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Different paths to massive higher spin scattering

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Colophon

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Abstract

This thesis reviews the construction of theories of higher spin in the canonical formalism as well as the spinor helicity formalism, and discusses the difficulties that arise when interactions with photons are considered in these frameworks, in particular the Compton scattering amplitude. We then consider an alternative formalism originally introduced by S. Weinberg, which involve fields in arbitrary representations, and discuss how this framework may by-pass the difficulties encountered in the previous formalism. In particular, we consider the construction of the off-shell higher spin propagator, which differs from the on-shell propagator found in the literature. Finally, the different formalisms are compared in the context of tree-level scattering processes.

Contents

Chapter 1

Introduction

The physics of elementary particles requires the combined features of quantum mechanics and special relativity, resulting in the framework known as quantum field theory. A key feature of this theory, is the notion that particles posses an intrinsic property called spin. This property governs the way particles interact, in particular in the presence of electromagnetic- and gravitational fields. The classical analogy of this notion is intrinsic angular momentum, and it is from this analogy that the quantum phenomenon derives its name, "spin". In addition to affecting fundamental interactions, spin is also central to our definition of particles and how we classify them. Our current knowledge of the elementary particles in nature, as captured by the Standard Model of particle physics, includes particles of spins 0 (such as the Higgs boson), $1/2$ (leptons and quarks) and 1 (such as photons and the W- and Z-bosons). In addition, the mediation of gravitational forces, as described by Einstein's theory of general relativity, includes a massless spin 2 particle (the graviton). Elementary particles of other spins are present in more exotic theories, such as supersymmetry which introduces a spin $\frac{3}{2}$ superpartner of the graviton, the gravitino[1].

While elementary particles of higher spin are yet to be seen in nature, the existence of higher spin composite particles is well known[1]. Such particles interact via the electromagnetic as well as gravitational forces, (otherwise, we would not detect them) and we would naturally like a quantum field theory describing such interactions, as we have for the elementary particles. It is reasonable to assume that such a theory exists, at least at length scales L where the composite nature of particles becomes irrelevant, i.e. $L \gg \lambda_m$, where $\lambda_m \sim 1/m$ denotes the deBroglie wavelength of a particle of mass m. In addition to describing the physics of elementary particles, an effective theory operating on such scales is also of interest to the field of gravitational physics. Spinning black holes (or Kerr black holes) are characterised entirely by their mass M and (classical) spin S[2], and at sufficiently great distances (much greater than the Schwarzchild radius) their interactions admit a particle like description[4]. This viewpoint, combined with a low energy effective field theory of gravity, allows for a description of black hole physics in terms of quantum field theory methods[5, 6, 7, 8, 9, 10, 11, 12, 13]. For this reason, understanding the scattering of massive higher spin particles in gravity has become relevant to the study of black hole physics.

The standard approach to describing any kinds of interaction in quantum field theory, is to perform an expansion in the coupling constant in terms of Feynman diagrams. The rules for evaluating such diagrams are typically obtained from a Lagrangian, which describes the fields included in the theory. For higher spin fields, it is possible construct a Lagrangian which describes a free, on-shell field of arbitrary spin and mass. Once interaction are considered however, problems arise. One problem is, that the number of degrees of freedom encoded by the field may change in the presence of e.g. an electromagnetic field. This pathology can be remedied by ensuring that the correct equations of motion are obtained from a Lagrangian. While such Lagrangians

1: E.g. the baryon $\Delta(2950)$, with spin $s = \frac{15}{2}$.[2]
2: This is the famous "No hair" theorem[3]

are constructible, they necessitate the introduction of a large number of auxiliary fields, making the arbitrary spin Lagrangians complicated[14, 15, 16]. Another problem, which appears for spins as low as $s = \frac{3}{2}$, is that minimal coupling to electromagnetic- or gravitational fields leads to solutions which display superluminal propagation, breaking causality (this is known as the Velo-Zwanziger problem[17]). The solution to this problem is well-known for spin $s = \frac{3}{2}$, and involves the introduction of a number of non-minimal coupling terms[18]. For higher spins however, the task of introducing such terms quickly becomes non-trivial, since the number and complexity of the necessary terms increase with the spin of the particle[19]. These obstacles indicate that the field theory description of Lagrangians and Feynman rules are not well-suited to the task of describing interacting, massive higher spin fields.

In addition to the Lagrangian formulation of field theory, there is the modern approach to scattering amplitudes which considers the kinematic properties of the amplitudes more directly, without the introduction of quantum fields[20, 21]. This line of research has seen tremendous success in recent years, and has become instrumental in the calculation of scattering amplitudes, in particular in gravity[22, 23, 24, 25, 26]. Interestingly, the application to massive higher spin scattering has recently been considered[22, 27, 28] with some success, although problems arise when the exchange of massive higher spin states are exchanged[27].

In addition to the two approaches mentioned, alternative formulations of massive higher spin theory exists, if somewhat obscurely[29, 30, 31, 32, 33]. One may hope that the difficulties encountered in the more mainstream directions are avoided by such formalisms.

The main purpose of this thesis is to review the difficulties associated with the traditional approach to higher spin, and consider some of the possible alternative formulations of higher spin theory, which may be better suited to describe scattering processes involving particles of arbitrary spins.

Outline of this thesis

The structure of this thesis is as follows. First, we will review the standard story of the Poincaré group and the definition of particles as irreducible representations thereof. This will set the conventions for the rest of the thesis. In chapter 2, we will review the description of massive higher spin particles in the canonical framework, in which higher spin states are described as Lorentz tensors and tensor spinors. This includes the construction of the simplest quadratic Lagrangian from which the necessary on-shell conditions can be derived through the equations of motion, and the problems one encounters when turning interactions by minimal coupling to a photon. In chapter 3 the massive spinor helicity formalism is introduced, and the construction of four-point amplitudes for massive spin s particles and photons is reviewed, including a discussion of the Compton amplitude (which, like in the canonical formalism, is accompanied by difficulties for higher spins). Following this, we return to a field-theory point of view in chapter 4, where we consider a bi-spinor description of massive higher spin, including the construction of Feynman rules. In chapter 5 we consider the generalisation of the spinor field description to arbitrary representations. Finally, in chapter 6 we discuss the merits of the different formalisms in the context of tree-level scattering.

1.1 Setup

In order to understand the challenges that lie in constructing a consistent formulation of higher spin theories, we should begin by defining what we mean by a "particle of higher spin". We will review the standard story, in which particles are defined as irreducible representations of the Poincaré group.

1.1.1 The Poincaré group and the Lorentz group

The Poincaré group,

$$P = SO(1,3) \ltimes \mathbb{R}^4, \tag{1.1}$$

consists of all the space-time isometries: spatial rotations, spatial translations and time translations. The Lorentz group $SO(1,3)$ is a subgroup of the Poincaré group which consists of elements (Lorentz transformations) Λ that leave the spacetime metric η invariant:

$$\Lambda \in SO(1,3), \quad \Lambda^T \eta \Lambda = \eta, \tag{1.2}$$

with the additional condition that

$$\det(\Lambda) = 1. \tag{1.3}$$

For the Minkowski spacetime metric we take the mostly positive convention:

$$\eta = \text{diag}\{-1,+1,+1,+1\}. \tag{1.4}$$

Rotations and boosts are included in the Lorentz group; they make up the connected part of the Lorentz group, $SO^+(1,3)$, whose elements fulfill the additional requirement:

$$\Lambda^0_{\ 0} > 0, \tag{1.5}$$

In addition to preserving the metric, the elements of $SO^+(1,3)$ also preserve orientations in spacetime. From this point forward, we will be referring to

$SO^+(1,3)$ simply as the Lorentz group. The elements $\Lambda \in P$ of the Poincaré group can be written in terms of the generators, P^μ, $M^{\mu\nu}$:

$$\Lambda = \exp[ia_\mu P^\mu]\exp[i\omega_{\mu\nu}M^{\mu\nu}]. \tag{1.6}$$

Here, a_μ is a constant four-vector parameterising the translations generated by P^μ and $\omega_{\mu\nu} = -\omega_{\nu\mu}$ is an anti-symmetric tensor which contains the angles parameterising the rotations and boosts, generated by $M^{\mu\nu}$. Specifically, the generators of rotations in three dimensions J^k and boosts K^i are:

$$\begin{aligned} M^{ij} &= \epsilon^{ijk}J^k, \\ M^{0i} &= K^i, \end{aligned} \tag{1.7}$$

where $\epsilon^{123} = +1$. The generators satisfy the Poincaré algebra[34]:

$$\begin{aligned} [P^\mu, P^\nu] &= 0, \\ [M^{\mu\nu}, P^\lambda] &= i\eta^{\lambda[\mu}P^{\nu]}, \\ [M^{\mu\nu}, M^{\rho\sigma}] &= iM^{[\mu\nu}\eta^{\rho\sigma]}. \end{aligned} \tag{1.8}$$

Here, we use the normalised antisymmetrisation convention:

$$T_{[\mu_1\cdots\mu_n]} = \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^{\text{sign}(\sigma)}T_{\mu_{\sigma(1)}\cdots\mu_{\sigma(n)}}. \tag{1.9}$$

The algebra satisfied by the generators J^i, K^i follow from the Poincaré algebra:

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk}J_k, \\ [J_i, K_j] &= i\epsilon_{ijk}K_k, \\ [K_i, K_j] &= -i\epsilon_{ijk}J_k. \end{aligned} \tag{1.10}$$

Note here that the above algebra depends on the metric signature, from our convention with $\eta_{00} = -1$ we get a sign on the boost commutator. We will need explicit expressions for the unitary representations of the Lorentz group when defining particle states later. Consider the generic four-momentum vector $p^\mu = (\mathbf{p}, \omega(\mathbf{p}))$, with:

$$\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}. \tag{1.11}$$

The unitary representation of a finite rotation about the axis defined by a vector $\mathbf{n}$ by an angle ϕ takes the familiar form seen in most introductory texts on quantum mechanics (e.g. [35]):

$$U(R) = \exp[-i\phi \cdot \mathbf{n} \times \mathbf{J}], \tag{1.12}$$

where $\mathbf{J}$ is the vector with indices J_1, J_2, J_3. Of particular interest to us will be the finite rotation which aligns the momentum vector $\mathbf{p}$ with the quantization axis[3], which we will take to be $\mathbf{e}_3$. In this case, the vector $\mathbf{n}$ is given by:

$$\mathbf{n} = \frac{\mathbf{p} \times \mathbf{e}_3}{|\mathbf{p} \times \mathbf{e}_3|} = \frac{\mathbf{p} \times \mathbf{e}_3}{\sqrt{p_1^2 + p_2^2}}, \tag{1.13}$$

while the angle ϕ is determined by:

$$\cos\phi = \frac{\mathbf{p} \cdot \mathbf{e}_z}{|\mathbf{p}||\mathbf{e}_z|} = \frac{p_z}{|\mathbf{p}|}, \quad |\mathbf{p}| = \sqrt{p_1^2 + p_2^2 + p_3^2}. \tag{1.14}$$

A boost $B^{\mu}_{\nu}(p)$ can be parameterised in terms of a hyperbolic angle θ:

$$\sinh(\theta) = \frac{\mathbf{p}}{m}, \; \cosh(\theta) = \omega(\mathbf{p})/m, \tag{1.15}$$

as[29]:

$$\begin{aligned} B^i_{\ j} &= \delta_{ij} + (\mathbf{e}_p)_i(\mathbf{e}_p)_j(\cosh(\theta) - 1), \\ B^i_{\ 0} &= B^0_{\ i} = (\mathbf{e}_p)_i \sinh(\theta), \\ B^0_{\ 0} &= \cosh(\theta), \end{aligned} \tag{1.16}$$

where we have defined the unit vector pointing in the direction of the momentum three-vector $\mathbf{p}$:

$$\mathbf{e}_p = \frac{\mathbf{p}}{|\mathbf{p}|}. \tag{1.17}$$

The unitary representation of the finite boost defined above is then given by:

$$U(B(p)) = \exp[-i\theta \mathbf{e}_p \cdot \mathbf{K}], \tag{1.18}$$

where $\mathbf{K}$ is the vector with components K_1, K_2, K_3, and with θ defined as above.

3: By quantization axis we mean the axis unto which we project angular momentum, such that the helicity σ denotes the projection of spin along this axis.

1.1.2 Particles

Next, we want to define what a particle is. In quantum mechanics, we like to think about particles in terms of one-particle states living in a Hilbert space, on which we can act with transformations through their unitary representations, $U(\Lambda)$ (we here make the assumption that for every Λ in the Lorentz group there exists a corresponding unitary representation $U(\Lambda)$, and that these representations have the property $U(\Lambda_1\Lambda_2) = U(\Lambda_1)U(\Lambda_2)$.) Following S. Weinberg [29, 34] we start by defining such one-particle states with momentum p^μ as an eigenstate of the operator P^μ:

$$P^\mu \left|p, \sigma\right\rangle = p^\mu \left|p, \sigma\right\rangle . \tag{1.19}$$

Here, σ is a label representing any other (discrete) degrees of freedom the particle state may carry, such as helicity. The one-particle state $\left|p, \sigma\right\rangle$ can be defined from the one-particle state in the rest frame ($p^\mu = (m, 0, 0, 0)$) via the boost $B(p)$ which takes $(m, 0, 0, 0) \to (m, \mathbf{p})$ as:

$$\left|p, \sigma\right\rangle = \sqrt{m/\omega(\mathbf{p})}U(B(p)) \left|0, \sigma\right\rangle . \tag{1.20}$$

Here, we have chosen the normalisation such that:

$$\langle p', \sigma | p, \sigma\rangle = \delta^3(\mathbf{p} - \mathbf{p}')\delta_{\sigma\sigma'}. \tag{1.21}$$

In principle, one does not have to start in the rest frame; we could simply work with some reference momentum p and define states with arbitrary momentum p' using the boost $B(p', p)$:

$$(p')^\mu = B^\mu_\nu(p', p)p^\nu. \tag{1.22}$$

Importantly, the boost $B^\mu_\nu(p', p)$ is not unique, since there exists a subset of the Lorentz group comprised of transformations $W(p)$ which leaves the momentum p^μ invariant:

$$W(p)^\mu_{\ \nu} p^\mu = p^\nu. \tag{1.23}$$

We call the group comprised of the transformations $W(p)$ the Little-group. For massive particles in the rest frame, this is simply the group comprised of spatial rotations, $SO(3)$. Next, we may consider how a general one-particle state $|p,\sigma\rangle$ behaves under a general Lorentz transformation $U(\Lambda)$. Write:

$$U(\Lambda)\,|p,\sigma\rangle = \sqrt{m/\omega(\mathbf{p})}U(\Lambda)\,|0,\sigma\rangle\,.$$

By multiplying with $1 = U(B(\Lambda p))U(B^{-1}(\Lambda p)$ and using the composition property of the unitary representations, this can be re-written as:

$$\begin{aligned} U(\Lambda)\,|p,\sigma\rangle &= \sqrt{m/\omega(\mathbf{p})}U(B(\Lambda p))U(B^{-1}(\Lambda p)U(\Lambda)\,|0,\sigma\rangle \\ &= U(B(\Lambda p))U(B^{-1}(\Lambda p)\Lambda B(p))\,|0,\sigma\rangle\,. \end{aligned}$$

Note that the transformation $U(B^{-1}(\Lambda p)\Lambda B(p)) = U(W(p))$ is an element of the Little group which leaves p invariant. Next, we can expand the above expression by inserting $\sum_{\sigma'}|\sigma'\rangle\langle\sigma'|$, such that:

$$\begin{aligned} U(\Lambda)\,|p,\sigma\rangle &= \sqrt{m/\omega(\mathbf{p})}\sum_{\sigma'}U(B(\Lambda p))\,|\sigma'\rangle\,\langle\sigma'|\,W(p)\,|\sigma\rangle \\ &= \sqrt{m/\omega(\mathbf{p})}\sum_{\sigma'}U(B(\Lambda p))\,|\sigma'\rangle\,D^{(s)}_{\sigma'\sigma}(W(p)), \end{aligned}$$

where we have defined the representation of the Little group,

$$D^{(s)}_{\sigma'\sigma}(W(p)) \equiv \langle\sigma'|\,U(W(p))\,|\sigma\rangle\,. \tag{1.24}$$

Setting $|\Lambda p,\sigma\rangle = \sqrt{\omega(\mathbf{p})/m}U(B(\Lambda))\,|0,\sigma\rangle$, we arrive at the transformation property:

$$U(\Lambda)\,|p,\sigma\rangle = \sqrt{\omega(\Lambda\mathbf{p})/\omega(\mathbf{p})}\sum_{\sigma'}D_{\sigma'\sigma}\,|\Lambda p,\sigma'\rangle\,. \tag{1.25}$$

The conclusion here is that a one-particle state transforms under the irreducible representations of the little group. In particular, massive particles of spin s transform under the irreducible representation of $SO(3)$, which is given by the spin s rotation matrices. With this definition, we can proceed to consider the description of fields into which we embed such particles.

Chapter 2

Lagrangian formulation of higher spins

The purpose of the vast majority of work on higher spin field theories (quantum and classical) has been to formulate a theory in the canonical framework, i.e. with a Lagrangian from which equations of motion may be derived. In this chapter, we will review the attempts at formulating such a theory, for both integer and half-integer spin. The first step is to pick a representation of the Lorentz group under which the higher spin fields transform. As such a theory must be Lorentz covariant, it makes sense to pick representations corresponding to Lorentz tensors (and tensor spinors for half-integer spin). The next step is then to construct a Lagrangian for these fields, from which the appropriate equations of motion may be derived. Once the Lagrangian is constructed, we will consider the introduction of interactions in the form of minimal coupling, and the difficulties that follow.

2.1 Covariant representations

Bosons: $D(\frac{s}{2},\frac{s}{2})$ representation

Consider a field of integer spin, s. As a spin 1 particle can be described in terms of a vector field A^{μ} (belonging to the representation $D(\frac{1}{2},\frac{1}{2})$) so can a particle of spin s be described by a totally symmetric and traceless Lorentz tensor of s, belonging to the representation:

$$D(\tfrac{s}{2},\tfrac{s}{2}), \tag{2.1}$$

of the Lorentz group. We will call such tensors describing integer spin particles $\Phi^{\mu_1\cdots\mu_s}$. The field $\Phi^{\mu_1\cdots\mu_s}$ must satisfy the Klein-Gordon equation:

$$(\partial^2+m^2)\Phi^{\mu_1\cdots\mu_s}=0. \tag{2.2}$$

To avoid a proliferation of indices, it is customary to introduce the shorthand notation[36, 37]:

$$\Phi^{\mu_1\cdots\mu_s}=\Phi^{(\mu_1\cdots\mu_s)}\equiv\Phi^{(s)}. \tag{2.3}$$

The suppression of indices is less confusing since the tensor is totally symmetric (and the ordering thus does not matter)[1]. In the same index-free spirit, the contraction of totally symmetric tensors may be written in the compact form:

$$A_{(\mu_1\cdots\mu_n)}B^{(\mu_1\cdots\mu_m)}=(A\cdot B). \tag{2.4}$$

(Note that the total symmetry of A and B makes this contraction unambiguous). Products of n similar operators also admits a shorthand notation. For example, we shall often encounter expressions involving several derivatives of numbered indices. These can be written as:

$$\partial_{\mu_1}\ldots\partial_{\mu_n}\equiv\partial_n, \tag{2.5}$$

1: We could denote totally anti-symmetric tensors of rank r in a similar fashion as $T^{[r]}$.

and as $\partial_\mu = \partial$ when $n = 1$. In this way, we can treat such repeated derivatives as a rank n tensor, and write contractions using the above defined notation:

$$\partial_{\mu_1} \dots \partial_{\mu_n} A^{\mu_1 \dots \mu_n} = \partial_n \cdot A^{(n)}. \tag{2.6}$$

This is not to be confused with terms of the form:

$$\partial_{\mu_1} \dots \partial_{\mu_n} A_{\nu_1 \dots \nu_n} = \partial_n A^{(n)}, \tag{2.7}$$

where the lack of a $[\cdot]$ indicates that the n indices are not being contracted. This choice of notation makes no discrimination between covariant and contravariant tensors, and so we will only employ it when there can be no cause of confusion. We will sometimes drop it to make explicit calculations easier to follow.

Using this notation, the tracelessness of the tensors $\Phi^{(s)}$ takes the form:

$$\eta \cdot \Phi^{(s)} = 0. \tag{2.8}$$

An important requirement to impose on tensors describing massive higher spin particles, is that they must encode the correct number of degrees of freedom. It is well-established that a massive particle of spin s has $2s + 1$ degrees of freedom [38]. Meanwhile, a totally symmetric tensor in d dimensions carries [36]:

$$n(s) = \begin{pmatrix} d - 1 + s \\ s \end{pmatrix}, \tag{2.9}$$

degrees of freedom. Setting $d = 4$, this is:

$$n(s) = \begin{pmatrix} 3 + s \\ s \end{pmatrix} = \frac{(s+3)!}{s!3!} > 2s + 1. \tag{2.10}$$

The tracelessness condition subtracts from this number by removing the degrees of freedom encoded in the totally symmetric rank $s - 2$ tensor that is the trace $\eta \cdot \Phi^{(s)}$:

$$n(s-2) = \begin{pmatrix} 3 + (s-2) \\ s - 2 \end{pmatrix} = \begin{pmatrix} s + 1 \\ s - 2 \end{pmatrix}. \tag{2.11}$$

The total number of degrees of freedom is then brought down to:

$$n = n(s) - n(s-2) = \begin{pmatrix} 3+s \\ s \end{pmatrix} - \begin{pmatrix} 1+s \\ s-2 \end{pmatrix},$$

which is still greater than $2s+1$. We therefore need additional constraints on the tensor $\Phi^{(s)}$ in order to remove the superfluous degrees of freedom. Suppose we, in addition to removing the trace $\eta \cdot \Phi^{(s)}$, also remove the divergence $\partial \cdot \Phi^{(s)}$. The divergence itself is a rank $s-1$ tensor (since we contract a single index) and is already traceless due to the tracelessness condition imposed on $\Phi^{(s)}$. The divergence therefore must carry:

$$n_{\partial \cdot \Phi^{(s)}} = \begin{pmatrix} 2+s \\ s-1 \end{pmatrix} - \begin{pmatrix} s \\ s-3 \end{pmatrix}, \tag{2.12}$$

since the tracelessness of the divergence removes the degrees of freedom corresponding to a totally symmetric rank $s-3$ tensor. With this additional constraint, we arrive at the number:

$$\begin{aligned} n &= n(s) - n(s-2) - n(s-1) - n(s-3) \\ &= \begin{pmatrix} 3+s \\ s \end{pmatrix} - \begin{pmatrix} 1+s \\ s-2 \end{pmatrix} - \begin{pmatrix} 2+s \\ s-1 \end{pmatrix} - \begin{pmatrix} s \\ s-3 \end{pmatrix} \\ &= \begin{pmatrix} s \\ s \end{pmatrix} + 2 \begin{pmatrix} s \\ s-1 \end{pmatrix} \\ &= 2s+1. \end{aligned}$$

This is the desired number of degrees of freedom. The removal of the divergence of $\Phi^{(s)}$ was originally motivated by the desired to remove negative energy solutions to the wave equation [14]. In addition to removing unwanted degrees of freedom from the tensor $\Phi^{(s)}$, the condition $\partial \cdot \Phi^{(s)}$ serves another purpose. The representation $D(\frac{s}{2}, \frac{s}{2})$ is reducible under the subgroup of spatial rotations, $SO(3)$ [15]:

$$D(\tfrac{s}{2}, \tfrac{s}{2}) = \sum_{s'=0}^{s} D(s'). \tag{2.13}$$

This is bad news, as we want our particle to be an irreducible representation of the Lorentz group. We therefore want to make it such that only the $s' = s$ piece of the above decomposition remains, and this can be achieved by removing the divergence of $\Phi^{(s)}$. To see why, consider the particle in the rest frame, in which the momentum is $p^\mu = (m, 0, 0, 0)$. We can imagine decomposing the tensor $\Phi^{(s)}$ as follows:

$$\Phi^{(s)} = \Phi^{\mu_1 \ldots \mu_s} = \Phi^{i_1 \ldots i_s} \oplus \Phi^{i_1 \ldots i_{s-1} 0} \oplus \cdots \oplus \Phi^{0 \ldots 0}, \tag{2.14}$$

where i denotes spatial indices and 0 denotes time indices. In the rest-frame, the first piece of the above decomposition corresponds to a rank s tensor, the next to a rank $s-1$ tensor, and so on. In the rest-frame, the vanishing of the divergence implies that all $\mu_k = 0$ components must vanish, since (in momentum space) taking the divergence corresponds to contracting with the four-vector momentum, p^μ.

Fermions: Rarita-Schwinger representation

Next, we consider the case where s is a half-integer, $s = k + 1/2$ with k an integer. Since the labels of a representation of the Lorentz group must consist of integers or half-integers, we can no longer use $D(\frac{s}{2}, \frac{s}{2})$. The natural way to represent the half-integer field is by constructing a tensor spinor (a tensor with an additional spinor index). Such objects belong to the representation [36]:

$$D(\tfrac{k+1}{2}, \tfrac{k}{2}) \oplus D(\tfrac{k}{2}, \tfrac{k+1}{2}). \tag{2.15}$$

For $k = 0$ we recover the familiar $D(\frac{1}{2}, 0) \oplus D(0, \frac{1}{2})$ representation to which the Dirac bi-spinor belongs[38]. Historically, the above kind of representation was first considered by W. Rarita and S. Schwinger in describing the $s = 3/2$ field using a vector spinor belonging to the representation:

$$D(\tfrac{1}{2}, 1) \oplus D(\tfrac{1}{2}, 0) \oplus D(1, \tfrac{1}{2}) \oplus D(0, \tfrac{1}{2}). \tag{2.16}$$

Schematically, this corresponds to a $4-$vector times a spinor:

$$D(\tfrac{1}{2},\tfrac{1}{2}) \otimes [D(\tfrac{1}{2},0) \oplus D(0,\tfrac{1}{2})] = D(\tfrac{1}{2},1) \oplus D(\tfrac{1}{2},0) \oplus D(1,\tfrac{1}{2}) \oplus D(0,\tfrac{1}{2}).$$

In our case, we will have a rank k tensor multiplied by a spinor. Such tensor spinors will be denoted by:

$$\Psi^{(s)} \equiv \Psi^{\mu_1 \dots \mu_k, \alpha}. \tag{2.17}$$

Often, the spinorial index α is suppressed altogether, but since our choice of notation suppresses all indices, we need not worry about this. As in the bosonic representation tracelessness is required, but this time with respect to the γ_μ matrices[2]:

$$\gamma \cdot \Psi^{(s)} = 0. \tag{2.18}$$

In the case of the Rarita Schwinger field, this condition removes the $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ from (2.16), leaving the $(1,\frac{1}{2}) \oplus (\frac{1}{2},1)$, which is the desired piece. In addition to $\gamma-$tracelessness we also need to still impose the condition:

$$\partial \cdot \Psi^{(s)} = 0. \tag{2.19}$$

For on-shell particles, $\Psi^{(s)}$ must obey the Dirac equation:

$$(i\not{\partial} - m)\Psi^{(s)} = 0. \tag{2.20}$$

(Note that this is sufficient, as the γ-tracelessness condition in combination with the Dirac equation yields the equation of motion (2.2)). The counting of degrees of freedom for higher spin fermions is analogous to that of higher spin bosons. The totally symmetric rank k tensor spinor $\Psi^{(s)}$ will (in $d=4$ spacetime dimensions) carry:

$$n(k) = 2 \times \begin{pmatrix} 3+n \\ n \end{pmatrix}, \tag{2.21}$$

2: We will use the Weyl basis throughout, so $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$. Conventions for the Pauli matrices will be introced in chapter 3.

degrees of freedom, the factor of 2 coming from the spinor. The $\gamma-$trace and the divergence conditions both substract from this, and the final superfluous degrees of freedom are removed by the Dirac equation, which reduces the above number by half[36]. Together, the $D(\frac{k+1}{2},\frac{k}{2})\oplus(\frac{k}{2},\frac{k+1}{2})$ and $D(\frac{s}{2},\frac{s}{2})$ representations make the starting point for the canonical description of massive higher spin particles of any spin, and the conditions:

$$\partial\cdot\Phi=\eta\cdot\Phi=\partial\cdot\Psi=\gamma\cdot\Psi=0, \tag{2.22}$$

$$(\partial^2-m^2)\Phi^{(s)}=(i\partial\!\!\!/-m)\Psi^{(s)}=0, \tag{2.23}$$

are collectively referred to as the Fierz-Pauli conditions, in reference to the early work on high spin fields by Fierz and Pauli [14]. We will see later how the Fierz-Pauli conditions can be derived from a variational principle at the cost of the introduction of auxiliary fields [14, 15, 16].

2.2 Wave functions for higher spin

Let us now consider the wave functions describing freely propagating massive particles of higher spin. These are the quantities which will enter into the evaluation of Feynman diagrams, as the on-shell external states. Since the totally symmetric tensors $\Phi^{(s)}$, $\Psi^{(s)}$ both satisfy the Klein-Gordon equations the wave functions will take the form of plane-wave solutions, such as:

$$\begin{aligned}&\tau^{(s),\sigma}(\mathbf{p})e^{ip\cdot x},\\&u^{(s),\sigma}(\mathbf{p})e^{ip\cdot x}.\end{aligned} \tag{2.24}$$

The objects:

$$\begin{aligned}\tau^{(s),\sigma}(\mathbf{p})&=\tau^{\sigma}_{\mu_1\ldots\mu_s}(\mathbf{p}),\\u^{(s),\sigma}(\mathbf{p})&=u^{\sigma}_{\mu_1\ldots\mu_k,\alpha}(\mathbf{p}),\end{aligned} \tag{2.25}$$

are a rank s totally symmetric tensor and a rank $k=s-1/2$ totally symmetric tensor spinor respectively. Both are subject to the Fierz-Pauli conditions:

$$p\cdot\tau^{(s),\sigma}(\mathbf{p})=\eta\cdot\tau^{(s)}(\mathbf{p})=\gamma\cdot u^{(s),\sigma}=\eta\cdot u^{(s),\sigma}=0. \tag{2.26}$$

(We have gone to momentum space, and so a vanishing divergence becomes transversality to momentum). In the following, we will see how these can be constructed as specific linear combinations of their lower spin counterparts, the $s = 1/2$ spinor and the $s = 1$ polarisation vector.

2.2.1 Spin $s = 1/2$ - Dirac spinors

Let us first quickly review the construction of wave functions for spin $s = 1/2$, which are the familiar spinor solutions, $u^\sigma(\mathbf{p})$ and $v^\sigma(\mathbf{p})$ (where σ takes the values $\pm\frac{1}{2}$). These spinors must satisfy the Dirac equation[38]:

$$\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\overline{\sigma} & -m \end{pmatrix} u^\sigma(\mathbf{p}) = \begin{pmatrix} -m & -p\cdot\sigma \\ -p\cdot\overline{\sigma} & -m \end{pmatrix} v^\sigma(\mathbf{p}) = 0. \tag{2.27}$$

In the rest-frame, this leads to the constant linearly independent solutions:

$$u^{+1/2}(\mathbf{0}) = \sqrt{m}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ u^{-1/2}(\mathbf{0}), = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \tag{2.28}$$

and

$$v^{+1/2}(\mathbf{0}) = \sqrt{m}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ v^{-1/2}(\mathbf{0}) = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}. \tag{2.29}$$

To obtain solutions for non-zero momenta, we can boost the above solutions. To be particular, we can perform the boost along the $z-$axis. The above solutions then become:

$$u^{+1/2}(\mathbf{0}) = \begin{pmatrix} \sqrt{\omega(\mathbf{p})+p_3} \\ 0 \\ \sqrt{\omega(\mathbf{p})-p_3} \\ 0 \end{pmatrix},\ u^{-1/2}(\mathbf{0}), = \begin{pmatrix} 0 \\ \sqrt{\omega(\mathbf{p})+p_3} \\ 0 \\ \sqrt{\omega(\mathbf{p})-p_3} \end{pmatrix}$$

and

$$v^{+1/2}(\mathbf{0}) = \begin{pmatrix} \sqrt{\omega(\mathbf{p}) - p_3} \\ 0 \\ -\sqrt{\omega(\mathbf{p}) + p_3} \\ 0 \end{pmatrix}, \; v^{-1/2}(\mathbf{0}) = \sqrt{m} \begin{pmatrix} 0 \\ \sqrt{\omega(\mathbf{p}) + p_3} \\ 0 \\ -\sqrt{\omega(\mathbf{p}) - p_3} \end{pmatrix}.$$

2.2.2 Spin 1 - polarisation vectors

For spin 1, we have the familiar polarisation vectors:

$$\tau^{\sigma}_{\mu}(\mathbf{p}), \quad \sigma = -1, 0, 1. \tag{2.30}$$

(We choose to label these as τ rather than ϵ in order avoid confusion with the polarisation vectors which describe photons, ϵ_{μ}). The Fierz-Pauli conditions implies that they are transverse with respect to the four-momentum p^{μ}:

$$p^{\mu}\tau^{\sigma}_{\mu}(\mathbf{p}) = 0. \tag{2.31}$$

To be explicit, we can pick the rest-frame polarisation vectors such that they represent a particle with spin pointing along the $z-$axis:

$$\begin{aligned} \tau^{0}_{\mu}(\mathbf{0}) &= (0, 0, 0, 1), \\ \tau^{\pm 1}_{\mu}(\mathbf{0}) &= \frac{1}{\sqrt{2}}(0, \mp 1, i, 0). \end{aligned} \tag{2.32}$$

Note that the above vectors are normalized such that:

$$\tau^{\sigma}(\mathbf{0}) \cdot \overline{\tau}^{\sigma'}(\mathbf{0}) = \delta_{\sigma\sigma'}. \tag{2.33}$$

As with the spinor wavefunctions, we can obtain the polarisation vectors for general momenta by acting on the rest-frame wavefunction with a boost. Boosting along the $z-$axis takes us from $(m, 0, 0, 0) \to (\omega(\mathbf{p}), 0, 0, p_z)$. In this case, the $\sigma = \pm 1$ polarisation vectors are left unaffected while the $\sigma = 0$ vector becomes:

$$\tau^{0}_{\mu}(\mathbf{p}) = (\omega(\mathbf{p})/m, 0, 0, p_z/m). \tag{2.34}$$

This is the longitudinal mode of the polarisation vectors[38].

2.2.3 $s = 3/2$ - polarisation vector spinor

Before we generalise the discussion to higher spin, it is instructive to consider the next steps in spins, which we can consider as the simplest higher spin cases, $s = 3/2$ and $s = 2$. From non-relativistic quantum mechanics, we are familiar with the notion of adding spins to create wave functions of higher spin:

$$|j_3, m_3\rangle = \sum_{m_1,m_2} \langle j_1 m_1, j_2 m_2 | j_3 m_3\rangle \, |j_1 m_1\rangle \, |j_2 m_2\rangle \,, \tag{2.35}$$

which defines the Clebsch-Gordan coefficients:

$$\langle j_1 m_1, j_2 m_2 | j_3 m_3\rangle. \tag{2.36}$$

The Clebsch-Gordan coefficients require that $m_1 + m_2 + m_3 = 0$ and that the spins j_1, j_2, j_3 satisfy the triangle inequalities. Since the rules of addition of angular momentum still apply in quantum field theory, we can follow the same procedure when constructing wave functions for particles of higher spin [39, 40]. To construct a $s = 3/2$ state we may couple a $s = 1$ state and a $s = 1/2$ state:

$$u^{\sigma}_{\mu,\alpha}(\mathbf{0}) = \sum_{\sigma_1,\sigma_2} \langle 1\sigma_1, \tfrac{1}{2}\sigma_2 | \tfrac{3}{2}\sigma\rangle \tau^{\sigma_1}_{\mu}(\mathbf{0}) u^{\sigma_2}_{\alpha}(\mathbf{0}), \tag{2.37}$$

with the polarisation vector and spinor defined above. Here, the sum is over all values of the helicities σ_1, σ_2 (which run over $-1, 0, 1$ and $\pm 1/2$ respectively) for which the Clebsch-Gordan coefficient does not vanish. The above sum gives rise to the vector spinors:

$$\begin{aligned} u^{\pm 3/2}_{\mu,\alpha}(\mathbf{0}) &= \tau^{\pm 1}_{\mu}(\mathbf{0}) u^{\pm 1/2}_{\alpha}(\mathbf{0}), \\ u^{\pm 1/2}_{\mu,\alpha}(\mathbf{0}) &= \frac{1}{\sqrt{3}} \tau^{\pm 1}_{\mu}(\mathbf{0}) u^{\mp 1/2}_{\alpha}(\mathbf{0}) + \sqrt{\frac{2}{3}} \tau^{0}_{\mu}(\mathbf{0}) u^{\pm 1/2}_{\alpha}(\mathbf{0}). \end{aligned} \tag{2.38}$$

Likewise, the corresponding $v_{\mu,\alpha}$ states are obtained by replacing u with v in the above. To obtain the vector spinors in a boosted frame, we may simply replace the rest-frame quantities in the above linear combination with their boosted counterparts. Since the above wavefunctions are constructed directly from the transverse polarisation vectors $\tau^{\sigma}_{\mu}(\mathbf{0})$, transversality follows trivially.

2.2.4 $s = 2$ - polarisation tensor

Consider a particle of spin $s = 2$ in its rest frame. The wave function is the symmetric polarisation tensor, $\tau^{\sigma}_{\mu\nu}(\mathbf{0})$. The construction of this polarisation tensor is analogous to the above procedure, this time involving the coupling of two $s = 1$ wave functions:

$$\tau^{\sigma}_{\mu\nu}(\mathbf{0}) = \sum_{\sigma_1,\sigma_2} \langle 1\sigma_1, 1\sigma_2 | 2\sigma \rangle \tau^{\sigma_1}_{\mu}(\mathbf{0}) \tau^{\sigma_2}_{\nu}(\mathbf{0}), \tag{2.39}$$

and from this, one obtains the five linear combinations:

$$\begin{aligned}
\tau^{\pm 2}_{\mu\nu}(\mathbf{0}) &= \tau^{\pm 1}_{\mu}(\mathbf{0}) \tau^{\pm 1}_{\nu}(\mathbf{0}), \\
\tau^{\pm 1}_{\mu\nu}(\mathbf{0}) &= \frac{1}{\sqrt{2}} [\tau^{\pm 1}_{\mu}(\mathbf{0}) \tau^{0}_{\nu}(\mathbf{0}) + \tau^{0}_{\mu}(\mathbf{0}) \tau^{\pm 1}_{\nu}(\mathbf{0})] \\
\tau^{0}_{\mu\nu}(\mathbf{0}) &= \frac{1}{\sqrt{6}} [\tau^{-1}_{\mu}(\mathbf{0}) \tau^{+1}_{\nu}(\mathbf{0}) + \tau^{+1}_{\mu}(\mathbf{0}) \tau^{-1}_{\nu}(\mathbf{0})] + \sqrt{\frac{2}{3}} \tau^{0}_{\mu}(\mathbf{0}) \tau^{0}_{\nu}(\mathbf{0}).
\end{aligned} \tag{2.40}$$

Note that $\tau^{\sigma}_{\mu\nu}(\mathbf{0})$ defined by the above construction is symmetric in the indices μ, ν. We also need $\tau^{\sigma}_{\mu\nu}$ to be traceless. A quick calculation using the above construction in combination with (2.32) proves this is the case.

2.2.5 Arbitrary spin s - polarisation tensors

We now turn to the construction of the wave functions for arbitrary s. The above arguments generalises easily. To start, we can construct a spin s wave function by coupling a $s - 1$ wave function to a $s = 1$ wave function:

$$\tau^{\sigma}_{\mu_1 \dots \mu_s}(\mathbf{0}) = \sum_{\sigma_1,\sigma_2} \langle 1\sigma_1, (s-1)\sigma_2 | s\sigma \rangle \tau^{\sigma_1}_{\mu_1}(\mathbf{0}) \tau^{\sigma_2}_{\mu_2 \dots \mu_s}(\mathbf{0}), \tag{2.41}$$

and likewise construct a half-integer wave function with $s = k + 1/2$ by coupling a $s = k$ integer spin wave function with a spinor:

$$u^{\sigma}_{\mu_1 \dots \mu_s, \alpha}(\mathbf{0}) = \sum_{\sigma_1,\sigma_2} \langle \tfrac{1}{2}\sigma_1, k\sigma_2 | s\sigma \rangle u^{\sigma_1}_{\alpha}(\mathbf{0}) \tau^{\sigma_2}_{\mu_1 \dots \mu_k}(\mathbf{0}). \tag{2.42}$$

Consider the integer case. In the above sum, we can insert the wave function given by the rank $s - 1$ polarisation tensor:

$$\tau^{\sigma_2}_{\mu_2 \dots \mu_s}(\mathbf{0}) = \sum_{\sigma_3,\sigma_4} \langle 1\sigma_3, (s-2)\sigma_4 | (s-1)\sigma_2 \rangle \tau^{\sigma_3}_{\mu_2}(\mathbf{0}) \tau^{\sigma_4}_{\mu_3 \dots \mu_s}(\mathbf{0}). \tag{2.43}$$

The lower rank tensors can be constructed similarly. Inserting the lower rank tensors, we see that the spin s wave function can be constructed by successively coupling the lower spins:

$$\tau^{\sigma}_{\mu_1\ldots\mu_s}(\mathbf{0}) = \sum_{\sigma_1\ldots\sigma_s} \langle 1\sigma_1, \sigma_2(s-1)|\sigma s\rangle\langle 1\sigma_3, (s-2)\sigma_4|\sigma_2(s-1)\rangle \ldots$$
$$\ldots\langle 1\sigma_{s-1}, 1\sigma_s|2\sigma_{s-2}\rangle\tau^{\sigma_1}_{\mu_1}(\mathbf{0})\ldots\tau^{\sigma_s}_{\mu_s}(\mathbf{0}). \quad (2.44)$$

Note that for the cases where $\sigma = \pm s$, the corresponding wave function is simply:

$$\tau^{\pm s}_{\mu_1\ldots\mu_s}(\mathbf{0}) = \tau^{\pm 1}_{\mu_1}(\mathbf{0})\ldots\tau^{\pm 1}_{\mu_s}(\mathbf{0}). \quad (2.45)$$

The procedure is analogous for the half-integer case. It follows from the expression (2.44) that the spin s wave function is transverse and totally symmetric. Furthermore, the tracelessness of spin s wave functions follows from the tracelessness of the $s = 2$ wave function.

2.3 Propagators

Next, we turn to consider the construction of propagators. The propagator connects two vertices, and describes the propagation of an off-shell (virtual) particle. It figures in tree level scattering processes such as Compton scattering, in which a virtual spin s particle is exchanged. We define the propagator via the vacuum expectation value of the time ordered product of two fields, $\Phi_1(y), \overline{\Phi}_2(x)$:

$$\Delta(x-y) \equiv \langle 0|T\{\Phi_1(y), \overline{\Phi}_2(x)\}|0\rangle, \tag{2.46}$$

where $|0\rangle$ denotes the vacuum state of the free theory. In momentum-space, the propagator becomes:

$$\Delta(q) = i\int d^4x e^{-iq(x-y)}\langle 0|T\{\Phi_1(y), \Phi_2(x)\}|0\rangle, \tag{2.47}$$

where q denotes the momentum of the propagating particle. Note that this definition is entirely independent of the representation of the two fields. For scalar fields, this gives the familiar Feynman propagator [38]:

$$\Delta(x-y) = \int \frac{d^4q}{(2\pi)^4}\frac{ie^{iq(x-y)}}{q^2+m^2-i\epsilon}. \tag{2.48}$$

(We will omit the $i\epsilon$ henceforth). We can also think of the propagator as the "amplitude" for a particle created at a point x to vanish at a later time at a point y, the propagator then represents the product of the amplitude for the creation at point x multiplied by the amplitude for annihilation at point y, summed over all possible helicities[41]. In momentum space, the summation over all polarisations is the spin sum:

$$\mathcal{P}_{\mu_1\ldots\mu_s,\nu_1,\ldots\nu_s}(p) = \sum_\sigma \tau^\sigma_{\mu_1\ldots\mu_s}(p)\overline{\tau}^\sigma_{\nu_1\ldots\nu_s}(p), \tag{2.49}$$

and one would expect the propagator to take the form:

$$\Delta_{\mu_1\ldots\mu_s,\nu_1\ldots\nu_s}(p) = \frac{i\mathcal{P}_{\mu_1\ldots\mu_s,\nu_1\ldots\nu_s}(p)}{p^2+m^2}. \tag{2.50}$$

This expression is well-known for spins $1/2, 1$, and we will see how the projection operator can be deduced for higher spins in the following. In addition to describing the numerator, this expression also defines the projection operator for a massive particle of spin s. This action of this operator on a general rank s tensor is to project it onto the space of tensors which satisfy the on-shell conditions for a massive particle of spin s. It should be noted however, that this definition is flawed, since we, when evaluating the spin sum, assume that the wave functions represent on-shell states which satisfy the appropriate on-shell conditions. Since the propagator describes an off-shell particle (a virtual particle exchanged by on-shell states), it should reflect the off-shell degrees of freedom present. A more correct ansatz for the general propagator would therefore be:

$$\Delta_{\mu_1\ldots\mu_s,\nu_1\ldots\nu_s} = \frac{i\mathcal{N}_{\mu_1\ldots\mu_s,\nu_1\ldots\nu_s}(p)}{p^2+m^2}, \tag{2.51}$$

where $\mathcal{N}(p)$ is a tensor which reduces to the projection operator when p^2 is on the mass shell:

$$\mathcal{P}_{\mu_1\ldots\mu_s,\nu_1,\ldots\nu_s}(p) = \mathcal{N}_{\mu_1\ldots\mu_s,\nu_1\ldots\nu_s}(p)|_{p^2=-m^2}. \tag{2.52}$$

For now, we will work with the on-shell case, and consider the numerator structure as given by the projection operator. In chapter 5, we will see how the general numerator structure may be constructed.

2.3.1 The spin-1 projection operator

Consider the $s=1$ case. The wave functions are given by the polarisation vectors, $\tau^\sigma_\mu(\mathbf{p})$ defined in (2.32). The projection operator is then given by the spin sum:

$$\mathcal{P}_{\mu\nu}(p) = \sum_\sigma \tau^\sigma_\mu(\mathbf{p})\overline{\tau}^\sigma_\nu(\mathbf{p}). \tag{2.53}$$

To evaluate this, one can first note that the three rest-frame polarisation vectors introduced in equation (2.32), in combination with the vector:

$$\tilde{\tau}_\mu = \frac{p_\mu}{m} = (1,0,0,0), \tag{2.54}$$

form a complete basis in Minkowski space. They must therefore satisfy the completion relation[38]:

$$\sum_\sigma \tau_\mu^\sigma(\mathbf{p})\overline{\tau}_\nu^\sigma(\mathbf{p}) - \tilde{\tau}_\mu\tilde{\tau}_\nu = \eta_{\mu\nu}. \tag{2.55}$$

To obtain the polarisation sum, we must therefore subtract the piece:

$$\tilde{\tau}_\mu\tilde{\tau}_{nu} = \frac{p_\mu p_\nu}{m^2}, \tag{2.56}$$

giving the well-known expression:

$$\mathcal{P}_{\mu\nu}(p) = \sum_\sigma \tau_\mu^\sigma(\mathbf{p})\overline{\tau}_\nu^\sigma(\mathbf{p}) = \eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}. \tag{2.57}$$

2.3.2 Higher spin projection operators - bosons

For higher spin states, we can define the projection operators in a similar fashion, via the spin sums:

$$\mathcal{P}^{(s,s)}(p) = \sum_\sigma \tau^{(s),\sigma}(\mathbf{p})\overline{\tau}^{(s),\sigma}(\mathbf{p}). \tag{2.58}$$

Since the polarisation tensors are symmetrized in their s indices each, we have upgraded our symmetric tensor notation; $\mathcal{P}^{(s,s)}$ is a rank $2s$ pairwise symmetric tensor, in the sense that:

$$\mathcal{P}^{(s,s)} = \mathcal{P}^{(\mu_1\dots\mu_s),(\nu_1\dots\nu_s)} = \mathcal{P}^{(\nu_1\dots\nu_s),(\mu_1\dots\mu_s)}. \tag{2.59}$$

From the definition (2.58), we also have the condition:

$$p \cdot \mathcal{P}^{(s,s)}(p) = \eta \cdot \mathcal{P}^{(s,s)}(p) = 0. \tag{2.60}$$

With these restrictions on the projection operator, it is possible to solve its structure without explicitly evaluating the spin sum (2.58). From the definition (2.58) we can surmise that it must be a polynomial in the spin 1 projection operators. An explicit form can be solved for, yielding the expression[42]:

$$\begin{aligned}\mathcal{P}^{(s,s)}(q) = &\sum_{r=1}^{\lfloor \frac{s}{2} \rfloor} c_r(s) \\ &\times \left[\mathcal{P}_{\mu_1\mu_2}\mathcal{P}_{\nu_1\nu_2}\dots\mathcal{P}_{\mu_{2r-1}\mu_{2r}}\mathcal{P}_{\nu_{2r-1}\nu_{2r}}\mathcal{P}_{\mu_{2r+1}\nu_{2r+1}}\dots\mathcal{P}_{\mu_s\nu_s}\right]_{(\mu,\nu)},\end{aligned} \tag{2.61}$$

where the subscript (μ, ν) indicates symmetrisation between all μ and ν indices (hence there is a hidden factor $1/s!$). The sum runs up to the highest integer bounded by $s/2$, as indicated by the floor function. The coefficient c_r can be determined by normalisation and is [42, 43]:

$$c_r(s) = (-1)^r \frac{(s!)^2(2s-2r)!}{(2s)!(s-r)!(s-2r)!r!}. \tag{2.62}$$

With this definition, it is easy to verify that $\mathcal{P}^{(s,s)}(p)$ reduces to the $s = 1$ projection operator. As a less trivial example, we can also consider the $s = 2$ projection operator. A quick calculation yields the result:

$$\mathcal{P}_{\mu_1\mu_2,\nu_1\nu_2}(p) = \frac{1}{2}\left(\mathcal{P}_{\mu_1\nu_1}\mathcal{P}_{\mu_2\nu_2} + \mathcal{P}_{\mu_1\nu_2}\mathcal{P}_{\mu_2\nu_1}\right) - \frac{1}{3}\mathcal{P}_{\mu_1\mu_2}\mathcal{P}_{\nu_1\nu_2}, \tag{2.63}$$

which is a well-known result in the literature[28, 42, 44].

2.3.3 Higher spin projection operators - fermions

Next, we turn to the projection operator for higher spin fermions. Starting with the polarisation sum,

$$\mathcal{P}^{(k,k)}(p) = \sum_\sigma u^{(s),\sigma}(\mathbf{p})\overline{u}^{(s),\sigma}(\mathbf{p}), \tag{2.64}$$

(to distinguish the fermionic projection operator from the bosonic projection operator, we denote the fermionic one with $k = s - 1/2$ instead of s) we see that it must necessarily satisfy the conditions:

$$\gamma \cdot \mathcal{P}^{(k,k)}(p) = \eta \cdot \mathcal{P}^{(k,k)} = p \cdot \mathcal{P}^{(k,k)}(p) = 0, \tag{2.65}$$

as well as pairwise total symmetry. In addition, the presence of on-shell spinors which satisfy the Dirac equation implies that $\mathcal{P}^{(s,s)}(p)$ must also obey the on-shell condition[45],

$$(\not{p} + m)\mathcal{P}^{(s,s)}(p) = \mathcal{P}^{(s,s)}(p)(\not{p} + m) = 0. \tag{2.66}$$

The procedure for its construction is analogous to the boson case. Here, $\mathcal{P}^{(k,k)}(p)$ can be determined to be[28]:

$$\mathcal{P}^{(k,k)}(p) = (\not{p} + m)A^{(k)} + \left(\gamma_{\mu_1} + \frac{p_{\mu_1}}{m}\right)\left(\gamma_{\nu_1} + \frac{p_{\nu_1}}{m}\right)(\not{p} - m)B^{(k)}, \quad (2.67)$$

with the tensors $A^{(s)}$ and $B^{(s)}$ defined recursively:

$$\begin{aligned} A^{(k)} &= \mathcal{P}_{\mu_1\nu_1}A^{(k-1)} - \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}A^{(k-2)}, \quad A^{(0)} = 1, \\ B^{(k)} &= \frac{1}{2k+1}A^{(k-1)}, \quad B^{(0)} = 0. \end{aligned} \quad (2.68)$$

For $s = 1/2$ ($k = 0$) it trivially reduces to the familiar spin sum $\not{p} + m$. Let us also consider the projection operator for $s = 3/2$, for a more interesting example. In this case, $k = 1$, and we have:

$$\mathcal{P}_{\mu\nu}(p) = (\not{p} + m)A^{(1)} + \left(\gamma_\mu + \frac{p_\mu}{m}\right)\left(\gamma_\nu + \frac{p_\nu}{m}\right)(\not{p} - m)B^{(1)}. \quad (2.69)$$

From (2.68) we have:

$$A^{(1)} = \eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}, \quad B^{(1)} = \frac{1}{3}, \quad (2.70)$$

gibing the projection operator:

$$\begin{aligned} \mathcal{P}_{\mu\nu}(p) = (\not{p} + m)\left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right) \\ + \frac{1}{3}\left(\gamma_\mu + \frac{p_\mu}{m}\right)\left(\gamma_\nu + \frac{p_\nu}{m}\right)(\not{p} - m). \end{aligned} \quad (2.71)$$

This is a well-known result in the literature. As one should expect, it can also be obtained by inverting the kinetic term of the RS Lagrangian[1].

2.4 Lagrangians

We have now seen the construction of wave functions and propagators in the free higher spin theory. The next step is to consider interactions, such as coupling to an electromagnetic background. Before we do this however, we will first consider the construction of Lagrangians for higher spin fields. A Lagrangian formulation holds many benefits, such as the ability to deduce Feynman rules which in the context of higher spin theory, turns out to be especially important in order to ensure consistent interactions. We will elaborate later on how the introduction of minimal coupling at the level of equations of motion, for higher spin, leads to physical inconsistencies. This issue may be alleviated by performing the minimal coupling procedure at the level of the Lagrangian[14]. We have seen that the higher spin fields are subject to the equations of motion:

$$
\begin{aligned}
&(-\partial^2 + m^2)\Phi^{(s)} = \partial \cdot \Phi^{(s)} = 0, \\
&(i\partial\!\!\!/ + m)\Psi^{(s)} = \partial \cdot \Psi^{(s)} = 0.
\end{aligned}
\tag{2.72}
$$

These are then to follow as the equations of motion from a Lagrangian, via the Euler-Lagrange equations:

$$
\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^s)} \right) = \frac{\partial \mathcal{L}}{\partial \Phi^s}. \tag{2.73}
$$

It is clear that the equations,

$$
\begin{aligned}
(-\partial^2 + m^2)\Phi^{(s)} &= 0, \\
(i\partial\!\!\!/ + m)\Psi^{(s)} &= 0,
\end{aligned}
\tag{2.74}
$$

can be obtained from Lagrangians of the form:

$$
\mathcal{L}_{\text{boson}} = \frac{1}{2}\overline{\Phi}^{(s)}(-\partial^2 + m^2)\Phi^{(s)}, \tag{2.75}
$$

and

$$
\mathcal{L}_{\text{fermion}} = \overline{\Psi}^{(s)}(i\partial\!\!\!/ + m)\Psi^{(s)}. \tag{2.76}
$$

The question is then how to obtain the subsidiary conditions,

$$\partial \cdot \Phi^{(s)} = \partial \cdot \Psi^{(s)} = 0, \tag{2.77}$$

from the Lagrangians. As one might expect, the complexity of the required procedure will scale with the spin of the field. Luckily, a pattern emerges, and a general form can be obtained for arbitrary spins[15, 16]. To see how the pattern emerges, we will start by considering known lower spin Lagrangians and show how the Fierz-Pauli conditions can be derived as equations of motion by introducing auxiliary fields, following [15]. As the Lagrangian structure differs for bosonic and fermionic fields, we will treat them separately, starting with bosonic fields.

2.4.1 s = 1 - the Proca Lagrangian

We start by considering the simplest non-trivial case, where $s = 1$ and $\Phi^{(s)} = \Phi^{\mu}$. Massive spin 1 fields are well-known in the literature, as the Standard Model of particle physics contains massive vector bosons such as the $W^{\pm}$. The Lagrangian describing such a massive spin 1 boson was introduced by A. Proca, and reads [38]:

$$\mathcal{L}_{s=1} = -\frac{1}{2}(\partial_{[\mu}\overline{\Phi}_{\nu]})(\partial^{[\mu}\Phi^{\nu]}) + m^2\overline{\Phi}_{\nu}\Phi^{\nu}. \tag{2.78}$$

For clarity, we will keep explicit indices presently. The Proca Lagrangian is simple in the sense that it is constructed from all the Lorentz invariant kinetic terms we can write [38]. Occasionally, it is written in terms of a field strength, analogous to the electromagnetic field strength tensor:

$$F_{\mu\nu} \equiv 2\partial_{[\mu}\Phi_{\nu]}, \tag{2.79}$$

such that the Lagrangian takes the neat form:

$$\mathcal{L}_{s=1} = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + m^2\overline{\Phi}_{\nu}\Phi^{\nu}. \tag{2.80}$$

The Euler-Lagrange equations yield:

$$-2\partial^\mu(\partial_{[\mu}\Phi_{\nu]}) + m^2\Phi_\nu = 0. \tag{2.81}$$

Note that this equation has the same symmetries as the Proca field; it is both symmetric and traceless. Contracting the above expression with ∂^ν this becomes:

$$\begin{aligned}0 &= -2\partial^\nu\partial^\mu(\partial_{[\mu}\Phi_{\nu]}) + m^2\partial^\nu\Phi_\nu \\ &= -\partial^\nu\partial^2\Phi_\nu + \partial^\mu\partial^2\Phi_\mu + m^2\partial^\nu\Phi_\nu \\ &= m^2\partial^\nu\Phi_\nu,\end{aligned}$$

from which one obtains:

$$\partial^\nu\Phi_\nu = \partial\cdot\Phi^{(1)} = 0, \tag{2.82}$$

since we assume that $m^2 \neq 0$. Now we can expand (2.81) and insert $\partial\cdot\Phi = 0$ to obtain:

$$\begin{aligned}0 &= -2\partial^\mu(\partial_{[\mu}\Phi_{\nu]}) + m^2\Phi_\nu \\ &= -\partial^\mu(\partial_\mu\Phi_\nu - \partial_\nu\Phi_\mu) + m^2\Phi_\nu \\ &= -\partial^2\Phi_\nu - \partial_\nu\partial\cdot\Phi_\mu + m^2\Phi_\nu \\ &= (-\partial^2 + m^2)\Phi_\nu,\end{aligned}$$

which is the Klein Gordon equation. Thus, the Proca Lagrangian reproduces both the Fierz-Pauli condtion $\partial\cdot\Phi = 0$ as well as the Klein Gordon equation via the equations of motion.

2.4.2 s = 2 - the Fierz-Pauli Lagrangian

For $s = 2$ the field is a symmetric and traceless rank two tensor, $\Phi^{(2)} = \Phi^{(\mu\nu)}$. While the Standard Model contains no massive $s = 2$ particles, the massive spin 2 plays important roles in several directions of research, such as massive gravity. The Lagrangian for a massive spin 2 field was first studied Fierz and Pauli in the early days of the HS program [14, 15, 38]. It consists of two parts:

$$\mathcal{L}_{s=2} = \mathcal{L}_{\Phi^{(2)}} + \mathcal{L}_{\Phi^{(0)}}, \tag{2.83}$$

with:

$$\begin{aligned} \mathcal{L}_{\Phi^{(2)}} &= \frac{1}{2}\Phi^{(2)}(\partial^2 + m^2)\Phi^{(2)} - (\partial\Phi^{(2)})^2, \\ \mathcal{L}_{\Phi^{(0)}} &= -\frac{1}{2}\Phi^{(0)}(a_0\partial^2 + b_0 m^2)\Phi^{(0)} + \Phi^{(0)}(\partial_2 \cdot \Phi^{(2)}). \end{aligned} \tag{2.84}$$

The first part, $\mathcal{L}_{\Phi^{(2)}}$ is simply the Lagrangian describing a free spin 2 field, and can be though of as the "direct" generalisation of the Proca Lagrangian to spin 2. The second part, $\mathcal{L}_{\Phi^{(0)}}$ describes as massive scalar field, which in addition to having a kinetic term, couples to the scalar quantity $\partial_2 \cdot \Phi^{(2)}$[3]. The quantities a_0, b_0 are numbers which are to be determined in the following. Varying the Lagrangian with respect to the two fields $\Phi^{(2)}$ and $\Phi^{(0)}$ yields the following two equations of motion:

$$\begin{aligned} (\partial^2 + m^2)\Phi^{(2)} - \partial_{(\nu}(\partial \cdot \Phi)_{\mu)} + \frac{1}{2}\eta_{\mu\nu}(\partial_2 \cdot \Phi) \\ - \partial_\mu\partial_\nu\Phi^{(0)} - \frac{1}{4}\eta_{\mu\nu}\partial^2\Phi^{(0)} = 0, \end{aligned} \tag{2.85}$$

and,

$$(a_0\partial^2 + b_0 m^2)\Phi^{(0)} + \partial_2\Phi^{(2)} = 0. \tag{2.86}$$

In the above, $(\partial\cdot\Phi^{(2)})_\nu$ is to be understood as $\partial^\mu\Phi_{\mu\nu}$. By taking the divergence of the first equation twice (i.e. contracting with $\partial^\mu\partial^\nu$), one arrives at:

$$\left(m^2 - \frac{1}{2}\partial^2\right)(\partial_2 \cdot \Phi^{(2)}) + \frac{3}{4}\partial^4\Phi^{(0)}. \tag{2.87}$$

This equation, together with the equation of motion for the auxiliary scalar field $\Phi^{(0)}$, comprises a linear homogenous system of equations in the variables

3: Often in the literature [28, 38], this scalar field is absorbed into the spin-2 field by a field redefinition: $\Phi^{\mu\nu} \to \tilde{\Phi}^{\mu\nu} = \Phi^{\mu\nu} + 1/4\eta^{\mu\nu}\Phi^{(0)}$, such that $\mathrm{Tr}\,\tilde{\Phi}^{\mu\nu} = \Phi^{(0)}$.

$\partial_2 \cdot \Phi^{(2)}$ and $\Phi^{(0)}$ [36, 46]. This system admits the unique algebraic solution $\partial_2 \cdot \Phi^{(2)} = \Phi^{(0)} = 0$ provided that:

$$a_0 = b_0 = \frac{3}{4}. \tag{2.88}$$

Inserting $\partial_2 \cdot \Phi^{(2)} = \Phi^{(0)}$ into the equation of motion for $\Phi^{(2)}$ and taking the divergence we obtain:

$$m^2 \partial \cdot \Phi^{(2)} = 0, \tag{2.89}$$

which implies the vanishing of $\partial \cdot \Phi^{(2)}$ condition for $m^2 \neq 0$. Note that obtaining this was only possible because the auxiliary field allowed for the elimination of the scalar $\partial_2 \cdot \Phi^{(2)}$; that is, in order to reproduce the Fierz-Pauli conditions from the Lagrangian describing a massive $s = 2$ field, one must introduce an auxiliary scalar field.

2.4.3 $s > 2$ - the Singh-Hagen Bosonic Lagrangian

The introduction of a dynamic, auxiliary scalar field in the $s = 2$ Lagrangians leads to equations of motion forming a system of linear homogenous equations in the variables $\Phi^{(0)}, \partial_2 \cdot \Phi^{(2)}$, with the unique solution $\Phi^{(0)} = \partial_2 \cdot \Phi^{(2)} = 0$, which in turn can be used to extract the condition $\partial \cdot \Phi^{(2)} = 0$. For arbitrary spin, a similar procedure can be performed, requiring the introduction of a total of $s - 1$ auxiliary fields $\Phi^{(k)}$ of ranks $k = s - 2, s - 3, \ldots, 0$ with each corresponding to the representations $D(k/2, k/2)$ of the Lorentz group[15]. The general Lagrangian for this tower of fields can be written in the systematic fashion:

$$\mathcal{L} = \mathcal{L}_s + c[\mathcal{L}_{s-2} + \mathcal{L}_k] \tag{2.90}$$

where:

$$\mathcal{L}_s = -\frac{1}{2}\overline{\Phi}^{(s)}(\partial^2 + m^2)\Phi^{(s)} + s(\partial \cdot \Phi^{(s)})^2, \tag{2.91}$$

describes the free, on-shell Lagrangian for a spin s field,

$$\mathcal{L}_{s-2} = -\frac{1}{2}\overline{\Phi}^{(s-2)}(\partial^2 - a_{s-2}m^2)\Phi^{(s-2)} \\ + c_{s-2}(\partial\cdot\Phi^{(s-2)})^2 + \Phi^{(s-2)}(\partial_2\Phi^{(s)}),$$

describes the Lagrangian for the highest rank auxiliary field which couples to $\partial_2\cdot\Phi^{(s)}$ (the scalar field in the case $s=2$). The final term is only present for $s\geq 3$ and describes the tower of lower rank auxiliary which couples "upwards" to the divergence of higher-rank fields:

$$\mathcal{L}_k = \sum_{q=3}^{s}\left(\prod_{k=2}^{q-1}c_k\right)\left[\tfrac{1}{2}\Phi^{(s-q)}(\partial^2 - a_q m^2)\Phi^{(s-q)} \right. \\ \left. -\tfrac{1}{2}b_q(\partial\cdot\Phi^{(s-q)})^2 - \Phi^{(s-q-1)}(\partial\cdot\Phi^{(s-q)})\right]$$

the Lagrangian for the tower of lower rank auxiliary fields. Note that we make this separation of the Lagrangians because fields of rank $s-3$ and lower couple upwards one step, while the $s-2$ couples upwards to steps (to $\partial_2\cdot\Phi^{(s)}$). It should also be noted that this is simply the simplest quadratic Lagrangian we can form of these fields. This is justified however, as the off-shell auxiliary field are not be imbued with physical significance, and as such we should only bother with the simplest possible form for their Lagrangians. The coefficients are determined uniquely by the tower of constraints and are given by [15]:

$$\begin{aligned} c &= \frac{s(s-1)^2}{(2s-1)} \\ a_q &= \frac{q(2s-q+1)(s-q+2)}{2(2s-2q+3)(s-q+1)} \\ b_{q-1} &= \frac{(s-q+1)^2}{2s-2q+5} \\ c_{q-1} &= \frac{(q-1)(s-q+1)^2(s-q+3)(2s-q+3)}{2(s-q+2)(2s-2q+3)(2s-2q+5)}. \end{aligned}$$

2.4.4 Fermion Lagrangians

Next, we consider Lagrangians for fermions. The story is similar here, although slightly more complicated. A hierarchy of auxiliary fields will be necessary again, although the number will differ from the bosonic case [16]. We will follow [16] in deriving the simplest $s = \frac{3}{2}$ Lagrangian, and then generalise the procedure to higher fermion spins.

s = 3/2 - the Rarita Schwinger Lagrangian

The first non-trivial case is $s = \frac{3}{2}$. The free fermion field is described by the vector spinor $\Psi^{(1)}$. The equation of motion,

$$(i\partial\!\!\!/ - m)\Psi^{(1)} = 0, \tag{2.92}$$

can be obtained from the unmodified Lagrangian. Following the logic from earlier, we can add an auxiliary field of rank comparable to $\partial \cdot \Psi^{(1)}$, which in this case is a rank 1 spinor, $\Psi^{(0)}$. The modified Lagrangian is then the most general Lagrangian containing the fields $\Psi^{(1)}$ and $\Psi^{(0)}$ is [16]:

$$\mathcal{L} = \frac{1}{2}\overline{\Psi}^{(1)}(i\partial\!\!\!/ - m)\Psi^{(1)} - b_0\frac{1}{2}\overline{\Psi}^{(0)}(i\partial\!\!\!/ - a_0 m)\Psi^{(0)} - b_0\Psi^{(0)}\partial \cdot \Psi^{(1)}, \tag{2.93}$$

where a_0 and b_0 are real constants to be determined. The Euler-Lagrange equations are:

$$(-i\partial\!\!\!/ + m)\Psi^{(1)} + \frac{1}{2}i\gamma_\mu(\partial \cdot \Psi^{(1)}) = b_0[\partial_\mu + \frac{1}{4}\gamma_\mu(\partial\!\!\!/)]\Psi^{(0)}, \tag{2.94}$$

$$\partial \cdot \Psi^{(1)} = -(i\partial\!\!\!/ + a_0 m)\Psi^{(0)}. \tag{2.95}$$

By contracting (2.95) with ∂^μ and substituting out $\partial \cdot \Psi^{(1)}$ using (2.94), one can obtain $\Psi^{(0)} = \partial \cdot \Psi^{(1)} = 0$ provided that,

$$b_0 = \frac{2}{3},\ a_0 = 2. \tag{2.96}$$

Thus, for $s = \frac{3}{2}$ it suffices to introduced a single auxiliary field of (tensor) rank 0. It should be noted that a different formulation of the $s = \frac{3}{2}$ Lagrangian exists in the literature, namely the Rarita-Schwinger Lagrangian [RS] after W. Rarita and J. Schwinger who first studied the construction of a free $s = \frac{3}{2}$ theory. The RS Lagrangian reads [28]:

$$\mathcal{L}_{\text{RS}} = -\frac{1}{2}\overline{\Psi}_\nu \gamma^{\mu\nu\rho}(i\partial_\nu - \tfrac{1}{2}im\gamma_\nu)\Psi_\rho \tag{2.97}$$

where $\gamma^{\mu\nu\rho} \equiv \gamma^{(\mu}\gamma^\nu\gamma^{\rho)}$. Alternatively, one can apply a number of $\gamma-$matrix identities (see Appendix) to obtain [1]:

$$\mathcal{L}_{\text{RS}} = \frac{1}{2}\overline{\Psi}_\nu(\epsilon^{\nu\mu\rho\sigma}\gamma_5\gamma_\mu\partial_\rho + im\sigma^{\nu\sigma})\Psi_\sigma, \tag{2.98}$$

where $\sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. The equation of motion which follows is:

$$-i\epsilon^{\nu\mu\rho\sigma}\gamma_5\gamma_\mu\partial_\rho\Psi_\sigma + im\sigma^{\nu\sigma}\Psi_\sigma = 0, \tag{2.99}$$

from which one can obtain the Fierz-Pauli condition $\partial \cdot \Psi = 0$ by contracting with γ_ν, which when inserted into the above yields the Dirac equation [1].

2.4.5 s > 3/2 - The Singh Hagen Fermion Lagrangian

For higher fermionic spins, a pattern similar to the bosonic case emerges. In order to obtain the Fierz-Pauli condition $\partial \cdot \Psi^{(s)} = 0$ one must introduce a rank $s-1$ auxiliary field $\Psi^{(s-1)}$ and obtain:

$$\partial_2 \cdot \Psi^{(s)} = \partial \cdot \Psi^{(s-1)} = 0. \tag{2.100}$$

The latter condition can be achieved by introducing a new pair of auxiliary tensor spinors, $\Psi_i^{(s-2)}$, $i = 1, 2$. This pattern then continues for auxiliary fields of ranks $s-3, \ldots 0$, leading to the introduction of a total of $2s-2$ auxiliary fields [SH2]. The explicit Lagrangian was derived in [16], and can be written as:

$$\mathcal{L} = \mathcal{L}_s + c_{s-1}\mathcal{L}_{s-1} + c_{s-2}\mathcal{L}_{s-2} + c_k\mathcal{L}_k. \tag{2.101}$$

As before, the first term is the free "on-shell" Lagrangian containing the bilinear of a rank $s - 1/2$ tensor spinor:

$$\mathcal{L}_s = \frac{1}{2}\overline{\Psi}^{(s)}(i\partial\!\!\!/ - m)\Psi^{(s)}, \tag{2.102}$$

while:

$$\mathcal{L}_{s-1} = \frac{1}{2}\overline{\Psi}^{(s-1)}(i\partial\!\!\!/ + a_1 m)\Psi^{(s-1)} + \Psi^{(s-1)}(\partial \cdot \Psi^{(s)}), \tag{2.103}$$

and:

$$\begin{aligned}\mathcal{L}_{s-2} = -\frac{1}{2}\overline{\Psi_1}^{(s-2)}(i\partial\!\!\!/ + d_2 m)\Psi_1^{(s-2)} \\ - \frac{1}{2}\overline{\Psi_2}^{(s-2)}(i\partial\!\!\!/ + a_2 m)\Psi_2^{(s-2)} - \frac{1}{2}mb_2\Psi_1^{(s-2)}\Psi_2^{(s-2)},\end{aligned}$$

and:

$$\begin{aligned}\mathcal{L}_q = \sum_{q=3}^{s-\frac{1}{2}}\left(\prod_{k=2}^{q-1}\right)\Big[\tfrac{1}{2}\Psi_1^{(s-q)}(i\partial\!\!\!/ + d_q m)\Psi_1^{(s-q)} + \tfrac{1}{2}b_q\Psi_2^{(s-q)}(i\partial\!\!\!/ + a_q m)\Psi_2^{(s-q)} \\ + \tfrac{1}{2}b_q m\Psi_1^{(s-q)}\Psi_2^{(s-q)} - \Psi_1^{(s-q)}(\partial \cdot \Psi^{(s-q+1)} - \Psi_2^{(s-q)}(\partial \cdot \Psi^{(s-q+1)}\Big].\end{aligned}$$

As in the bosonic case, the coefficients can be solved for by inspecting the equations of motion[4]. They are:

$$\begin{aligned}c &= \frac{2s^2}{2s+1} \\ a_q = d_q &= \frac{s+1}{s+1-q} \\ b_q &= -\frac{(q-1)(2s-q+3)}{2s-2q+3} \\ c_{q-1} &= \frac{2(s-q+3)^2}{(2s-2q+5)}.\end{aligned}$$

4: And generalising to arbitrary s via induction[16]

2.5 Electromagnetic interactions

So far, we have considered only the free higher spin theory, which is not conceptually different from simpler theories, e.g. a theory of free scalar fields. The true difficulties with treating higher spin will first become evident when we introduce interactions. For $s = 1/2$, electromagnetic interactions may be considered by directly covariantising the Dirac equation:

$$(i\not{D} - m)\Psi = 0, \tag{2.104}$$

where we have made the substitution:

$$\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu. \tag{2.105}$$

Multiplying with $(i\not{D} + m)$, one obtains:

$$(\not{D}^2 + m^2)\Psi = \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}\Psi^{(s)} + D^2 + m^2\Psi = 0. \tag{2.106}$$

Here, the piece $\frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}$ corresponds to the magnetic dipole moment, which in the non-relativistic limit leads to the interaction Hamiltonian [38]:

$$\mathcal{L}_{\text{int}} = \frac{eg}{2m}\langle \mathbf{B} \cdot \mathbf{S} \rangle. \tag{2.107}$$

Here g denotes the gyro-magnetic ratio, which at tree level is $g = 2$, and the vectors **B** and S denote the magnetic field and spin vector of the particle, respectively. Unfortunately, this procedure leads to difficulties in the general spin case, due to the subsidiary condition:

$$D \cdot \Psi^{(s)} = 0, \tag{2.108}$$

which must also be taken into account when covariantising the equations of motion. This above condition may be combined with the Klein-Gordon equation to yield the relation:

$$[D^2 - m^2, D_\mu]\Psi^{(s)} = 0, \tag{2.109}$$

which can be simplified using commutation rules:

$$[D^2 - m^2, D_\mu] = [D^2, D_\mu] \tag{2.110}$$

$$= [D_\nu D^\nu, D_\mu] \tag{2.111}$$

$$= [D^\nu, D_\mu]D_\nu + D^\nu[D_\mu, D_\nu] \tag{2.112}$$

$$= -ieD^\nu F_{\nu\mu}. \tag{2.113}$$

Here, the tracelessness of $F_{\nu\mu} = 2\partial_{(\mu}A_{\nu)}$ has been used in the final step. Thus, introducing minimal coupling at the level of equations of motion for general spins introduces the additional condition:

$$-ieD^\nu F_{\nu\mu}\Psi^{(s)} = 0, \tag{2.114}$$

which follow from the fact that the covariant derivatives do not commute. This means that, if we switch on minimal coupling without modifying the Lagrangian, additional conditions arise, altering the number of dynamic degrees of freedom contained in $\Psi^{(s)}$[14, 47, 48, 49].

Instead of performing the substitution $\partial_\mu \to D_\mu$ at the level of equations of motion, we may instead introduce them into the Lagrangians discussed in the previous section. Since the full equations of motion follow from the Lagrangians discussed earlier, this procedure eliminates the degree of freedom problem. Introducing minimal coupling will in general lead to current terms of the form:

$$\mathcal{L}^{\text{EM}}_{\text{int}} = eA_\mu j^\mu, \tag{2.115}$$

where j^μ is the current to which the electromagnetic field A_μ couples. Importantly, the current is conserved, so:

$$\partial_\mu j^\mu = 0. \tag{2.116}$$

The above interaction Lagrangian can then be used to deduce the three-point interaction vertex, describing two massive particles coupling to a photon. We will refer to the interaction Lagrangian (2.115) as the minimal coupling interaction, $\mathcal{L}^{\text{EM}}_{\text{min}} = eA_\mu j^\mu$. In the following we will first consider the coupling of massive $s = 1, 3/2$ particles to the electromagnetic field A^μ, as these cases serve to demonstrate some of the difficulties related to the general cases. We will then consider the interaction terms obtained from the Singh-Hagen Lagrangians.

2.5.1 Spin-1

Electromagnetic coupling to massive $s = 1$ particles is a well known phenomenon, e.g. in scattering processes involving the charged $W^\pm$ bosons. We will start by introducing the covariant derivative in the Proca Lagrangian (equation (2.78)):

$$\mathcal{L}_{s=1} = -\frac{1}{2}(D_{[\mu}\overline{\Phi}_{\nu]})(D^{[\mu}\Phi^{\nu]}) + m^2\overline{\Phi}_\nu\Phi^\nu. \tag{2.117}$$

The term including derivatives then becomes:

$$-\frac{1}{2}(D_{[\mu}\overline{\Phi}_{\nu]})(D^{[\mu}\Phi^{\nu]}) \;=\; -\frac{1}{2}(\partial_{[\mu}\overline{\Phi}_{\nu]} + ieA_{[\mu}\overline{\Phi}_{\nu]})(\partial^{[\mu}\Phi^{\nu]} - ieA^{[\mu}\Phi^{\nu]}).$$

Multiplying everything together and discarding the terms that corresponds to the free kinetic terms, we find the following:

$$\mathcal{L}_{\text{min}} = \frac{1}{2}ie(\partial_{[\mu}\overline{\Phi}_{\nu]}A^{[\mu}\Phi^{\nu]} - \partial^{[\mu}\Phi^{\nu]}A_{[\mu}\overline{\Phi}_{\nu]}) - \frac{1}{2}e^2A_{[\mu}\overline{\Phi}_{\nu]}A^{[\mu}\Phi^{\nu]}. \tag{2.118}$$

Terms linear and quadratic in A^μ will give rise to the one- and two-photon vertices respectively. It is useful to write the one-photon vertex using the notation $A\overleftrightarrow{\partial}B = A\partial B - (\partial A)B$ [50]:

$$\mathcal{L}_{A\Phi\overline{\Phi}} = ieA^\mu\overline{\Phi}^\nu(\eta_{\nu\lambda}\overleftrightarrow{\partial}_\mu - \eta_{\lambda\mu}\overleftrightarrow{\partial}_\nu)\Phi^\lambda, \tag{2.119}$$

and the two-photon term is conveniently written by raising all indices:

$$\mathcal{L}_{AA\Phi\overline{\Phi}} = -e^2A^\mu A^\nu\overline{\Phi}^\lambda\Phi^\rho(\eta_{\mu\nu}\eta_{\lambda\rho} - \eta_{\mu\lambda}\eta_{\nu\rho}). \tag{2.120}$$

Together, these terms make up the minimal coupling interaction Lagrangian for the Proca boson:

$$\mathcal{L}_{\text{min}} = ieA^{\mu}\overline{\Phi}^{\nu}(\eta_{\nu\lambda}\overleftrightarrow{\partial}_{\mu} - \eta_{\lambda\mu}\overleftrightarrow{\partial}_{\nu})\Phi^{\lambda} - e^{2}A^{\mu}A^{\nu}\overline{\Phi}^{\lambda}\Phi^{\rho}(\eta_{\mu\nu}\eta_{\lambda\rho} - \eta_{\mu\lambda}\eta_{\nu\rho}). \tag{2.121}$$

The one-photon vertex will enter into tree-level amplitudes such as those describing Compton scattering and photon exchange. Of these, the Compton scattering amplitude is interesting as it involves the exchange of the massive spin 1, which then involves the spin 1 propagator from (2.57). Referring to equation (2.57), we see that the numerator contains the term,

$$\frac{q_{\mu}q_{\nu}}{m^{2}}, \tag{2.122}$$

where q_{μ} denotes the momentum of the virtual spin 1 particle. Such a term is badly behaved at high energies, when the center of mass energy greatly exceeds the mass, $E^{2} \gg m^{2}$. In order to cancel the divergent term, a non-minimal (NM) coupling term is introduced[18, 19, 28, 51]:

$$\mathcal{L}_{s=1}^{\text{NM}} = \overline{\Phi}^{\mu}F_{\mu\nu}\Phi^{\nu}. \tag{2.123}$$

Writing out the NM term yields an addition to the one-photon interaction term:

$$\mathcal{L}_{s=1}^{\text{NM}} = A^{\lambda}\overline{\Phi}^{\mu}(\partial_{\mu}\eta_{\lambda\nu} - \partial_{\nu}\eta_{\nu\lambda})\Phi^{\nu}, \tag{2.124}$$

which will modify the one-photon vertex. To deduce the momentum space Feynman rules from the Lagrangian $\mathcal{L}_{\text{int}}^{s=1,EM} = \mathcal{L}_{\text{min}} + \mathcal{L}_{\text{NM}}$ we perform a Fourier transformation to momentum-space, from which derivative acting on Φ^{μ} will yield a factor $-ip_{1}^{\mu}$ while a derivative acting on $\overline{\Phi}^{\nu}$ will yield $+ip_{2}^{\nu}$. The fields $A^{\mu}(x), \Phi^{\mu}(x)$ become the polarisation vectors $\epsilon_{\mu}(k), \tau_{\mu}(p_{i})$. The one- and two-photon vertices then become:

$$\begin{aligned}(V_{s=1}^{\text{EM}})_{\mu} &= -i\tau_{2}^{*\nu}[(p_{1}+p_{2})_{\mu}\eta_{\nu\lambda} - (p_{1}+p_{2})_{\nu}\eta_{\mu\lambda}]\tau_{1}^{\lambda},\\ (V_{s=1}^{\text{EM}})_{\mu\nu} &= i\tau_{2}^{*\lambda}\tau_{1}^{\rho}(\eta_{\mu\nu}\eta_{\lambda\rho} - \eta_{\mu\lambda}\eta_{\nu\rho}),\end{aligned} \tag{2.125}$$

where we have introduced the shorthand notation $\tau^\mu(p_i) \equiv \tau_i^\mu$. Here, the one-photon vertex assumes that the two massive particles are on-shell (i.e. this is the vertex that would enter in a photon-exchange amplitude). For the vertex in which the photon is on-shell (used in Compton scattering) one would substitute a massive external state for a photon, e.g.

$$\tau_1^\lambda \to \epsilon_1^\lambda, \quad p_1 \to k_1, \tag{2.126}$$

where we use k to denote the momentum of a massless particle, satisfying the on-shell condition $k^2 = 0$. Interestingly, the inclusion of the NM term has effects the value of the gyro-magnetic coupling for the Proca boson; the inclusion of the NM coupling term changes it from the value $g = 1$ to $g = 2$[5, 19, 28].

2.5.2 Spin$-3/2$

For $s = \frac{3}{2}$ we may start with the RS Lagrangian in the compact form:

$$\mathcal{L}_{s=3/2} = -\frac{1}{2}\overline{\Psi}_\nu \gamma^{\mu\nu\rho}(iD_\nu - \tfrac{1}{2}im\gamma_\nu)\Psi_\rho. \tag{2.127}$$

The minimal coupling prescription then leads to the interaction term:

$$\mathcal{L}^{\text{min}}_{s=3/2} = -e\frac{1}{2}\overline{\Psi}_\mu \gamma^{\mu\nu\rho} A_\nu \Psi_\rho. \tag{2.128}$$

It was shown in [17] that the equations of motion derived from the minimally coupled RS Lagrangian, for sufficiently weak electromagnetic fields, admits solution which propagate superluminally. As a remedy, one may introduce additional non-minimal coupling terms. These are conventionally parameterised by a set of numbers, ℓ_i, $i = 1 - 5$, and are as follows [18, 52]:

$$\begin{aligned}\mathcal{L}^{\text{NM}}_{s=3/2} &= \ell_1 \overline{\Psi}_\mu F^{\mu\nu}\Psi_\nu + \ell_2 \overline{\Psi}_\mu F_{\rho\sigma}\gamma^\rho\gamma^\sigma\Psi^\mu \\ &+ \ell_3 F^{\mu\nu}(\overline{\Psi}_\mu \gamma_\nu(\gamma\cdot\Psi) + (\gamma\cdot\overline{\Psi})\gamma_\mu\Psi_\nu) + \ell_4(\gamma\cdot\overline{\Psi})(\gamma\cdot\Psi)F_{\rho\sigma}\gamma^\rho\gamma^\sigma \\ &+ i\ell_5 F_{\mu\nu}(\overline{\Psi}_\nu\gamma_\mu(\gamma\cdot\Psi) - (\gamma\cdot\overline{\Psi})\gamma_\mu\Psi_\nu)\end{aligned}$$

where we have omitted an overall factor $(-ie)/m$. Due to the Fierz-Pauli condition $\gamma \cdot \Psi = 0$, the terms for ℓ_3, ℓ_4, ℓ_5 do not contribute to on-shell vertices, and we shall therefore not consider these. This leaves the (on-shell) NM interaction term:

$$\mathcal{L}^{\text{NM}}_{s=3/2} = \ell_1 \overline{\Psi}^\mu F_{\mu\nu} \Psi^\nu + \ell_2 \overline{\Psi}_\mu F_{\rho\sigma} \gamma^\rho \gamma^\sigma \Psi^\mu. \tag{2.129}$$

Following the same procedure as before (and restoring factors of $(-ie)/m$), one obtains the interaction vertex for one photon and two massive spin $s = 3/2$ [28]:

$$\begin{aligned}(V^{\text{EM}}_{s=3/2})_\mu = -ie(\overline{u}_2)^\nu \gamma_{\nu\mu\rho}(v_1)^\rho - ie\frac{\ell_1}{m}(\overline{u}_2)^\lambda(\eta_{\mu\lambda}k_\nu - \eta_{\mu\nu}k_\lambda)(v_1)^\nu \\ - ie\frac{\ell_2}{m}(\overline{u}_2)_\rho(\eta_{\mu\lambda}k_\sigma - \eta_{\mu\sigma}k_\rho)\gamma^\lambda\gamma^\sigma(v_1)^\rho,\end{aligned}$$

where $k_\mu = (p_1 - p_2)_\mu$ and $(u_i)_\mu = u_\mu(p_i)$ are the spin $3/2$ external states constructed earlier. As in the $s = 1$ case, these NM terms alter the value of g to $g = 2$. From minimal coupling alone, one obtains $g = 2/3$[53], which, in combination previous results for $s = 1/2, 1$ suggests that minimal coupling alone provides $g = 1/s$, while the non-minimally coupled theory provides $g = 2$, independent of s[19].

2.5.3 Higher spin bosons

For particles of arbitrary integer spin, we can extract the minimal coupling interaction by considering the free (on-shell) part of the bosonic Singh-Hagen Lagrangian:

$$\mathcal{L}_s = \frac{1}{2}\overline{\Phi}^{(s)}(\partial^2 - m^2)\Phi^{(s)} + s(\partial \cdot \overline{\Phi}^{(s)})(\partial \cdot \Phi^{(s)}). \tag{2.130}$$

Performing the minimal coupling procedure gives rise to the one-photon term:

$$\mathcal{L}^{\text{min}}_s = ieA^\mu(\overline{\Phi}^{(s)} \overleftrightarrow{\partial} \Phi^{(s)})_\mu, \tag{2.131}$$

from which an on-shell current[5] can be deduced:

$$j_\mu = ie(\overline{\Phi}^{(s)}\overleftrightarrow{\partial}\Phi^{(s)})_\mu. \tag{2.132}$$

The current interactions then leads to the interaction vertex for two massive higher spin bosons and one photon:

$$(V_s^{\text{min}})_\mu = -ie(p_1 + p_2)_\mu(\tau_1^{(s)} \cdot \overline{\tau}_2^{(s)}), \tag{2.133}$$

In addition to the minimal coupling interaction, we can again consider NM interaction terms. The argument for introducing these terms is analogous to that for the Proca boson. The higher spin generalisation of the spin 1 NM interaction term is simply [18, 19]:

$$\mathcal{L}^{\text{NM}}(\alpha) = ie\alpha(\overline{\Phi}^{(s)} \cdot F \cdot \Phi^{(s)}), \tag{2.134}$$

where α is a number which parameterises the NM term. For $\alpha = 2s$, one may obtain $g = 2$ for arbitrary integer spin[19]. The contraction in the above term is to be understood as $\overline{\Phi}^{\mu}{}_{\cdots} F_{\mu\nu}\Phi^{\nu\cdots}$. In addition to this term, a number of increasingly complicated NM terms exist, corresponding to higher multipole terms[19]. These terms are important for the fixing of the high-energy limit of the Compton scattering amplitude[6]. We will not consider these terms here, but will instead see how they may be incorporated in other frameworks.

2.5.4 Higher spin fermions

Finally, we consider the the interactions of higher spin fermions. The procedure for extracting the electromagnetic current for higher spin fermions is similar to that for higher spin bosons. Starting with the free Lagrangian with the on-shell part of the Singh Hagen fermion Lagrangian:

$$\mathcal{L}_s = \frac{1}{2}\overline{\Psi}^{(s)}(\partial\!\!\!/ - m)\Psi^{(s)}, \tag{2.135}$$

5: Note that this current is not unique, as a different form is obtainable by first performing integration by parts on the Lagrangian and then introducing the covariant derivative [19].

6: In the low energy limit, these terms become increasingly small since they contain several derivatives, leading to factors of small p.

one can perform the minimal coupling substitution, which leads to the interaction Lagrangian[19]:

$$\mathcal{L}_{\text{int}} = -ie\overline{\Psi}^{(s)} \not{A}^{\mu} \Psi^{(s)}, \tag{2.136}$$

leading to the conserved on-shell current:

$$j^{\mu} = -i\overline{\Psi}^{(s)} \gamma^{\mu} \Psi^{(s)}. \tag{2.137}$$

The corresponding vertex is:

$$(V_s^{\min})^{\mu} = -ie(\overline{u}_2)^{(s)} \gamma^{\mu} (v_1)^{(s)}. \tag{2.138}$$

Here, as in the integer spin case, we may consider the simplest NM terms (which effect the value of g). These are the direct generalisation of the $s = \frac{3}{2}$ NM terms[18]:

$$\mathcal{L}_s^{\text{NM}} = \ell_1 \overline{\Psi}^{(s)} \cdot F \cdot \Psi^{(s)} + \ell_2 \overline{\Psi}^{(s)} F \cdot \gamma_2 \Psi^{(s)}. \tag{2.139}$$

The corresponding NM addition to the minimal coupling vertex for a higher spin fermion coupling to a photon may then be deduced analogously to the $s = \frac{3}{2}$ case. As in the integer spin case, higher multipole terms may be added, and are needed in order to fix the high-energy limit of certain amplitudes.

2.6 Summary

Before we move on to different formalisms, let us briefly summarize the Lagrangian formulation of higher spin theory. Higher massive spin fields are represented by tensors and spinor tensors, both of which are subject to extra on-shell conditions which fix the physical degrees of freedom to be $2s + 1$. These extra conditions, must be obtainable from a Lagrangian in order to immediate pathologies when electromagnetic fields are introduced. Constructing such Lagrangians necessitates the introduction of a tower of lower rank auxiliary field, which vanish on-shell. Even so, consistent electromagnetic interactions requires the introduction of non-minimal coupling terms, in addition to the minimal coupling current, j^{μ}. The key difficult of the Lagrangian formalism is thus the construction of necessary NM terms, in addition to the construction of the off-shell propagator.

Chapter 3

The spinor helicity formalism

An alternative route for describing the interactions of massive higher spin particles is via the "modern" approach to scattering amplitudes, which has seen tremendous success in application to strong interactions as well as gravity. In contrast to the off-shell approach involving Feynman diagrams and Lagrangians, this approach centers on the analytical properties of scattering amplitudes, and is entirely on-shell. For a process involving an initial state of momentum p_i scattering to a final state of momentum p_f, the scattering amplitude $\mathcal{A}$ is the kinematically interesting part of the $T-$matrix:

$$T = \delta^{(4)}\left(\sum_i p_i - \sum_f p_f\right)\mathcal{A}(p_i \to p_f), \tag{3.1}$$

which encodes the interacting part of the $S-$matrix[1]:

$$S = 1 + iT. \tag{3.2}$$

For a process involving n particles, it is customary to label the scattering amplitude as $\mathcal{A}_n$. We will denote scattering amplitudes with arguments corresponding to the external states of the process, labelling the external states with numbers going around clockwise, e.g. $\mathcal{A}_n(1, 2, \ldots, n)$. For massless

1: If no interactions are present, T vanishes and S reduces to the identity operator.

particles, the legs are labelled by the corresponding particles helicity h_i, e.g

$$\mathcal{A}(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4}). \tag{3.3}$$

The subject of the next section is to consider a special set of kinematic variables from which one may build the scattering amplitudes.

3.1 Converting between spinors and vectors

When calculating scattering amplitudes, it is useful to work with a set of variables that simplifies computations as much as possible, while also ensuring that the calculated amplitude retains the desired transformation properties, e.g. under little group transformations. A particularly convenient set of kinematic variables can be obtained by extending our formalism to the double cover of the Lorentz group, $SL(2,\mathbb{C})$. The Pauli matrices[2],

$$\sigma^{\mu}_{\alpha\dot{\alpha}} = (-\delta_{\alpha\dot{\alpha}}, \sigma^{i}_{\alpha\dot{\alpha}}), \quad \overline{\sigma}^{\mu}_{\dot{\alpha}\alpha} = (-\delta_{\dot{\alpha}\alpha}, -\sigma^{i}_{\dot{\alpha}\alpha}), \tag{3.4}$$

provide a homomorphism between $SO(1,3)$ and $SL(2,\mathbb{C})$, in the sense that a four-vector x^{μ} can be converted into a complex 2×2 matrix with $SL(2,\mathbb{C})$ indices as follows:

$$x_{\mu}\sigma^{\mu}_{\alpha\dot{\alpha}} = x_{\alpha\dot{\alpha}}. \tag{3.5}$$

Here, the two-valued $SL(2,\mathbb{C})$ indices $\alpha, \dot{\alpha}$ take the values $0, 1$ and $\dot{0}, \dot{1}$ respectively. If the components of the four vector x_{μ} are real, the matrix $x_{\alpha\dot{\alpha}}$ is hermitian. Thus, the Pauli matrices provide a way of relating a hermitian 2×2 matrix to a given four vector. For the Pauli matrices, we will use the convention:

$$\sigma^{1}_{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^{2}_{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^{3}_{\alpha\dot{\alpha}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.6}$$

The 2×2 matrix $x_{\alpha\dot{\alpha}}$ then takes the form:

$$x_{\alpha\dot{\alpha}} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}. \tag{3.7}$$

The $SL(2,\mathbb{C})$ are raised and lowered using the anti-symmetric objects,

$$\epsilon_{\alpha\beta}, \quad \epsilon_{\dot{\alpha}\dot{\beta}}, \quad \epsilon^{\alpha\beta}, \quad \epsilon^{\dot{\alpha}\dot{\beta}}, \tag{3.8}$$

2: We use this convention due to the choice of metric signature (see equation (3.11)). For a mostly minus signature, $\eta_{\mu\nu} = \text{diag}(+,-,-,-)$, one would choose $\sigma^{0}_{\alpha\dot{\alpha}} = +\delta_{\alpha\dot{\alpha}}$.

which take the values

$$\begin{aligned} \epsilon_{01} &= -\epsilon_{10} = 1, \quad \epsilon^{01} = -\epsilon^{10} = -1 \\ \epsilon_{\dot{0}\dot{1}} &= -\epsilon_{\dot{1}\dot{0}} = -1, \quad \epsilon^{\dot{0}\dot{1}} = -\epsilon^{\dot{1}\dot{0}} = 1. \end{aligned} \tag{3.9}$$

The dotted- and un-dotted symbols are related via complex conjugation, $\overline{(\epsilon_{\alpha\beta})} = -\epsilon_{\dot{\alpha}\dot{\beta}}$. The values listed above also imply that,

$$\epsilon^{\alpha}_{\ \beta} = -\epsilon_{\beta}^{\ \alpha} = \delta^{\alpha}_{\beta}, \quad \epsilon^{\dot{\alpha}}_{\ \dot{\beta}} = -\epsilon_{\dot{\beta}}^{\ \dot{\alpha}} = \delta^{\dot{\alpha}}_{\dot{\beta}}, \tag{3.10}$$

and hence the ordering of indices is important to keep in mind when raising and lowering. Finally, the anti-symmetric matrices are related to the Minwkoski spacetime metric:

$$\eta_{\mu\nu} = \sigma^{\alpha\dot{\alpha}}_{\mu}\sigma^{\beta\dot{\beta}}_{\nu}\eta_{\alpha\beta,\dot{\alpha}\dot{\beta}}, \quad \eta_{\alpha\beta,\dot{\alpha}\dot{\beta}} = \frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}. \tag{3.11}$$

Consider now a real two-component spinor, λ_α, such as might be a Weyl spinor. We raise and lower indices with the $\epsilon_{\alpha\beta}$ symbols,

$$\lambda_\alpha = \lambda^\beta \epsilon_{\alpha\beta}, \quad \lambda^\alpha = \epsilon^{\alpha\beta}\lambda_\beta, \tag{3.12}$$

and as a result, the inner product is also anti-symmetric:

$$\lambda^\alpha \xi_\alpha = -\lambda_\alpha \xi^\alpha, \tag{3.13}$$

and likewise in the dotted case. Note that, for $\xi_\alpha = \lambda_\alpha$, this implies that:

$$\lambda^\alpha \lambda_\alpha = \lambda_\alpha \lambda^\alpha = 0. \tag{3.14}$$

Having laid down the conventions for handling $SL(2,\mathbb{C})$ objects, we are now ready to consider the use of spinor helicity variables.

3.2 Spinor helicity variables

Using the $SL(2,\mathbb{C})$ map, it is possible to construct a useful set of kinematic variables from which one can construct scattering amplitudes. Consider converting a four vector momentum into a 2×2 matrix:

$$p_\mu \to p_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} p_\mu = p_{\alpha\dot{\alpha}} = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}. \tag{3.15}$$

Note that the determinant of this matrix gives the norm of the four momentum:

$$\det(p_{\alpha\dot{\alpha}}) = p_0^2 - p_1^2 - p_2^2 - p_3^2 = -p^2. \tag{3.16}$$

Depending on whether p^2 vanishes or not, the procedure for constructing spinor helicity variables varies. We will treat the simpler case first, in which $p^2 = 0$, corresponding to kinematics for massless particles.

3.2.1 Massless particles

For massless particles, the momentum is a null vector:

$$p^2 = 0. \tag{3.17}$$

For the matrix $p_{\alpha\dot{\alpha}}$, this translates into:

$$\det(p_{\alpha\dot{\alpha}}) = 0. \tag{3.18}$$

A 2×2 matrix with vanishing determinant can be decomposed into two 2 vectors, i.e. spinors:

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}. \tag{3.19}$$

We name $\lambda, \tilde{\lambda}$ spinor helicity variables, and they will act as our kinematic variables from which we can construct the kinematic part of scattering amplitudes. Scattering amplitudes will in general be Lorentz invariant functions of $\{\lambda, \tilde{\lambda}\}$.

Physically, we can think of the massless spinors $\lambda, \tilde{\lambda}$ as Weyl spinors; λ_α corresponds to a left-handed Weyl spinor while $\tilde{\lambda}^{\dot{\alpha}}$ corresponds to a right-handed Weyl spinor. They each satisfy the on-shell conditions:

$$p_{\alpha\dot{\alpha}} \lambda^\alpha = 0, \tag{3.20}$$

$$p^{\alpha\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}} = 0. \tag{3.21}$$

The association of $\{\lambda, \tilde{\lambda}\}$ with a given four-momentum $p_{\alpha\dot{\alpha}}$ is not unique, since one can always perform a re-scaling:

$$\{\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}\} \to \{z\lambda_\alpha, z^{-1}\tilde{\lambda}_{\dot{\alpha}}\}, \tag{3.22}$$

such that $p_{\alpha\dot{\alpha}}$ is left invariant. By writing the re-scaling in a suggestive way,

$$z = e^{i\phi} \in U(1), \tag{3.23}$$

one sees that the freedom of spinor helicity variables is simply a manifestation of the invariance of $p_{\alpha\dot{\alpha}}$ under the Little group $U(1)$ (the scaling in this case is sometimes referred to as little group scaling for this reason[21]). This also means that the spinor helicity variables $\{\lambda, \tilde{\lambda}\}$ have little group weights ± 1 respectively. This places a restriction on the structure of scattering amplitudes, as they must transform accordingly when the spinors are scaled. Specifically, if we consider an amplitude of n massless legs,

$$\mathcal{A}_n(1^{h_1}, \ldots, n^{h_n}), \tag{3.24}$$

and scale a leg, e.g. $1 \to 1_t$ by letting $\lambda_1, \tilde{\lambda_1} \to t\lambda_1, t^{-1}\tilde{\lambda}_1$, then we must have[21]:

$$\mathcal{A}_n(1_t^{h_1}, \ldots n^{h_n}) = t^{-2h_1}\mathcal{A}(1^{h_1}, \ldots, n^{h_n}). \tag{3.25}$$

This places extremely powerful restrictions on the structure of scattering amplitudes[3], and we will see later how it can be used to fix the structure of three-point amplitudes.

The spinor helicity variables come with a neat, index-free notation which will reduce the clutter of $SL(2,\mathbb{C})$ indices later. For a scattering amplitude involving n particles of momenta p_i^μ (with $i \in \{1, n\}$), we can label each pair of spinors corresponding to the momentum four-vector p_i^μ as:

$$\lambda(p_i)_\alpha \equiv |i\rangle, \;\; \tilde{\lambda}(p_i)^{\dot{\alpha}} \equiv |i], \tag{3.26}$$

3: The textbook example being the three-point massless scattering amplitudes which are fixed uniquely by this scalig law[38].

$$\lambda(p_i)^\alpha \equiv \langle i|,\ \tilde{\lambda}(p_i)^{\dot{\alpha}} \equiv [i|. \tag{3.27}$$

Inner products between variables are denoted as:

$$\lambda^\alpha(p_i)\lambda_\alpha(p_j) = \langle ij\rangle, \tag{3.28}$$

$$\tilde{\lambda}^{\dot{\alpha}}(p_i)\tilde{\lambda}(p_j)_{\dot{\alpha}} = [ij]. \tag{3.29}$$

Since raising and lowering indices is an antisymmetric action, we have that $\langle ij\rangle = -\langle ji\rangle$, and likewise for squares. In particular, inner products with repeated numbers vanish:

$$\langle ii\rangle = [ii] = 0. \tag{3.30}$$

In addition, we can use these spinor product to express products such as:

$$p_i \cdot p_j = 2\langle ij\rangle[ji]. \tag{3.31}$$

We will now list a number of properties and definitions that carry over using the expansion $p_i = i\rangle[i$. Momentum conservation becomes[4]:

$$\sum_n \langle ij\rangle[jk] = 0. \tag{3.32}$$

The mandelstam variables are:

$$s_{ij} = (k_i + k_j)^2 = \frac{1}{2}\langle ij\rangle[ji]. \tag{3.33}$$

A number of useful identities for the spinor products exists, that are helpful in reducing expressions for scattering amplitudes. First, it is useful to note that for real momenta, the spinor helicity variables are related by complex conjugation, i.e:

$$\overline{\langle ij\rangle} = [ji]. \tag{3.34}$$

In addition, we have the two Schouten identities:

$$\begin{aligned} &\langle ij\rangle\langle kl\rangle + \langle ik\rangle\langle lj\rangle + \langle il\rangle\langle jk\rangle = 0, \\ &[ij][kl] + [ik][lj] + [il][jk] = 0. \end{aligned} \tag{3.35}$$

4: We use the convention that all momenta in any amplitude are *outgoing*, such that $\sum_i p_i = 0$.

3.2.2 Massive particles

So far we have only treated the case for which $p^2 = 0$, i.e. massless particles. For on-shell massive particles, we have $p^2 = m^2$ and so the matrix $p_{\alpha\dot{\alpha}}$ is no longer degenerate. Rather than decomposing $p_{\alpha\dot{\alpha}}$ into a direct product of spinors, we can write it as a sum of two degenerate matrices:

$$p_{\alpha\dot{\alpha}} = q^1_{\alpha\dot{\alpha}} + q^2_{\alpha\dot{\alpha}}. \tag{3.36}$$

The matrices q^1, q^2 can then be written in terms of spinors[27, 54]:

$$q^I = \lambda^I_\alpha \tilde{\lambda}^I_{\dot{\alpha}}, \quad I = \{1, 2\}. \tag{3.37}$$

More compactly, we can write $p_{\alpha\dot{\alpha}}$ by contracting the index I:

$$p_{\alpha\dot{\alpha}} = \lambda^I_\alpha \tilde{\lambda}_{\dot{\alpha}I} = \epsilon^{IJ} \lambda_{\alpha J} \tilde{\lambda}_{\dot{\alpha}I}. \tag{3.38}$$

The indices I, J come from the little group $SU(2)$, and are therefore raised and lowered using the anti-symmetric symbols $\epsilon_{IJ}, \epsilon^{IJ}$. In this sense, each pair λ^I_α and $\tilde{\lambda}^I_{\dot{\alpha}}$ form a doublet under the $SU(2)$ group. As with the massive SH variables, this means that the association of a set of variables to a given four momentum $p_{\alpha\dot{\alpha}}$ is not unique, as the momentum is left invariant under transformations $W^I_J \in SU(2)$ of the form:

$$\begin{aligned} \lambda^I &\to W^I_J \lambda^J, \\ \tilde{\lambda}^I &\to (W^{-1})^I_J \tilde{\lambda}^J. \end{aligned} \tag{3.39}$$

The massive spinors obey the on-shell conditions:

$$\begin{aligned} p^{\dot{\alpha}\alpha} \lambda^I_\alpha(p) &= m \tilde{\lambda}^{\dot{\alpha}I}(p), \\ p^{\dot{\alpha}\alpha} \tilde{\lambda}^I_{\dot{\alpha}}(p) &= -m \lambda^{\alpha I}(p). \end{aligned} \tag{3.40}$$

Together, the above condition make up the components of the Dirac equation[54]:

$$u^I(p) = \begin{pmatrix} \lambda^I_\alpha(p) \\ \tilde{\lambda}^{\dot{\alpha}I}(p) \end{pmatrix}, \; v^I(p) = \begin{pmatrix} -\lambda^I_\alpha(p) \\ \tilde{\lambda}^{\dot{\alpha}I}(p) \end{pmatrix}, \tag{3.41}$$

and:

$$(\not{p} - m)u^I(p) = (\not{p} + m)v^I(p) = 0. \tag{3.42}$$

So, while the massless spinor helicity variables can be interpreted as Weyl spinors each satisfying the appropriate on-shell conditions, the massive spinor helicity variables combine into Dirac spinors that obey the Dirac equation.

A more direct way to obtain the massive spinor helicity variables is by starting with the time-like momentum vector p_μ and performing a light-cone decomposition[28]:

$$p^\mu = k^\mu + \frac{m^2}{2k \cdot q} q^\mu, \tag{3.43}$$

where $k^2 = q^2 = 0$ are null vectors. We can take q^μ to be an arbitrary reference vector, k^μ is then determined by the decomposition. We can obtain the massive spinor helicity variables by contracting the momentum vector p_μ with the Pauli matrices:

$$\sigma \cdot p = \sigma \cdot k + \frac{m^2}{2k \cdot q} \sigma \cdot q. \tag{3.44}$$

Since k^μ and q^μ are null vectors, we can write the contractions $\sigma \cdot k,\ \sigma \cdot q$ in terms of bi-spinor brackets and angles, and thus define our massive spinor helicity variable through the decomposition:

$$|p_J\rangle[p^J| \equiv |k\rangle[k| + \frac{m^2}{\langle kq\rangle} |q\rangle[q|, \tag{3.45}$$

where it has been used that $\langle kq\rangle = 2k \cdot q$. Here, the index J is the little-group index of our massive momentum vector, for which the little-group is $SU(2)$. Thus, we can interpret $|p^J\rangle$ and $|p^J]$ as spinors with two components:

$$|p^J\rangle = \begin{pmatrix} |q\rangle \frac{m}{\langle kq\rangle} \\ |k\rangle \end{pmatrix}. \tag{3.46}$$

The elements of $|p^J]$ follows by flipping the entries and replacing angles with brackets. These little-group indices J behave as normal, and are raised and lowered using ϵ_{IJ}. Just like the scattering amplitudes for massless particles carry the little-group helicities and most transform appropriately, a scattering amplitude constructed out of massive spinor helicity variables will carry the little-group indices I, J. In addition, the scattering amplitudes will be completely symmetrised in the little-group indices. This fact allows to suppress the little-group indices completely when constructing amplitudes from massive spinor helicity variables. In order to distinguish massive spinor-helicity variables from massless ones, it is customary to denote them with bold font:

$$|i^J\rangle \equiv |\mathbf{i}\rangle, \; |i^J] \equiv |\mathbf{i}]. \tag{3.47}$$

With the indices suppressed, the rule of antisymmetric inner products and other identities for massless spinor-helicity variables, such as the Fierz identity still applies.

3.2.3 Polarisation vectors

Another important feature of the spinor-helicity formalism is the incorporation of polarisation vectors in terms of the spinor-helicity variables. For massless particles of helicities $h = \pm$ we define:

$$\epsilon^+(p_i) = \frac{\langle i|\sigma^\mu|q]}{\langle iq\rangle} \equiv \epsilon_i^+(q), \tag{3.48}$$

$$\epsilon^-(p_i) = \frac{[i|\sigma^\mu|q\rangle}{[iq]} \equiv \epsilon_i^-(q). \tag{3.49}$$

Here, q is a reference spinor that may be chosen freely. Unless the specific choice of q is relevant, we will suppress the dependency and simply write $\epsilon_i^\pm(q) = \epsilon_i^\pm$. The polarisation vector of the opposite helicity can be obtained through conjugation:

$$(\epsilon_i^\pm)^* = \epsilon_i^\mp, \tag{3.50}$$

and the polarisation vectors are normalized such that:

$$\epsilon_i^+ \cdot \epsilon_i^- = +1, \tag{3.51}$$

In addition, they obey the transversality condition:

$$p_i \cdot \epsilon_i^{\pm} = q \cdot \epsilon_i^{\pm} = 0. \tag{3.52}$$

Finally, it is also required that:

$$(\epsilon_i^{\pm})^2 = 0. \tag{3.53}$$

The freedom of choice for the reference spinor q will often be employed in combination with the transversality condition to eliminate expressions such as:

$$\epsilon_i^{+}(p_j) \cdot \epsilon_j^{-}(q) = 0. \tag{3.54}$$

For massive external states, we can construct polarisation vectors in an analogous fashion[28, 54]:

$$\epsilon(p) = \frac{|\mathbf{p}\rangle[\mathbf{p}|}{m}. \tag{3.55}$$

These polarisation vectors play the role of on-shell external states in the spinor-helicity formalism.

3.3 Three-point amplitudes

Having introduced the kinematic variables of the massive spinor helicity formalism, we are now in a position to review the simplest of amplitudes for massive particles of arbitrary spin, three-point amplitudes. In the language of Feynman rules, these amplitudes corresponds to a vertex dressed with external states, and as such they tell us about the possible couplings that HS particles may make with other particle species. Thus, the primary object of interest to us is the "stripped" three-point amplitude, i.e. the amplitude where external spinor helicity variables have been removed. We will denote the stripped amplitude by A:

$$\mathcal{A} = \Lambda_{(\alpha\dot{\alpha})} A^{(\alpha\dot{\alpha})}, \tag{3.56}$$

where $\Lambda_{(\alpha\dot{\alpha})}$ is short-hand for all relevant external spinor helicity variables. This form makes it possible to remove the dependency on the $SU(2)$ group indices, and allows us to express the amplitude in purely dotted- or undotted indices[5]. In addition to spinor indices, the stripped amplitude will carry the little-group indices of the participating particles. An amplitude describing the scattering of n massive particles of spin s_i and m massless particles will carry the indices:

$$\mathcal{A}^{I_1 \ldots I_n, h_1 \ldots h_m}(\mathbf{1}, \ldots \mathbf{n}, 1, \ldots, m), \tag{3.57}$$

while the corresponding stripped amplitude will carry spinor indices in addition. In general when we refer to the stripped three-point amplitude, we will refer to the amplitude where the massive external states have been stripped, and the stripped amplitude will therefore not carry any free $SU(2)$ indices. The complexity and structure of these three-point amplitudes depend on which particles participate, e.g., the structure of the three-point amplitude for two massless scalars and a massive spin-s will be vastly more restricted than

5: The un-dotted choice is sometimes referred to as the chiral basis.

that of one massless scalar and two massive spin-s[27]. If we consider for example the amplitude for one massive spin s particle coupling to two massless particles of helicities h_1, h_2, the little group indices of the full amplitude are h_1, h_2 in addition to $2s$ $SU(2)$ indices:

$$\mathcal{A}^{I_1 \dots I_{2s}, h_1, h_2} = (\lambda)^{I_1 \dots I_{2s}}_{\alpha_1 \dots \alpha_{2s}} A^{\alpha_1 \dots \alpha_{2s}, h_1, h_2}. \tag{3.58}$$

Since the stripped amplitude only carries spinorial indices, the problem of figuring out is structure is reduced to picking a suitable basis of rank 1 spinors. The available options depend on the configuration of the amplitude, i.e. how many massless and massive legs it contains. Of relevance to us are the cases with one- and two massive legs, so we will review these cases in some detail in what follows.

3.3.1 One massive leg

As a warm-up before we consider the interesting three-point amplitude with two massive legs, we consider the simpler case in which only one leg is massive. The stripped amplitude is a rank s spinorial tensor:

$$A^{h_1, h_2}_{\alpha_1 \dots \alpha_s}. \tag{3.59}$$

The massless spinor helicity variables make a naturual choice for a basis in which to express the amplitude. Temporarily ignoring any coupling constants and factors of mass, take the anzats for the stripped amplitude (in un-dotted indices):

$$A^{\alpha_1 \dots \alpha_{2s}} = [(\lambda_1)^a (\lambda_2)^b]^{\alpha_1 \dots \alpha_{2s}} (\tilde{\lambda}_1 \cdot \tilde{\lambda}_2)^c. \tag{3.60}$$

The conjugate spinors must be contracted with each other since the stripped amplitude only carries un-dotted spinor indices. We therefore only need to determine the numbers a, b, c. All we need to do is count the helicity weights of each massless particle; recall that an amplitude involving massless particles of helicities h_i must scale correctly under little-group re-scalings, leading to the constraint:

$$|N(\lambda) - N(\tilde{\lambda})| = 2h_i. \tag{3.61}$$

In our case, this leads to $a - c = b - c = 2$, and since the particles may carry different helicities this fixes the constants as:

$$\begin{aligned} a &= s + h_2 - h_1, \\ b &= s + h_1 - h_2, \\ c &= s + h_1 + h_2. \end{aligned} \tag{3.62}$$

The stripped amplitude is thus uniquely constrained by the helicity weights of the massless particles:

$$A^{\alpha_1 \ldots \alpha_{2s}} = ((\lambda_1)^{s+h_2-h_1}(\lambda_2)^{s+h_1-h_2})^{\alpha_1 \ldots \alpha_{2s}}[12]^{s-h_1-h_2}. \tag{3.63}$$

The full amplitude can now be obtained by contracting the above with the massive spinor helicity variables, $(\lambda_3)^{\alpha}_{I}$. When doing this however, we must be careful to properly symmetrize in the SU(2) indices. Numbering the massive leg as 3 and making the little group indices explicit, we have:

$$\mathcal{A}(1, 2, \mathbf{3}) = \prod_{k=1}^{s+h_2-h_1} \langle 1\mathbf{3}^{(I_k}\rangle \prod_{\ell=s+h_2-h_1+1}^{2s} \langle 2\mathbf{3}^{I_\ell)}\rangle [12]^{2s}. \tag{3.64}$$

Technically this is incorrect, as an amplitude in 4d should carry mass-dimension 1. The above expression carries a total mass dimension of $2s + h_1 + h_2$, and so it should be divided by a factor $m^{-(2s+h_1+h_2-1)}$ [27].

3.3.2 Two massive legs of equal mass

Next, we consider the more complicated where two of the three external states are massive particles, we will let the legs 1 and 2 be massive and leg 3 be massless. We will restrict our attention to the case in which the massive particles have equal mass, $m_1 = m_2 = m$ and equal spin, $s_1 = s_2 = s$. The full amplitude carries two sets of SU(2) indices:

$$\mathcal{A}^{I_1 \ldots I_{2s}, J_1 \ldots J_{2s}, h} = (\lambda_1)^{I_1 \ldots I_{2s}}_{\alpha_1 \ldots \alpha_{2s}} (\lambda_2)^{J_1 \ldots J_{2s}}_{\beta_1 \ldots \beta_{2s}} A^{\alpha_1 \ldots \alpha_{2s}, \beta_1 \ldots \beta_{2s}, h}. \tag{3.65}$$

For the basis one can pick the massless leg, $(\lambda_3)_\alpha$. The spinors $(\lambda_3)_\alpha$ and $(p_1)_{\alpha\dot{\alpha}}(\lambda_3)^\alpha$ make a natural choice. However, when $m_1 = m_2 = m$, the kinematics become degenerate. Momentum conservation gives (all out convention):

$$\sum_i p_i = 0 \implies p_1 + p_3 = -p_2, \tag{3.66}$$

squaring both sides with $p_1^2 = p_2^2 = m^2$ and $p_3^2 = 0$ gives:

$$2p_1 \cdot p_3 = \langle 3|p_1|3] = 0. \tag{3.67}$$

But:

$$\langle 3|p_1|3] = \lambda^\alpha (p_1)_{\alpha\dot{\alpha}} \tilde{\lambda}_3^{\dot{\alpha}}, \tag{3.68}$$

and

$$(p_1)_{\alpha\dot{\alpha}} \tilde{\lambda}_3^{\dot{\alpha}} = m(p_1)_{\alpha\dot{\alpha}} \lambda_3^\alpha, \tag{3.69}$$

so $\lambda_3^\alpha \propto (p_1)_{\alpha\dot{\alpha}}(\lambda_3)^\alpha$. It is useful to define the constant of proportionality [27]:

$$x\lambda_3^\alpha \equiv \tilde{\lambda}_3^{\dot{\alpha}} \frac{(p_1)_{\alpha\dot{\alpha}}}{m}. \tag{3.70}$$

Dividing with λ_3^α and contracting with a reference spinor ξ_α, this gives:

$$x \equiv \frac{[3|p_1|\xi\rangle}{m\langle 3\xi\rangle}. \tag{3.71}$$

By inspection, we see that the above can be written in terms of a massless polarisation vector of helicity $+1$:

$$x = (p_1 - p_2) \cdot \epsilon^+. \tag{3.72}$$

Including p_2 reminds us that the x can always be defined with p_2 instead of p_1, and either can be eliminated by an appropriate choice of the reference spinor ξ. In analogy to (3.72), we define the "inverse" $x-$factor:

$$x^{-1} = (p_1 - p_2) \cdot \epsilon^-, \tag{3.73}$$

to be used when the massless leg has negative helicity. With λ_3^α and x, the $m_1 = m_2 = m$ three-point amplitude can be expressed as[27, 55]:

$$A^h_{\alpha_1\ldots\alpha_{2s},\beta_1\ldots\beta_{2s}} = x^h \left[\sum_{a=0}^{2s} \epsilon_{2s-a} \left(\frac{\lambda_3(p_1)\tilde{\lambda}_3}{m}\right)_a\right]_{\alpha_1\ldots\alpha_{2s},\beta_1\ldots\beta_{2s}},$$

where ϵ_{2s-a} denotes a product of $2s - a$ $\epsilon_{\alpha\beta}$ symbols. This expression is complicated in comparison with (3.63) due to the $2s+1$ different structures allowed by helicity weight constraints. Of these structures, the $a = 0$ was identified in [27] as special, due to its nice high-energy limit. In this case, all the $4s$ $SL(2,\mathbb{C})$ indices are carried by the $\epsilon_{\alpha\beta}$ symbols, and for this reason this simple structure is sometimes referred to as the "minimal coupling" even though they strictly speaking are not related to what we refer to as minimal coupling in the language of Lagrangians and Feynman rules[27]. Dressing the $a = 0$ with massive spinor helicity variables and setting $h = 1$, one obtains the minimal coupling amplitudes for a massive higher spin field coupling to a photon:

$$\begin{aligned}\mathcal{A}_3^0(\mathbf{1},\mathbf{2},3^+) &= mx\left(\frac{\langle\mathbf{12}\rangle}{m}\right)^{2s}\\ \mathcal{A}_3^0(\mathbf{1},\mathbf{2},3^-) &= \frac{m}{x}\left(\frac{[\mathbf{12}]}{m}\right)^{2s},\end{aligned} \tag{3.74}$$

where we let the 0 indicate that these are the special amplitudes corresponding to the $a = 0$ term of the general expression. In addition to the photon minimal coupling, we can obtain the amplitudes for minimal coupling to gravity by setting $h = \pm 2$[27]:

$$\begin{aligned}\mathcal{M}_3^0(\mathbf{1},\mathbf{2},3^+) &= \frac{(mx)^2}{M_{pl}}\left(\frac{\langle\mathbf{12}\rangle}{m}\right)^{2s},\\ \mathcal{M}_3^0(\mathbf{1},\mathbf{2},3^-) &= \frac{(mx^{-1})^2}{M_{pl}}\left(\frac{[\mathbf{12}]}{m}\right)^{2s},\end{aligned} \tag{3.75}$$

where a factor of planck mass M_{pl} is added to keep the units right. It is important to note that these three-point amplitudes contain non-trivial pieces, such as the NM coupling terms discussed in chapter 2. For example it can be shown, the $s = 1$ and $s = \frac{3}{2}$ photon amplitudes contain excactly the NM terms introduced earlier[28].

3.4 Higher spin Compton scattering

The most interesting application of the higher-spin three point amplitudes is the construction of higher spin Compton amplitudes. This is particularly interesting, as the Compton amplitude involves the exchange of a massive higher spin particle, which in the language of Feynman diagrams lead to the necessary introduction of non-minimal couplings in order to resolve tree-level unitarity violation. While the evaluation of this (relatively simple) expression is well known by now (see e.g. [22, 27, 56]) it is worth while to analyse its derivation here, if but for completeness. Let us therefore consider the amplitude describing Compton scattering for two massive spin s particles and two photons. Let the legs $1, 4$ be massive and $2, 3$ massless, and let us consider the contribution from the s-channel. We have:

$$\mathcal{A}_4(\mathbf{1}, 2^{\pm}, 3^{\mp}, \mathbf{4}) = \mathcal{A}_3(\mathbf{1}, 2^{\pm}, \hat{\mathbf{P}}_{12})\frac{i}{s}\mathcal{A}_3(-\hat{\mathbf{P}}_{12}, 3^{\mp}, \mathbf{4}) \tag{3.76}$$

For now, we fix the helicities to be $2^+, 3^-$. Inserting the three-point amplitudes (3.74) this becomes:

$$\begin{aligned}\mathcal{A}_4(\mathbf{1}, 2^{\pm}, 3^{\mp}, \mathbf{4}) &= m x_{12}\left(\frac{\langle \mathbf{1}\hat{\mathbf{P}}_{12}\rangle}{m}\right)^{2s}\frac{i}{s+m^2}\frac{m}{x_{34}}\left(\frac{[-\hat{\mathbf{P}}_{12}\mathbf{4}]}{m}\right)^{2s} \\ &= \frac{x_{12}}{x_{34}}\frac{i}{s+m^2}m^{-2s+2}(\langle \mathbf{1}\hat{\mathbf{P}}_{12}\rangle[\hat{\mathbf{P}}_{12}\mathbf{4}])^{2s}.\end{aligned}$$

To proceed, we first need the residue at the pole:

$$\begin{aligned}z &= \frac{-s_{12}}{\langle 3|\mathbf{P}_{12}|2]} \\ &= \frac{-m^2}{\langle 3|(\mathbf{1}+2)|2]} \\ &= -\frac{m^2}{\langle 3|\mathbf{1}|2]}.\end{aligned}$$

We can now evaluate the term involving the shifted pole $\hat{\mathbf{P}}_{12}$:

$$\begin{aligned}\langle \mathbf{1}|\hat{\mathbf{P}}_{12}|\mathbf{4}] &= \langle \mathbf{1}|(\mathbf{1}+2+|3\rangle\, z[2|)|\mathbf{4}] \\ &= \langle \mathbf{1}|(2-|3\rangle\frac{m^2}{\langle 3|\mathbf{1}|2]}[2|)|\mathbf{4}] \\ &= \frac{-m^2}{\langle 3|\mathbf{1}|2]}(\langle \mathbf{1}3\rangle[2\mathbf{4}]+\langle \mathbf{1}2\rangle[3\mathbf{4}]).\end{aligned}$$

We also need to evaluate the fraction x_{12}/x_{34}. Referring back to their definition (3.72), the $x-$factors in (3.77) are:

$$\begin{aligned} m x_{12} &= \frac{\langle \xi_{12}|\mathbf{1}|2]}{\langle 2\xi_{12}\rangle}, \\ m x_{34}^{-1} &= \frac{[\xi_{34}|\mathbf{3}|4\rangle}{\langle 3\xi_{34}\rangle}. \end{aligned} \tag{3.77}$$

Choosing the reference spinors to be:

$$\xi_{12} = 3, \quad \xi_{34} = 2,$$

this yields the expression:

$$m^2 \frac{x_{12}}{x_{34}} = \frac{(\langle 3|\mathbf{1}|2])^2}{\langle 23\rangle[32]} = \frac{(\langle 3|\mathbf{1}|2])^2}{t}. \tag{3.78}$$

Inserting back into (3.77):

$$\begin{aligned} \mathcal{A}_4(\mathbf{1}, 2^{\pm}, 3^{\mp}, \mathbf{4}) &= \frac{i}{t(s+m^2)} m^{-2s} (\langle 3|\mathbf{1}|2])^2 \left(\frac{-m^2}{\langle 3|\mathbf{1}|2]}\right)^{2s} (\langle \mathbf{1}3\rangle[2\mathbf{4}] + \langle 12\rangle[3\mathbf{4}])^{2s} \\ &= \frac{i}{t(s+m^2)} (\langle 3|\mathbf{1}|2])^{2-2s} (\langle \mathbf{1}3\rangle[2\mathbf{4}] + \langle \mathbf{1}2\rangle[3\mathbf{4}])^{2s}. \end{aligned}$$

For $2s > 2$, the above expression develops a spurious pole (i.e. a non-physical singularity). Since the three-point amplitudes appear to contain the desirable NM coupling terms required for consistent coupling to the photon, we surmise that the issue here is related to the on-shell construction of the four point amplitude. This can be interpreted as an issue related to propagator of the massive higher spin particle. In a full theory, one would expect contact terms which may cancel the spurious pole. For lower spins however (i.e. $s = 0, 1/2, 1$) the above does indeed reproduce the expected results[22].

3.5 Summary

The spinor helicity formalism offers a compact and somewhat more direct approach to the higher spin formalism, and avoids much of the complicated mess of the canonical formalism. The particularly nice feature is the construction of special minimal coupling amplitudes, which seems to by-pass the difficulty of constructing consistent interactions in the canonical language. This approach is however not bullet-proof, as is displayed by the spurious poles developed by the Compton amplitude. Indeed, the core problem of the spinor helicity approach, is the lack of theory on which to build this formalism for higher spin. As suspected, the problem is primarily related to the exchange of massive higher spin. We will see in the following chapters how this may be alleviated.

Chapter 4

Spinor fields

In this chapter, we will consider a somewhat different approach to higher spin which employs spinor fields to describe spin s states. It is somewhat in between the previous two frameworks, in that it may be considered a field theoretical approach, but avoids the introduction of Lagrangians. This approach is not new; we will in this chapter closely follow the approach as put forward by S. Weinberg in the 1960's[29].

The starting point is to consider the rank $2s$ spinorial tensors:

$$\varphi^{\alpha_1 \dots \alpha_{2s}}, \quad \overline{\varphi}^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}. \tag{4.1}$$

Such objects belong to the representations:

$$D(s, 0), \quad D(0, s), \tag{4.2}$$

describing the un-dotted and dotted spinorial tensors respectively. The two tensors are related by complex conjugation:

$$\overline{\varphi^{\alpha_1 \dots \alpha_{2s}}} = \overline{\varphi}^{\dot{\alpha}_1 \dots \dot{\alpha}_{2s}}. \tag{4.3}$$

This particular choice of representation has a number of advantages over the traditional representations presented in the chapter 2. One obvious advantage is that a totally symmetric rank $2s$ spinorial tensor in $d = 4$

space-time dimensions encodes exactly $2s+1$ degrees of freedom, and so no auxiliary conditions are needed. Like the Lorentz tensors from 2 the spinorial tensors are totally symmetric, and so we will extend the index-free notation from earlier by writing:

$$\varphi^{\alpha_1 \ldots \alpha_{2s}} = \varphi^{(s)}, \tag{4.4}$$

$$\overline{\varphi}^{\dot{\alpha}_1 \ldots \dot{\alpha}_{2s}} = \overline{\varphi}^{(s)}. \tag{4.5}$$

The spinor $\varphi^{(s)}$ and its conjugate live in spinor space, and so we can raise and lower indices using the anti-symmetric symbols:

$$\epsilon_{\alpha\beta}, \quad \overline{\epsilon}_{\dot{\alpha}\dot{\beta}}. \tag{4.6}$$

A consequence of this, is that, by virtue of being totally symmetric, $\varphi^{(s)}, \overline{\varphi}^{(s)}$ are both automatically traceless:

$$\epsilon \cdot \varphi^{(s)} = \overline{\epsilon} \cdot \overline{\varphi}^{(s)} = 0. \tag{4.7}$$

(Since the contraction of a totally symmetric with a totally anti-symmetric tensor vanishes).

4.1 Equations of motion

Like in the covariant representation, the tensors representing on-shell particles satisfy the equation of motion[29, 57]:

$$({}^2 + m^2)\varphi^{(s)} = 0. \tag{4.8}$$

Where $\partial^2 = \partial_{\dot{\alpha}\alpha}\partial^{\dot{\alpha}\alpha}$ and $\partial_{\alpha\dot{\alpha}} = \sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu}$. This equation of motion can also written in the form of a system of two equations, by introducing an auxiliary field[57]:

$$\partial \cdot \varphi^{(s)} = m\rho^{(s-1)}, \tag{4.9}$$

$$\partial \cdot \rho^{(s-1)} = \frac{1}{m}\varphi^{(s)}. \tag{4.10}$$

where:

$$\rho^{(s-1)} \equiv \rho^{\alpha_2 \dots \alpha_{2s}}_{\dot{\alpha}_1}, \tag{4.11}$$

is an auxiliary field. For $s = 1/2$, the fields φ^α and $\rho_{\dot{\alpha}}$ are fields in the representations $D(\frac{1}{2}, 0)$ and $D(0, \frac{1}{2})$ respectively, and they can be combined to form a Dirac spinor $\Psi = (\varphi^\alpha, \rho_{\dot{\alpha}})$. The above system of equations then becomes the Dirac equation. Using the identity[57]:

$$\frac{1}{2}\epsilon_{\alpha\beta}\partial^2 = \partial_{\alpha\dot{\alpha}}\partial^{\dot{\alpha}}_{\beta} \tag{4.12}$$

in combination with (4.9) and (4.11) yields the equation of motion (4.8), and so the system of equations (4.9), (4.11) is equivalent to (4.8). Since $\varphi^{(s)}$ only contains the $2s + 1$ degrees of freedom, these are all the equations we can construct for $\varphi^{(s)}$ (or equivalently $\overline{\varphi}^{(s)}$) alone.

On their own, the fields $\varphi^{(s)}$ and $\overline{\varphi}^{(s)}$ cannot partake in parity conserving interactions[29]. A parity invariant theory can be obtained by combining the two fields in a rank-s bi-spinor,

$$\xi^{(s)} = \begin{pmatrix} \varphi^{(s)} \\ \overline{\varphi}^{(s)} \end{pmatrix}, \tag{4.13}$$

which belongs to the representation:

$$D(s, 0) \oplus D(0, s). \tag{4.14}$$

This is just the generalisation of the representation $D(1/2, 0) \oplus D(0, 1/2)$ under which the Dirac bi-spinor transforms, and so we think of $\xi^{(s)}$ as the spin s version of the Dirac bi-spinor. This bi-spinor constitutes a $2(2s + 1)$ component field, transforming under the $D(\frac{s}{2}, 0) \oplus D(0, \frac{s}{2})$ representation. Like the $2s + 1$ component fields, $\varphi^{(s)}$ and $\overline{\varphi}^{(s)}$, it satisfies an equation of motion of the form:

$$(\partial^2 + m^2)\xi^{(s)} = 0. \tag{4.15}$$

In addition, the two-equation system (4.9), (4.11) can be combined for $\xi^{(s)}$ to form a generalisation of the Dirac equation:

$$[\Gamma^{(s)} \cdot \partial_s + m^{2s}]\xi^{(s)} = 0. \tag{4.16}$$

This equation is known as the Joos-Weinberg (JW) equation, discovered independently in 1962 and 1964 by H. Joos[58] and S. Weinberg[29] respectively. Here,

$$\Gamma^{(s)} = \Gamma^{\mu_1 \dots \mu_s}, \tag{4.17}$$

is a generalisation of the γ_μ-matrices defined in[29] (with $\Gamma^\mu = \gamma^\mu$). They are symmetric and traceless,

$$\Gamma^{(s)} = \Gamma^{(\mu_1 \dots \mu_s)}, \quad \eta \cdot \Gamma^{(s)} = 0, \tag{4.18}$$

and they can be thought of a generalisation of the γ^μ in the sense that their symmetrised products give rise to symmetrised products of the metric. For example, for $s = 2$[29],

$$\{\Gamma^{\mu\nu}, \Gamma^{\rho\sigma}\} + \{\Gamma^{\mu\rho}, \Gamma^{\sigma\nu}\} + \{\Gamma^{\mu\sigma}, \Gamma^{\nu\rho}\} \\ = 2(\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\sigma}),$$

and likewise for higher s. For $s = 1/2$, the JW equation reduces to the Dirac equation.

4.2 Transformation properties

Before we can begin considering interactions and Feynman rules with the spinorial fields $\varphi^{(s)}, \overline{\varphi}^{(s)}$ it is instructive to consider their transformation properties in more detail. First, we note that the $2s + 1$ component field can be considered by labelling it by its helicity:

$$\varphi^{\alpha_1 \dots \alpha_{2s}}(x) \to \varphi_\sigma(x). \tag{4.19}$$

Since $\sigma = -s, -s+1, \dots s-1, s$, it runs over exactly $2s+1$ values. More precisely, the components are related as in the following sense:

$$\begin{aligned}
\varphi_s &= \varphi_{11\dots1},\\
&\vdots\\
\varphi_0 &= \varphi_{1\dots10\dots0}\\
&\vdots\\
\varphi_{-s} &= \varphi_{00\dots0}.
\end{aligned} \tag{4.20}$$

And likewise for $\overline{\varphi}^{(s)}$. The point of representing the field in this fashion is that we can consider the transformation properties of $\varphi_\sigma(x)$ in terms of the representations $D^{(s)}_{\sigma\sigma'}$. Recall that the field $\varphi_\sigma(x)$ describes a massive particle of spin s and helicity σ. Such a particle may be created from the vacuum state $|0\rangle$ with a creation operator $b^\dagger(p,\sigma)$:

$$|p,\sigma\rangle = b^\dagger(p,\sigma)\,|0\rangle\,. \tag{4.21}$$

The particle transforms according to the representation $D^{(s)}_{\sigma\sigma'}$:

$$U(\Lambda)\,|p,\sigma\rangle = \sqrt{\omega(\Lambda p)/\omega(p)}\sum_{\sigma'} D^{(s)}_{\sigma\sigma'}(W(p))\,|\Lambda p,\sigma\rangle\,. \tag{4.22}$$

Likewise, the creation operator transforms as:

$$U(\Lambda)a(p,\sigma)U(\Lambda^{-1}) = \sqrt{\omega(\Lambda\mathbf{p})/\omega(p)}\sum_{\sigma'} D^{(s)}_{\sigma\sigma'}(W(p))a^\dagger(\Lambda p,\sigma). \tag{4.23}$$

Using the transformation properties listed above, one can construct the $(s,0)$ representation field as[29]:

$$\varphi_\sigma(x) = \int \frac{d^3\mathbf{p}}{\sqrt{2\omega(\mathbf{p})}(2\pi)^{3/2}}\sum_{\sigma'}\left[D^{(s)}_{\sigma\sigma'}(B(p))a(p,\sigma)e^{ip\cdot x} + \text{h.c.}\right], \tag{4.24}$$

which has the transformation property:

$$U(\Lambda)\varphi_\sigma(x)U^{-1}(\Lambda) = \sum_{\sigma'} D^{(s)}_{\sigma\sigma'}(\Lambda^{-1})\varphi_{\sigma'}(\Lambda x). \tag{4.25}$$

It is interesting to note that the wave functions appearing in the above field are given by the representations of the boost,

$$u_{\sigma\sigma'}(p) = D^{(s)}_{\sigma\sigma'}(B(p))e^{ip\cdot x}. \tag{4.26}$$

4.3 Propagators

The propagator for the $2s+1$ component fields can be derived in the same fashion as was done in chapter 2. The quantity analogous to the spin-sum is now given by:

$$\pi^{(s)}_{\sigma\sigma'}(p) \equiv D^{(s)}_{\sigma\sigma'}(B(p))\overline{D^{(s)}_{\sigma\sigma'}(B(p))}. \tag{4.27}$$

Note that this is a $(2s+1)\times(2s+1)$ matrix. To evaluate this expression, we need to consider the boost in the $(s,0)$ and $(0,s)$ representations of the Lorentz group. These take the particularly simple form:

$$\begin{aligned} D^{(s)}_{\sigma\sigma'}(B(p)) &= \exp\left[-\theta\mathbf{e}_p\cdot\mathbf{J}^{(s)}\right], \\ \overline{D}^{(s)}_{\sigma\sigma'}(B(p)) &= \exp\left[\theta\mathbf{e}_p\cdot\mathbf{J}^{(s)}\right]. \end{aligned} \tag{4.28}$$

Furthermore, the above matrices are related by[29]:

$$D^{(s)}_{\sigma\sigma'}(B(p)) = \overline{\overline{D}^{(s)}_{\sigma\sigma'}(B(p)^{-1})}. \tag{4.29}$$

The matrix $\pi^{(s)}_{\sigma\sigma'}(p)$ then becomes:

$$\pi^{(s)}_{\sigma\sigma'}(p) = \exp\left[-2\theta\mathbf{e}_p\cdot\mathbf{J}^{(s)}\right]. \tag{4.30}$$

For integer- and half-integer values of s, the above expression can be expressed in terms of a series expansion[29]:

$$\begin{aligned} \pi^{(s)}_{\sigma\sigma'}(q) = (-q^2)^s + \sum_{n=0}^{s-1}\frac{(-q^2)^{s-1-n}}{(2n+2)!}(2\mathbf{q}\cdot\mathbf{J})[(2\mathbf{q}\cdot\mathbf{J})^2-(2\mathbf{q})^2][(2\mathbf{q}\cdot\mathbf{J})^2-(4\mathbf{q})^2]\dots \\ \times[(2\mathbf{q}\cdot\mathbf{J})^2-(2n\mathbf{q})^2][2\mathbf{q}\cdot\mathbf{J}-(2n+2)q_0], \end{aligned} \tag{4.31}$$

for s an integer, and:

$$\begin{aligned} \pi^{(s)}_{\sigma\sigma'}(q) = (-q^2)^k[q_0-2\mathbf{q}\cdot\mathbf{J}] + \sum_{n=1}^{k}\frac{(-q^2)^{k-n}}{(2n+1)!}[2\mathbf{q}\cdot\mathbf{J})^2-\mathbf{q}^2][(2\mathbf{q}\cdot\mathbf{J})^2-(3\mathbf{q})^2]\dots \\ \times[(2\mathbf{q}\cdot\mathbf{J})^2-([2n-1]\mathbf{q})^2][(2n+1)q_0-2\mathbf{q}\cdot\mathbf{J}], \end{aligned} \tag{4.32}$$

for $s=k+1/2$ a half-integer. When no matrix is present, the $(2s+1)\times(2s+1)$ identity matrix is implied. Note that, since the matrix $\pi^{(s)}(q)$ originates from the direct product of the $D(s,0)$ and $D(0,s)$ fields, it belongs itself to the $D(s,s)$ representation of the Lorentz group.

Bi-spinor propagator

The propagator discussed above describes the fields in the $D(s,0)$ and $D(0,s)$ representations respectively. As mentioned earlier, we may combine such fields to form a $2(2s+1)$ component bi-spinor, Ψ_σ. For such a $2(2s+1)$ component field, the following expression was presented in [29]:

$$\Delta^{(s)}_{\sigma\sigma'}(q) = \frac{\Pi^{(s)}_{\sigma\sigma'}(q) + m^{2s}}{q^2+m^2}, \tag{4.33}$$

where the matrix $\Pi^{(s)}_{\sigma\sigma'}(q)$ is given by:

$$\Pi^{(s)}_{\sigma\sigma'}(q) = \begin{pmatrix} 0 & \pi(q) \\ \overline{\pi}(q) & 0 \end{pmatrix}. \tag{4.34}$$

Let us consider a concrete example. For $s = \frac{1}{2}$, the quantity $\pi^{(s)}_{\sigma\sigma'}(q)$ reads:

$$\pi^{(1/2)}_{\sigma\sigma'}(q) = (q_0 - 2\mathbf{q}\cdot\mathbf{J}^{(1/2)}). \tag{4.35}$$

The $s = \frac{1}{2}$ rotation matrices are given by the Pauli matrices:

$$\mathbf{J}^{(1/2)} = \frac{1}{2}\boldsymbol{\sigma}, \tag{4.36}$$

and so:

$$\pi^{(1/2)}_{\sigma\sigma'}(q) = \sigma^\mu_{\sigma\sigma'} q_\mu = q_{\sigma\sigma'}. \tag{4.37}$$

In this case then, the bi-spinor quantity $\Pi^{(1/2)}_{\sigma\sigma'}(q)$ becomes:

$$\Pi^{(1/2)}_{\sigma\sigma'}(q) = \begin{pmatrix} 0 & \sigma^\mu_{\sigma\sigma'} q_\mu \\ \overline{\sigma^\mu}_{\sigma\sigma'} q_\mu & 0 \end{pmatrix} = \gamma^\mu_{\sigma\sigma'} q_\mu = \not{q}, \tag{4.38}$$

giving the familiar result:

$$\Delta^{(1/2)}_{\sigma\sigma'}(q) = \frac{\not{q} + m}{q^2+m^2}. \tag{4.39}$$

The expression (4.37) can be generalised to higher spins by writing,

$$\pi^{(s)}_{\sigma\sigma'}(q) = \Sigma^{\mu_1\ldots\mu_{2s}}_{\sigma\sigma'} q_{\mu_1} \ldots q_{\mu_{2s}}. \tag{4.40}$$

The object $\Sigma^{\mu_1\cdots\mu_{2s}}$ can be thought of as a generalisation of the Pauli matrices. For the bi-spinor, this then defines the generalised $\gamma-$matrices, $\Gamma^{(s)}$:

$$\Gamma^{(s)} = \begin{pmatrix} 0 & \Sigma^{(s)} \\ \overline{\Sigma}^{(s)} & 0 \end{pmatrix}. \tag{4.41}$$

Indeed, this is the same $\Gamma^{(s)}$ as appears in the equations of motion for the bi-spinor.

4.4 Interactions

Let us now consider interactions formed by spinor fields. For simplicity, let us consider interactions not involving any derivatives. In this case, we have to couple the three fields $\varphi_{\sigma_i}(x),\ i = 1, 2, 3$, of spins s_i and helicity σ_i. Here, it helps to think of this problem as coupling three wave functions describing particles of spin s_i and helicity σ_i. Following the rules of addition of angular momentum, such coupling can be achieved using Clebsch-Gordan coefficients. In this particular case, we want to couple the three wave functions in such a way so that they form a singlet state, since we want our interaction to transform as a Lorentz scalar (and therefore also as a rotational scalar, hence a singlet). In order to form a singlet, we most impose the condition:

$$\sum_i \sigma_i = 0, \tag{4.42}$$

and in addition, the spins s_i must satisfy the triangle inequalities:

$$\begin{aligned} |s_1 - s_2| &\leq s_3 \leq s_1 + s_2 \\ |s_1 - s_3| &\leq s_2 \leq s_1 + s_3 \\ |s_2 - s_3| &\leq s_1 \leq s_2 + s_3. \end{aligned} \tag{4.43}$$

These conditions are encoded in the Wigner three-J symbol (see appendix A):

$$\begin{pmatrix} s_1 & s_2 & s_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}, \tag{4.44}$$

which vanishes when the conditions (4.42) and (4.43) are not met. The general three-field interaction involving no derivatives can then be written as[29]:

$$\mathcal{L}_{\text{int}} = g_{123} \sum_{\sigma_i} \begin{pmatrix} s_1 & s_2 & s_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} \varphi^{(s_1)}_{\sigma_1} \varphi^{(s_2)}_{\sigma_2} \varphi^{(s_3)}_{\sigma_3} + \text{h.c.}, \tag{4.45}$$

where g_{123} is a coupling constant and $h.c.$ denotes the hermitian conjugate.

Chapter 5

The Weinberg Formalism

In the previous chapter we considered the construction of Feynman rules for spinor fields in the $D(s,0)$ and $D(0,s)$ representation, and in chapter 2 we considered the construction in the covariant representations. In this chapter, we will consider a formalism due to S. Weinberg[31] in which one constructs Feynman rules for fields in *arbitrary* representations with labels (A,B), $D(A,B)$. While significantly more abstract than the previous frameworks we have conisdered so far, the generality of this formalism makes it powerful.

5.1 General representations of the Lorentz group

Before we can consider the construction of fields and Feynman rules in terms of general representations, we need to properly define what we mean by such representations. To this end, we start by considering the generators of the Lorentz group, J^i and K^i, which generate rotations and boosts respectively. These generators can be combined into a new set of quantities which we will call A^i, B^i:

$$\begin{aligned} A_i &\equiv \frac{1}{2}(J_i + iK_i), \\ B_i &\equiv \frac{1}{2}(J_i - iK_i). \end{aligned} \tag{5.1}$$

From the Lorentz algebra, we can deduce that the above generators must satisfy the algebra:

$$\begin{aligned} [A_i, A_j] &= i\epsilon_{ijk}A_k, \\ [B_i, B_j] &= i\epsilon_{ijk}B_k, \\ [A_i, B_j] &= 0. \end{aligned} \tag{5.2}$$

Since A_i, B_i commutes with each other, the above corresponds to two decoupled algebras of $su(2)$. Following [34], the matrices that satisfy (5.2) can be labelled by indices a, b that run in unit steps from $-A, A$ and $-B, B$ respectively. They are conveniently written as:

$$\begin{aligned} \mathbf{A}_{aba'b'} &= \delta_{bb'}\mathbf{J}^{(A)}_{aa'}, \\ \mathbf{B}_{aba'b'} &= \delta_{aa'}\mathbf{J}^{(B)}_{bb'}. \end{aligned} \tag{5.3}$$

Here, $\mathbf{J}^{(A)} = \{J_1^{(A)}, J_2^{(A)}, J_3^{(A)}\}$ are the spin-matrices which appear e.g. in the quantum mechanical description of a particle of spin A. In terms of the a, b indices, they are given by [34]:

$$(J_3)_{aa'} = a\delta_{aa'}, \tag{5.4}$$

$$(J_1 \pm iJ_2)_{aa'} = \delta_{a',a\pm 1}\sqrt{(A \mp a)(A \pm a + 1)}, \tag{5.5}$$

Since $a = -A, \ldots A$, the above matrices are of dimension $(2A + 1)$, and the matrices which furnish the (A, B) representation of the Lorentz group are of dimension $(2A + 1) \times (2B + 1)$.

A massive particle of spin s can then be embedded into a field $\psi_{ab}^{(AB)}(x)$ which transforms under the (A, B) representation of the Lorentz group, with $A + B = s$. The field is subject to the transformation rule[29]:

$$U(\Lambda)\psi_{ab}^{(AB)}(x)U^{-1}(\Lambda) = \sum_{a'b'} D^{(A)}_{aa'}(\Lambda^{-1})D^{(B)}_{bb'}(\Lambda^{-1})\psi^{(AB)}_{a'b'}(\Lambda x). \tag{5.6}$$

Here, $U(\Lambda)$ is a unitary representation of the Lorentz transformation Λ, and $D^{(A)}_{aa'}(\Lambda)$ is the finite-dimensional $(A,0)$ representation of the Lorentz group (which is just the spin matrices for spins A, B respectively). In the above statement, we have made use of the fact that $(A,B) = (A,0)\otimes(0,B)$. The field can be written as a linear combination of creation- and annihilation operators in the usual way[31]:

$$\psi^{(AB)}_{ab}(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega(p)}} \times \sum_{\sigma}\left(\tau_{ab}(\mathbf{p},\sigma)a(\mathbf{p},\sigma)e^{-ip\cdot x} + \overline{\tau}_{ab}(\mathbf{p},\sigma)b^{\dagger}(\mathbf{p},\sigma)e^{ip\cdot x}\right) \quad (5.7)$$

where the sum is over values of the projection of the spin along the $z-$axis, σ. Here, $a(\mathbf{p},\sigma)$ and $b^{\dagger}(\mathbf{p},\sigma)$ are the creation- and annihilation operators respectively. The object $\tau_{ab}(\mathbf{p},\sigma)$ and its conjugate are simply the generalized notion of a polarisation vector to arbitrary spin and representations.

5.2 External states

Consider the construction of the external states for fields in the general (A,B) representations. To determine them, we can use the transformation properties of the creation- and annihilation operators [31]:

$$U(\Lambda)a(\mathbf{p},\sigma)U^{-1}(\Lambda) = N(\mathbf{p})\sum_{\sigma'} D^{(s)}_{\sigma\sigma'}(W^{-1})a(\Lambda\mathbf{p},\sigma'),$$
$$U(\Lambda)a^{\dagger}(\mathbf{p},\sigma)U^{-1}(\Lambda) = (-1)^{\sigma-s}N(\mathbf{p})\sum_{\sigma'} D^{(s)}_{\sigma\sigma'}(W^{-1})a^{\dagger}(\Lambda\mathbf{p},-\sigma'),$$

where:

$$N(\mathbf{p}) \equiv (\omega(\Lambda\mathbf{p})/\omega(\mathbf{p}))^{1/2}, \quad (5.8)$$

is a normalisation factor and W is the Wigner rotation[31]:

$$W(p) = B^{-1}(\Lambda\mathbf{p})R^{-1}(\Lambda\mathbf{p})\Lambda R(\mathbf{p})B(\mathbf{p}), \quad (5.9)$$

with $B(\mathbf{p})$ a boost along the $z-$axis and $R(\mathbf{p})$ the rotation which aligns the momentum vector $\mathbf{p}$ with the $z-$axis. Inserting into the expansion (5.7) of the field and recalling the transformation rule:

$$U(\Lambda)\psi_{ab}^{(AB)}(x)U^{-1}(\Lambda) = \sum_{a'b'} D_{aa'}^{(A)}(\Lambda^{-1})D_{bb'}^{(B)}(\Lambda^{-1})\psi_{a'b'}^{(AB)}(x),$$

one arrives at the following transformation properties of the function $\tau_{ab}(\mathbf{p}, \sigma)$:

$$\sum_{\sigma'} D_{\sigma\sigma'}^{(s)}(R)\tau_{a'b'}^{\sigma'}(\mathbf{p}) = \sum_{a'b'} D_{aa'}^{(A)}(R)D_{bb'}^{(B)}(R)\tau_{a'b'}^{\sigma}(\mathbf{p}). \tag{5.10}$$

Here, we recognize the definition of the Clebsch-Gordan coefficients:

$$\sum_{\sigma'} D_{\sigma\sigma'}^{(s)}(R)\langle Aa', Bb'|s\sigma'\rangle = \sum_{a'b'} D_{aa'}^{(A)}(R)D_{bb'}^{(B)}(R)\langle Aa', Bb'|s\sigma\rangle, \tag{5.11}$$

and hence:

$$\tau_{ab}^{\sigma}(\mathbf{p}) = \sum_{a',b'} f_{aa',bb'}(\mathbf{p})\langle Aa, Bb|s\sigma\rangle. \tag{5.12}$$

Thus, in the rest frame with $\mathbf{p} = 0$, we have:

$$\tau_{ab}^{\sigma}(\mathbf{0}) = \langle Aa, Bb|s\sigma\rangle, \tag{5.13}$$

To obtain $\tau_{ab}^{\sigma}(\mathbf{p})$ we simply boost, using the general (AB) representation for a boost B, $D_{ab,a'b'}^{(AB)}(B) = D_{aa'}^{(A)}(B)D_{bb'}^{(B)}(B)$:

$$D_{ab,a'b'}^{(AB)}(B) = \left[\exp(-\theta\mathbf{e}_p \cdot J^{(A)})\right]_{aa'} \times \left[\exp(\theta\mathbf{e}_p \cdot J^{(B)})\right]_{bb'}. \tag{5.14}$$

In particular, if $\mathbf{p}$ is in the $z-$direction such that $\mathbf{e}_p = \mathbf{e}_z$, then $\mathbf{e}_p \cdot J^{(A)} = J_3^{(A)} = a\delta_{aa'}$, and the exponential reduces to:

$$D_{ab,a'b'}^{(AB)}(B) = \delta_{aa'}\delta_{bb'}\exp[\theta(b-a)]. \tag{5.15}$$

Using the parameterisation,

$$\begin{aligned} \cosh\theta &= \frac{\omega(\mathbf{p})}{m}, \\ \sinh\theta &= \frac{|\mathbf{p}|}{m}, \end{aligned} \tag{5.16}$$

this becomes:

$$D^{(AB)}_{ab,a'b'} = \delta_{aa'}\delta_{bb'}\left(\frac{|\mathbf{p}| + \omega(\mathbf{p})}{m}\right)^{b-a}. \qquad (5.17)$$

Thus, the function describing the external state of a particle of mass m, spin s and helicity σ (moving in the $z-$direction) is given by the expression:

$$\tau^{\sigma}_{ab}(\mathbf{p}) = \sum_{a'b'}\left(\frac{|\mathbf{p}| + \omega(\mathbf{p})}{m}\right)^{b'-a'} \langle Aa', Bb'|s\sigma\rangle. \qquad (5.18)$$

5.3 Interactions

Consider three massive fields, $\psi^{(A_iB_i)}_{a_ib_i}(x)$, with spins $s_i = A_i + B_i$ and masses m_i ($i = 1, 2, 3$). One can easily generalise the construction of the singlet state discussed in chapter 4 to general representations easily, by imposing the conditions:

$$\sum_i a_i = \sum_i b_i = 0, \qquad (5.19)$$

and:

$$|A_1 - A_2| \leq A_3 \leq A_1 + A_2, \qquad (5.20)$$

etc. The interaction will now include two Wigner-$3J$ symbols[31]:

$$\mathcal{L}_{\text{int}} = g_{123}\sum_{a_ib_i}\begin{pmatrix} A_1 & A_2 & A_3 \\ a_1 & a_2 & a_3 \end{pmatrix}\begin{pmatrix} B_1 & B_2 & B_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \times \psi^{A_1B_1}_{a_1b_1}\psi^{A_2B_2}_{a_2b_2}\psi^{A_3B_3}_{a_3b_3} + \text{h.c.} \qquad (5.21)$$

At this point, we have not specified the properties of the fields, and the above interaction holds for massless as well as massive fields[31]. There is an additional benefit to this interaction, which comes from the general nature of the fields $\psi^{(AB)}_{ab}(x)$, namely that the above interaction also includes any derivative couplings we might construct. To see this, take n derivatives of the field $\psi^{(AB)}_{ab}$:

$$\partial_{\mu_1}\ldots\partial_{\mu_n}\psi^{(AB)}_{ab}(x). \qquad (5.22)$$

The rank n tensor, $\partial_n = \partial_{\mu_1} \ldots \partial_{\mu_n}$ is totally symmetric. Removing its trace, we get a traceless symmetric tensor of rank n, $\tilde{\partial}_n$, which belongs to the $D(\frac{n}{2}, \frac{n}{2})$ representation of the Lorentz group. The quantity,

$$\tilde{\partial}_n \psi_{ab}^{(AB)}(x), \tag{5.23}$$

then belongs to the reducible representation,

$$D(\tfrac{n}{2}, \tfrac{n}{2}) \otimes D(A, B) = \bigoplus_{A'=|A-\frac{n}{2}|}^{A+\frac{n}{2}} \bigoplus_{B'=|B-\frac{n}{2}|}^{B+\frac{n}{2}} D(A', B'). \tag{5.24}$$

If we therefore write the interaction in terms of the representations (A', B') and include a sum over all values for (A', B') which are allowed by the decomposition, the interaction will then include any fields constructed from derivatives. That is, the interaction term of the form,

$$\mathcal{L} = g_{123} \sum_{A_i, B_i} \sum_{a_i, b_i} \begin{pmatrix} A_1 & A_2 & A_3 \\ a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} B_1 & B_2 & B_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \times \psi_{a_1 b_1}^{A_1 B_1} \psi_{a_2 b_2}^{A_2 B_2} \psi_{a_3 b_3}^{A_3 B_3} + \text{h.c.} \tag{5.25}$$

is the most general interaction possible. Note that some of the fields included most necessarily vanish; some vanish on-shell due to the equations of motion imposed on the fields.

5.4 Propagators: Off the mass-shell

We have considered the on-shell states of fields in arbitrary representations as well as the most general constructable interactions including such fields and their derivatives. It remains for us to consider the propagator for such general representation fields. Such a propagator will describe the propagation of a particle created at one point by a field in the representation (A_1, B_1) and annihilated at another point by a field in the representation (A_2, B_2). If the propagating particle is on-shell, the propagator will simply be given by the spin sum:

$$\Delta^{(A_1B_1,A_2B_2)}_{a_1b_1,a_2b_2}(p,s) = i\frac{\mathcal{P}^{(A_1B_1,A_2B_2)}_{a_1b_1,a_2b_2}(p,s)}{p^2+m^2}, \tag{5.26}$$

with the projection operator defined for general representations:

$$\mathcal{P}^{(A_1B_1,A_2B_2)}_{a_1b_1,a_2b_2}(p,s) \equiv \sum_\sigma \tau^{(A_1B_1),\sigma}_{a_1b_1}(\mathbf{p})\overline{\tau}^{(A_2B_2),\sigma}_{a_2b_2}(\mathbf{p}), \tag{5.27}$$

where the superscript s indicates that this describes a particle of spin $s = A_1 + B_1 = A_2 + B_2$ and helicity $\sigma = -s, \cdots + s$. If the propagating particle is not on the mass shell, the propagator is:

$$\Delta^{(A_1B_1,A_2B_2)}_{a_1b_1,a_2b_2}(q) = i\frac{\mathcal{N}^{(A_1B_1,A_2B_2)}_{a_1b_1,a_2b_2}(q)}{q^2+m^2}, \tag{5.28}$$

where $\mathcal{N}^{(A_1B_1,A_2B_2)}_{a_1b_1,a_2b_2}(q)$ is a polynomial in q which reduces to the projection operator for q on the mass shell:

$$\mathcal{N}^{(A_1B_1,A_2B_2)}_{a_1b_1,a_2b_2}(q,s)|_{q^2=m^2} = \mathcal{P}^{(A_1B_1,A_2B_2)}_{a_1b_1,a_2b_2}(q,s). \tag{5.29}$$

More specifically, the above quantity must be a polynomial of rank $2n \leq s$ in the four vector q_μ, and can without loss of generality be expressed in terms of an irreducible, symmetric tensor, belonging to the (n,n) representation of the Lorentz group. Thus, to re-cast the numerator in terms of an object which reduces to the projection operator for q on the mass shell, we may re-couple the representations (A_1, B_1) and (A_2, B_2) to an "off-shell" representation, (n,n) (with corresponding indices $\pm\nu$). First, one can take the projection operator and expand the wave functions according to the expression (5.18) to obtain:

$$\begin{aligned}\mathcal{P}^{(A_1B_1,A_2B_2)}_{a_1b_1,a_2b_2}(q,s) = &\sum_\sigma \sum_{a_1'b_1',a_2'b_2'} D^{(A_1)}_{a_1a_1'}D^{(B_1)}_{b_1b_1'}D^{(A_2)}_{a_2a_2'}D^{(B_2)}_{b_2b_2'} \\ &\times \left(\frac{|\mathbf{p}|+\omega(\mathbf{p})}{m}\right)^{b_1'+b_2'-a_1'-a_2'} \langle A_1a_1', B_1b_1'|s\sigma\rangle\langle A_2a_2', B_2b_2'|s(-\sigma)\rangle.\end{aligned}$$

(The argument of the representation matrices have been dropped for compactness). Re-coupling this expression can then be done applying the re-coupling rules for the representation matrices and Clebsch-Gordan coefficients[59]:

$$\begin{aligned}\sum_{\sigma}(-1)^{s+\sigma}\langle A_1a_1', B_1b_1'|s\sigma\rangle\langle A_2a_2', B_2b_2'|s(-\sigma)\rangle \\ = (2s+1)\sum_{j,\lambda}(-1)^{A_1+B_2+s-\lambda}W(A_1B_1A_2B_2;sj) \\ \times \langle A_1a_1', A_2a_2'|j\lambda\rangle\langle B_1b_1', B_2b_2'|j(-\lambda)\rangle.\end{aligned}$$

where we have defined the Racah-W coefficient (see appendix A). The rotation matrices also need to be expressed using the new representation, and this is done via the Clebsch-Gordan-series (see e.g. [35] p. 230):

$$\sum_{a_1'a_2'}\langle A_1a_1', A_2a_2'|j\lambda\rangle D^{(A_1)}_{a_1a_1'}D^{(A_2)}_{a_2a_2'} = \sum_{\lambda'}\langle A_1a_1, A_2a_2|j\lambda'\rangle D^{(j)}_{\lambda\lambda'}, \tag{5.30}$$

and likewise for $A \to B$ and $\lambda, \lambda' \to -\lambda, -\lambda'$. Inserting into the projection operator yields the off-shell polynomial:

$$\begin{aligned}\mathcal{N}^{(A_1B_1,A_2B_2)}_{a_1b_1,a_2b_2}(q) = (2s+1)\sum_{j,\lambda,\lambda',\lambda''} W(A_1B_1A_2B_2;sj) \\ \times (-1)^{A_1-B_2-s-\lambda}\langle A_1a_1, A_2a_2|j\lambda'\rangle\langle B_1b_1, B_2b_2|j\lambda''\rangle \\ \times \left(\frac{|\mathbf{q}|+q_0}{\sqrt{-q^2}}\right)^{-2\lambda} D^{(j)}_{\lambda'\lambda}D^{(j)}_{\lambda''(-\lambda)}.\end{aligned}$$

Here, we have taken the momentum off-shell, $\mathbf{p} \to \mathbf{q}$, $\omega(\mathbf{p}) \to q_0$, and therefore replaced the mass m^2 with $-q^2$. In addition, we have used that $b_1'+b_2' = -(a_1'+a_2') = -\lambda$ (enforced by the Clebsch-Gordan coefficients in the re-coupling rule). In the above form, the numerator structure can be directly calculated, provided that we have the representation matrices, $D^{(j)}_{\lambda\lambda'}(R)$. For spins $s = 1/2, 1$ these are well-known expressions (see appendix A). For higher spins it is possible to either construct them recursively (using the Clebsch-Gordan series[59]), and so it is in principle possible, though in a cumbersome way[1]. Luckily, it is possible to relate the $\mathcal{N}$ numerator structure to the on-shell projection operators considered in chapter 2. To do this, one may exploit the orthogonality condition of the Racah-coefficients and solve to jth order in q to obtain[31]:

$$\mathcal{N}^{(A_1B_1,A_2B_2)}_{a_1b_1,a_2b_2}(q) = \sum_{s'} c_s(q,s')\mathcal{P}^{(A_1B_1,A_2B_2)}_{a_1b_1,a_2b_2}(q,s'), \tag{5.31}$$

where the projection operator $\mathcal{P}$ is evaluated in the off-shell momentum q, and the coefficients $c_s(q,s')$ are given by the expression[31]:

$$\begin{aligned} c_s(q,s') = (-1)^{s'-s}(2s+1) \\ \times \sum_j W(A_1B_1A_2B_2;sj)W(A_1B_1A_2B_2;s'j)\left(\frac{-q^2}{m^2}\right)^j . \end{aligned} \tag{5.32}$$

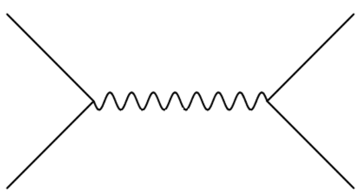
As always, the sum here is over all values of j for which the Racah-W coefficients do not disappear. The up-shot of the expression (5.31) is that the numerator of the off-shell propagator is given by a linear combination of lower spin projection operators, with the coefficients given by $c_s(q,s')$[31]. Importantly, the coefficients are such that, for $q^2 = -m^2$, the expression (5.31) reduces to $\mathcal{P}^s(q)$.

1: Explicit versions of the rotation formula analogous to the $s = 1/2, 1$ cases exists, see e.g. [60].

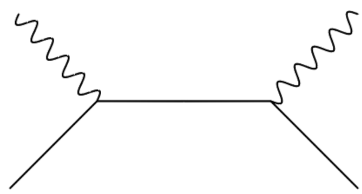
Chapter 6

Discussion

We will end this review of higher spin formalisms by briefly discussing how they compare, particularly in the application of calculating tree level scattering amplitudes. In general, we have two distinct types of diagrams; those that involve the exchange of a photon and those that involve the exchange of a massive spin s particle, i.e. Compton scattering. Such diagrams



((A)) Exchange diagram.



((B)) Compton scattering diagram.

FIGURE 6.1: Generic tree level diagrams for massive spin s particles scattering with photons. Solid lines represent massive spin s particles, while wavy lines represent a photon.

can, in principle, be evaluated using the Feynman rules introduced in chapter 2. Indeed, for lower spins (up to and including $s = \frac{3}{2}$), the above diagrams may be evaluated, and yield sensible results[28]. For arbitrary spins however, we should expect to encounter problems in the high-energy limit, particularly for the Compton amplitude, in which the projection operator is involved.

For the spinor helicity formalism, the issue persists regarding the Compton scattering diagram, where spurious poles develop for spins $s > \frac{3}{2}$. The exchange diagram however, may be evaluated using BCFW recursion, and for lower spins the results can be shown to agree with amplitudes calculated using the covariant formalism. This is a good indication that the general three-point amplitudes encode the correct non-minimal coupling terms for all spins. An interesting example of this is the application to scattering amplitudes in gravity: the amplitude for the process in which four spin s particles exchange a graviton,

$$\mathcal{M}_4 = \mathcal{M}_3(\mathbf{1}, \mathbf{2}, \hat{P}^{\pm}_{12}) \frac{i}{s} \mathcal{M}_3(-\hat{P}^{\mp}_{12}, \mathbf{3}, \mathbf{4}), \tag{6.1}$$

contains information on the scattering of two spinning black holes (of equal masses and spin) to first order in pertubation theory (i.e. order $\mathcal{O}(G)$). The classical information is extracted by taking a specific limit; this procedure requires taking the particle spin $s \to \infty$[1] potential[24, 25, 26].

In terms of the general representation scheme, all fields involved will be labelled by their representations, (A_i, B_i) (see figure 6.2). The procedure

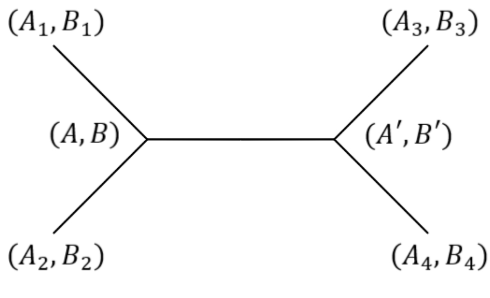


FIGURE 6.2: Generic tree level diagram involving general representations.

for evaluating such a diagram is then analogous to the covariant formalism, except we use the general coupling expressions discussed in chapter 5. More

1: For this reason, it does not matter whether the particle spin s is an integer or half-integer in the context of black hole physics.

explicitly, the diagram in figure 6.2 reads:

$$\sum_{a_i,b_i} \psi_1\psi_2 V_{12}(A,B)\frac{i\mathcal{N}^{(AB,A'B')}_{ab,a'b'}(q)}{q^2+m^2} V_{34}(A',B')\psi_3\psi_4, \tag{6.2}$$

where $\psi_i = \psi^{(A_iB_i)}_{a_ib_i}$ and we have introduced the shorthand notation:

$$V_{ij}(A,B) = \begin{pmatrix} A_i & A_j & A \\ a_i & a_j & a \end{pmatrix} \begin{pmatrix} B_i & B_j & B \\ b_i & b_j & b \end{pmatrix}. \tag{6.3}$$

As discussed in the previous chapter, this vertex contains all possible coupling terms one may construct with the fields, $\psi^{(AB)}_{ab}$. Since we assume that the difficulties associated with coupling higher spin fields to e.g. electromagnetism may be solved by adding sufficiently complicated interaction terms, difficulties such as the Velo-Zwanziger problem are not relevant to this formalism. In addition, since the off-shell numerator structure $\mathcal{N}$ contains off-shell terms not present in the projection operator $\mathcal{P}$, given by the linear combination in equation 5.31. Inserting this expression, one may rewrite the diagram in figure 6.2 as a decomposition into lower spin diagrams, corresponding to the off-shell lower spin components of the propagator.

It is worth noting here, that propagators with the correct off-shell terms can be obtained in the covariant formalism, examples in the literature include the $s = 2$[44] and $s = \frac{5}{2}$[28]. The procedure for arbitrary spin however, seems to be overly complicated due to the nature of the Singh Hagen Lagrangians. It would however be interesting to compare the Weinberg propagator for e.g. $s = 2$ with the result found in the covariant literature.

Chapter 7

Conclusion and outlook

In this thesis, we have reviewed a number of different frameworks for interacting massive higher spin theories. These frameworks largely differ by which choice of representation of the Lorentz group one picks. In the Lagrangian formulation, related to the $D(\frac{s}{2}, \frac{s}{2})$ and $D(\frac{s+1}{2}, \frac{s}{2}) \oplus D(\frac{s}{2}, \frac{s+1}{2})$ representations, we saw that massive higher spin fields are described by totally symmetric rank s ($s - 1/2$) tensors (tensor spinors). Superfluos degrees of freedom are removed by additional on-shell conditions, the introduction of which necessitate the presence of auxiliary fields in the Lagrangian. These auxiliary fields complicates the procedure of obtaining the full, off-shell propagator for massive higher spin fields. When interactions are considered, the minimal coupling procedure is insufficient, and non-minimal interactions must be introduced. For spins $s = 1, \frac{3}{2}$ we considered such interactions explicitly.

Using the massive spinor helicity formalism, we saw how the construction of three-point amplitudes is possible. The simplest possible structure for two massive spin s coupling to a photon (or graviton) was considered, and we saw how these special three-point amplitudes can be used to construct the Compton scattering amplitude, which is badly behaved for spins $s > \frac{3}{2}$. We then considered another approach which describes massive spin s states in terms of spinor fields belonging to the $D(s, 0)$ and $D(0, s)$ representations, that encode exactly $2s + 1$ degrees of freedom. We reviewed how Feynman rules may be constructed for such fields, including an off-shell propagator. Importantly, when combined into bi-spinors, we saw how the $s = \frac{1}{2}$ propagator reproduces the well-known fermion propagator.

The generalisation of the spinor field formalism to arbitrary $D(A, B)$ representations of the Lorentz group was then considered. Since the representations are kept arbitrary, interactions in this formalism may be constructed to automatically incorporate all possible non-minimal coupling terms, of arbitrary complexity. By re-coupling the representations in the projection operator for fields in the representations $D(A, B)$ and $D(A', B')$, we saw how the general off-shell propagator for a massive spin s field may be written as a linear combination of lower spin field projection operators.

The general representation framework suffers none of the drawbacks of the previous formalisms: interactions automatically contain all relevant terms, and the full, off-shell propagator may be readily constructed. The framework does however present its own challenges: the interactions contain more terms than necessary (e.g. terms that vanish due to the equations of motion), and the relation to e.g. the covariant language is not trivial.

For future developments in this framework, one would ideally want a complete map between e.g. the covariant formulation and the Weinberg formalism. Once this is achieved, it may be possible to apply the Weinberg formalism to problems such as the graviton Compton scattering amplitude for two massive spin s particles.

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Appendix A

Coupling coefficients

In this appendix we give some details on the conventions and properties related to the Wigner-3j symbols and Racah-Coefficients used throughout the thesis.

Clebsch-Gordan coefficients

The Clebsch-Gordan coefficients appear when we couple to states $|s_1, \sigma_1\rangle$, $|s_2, \sigma_2\rangle$ to form a third, $|s, \sigma\rangle$[35]:

$$|s_3, \sigma_3\rangle = \sum_{\sigma_1,\sigma_2} |s_1, \sigma_1\rangle |s_2, \sigma_2\rangle \underbrace{\langle s_1\sigma_1, s_2\sigma_2| \, s_3\sigma_3\rangle}_{\text{Clebsch-Gordan coefficient}} . \tag{A.1}$$

They are orthonormal, in the sense that:

$$\begin{aligned}
&\sum_{\sigma_1,\sigma_2} \langle s_1\sigma_1, s_2\sigma_2| \, s\sigma\rangle \langle s_1\sigma_1, s_2\sigma_2| \, s'\sigma'\rangle = \delta_{ss'}\delta_{\sigma\sigma'} \\
&\sum_{s_3,\sigma_3} \langle s_1\sigma_1, s_2\sigma_2| \, s_3\sigma_3\rangle \langle s_1\sigma_1', s_2\sigma_2'| \, s_3\sigma_3\rangle = \delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'}, \\
&\sum_{\sigma_1,\sigma_2} | \langle s_1\sigma_1', s_2\sigma_2'| \, s_3\sigma_3\rangle|^2 = 1,
\end{aligned} \tag{A.2}$$

and obey the symmetry relation[34]:

$$\langle s_1\sigma_1, s_2\sigma_2|s_3\sigma_3\rangle = (-1)^{s_1+s_2-s}\langle s_1(-\sigma_1)s_2(-\sigma_2)|s_3(-\sigma_3)\rangle. \tag{A.3}$$

Since the Clebsch-Gordan coefficients vanish unless $\sigma_1 + \sigma_2 + \sigma_3 = 0$, the condition $\sigma_3 = -(\sigma_1 + \sigma_2)$ is always implied, and we may suppress the σ_3 index for convenience:

$$\langle s_1\sigma_1, s_2\sigma_2|s_3\sigma_3\rangle = \langle s_1\sigma_1, s_2\sigma_2|s_3\rangle. \tag{A.4}$$

Wigner 3j symbols

The Wigner 3j symbol is a quantity which describes the coupling of three spins to form a rotational scalar. In terms of Clebsch-Gordan coefficients, it reads:

$$\begin{pmatrix} s_1 & s_2 & s_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} = \frac{(-1)^{s_2 - s_1 + \sigma_3}}{\sqrt{2s+1}} \langle s_1\sigma_1, s_2\sigma_2|s_3\sigma_3\rangle. \tag{A.5}$$

Note that some sources, e.g. [35] defines the Wigner 3j symbol such that $\sigma_1 + \sigma_2 = \sigma$. We use the convention $\sigma_1 + \sigma_2 = -\sigma$. The Wigner 3j symbols are symmetric under even permutations of the columns:

$$\begin{pmatrix} s_1 & s_2 & s_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} = \begin{pmatrix} s_3 & s_1 & s_2 \\ \sigma_3 & \sigma_1 & \sigma_2 \end{pmatrix} = \begin{pmatrix} s_2 & s_3 & s_1 \\ \sigma_2 & \sigma_3 & \sigma_1 \end{pmatrix}, \tag{A.6}$$

and symmetric under odd permutations up to a phase:

$$\begin{pmatrix} s_1 & s_2 & s_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} = (-1)^{s_1+s_2+s_3} \begin{pmatrix} s_2 & s_1 & s_3 \\ \sigma_2 & \sigma_1 & \sigma_3 \end{pmatrix}, \tag{A.7}$$

and so on.

Racah W-coefficients

The Racah W-coefficients arise in the coupling of three angular momenta (as opposed to two). The coefficient $W(s_1, s_2, s_3, s_4|s_5, s_6)$ is defined in terms of the Clebsch-Gordan coefficients[59]:

$$W(s_1 s_2 s_3 s_4|s_5 s_6) = \frac{1}{\sqrt{(2s_5+1)(2s_6+1)}}$$
$$\times \sum_{\sigma,\sigma'} \langle s_1\sigma s_2\sigma'|s_5\rangle \langle s_5(\sigma+\sigma'), s_4\sigma''|s_3\rangle \langle s_2\sigma', s_4\sigma''|s_6\rangle \langle s_1\sigma, s_6(\sigma'+\sigma'')|s_3\rangle.$$

Here, $\sigma, \sigma', \sigma''$ are all dummy labels. The Racah W-coefficients also satisfy the orthogonality condition:

$$(2n+1)\sum_m (2m+1) W(s_1 s_2 s_3 s_4|mn) W(s_1 s_2 s_4 s_4|mn') = \delta_{nn'}. \tag{A.8}$$

As an alternative representation, one may also express the Racah W-coefficient in terms of the Wigner 6j symbol:

$$W(s_1 s_2 s_3 s_4|s_5 s_6) = (-1)^{s_1+s_2+s_3+s_4} \underbrace{\begin{Bmatrix} s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 \end{Bmatrix}}_{\text{Wigner 6j symbol}}. \tag{A.9}$$

Appendix B

Explicit rotation matrices

The rotation matrices appear in the definition of projection operators in the Weinberg formalism. We here give the explicit forms for $s = 1/2, 1$.

s = 1/2

For $s = 1/2$, the rotation matrix $D(R(p))$ which aligns the momentum vector $\mathbf{p}$ with the $3-$axis is given by the expression:

$$D^{(1/2)}_{\sigma\sigma'}(R(p)) = \exp\big[-i\phi\mathbf{n}\cdot\mathbf{J}^{(1/2)}\big]_{\sigma\sigma'}, \tag{B.1}$$

with,

$$\begin{aligned} \mathbf{J}^{(1/2)} &= \frac{1}{2}\boldsymbol{\sigma}, \\ \mathbf{n} &= \frac{\mathbf{e}_1 p_2 - \mathbf{e}_2 p_1}{\sqrt{p_1^2+p_2^2}}, \\ \cos\phi &= \frac{p_3}{|\mathbf{p}|}. \end{aligned} \tag{B.2}$$

The exponential can be expanded as[35]:

$$D^{(1/2)}_{\sigma\sigma'}(R(p)) = \cos\tfrac{\phi}{2}\delta_{\sigma\sigma'} - i\sin\tfrac{\phi}{2}(\mathbf{n}\cdot\boldsymbol{\sigma})_{\sigma\sigma'}. \tag{B.3}$$

Inserting $\phi = \cos^{-1}\frac{p_3}{|\mathbf{p}|}$ and using the trigonometric identities,

$$\begin{aligned} \cos\big(\tfrac{1}{2}\cos^{-1}x\big) &= \sqrt{\tfrac{1}{2}(1+x)}, \\ \sin\big(\tfrac{1}{2}\cos^{-1}x\big) &= \sqrt{\tfrac{1}{2}(1-x)}, \end{aligned} \tag{B.4}$$

one obtains:

$$D^{(1/2)}_{\sigma\sigma'}(R(p)) = \\ \sqrt{\tfrac{1}{2}(1+\tfrac{p_3}{|\mathbf{p}|})}\delta_{\sigma\sigma'} - \frac{i}{\sqrt{p_1^2+p_2^2}}\sqrt{\tfrac{1}{2}(1-\tfrac{p_3}{|\mathbf{p}|})}(\sigma_1 p_2 - \sigma_2 p_1)_{\sigma\sigma'}.$$

The indices σ, σ' take the values $\pm\frac{1}{2}$, so the above expressions denotes the elements of a 2×2 matrix.

s=1

The spin matrices for $s = 1$ are (in the basis in which J_3 is diagonal [59]):

$$J_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Like in the $s = 1/2$, the exponential can be expanded, giving:

$$D^{(1)}_{\sigma\sigma'}(R(p)) = \delta_{\sigma\sigma'} + i\sin\phi\,\mathbf{n}\cdot\mathbf{J}^{(1)} - (\mathbf{n}\cdot\mathbf{J}^{(1)})^2(1-\cos\phi). \tag{B.5}$$

With:

$$\sin\phi = \frac{\sqrt{p_1^2+p_2^2}}{|\mathbf{p}|}, \quad \cos\phi = \frac{p_3}{|\mathbf{p}|}, \tag{B.6}$$

this becomes:

$$D^{(1)}_{\sigma\sigma'}(R(p)) = \delta_{\sigma\sigma'} + i\frac{\sqrt{p_1^2+p_2^2}}{|\mathbf{p}|}(J_1 p_2 - J_2 p_1)_{\sigma\sigma'} \\ - \left(1 - \frac{p_3^2}{|\mathbf{p}|^2}\right)(J_1 p_2 - J_2 p_1)^2_{\sigma\sigma'}.$$

Appendix C

Contracted propagators and center of mass decomposition

The on-shell projection operator for higher spin particles displays a particularly nice feature in the center of mass frame, which we display here. To start, a way of displaying the properties of the higher spin projection operators is by considering them with all indices contracted with auxiliary vectors k, k' [28, 45]:

$$\mathcal{P}(k,k') \equiv k_{\mu_1} \dots k_{\mu_s} k'_{\nu_1} \dots k'_{\nu_s} \mathcal{P}^{\mu_1 \dots \mu_s, \nu_1 \dots \nu_s}. \tag{C.1}$$

In this way, the bosonic propagator reduces to a scalar while the fermionic propagator reduces to a 2×2 matrix. The symmetry of $\mathcal{P}$ in the indices μ, ν then allows us to express the un-contracted projector as:

$$\mathcal{P}^{\mu_1 \dots \mu_s, \nu_1 \dots \nu_s} = \frac{1}{(s!)^2} \frac{\partial}{\partial k_{\mu_1}} \dots \frac{\partial}{\partial k_{\mu_s}} \frac{\partial}{\partial k'_{\nu_1}} \dots \frac{\partial}{\partial k'_{\nu_s}} \mathcal{P}(k,k'). \tag{C.2}$$

If we want to, we can also obtain a partially contracted projection operator with arbitrary amounts of indices by differentiating an appropriate amount of times. In particular, the trace of the projection operator becomes:

$$\eta \cdot \mathcal{P}^{(s,s)} \to \partial_k^2 \mathcal{P}(k,k'). \tag{C.3}$$

Working with $\mathcal{P}(k,k')$ is somewhat more efficient, as the projection operator (when appearing in on-shell amplitudes) will always be contracted on either side by momenta and polarisation vectors. In addition, it makes certain calculations easier. As an example of this, we can consider the expansion of the projection operator in the rest frame in terms of Legendre polynomials. In the rest-frame, we have $p_\mu = (\mathbf{0}, m)$. The transversality condition then means that all time-components of $\mathcal{P}^{(s,s)}$ must vanish. Denote the rest-frame projection operator by π. Then:

$$\mathcal{P}^{\mu_1 \ldots \mu_s, \nu_1 \ldots \nu_s} \to \pi^{m_1 \ldots m_s, n_1 \ldots n_s}, \tag{C.4}$$

where $m, n = 1, 2, 3$. While the contracted bosonic projection operator is a lorentz scalar, the contracted rest-frame projection operator:

$$\pi(\mathbf{k}, \mathbf{k}'), \tag{C.5}$$

(note that the vectors are now three-vectors) is a rotational scalar. Because it is a rotational scalar, we can make the ansatz:

$$\pi(\mathbf{k}, \mathbf{k}') = (|\mathbf{k}||\mathbf{k}'|)^s \pi_s(\cos\theta), \tag{C.6}$$

where π_s is a degree-s polynomial in the variable:

$$\cos\theta = \frac{\mathbf{k} \cdot \mathbf{k}'}{|\mathbf{k}||\mathbf{k}'|}. \tag{C.7}$$

The tracelessness condition now implies:

$$\nabla_k^2 \pi_s = 0. \tag{C.8}$$

Writing ∇^2 in polar coordinates, we are then led to conclude that π_s is proportional to the Legendre polynomials:

$$\pi_s(\cos\theta) = a_s P_s(\cos\theta). \tag{C.9}$$

The constant a_s can be determined by imposing the projection operator condition $\pi^2 = \pi$[45]:

$$a_s = \frac{s!}{(2s-1)!!}. \tag{C.10}$$