 2\check{\nabla}_{(\mu}\zeta_{\nu)}, \tag{4.69b}$$

where a_μ is the torsion vector for the connection (4.62), i. e.,

$$a_\mu := 2\check{C}^\rho_{[\mu\rho]} = -2v^\rho \partial_{[\mu}\tau_{\rho]} = \mathcal{L}_v \tau_\mu, \tag{4.70}$$

and we have introduced the extrinsic curvature

$$K_{\mu\nu} := -\frac{1}{2}\mathcal{L}_v h_{\mu\nu}. \tag{4.71}$$

Therefore, the result of expanding Lorentzian geometry at NLO is a geometry realised by the fields

$$(\tau_\mu, h^{\mu\nu}, m_\mu, \pi_\mu{}^a),$$

or alternatively,

$$(\tau_\mu, h^{\mu\nu}, m_\mu, \Phi_{\mu\nu}),$$

transforming as described above. If we compare this to type I TNC, we immediately realise that the expansion gives rise to one extra field, namely $\pi_\mu{}^a$ or $\Phi_{\mu\nu}$. In addition, the transformation rule for m_μ in (4.69a) does not coincide with the one in (3.65) corresponding to type I TNC geometry, unless τ_μ is closed. Indeed, notice that in this case the last two terms in (4.69a) vanish due to $\mathrm{d}\tau = 0$ and using (4.70). The geometry arising from the large speed of light expansion of Lorentzian geometry at NLO is therefore different from type I TNC and it is precisely the one we introduced as type II TNC geometry in Section 3.3.3. Next, we shall elucidate what is the underlying symmetry algebra of this geometry.

4.3.3 Poincaré algebra

We have seen in Sections 2.4 and 3.3.1 how Lorentzian and NC geometries can be elegantly obtained by gauging their corresponding symmetry algebras. It is then natural to study the large c expansion of the Poincaré algebra itself, which was first considered in [44] and subsequently in [10, 46], by means of the method of Lie algebra expansions [23, 60]. We will perform such expansion and then show that the gauging of the level one expanded algebra yields the relevant geometric fields of the type II TNC geometry emerging from the expansion of Lorentzian geometry at subleading order.

We start by considering the Poincaré generators $T_I = \{H, P_a, K_a, J_{ab}\}$ as in Section 3.1. It is now convenient for our purposes to make the factors of σ explicit in the structure constants of the Poincaré algebra. With this consideration, the commutation relations (2.72) become

$$\begin{aligned} &[H, K_a] = P_a, &&[J_{ab}, P_c] = \delta_{ac}P_b - \delta_{bc}P_a, \\ &[P_a, K_b] = \sigma\delta_{ab}H, &&[J_{ab}, K_c] = \delta_{ac}K_b - \delta_{bc}K_a, \\ &[K_a, K_b] = -\sigma J_{ab}, &&[J_{ab}, J_{cd}] = \delta_{ac}J_{bd} - \delta_{bc}J_{ad} - \delta_{ad}J_{bc} + \delta_{bd}J_{ac}. \end{aligned} \tag{4.72}$$

We can then write the Cartan connection taking values in the Poincaré algebra as

$$\mathcal{A}_\mu \equiv T_I\mathcal{A}_\mu^I = HT_\mu + P_aE_\mu{}^a + K_a\omega_\mu{}^a + \frac{1}{2}J_{ab}\omega_\mu{}^{ab}, \tag{4.73}$$

where the Lorentz boost connection $\omega_\mu{}^a$ and the rotation connection $\omega_\mu{}^{ab}$ correspond to the splitting of the usual Lorentz connection $\omega_\mu{}^{AB}$ appearing in (2.73). Assuming that they admit a Taylor expansion like (4.53), we can now expand in large c each of the components $\mathcal{A}_\mu^I$,

$$\mathcal{A}_\mu^I = \sum_{n=0}^{\infty}\sigma^n \overset{(n)}{\mathcal{A}_\mu}{}^I. \tag{4.74}$$

On the algebraic level, this means that we obtain the new generators

$$T_I^{(n)} := T_I \otimes \sigma^n, \tag{4.75}$$

where $n \geq 0$, $n \in \mathbb{N}$, is called the *level*. The set $\{T_I^{(n)}\}_{n\geq 0}$ generates a graded Lie algebra $\mathfrak{g} = \bigoplus_{n\geq 0}\mathfrak{g}_n$ in which $\mathcal{A}_\mu$ takes values,

$$\mathcal{A}_\mu = \sum_{n=0}^{\infty} T_I^{(n)}\overset{(n)}{\mathcal{A}_\mu}{}^I. \tag{4.76}$$

The graded algebra $\mathfrak{g}$ has the following non-zero commutation relations

$$\begin{aligned} &\left[H^{(m)}, K_a^{(n)}\right] = P_a^{(m+n)}, &&\left[J_{ab}^{(m)}, P_c^{(n)}\right] = \delta_{ac}P_b^{(m+n)} - \delta_{bc}P_a^{(m+n)}, \\ &\left[P_a^{(m)}, K_b^{(n)}\right] = \delta_{ab}H^{(m+n+1)}, &&\left[J_{ab}^{(m)}, K_c^{(n)}\right] = \delta_{ac}K_b^{(m+n)} - \delta_{bc}K_a^{(m+n)}, \\ &\left[K_a^{(m)}, K_b^{(n)}\right] = -J_{ab}^{(m+n+1)}, &&\left[J_{ab}^{(m)}, J_{cd}^{(n)}\right] = 2\delta_{a[c}J_{bd]}^{(m+n)} + 2\delta_{b[d}J_{ac]}^{(m+n)}. \end{aligned} \tag{4.77}$$

For a given level L, the subalgebra $\bigoplus_{n>L}\mathfrak{g}_n \subset \mathfrak{g}$ is an ideal of $\mathfrak{g}$, and we can define

$$\mathfrak{g}_L := \mathfrak{g}\Big/\bigoplus_{n>L}\mathfrak{g}_n, \tag{4.78}$$

which is a graded Lie algebra itself. Notice that quotienting out all the generators with level $n > L$ amounts to truncating the small σ expansion of the Poincaré algebra at order L. In this way, we find from (4.77) that $\mathfrak{g}_0$ has the following commutation relations

$$\begin{aligned} &\left[H^{(0)}, K_a^{(0)}\right] = P_a^{(0)}, &&\left[J_{ab}^{(0)}, P_c^{(0)}\right] = \delta_{ac}P_b^{(0)} - \delta_{bc}P_a^{(0)}, \\ &\left[P_a^{(0)}, K_b^{(0)}\right] = 0, &&\left[J_{ab}^{(0)}, K_c^{(0)}\right] = \delta_{ac}K_b^{(0)} - \delta_{bc}K_a^{(0)}, \\ &\left[K_a^{(0)}, K_b^{(0)}\right] = 0, &&\left[J_{ab}^{(0)}, J_{cd}^{(0)}\right] = \delta_{ac}J_{bd}^{(0)} - \delta_{bc}J_{ad}^{(0)} - \delta_{ad}J_{bc}^{(0)} + \delta_{bd}J_{ac}^{(0)}, \end{aligned} \tag{4.79}$$

thus being isomorphic to the Galilean algebra (3.2) through

$$H^{(0)} \mapsto H, \qquad P_a^{(0)} \mapsto P_a, \qquad K_a^{(0)} \mapsto G_a, \qquad J_{ab}^{(0)} \mapsto J_{ab}. \tag{4.80}$$

At the next level, we find that $\mathfrak{g}_1$ is generated by the four generators of $\mathfrak{g}_0$ plus four extra generators:

$$N := H^{(1)}, \qquad T_a := P_a^{(1)}, \qquad B_a := K_a^{(1)}, \qquad S_{ab} := J_{ab}^{(1)}, \tag{4.81}$$

satisfying the following non-zero commutation relations

$$\begin{aligned}
[H, G_a] &= P_a, & [J_{ab}, X_c] &= \delta_{ac} X_b - \delta_{bc} X_a, \\
[P_a, G_b] &= \delta_{ab} N, & [S_{ab}, P_c] &= \delta_{ac} T_b - \delta_{bc} T_a, \\
[G_a, G_b] &= -S_{ab}, & [S_{ab}, G_c] &= \delta_{ac} B_b - \delta_{bc} B_a, \\
[H, B_a] &= T_a & [J_{ab}, J_{cd}] &= \delta_{ac} J_{bd} - \delta_{bc} J_{ad} - \delta_{ad} J_{bc} + \delta_{bd} J_{ac}, \\
[N, G_a] &= T_a & [J_{ab}, S_{cd}] &= \delta_{ac} S_{bd} - \delta_{bc} S_{ad} - \delta_{ad} S_{bc} + \delta_{bd} S_{ac},
\end{aligned} \tag{4.82}$$

where $X_a \in \{P_a, T_a, G_a, B_a\}$. This is exactly the algebra (3.78) that we introduced without any further motivation as the one encoding the local symmetries of type II TNC geometry. Notice that N enters the commutator of P_a and G_b as in the Bargmann algebra (3.9). However, N is not central in this case as follows from $[N, G_a] = T_a$. In particular, the Bargmann algebra, which is the underlying local symmetry algebra of type I Newton-Cartan geometry, is not a subalgebra of $\mathfrak{g}_1$. As we anticipated in Section 3.3.3 and shall now see in detail, gauging the algebra $\mathfrak{g}_1$ yields the relevant fields and transformations properties of type II TNC geometry [43]. Altogether this yields a more algebraic perspective on the difference between type I and type II TNC geometry.

We start the gauging procedure as always, by taking a cartan Connection A_μ taking values in $\mathfrak{g}_1$:

$$A_\mu = H\tau_\mu + P_a e_\mu{}^a + N m_\mu + T_a \pi_\mu{}^a + G_a \omega_\mu{}^a + B_a \Omega_\mu{}^a + \frac{1}{2} J_{ab} \omega_\mu{}^{ab} + \frac{1}{2} S_{ab} \Omega_\mu{}^{ab}, \tag{4.83}$$

with associated curvature given by

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \\
&= H R_{\mu\nu}(H) + P_a R_{\mu\nu}{}^a(P) + N R_{\mu\nu}(N) + T_a R_{\mu\nu}{}^a(T) + \ldots
\end{aligned} \tag{4.84}$$

The connection transforms according to

$$\delta_\Lambda A_\mu = \partial_\mu \Lambda + [A_\mu, \Lambda] = H \delta_\Lambda \tau_\mu + P_a \delta_\Lambda e_\mu{}^a + N \delta_\Lambda m_\mu + T_a \delta_\Lambda \pi_\mu{}^a + \ldots, \tag{4.85}$$

where $\Lambda \in \mathfrak{g}_1$ is infinitesimal and can be written as

$$\begin{aligned}
\Lambda &= H\xi^\mu \tau_\mu + P_a \xi^\mu e_\mu{}^a + N(\xi^\mu m_\mu + \zeta^\mu \tau_\mu) + T_a(\xi^\mu \pi_\mu{}^a + \zeta^\mu e_\mu{}^a) + G_a(\xi^\mu \omega_\mu{}^a + \lambda^a) \\
&+ B_a(\xi^\mu \Omega_\mu{}^a + \zeta^\mu \omega_\mu{}^a + \eta^a) + \frac{1}{2} J_{ab}(\xi^\mu \omega_\mu{}^{ab} + \lambda^{ab}) + \frac{1}{2} S_{ab}(\xi^\mu \Omega_\mu{}^{ab} + \zeta^\mu \omega_\mu{}^{ab} + \rho^{ab}),
\end{aligned} \tag{4.86}$$

where ξ^μ and ζ^μ are two non-zero vectors. Notice that there is no loss of generality here, since this is a linear combination of the generators with general coefficients. These are however written in a way adequate to our interests, as will now become clear. Indeed, we can now define a new set of transformations by

$$\delta\tau_\mu := \delta_\Lambda \tau_\mu - \xi^\nu R_{\mu\nu}(H), \tag{4.87a}$$
$$\delta e_\mu{}^a := \delta_\Lambda e_\mu{}^a - \xi^\nu R_{\mu\nu}{}^a(P), \tag{4.87b}$$
$$\delta m_\mu := \delta_\Lambda m_\mu - \xi^\nu R_{\mu\nu}(N) - \zeta^\nu R_{\mu\nu}(H), \tag{4.87c}$$
$$\delta \pi_\mu{}^a := \delta_\Lambda \pi_\mu{}^a - \xi^\nu R_{\mu\nu}{}^a(T) - \zeta^\nu R_{\mu\nu}{}^a(P). \tag{4.87d}$$

We then use (4.86) to compute the adjoint transformation of A_μ in (4.85) and collect the corresponding adjoint transformations δ_Λ of τ_μ, m_μ, $e_\mu{}^a$ and $\pi_\mu{}^a$. Plugging these in (4.87) yields

$$\delta\tau_\mu = \mathcal{L}_\xi \tau_\mu, \tag{4.88a}$$
$$\delta m_\mu = \mathcal{L}_\xi m_\mu + \mathcal{L}_\zeta \tau_\mu + \lambda_a e_\mu{}^a, \tag{4.88b}$$
$$\delta e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a + \lambda^a \tau_\mu + \lambda^a{}_b e_\mu{}^b, \tag{4.88c}$$
$$\delta \pi_\mu{}^a = \mathcal{L}_\xi \pi_\mu{}^a + \mathcal{L}_\zeta \pi_\mu{}^a + \lambda^a m_\mu + \eta^a \tau_\mu + \lambda^a{}_b \pi_\mu{}^b + \rho^a{}_b e_\mu{}^b, \tag{4.88d}$$

which are exactly the transformations (4.64a), (4.65a) and (4.66) expected for the corresponding fields of type II TNC.

We have thus seen that type II TNC can be understood as the geometry arising from gauging the non-relativistic algebra $\mathfrak{g}_1$, in the same way that NC geometry can be understood as the geometry arising from gauging the non-relativistic algebra $\mathfrak{g}_0$, i. e., the Galilei algebra. It seems reasonable to conclude that this holds at any order in the expansion, namely that the local symmetry underlying the geometry of the N^{n}LO expansion of Lorentzian geometry is given by the algebra obtained from the level n expansion of the Poincaré algebra.

4.4 Non-relativistic gravity

In this section, we apply the results discussed above as well as those discussed in Section 4.1 to the EH Lagrangian of GR. In the same way that making Lorentzian geometry dynamical yields GR, we refer to the theories arising from making non-relativistic geometries dynamical as non-relativistic theories of gravity. The expansion of the EH Lagrangian up to NNLO, as well as the obtention of the corresponding EOM and the study of matter couplings was presented in [46]. Here, we shall mainly focus in the LO theory, whose EOM will simply constrain the underlying NC geometry to be TTNC geometry, and write down the NLO and NNLO Lagrangians as follow from the general framework of Section 4.1. The procedure, however, can in principle be extended to any desired order up to computational complexity, so we shall highlight some of its general aspects.

Our goal is therefore to study the Lagrangian arising from expanding according to (4.54) the PNR fields appearing in the PNR form (4.45) of the EH Lagrangian. Following the notation introduced in Section 4.1, setting $\alpha = \sigma$, the small σ expansion of the EH Lagrangian will yield a theory

$$\mathcal{L}_{\text{EH}} = c^6 \left(\mathcal{L}_{\text{LO}} + \sigma \mathcal{L}_{\text{NLO}} + \mathcal{L}_{\text{NNLO}} + \mathcal{O}(\sigma^3) \right) \tag{4.89}$$

depending on the fields

$$\phi_0^I = \{\tau_\mu, h_{\mu\nu}\}, \qquad \phi_1^I = \{m_\mu, \Phi_{\mu\nu}\}, \qquad \phi_2^I = \{B_\mu, \psi_{\mu\nu}\}, \tag{4.90}$$

with $I \in \{1,2\}$. In analogy with (4.10), we define the EOM of τ_α and $h_{\alpha\beta}$ in the N^{n}LO Lagrangian ($n \in \mathbb{N}$) as[2]

$$\overset{(\mathrm{N}^n\mathrm{LO})}{G^{\alpha}_{\tau}} := -\frac{8\pi G_N}{e}\left[\frac{\partial\mathcal{L}_{\mathrm{N}^n\mathrm{LO}}}{\partial\tau_\alpha} - \partial_\mu\left(\frac{\partial\mathcal{L}_{\mathrm{N}^n\mathrm{LO}}}{\partial(\partial_\mu\tau_\alpha)}\right)\right], \tag{4.91a}$$

$$\overset{(\mathrm{N}^n\mathrm{LO})}{G^{\alpha\beta}_{h}} := -\frac{16\pi G_N}{e}\left[\frac{\partial\mathcal{L}_{\mathrm{N}^n\mathrm{LO}}}{\partial h_{\alpha\beta}} - \partial_\mu\left(\frac{\partial\mathcal{L}_{\mathrm{N}^n\mathrm{LO}}}{\partial(\partial_\mu h_{\alpha\beta})}\right)\right], \tag{4.91b}$$

where e is the leading order term in the expansion of the PNR integration measure (4.21) given by

$$e := \sqrt{-\det(-\tau_\mu\tau_\nu + h_{\mu\nu})}, \tag{4.92}$$

and acting as the Galilean boost invariant measure of both type I and type II TNC geometry. Similarly, for the subleading fields m_α and $\Phi_{\mu\nu}$ we define

$$\overset{(\mathrm{N}^n\mathrm{LO})}{G^{\alpha}_{m}} := -\frac{8\pi G_N}{e}\left[\frac{\partial\mathcal{L}_{\mathrm{N}^n\mathrm{LO}}}{\partial m_\alpha} - \partial_\mu\left(\frac{\partial\mathcal{L}_{\mathrm{N}^n\mathrm{LO}}}{\partial(\partial_\mu m_\alpha)}\right)\right], \tag{4.93a}$$

$$\overset{(\mathrm{N}^n\mathrm{LO})}{G^{\alpha\beta}_{\Phi}} := -\frac{16\pi G_N}{e}\left[\frac{\partial\mathcal{L}_{\mathrm{N}^n\mathrm{LO}}}{\partial \Phi_{\alpha\beta}} - \partial_\mu\left(\frac{\partial\mathcal{L}_{\mathrm{N}^n\mathrm{LO}}}{\partial(\partial_\mu \Phi_{\alpha\beta})}\right)\right], \tag{4.93b}$$

and analogously for the NNLO fields B_μ and $\psi_{\mu\nu}$.

The LO Lagrangian $\mathcal{L}_{\mathrm{LO}}$ will depend only on the LO fields τ_μ and $h_{\mu\nu}$, while the NLO Lagrangian will also depend on the subleading fields m_μ and $\Phi_{\mu\nu}$. In general, at N^{n}LO there are $2(n+1)$ fields, and hence $2(n+1)$ EOM. The recursive structure of the EOM discussed in Section 4.1, however, implies that only 2 new EOM appear at every order in the expansion, the remaining $2n$ being those corresponding to the EOM of the N^{n-1}LO Lagrangian. More precisely, at every order n the only new pair of EOM to solve is

$$\overset{(\mathrm{N}^n\mathrm{LO})}{G^{\alpha}_{\tau}} := 0, \qquad \overset{(\mathrm{N}^n\mathrm{LO})}{G^{\alpha\beta}_{h}} := 0. \tag{4.94}$$

The recursive structure of the EOM up to NNLO is summarised in the following diagram:

$$\begin{array}{ccccccccccc}
 & & & & \overset{(\mathrm{LO})}{G^{\alpha\beta}_{h}} & & \overset{(\mathrm{LO})}{G^{\alpha}_{\tau}} & & & & \\
 & & & // & & & & \backslash\backslash & & & \\
 & & \overset{(\mathrm{NLO})}{G^{\alpha\beta}_{\Phi}} & & \overset{(\mathrm{NLO})}{G^{\alpha\beta}_{h}} & & \overset{(\mathrm{NLO})}{G^{\alpha}_{\tau}} & & \overset{(\mathrm{NLO})}{G^{\alpha}_{m}} & & \\
 & // & & // & & & & \backslash\backslash & & \backslash\backslash & \\
\overset{(\mathrm{NNLO})}{G^{\alpha\beta}_{\psi}} & & \overset{(\mathrm{NNLO})}{G^{\alpha\beta}_{\Phi}} & & \overset{(\mathrm{NNLO})}{G^{\alpha\beta}_{h}} & & \overset{(\mathrm{NNLO})}{G^{\alpha}_{\tau}} & & \overset{(\mathrm{NNLO})}{G^{\alpha}_{m}} & & \overset{(\mathrm{NNLO})}{G^{\alpha}_{B}}
\end{array}$$

4.4.1 LO theory: on shell TTNC geometry

Let us now study the LO Lagrangian. Using (4.7) and (4.46), we find

$$\mathcal{L}_{\mathrm{LO}} = \mathcal{L}_{\mathrm{LO}}(\tau, h, \partial) = \frac{E}{16\pi G_N}\frac{1}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma}T_{\mu\rho}T_{\nu\sigma}\bigg|_{\sigma=0} = \frac{e}{16\pi G_N}\frac{1}{4}h^{\mu\nu}h^{\rho\sigma}\tau_{\mu\rho}\tau_{\nu\sigma}, \tag{4.95}$$

[2]Notice here the inclusion of some prefactors in the definitions of the EOM, following the notation introduced in [46].

where we have defined

$$\tau_{\mu\nu} := 2\partial_{[\mu}\tau_{\nu]}. \tag{4.96}$$

The corresponding EOM are then given by

$$\overset{\text{(LO)}}{G}{}^{\alpha}_{\tau} = \frac{1}{8}h^{\mu\nu}h^{\rho\sigma}\tau_{\mu\rho}\tau_{\nu\sigma}v_\alpha + \frac{1}{2}a_\mu h^{\mu\nu}h^{\rho\alpha}\tau_{\nu\rho} + \frac{1}{2}e^{-1}\partial_\mu(eh^{\mu\nu}h^{\rho\alpha}\tau_{\nu\rho}) = 0, \tag{4.97a}$$

$$\overset{\text{(LO)}}{G}{}^{\alpha\beta}_{h} = -\frac{1}{8}h^{\mu\nu}h^{\rho\sigma}\tau_{\mu\rho}\tau_{\nu\sigma}h^{\alpha\beta} + \frac{1}{2}h^{\mu\alpha}h^{\nu\beta}h^{\rho\sigma}\tau_{\mu\rho}\tau_{\nu\sigma} = 0, \tag{4.97b}$$

where we have used

$$\frac{\partial e}{\partial \tau_\mu} = -ev^\mu, \qquad \frac{\partial e}{\partial h_{\mu\nu}} = \frac{e}{2}h^{\mu\nu}, \tag{4.98}$$

as well as

$$\frac{\partial h_{\mu\nu}}{\partial \tau_\alpha} = 2v^{(\mu}h^{\alpha\nu)}, \qquad \frac{\partial h^{\mu\nu}}{\partial h_{\alpha\beta}} = -h^{\mu(\alpha}h^{\beta)\nu}. \tag{4.99}$$

If we now contract (5.94a) with τ_α we find

$$0 = \overset{\text{(LO)}}{G}{}^{\alpha}_{\tau}\tau_\alpha \propto h^{\mu\nu}h^{\rho\sigma}\tau_{\mu\rho}\tau_{\nu\sigma}, \tag{4.100}$$

which can be written as a sum of squares. Indeed, since

$$\tau_{\mu\nu} = (\mathrm{d}\tau)_{\mu\nu}, \tag{4.101}$$

we can write, for any spacetime vector fields X^μ and Y^μ,

$$\mathrm{d}\tau(X,Y) := (\mathrm{d}\tau)_{\mu\nu}X^\mu Y^\nu. \tag{4.102}$$

In this component-free notation, we can expand the RHS of (4.100) as

$$h^{\mu\nu}h^{\rho\sigma}\tau_{\mu\rho}\tau_{\nu\sigma} \equiv \delta^{ab}\delta^{cd}\mathrm{d}\tau(e_a,e_c)\mathrm{d}\tau(e_b,e_d) = \mathrm{d}\tau(e_a,e_b)\mathrm{d}\tau(e^a,e^b), \tag{4.103}$$

after renaming dummy indices. Therefore,

$$h^{\mu\nu}h^{\rho\sigma}\tau_{\mu\rho}\tau_{\nu\sigma} = 0 \quad\Longrightarrow\quad \mathrm{d}\tau(e_a,e_b) = 0,$$

which, upon multiplication by $e^{\nu a}e^{\sigma b}$, yields

$$h^{\mu\nu}h^{\rho\sigma}\tau_{\mu\rho} = 0. \tag{4.104}$$

Notice that this automatically sets $\overset{\text{(LO)}}{G}{}^{\alpha\beta}_{h}$ to zero.

We will now show that the condition above is equivalent to the TTNC condition $\tau\wedge\mathrm{d}\tau = 0$. With this purpose, let us first note the following equivalence:

$$\mathrm{d}\tau = a\wedge\tau \quad\Longleftrightarrow\quad \tau\wedge\mathrm{d}\tau = 0, \tag{4.105}$$

where the first implication is straightforward from the anticommutativity and associativity of the wedge product and the converse can be obtained by writing down each side of the equivalence in components. Now, we use that any given two-form $A_{\mu\nu}$ can be written as

$$A_{\mu\nu} = -2\tau_{[\mu}v^\alpha A_{\alpha\nu]} + h_{\mu\rho}h_{\nu\sigma}A_{\mathrm{s}}^{\rho\sigma}, \tag{4.106}$$

with

$$A_{\mathrm{s}}^{\rho\sigma} := h^{\alpha\rho}h^{\beta\sigma}A_{\alpha\beta}, \tag{4.107}$$

as follows from the last equality in (4.59). In particular, for $A_{\mu\nu} = \tau_{\mu\nu}$ this means

$$\tau_{\mu\nu} = 2a_{[\mu}\tau_{\nu]} + h_{\mu\rho}h_{\nu\sigma}\tau_{\mathrm{s}}^{\rho\sigma}, \tag{4.108}$$

where we have also used (4.70). The LO EOM then imposes $\tau_{\mathrm{s}}^{\rho\sigma} = 0$ and we have proved the first implication in

$$h^{\mu\nu}h^{\rho\sigma}\tau_{\mu\nu} = 0 \quad \iff \quad \partial_{[\mu}\tau_{\nu]} = a_{[\mu}\tau_{\nu]}, \tag{4.109}$$

the converse being obvious from the orthogonality of $h^{\mu\nu}$ and τ_μ. Of course, the RHS here is simply the condition $\mathrm{d}\tau = a \wedge \tau$ written in components. Putting everything together, we have proved the following chain of equivalences:

$$h^{\mu\nu}h^{\rho\sigma}\tau_{\mu\nu} = 0 \quad \iff \quad \mathrm{d}\tau = a \wedge \tau \quad \iff \quad \tau \wedge \mathrm{d}\tau = 0. \tag{4.110}$$

We conclude that the EOM of the LO Lagrangian constrain the NC geometry resulting from the LO expansion to be TTNC geometry. Therefore, on shell we have a foliation of the NC spacetime in hypersurfaces of simultaneity.

4.4.2 NLO and NNLO Lagrangians

We can now write down the Lagrangian of the NLO theory as follows from (4.8). First, we need to calculate the partial derivative of (4.46) with respect to σ evaluated at $\sigma = 0$, which reads

$$\left.\frac{\partial\tilde{\mathcal{L}}_{\mathrm{EH}}}{\partial\sigma}\right|_{\sigma=0} = \frac{E}{16\pi G_N}\Pi^{\mu\nu}\left.\overset{(C)}{R}_{\mu\nu}\right|_{\sigma=0} = \frac{e}{16\pi G_N}h^{\mu\nu}\check{R}_{\mu\nu}. \tag{4.111}$$

The remaining terms in (4.8) are just the EOM for the LO Lagrangian of the leading order fields contracted with the corresponding subleading fields (together with the adequate changes to account for the prefactors introduced in (4.91)), and we obtain

$$\mathcal{L}_{\mathrm{NLO}} = \frac{e}{8\pi G_N}\left(\frac{1}{2}h^{\mu\nu}\check{R}_{\mu\nu} - m_\mu \overset{(\mathrm{LO})}{G_\tau^\mu} - \frac{1}{2}\Phi_{\mu\nu}\overset{(\mathrm{LO})}{G_h^{\mu\nu}}\right). \tag{4.112}$$

This theory is referred to as *Galilean gravity* [46] and was previously studied in [8] with a first-order formalism. As a first approach to the study of its dynamics, it can be instructive to consider a truncation of the NLO theory obtained by setting the subleading fields m_μ and $\Phi_{\mu\nu}$ to zero by hand:

$$\left.\mathcal{L}_{\mathrm{NLO}}\right|_{m_\mu=\Phi_{\mu\nu}=0} = \frac{e}{16\pi G_N}h^{\mu\nu}\check{R}_{\mu\nu}. \tag{4.113}$$

This truncation no longer reproduces the EOM of the LO theory, but the resulting EOM for τ_μ and $h_{\mu\nu}$ are simpler than those for the full NLO theory. We shall come back to this truncated NLO theory in Section 4.4.3.

As for the NNLO theory, we can now write down what (4.12) means for the EH Lagrangian (4.46). In this case, we need the second derivative

$$\left.\frac{\partial^2\mathcal{L}_{\mathrm{LO}}}{\partial\sigma^2}\right|_{\sigma=0} = -\frac{E}{8\pi G_N}V^\mu V^\nu\left.\overset{(C)}{R}_{\mu\nu}\right|_{\sigma=0} = -\frac{e}{8\pi G_N}v^\mu v^\nu\check{R}_{\mu\nu}. \tag{4.114}$$

Now, using that $\mathcal{L}_{\rm LO}$ does not depend on the derivatives of $h_{\mu\nu}$, we get

$$
\begin{aligned}
\mathcal{L}_{\rm NNLO} = &-\frac{e}{8\pi G_N}\left(\frac{1}{2}v^\mu v^\nu \check{R}_{\mu\nu} + B_\mu \overset{\rm (LO)}{G^\mu_\tau} + \frac{1}{2}\psi_{\mu\nu}\overset{\rm (LO)}{G^{\mu\nu}_h}\right)\\
&+\frac{1}{16\pi G_N}\left[m_\mu\left(\frac{\partial}{\partial\tau_\mu} - \partial_\rho\left(\frac{\partial}{\partial(\partial_\rho\tau_\mu)}\right)\right) + \Phi_{\mu\nu}\left(\frac{\partial}{\partial h_{\mu\nu}} - \partial_\rho\left(\frac{\partial}{\partial(\partial_\rho h_{\mu\nu})}\right)\right)\right] e h^{\mu\nu}\check{R}_{\mu\nu}\\
&+\frac{1}{2}m_\mu m_\nu \frac{\partial^2\mathcal{L}_{\rm LO}}{\partial\tau_\mu\partial\tau_\nu} + m_\mu\Phi_{\nu\rho}\frac{\partial^2\mathcal{L}_{\rm LO}}{\partial\tau_\mu\partial h_{\nu\rho}} + \frac{1}{2}\Phi_{\mu\nu}\Phi_{\rho\sigma}\frac{\partial^2\mathcal{L}_{\rm LO}}{\partial h_{\mu\nu}\partial h_{\rho\sigma}} + m_\mu\partial_\rho m_\nu\frac{\partial^2\mathcal{L}_{\rm LO}}{\partial\tau_\mu\partial(\partial_\rho\tau_\nu)}\\
&+\Phi_{\mu\sigma}\partial_\rho m_\nu\frac{\partial^2\mathcal{L}_{\rm LO}}{\partial h_{\mu\sigma}\partial(\partial_\rho\tau_\nu)} + \frac{1}{2}\partial_\mu m_\nu\partial_\rho m_\sigma\frac{\partial^2\mathcal{L}_{\rm LO}}{\partial(\partial_\mu\tau_\nu)\partial(\partial_\rho\tau_\sigma)}.
\end{aligned}
\tag{4.115}
$$

The study of this theory is beyond the scope of this work, but we would like to record some of the additional results determined in the original work [46], in the sake of completeness. The last two lines in (4.115) can be written as

$$
\cdots = \frac{e}{16\pi G_N}\left(\frac{1}{4}F_{\mu\nu}F_{\rho\sigma} + h^{\mu\rho}h^{\nu\sigma}\tau_{\mu\nu}X_{\rho\sigma}\right), \tag{4.116}
$$

where $X_{\rho\sigma}$ is some arbitrary tensor and we have defined

$$
F_{\mu\nu} := 2\left(\partial_{[\mu}m_{\nu]} - a_{[\mu}m_{\nu]}\right). \tag{4.117}
$$

Notice that any variation of $eh^{\mu\rho}h^{\nu\sigma}\tau_{\mu\nu}X_{\rho\sigma}$ that is proportional to $h^{\mu\rho}h^{\nu\sigma}\tau_{\mu\nu}$ will not contribute on shell. It turns out that if one is only interested in the EOM for the LO and NLO fields, the term involving $X_{\rho\sigma}$ as well as the terms involving the NNLO fields can be ignored. This amounts to imposing the TTNC condition $\tau\wedge \mathrm{d}\tau = 0$ off shell, by means of a Lagrange multiplier $L_{\rho\sigma}$. The Lagrangian resulting from this considerations is known as the *non-relativistic gravity Lagrangian*:

$$
\mathcal{L}_{\rm NRG} := \mathcal{L}_{\rm NNLO}\Big|_{\tau\wedge\mathrm{d}\tau=0} + \frac{e}{32\pi G_N}L_{\rho\sigma}h^{\mu\rho}h^{\nu\sigma}\tau_{\mu\nu}. \tag{4.118}
$$

This Lagrangian was first considered in [43], where it was obtained not from a large speed of light expansion of the EH Lagrangian, but from gauge symmetry principles. More precisely, it was there determined to be the unique two-derivative Lagrangian featuring all the gauge invariances of type II TNC geometry.

Equipped with these Lagrangians, one can then find the corresponding EOM and study possible solutions. It turns out that many of the canonical solutions of GR, such as FLRW's or Schwarszchild's, can also be viewed as solutions of the NRG theory, showing that the latter is much richer than Newtonian gravity. Moreover, one can use the methods presented in Section 4.1 here to perform a large speed of light expansion of a generic matter Lagrangian, in order to find the EOM of NRG in the presence of matter. One can then focus on specific matter Lagrangians like those corresponding to point particles, scalar fields, fluids or electrodynamics, and study how they couple to non-relativistic gravity. Once again, we refer the reader to the original work [46] for the details.

4.4.3 NLO theory and non-relativistic magnetic limit of GR

It has been known for many years that electromagnetism admits two distinct non-relativistic limits accounting for electric and magnetic effects [7]. These are known as the electric and the magnetic non-relativistic limits, respectively. More recently, it was determined that electromagnetism also admits two distinct ultra-local Carroll limits retaining the electric and magnetic sectors, respectively [30]. Such two inequivalent Carroll limits were then also studied from a Hamiltonian perspective for Lorentz-invariant field theories other than electromagnetism, as well as for gravity[3] [55]. These were called electric and magnetic Carroll limits, following the original terminology. Subsequently, they were considered for field theories from a Lagrangian perspective in [24]. Finally, this picture was completed in [48] by a successful Lagrangian approach to both the electric and magnetic Carroll limits of gravity. In particular, it was shown there that the leading order action in the ultra-local expansion of GR corresponds to its electric Carroll limit, while its magnetic Carroll limit is equivalent to a truncated sector of the NLO theory.

Regarding the non-relativistic case, however, the picture of the corresponding electric and magnetic limits of GR is still open. As we mentioned earlier, however, the NLO theory (4.112) can be related to the action obtained from a first-order perspetive in [8] by taking an appropriate non-relativistic limit of the EH action. Our goal is then to contribute to this picture by proposing an interpretation of the truncated NLO theory (4.113) as the non-relativistic magnetic limit of GR. More precisely, we show that the latter is equivalent to a Galilei-invariant action obtained from a non-relativistic limit of the EH Lagrangian in PNR form, which is obtained by mirroring the analogous procedure described in [48] for the ultra-local case. This, in turn, is based on the methods considered in [24] for building Carroll invariant actions from ultra-local Carroll limits of relativistic field theories.

We start by rewriting the PNR form (4.45) of the EH Lagrangian as

$$\mathcal{L}_{EH} = \frac{c^6}{16\pi G_N} E\left(\frac{1}{4} G^{\mu\rho,\nu\sigma} T_{\mu\rho} T_{\nu\sigma} + \sigma \Pi^{\mu\nu} \overset{(C)}{R}_{\mu\nu} - \sigma^2 V^\mu V^\nu \overset{(C)}{R}_{\mu\nu}\right), \tag{4.119}$$

where we have introduced

$$G^{\mu\rho,\nu\sigma} := \Pi^{\mu[\nu}\Pi^{\rho\sigma]} = \Pi^{[\mu\nu}\Pi^{\rho]\sigma}. \tag{4.120}$$

The latter can be thought of as a symmetric bilinear form in the space of antisymmetrised indices. Indeed, if we put

$$\dot{A} = [\mu\rho], \quad \mu < \rho$$

then $\dot{A} = 0, \ldots, \frac{1}{2}D(D-1) - 1$ and we can write

$$G^{\mu\rho,\nu\sigma} T_{\mu\rho} T_{\nu\sigma} = G^{\dot{A}\dot{B}} T_{\dot{A}} T_{\dot{B}}.$$

The symmetry of $G^{\dot{A}\dot{B}}$ follows from the symmetry of $\Pi^{\mu\nu}$.

$$G^{\dot{B}\dot{A}} = G^{\nu\sigma,\mu\rho} = \Pi^{\nu[\mu}\Pi^{\sigma\rho]} = \Pi^{[\mu\nu}\Pi^{\rho]\sigma} = G^{\dot{A}\dot{B}}.$$

We can define a projective inverse $G_{\mu\nu,\rho\sigma}$ of $G^{\mu\rho,\nu\sigma}$ by

$$G_{\mu\nu,\rho\sigma} := \Pi_{\mu[\rho}\Pi_{\nu\sigma]} = \Pi_{[\mu\rho}\Pi_{\nu]\sigma}, \tag{4.121}$$

so that

$$G_{\mu\nu,\rho\sigma} G^{\rho\sigma,\lambda\kappa} = P^{[\lambda}_{\mu} P^{\kappa]}_{\nu}, \qquad P^\mu_\nu = \delta^\mu_\nu + V^\mu T_\nu. \tag{4.122}$$

[3]Similar Carroll-invariant theories of gravity were obtained from gauging procedures in [50, 8]

We can introduce an auxiliary antisymmetric tensor $\zeta^{\mu\nu}$, which we take to be purely spatial, i. e., such that $\zeta^{\mu\nu}T_\mu = 0$. This allows us to rewrite the EH Lagrangian as

$$\mathcal{L}^{\zeta}_{EH} = \frac{c^4}{16\pi G_N} E \left[\zeta^{\mu\rho} T_{\mu\rho} + \Pi^{\mu\nu} \overset{(C)}{R}_{\mu\nu} - \frac{1}{c^2} \left(G_{\mu\rho,\nu\sigma} \chi^{\mu\rho} \chi^{\nu\sigma} + V^\mu V^\nu \overset{(C)}{R}_{\mu\nu} \right) \right], \tag{4.123}$$

with $\zeta^{\mu\nu}$ acting as a Lagrange multiplier. Indeed, one can verify that the EOM for $\zeta^{\mu\nu}$ sets

$$\zeta^{\mu\rho} = \frac{c^2}{2} G^{\mu\rho,\lambda\kappa} T_{\lambda\kappa}. \tag{4.124}$$

from which one recovers the EH Lagrangian in (4.119). Moreover, if we now take the non-relativistic $c \to \infty$ limit of (4.123) we find

$$\mathcal{L}_{\text{mag}} := \lim_{c\to\infty} c^{-4} \mathcal{L}^{\zeta}_{EH} = \frac{e}{16\pi G_N} \left(\zeta^{\mu\rho} \tau_{\mu\rho} + h^{\mu\nu} \check{R}_{\mu\nu} \right) = \mathcal{L}_{\text{NLO}} \Big|_{m_\mu = \Phi_{\mu\nu} = 0} + \frac{e}{16\pi G_N} \zeta^{\mu\rho} \tau_{\mu\rho}. \tag{4.125}$$

Therefore, the non-relativistic limit yields the truncated NLO theory (4.113) plus a Lagrange multiplier term that on shell sets $\tau_{\mu\rho}$ to zero. But imposing this constraint in the full NLO theory is equivalent to setting the subleading fields to zero by hand, as follows from the fact that

$$\tau_{\mu\rho} = 0 \quad \Longrightarrow \quad \overset{(\text{LO})}{G}{}^{\alpha}_{\tau} = \overset{(\text{LO})}{G}{}^{\alpha\beta}_{h} = 0. \tag{4.126}$$

This means that the theory $\mathcal{L}_{\text{mag}}$ obtained from the non-relativistic limit of $\mathcal{L}^{\zeta}_{\text{EH}}$ is equivalent on shell to the truncated NLO theory (4.113), but with the advantage that it can be made Galilei-invariant by an appropriate transformation law for $\zeta^{\mu\nu}$ under Galilean boosts. In particular, when the latter is given by

$$\delta\chi^{\mu\rho} = -h^{[\mu\nu} h^{\rho]\sigma} v^\kappa \lambda_\nu \tau_{\kappa\sigma} + 2 v^{[\mu} h^{\kappa\nu} h^{\rho]\sigma} \lambda_\kappa \tau_{\nu\sigma}, \tag{4.127}$$

then $\mathcal{L}_{\text{mag}}$ transforms into a total derivative under Galilean boosts. Then, the action

$$S_{\text{mag}} := \int d^D x \mathcal{L}_{\text{mag}} = \frac{1}{16\pi G_N} \int d^D x e (\zeta^{\mu\rho} \tau_{\mu\rho} + h^{\mu\nu} \check{R}_{\mu\nu}) \tag{4.128}$$

is Galilei-invariant.

Chapter 5

Weak field limit of non-relativistic gravity

In the previous chapter, we have shown how covariant formulations of different non-relativistic theories of gravity[1] can be obtained from a non-relativistic expansion of GR, truncated at the desired order. An interesting and natural subsequent step is to consider non-relativistic gravity in the weak field limit, in analogy with the well-known weak field limit of GR. The latter is obtained by considering the metric tensor to be a small perturbation around the flat Minkowski metric and linearising the theory at first order in the perturbation. As a consequence of this linearised analysis[2], the Einstein field equations (EFEs) are greatly simplified, which allows to obtain gravitational waves as an exact solution of linearised GR (LinGR).

As argued in Section 1.2, the weak field limit of NRG provides a new framework for the study of gravity under the two assumptions that relativistic effects are small and the gravitational field is weak. Starting from GR, these assumptions correspond to taking simultaneously the following two *independent* limiting cases: non-relativistic expansion and weak-field limit through a linearisation. This can be summarised as follows:

$$
\begin{tikzcd}
\text{GR} \arrow[r, "\text{NR expansion}"] \arrow[d, "\text{Linearisation}"'] & \text{NRG} \arrow[d, "\text{Linearisation}"'] \\
\text{LinGR} \arrow[r, "\text{NR expansion}"] & \text{NRLinGR/LinNRG}
\end{tikzcd}
$$

[1]These correspond to the different Lagrangians obtained from the non-relativistic expansion of the EH Lagrangian. In what follows, we shall use the term NRG in a wide sense, as the physics described by any of these theories, and not as the name of the theory with Lagrangian (4.118).

[2]One can of course perform a linearisation of GR around any given solution of the latter. From now on, unless otherwise stated, it will be implied when we talk about linearising GR that we are referring to linearising GR around Minkowski's solution.

The goal of this chapter is to explore the lower right corner of the diagram above, by following the two natural routes towards it. The first one corresponds to considering the non-relativistic expansion of LinGR, resulting in a theory that we call non-relativistic linearised GR (NRLinGR). The second one corresponds to the linearisation of NRG around a flat NC background, resulting in a theory of linearised NRG (LinNRG). Conceptually, we expect the diagram to be commutative and the two theories equivalent, since they describe the same physics.

With the ultimate goal of finding a formulation of the weak field limit of NRG that is consistent with the two approaches, we start by studying the non-relativistic expansion of LinGR at the Lagrangian level. This involves writing the usual metric perturbation in PNR form and finding what we call a *perturbative pre-non-relativistic* (PPNR) parametrisation of the Lagrangian of LinGR. We then find the resulting Lagrangians at LO and NLO theories in the non-relativistic expansion, as well as their EOM. Finally, we also comment on how the usual choice of harmonic gauge in its PPNR form can be implemented to simplify the EOM of the LO theory.

Next, we study the linearisation of the LO theory (4.95) around a flat NC background. We show that the resulting theory is precisely the LO Lagrangian of NRLinGR. We consider this a central result, and conjecture that the equivalence holds also beyond LO, although it is then no longer manifest. Therefore, this result also suggests that our formulation of NRLinGR provides an adequate framework for the study of the weak field limit of NRG.

5.1 Non-relativistic expansion of linearised GR

5.1.1 Linearised GR

Let us start by reviewing the linearisation of GR around the flat Minkowski solution. This procedure, as well as the study of the resulting theory, can be found in any GR textbook. There are therefore countless references on the subject, but we shall mainly follow [13, 15, 71].

As mentioned above, the starting point for the linearised analysis of GR is the assumption that the gravitational field is weak enough as to be described by a small perturbation around the flat Minkowski metric[3],

$$g_{\mu\nu} = \eta_{\mu\nu} + \chi_{\mu\nu}, \qquad |\chi_{\mu\nu}| \ll 1. \tag{5.1}$$

Here, we restrict ourselves to coordinate systems in which $\eta_{\mu\nu}$ takes the form:

$$\eta_{\mu\nu} = \mathrm{diag}(-c^2, 1, 1, 1). \tag{5.2}$$

The idea is then that, given the smallness of $\chi_{\mu\nu}$, we can ignore anything that is higher than first order in this quantity or in its derivatives. In particular, this means that we can take the inverse metric to be

$$g^{\mu\nu} = \eta^{\mu\nu} - \chi^{\mu\nu}, \tag{5.3}$$

with $\chi^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}\chi_{\rho\sigma}$. It then follows that coordinate indices are raised and lowered using $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, respectively. This means that we can interpret LinGR as a theory describing

[3]We depart from the usual notation for the perturbation of the metric found in the literature, in order to avoid confusion with the leading order term $h_{\mu\nu}$ in the expansion of $\Pi_{\mu\nu}$.

a symmetric tensor field $\chi_{\mu\nu}$ propagating on a flat background spacetime. In particular, the theory is Lorentz invariant and the perturbation of the metric transforms as

$$\chi_{\mu\nu} \to \chi_{\mu'\nu'} = (\Lambda^{-1})^{\mu}{}_{\mu'}(\Lambda^{-1})^{\nu}{}_{\nu'}\chi_{\mu\nu}, \tag{5.4}$$

under a Lorentz transformation $x^{\mu} \to x^{\mu'} = \Lambda^{\mu'}{}_{\mu}x^{\mu}$.

With the considerations above, the Christoffel symbols of the Levi-Civita connection are found to be

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}(\partial_{\mu}\chi_{\nu\lambda} + \partial_{\nu}\chi_{\lambda\mu} - \partial_{\lambda}\chi_{\mu\nu}). \tag{5.5}$$

In particular, they are already linear in the perturbation. The terms of the form Γ^2 in the definition (2.4) of the Riemann tensor can then be ignored and we have

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \eta_{\mu\lambda}(\partial_{\rho}\Gamma^{\lambda}_{\sigma\nu} - \partial_{\sigma}\Gamma^{\lambda}_{\rho\nu}) \\ &= \frac{1}{2}(\partial_{\rho}\partial_{\nu}\chi_{\mu\sigma} + \partial_{\sigma}\partial_{\mu}\chi_{\rho\nu} - \partial_{\sigma}\partial_{\nu}\chi_{\mu\rho} - \partial_{\rho}\partial_{\mu}\chi_{\sigma\nu}). \end{aligned} \tag{5.6}$$

The Ricci tensor then becomes

$$\begin{aligned} R_{\mu\nu} &= \eta^{\rho\sigma}R_{\mu\rho\nu\sigma} \\ &= \frac{1}{2}(\partial_{\sigma}\partial_{\mu}\chi^{\sigma}{}_{\nu} + \partial_{\sigma}\partial_{\nu}\chi^{\sigma}{}_{\mu} - \partial_{\mu}\partial_{\nu}\chi - \Box\chi_{\mu\nu}), \end{aligned} \tag{5.7}$$

where $\chi := \eta^{\mu\nu}\chi_{\mu\nu}$ is the trace of the perturbation and $\Box := \partial^{\alpha}\partial_{\alpha}$ is the usual d'Alembertian operator in flat space. Contracting again with $\eta^{\mu\nu}$ yields the linearised Ricci scalar,

$$R = \partial_{\mu}\partial_{\nu}\chi^{\mu\nu} - \Box\chi. \tag{5.8}$$

Finally, we can build the linearised Einstein tensor, which reads

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \\ &= \frac{1}{2}(\partial_{\mu}\partial_{\sigma}\chi^{\sigma}{}_{\nu} + \partial_{\sigma}\partial_{\nu}\chi^{\sigma}{}_{\mu} - \partial_{\mu}\partial_{\nu}\chi - \Box\chi_{\mu\nu} - \eta_{\mu\nu}\partial_{\rho}\partial_{\sigma}\chi^{\rho\sigma} + \eta_{\mu\nu}\Box\chi), \end{aligned} \tag{5.9}$$

and satisfies the linearised version of the Bianchi identity for the full Einstein tensor,

$$\partial^{\mu}G_{\mu\nu} = 0. \tag{5.10}$$

The linearised vacuum EFEs then read

$$\frac{1}{2}(\partial_{\mu}\partial_{\sigma}\chi^{\sigma}{}_{\nu} + \partial_{\sigma}\partial_{\nu}\chi^{\sigma}{}_{\mu} - \partial_{\mu}\partial_{\nu}\chi - \Box\chi_{\mu\nu} - \eta_{\mu\nu}\partial_{\rho}\partial_{\sigma}\chi^{\rho\sigma} + \eta_{\mu\nu}\Box\chi) = 0. \tag{5.11}$$

These EOM can also be obtained from an action principle, by varying with respect to $\chi^{\mu\nu}$ the action

$$S[\chi] = \frac{c^3}{16\pi G_N}\int dt d^d x\sqrt{-\eta}\mathcal{L}(\chi, \partial\chi), \tag{5.12}$$

with

$$\mathcal{L}(\chi, \partial\chi) = -\frac{1}{2}(\partial_{\mu}\chi^{\mu\nu})(\partial_{\nu}\chi) + \frac{1}{2}(\partial_{\mu}\chi^{\rho\sigma})(\partial_{\rho}\chi^{\mu}{}_{\sigma}) - \frac{1}{4}(\partial_{\mu}\chi^{\rho\sigma})(\partial^{\mu}\chi_{\rho\sigma}) + \frac{1}{4}(\partial^{\mu}\chi)(\partial_{\mu}\chi). \tag{5.13}$$

This Lagrangian is known as the Fierz-Pauli Lagrangian [32] for a free masless spin-2 field $\chi_{\mu\nu}$ propagating on a flat background. Notice that we can take the integration measure to be the Minkowski one since any contribution of $\chi_{\mu\nu}$ to the measure would yield subleading contributions to the Lagrangian. Notice also that we have explicitly included such measure to stress the fact that it carries a power of c. More precisely, following our convention (5.2) we have $\sqrt{-\eta} = c$, so that we can ignore the integration measure by adding an extra power of c in (5.12). For the sake of definiteness and consistency with our notation in Section 4.2.3, let us define the theory of LinGR as the one given by

$$S_{\text{LinGR}}[\chi] = \int dt d^d x \mathcal{L}_{\text{LinGR}}, \tag{5.14}$$

with

$$\mathcal{L}_{\text{LinGR}} = \frac{c^4}{32\pi G_N}\left[-(\partial_\mu \chi^{\mu\nu})(\partial_\nu \chi) + (\partial_\mu \chi^{\rho\sigma})(\partial_\rho \chi^\mu{}_\sigma) - \frac{1}{2}(\partial_\mu \chi^{\rho\sigma})(\partial^\mu \chi_{\rho\sigma}) + \frac{1}{2}(\partial^\mu \chi)(\partial_\mu \chi)\right]. \tag{5.15}$$

It is precisely this Lagrangian that we will expand in powers of c^{-2} to study the non-relativistic expansion of LinGR. Note that, besides the powers in the prefactor, (5.15) depends on c through the implicit presence of $\eta_{\mu\nu}$ and its inverse, that accounts for all the lowered and raised indices.

With these conventions, one can show that a variation

$$\chi_{\mu\nu} \to \chi_{\mu\nu} + \delta\chi_{\mu\nu} \tag{5.16}$$

leads to

$$\delta S_{\text{LinGR}}[\chi] = -\frac{c^4}{16\pi G_N}\int dt d^d x G_{\mu\nu}\delta\chi^{\mu\nu}, \tag{5.17}$$

with $G_{\mu\nu}$ as defined in (5.9). Notice that the minus sign here comes from the fact that

$$\delta g^{\mu\nu} = -\delta\chi^{\mu\nu}, \tag{5.18}$$

as follows from (5.3). In this way, the expression (5.17) is consistent with the usual definition of the Einstein tensor as the response to the variation of the EH Lagrangian with respect to the metric tensor[4].

Besides a manifest Lorentz symmetry, the theory (5.14) has a gauge symmetry inherited from the diffeomorphism invariance of GR. Indeed, under an infinitesimal change of coordinates

$$x^\mu \to x^\mu - \xi^\mu, \qquad |\xi^\mu| \ll 1, \tag{5.19}$$

the metric (5.1) changes by

$$\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi \eta_{\mu\nu} + \mathcal{L}_\xi \chi_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{5.20}$$

where in the last equality we have used that the contribution of $\mathcal{L}_\xi \chi_{\mu\nu}$ is subleading. We can therefore interpret the infinitesimal coordinate transformation (5.19) as the following transformation of the metric perturbation:

$$\chi_{\mu\nu} \to \chi_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{5.21}$$

so that

$$\delta\chi_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)}. \tag{5.22}$$

[4]This sign convention is followed by [13, 71], but not by many references in the literature, e.g. [15], where the Lagrangian in (5.13) is taken to have the opposite sign.

Notice that the smallness of ξ^μ ensures that such a transformation is consistent with the weak field approximation. It is straightforward to check that the linearised Riemann tensor (5.6) is invariant under the transformation (5.21). This is in complete analogy with the gauge transformations of electrodynamics, where a shift in the gauge potential of the form

$$A_\mu \to A_\mu + \partial_\mu \alpha \tag{5.23}$$

leaves the field strength $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ unchanged. At the level of the action, it follows from (5.17) and (5.22) that under a gauge transformation like (5.21),

$$\delta S_{\text{LinGR}} = -\frac{c^4}{16\pi G_N}\int dt d^d x 2 G_{\mu\nu}\partial^\mu \xi^\nu = \frac{c^4}{16\pi G_N}\int dt d^d x 2(\partial^\mu G_{\mu\nu})\xi^\nu = 0, \tag{5.24}$$

where the second equality holds up to boundary terms and in the last one we have used the linearised Bianchi identity (5.10).

Returning to the analogy with electromagnetism, we know that Maxwell's equations,

$$\partial^\mu F_{\mu\nu} = j_\nu, \tag{5.25}$$

take the particularly nice form

$$\Box A_\nu = j_\nu, \tag{5.26}$$

after imposing the Lorenz gauge condition

$$\partial^\mu A_\mu = 0. \tag{5.27}$$

The analogue of the Lorenz gauge in GR is the so-called *harmonic*[5] or *de Donder gauge*, which in its linearised version is given by the condition

$$\partial^\mu \chi_{\mu\nu} - \frac{1}{2}\partial_\nu \chi = 0. \tag{5.28}$$

The harmonic gauge condition above simplifies significantly the linearised vacuum EFEs (5.11), which reduce to the set of wave equations

$$-\frac{1}{2}\Box \bar{\chi}_{\mu\nu} = 0, \tag{5.29}$$

where we have defined

$$\bar{\chi}_{\mu\nu} = \chi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\chi. \tag{5.30}$$

5.1.2 PPNR parametrisation

The goal of this section is to write the Lagrangian (5.15) in a form that is suitable for its non-relativistic expansion. Since we are interested in making contact with the theory obtained when we linearise NRG, we shall rely heavily on the PNR formulation of GR that we studied in Section 4.2. In particular, we will consider the PNR parametrisations (4.20) of a general metric and its inverse in the specific case of the weak field metric (5.1), in order to relate its perturbation $\chi_{\mu\nu}$ to perturbations of the PNR fields, and express $\mathcal{L}_{\text{LinGR}}$ in terms of these.

[5]The name stems from the fact that in this gauge the coordinates are harmonic, i.e., they satisfy the equation $\nabla^\alpha \nabla_\alpha x^\mu = 0$.

As a first step, it is convenient to rewrite $\mathcal{L}_{\text{LinGR}}$ with the partial derivatives and the perturbation of the metric appearing only with lower indices, as

$$\begin{aligned}\mathcal{L}_{\text{LinGR}}(\chi,\partial\chi) = \frac{c^4}{32\pi G_N}\eta^{\mu\rho}\eta^{\nu\sigma}\eta^{\lambda\kappa}\bigg[& -(\partial_\mu\chi_{\rho\sigma})(\partial_\nu\chi_{\lambda\kappa}) + (\partial_\lambda\chi_{\mu\nu})(\partial_\rho\chi_{\kappa\sigma}) \\ & - \frac{1}{2}(\partial_\mu\chi_{\kappa\sigma})(\partial_\rho\chi_{\nu\lambda}) + \frac{1}{2}(\partial_\mu\chi_{\nu\sigma})(\partial_\rho\chi_{\lambda\kappa})\bigg].\end{aligned} \tag{5.31}$$

In analogy with (4.20) and using (5.2), we can write the PNR parametrisations of $\eta_{\mu\nu}$ and its inverse as

$$\eta_{\mu\nu} = -c^2\delta^0_\mu\delta^0_\nu + s_{\mu\nu}, \tag{5.32a}$$

$$\eta^{\mu\nu} = -\frac{1}{c^2}\delta^\mu_0\delta^\nu_0 + s^{\mu\nu}, \tag{5.32b}$$

where we have introduced

$$s_{\mu\nu} = \delta_{ab}\delta^a_\mu\delta^b_\nu, \quad s^{\mu\nu} = \delta^{ab}\delta^\mu_a\delta^\nu_b, \tag{5.33}$$

satisfying the following relations:

$$s_{\mu\nu}\delta^\nu_0 = 0, \qquad s^{\mu\nu}\delta^0_\nu = 0, \qquad s_{\mu\rho}s^{\rho\nu} = \delta^\nu_\mu - \delta^\nu_0\delta^0_\mu = \delta^\nu_a\delta^a_\mu. \tag{5.34}$$

Our goal is then to express the perturbation $\chi_{\mu\nu}$ of the metric around $\eta_{\mu\nu}$ in terms of perturbations of the PNR fields T_μ and $\Pi_{\mu\nu}$ around δ^0_μ and $s_{\mu\nu}$, respectively. And similarly for the corresponding inverses and projective inverses. To this end, let us consider the timelike and spatial vielbeine (and their inverses) to be given by

$$T_\mu = \delta^0_\mu + \hat{T}_\mu, \tag{5.35a}$$

$$E_\mu{}^a = \delta^a_\mu + \hat{E}_\mu{}^a, \tag{5.35b}$$

$$V^\mu = -\delta^\mu_0 + \hat{V}^\mu, \tag{5.35c}$$

$$E^\mu{}_a = \delta^\mu_a + \hat{E}^\mu{}_a, \tag{5.35d}$$

where hatted variables denote the corresponding perturbations and hence satisfy

$$|\hat{T}_\mu|, |\hat{V}^\mu|, |\hat{E}_\mu{}^a|, |\hat{E}^\mu{}_a| \ll 1.$$

If we now impose the orthogonality and completeness relations (4.16) to hold up to linear order in the perturbations, we get

$$\hat{V}^0 = \hat{T}_0, \quad \hat{V}^a = \hat{E}_0{}^a, \quad \hat{E}^0{}_a = -\hat{T}_a, \quad \hat{E}^a{}_b = -\hat{E}_b{}^a, \quad \delta^0_\mu\hat{V}^\nu - \delta^a_\mu\hat{E}^\nu{}_a = \delta^\nu_0\hat{T}_\mu + \delta^\nu_a\hat{E}_\mu{}^a. \tag{5.36}$$

In particular, this means that the perturbations of the inverse vielbeine are completely determined by the perturbations of the vielbeine, since we can write

$$\hat{V}^\mu = \delta^\mu_0\hat{T}_0 + \delta^\mu_a\hat{E}_0{}^a, \tag{5.37a}$$

$$\hat{E}^\mu{}_a = -\delta^\mu_0\hat{T}_a - \delta^\mu_b\hat{E}_a{}^b. \tag{5.37b}$$

Ignoring terms that are higher than linear in the perturbations, we can now write the PNR fields and their projective inverses as

$$T_\mu = \delta^0_\mu + \hat{T}_\mu, \tag{5.38a}$$

$$\Pi_{\mu\nu} = s_{\mu\nu} + \hat{\Pi}_{\mu\nu}, \tag{5.38b}$$

$$V^\mu = -\delta^\mu_0 + \hat{V}^\mu, \tag{5.38c}$$

$$\Pi^{\mu\nu} = s^{\mu\nu} + \hat{\Pi}^{\mu\nu}, \tag{5.38d}$$

where we have defined

$$\hat{\Pi}_{\mu\nu} := 2\delta_{ab}\delta^a_{(\mu}\hat{E}_{\nu)}{}^b, \qquad \hat{\Pi}^{\mu\nu} := 2\delta^{ab}\delta^{(\mu}_a\hat{E}^{\nu)}{}_b, \tag{5.39}$$

so that $|\hat{\Pi}_{\mu\nu}|, |\hat{\Pi}^{\mu\nu}| \ll 1$. We shall refer to the perturbations of the PNR fields as the *PPNR fields*. Note that, imposing that the orthogonality and completeness relations (4.18) hold at linear order in the PPNR fields, one has

$$\hat{V}^0 = \hat{T}_0, \qquad \hat{\Pi}^{0\nu} = -\hat{T}_\mu s^{\mu\nu}, \qquad \hat{\Pi}_{0\nu} = \hat{V}^\mu s_{\mu\nu}, \qquad \delta^0_\mu\hat{V}^\nu - s_{\mu\rho}\hat{\Pi}^{\rho\nu} = \delta^\nu_0\hat{T}_\mu + s^{\rho\nu}\hat{\Pi}_{\mu\rho}. \tag{5.40}$$

It is also useful to realise that

$$\hat{\Pi}_{00} = \hat{\Pi}^{00} = 0, \tag{5.41}$$

which follows from (5.40) by using $s_{\mu 0} = 0$ and $s^{\mu 0} = 0$.

If the PNR fields are given by (5.38), then it follows from (4.20a) and (5.32a) that the metric can be written, up to terms quadratic in the perturbations, as

$$g_{\mu\nu} = \eta_{\mu\nu} - 2c^2\delta^0_{(\mu}\hat{T}_{\nu)} + \hat{\Pi}_{\mu\nu}. \tag{5.42}$$

We shall refer to the expression above as the PPNR parametrisation of the metric. If we now compare it with (5.1), we find

$$\chi_{\mu\nu} = -2c^2\delta^0_{(\mu}\hat{T}_{\nu)} + \hat{\Pi}_{\mu\nu}, \qquad |\chi_{\mu\nu}| \ll 1. \tag{5.43}$$

This relation is very important for our purposes. It can be interpreted as a PNR parametrisation of the metric perturbation and will allow us to write $\mathcal{L}_{\text{LinGR}}$ in terms of the PPNR fields. Notice that there is no loss of generality here, the expression above is just the statement that any metric perturbation $\chi_{\mu\nu}$ as defined by (5.1) can be interpreted as tracing back to perturbations on the vielbeine $E_\mu{}^A$, which is ultimately a consequence of (2.26).

Similarly, using (4.20b) and (5.32b), we can write the PPNR parametrisation of the inverse metric,

$$g^{\mu\nu} = \eta^{\mu\nu} + \frac{2}{c^2}\delta^{(\mu}_0\hat{V}^{\nu)} + \hat{\Pi}^{\mu\nu}. \tag{5.44}$$

In this case, a comparison with (5.3) yields

$$\chi^{\mu\nu} = -\frac{2}{c^2}\delta^{(\mu}_0\hat{V}^{\nu)} - \hat{\Pi}^{\mu\nu}, \qquad |\chi^{\mu\nu}| \ll 1. \tag{5.45}$$

We have now all the ingredients to write the Lagrangian of LinGR as a function of the PPNR fields. Indeed, plugging (5.32b) and (5.43) in (5.31), we have

$$\mathcal{L}_{\text{LinGR}} = c^6\tilde{\mathcal{L}}_{\text{LinGR}}(\sigma; \hat{T}, \hat{\Pi}, \partial), \tag{5.46}$$

with

$$\tilde{\mathcal{L}}_{\text{LinGR}}(\sigma; \hat{T}, \hat{\Pi}, \partial) = \frac{1}{16\pi G_N}\left(\tilde{\mathcal{L}}_0 + \sigma\tilde{\mathcal{L}}_1 + \sigma^2\tilde{\mathcal{L}}_2 + \sigma^3\tilde{\mathcal{L}}_3\right), \tag{5.47}$$

and

$$\tilde{\mathcal{L}}_0 = \frac{1}{4}s^{\mu\nu}s^{\rho\sigma}\hat{T}_{\mu\rho}\hat{T}_{\nu\sigma}, \tag{5.48a}$$

$$\begin{aligned}\tilde{\mathcal{L}}_1 = s^{\mu\nu}s^{\rho\sigma}\Big\{&2(\partial_\mu\hat{T}_{[\sigma})(\partial_\rho\hat{\Pi}_{0]\nu}) + (\partial_\mu\hat{T}_{[\rho})(\partial_0\hat{\Pi}_{\nu]\sigma})\\ &+ (\partial_{[\mu}\hat{T}_{0]})(\partial_\nu\hat{\Pi}_{\rho\sigma}) + (\partial_\mu\hat{T}_\rho)(\partial_{[0}\hat{\Pi}_{\nu]\sigma}) + (\partial_\mu\hat{T}_{[0})(\partial_\nu\hat{\Pi}_{\rho]\sigma})\\ &+ \frac{1}{2}s^{\lambda\kappa}\Big[(\partial_\mu\hat{\Pi}_{\rho[\sigma})(\partial_{\nu]}\hat{\Pi}_{\lambda\kappa}) + (\partial_\mu\hat{\Pi}_{\rho\lambda})(\partial_{[\kappa}\hat{\Pi}_{\nu]\sigma}) + (\partial_\mu\hat{\Pi}_{\rho[\lambda})(\partial_{\sigma]}\hat{\Pi}_{\nu\kappa})\Big]\Big\},\end{aligned} \tag{5.48b}$$

$$\tilde{\mathcal{L}}_2 = s^{\mu\nu}s^{\rho\sigma}\Big[(\partial_\nu\hat{\Pi}_{\sigma 0})(\partial_{[\mu}\hat{\Pi}_{\rho]0}) + \frac{1}{2}(\partial_0\hat{\Pi}_{\mu[\rho})(\partial_0\hat{\Pi}_{\nu]\sigma})\Big] - s^{\mu\nu}(\partial_0\hat{T}_0)(\partial_\mu\hat{\Pi}_{\nu 0}), \tag{5.48c}$$

$$\tilde{\mathcal{L}}_3 = 0. \tag{5.48d}$$

Note also that we have defined

$$\hat{T}_{\mu\nu} = 2\partial_{[\mu}\hat{T}_{\nu]}. \tag{5.49}$$

Some of the equalities above hold up to total derivative. In particular, we have used that the product of two partial derivatives with antisymmetrised indices yields only total derivatives and hence can be ignored. For example, we have

$$s^{\mu\nu}(\partial_{[\mu}\hat{T}_\nu)(\partial_{0]}\hat{T}_0) = s^{\mu\nu}\partial_{[\mu}\left(\hat{T}_\nu\partial_{0]}\hat{T}_0\right) - s^{\mu\nu}\hat{T}_\nu\partial_{[\mu}\partial_{0]}\hat{T}_0 = \partial_{[\mu}\left(s^{\mu\nu}\hat{T}_\nu\partial_{0]}\hat{T}_0\right), \tag{5.50}$$

where in the last equality we have used that $s^{\mu\nu}$ is constant.

The theory above, with the PPNR fields as dynamic variables, is at this stage completely equivalent to LinGR. In particular, it has the same symmetries. This means that under a Lorentz transformation $x^\mu \to x^{\mu'} = \Lambda^{\mu'}{}_\mu x^\mu$, the PPNR fields transform as

$$\hat{T}_\mu \to \hat{T}_{\mu'} = (\Lambda^{-1})^\mu{}_{\mu'}\hat{T}_\mu, \qquad \hat{\Pi}_{\mu\nu} \to \hat{\Pi}_{\mu'\nu'} = (\Lambda^{-1})^\mu{}_{\mu'}(\Lambda^{-1})^\nu{}_{\nu'}\hat{\Pi}_{\mu\nu}, \tag{5.51}$$

as follows from plugging (5.43) in (5.4). Similarly, the gauge symmetry (5.21) results in

$$\hat{\Pi}_{\mu\nu} \to \hat{\Pi}_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \tag{5.52}$$

upon the same substitution and assuming that the dependence on c of ξ^μ is of the type (4.53).

In what follows, we shall carry out the large speed of light expansion of the PPNR fields with the ultimate goal of obtaining the non-relativistic expansion of the Lagrangian of LinGR, by applying the general results of Section 4.1 to the Lagrangian (5.46).

5.1.3 Large speed of light expansion of the PPNR fields

Assuming that the vielbeine perturbations admit a Taylor expansion around $\sigma = 0$ of the form (4.53), we can write

$$\hat{T}_\mu = \hat{\tau}_\mu + \sigma\hat{m}_\mu + \sigma\hat{B}_\mu + \mathcal{O}(\sigma^3), \tag{5.53a}$$

$$\hat{E}_\mu{}^a = \hat{e}_\mu{}^a + \sigma\hat{\pi}_\mu{}^a + \mathcal{O}(\sigma^2), \tag{5.53b}$$

$$\hat{V}^\mu = \hat{v}^\mu + \sigma\hat{n}^\mu + \mathcal{O}(\sigma^2), \tag{5.53c}$$

$$\hat{E}^\mu{}_a = \hat{e}^\mu{}_a + \sigma\hat{\rho}^\mu{}_a + \mathcal{O}(\sigma^2). \tag{5.53d}$$

Substituting in (5.36) and collecting terms in powers of σ we find

$$\hat{\upsilon}^0 = \hat{\tau}_0, \quad \hat{\upsilon}^a = \hat{e}_0{}^a, \quad \hat{e}^0{}_a = -\hat{\tau}_a, \quad \hat{e}^a{}_b = -\hat{e}_b{}^a, \quad \delta^0_\mu \hat{\upsilon}^\nu - \delta^a_\mu \hat{e}^\nu{}_a = \delta^\nu_0 \hat{\tau}_\mu + \delta^\nu_a \hat{e}_\mu{}^a, \tag{5.54}$$

and

$$\hat{n}^0 = \hat{m}_0, \quad \hat{n}^a = \hat{\pi}_0{}^a, \quad \hat{\rho}^0{}_a = -\hat{m}_a, \quad \hat{\rho}^a{}_b = -\hat{\pi}_b{}^a, \quad \delta^0_\mu \hat{n}^\nu - \delta^a_\mu \hat{\rho}^\nu{}_a = \delta^\nu_0 \hat{m}_\mu + \delta^\nu_a \hat{\pi}_\mu{}^a. \tag{5.55}$$

It then follows that

$$\hat{\upsilon}^\mu = \delta^\mu_0 \hat{\tau}_0 + \delta^\mu_a \hat{e}_0{}^a, \tag{5.56a}$$

$$\hat{e}^\mu{}_a = -\delta^\mu_0 \hat{\tau}_a - \delta^\mu_b \hat{e}_a{}^b, \tag{5.56b}$$

$$\hat{n}^\mu = \delta^\mu_0 \hat{m}_0 + \delta^\mu_a \hat{\pi}_0{}^a, \tag{5.56c}$$

$$\hat{\rho}^\mu{}_a = -\delta^\mu_0 \hat{m}_a - \delta^\mu_b \hat{\pi}_a{}^b, \tag{5.56d}$$

which implies that the fields appearing in the expansion of the perturbations of the inverse veilbeine at a certain order in σ are completely determined by those appearing in the expansion of the vielbeine at the same order in σ.

The expansion of the PPNR fields is found to be given by

$$\hat{T}_\mu = \hat{\tau}_\mu + \sigma \hat{m}_\mu + \sigma \hat{B}_\mu + \mathcal{O}(\sigma^3), \tag{5.57a}$$

$$\hat{\Pi}_{\mu\nu} = \hat{h}_{\mu\nu} + \sigma \hat{\Phi}_{\mu\nu} + \mathcal{O}(\sigma^2), \tag{5.57b}$$

$$\hat{V}^\mu = \hat{\upsilon}^\mu + \sigma \left(\delta^\mu_0 \hat{m}_0 + \delta^\mu_a \hat{\pi}_0{}^a\right) + \mathcal{O}(\sigma^2), \tag{5.57c}$$

$$\hat{\Pi}^{\mu\nu} = \hat{h}^{\mu\nu} - \sigma \left(2 s^{\rho(\mu} \delta^{\nu)}_0 \hat{m}_\rho + s^{\mu\rho} s^{\nu\sigma} \hat{\Phi}_{\rho\sigma}\right) + \mathcal{O}(\sigma^2), \tag{5.57d}$$

where we have used (5.39) and defined

$$\hat{h}_{\mu\nu} := 2\delta_{ab} \delta^a_{(\mu} \hat{e}_{\nu)}{}^b, \tag{5.58a}$$

$$\hat{h}^{\mu\nu} := 2\delta^{ab} \delta^{(\mu}_a \hat{e}^{\nu)}{}_b, \tag{5.58b}$$

$$\hat{\Phi}_{\mu\nu} := 2\delta_{ab} \delta^a_{(\mu} \hat{\pi}_{\nu)}{}^b. \tag{5.58c}$$

The leading order fields satisfy the following relations:

$$\hat{\upsilon}^0 = \hat{\tau}_0, \qquad \hat{h}^{0\nu} = -\hat{\tau}_\mu s^{\mu\nu}, \qquad \hat{h}_{0\nu} = \hat{\upsilon}^\mu s_{\mu\nu}, \qquad \delta^0_\mu \hat{\upsilon}^\nu - s_{\mu\rho} \hat{h}^{\rho\nu} = \delta^\nu_0 \hat{\tau}_\mu + s^{\rho\nu} \hat{h}_{\mu\rho}. \tag{5.59}$$

The transformation properties of the fields resulting from the expansion of the PPNR fields are obtained from expanding the Lorentz and gauge transformations of the latter. At leading order, the expansion of (5.51) yields the usual Galilean symmetry. This is all we will need in order to make contact with the subsequent linearisation of the LO theory of NRG.

Notice that the expansions in (5.57) are all we need to carry out the expansion of the Lagrangian (5.46). This is of course because $\eta_{\mu\nu}$ and its inverse don't carry any further powers of c than the ones appearing in (5.32). Even if it is not necessary for our purposes, let us write down the corresponding expansion of the metric perturbation and its inverse, resulting from plugging (5.57) in (5.43) and (5.45), respectively:

$$\chi_{\mu\nu} = -2c^2 \delta^0_{(\mu} \hat{\tau}_{\nu)} + \hat{\hat{h}}_{\mu\nu} + \sigma \hat{\hat{\Phi}}_{\mu\nu} + \mathcal{O}(\sigma^2), \tag{5.60a}$$

$$\chi^{\mu\nu} = -\hat{h}^{\mu\nu} - \sigma (2\delta^{(\mu}_0 \hat{\hat{\upsilon}}^{\nu)} - s^{\mu\rho} s^{\nu\sigma} \hat{\hat{\Phi}}_{\rho\sigma}) + \mathcal{O}(\sigma^2), \tag{5.60b}$$

where we have defined

$$\hat{\hat{h}}_{\mu\nu} := \hat{h}_{\mu\nu} - 2\delta^0_{(\mu}\hat{m}_{\nu)}, \tag{5.61a}$$

$$\hat{\hat{v}}^\mu := \hat{v}^\mu - s^{\mu\rho}m_\rho, \tag{5.61b}$$

$$\hat{\hat{\Phi}}_{\rho\sigma} = \hat{\Phi}_{\rho\sigma} - 2\hat{B}_{(\mu}\delta^0_{\nu)}. \tag{5.61c}$$

Let us finish this section with a comment about how the results above can be related to those of Section 4.3. The expressions in (5.57) and (5.60) follow from our definitions of the PPNR fields and their expansions, as explained above. However, they can also be obtained in a more straightforward way by using the general results of Section 4.3 in the particular case where the PNR fields take the form (5.38). Indeed, if we substitute there the corresponding expansions at each side of the equalities, we can identify

$$\tau_\mu = \delta^0_\mu + \hat{\tau}_\mu, \qquad h_{\mu\nu} = s_{\mu\nu} + \hat{h}_{\mu\nu}, \tag{5.62a}$$

$$v^\mu = -\delta^\mu_0 + \hat{v}^\mu, \qquad h^{\mu\nu} = s^{\mu\nu} + \hat{h}^{\mu\nu}, \tag{5.62b}$$

$$m_\mu = \hat{m}_\mu, \qquad \Phi_{\mu\nu} = \hat{\Phi}_{\mu\nu}. \tag{5.62c}$$

One can then verify that (5.57) and (5.60) can be obtained by substituting (5.62) in (4.54) and (4.57), respectively, and keeping only terms that are at most linear in the perturbations. In the last case, one needs to keep in mind that $g^{\mu\nu}$ and $\chi^{\mu\nu}$ carry different signs, as follows from (5.3).

5.1.4 LO and NLO theories

We shall now carry out the large speed of light expansion up to next-to-leading order of the Lagrangian $\mathcal{L}_{\text{LinGR}}$ in its PPNR form given by (5.46). The procedure again relies on the general results of Section 4.1, and will therefore be completely analogous to the non-relativistic expansion of the EH Lagrangian in Section 4.4. In this case, the expansion will yield a theory

$$\mathcal{L}_{\text{LinGR}} = c^6\left(\hat{\mathcal{L}}_{\text{LO}} + \sigma\hat{\mathcal{L}}_{\text{NLO}} + \mathcal{O}(\sigma^2)\right), \tag{5.63}$$

depending *a priori* on the fields

$$\phi^I_0 = \{\hat{\tau}_\mu, \hat{h}_{\mu\nu}\}, \qquad \phi^I_1 = \{\hat{m}_\mu, \hat{\Phi}_{\mu\nu}\}, \tag{5.64}$$

with $I \in \{1,2\}$. We also define the EOM of these fields with respect to the N^{n}LO ($n \in \mathbb{N}$) Lagrangian by

$$\overset{(\text{N}^n\text{LO})}{G^\alpha_{\hat{\tau}}} := -8\pi G_N\left[\frac{\partial\hat{\mathcal{L}}_{\text{N}^n\text{LO}}}{\partial\hat{\tau}_\alpha} - \partial_\mu\left(\frac{\partial\hat{\mathcal{L}}_{\text{N}^n\text{LO}}}{\partial(\partial_\mu\hat{\tau}_\alpha)}\right)\right], \tag{5.65a}$$

$$\overset{(\text{N}^n\text{LO})}{G^{\alpha\beta}_{\hat{h}}} := -16\pi G_N\left[\frac{\partial\hat{\mathcal{L}}_{\text{N}^n\text{LO}}}{\partial\hat{h}_{\alpha\beta}} - \partial_\mu\left(\frac{\partial\hat{\mathcal{L}}_{\text{N}^n\text{LO}}}{\partial(\partial_\mu\hat{h}_{\alpha\beta})}\right)\right], \tag{5.65b}$$

$$\overset{(\text{N}^n\text{LO})}{G^\alpha_{\hat{m}}} := -8\pi G_N\left[\frac{\partial\hat{\mathcal{L}}_{\text{N}^n\text{LO}}}{\partial\hat{m}_\alpha} - \partial_\mu\left(\frac{\partial\hat{\mathcal{L}}_{\text{N}^n\text{LO}}}{\partial(\partial_\mu\hat{m}_\alpha)}\right)\right], \tag{5.65c}$$

$$\overset{(\text{N}^n\text{LO})}{G^{\alpha\beta}_{\hat{\Phi}}} := -16\pi G_N\left[\frac{\partial\hat{\mathcal{L}}_{\text{N}^n\text{LO}}}{\partial\hat{\Phi}_{\alpha\beta}} - \partial_\mu\left(\frac{\partial\hat{\mathcal{L}}_{\text{N}^n\text{LO}}}{\partial(\partial_\mu\hat{\Phi}_{\alpha\beta})}\right)\right], \tag{5.65d}$$

in analogy with (4.91) and (4.93). The recursive structure of the EOM up to NLO can be summarised as

$$
\begin{array}{cccc}
 & \overset{(\mathrm{LO})}{G}{}^{\alpha\beta}_{\hat{h}} & \overset{(\mathrm{LO})}{G}{}^{\alpha}_{\hat{\tau}} & \\
 / \! / & & & \backslash\!\backslash \\
\overset{(\mathrm{NLO})}{G}{}^{\alpha\beta}_{\hat{\Phi}} & \overset{(\mathrm{NLO})}{G}{}^{\alpha\beta}_{\hat{h}} & \overset{(\mathrm{NLO})}{G}{}^{\alpha}_{\hat{\tau}} & \overset{(\mathrm{NLO})}{G}{}^{\alpha}_{\hat{m}}
\end{array}
$$

Using (4.7) and (5.47), the LO Lagrangian reads

$$
\hat{\mathcal{L}}_{\mathrm{LO}} = \tilde{\mathcal{L}}_{\mathrm{LinGR}}(0) = \frac{1}{16\pi G_N}\frac{1}{4}s^{\mu\nu}s^{\rho\sigma}\hat{T}_{\mu\rho}\hat{T}_{\nu\sigma}\bigg|_{\sigma=0} = \frac{1}{16\pi G_N}\frac{1}{4}s^{\mu\nu}s^{\rho\sigma}\hat{\tau}_{\mu\rho}\hat{\tau}_{\nu\sigma}, \tag{5.66}
$$

where we have defined

$$
\hat{\tau}_{\mu\nu} := 2\partial_{[\mu}\hat{\tau}_{\nu]}. \tag{5.67}
$$

Note that the LO Lagrangian is independent of $\hat{h}^{\mu\nu}$ and its derivatives, and in fact depends only on the derivatives of $\hat{\tau}_\mu$. Of course, this comes as no surprise since $\tilde{\mathcal{L}}_0$ in (5.48a) only depends on the derivatives of $\hat{T}_\mu$. The corresponding EOM are then given by

$$
\overset{(\mathrm{LO})}{G}{}^{\alpha}_{\hat{\tau}} := 8\pi G_N \partial_\mu \left(\frac{\partial \hat{\mathcal{L}}_{\mathrm{LO}}}{\partial(\partial_\mu \hat{\tau}_\alpha)} \right) = \frac{1}{2}s^{\mu\nu}s^{\rho\alpha}\partial_\mu\hat{\tau}_{\nu\rho}, \tag{5.68a}
$$

$$
\overset{(\mathrm{LO})}{G}{}^{\alpha\beta}_{\hat{h}} := 0. \tag{5.68b}
$$

Let us now move to the NLO theory. Using (4.8) with the appropriate changes to account for the prefactors in (5.65), the NLO Lagrangian is given by

$$
\hat{\mathcal{L}}_{\mathrm{NLO}} = \tilde{\mathcal{L}}'_{\mathrm{LinGR}}(0) = \frac{\partial \tilde{\mathcal{L}}_{\mathrm{LinGR}}}{\partial \sigma}\bigg|_{\sigma=0} - \frac{1}{8\pi G_N}\hat{m}_\mu \overset{(\mathrm{LO})}{G}{}^{\alpha}_{\hat{\tau}} = \frac{1}{8\pi G_N}\left(\frac{1}{2}\tilde{\mathcal{L}}_1\Big|_{\sigma=0} - \hat{m}_\mu \overset{(\mathrm{LO})}{G}{}^{\mu}_{\hat{\tau}} \right), \tag{5.69}
$$

where in the first equality we have used (5.68b) and in the second one that

$$
\frac{\partial \tilde{\mathcal{L}}_{\mathrm{LinGR}}}{\partial \sigma}\bigg|_{\sigma=0} = \frac{1}{16\pi G_N}\tilde{\mathcal{L}}_1\Big|_{\sigma=0}. \tag{5.70}
$$

In particular, the NLO Lagrangian does not depend on $\hat{\Phi}_{\mu\nu}$, as expected. With these considerations, one finds the latter to be given by

$$
\begin{aligned}
\hat{\mathcal{L}}_{\mathrm{NLO}} = \frac{1}{16\pi G_N} s^{\mu\nu}s^{\rho\sigma}\Big\{ & 2(\partial_\mu\hat{m}_\rho)(\partial_{[\nu}\hat{\tau}_{\sigma]}) + 2(\partial_\mu\hat{\tau}_{[\sigma})(\partial_\rho\hat{h}_{0]\nu}) + (\partial_\mu\hat{\tau}_{[\rho})(\partial_0\hat{h}_{\nu]\sigma}) \\
& + (\partial_{[\mu}\hat{\tau}_{0]})(\partial_\nu\hat{h}_{\rho\sigma}) + (\partial_\mu\hat{\tau}_\rho)(\partial_{[0}\hat{h}_{\nu]\sigma}) + (\partial_\mu\hat{\tau}_{[0})(\partial_\nu\hat{h}_{\rho]\sigma}) \\
& + \frac{1}{2}s^{\lambda\kappa}\Big[(\partial_\mu\hat{h}_{\rho[\sigma})(\partial_{\nu]}\hat{h}_{\lambda\kappa}) + (\partial_\mu\hat{h}_{\rho\lambda})(\partial_{[\kappa}\hat{h}_{\nu]\sigma}) + (\partial_\mu\hat{h}_{\rho[\lambda})(\partial_{\sigma]}\hat{h}_{\nu\kappa})\Big]\Big\},
\end{aligned} \tag{5.71}
$$

up to a total derivative containing $\hat{m}_\mu$. Its EOM are then found to be given by

$$
\begin{aligned}
\overset{\text{(NLO)}}{G}{}^{\alpha}_{\hat{\tau}} =& -\delta^{\alpha}_{[0} s^{\mu\nu} s^{\rho\sigma} \left[\partial_\mu \partial_\rho \hat{h}_{\sigma]\nu} - \frac{1}{2}\left(\partial_{\mu]}\partial_\nu \hat{h}_{\rho\sigma} + \partial_\mu \partial_\nu \hat{h}_{\rho]\sigma}\right)\right] \\
& + s^{\alpha\mu} s^{\rho\sigma}\left(\partial_\rho \partial_{[\mu} \hat{m}_{\sigma]} + \frac{1}{2}\partial_\rho \partial_{[0} \hat{h}_{\sigma]\mu}\right),
\end{aligned}
\tag{5.72a}
$$

$$
\begin{aligned}
\overset{\text{(NLO)}}{G}{}^{\alpha\beta}_{\hat{h}} =& \, \delta^{(\alpha}_{[0} s^{\beta)\rho} s^{\mu\nu}\Big(2\partial_\mu \partial_\rho \hat{\tau}_{\nu]} - \partial_{\mu]}\partial_\nu \hat{\tau}_\rho - \partial_\mu \partial_\nu \hat{\tau}_{\rho]}\Big) + s^{\alpha\beta} s^{\mu\nu} \partial_{[\nu}\partial_\mu \hat{\tau}_{0]} \\
& - s^{(\alpha\left[\beta)\right.} s^{\left.\mu\right]\nu}\Big(2\partial_0 \partial_\mu \hat{\tau}_\nu - \frac{1}{2} s^{\rho\sigma}\partial_\mu \partial_\nu \hat{h}_{\rho\sigma} + \frac{1}{4} s^{\rho\sigma} \partial_\mu \partial_\rho \hat{h}_{\nu\sigma}\Big),
\end{aligned}
\tag{5.72b}
$$

$$
\overset{\text{(NLO)}}{G}{}^{\alpha}_{\hat{m}} = \overset{\text{(LO)}}{G}{}^{\alpha}_{\hat{\tau}}, \tag{5.72c}
$$

$$
\overset{\text{(NLO)}}{G}{}^{\alpha\beta}_{\hat{\Phi}} = \overset{\text{(LO)}}{G}{}^{\alpha\beta}_{\hat{h}}. \tag{5.72d}
$$

Notice that in order to complete the description of the NLO theory, we would like to have a better understanding of its underlying symmetry resulting from the expansion of the Lorentz transformations and diffeomorphism generating vector field in (5.51) and (5.52), respectively. As argued in Chapter 6, this is left for future work.

5.1.5 Harmonic gauge

As argued before, the gauge freedom of LinGR inherited from the diffeomorphism invariance of the full theory can be exploited to simplify the linearised EOM. It is therefore reasonable to study how such gauge freedom can be used to simplify the EOM of the expanded theory. In particular, we choose the harmonic gauge (HG) given by the condition (5.28), and focus on how it can simplify the leading order EOM (5.68a). As shown in the diagram above, there are two natural ways of implementing the choice of gauge to our formalism, depending on whether we impose the HG condition before or after the PPNR parametrisation of $\mathcal{L}_{\text{LinGR}}$. As expected, they turn out to be equivalent, and we shall comment on both.

$$
\begin{tikzcd}
\mathcal{L}_{\text{LinGR}}(\chi, \partial\chi) \arrow[r, "\text{HG}"] \arrow[d, "\text{PPNR}"'] & \mathcal{L}^{\text{HG}}_{\text{LinGR}}(\chi, \partial\chi) \arrow[d, "\text{PPNR}"'] \\
\mathcal{L}_{\text{LinGR}}(\hat{T}, \hat{\Pi}, \partial) \arrow[r, "\text{HG}"] & \mathcal{L}^{\text{HG}}_{\text{LinGR}}(\hat{T}, \hat{\Pi}, \partial)
\end{tikzcd}
$$

Let us start by considering the PPNR version of the HG condition. To this end, let us start rewriting the latter as

$$
\eta^{\mu\rho}\partial_\rho \chi_{\mu\nu} = \frac{1}{2}\eta^{\mu\rho}\partial_\nu \chi_{\mu\rho}. \tag{5.73}
$$

Substituting (5.32b) and (5.43) and collecting terms in powers of c^2 yields the following equalities,

$$
\delta^0_\nu s^{\mu\rho}\partial_\mu \hat{T}_\rho = 0 \quad \Longrightarrow \quad s^{\mu\rho}\partial_\mu \hat{T}_\rho = 0, \tag{5.74a}
$$

$$
s^{\mu\rho}\partial_\mu \hat{\Pi}_{\rho\nu} = 2\partial_{[\nu}\hat{T}_{0]} - \delta^0_\nu \partial_0 \hat{T}_0 + \frac{1}{2} s^{\mu\rho}\partial_\nu \hat{\Pi}_{\mu\rho} \tag{5.74b}
$$

$$
\partial_0 \hat{\Pi}_{0\nu} = 0, \tag{5.74c}
$$

which we interpret as the PPNR version of the HG condition. These can be plugged in (5.46) to obtain the Lagrangian $\mathcal{L}^{\rm HG}_{\rm LinGR}(\hat{T}, \hat{\Pi}, \partial)$ of LinGR in HG and PPNR form. Since we are interested in the LO theory, however, we shall focus on the term (5.48a), as it is the only one contributing to the LO EOM. In this case, we have

$$\mathcal{L}^{\rm HG}_{\rm LinGR}(\hat{T}, \hat{\Pi}, \partial) = \frac{c^6}{16\pi G_N}\left(\tilde{\mathcal{L}}^{\rm HG}_0 + \mathcal{O}(\sigma)\right), \tag{5.75}$$

and we can use (5.74a) to find

$$\tilde{\mathcal{L}}^{\rm HG}_0 = s^{\mu\nu}s^{\rho\sigma}\partial_{[\mu}\hat{T}_{\rho]}\partial_{[\nu}\hat{T}_{\sigma]}\Big|_{\rm HG} = \frac{1}{2}s^{\mu\nu}s^{\rho\sigma}(\partial_\mu\hat{T}_\rho)(\partial_\nu\hat{T}_\sigma), \tag{5.76}$$

where the last equality holds up to a total derivative. It follows that we can then write the large speed of light expansion of (5.75) as

$$\mathcal{L}^{\rm HG}_{\rm LinGR} = c^6\hat{\mathcal{L}}^{\rm HG}_{\rm LO} + \mathcal{O}(c^4), \tag{5.77}$$

where

$$\hat{\mathcal{L}}^{\rm HG}_{\rm LO} = \frac{1}{16\pi G_N}\tilde{\mathcal{L}}^{\rm HG}_0\Big|_{\sigma=0} = \frac{1}{16\pi G_N}\frac{1}{2}s^{\mu\nu}s^{\rho\sigma}(\partial_\mu\hat{\tau}_\rho)(\partial_\nu\hat{\tau}_\sigma). \tag{5.78}$$

The non-vanishing EOM at LO then read

$$\overset{(\rm LO)}{G^{\alpha}_{\hat{\tau}}}\Big|_{\rm HG} := -8\pi G_N\left[\frac{\partial\hat{\mathcal{L}}^{\rm HG}_{\rm LO}}{\partial\hat{\tau}_\alpha} - \partial_\mu\left(\frac{\partial\hat{\mathcal{L}}^{\rm HG}_{\rm LO}}{\partial(\partial_\mu\hat{\tau}_\alpha)}\right)\right] = \frac{1}{2}s^{\mu\nu}s^{\rho\alpha}\partial_\mu\partial_\nu\hat{\tau}_\rho = 0. \tag{5.79}$$

Notice that we could have reached the same result in a more straightforward way by simply imposing the relation (5.74a) in (5.68a). The approach at the level of the Lagrangian is however necessary to compare to the situation resulting from imposing the HG condition before the PPNR parametrisation. This is done by imposing (5.28) on the Lagrangian (5.15), which up to a total derivative results in

$$\mathcal{L}^{\rm HG}_{\rm LinGR}(\chi, \partial\chi) = \frac{c^4}{32\pi G_N}\left[\frac{1}{4}(\partial^\mu\chi)(\partial_\mu\chi) - \frac{1}{2}(\partial_\mu\chi^{\rho\sigma})(\partial^\mu\chi_{\rho\sigma})\right]. \tag{5.80}$$

Indeed, one can verify that a variation of this Lagrangian with respect to $-\chi^{\alpha\beta}$ yields the linearised vacuum EFEs in HG as given in (5.29). By writing (5.80) in PPNR form, one recovers the expression

$$\mathcal{L}^{\rm HG}_{\rm LinGR}(\hat{T}, \hat{\Pi}, \partial) = \frac{c^6}{16\pi G_N}\left(\tilde{\mathcal{L}}^{\rm HG}_0 + \mathcal{O}(\sigma)\right), \tag{5.81}$$

with $\tilde{\mathcal{L}}^{\rm HG}_0$ given as in (5.76). In particular, after the corresponding large speed of light expansion, the EOM for the LO theory is given again by (5.79), showing that it does not make a difference whether the HG condition is imposed before or after the PPNR parametrisation.

In summary, we have shown that the HG conditon allows to simplify the EOM of the LO theory to

$$s^{\mu\nu}s^{\rho\alpha}\partial_\mu\partial_\nu\hat{\tau}_\rho = 0. \tag{5.82}$$

Noticing that

$$s^{\mu\nu}\partial_\mu\partial_\nu = \partial^a\partial_a = \vec{\nabla}^2, \tag{5.83}$$

and using (5.59), the EOM (5.82) reduces to

$$\vec{\nabla}^2\hat{h}^{\alpha 0} = 0. \tag{5.84}$$

5.2 Linearisation of NRG

Let us now address the linearisation of the theory resulting from the non-relativistic expansion of GR at LO,

$$\mathcal{L}_{\mathrm{LO}} = \frac{e}{16\pi G_N}\frac{1}{4}h^{\mu\nu}h^{\rho\sigma}\tau_{\mu\rho}\tau_{\nu\sigma}, \tag{5.85}$$

More precisely, we consider a linearisation around a flat NC background[6]. Flat NC geometry is characterised by the existence of global inertial coordinates [51], such that the clock form and the inverse spatial metric can be taken to be

$$(\delta^0_\mu, s^{\mu\nu}), \tag{5.86}$$

where $s^{\mu\nu} = \delta^{ab}\delta^\mu_a\delta^\nu_b$. Alternatively, one can also think of it as the geometry arising from the non-relativistic expansion of the flat space solution $\eta_{\mu\nu}$ of GR at LO. The projective inverses of (5.86) are given by

$$(-\delta^\mu_0, s_{\mu\nu}),$$

with $s_{\mu\nu} = \delta_{ab}\delta^a_\mu\delta^b_\nu$.

Note that the pair (5.86) is a (trivial) solution of the EOM (4.97) of the LO theory. In order to carry out the linearisation around this solution, we therefore consider a NC pair

$$(\tau_\mu, h^{\mu\nu})$$

with corresponding projective inverses $(v^\mu, h_{\mu\nu})$, given by

$$\tau_\mu = \delta^0_\mu + \hat{\tau}_\mu, \tag{5.87a}$$
$$h^{\mu\nu} = s^{\mu\nu} + \hat{h}^{\mu\nu}, \tag{5.87b}$$
$$v^\mu = -\delta^\mu_0 + \hat{v}^\mu, \tag{5.87c}$$
$$h_{\mu\nu} = s_{\mu\nu} + \hat{h}_{\mu\nu}, \tag{5.87d}$$

with $|\hat{\tau}_\mu|, |\hat{h}^{\mu\nu}|, |\hat{v}^\mu|, |\hat{h}_{\mu\nu}| \ll 1$. Imposing the orthogonality and completeness relations (4.59) one obtains again the constraints (4.59), implying that also in this case the projective inverses $\hat{v}^\mu$ and $\hat{h}_{\mu\nu}$ are determined by those of $\hat{\tau}_\mu$ and $\hat{h}^{\mu\nu}$, respectively. The perturbations on the spatial metrics can be related to the following perturbations on the spatial vielbeine,

$${e_\mu}^a = \delta^a_\mu + {\hat{e}_\mu}{}^a, \tag{5.88a}$$
$${e^\mu}_a = \delta^\mu_a + {\hat{e}^\mu}{}_a, \tag{5.88b}$$

with $|{\hat{e}_\mu}{}^a|, |{\hat{e}^\mu}{}_a| \ll 1$, by

$$\hat{h}^{\mu\nu} = 2\delta^{ab}\delta^{(\mu}_a{\hat{e}^{\nu)}}_b, \qquad \hat{h}_{\mu\nu} = 2\delta_{ab}\delta^a_{(\mu}{\hat{e}_{\nu)}}^b. \tag{5.89}$$

Before proceeding to the linearisation of the EOM of (5.85), let us first address address the linearisation of the integration measure e, defined in (4.92). Up to first order in the perturbations, we have

$$\det(-\tau_\mu\tau_\nu + h_{\mu\nu}) = \det(\bar{\eta}_{\mu\nu} - 2\delta^0_{(\mu}\hat{\tau}_{\nu)} + \hat{h}_{\mu\nu}), \tag{5.90}$$

[6] The linearisation of non-dynamical NC geometry around a flat NC background was already considered in [41]

where $\bar{\eta}_{\mu\nu} = -\delta^0_\mu \delta^0_\nu + s_{\mu\nu}$. Putting $A_{\mu\nu} := -2\delta^0_{(\mu}\hat{\tau}_{\nu)} + \hat{h}_{\mu\nu}$ and using that $\|A_{\mu\nu}\| \ll 1$, we get

$$\det(\bar{\eta}_{\alpha\beta} + A_{\alpha\beta}) = -1 - \bar{\eta}^{\mu\nu} A_{\mu\nu} + \mathcal{O}(A^2), \tag{5.91}$$

which follows from Taylor expanding the matrix identity

$$\det M = e^{\mathrm{Tr}\log M}, \tag{5.92}$$

for $M = \bar{\eta} + A$. We then have

$$e = \left(1 + \bar{\eta}^{\mu\nu} A_{\mu\nu} + \mathcal{O}(A^2)\right)^{1/2} = 1 + \frac{1}{2}\bar{\eta}^{\mu\nu} A_{\mu\nu} + \mathcal{O}(A^2) = 1 - \hat{\tau}_0 + \hat{e}_a{}^a + \mathcal{O}(\hat{\tau}^2, \hat{e}^2). \tag{5.93}$$

We can now obtain a linearised version of the EOM (4.97), simply by substituting the expressions (5.87) and ignoring terms that are higher than linear in the perturbations. In particular, it then follows from the discussion above that we can take the integration measure to be the flat NC one $e = 1$, since any contribution of the perturbations to the measure will yield terms at least quadratic in the perturbations. In this way, we find

$$\overset{\text{(LO)}}{G}{}^{\alpha}_{\tau}\Big|_{\text{lin.}} = \frac{1}{2} s^{\mu\nu} s^{\rho\alpha} \partial_\mu \tau_{\nu\rho}, \tag{5.94a}$$

$$\overset{\text{(LO)}}{G}{}^{\alpha\beta}_{h}\Big|_{\text{lin.}} = 0, \tag{5.94b}$$

Note that these are precisely the EOM (5.68a) obtained from the non-relativistic expansion of LinGR at leading order, showing that the two theories are equivalent on shell. We would however like to show the two theories to be equivalent also off shell. To do this, we simply need to consider the leading term in the perturbations of $\hat{\mathcal{L}}_{\text{LO}}$, which is also straightforward. Indeed, substituting (5.87) in (5.85), we find

$$\mathcal{L}_{\text{LO}} = \frac{1}{16\pi G_N} \frac{1}{4} s^{\mu\nu} s^{\rho\sigma} \hat{\tau}_{\mu\rho} \hat{\tau}_{\nu\sigma} + \text{subleading terms}, \tag{5.95}$$

where we have again used that the contributions of the perturbations to the measure yield only subleading terms. We see that the leading order contribution of the linearised theory is precisely the leading order Lagrangian (5.66) of NRLinGR.

Chapter 6

Discussion and outlook

The main purpose of this thesis has been the study of the weak field limit of non-relativistic gravity. To this end, we have presented a description of some non-Lorentzian geometries that arise from non-Lorentzian local symmetries. In particular, we have studied NC geometry and its extensions as the adequate geometric framework of non-relativistic physics, as well as Carrollian geometry, as the appropriate geometric framework of ultra-local physics. We have payed special attention to understanding both geometries in the more general context of frame bundles. The generality of this approach blurs the initial conceptual barrier between Lorentzian and non-Lorentzian geometry, since both can be described in terms of G-structures but for different subgroups $G < \mathrm{GL}(D, \mathbb{R})$. The topic inevitably lies in the interface between theoretical physics and mathematics, and as such its further development from any of the two sides would most likely benefit the other.

We have presented a detailed description of the non-relativistic expansion of GR, and how it gives rise to the recently discovered covariant formulation of non-relativistic gravity. In passing, we have benefited from the duality between NC and Carroll geometry to gain valuable insight on non-relativistic gravity by studying the even more recent covariant formulation of Carroll gravity from an ultra-local expansion of GR. In particular, this has led us to propose an interpretation of a truncated sector of the NLO theory in the non-relativistic expansion as the non-relativistic magnetic limit of GR.

The main contribution of this thesis is the investigation of the weak field limit of non-relativistic gravity. As we have previously argued, the latter can be approached from two routes: a non-relativistic expansion of linearised GR and a linearisation of NRG. We have explored both of them, giving rise to the theories of non-relativistic linearised GR (NRLinGR) and linearised NRG (LinNRG). The equivalence between the two was expected, but its non-trivial explicit realisation has been one of the main goals of this work. We have succeeded in showing that the equivalence holds at leading order. More precisely, our formulation of NRLinGR yields a LO Lagrangian that coincides with the one obtained by considering small perturbations of the geometric fields of NRG at LO around a flat NC background. We conjecture that the equivalence holds also beyond LO, although in this case it is no longer manifest. The result therefore suggests that our formulation of NRLinGR provides an adequate framework for the study of the weak field limit of NRG. This is very relevant in that the study of NRLinGR beyond leading order seems to be simpler than that of LinNRG. In particular, we have obtained the LO and NLO Lagrangians of the latter as well as their corresponding EOM. These are also among the main contributions of this work. Finally, we have also found a simplification of the EOM of its LO theory by considering a non-relativistic version of the harmonic gauge condition of linearised GR.

The results explained above open up a variety of interesting future research directions. As a first further analysis, we would of course like to explicitly show the equivalence beyond LO of the two paths to the weak field limit of NRG. As a first step in this direction, it could be instructive to consider the linearisation of the truncated sector of the NLO theory in the non-relativistic expansion of GR, that we have interpreted as its non-relativistic magnetic limit. The obtention of the EOM and solutions thereof for this theory (outside the weak field limit) is also something we would like to explore. Regarding the non-relativistic expansion of LinGR, we would like to look closely on the symmetries of the NLO theory, which requires the large speed of light expansion of the Lorentz and gauge transformations of LinGR. Another obvious direction is the study of possible solutions to the EOM of the LO and NLO theories. In this sense, the gauge fixed simplified version of the LO EOM is a natural start. In particular, we would like to study solutions of NRLinGR by considering the linearised versions of non-trivial solutions of NRG, such as the Schwarzschild solution of [44].

Another interesting problem to address is the study of the weak field limit of Carroll gravity, to which we believe that the methods developed here should readily apply. Indeed, the duality between the non-relativistic and ultra-local expansions of GR from which NRG and Carroll gravity are obtained implies that our PPNR parametrisation of the EH Lagrangian could be mirrored in order to obtain a perturbative pre-ultra-local (PPUL) parametrisation of the latter. This would eventually lead to an ultra-local expansion of LinGR, that could then be compared with a linearised version of Carroll gravity.

On a conceptual level, we would like to understand better the relation between Newtonian gravity and the weak field limit of NRG. The former is obtained from GR by the so-called Newtonian limit, which is characterised not only by the assumption that the field is weak, but also that it is static and that test particles are moving slowly with respect to the speed of light. The last condition effectively removes relativistic effects from the picture. Therefore, we expect the weak field limit of NRG to go beyond Newtonian gravity as it allows for a time dependence of the perturbations of the geometric fields. Moreover, the perturbation $\hat{\tau}_\mu$ of the clock-form can in principle give rise to a small amount of torsion and hence local time dilation, which would also go beyond the absolute time of Newtonian gravity.

Finally, a good understanding of the weak field limit of NRG could lead to an eventual study of its quantum version, thus providing a new way to access the non-relativistic quantum gravity corner of the Bronstein cube in Fig. 1.1.

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