Effective Field Theories of Post-Newtonian Gravity

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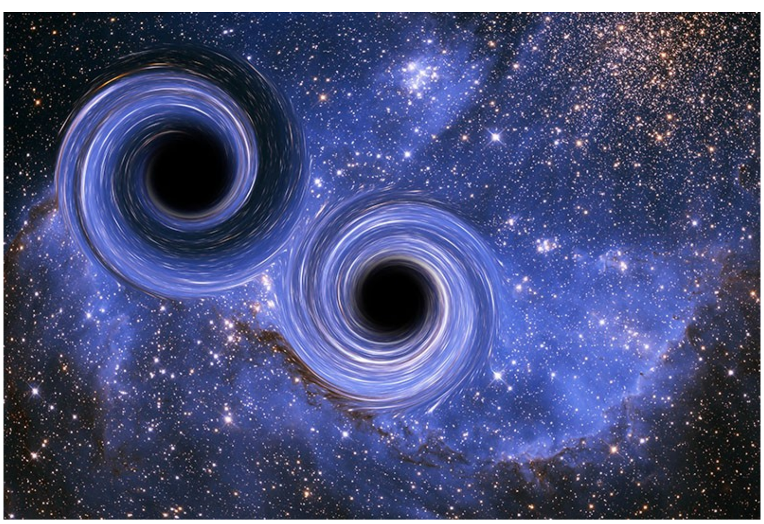

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Summary

Chapter 1

Introduction

In this work we tackle the physical problem of a coalescent binary system of compact objects, namely black holes (BHs) and neutron stars. This physical problem has gained a remarkable attention because of its importance to the testing of theories in gravity, and the predictions of gravitational waves (GWs), especially since the detection of GWs made by the Advanced LIGO detectors in 2015, [1]. Because of that, it has been of great importance to push the frontiers of high precision theoretical computations to help with the predictions which models the GWs templates.

The binary system evolution comprises three distinguished phases: the inspiral phase, when the two objects are inspiralling towards each other and have non-relativistic velocities, the merger, when the two objects get closer beyond approximately the innermost stable orbit of a BH and reach relativistic velocities, and the ringdown phase, in which, for BHs through quasinormal modes of oscillations, the two BH settle down to a rotating Kerr BH. The phase we will mostly be concerned in this work is the inspiral phase, to which the post-Newtonian approximation is applied with great success, meanwhile for the other two phases the PN approximation breaks down since we are not dealing with small velocities anymore, and we need to resort to other methods.

There are different formulations used to study the other phases of the binary i.e the merger and ringdown phases. For these two phases it is not viable to use the post-Newtonian approximation, as mentioned before. Therefore they are currently treated via numerical simulations, and, for the ringdown phase, we can also use BH perturbation theory and self-force formalism. In figure 1.1 we show a graph with the relevant methodologies used for the study of a coalescent binary system depending on the mass ratio of the two compact objects and the compactness of the system, which is evaluated by $\frac{r}{M}$, with $M = m_1 + m_2$ and r being the typical size of the system.

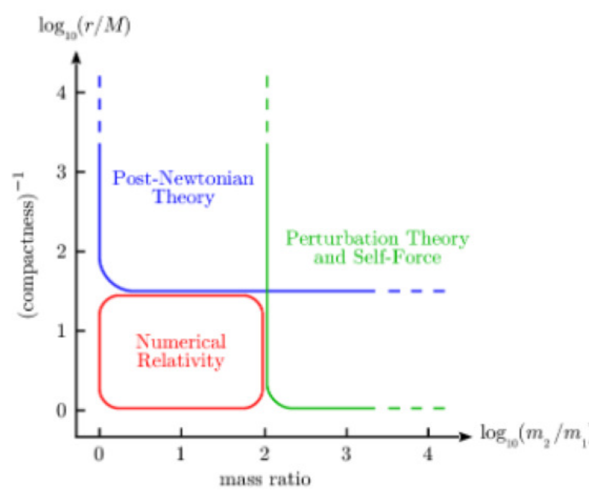

Figure 1.1: The different methods to model GW templates of a coalescent binary according to the mass ratio and compactness. Credit: A. Le Tiec.

To what concerns the predictions of the inspiral phase using a PN approximation, a good approach was that of the effective field theory (EFT). The EFT framework is a universal one, which is great for describing systems in which we have a distinguishable separation of scales, as it is the case for our binary system of compact objects. Even though EFT was conceived in the realms of quantum field theory (QFT), we may use it whether or not the small perturbative parameter is related to the quantum parameter, $\hbar$. In fact, EFT shares the powerful toolbox of that in QFT, as Feynman diagrams, renormalization, regularization and such, which will be of great efficacy when computing the interaction between the two compact objects in the binary.

The EFT approach for the inspiralling binary was formulated in [2], see also [3], where it did not yet take the spin component of the BH into consideration. The description of spin effects are though of great relevance to gravity, quantum chromodynamics (QCD) and astrophysics. Following that, a first attempt to develop a formulation to spinning objects was in [4]. Based on [5], [6], [7], [8] a formulation for EFTs including spins was successfully developed in [9], [10], which enabled a significant push in high precision computations with spins to many sectors, as we can see in figure 1.2, which shows the state-of-the-art PN high precision conservative dynamics for inspirally compact binaries. The EFT approach including the formulation in [9] was comprehensively reviewed in [10], which presented the big perspective of the field in a didactic way. Subsequently, a sequence of works done in the spin conservative sectors were computed; the complete up to NLO and up to quadratic-in-spin sector was presented in [9], the NNLO spin-orbit and spin-squared sectors were presented in [11], [12], respectively, and the complete state-of-the-art with spins at 4PN order was presented in [13].

Following up from [14], that presents the full LO sector for cubic-in-spin and quartic-in-spin sectors, we computed during this thesis the cubic-in-spin at NLO interaction, that enters at 4.5PN order, [15]. The section was computed manually and also by using the code EFTofPNG available at `https://github.com/miche-levi/pncbc-eftofpng`, [16].

Several recently published works were computed after the cubic-in-spin NLO computation, pushing the frontiers in both PN orders and spin orders. The NNNLO spin-orbit and spin-squared interactions were presented in [17] and [18], and the NLO quartic-in-spin interaction sector was presented in [6].

l \ n	(N^0)LO	$N^{(1)}$LO	N^2LO	N^3LO	N^4LO
S^0	1	0	3	0	25
S^1	2	7	32	**174**	
S^2	2	2	**18**	**52**	
S^3	4	**24**			
S^4	3	**5**			

Figure 1.2: The complete state-of-the-art of PN orbital dynamics of generic compact binaries. The PN corrections enter at the order $n + l + Parity(l)/2$, with the parity 0 or 1 for even or odd l, respectively. The sectors with the entries in boldface have been addressed in [15], [17], [18], [12] and [6], and were all derived for the first time based on the formulation from [9]. The quartic in spin at LO was computed in [14]. The entries in the table indicate the loop computational scale within our framework. Credit: M. Levi, [6].

The computation of the Feynman rules and diagrams are made using the EFT theories of PN gravity, which makes use of the Kaluza-Klein reduction of spacetime, [19], [20]. All the computations in the theory that will follow were made within the GR for classical gravity framework.

In the work that follows we will work with the convention of a mostly negative Minkowski metric $\eta = diagonal(1, -1, -1, -1)$, the velocity of light as well as the reduced Planck constant are taken as $c = \hbar = 1$, such that the Planck mass, m_p, and the gravitational constant G are normalized as $m_p^2 \equiv \frac{1}{32\pi G}$. Normally the Greek indices will refer to general spacetime coordinates, while Latin indices will refer to locally flat frames.

In a less straightforward sense, the attempt of arriving at a complete theory of gravity across all scales in order to eventually facilitate us reach a theory of quantum gravity has been also a motivating factor. QFT has been an amazing tool for the treatment and predictions of the Standard Model. Through the years, the matching between experimental data and the theoretical results QFT provided us has been a strong motivation to study this computational tool and develop it even more. The Standard Model is a theory of fundamental particles and their interactions, to be more specific, the study of the interaction of particles that obeys the Electro-Weak and the Strong forces. Each of those has its own theory, respectively, Quantum Electrodynamics and Quantum Chromodynamics. Unfortunately there is a missing piece in this theory, the gravitational force. The Standard Model doesn't seem to give a good description of gravity and that leaves a hard task for us, theoretical physicists, that is to find a theory that unifies the description of all known forces in nature.

It is well known today that the theory of gravity is a non-renormalizable theory. For a theory to be what is called a *renormalizable* theory, it means that it is possible to deal with the divergences that arise in the Lagrangian in a certain limit of energy, the UV. We can do this by either redefining the coupling constants in the Lagrangian, or by adding terms to the Lagrangian, known as the counter terms. If we can handle all the divergences by adding a finite number of counter terms, the theory is still renormalizable, but if we need an infinite number of counter terms, the theory is said to be non-renormalizable, [21], [22], [23]. For the latter case, effective field theories can be used by means of an ultraviolet cutoff to establish an energy scale where our theory will work, enabling us to make predictions for our theory with high accuracy. Exploring this EFT frame could be a possible path to arrive at a complete theory of gravity and perhaps all fundamental forces.

This thesis is organized as follows. In section 2.1 we start by analyzing the construction of a non-spinning point-particle EFT through the bottom-up approach, where the Wilson coefficients will be introduced as well as the new operators after the effective action construction procedure is made as we integrate out the strong modes of the gravitational field $g_{\mu\nu} = g^s_{\mu\nu} + \bar{g}_{\mu\nu}$, where $g^s_{\mu\nu}$ denotes the strong modes. We discuss both minimal and non-minimal coupling terms of the action, the latter accounts for the finite size effects of the object and is build with the higher mass-induced multipoles.

In section 2.2 we introduce the spin of the compact object to the problem, and start by discussing the degrees of freedom (DOFs) and symmetries of our system. Afterwards we move on to build the minimal and non-minimal coupling of the spinning case effective action of one single compact object. In order to do that we also discuss the gauge fixing of the rotational variables that were introduced.

In section 2.3 we finally arrive to the effective action of the composite particle, which is composed by the two compact objects. In this section we are interested in integrating out the orbital modes of the metric, $H_{\mu\nu}$, where we decomposed the metric as $\bar{g}_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu} + \tilde{h}_{\mu\nu}$, and where $H_{\mu\nu}$ denotes the orbital modes and $\tilde{h}_{\mu\nu}$ denotes the radiation modes. To do that we must first disentangle the field DOFs from the particle worldline DOFs, and to fix all rotational gauges. That is done with a factorization of the worldline tetrad introduced in section 2.2.

Section 3 refers to the results obtained in the master's project, which was the full computation for the NLO cubic-in-spin interaction that enters at the 4.5PN order. There we present the paper which was the essence of this work, [15], containing all the Feynman diagrams for the interactions of the two compact objects, the new type of cubic-in-spin couplings that appear at this order because of dependence in spin in the four-momentum of the objects, and the total potential of this sector.

Finally in section 4, we discuss the conclusions of the work, the relevance of the results and what could come next in terms of research in this field.

1.1 EFT for binary system and hierarchy of scales

As mentioned before, EFT is a great approach for our gravitational binary system since we can distinguish three very different scales in the problem. The difference in scale is characterised by the ratio between scales. The smaller scale would be the scale of a single object (say a black-hole or a neutron star by itself), r_s. Following next we have the scale of the orbital separation between the two objects of our binary system, the composite, denoted by r. And finally we have the scale of the wavelength of radiation that is emitted by the inspiraling binary, λ. The scaling follows as $r_s <<< r << \lambda$. Each compact object will have a characteristic mass, m and a Schwarzschild radius $Gm \sim r_s$. Also, according to the Virial theorem, for non-relativistic velocities $v << 1$, we have that $\frac{Gm}{r} \sim v^2$, with r being the radius of the orbital scale. At the radiation scale, we have that the binary system emits gravitational radiation for which the frequency is fixed by the orbital frequency of the binary ω. Apart from that, the radiation modes consist of on-shell gravitons emitted from the system, meaning that the momentum satisfies $k_0 = |\vec{k}| \equiv k$, and the radiation wavelength λ will scale as:

$$\lambda^{-1} \sim k \sim \omega \sim \frac{v}{r} \tag{1.1}$$

Taking all of the discussion above into consideration, we end up with the following relation for the characteristic scales of the binary:

$$r_s \sim rv^2 \sim \lambda v^3 \tag{1.2}$$

such that the mass m will be the only scale of the full theory, and v the small parameter of the perturbative approach of the EFT.

Given this hierarchy of scales, we are able to construct a tower of EFTs accordingly, to integrate out each scale in turn.

There are two distinguished approaches for constructing an effective action, the top-down approach, and the bottom-up approach. We use both of them for the construction of the final effective field theory for this work.

The top-down approach consists of eliminating the DOFs from the full action of the highest energy scale, or smallest scale, by standard QFT perturbative methods. Whereas the bottom-up approach consists of constructing the action from scratch with an infinite series of operators which will contain DOFs that should be constrained by the symmetries of the system at the relevant scale.

To arrive at the effective action, we integrate out the modes of the scale we want to get rid of, obtained through an expansion of new operators accompanied by new coupling constants, the *Wilson coefficients*, that will encode the UV information that was suppressed from the effective action.

The two types of approach can be used in parallel, and that is in fact a way to do the fixing, or matching, of the Wilson coefficients. Otherwise they can be fixed using experimental data.

Chapter 2

EFTs of post-Newtonian Gravity

2.1 One-particle EFT

In this section we will discuss the effective action of the smallest scale of the binary system, the scale corresponding to that of a single compact object such as a black-hole or neutron star. We will discuss how to obtain an effective action for the single object using the bottom-up approach to effective field theory.

In the realm of GR, the full pure gravitational theory that describes the full theory is given by Einstein-Hilbert's action:

$$S[g_{\mu\nu}] = -\frac{1}{16\pi G}\int d^4x\sqrt{g}R[g_{\mu\nu}] \tag{2.1}$$

where $g_{\mu\nu}(x)$ is the gravitational field. In order to remove the scale of the single object, we will decompose the metric two distinct Fourier modes $g_{\mu\nu} = g^s_{\mu\nu} + \bar{g}_{\mu\nu}$ where $g^s_{\mu\nu}$ are the strong modes of the theory that we want to remove in order to remove the r_s scale. We will remove the DOFs associated with these strong field modes, by introducing new interactions and new coefficients in our theory, following the bottom-up approach. After removing the strong field modes, we will describe the theory with the remaining modes of the gravitational field, $\bar{g}_{\mu\nu}$, plus an infinite series of operators which compensate for the removal of the strong field modes DOFs. After integrating out the strong modes, we will have:

$$S[g_{\mu\nu}] \to S[\bar{g}_{\mu\nu}] + \textit{new interaction terms} \tag{2.2}$$

These new interaction terms will include new DOFs in our theory, namely new generic worldline DOFs that we will introduce to the theory, corresponding to the position and the rotation of the object (in the case of a spinning particle). These new interaction terms will all be part of what we will call the point-particle action, S_{pp}. In more details, we will have a new worldline parameter σ, on which the new worldline DOFs of position, $y^\mu(\sigma)$, and rotation, $e^\mu_A(\sigma)$ will depend. Then, we will have:

$$S[\bar{g}_{\mu\nu}, y^\mu(\sigma), e^\mu_A(\sigma)] = S[\bar{g}_{\mu\nu}] + S_{pp}[\bar{g}_{\mu\nu}, y^\mu(\sigma), e^\mu_A(\sigma)] \tag{2.3}$$

Following from section 1.1, we have that the new operators in the point-particle action, $\mathcal{O}_i(\sigma)$, will be accompanied by the Wilson coefficients, $C_i(r_s)$. It follows:

$$S[\bar{g}_{\mu\nu}, y^\mu(\sigma), e^\mu_A(\sigma)] = -\frac{1}{16\pi G}\int d^4x\sqrt{\bar{g}}R[\bar{g}_{\mu\nu}] + \sum_{i=1}^{\infty} C_i(r_s) \int d\sigma \mathcal{O}_i(\sigma) \tag{2.4}$$

As we can see from the expression above, the Wilson coefficients will depend on the scale we are removing, r_s, and will contain all the UV physics that was removed. As for the new generic operators, they will depend on the scale of the effective field theory remaining, which is r, the scale of the orbital separation.

2.1.1 Point-particle action: Minimal coupling

For the case of a non-spinning massive particle, we deal with the coupling of a massive object to the gravitational field. The addition of new DOFs to the theory will be given by the new DOF worldline, $y^\mu(\sigma)$. What we need to do now is to identify the symmetries of both the spacetime and the object that we need to incorporate in our new operators, and with that construct all possible worldline operators which couple the worldline DOFs to the gravitational field in all possible ways allowed by the symmetries. The relevant symmetries for our theory will be general coordinate invariance, worldline reparametrization invariance and internal Lorentz invariance of the local frame field. The latter corresponds to a gauge freedom in the tetrad field and it will be approached later.

Say τ is the proper time along the worldline, m is the mass of the object. The action for the minimal coupling term of the point-particle action will read:

$$\begin{aligned} S_{Mpp} = -m\int d\tau = -m\int \sqrt{\bar{g}_{\mu\nu}dy^\mu dy^\nu} = \\ -m\int d\sigma\sqrt{\bar{g}_{\mu\nu}\frac{dy^\mu}{d\sigma}\frac{dy^\nu}{d\sigma}} = -m\int d\sigma\sqrt{u^2} \end{aligned} \tag{2.5}$$

where $u^\mu \equiv \frac{dy^\mu}{d\sigma}$ is the four-velocity of the object. In EFT, by point-particle action we are actually meaning all terms in the action that are induced by the mass. This takes into account the simplest term, corresponding to the leading term of 0PN order in the post-Newtonian approximation, the one above, but also higher multipoles that are mass-induced. We follow then to the non-minimal couplings of the point-particle action.

2.1.2 Point-particle action: Non-minimal coupling

The mass-induced higher multipoles terms accounts for the finite size effects of the object in question, and will be the subleading terms in our point-mass action. Our goal now is to obtain higher PN order terms for which the mass contributes to. In order to do that, we start with the most trivial guess for operators that depend on the Riemann tensor and that has covariant derivatives. The first one that comes to mind is the Ricci tensor, $R_{\mu\nu}$. But since we are dealing with a vacuum spacetime, the Ricci tensor is null, $R_{\mu\nu} = 0$. We call that a redundant operator, meaning that it can be omitted from the effective action with no physical effect. Next step is to consider operators containing the Riemann tensor. For the vacuum solutions we have that the curvature of spacetime is given by the Weyl tensor. It is straightforward to see that from the fact that a null Ricci tensor and Ricci scalar makes the Weyl tensor equivalent to the Riemann tensor, from the Weyl tensor definition:

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} - \frac{2}{n-2}(g_{\mu[\alpha}R_{\beta]\nu} - g_{\nu[\alpha}R_{\beta]\mu}) + \frac{2}{(n-1)(n-2)}Rg_{\mu[\alpha}g_{\beta]\nu} \tag{2.6}$$

We can decompose the Weyl tensor into two definite parity components an electric and magnetic component that are defined as:

$$E_{\mu\nu} = C_{\mu\alpha\nu\beta}u^{\alpha}u^{\beta} \qquad B_{\mu\nu} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\mu}C^{\alpha\beta}{}_{\delta\nu}u^{\gamma}u^{\delta}$$

Since the Weyl tensor and Riemann tensor are equivalent for vacuum solutions we may write the components as:

$$E_{\mu\nu} = R_{\mu\alpha\nu\beta}u^{\alpha}u^{\beta} \qquad B_{\mu\nu} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\mu}R^{\alpha\beta}{}_{\delta\nu}u^{\gamma}u^{\delta}$$

They are of even and odd parity, respectively. The definite parities are important when taking in consideration that we need our action to be parity invariant, facilitating the construction of higher order operators in our action. Taking this into consideration, and the general coordinate invariance, the point-particle action up to leading order in the non-minimal coupling will take the following form:

$$S_{pp} = -m \int d\sigma \sqrt{u^2} + c_E \int d\sigma \frac{E^2_{\mu\nu}(y^\alpha(\sigma))}{(\sqrt{u^2})^3} + c_B \int d\sigma \frac{B^2_{\mu\nu}(y^\alpha(\sigma))}{(\sqrt{u^2})^3} \tag{2.7}$$

Where the coefficients c_E and c_B are the Wilson coefficients for each of these operators in the action.

2.2 Spinning particle EFT

In this section we will upgrade the description of our system to the case of an actual spinning gravitating object, which is the real deal. A first point to note is that taking the spin into consideration requires the object to be an extended body, and that is where the complexity of this description lies, since it enters in conflict with the point-particle perspective.

2.2.1 DOFs and Symmetries

In order to construct the effective field theory, is it essential to note all the DOFs and symmetries we will be dealing with from now on. The three DOFs we will have in our theory are:

1. The gravitational field DOFs, $g_{\mu\nu}$. For higher orders of the coupling, beyond the mass monopole and already considering spins, we will define a tetrad field, $\eta^{ab}\tilde{e}^\mu_a\tilde{e}^\nu_b = g^{\mu\nu}(x)$ which will also represents the field DOFs.

2. The particle position worldline DOFs, $y^\mu(\sigma)$, where σ is an arbitrary affine parameter, later on set to be the time coordinate, t.

3. The particle worldline rotating DOFs, we consider at first the worldline tetrad in an orthonormal frame $\eta^{AB}\hat{e}^\mu_A(\sigma)\hat{e}^\nu_B(\sigma) = g^{\mu\nu}$, and later we will disentangle the tetrad field DOFs from the particle worldline DOFs by doing $\hat{e}^\mu_A = \hat{\Lambda}^b_A\tilde{e}^\mu_b$.

As for the symmetries of the system, they will be as follows:

1. General coordinate invariance, in particular *parity invariance*. Which is present in a general way in General Relativity.

2. Worldline reparametrization invariance. Used to construct the minimal and non-minimal couplings parts of the point-particle action.

3. Internal Lorentz invariance of the local frame field. In general, the local Lorentz transformations will have $3 + 3$ DOFs to fix the gauge of the tetrad field, which in general has 16 DOFs.

4. SO(3) invariance of the body-fixed spatial triad, e^μ_i. The triad consist of three spacelike vectors, which comes from the 3 DOFs the particle has in respect to spatial orientation in the body-fixed frame. The consequence is that the worldline spin DOF ends up also being a SO(3) tensor in the body-fixed frame.

5. Spin gauge invariance. There will be a invariance under the choice of the timelike vector of the worldline tetrad, related to the body-fixed spatial triad discussed above. We will refer to this as the rotational gauge in what follows.

It is also assumed that the compact objects have no intrinsic permanent multipole moments beyond the mass monopole and the spin dipole. Time-reversal symmetry is not assumed, but the terms which violate this symmetry are shown to not contribute, [9].

2.2.2 Minimal-coupling

Now let us construct the minimal-coupling part of the action of the spinning particle. We will make use of the tetrad formalism and consider an orthonormal frame $e^{\mu}_{A}(\sigma)$, that is localized on the particle worldline and connects the body-fixed frame with the general coordinates frame, such that $\eta^{AB}e^{\mu}_{A}e^{\nu}_{B} = g^{\mu\nu}$, where η^{AB} is the Minskowski metric, reinforcing that the upper case Latin letters labels the body-fixed frame, and the Greek letters labels the general coordinates frame. Given that, the reciprocal tetrad will be given by $e^{\mu A} \equiv \eta^{AB}e^{\mu}_{B}$, the projection of any tensor onto the tetrad frame, and the corresponding inverse projection will be, taking a vector as example, $V_A \equiv e^{\mu}_{A}V^{A}$ and $V_{\mu} \equiv e^{A}_{\mu}V_{A}$, respectively.

Now, making use of the tetrad and general coordinates frame, we are able to define an antisymmetric tensor of angular velocity, in a generalized definition of the flat spacetime definition given by $\Omega^{ab} \equiv \Lambda^{a}_{A}\frac{D\Lambda^{Ab}}{D\sigma}$, [24], [25], which follows as:

$$\Omega^{\mu\nu} \equiv e^{\mu}_{A}\frac{De^{A\nu}}{D\sigma} \tag{2.8}$$

where $\frac{D}{D\sigma}$ is the covariant derivative with respect to the worldline parameter σ. Since we want our Lagrangian to be reparametrization invariant, the Lagrangian is required to be an homogeneous function of degree 1 and linear in the four-velocity u^{μ}, the angular velocity $\Omega^{\mu\nu}$, and dependent of the metric, whose dependence is extended beyond minimal coupling to include the Riemann tensor and further covariant derivatives, that is $L_{pp}[u^{\mu}, \Omega_{\mu\nu}, g^{\mu\nu}]$.

Then we can define the spin variable which is conjugate to the antisymmetric angular velocity tensor in the following way:

$$S_{\mu\nu} = -2\frac{\partial L}{\partial \Omega^{\mu\nu}} \tag{2.9}$$

where the minus sign is chosen to give the correct form in the nonrelativistic limit. We will handle this spin variable as a further worldline degree of freedom, seeing the spin dipole moment as a graviton source, analogous to the mass monopole being a classical source on the worldline from a QFT perspective. Other than that, we also have that the spin will serve as an independent variational variable in the Lagrangian, and its equations of motions will be straightforwardly provided by the variation of the effective action.

As for the linear momentum, it is defined as:

$$p_\mu = -\frac{\partial L}{\partial u^\mu} \tag{2.10}$$

We note that its dependence on the Lagrangian will contribute for higher orders corrections in the momentum, satisfying $p^\mu = m\frac{u^\mu}{\sqrt{u^2}} + \mathcal{O}(S^2)$. To the order we computed in the paper [15], which is presented at the results section of this thesis, we have the first contribution of these higher order terms from the linear momentum, as we it will be shown.

With all definitions in hand, we get the **minimal-coupling** term of the point-particle action for a spinning object by means of Euler's theorem:

$$L_{pp} = -p_\mu u^\mu - \frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu} \tag{2.11}$$

hence, the action of equation 2.4 reads as follows:

$$S_{pp} = \int d\sigma[-m\sqrt{u^2} - \frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu} + L_{NM}[u^\mu, S_{\mu\nu}, g_{\mu\nu}(y^\mu)]] \tag{2.12}$$

where L_{NM} stands for the non-minimal coupling part of the action induced by the presence of spin.

2.2.3 Gauge-freedom

In order to have afterwards an effective action free of orbital scale DOFs, it is necessary to implement the rotational gauge fixing at the level of the point-particle action. But in order to fix the gauge, we first need to have a generic point-particle action in which the rotational gauge is not yet fixed. In the following section, we see that the way to do this is to start from a covariant gauge and then make a transformation to a generic spin variable. We do this by applying a covariant boost to the worldline tetrad and considering how the rotational minimal coupling term, $\frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu}$ and the non-minimal coupling term will be affected by it. We should also comment that an important feature of our theory is the SO(3) invariance of the worldline spatial tetrad rather than the SO(1,3) invariance of the worldline tetrad. Since the only physical feature we have is the spatial orientation of the object, it means that the action will be formulated in terms of the spatial components of the tetrads, e_i^μ, which has SO(3) indices i, and the timelike components of the tetrad, e_0^μ will be related to the gauge choice of the spin variable or SSC, which stands for 'spin supplementary condition'.

2.2.4 Unfixing the gauge

Following the paper [9], we apply a boost, in its 4-dimensional covariant form, $L^\mu_\nu(w,q)$, to the body-fixed tetrad e_A^μ from some gauge $e_0^\mu = q^\mu$ to a generic gauge:

$$\hat{e}_0^\mu = w^\mu, \tag{2.13}$$

with the following transformation:

$$\hat{e}_A^\mu = L^\mu_\nu(w,q)e_A^\nu \tag{2.14}$$

A unique covariant gauge that eliminates the unphysical DOFs was provided by Tulczyjew in [26], which is the covariant gauge:

$$q^{\mu} = \frac{p^{\mu}}{\sqrt{p^2}} \tag{2.15}$$

which corresponds to the SSC given by:

$$S_{\mu\nu} p^{\nu} = 0. \tag{2.16}$$

This gauge was shown to be the unique gauge that ensures the unique "center" of the spinning object.

By applying the boost and looking at the effect on the rotational minimal coupling term $\frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu}$, it is possible to identify a generic spin variable, $\hat{S}_{\mu\nu}$, related to $S_{\mu\nu}$, as:

$$\hat{S}_{\mu\nu} = S_{\mu\nu} - \frac{S_{\mu\rho} w^{\rho}}{\sqrt{p^2} + pw} p_{\nu} + \frac{S_{\nu\rho} w^{\rho}}{\sqrt{p^2} + pw} p_{\mu}, \tag{2.17}$$

which satisfies the following generic SSC:

$$\hat{S}^{\mu\nu} (p_{\nu} + \sqrt{p^2} \hat{e}_{0\nu}) = 0. \tag{2.18}$$

That together with the gauge choice for the time component of the tetrad, $\hat{e}_0^{\mu} = w^{\mu}$, enables us to get rid of all unphysical redundant DOFs on both the angular velocity and the spin tensors.

We can also express the generic spin variable in respect to a shift to the 'center' of the object, $\delta z^{\mu} \equiv \hat{z}^{\mu} - y^{\mu}$, as it follows:

$$\hat{S}_{\mu\nu} = S_{\mu\nu} - \delta z^{\mu} p^{\nu} + \delta z^{\nu} p^{\mu} \tag{2.19}$$

for which an identification with equation 2.17 leads us to:

$$\delta z^{\mu} = \frac{S^{\mu\rho} w_{\rho}}{\sqrt{p^2} + pw}. \tag{2.20}$$

It can be easily seen through the initial SSC, $S_{\mu\nu}p^{\nu} = 0$, that the shift from the 'center' position is also orthogonal to the momentum, $\delta z^{\mu}p_{\mu} = 0$, meaning that it is indeed spacelike. And in addition we see that the 'center' was shifted from the worldline position, y^{μ}, to a non-covariant gauge of the rotating DOFs. As for the minimal coupling, we have then, in terms of the new generic spin variable and the shift of position:

$$\frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu} = \frac{1}{2}\hat{S}_{\mu\nu}\hat{\Omega}^{\mu\nu} - \delta z^{\mu}\frac{Dp_{\mu}}{D\sigma} \tag{2.21}$$

which shows us that we have introduced a gauge freedom in the rotational minimal-coupling term of the action 2.12. The second term in the equation above is only taking into account the finite size on the case of a spinning particle. It does not however carry any Wilson coefficients, as it does not encapsulate any UV physics of the object.

By contracting equation 2.17 with the momentum p^{μ} we get that the shift in position may be written as:

$$\delta z^{\mu} = -\frac{\hat{S}^{\mu\rho}p_{\rho}}{p^2} \tag{2.22}$$

hence, the spin variable can be written in terms of the generic spin variable as:

$$S_{\mu\nu} = \hat{S}^{\mu\nu} - \frac{\hat{S}^{\mu\rho}p^{\rho}p_{\nu}}{p^2} + \frac{\hat{S}_{\nu\rho}p^{\rho}p_{\mu}}{p^2} \tag{2.23}$$

We will need this expression to restore the gauge invariance in the non-minimal coupling term of the action. This expression is a projection of the generic spin variable onto the spatial hypersurface of the rest frame, and it removes all the spin gauge dependence from the spin variable, since in the rest frame all spin gauges agree.

2.2.5 Non-minimal coupling

As mentioned before, we need to keep our action parity invariant, and it is trickier to keep track of that when dealing with the non-minimal coupling terms of the action. With that in mind, it was defined the odd-parity spin vector variable, S^μ, which is given in terms of the spin tensor as:

$$S^\mu \equiv *S^\mu_\nu \frac{p^\nu}{\sqrt{p^2}} \tag{2.24}$$

where

$$*S_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}S^{\mu\nu} \tag{2.25}$$

and we can also define the spin length scalar, S^2 as $S^2 \equiv -S_\mu S^\mu = \frac{1}{2}S_{\mu\nu}S^{\mu\nu}$.

According to the theorem of Cayley-Hamilton, higher powers of the spin tensor are expected to be dependent, resulting that the only independent combinations of spin tensors and spin vectors that we can use to build the action will be the even parity spin tensor, $S^{\mu\nu}$, the odd-parity spin vector S^μ and the even parity contraction of the spin tensor, $S^\mu{}_\rho S^\rho{}_\nu$, or vector $S^{^\nu}$, the latter when indices are not contracted among themselves [9]. In more detail, looking at the higher powers of the spins tensors, we have:

$$\begin{aligned} S^\mu{}_\rho S^\rho{}_\nu &= -S^\mu S_\nu + S^2\big(\delta^\mu{}_\nu - \frac{u^\mu u_\nu}{u^2}\big) \\ S^\mu{}_\alpha S^\alpha{}_\beta S^\beta{}_\nu &= -S^2 S^\mu{}_\nu \end{aligned} \tag{2.26}$$

From the last equation we can get the minimal polynomial of the spin matrix $S^\mu{}_\nu$, and find that the determinant of the spin matrix is zero. A similar analysis can be made for the square of the spin tensor, for more detailed discussion see section 4.1 of [9].

Another important conclusion we arrive at is that the spin-induced multipoles are symmetric, traceless, constant and spatial in the body-fixed frame, which is also where they should naturally be considered. That is quite intuitive since the spin only makes physical sense in flat spatial components, and indeed when computing the Feynman rules we will always project the spin from general coordinates to the locally flat frame. The spin tensors can be shown to be irreducible representation tensors of the SO(3) group, [27].

Now, recalling the electric and magnetic decomposition of the curvature operator defined in equation 2.7, we can conclude through the symmetries of the Riemann tensor, the first Bianchi identity and the leading vacuum equation that they are both symmetric, traceless and orthogonal to u^μ. And, just as the spin-tensors, they are SO(3) tensors, which only have spatial components in the body-fixed frame.

We note that we only take linear on the Riemann curvature tensor terms into account, since we are not interested in the tidal effects of external gravitational fields in this work.

Now we are left in a position where we can use either a product of two spin vectors S^μ, or the square of the spin tensor $S^\mu{}_\rho S^\rho{}_\nu$, to build our action term. But following from equation 2.26 they differ by a trace, S^2 and terms that depend on the four-velocity u^μ, which will vanish when contracting to the traceless and orthogonal to u^μ curvature tensor decomposition, so we can use either one of them.

Then, to build our non-minimal coupling terms for the spinning particle action, we match the odd and even parity spin vector products with the curvature components from equation 2.7, $E_{\mu\nu}$ and $B_{\mu\nu}$, respectively, in order to obtain a parity invariant action. Our higher order spin-induced terms will also be coupled to the covariant derivatives that will be added to the curvature tensors.

Somewhat analogous to a multipole expansion in Electrodynamics, we will add covariant derivatives preceding the curvature components $E_{\mu\nu}$ and $B_{\mu\nu}$

in order to get all possible ways of contractions to obtain the scalars in our action. The covariant derivatives are also projected to the body-fixed frame, and, from the differential Bianchi identity, we can deduce that the indices of the covariant derivatives are symmetric among themselves and also with respect to the curvature tensor components.

In a further analysis, we can also see that that traces involving the covariant derivatives vanish:

$$D_i E_{ij} = D_i B_{ij} = 0 \tag{2.27}$$

Therefore we can conclude that the electric and magnetic components of the curvature tensor together with their covariant derivatives form SO(3) tensors as well, which will couple to the SO(3) spin-induced higher multipoles in a way to keep the action parity invariant.

Based on all that was discussed above, we may infer a shape for our non-minimal coupling Lagrangian for all orders in spin as follows:

$$L_{NM} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \frac{C_{ES^{2n}}}{m^{2n-1}} D_{\mu_{2n}} ... D_{\mu_3} \frac{E_{\mu_1 \mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} ... S^{\mu_{2n}} +$$
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{C_{BS^{2n+1}}}{m^{2n}} D_{\mu_{2n+1}} ... D_{\mu_3} \frac{E_{\mu_1 \mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} ... S^{\mu_{2n+1}} \tag{2.28}$$

Note $C_{ES^{2n}}$ and $C_{BS^{n+1}}$, our new Wilson coefficients for each operator. These are formally matched using the full UV theory, but in principle the numerical factors are set so that they are unit for the Kerr black-hole. The mass exponent and the numerical factors are all set so that the Wilson coefficients are dimensionless, and the factorial in the numerical factor comes from the symmetry of the spin-induced multipoles. The $\sqrt{u^2}$ factor that comes along with the electric and magnetic components are present so that we can maintain the action raparametrization invariant, considering the velocities on the definitions of the curvature components.

We may then write some specific terms of the total non-minimal coupling Lagrangian, as for example the quadrupole and octupole terms:

$$\begin{aligned} L_{ES^2} &= -\frac{C_{ES^2}}{2m}\frac{E_{\mu\nu}}{\sqrt{u^2}}S^\mu S^\nu, \\ L_{BS^3} &= -\frac{C_{BS^2}}{6m^2}D_\lambda\frac{B_{\mu\nu}}{\sqrt{u^2}}S^\mu S^\nu S^\lambda \end{aligned} \tag{2.29}$$

During this master's in which we worked particularly with the cubic-in-spin interaction that enters at the 4.5PN order, both terms above are required for the extraction of Feynman rules that were used for this sector.

For a discussion about dissipative DOFs that enters, for instance, at the single object scale, of dissipative tidal effects I refer to section 3.3 of [10].

2.3 EFT of a composite particle

In this section we consider the effective field theory of the composite binary system of two compact objects (as mentioned before, either with BHs or neutron stars). Here we want to get rid of the orbital scale r. We will get to the composite action building on the point-particle action. We get to the EFT using the top-down approach, which is then matched with the bottom-up constructed point-particle action. We will make use of the standard perturbative techniques of QFT, which will lead us to an expansion of Feynman diagrams.

First of all, we decompose the gravitational field, $\bar{g}_{\mu\nu}$, once more. In this case we decompose it on the asymptotic flat spacetime, as follows:

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu} + \tilde{h}_{\mu\nu} \tag{2.30}$$

Where $\eta_{\mu\nu}$ is the Minkowski's flat metric, $H_{\mu\nu}$ denotes the orbital modes and $\tilde{h}_{\mu\nu}$ denotes the radiation modes. The orbital modes are the ones related to the gravitational interaction between the two objects, and the radiation modes are the ones related to the gravitational waves emitted from the whole binary system. The scale dependence to the velocity of these two modes are:

$$\partial_t H_{\mu\nu} \sim \frac{v}{r} H_{\mu\nu},\ \partial_i H_{\mu\nu} \sim \frac{1}{r} H_{\mu\nu},\ \partial_\mu \tilde{h}_{\mu\nu} \sim \frac{v}{r} \tilde{h}_{\mu\nu} \tag{2.31}$$

The full composite particle action at this scale will be of the kind:

$$S_{pp}[\bar{g}_{\mu\nu}, y_1^\mu, y_2^\mu, e_{1A}^\mu, e_{2A}^\mu] = S[\bar{g}^{\mu\nu}] + \sum_{a=1}^{2} S_{pp}[\bar{g}_{\mu\nu}, y_a^\mu, e_{aA}^\mu](\sigma_a) \tag{2.32}$$

where the indices 1 and 2 are respective to each single object on the binary, and $S_{pp}[_{\mu\nu}, y_a^\mu, e_{aA}^\mu](\sigma_a)$ corresponds to the action of a single spinning particle, discussed in section 2.2. From that we follow to the effective action with the orbital scale removed by integrating out the orbital modes $H_{\mu\nu}$, which according to standard QFT will be given by:

$$e^{iS_{eff}[\tilde{g}_{\mu\nu}, y_c^\mu, e_{cA}^\mu]} \equiv \int \mathcal{D}H_{\mu\nu} e^{iS_{eff}[\bar{g}_{\mu\nu}, y_1^\mu, y_2^\mu, e_{1A}^\mu, e_{2A}^\mu]} \tag{2.33}$$

where $\tilde{g}_{\mu\nu} \equiv \eta_{\mu\nu} + \tilde{h}_{\mu\nu}$. Here the subscript '$c$' corresponds to the generic worldline DOFs of the composite object. Along this section and thesis we focus mainly on the Feynman graphs containing the orbital field modes, corresponding to the conservative sector of the theory.

To get to the next effective theory in the tower, following the bottom-up approach, the effective action at the radiation scale is defined as:

$$S_{eff}[\tilde{g}_{\mu\nu}, y_c^\mu, e_{cA}^\mu] = -\frac{1}{16\pi G} \int d^4x \sqrt{\tilde{g}} R[\tilde{g}_{\mu\nu}] + S_{pp(rad)}[\bar{g}_{\mu\nu}, y_c^\mu, e_{cA}^\mu](\sigma_c) \tag{2.34}$$

where $S_{pp(rad)}$ above corresponds to the effective point-particle action at the radiation scale, with worldline DOFs y_c^μ and e_{cA}^μ.

This scale we make an expansion of Feynman graphs containing a single external radiation field mode, that are matched onto the radiative sector of the theory, which is not approached in this thesis. For more discussions on the radiation scale check section 5 of [10].

2.3.1 Integrating out the orbital mode

To proceed to obtain the two-body interaction by integrating out the orbital modes of the EFT, we need first to disentangle the field DOFs from the particle DOFs and then fix all the rotational gauges.

We follow then to the disentanglement of the DOFs. The generic gauge worldline tetrad we defined in equation 2.13, $\hat{e}^{\mu}_{A}$, has both rotational particle DOFs and field DOFs. For that matter, we do the following factorization:

$$\hat{e}^{\mu}_{A} = \hat{\Lambda}^{b}_{A}\tilde{e}^{\mu}_{b} \tag{2.35}$$

in order to decompose the worldline tetrad. Here $\hat{\Lambda}^{b}_{A}$ are the Lorentz matrices in the locally flat frames and it is defined by:

$$\eta^{AB}\hat{\Lambda}^{a}_{A}\hat{\Lambda}^{b}_{B} = \eta^{ab} \tag{2.36}$$

and the tetrad $\tilde{e}^{\mu}_{b}$, the tetrad the covers the manifold, is defined by:

$$\eta_{ab}\tilde{e}^{a}_{\mu}\tilde{e}^{b}_{\nu} = g_{\mu\nu} \tag{2.37}$$

When we do that we make a change in the representation of the field DOFs, which will be encoded in the new tetrad field $\tilde{e}^{a}_{\mu}$, while the rotational DOFs will be expressed in the Lorentz matrices Λ^{b}_{A}. Then we can express the rotational minimal coupling of equation 2.21 as:

$$\frac{1}{2}\hat{S}_{\mu\nu}\hat{\Omega}^{\mu\nu} = \frac{1}{2}\hat{S}_{ab}\hat{\Omega}^{ab}_{LF} + \frac{1}{2}\hat{S}_{ab}\omega^{ab}_{\mu}u^{\mu} \tag{2.38}$$

where $\hat{\Omega}^{ab}_{LF} \equiv \hat{\Lambda}^{aA}\frac{d\hat{\Lambda}^{b}_{A}}{d\sigma}$ is defined as the locally flat angular velocity, ω^{ab}_{μ} are the Ricci rotation coefficients and $\hat{S}_{ab} = \tilde{e}^{\mu}_{a}\tilde{e}^{\nu}_{b}\hat{S}_{\mu\nu}$ is the spin projected to the locally flat frame. In order to fully disentangle the rotational particle and field DOFs, we must also look at the generic gauge, specified by equations 2.13 and 2.18. We will have then:

$$\hat{\Lambda}_0^b = w^a = \tilde{e}^a_\mu w^\mu \tag{2.39}$$

and

$$\hat{S}^{ab}\big(p_b + \sqrt{p^2}\hat{\Lambda}_{0b}\big) = 0. \tag{2.40}$$

However, the disentanglement is not yet all done since the gauge of the temporal Lorentz matrix, $\hat{\Lambda}_0^a$ and the temporal spin component $\hat{S}^{0i}$ may contain further field dependence. As it was put forward in [5] we must fix the gauge of the rotational variables in order to fully disentangle the field and particle DOFs at the level of the Feynman rules, and finally be able to integrate out the orbital modes.

2.3.2 Kaluza-Klein parametrization and Tetrad field gauge

Before we proceed to the fixing of the rotational gauge and the further Feynman calculus, we will make a Kaluza-Klein reduction on time dimension. That makes sense since it can be shown that the propagator is actually instantaneous at leading order on the post-Newtonian approximation. We can see it from the relations $k_0 \simeq \frac{v}{r}$, $|\vec{k}| \simeq \frac{1}{r}$ and the propagator in momentum space:

$$\int \frac{d^4k}{(2\pi)^4} e^{-ikx} \frac{1}{k^2} = \int \frac{d^4k}{(2\pi)^4} e^{-ik_0 t - i\vec{k}\cdot\vec{x}} \frac{1}{k_0^2 - \vec{k}^2} \tag{2.41}$$

then the denominator can be expressed in the PN approximation as:

$$\frac{1}{k_0^2 - \vec{k}^2} = -\frac{1}{\vec{k}^2}\big(1 + \frac{k_0^2}{\vec{k}^2} + ...\big) = -\frac{1}{\vec{k}^2}(1 + \mathcal{O}(v^2)) \tag{2.42}$$

and the propagator of the orbital scale can be written as:

$$-\int \frac{dk_0}{2\pi} e^{-ik_0 t} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2} = -\delta(t) \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2}. \tag{2.43}$$

Therefore, in the non-relativistic limit it makes sense to use a KK time reduction, as the time component can be regarded as compact in respect to the other spatial components. Then, we define the metric in terms of the Kaluza-Klein fields as follows:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \equiv e^{2\phi}(dt - A_i dx^i)^2 - e^{-2\phi}\gamma_{ij} dx^i dx^j \tag{2.44}$$

Here $\phi, A_i, \gamma_{ij} \equiv \delta_{ij} + \sigma_{ij}$ are the Newtonian scalar, the gravito-magnetic vector and the tensor fields, respectively. The latter ones also obey to $\gamma^{ij}\gamma_{jk} \equiv \delta^i_k$ and $A^i \equiv \gamma^{ij} A_j$. Using Cartan's approach 2-form method we are able to compute the full gravitational action in terms of the KK parametrization, and with further computations the propagators and vertices Feynman rules, as we will see further ahead.

In order to fix the gauge of the tetrad field, it is convenient to choose Schwinger's time gauge [28], which is given by:

$$\tilde{e}^0_i(x) = 0. \tag{2.45}$$

With that, the tetrad field is expressed in terms of the KK fields as:

$$\tilde{e^a}_\mu(x) = \begin{pmatrix} e^\phi & -e^\phi A_i \\ 0 & e^{-\phi}\sqrt{\gamma_{ij}} \end{pmatrix}. \tag{2.46}$$

Here $\sqrt{\gamma_{ij}}$ is the symmetric square root of γ_{ij}, for which we should solve to the order we are interested in when doing actual computations.

For the following computations we will choose the worldline parameter to be the time coordinate $y^0 = t$, such that $\sigma = t$, $u^0 = 1$ and $u^i = \frac{dy^i}{dt} \equiv v^i$.

Notice that in terms of the KK fields we may express the mass coupling of the action in the following way:

$$-m\int dt\sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} = -m\int dt\Big[e^{\phi}\sqrt{(1-A_iv^i)^2 - e^{-4\phi}\gamma_{ij}v^iv^j}\Big] \tag{2.47}$$

and the modulus of the four-velocity, up to second order on velocity, as:

$$\begin{aligned}\sqrt{u^2} &= \sqrt{g_{\mu\nu}u^\mu u^\nu} = \sqrt{g_{00} + 2v^ig_{0i} + v^iv^jg_{ij}} \\ &= \sqrt{e^{2\phi} - 2v^iA_ie^{2\phi} - v^iv^j(\delta_{ij}+\sigma_{ij})e^{-2\phi} + v^iv^jA_{ij}e^{2\phi})}\end{aligned} \tag{2.48}$$

2.3.3 Fixing rotational gauge

Finally, in order to get ready for integrating out the orbital scale and tackle the Feynman calculus, we will fix the rotational gauge. There are different choices of gauges we can make, but in the following work we chose the canonical gauge, which is the generalization of the Pryce-Newton-Wigner SSC in special relativity, [29], [30]. That is:

$$\hat{\Lambda}^a_0 = \delta^a_0 \tag{2.49}$$

which also fixes the spin gauge, or SSC:

$$\hat{S}^{ab}\big(p_b + \sqrt{p^2}\delta_{0b}\big) = 0 \tag{2.50}$$

This means that we boosted the Lorentz matrices Λ^a_A to the locally rest frame, decoupled from the linear motion DOFs and also the field DOFs. This is the optimum we can get to start our integration of the orbital modes.

But before doing that, it is going to be useful to express the spin variable components, which were defined at first in the covariant gauge in section 2.2.4, in the canonical gauge. From the SSC in equation 2.50 we get to:

$$\hat{S}_{a0} = \frac{-S_{ai}u^i}{u^0+u} = -\frac{\hat{S}_{ab}u^b}{u}. \tag{2.51}$$

From equations 2.23 and 2.24 we can expand the definition of the spin vector in time and spatial components, such that we find:

$$S_0 = \frac{1}{2}\frac{1}{\sqrt{u^2}}\epsilon_{0ijk}S^{ij}v^k$$
$$S_k = \frac{1}{2}\frac{1}{\sqrt{u^2}}\epsilon_{0ijk}(2S^{0i}v^j + S^{ij}) \tag{2.52}$$

From the equations above we are able to find the spin vectors in the canonical gauge. We do that by using equation 2.50 and equation 2.52:

$$S^0 = \hat{S}^k v^k$$
$$S^l = \hat{S}^l + \frac{1}{2}\hat{S}^n v^n v^l \tag{2.53}$$

where we expanded only to second order in velocity, which was the order required for this work.

Further discussion on the treatment of the spins in the locally flat frame with the canonical gauge implemented, is found in section 5 of [10].

2.3.4 Feynman rules

With the particle and field DOFs disentangled and all the gauges fixed, we may proceed to the removal of the orbital scale, which will start with the Feynman rules for our diagrammatic expansion, and afterwards with the Feynman diagrams for the interaction of the two objects.

In this work we obtained the new Feynman rules for the cubic-in-spin interaction up to the next-to-leading order, as it will be shown in section 5. The other Feynman rules required for the computation of the 4.5PN Feynman diagrams were previously computed, and can be found in [11], and [12].

Formally, all the computations are made using the KK parametrization of the metric. For instance, the propagators are obtained following from equation 2.43 using the NR fields, and read of as:

$$
\begin{aligned}
< \phi(x_1)\ \phi(x_2) > &= 4\pi G\delta(t_1 - t_2) \int_{\vec{k}} \frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{\vec{k}^2} \\
< A_i(x_1)\ A_j(x_2) > &= -16\pi G\delta(t_1 - t_2) \int_{\vec{k}} \frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{\vec{k}^2}\delta_{ij} \\
< \sigma_{ij}(x_1)\ \sigma_{kl}(x_2) > &= 32\pi G\delta(t_1 - t_2) \int_{\vec{k}} \frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{\vec{k}^2}\ P_{ij,kl}
\end{aligned}
\tag{2.54}
$$

where $P_{ij,kl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl})$ and $\int_{\vec{k}}$ is an abbreviation of $\int \frac{d^3\vec{k}}{(2\pi)^3}$.

When extracting the Feynman rules of the worldline couplings, we want to project the spin variables onto the locally flat frame and implement the canonical gauge. To do that we will make use of the tetrad field, $\tilde{e}^{\mu}_{a}$ following from equation 2.46. In our work it was required the approximation up to the first order on the KK fields. The tetrad field to first order approximation on the KK fields reads as:

$$
\tilde{e_a}^{\mu}(x) = \begin{pmatrix} 1-\phi & 0 \\ A_i & (1+\phi)\delta_{ij} \end{pmatrix} \tag{2.55}
$$

That, together with the implementation of the canonical gauge on the spin vectors expressed in equations 2.52 enable us to expand the non-minimal coupling of the action, to the order we were concerned in this work which are given by:

$$
L = L_{ES^2} + L_{BS^3} = -\frac{C_{ES^2}}{2m}\frac{E_{\mu\nu}}{\sqrt{u^2}}S^{\mu}S^{\nu} - \frac{C_{BS^3}}{6m^2}D_{\lambda}\frac{B_{\mu\nu}}{\sqrt{u^2}}S^{\mu}S^{\nu}S^{\lambda} \tag{2.56}
$$

up to the order we want in the velocities and gravitons (the KK fields), so that we get each coupling vertex for each KK field in our set of Feynman rules.

Encrypted in the curvature tensor components $E_{\mu\nu}$ and $B_{\mu\nu}$ we will have a Riemann tensor, as we can see from equations 2.7. The Riemann curvature tensor and the covariant derivative are given by common GR literature, [31], as:

$$R_{\lambda\mu\nu\kappa} = -\frac{1}{2}\Big[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu d\partial x^\lambda}\Big] + g_{\eta\sigma}\big[\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - \Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}\big] \tag{2.57}$$

where the minus sign is present because of the different convention for the sign of the flat metric. And, for a tensor of rank (0, 4), the covariant derivative is applied as:

$$D_\sigma R_{\alpha\beta\mu\nu} = \partial_\sigma - \Gamma^\rho_{\sigma\rho} R_{\rho\beta\mu\nu} - \Gamma^\rho_{\sigma\rho} R_{\alpha\rho\mu\nu} - \Gamma^\rho_{\sigma\rho} R_{\alpha\rho\rho\nu} - \Gamma^\rho_{\sigma\rho} R_{\alpha\beta\mu\rho} \tag{2.58}$$

Those two expressions will be handy when extracting the Feynman rules. We recall that at this stage the orbital modes $H_{\mu\nu}$ are being integrated out of the theory, following equation 2.33. Then, in order to make a diagrammatic expansion of the interactions at the orbital scale, we express the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is the graviton field, which in our computations is being expanded in terms of the KK fields, and can be read off from equation 2.44, or, in a matrix notation:

$$g_{\mu\nu} = \begin{pmatrix} e^{2\phi} & -e^{2\phi}A_j \\ -e^{2\phi}A_i & -e^{-2\phi}A_j\gamma_{ij} + e^{2\phi}A_iA_j \end{pmatrix}$$
$$\simeq \begin{pmatrix} 1 + 2\phi + 2\phi^2 & -A_j - 2A_j\phi \\ -A_i - 2A_i\phi & -\delta_{ij} + 2\phi\delta_{ij} - \sigma_{ij} - 2\phi^2\delta_{ij} + 2\phi\sigma_{ij} + A_iA_j \end{pmatrix} \tag{2.59}$$

up to second order on the KK fields.

2.3.5 Feynman diagrams

Taking into account all the ingredients discussed above, we are ready to move on to the construction and computation of the Feynman diagrams for a specific PN and spin order.

The graphs will be constructed using each object worldlines and the gravitons that mediate the interaction between them. A crucial element for the construction of the graphs is the power counting in the small perturbative parameter v.

From equation 2.31 we see that each partial time derivative scales as v^1, while the partial spatial derivatives scale as v^0. A further discussion of power counting can be found in [2], [3].

As for the power counting of the spin variables, the spin of an object scales as:

$$S \sim m v_{rot} r_s \tag{2.60}$$

where v_{rot} is taken to be 1 for maximally rotating objects, which is what we assume for this work. Using the Virial theorem, where $\frac{Gm}{r} = v^2$, the Schwarzschild radius of the object obeying $r_s \sim Gm$ and $v_{rot} = 1$ we get:

$$S \sim m r_s \sim m v^2 r \sim L v \tag{2.61}$$

hence, each spin variable is taken to scale with v^1 in the Feynman graphs.

We may also consider the bare graph topologies at each order of the gravitational constant, G, when analyzing the power counting of a Feynman diagram. Each n-graviton self-interaction vertex will scale with $G^{\frac{n}{2}-1}$, while each n-graviton worldline coupling scales as $G^{\frac{n}{2}}$. In respect to the PN order notation and the amount of topologies we will have, for a order of G^n, which corresponds to the $(n-1)$PN order, we encounter $(n-1)$-loop topologies, where the n-PN order is the $(v/c)^{2n}$ order correction in GR to Newtonian gravity.

In this work we tackled for the first time the 4.5PN order, which accounts for the Feynman diagrams that scale with v^9. There were 49 graphs plus 4 special graphs in total, where 10 were one-graviton exchange, half of them including the newly computed spin octupole, 15 of them were two-graviton exchange diagrams, and 24 three-graviton interaction, or one loop graphs, which is the cubic self interacting graphs.

It is relevant to note that when we mention 'one-loop' graphs, it is respective to the self-interacting graphs in classical gravity due to the nonlinearity of GR in the gravitational constant, G. But, graviton loops on the other hand will be neglected, as the expansion in Feynman diagrams will only include tree level graphs in the field.

To get a better feeling of the computation of the diagrams, let us take an example of the Newtonian interaction, which can be seen in figure 2.1 below.

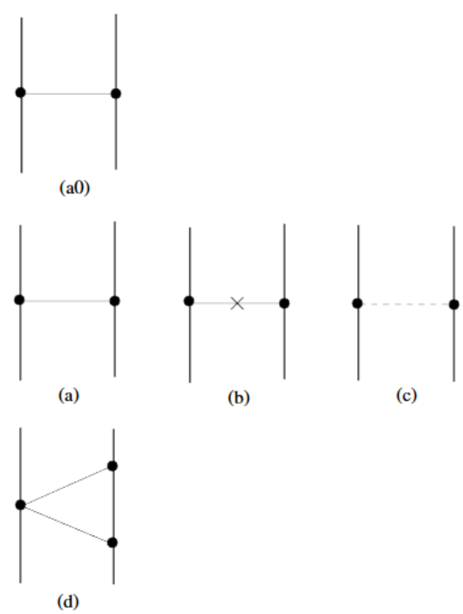

Figure 2.1: The Newtonian (0PN) and first post-Newtonian (1PN) interactions. (a0) represents the Newtonian interaction while (a)-(d) represent the first post-Newtonian interaction. Figure (b) present the first relativistic correction to the propagator and (d) is the first two-graviton exchange graph at lowest PN order. Credit: M. Levi, [10]

Here, and in all the diagrams we compute in this theory, the time direction goes up, differently from the convention in particle physics where it usually flows from left to right. The vertical lines on both ends of the diagram corresponds to the worldlines of each one of the two objects. The diagrammatic correspondence of each propagator can be seen in [10] or [9]. The black dot on each worldline is representing the mass coupling, and to first PN order we compute this graph to be:

$$\begin{aligned} Fig.(1) =& \Big(- m_1 \int dt_1 \phi(x_1) \Big) \times \Big(- m_2 \int dt_1 \phi(x_2) \Big) \\ &= 4\pi G m_1 m_2 \int dt_1 dt_2 \delta(t_1 - t_2) \int_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}_1(t_1) - \vec{x}_2(t_2))} \frac{1}{\vec{k}^2} \\ &= \int dt \frac{G m_1 m_2}{r} \end{aligned} \tag{2.62}$$

where $r \equiv |\vec{r}| \equiv |\vec{x}_1 - \vec{x}_2|$. As we can see this result corresponds to the well known action of the Newtonian interaction, which corresponds to the 0PN order. For the computation we performed a Wick contraction using the propagators defined on equation 2.54, and afterwards we integrated over time with the delta function. For the Fourier integral we use the following master integral:

$$I \equiv \int \frac{d^d \vec{k}}{(2\pi)^d} \frac{e^{\vec{k}\cdot\vec{r}}}{(\vec{k}^2)^\alpha} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\frac{d}{2} - \alpha)}{\Gamma(\alpha)} \Big(\frac{r^2}{4} \Big)^{\alpha - \frac{d}{2}} \tag{2.63}$$

where dimensional regularization is used, and eventually the limit $d \to 3$ is taken.

Now, as for time derivative treatment the usual procedure is to use integration by parts to take the time derivative off of the Dirac delta's, which are dependent on time, and drop it on the remaining variables of the respective worldline. For instance, beginning from a term in a graph computation of the kind:

$$S_{1a} \; S_{1b} \; S_{2c} \; v_2^d \; \partial_{t_1} \partial_{t_2} \delta(t_1 - t_2) \tag{2.64}$$

we would drop the time derivatives as it follows:

$$\big(-\partial_{t_1}(S_{1a}S_{1b})\big)\ \big(-\partial_{t_2}(S_{2c}v_2^d)\big)\ \delta(t_1-t_2). \tag{2.65}$$

We may also take advantage of the following identity:

$$\int dt_1 dt_2\ \partial_{t_1}\delta(t_1-t_2)f(t_1)g(t_2) = -\int dt_1 dt_2\ \partial_{t_2}\delta(t_1-t_2)f(t_1)g(t_2) \tag{2.66}$$

when treating the time derivatives. In general each integral in position space that comes from the bulk vertices will transform into a delta function conserving momenta in the vertex.

For the one-loop graphs, the following master integral was needed:

$$J \equiv \int \frac{d^d\vec{k}}{(2\pi)^d}\frac{1}{[\vec{k}]^\alpha[(\vec{k}-\vec{q})^2]^\beta} = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma(\alpha+\beta-d/2)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(d/2-\alpha)\Gamma(d/2-\beta)}{\Gamma(d-\alpha-\beta)}(q^2)^{\frac{d}{2}-\alpha\beta} \tag{2.67}$$

and as for further one-loop integrals that were needed to compute the NLO cubic-in-spin sector I refer to Appendix A of [7].

Another important ingredient of the Feynman diagrams is the symmetry factor. For each Feynman graph computation we must compute the following symmetry factor as well, which will account for the factor coming from the action and also the many different ways we can contract the KK fields.

Let's take an example to see how the symmetry factor of a graph is computed. Below we show diagram b2 from figure 6 in the paper [12]. This diagram is computed in the paper to G^3 order, or equivalently, 2PN order, and as we can see it is a two-loop diagram.

Writing the expression for the diagram in a schematic way we will have the following terms:

$$S_1\ A_1, S_1\ A_1', m_1\phi, A\ A\ \sigma, \sigma'\phi'\phi', m_2\phi_2 \tag{2.68}$$

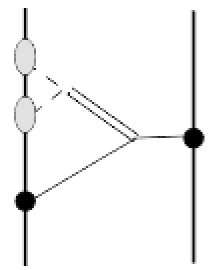

Figure 2.2: NNLO spin-squared example at 4PN order. Credit: M. Levi, [12].

Each field A from worldline 1, A_1 or A_1' can contract with each of the fields A in the bulk vertex. Meaning that we have 2 different kinds of contractions for the gravito-magnetic field. In the same analysis we have that each field ϕ' from the bulk vertex may contract with either ϕ_1 or ϕ_2, given us two more possibilities of contraction. And finally the tensor can only contract in one way. Taken all of that into account we have:

$$2 \times 2 \times 1 = 4 \tag{2.69}$$

identical ways of contracting the graviton fields.

Then, we must also take into account the factors that come in the expansion of the effective action from which we collect the Feynman graphs. That is, in a schematic way:

$$\begin{aligned} &e^{S_{eff}} \sim e^{S_{EH}+S_{1pp}+S_{2pp}} \\ &\sim \left(S_{EH} + \frac{(S_{EH})^2}{2!} + \frac{(S_{EH})^3}{3!} + ...\right) \times \left(S_{1pp} + \frac{(S_{1pp})^2}{2!} + ...\right) \times \left(S_{2pp} + \frac{(S_{2pp})^2}{2!} + ...\right) \end{aligned} \tag{2.70}$$

where S_{EH} denotes the terms from the bulk vertices, and S_{1pp} and S_{2pp} denotes the terms for each worldline.

Analyzing the diagram from left to right, we have 3 vertices for the first worldline, two bulk vertices and 1 term for the second worldline. The contribution from worldline 1 comes from a term of the kind $(m\phi+S_1A_1)^3$ from the action expansion, more specifically the one with two gravito-magnetic fields, A^2 and one scalar field ϕ. According to the expansion $(m\phi + S_1A_1)^3 = (m\phi)^3 + 3(m\phi)^2 S_1A_1 + 3m\phi(S_1A_1)^2 + (S_1A_1)^3$, we spot our term to be $3m\phi(A)^2$, and hence the coefficient from the first worldline is $\frac{3}{3!}$.

The same analysis is made to the two bulk vertices, leading us to a prefactor of $\frac{2}{2!}$, and to the second worldline, which is $\frac{1}{1!}$.

The total factor for this particular analysis above will be:

$$\frac{3}{3!} \times \frac{2}{2!} \times \frac{1}{1!} = \frac{1}{2} \tag{2.71}$$

and then the total symmetry factor of this diagram will be:

$$4 \times \frac{1}{2} = 2 \tag{2.72}$$

The following step after computing all Feynman diagrams is to obtain the interaction potential for the sector which will contain only physical DOFs. We note that all diagrams should be included together with their mirror diagrams. All the present higher-order time derivatives in the potential can be eliminated if we redefine the position and spin variables already at the level of the action. The procedure consists remaining the higher order time derivatives through a substitution of variables in the action, as developed in [8].

Chapter 3

Effective action of cubic-in-spin interaction at NLO

The sector for the cubic-in-spin interaction at 4.5PN order for a coalescent binary system of BHs or neutron-stars was fully computed during this master's project. They were first presented in [15], which was prepared for submission to JHEP and is under review at the current time. The results for each Feynman diagram and the total potential were computed manually and automatically, using the EFTofPNG code, [16]. The paper is presented below.

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Prepared for submission to JHEP

Gravitational cubic-in-spin interaction at the next-to-leading post-Newtonian order

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Abstract: In this work we derive for the first time the complete gravitational cubic-in-spin effective action at the next-to-leading order for the interaction of generic compact binaries via the effective field theory for gravitating spinning objects and its extension to this sector. This sector, which enters at the fourth and a half post-Newtonian (4.5PN) order for rapidly rotating compact objects, completes finite size effects up to this order, and is the first sector completed beyond the current state of the art for generic compact binary dynamics at the 4PN order. At this order in spins with gravitational nonlinearities we have to take into account additional terms, which arise from a new type of worldline couplings, due to the fact that at this order the Tulczyjew gauge for the rotational degrees of freedom, which involves the linear momentum, can no longer be approximated only in terms of the four-velocity. One of the main motivations for us to tackle this sector is also to see what happens when we go to a sector, which corresponds to the gravitational Compton scattering with quantum spins of three halves, and maybe possibly also get an insight on the inability to uniquely fix its amplitude from factorization when spins of five halves and higher are involved. A general observation that we can clearly make already is that even-parity sectors in the order of the spin are easier to handle than odd ones. In the quantum context this corresponds to the greater ease of dealing with bosons compared to fermions.

Contents

1 Introduction

Since the first detection of gravitational waves (GWs) from a binary black hole coalescence was announced in 2016 it has become increasingly pressing to provide high precision theoretical predictions for the modeling of GW templates. The latter significantly rely on implementing analytical results obtained within the post-Newtonian (PN) approximation of classical Gravity [1] via the Effective-One-Body approach [2]. In particular in recent years we have made a remarkable progress in pushing the precision frontier for the orbital dynamics of compact binaries, i.e. whose components are generic compact objects. The complete state of the art to date for the orbital dynamics of a generic compact binary is shown in table 1.

As a measure for the loop computational scale we show in table 1 the number of n-loop graphs that enter at the N^{n}LO in l powers of the spin, i.e. up to the lth spin-induced multipole moment, in the sectors completed to date. The count is based on computations carried out with the effective field theory (EFT) of PN Gravity [3], which use the Kaluza-Klein decomposition of the field from [4], that has considerably facilitated high precision computations within the EFT approach [4–16]. As can be seen the current complete state of the art is at the 4PN order, whereas the next-to-leading order (NLO) cubic-in-spin sector that enters at the 4.5PN order is evaluated in this paper. All of the sectors at the current state of the art (but the top right entry at the 4PN order for the non-rotating case) are available in the public "EFTofPNG" code at `https://github.com/miche-levi/pncbc-eftofpng` [17].

Let us stress that in order to attain a certain level of PN accuracy, the various sectors should be tackled across the diagonals of table 1, rather than along the axes, namely

l \ n	(N^0)LO	$N^{(1)}$LO	N^2LO	N^3LO	N^4LO
S^0	1	0	3	0	25
S^1	2	7	32	174	
S^2	2	2	18	52	
S^3	4	24			
S^4	3	5			

Table 1. The complete state-of-the-art of PN Gravity theory for the orbital dynamics of generic compact binaries. Each PN correction enters at the order $n + l + \text{Parity}(l)/2$, where the parity is 0 or 1 for even or odd l, respectively. We elaborate on the meaning of the numerical entries and the gray area in the text below.

progress must be made by going in parallel to higher loops and to higher orders of the spin. In general, the former involves more challenges of computational loop technology and tackling associated divergences, whereas the latter necessitates an improvement of the fundamental understanding of spin in gravity, and tackling finite size effects with spin [18]. These enter first at the 2PN order [19] from the LO spin-induced quadrupole. Within the EFT approach whose extension to the spinning case was first approached in [20], finite size effects include as additional parameters the Wilson coefficients, that correspond to the multipole deformations of the object due to its spin, as in [21] for the quadrupole.

With a considerable time gap from the LO result, the NLO spin-squared interaction was treated in a series of works [11, 22–25], where the result in [11] was derived within the formulation of the EFT for gravitating spinning objects introduced there. The LO cubic- and quartic-in-spin interactions were first tackled in [24, 26] for black holes. In [10], based on the formulation presented in [11], these were derived for generic compact objects, where also the quartic-in-spin interaction was completed. Only specific pieces of the latter results were recovered in [27] via S-matrix combined with EFT techniques, whereas [28] which treated only cubic-in-spin effects, also provided the LO effects in the energy flux. The work in [29] then also derived for the case of black holes the LO sectors to all orders in spin. Finally, the NNLO spin-squared interaction was derived in [13]. Notably the latter results together with the complete quartic-in-spin results for generic compact objects in [10], both at the 4PN order, were derived so far exclusively within the EFT formulation of spinning gravitating objects [11].

Recently, there has also been a surge of interest in harnessing modern advances in scattering amplitudes to the problem of a coalescence of a compact binary. Notably, a new implementation for the non-rotating case to the derivation of classical potentials was carried out in [30, 31]. Further, based on a new quantum formalism introduced in [32] for massive particles of any spin, new approaches to the computation of spin effects of black holes in the classical potential were put forward in [33, 34] and then in [35, 36]. In these approaches classical effects with spin to the lth order correspond to amplitudes involving a quantum spin of $s = l/2$. In particular as of the one-loop level the gravitational Compton

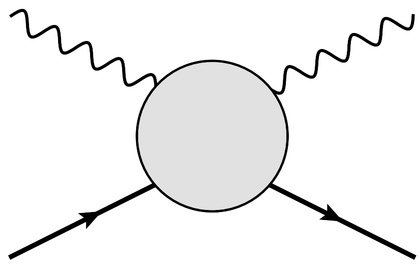


Figure 1. The gravitational Compton scattering relevant as of the one-loop level. The gravitational Compton amplitude involves two massive spinning particles and two gravitons, where factorization constraints do not uniquely determine the amplitude for $s > 2$ [32].

amplitude shown in figure 1 is required, where factorization constraints do not uniquely determine the amplitude for $s > 2$ [32]. The gray area in table 1 then corresponds in the quantum context to where the gravitational Compton scattering with a spin $s > 1$ is required.

Notably, the gray area in table 1 also corresponds to, as was already pointed out in [11], where we can no longer take the linear momentum p_μ, with which the generic formulation in [11] was derived, to be its leading approximation given by $m\frac{u^\mu}{\sqrt{u^2}}$, as was done in all past spin sectors tackled, but we have to take into account corrections to the linear momentum from the non-minimal coupling part of the spinning particle action. Can we then get a well-defined result? Can we get an insight from examining this new feature at the classical level on the non-uniqueness of fixing the graviton Compton amplitude with $s > 2$?

This work builds on the formalism of the EFT for gravitating spinning objects introduced in [11] and the implementation on [10] to compute the cubic-in-spin interaction at the NLO, that enters at the 4.5PN order for maximally-rotating compact objects, beyond the current state of the art of PN theory in general and with spins in particular [37], and is the leading sector in the intriguing gray area of table 1. We compute the complete sector, taking into account all interactions that include all possible multipoles up to the octupole. Thus beyond pushing the state of the art in PN theory, there are two conceptual objectives that we get to address in this work: 1. To learn how the difference from the leading linear momentum to its correction affects the results; 2. To see whether this difference is related with the non-uniqueness of the gravitational Compton amplitude of higher spin states, or to get any possible insight on this non-uniqueness.

The paper is organized as follows. In section 2 we go over the formulation from [11], and the necessary ingredients to evaluate this sector. In section 3 we present the essential computation, where the linear momentum assumes its leading approximation in terms of the four-velocity, as done in all past evaluations of spin sectors. In section 4 we find the new contributions arising from the correction to the leading linear momentum, which matters as of this sector, and gives rise to a new type of worldline-graviton coupling. In section 5 we compute the final action of this sector, and finally we conclude in section 6 with some observations and questions.

2 The EFT of gravitating spinning objects

Let us consider the ingredients that are required in order to carry out the evaluation of this sector, that contains spins up to cubic order along with first gravitational nonlinearities. This evaluation will build on the EFT of gravitating spinning objects formulated in [11], and its implementation from LO up to the state of the art at the 4PN order in [10–13, 37]. We will also use here the Kaluza-Klein decomposition of the metric [4, 38] to scalar, vector and symmetric tensor components, which was adopted in all high order PN computations both with and without spins for its facilitating virtues [18], and follow conventions consistent with the abovementioned works. Further, we follow similar gauge choices, notational and pictorial conventions as presented in [11].

The effective action we start from is that of a two-particle system [18], with each of the particles described by the one-particle effective action of a spinning particle, that was provided in [11]. This effective action contains a pure gravitational piece, from which the propagators and self-interacting vertices are derived. The Feynman rules for the propagator and the time insertions on the propagators are given e.g. in eqs. (5)-(10) of [9], and for the cubic gravitational vertices in eqs. (2.10)-(2.13), and (2.15) of [12]. Further, for each of the two particles the worldline action of a spinning particle is considered from [11], where its spin-induced non-minimal coupling part was constructed, and then gauge freedom of the rotational DOFs is incorporated into the action. We recall that this action has the following form:

$$S_{\text{pp}}(\sigma) = \int d\sigma \left[-m\sqrt{u^2} - \frac{1}{2}\hat{S}_{\mu\nu}\hat{\Omega}^{\mu\nu} - \frac{\hat{S}^{\mu\nu}p_\nu}{p^2}\frac{Dp_\mu}{D\sigma} + L_{\text{SI}} \right], \tag{2.1}$$

given in terms of four velocity u_μ, the linear momentum p_μ and the generic rotational DOFs, denoted with a hat e.g. $\hat{S}_{\mu\nu}$, and where the label "SI" stands for the spin-induced part of the action, which for the sector evaluated here will consist of its two leading terms given by

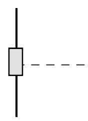
$$L_{\text{SI}} = -\frac{C_{ES^2}}{2m}\frac{E_{\mu\nu}}{\sqrt{u^2}}S^\mu S^\nu - \frac{C_{BS^3}}{6m^2}D_\lambda \frac{B_{\mu\nu}}{\sqrt{u^2}}S^\mu S^\nu S^\lambda, \tag{2.2}$$

where here it is the spin vector S^μ that is used, as described in detail in [11, 18]. We recall that in eq. (2.1) there is an extra term, which appears in the action from the restoration of gauge freedom of the rotational DOFs. This term, which is essentially the Thomas precession as discussed in detail in [11] (and recovered recently as "Hilbert space matching" in [36, 39]), contributes to all orders in the spin as of the LO spin-orbit sector, and in particular also to finite size spin effects, though it does not encode any UV physics, but rather in the context of an effective action just accounts for the fact that a relativistic gravitating object has an extended measure.

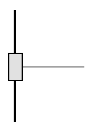
Since we compute here the *complete* NLO cubic-in-spin sector our graphs will contain all multipoles in the presence of spin up to the spin-induced octupole, i.e. also including the mass, spin and spin-induced quadrupole. For this reason we need to use Feynman rules of worldline-graviton coupling to NLO for all of these multipoles, where in this work we need to derive further new rules for the octupole couplings. The Feynman rules required for the mass couplings are given in eqs. (64), (67), (68), (79), (81), (93), (95) of [7]. Next, we approach the Feynman rules linear in spin, noting we that we have first kinematic contributions as noted in eq. (5.28) of [11], that are linear in the spin but have no field coupling, which we will take into account in section 5.

The Feynman rules required for the linear-in-spin couplings are given in eqs. (5.29)-(5.31) of [11], and eqs. (2.31)-(2.34) of [12]. For the spin quadrupole couplings the rules are given in eqs. (2.18)-(2.24) of [13], and for the LO spin octupole couplings they are found in eqs. (2.19),(2.20) of [10]. As we noted in addition to the abovementioned Feynman rules, further rules are required here for the spin-induced octupole worldline-graviton coupling. The two Feynman rules of the scalar and vector components of the KK decomposition, which appeared already at LO in [10] should be extended to higher PN order, and further we will have new rules that enter for the one-graviton coupling of the tensor component of the KK fields, and a couple of two-graviton couplings, involving again the scalar and the vector components of the KK fields, which appeared at LO.

The extended rules for the one-graviton couplings are then given as follows:



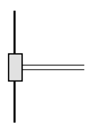
$$= \int dt \left[\frac{C_{\rm BS^3}}{12m^2} S_i S_j \epsilon_{klm}\Big[A_{k,ijl}\Big(S_m\Big(1+\frac{1}{2}v^2\Big)-\frac{1}{2}v^m S_n v^n\Big) + S_m\Big(v^l v^n\big(A_{i,njk}-A_{n,ijk}\big)+v^l\big(\partial_t A_{k,ij}+\partial_t A_{i,jk}\big)+v^i\partial_t A_{k,jl}\Big)\Big]\right], \tag{2.3}$$



$$= \int dt \left[\frac{C_{\rm BS^3}}{3m^2} S_i S_j \epsilon_{klm} S_m v^l \Big(\phi_{,ijk}\Big(1+\frac{v^2}{2}\Big)+v^i\partial_t\phi_{,jk}\Big)\right], \tag{2.4}$$

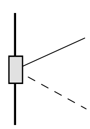
where the rectangular boxes represent the spin-induced octupole.

The new Feynman rules required here are given as follows:

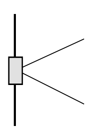


$$= \int dt \left[\frac{C_{\rm BS^3}}{12m^2} S_i S_j \epsilon_{klm} S_m \partial_i\partial_l\Big(\big(\partial_j\sigma_{kn}-\partial_n\sigma_{jk}\big)v^n-\partial_t\sigma_{jk}\Big)\right], \tag{2.5}$$

for the one-graviton coupling, whereas for the two-graviton couplings we get:



$$= \int dt \left[\frac{C_{\rm BS^3}}{12m^2} S_i S_j \epsilon_{klm} S_m \Big(6\phi A_{k,ijl}+9\phi_{,i}A_{k,jl}+3\phi_{,k}\partial_j\big(A_{i,l}-A_{l,i}\big) + 4\phi_{,ij}A_{k,l}+4\phi_{,jk}\big(A_{i,l}-A_{l,i}\big)+\delta_{ij}\phi_{,n}A_{l,kn}\Big)\right], \tag{2.6}$$



$$= \int dt \left[\frac{C_{\rm BS^3}}{3m^2} S_i S_j \epsilon_{klm} S_m \Big[v^l\Big(2\phi_{,ijk}\phi+3\phi_{,ij}\phi_{,k}+5\phi_{,i}\phi_{,jk}-\delta_{ij}\phi_{,n}\phi_{,kn}\Big)+v^i\phi_{,lj}\phi_{,k}\Big]\right]. \tag{2.7}$$

We note that in these rules the spin is already fixed to the canonical gauge and all indices are Euclidean. Notice the complexity of these couplings with respect to the other worldline couplings at the NLO level, and notice also the dominant role that the gravitomagnetic vector plays in the coupling to the odd-parity octupole, similar to the situation in the coupling to the spin dipole. Note that this is the first sector which necessitates to take the curved Levi-Civita tensor into account.

For this sector there is no need to extend the non-minimal coupling part of the spinning particle action and add higher dimensional operators beyond what was provided in [11], but we need to pay special attention to the new feature that differentiates this specific sector from all the spin sectors which were tackled in the past. In this sector it is no longer sufficient to use the leading approximation for the linear momentum p_μ in terms of the four-velocity u_ν all throughout, rather one has to take into account the subleading term in the linear momentum, which is linear in Riemann and quadratic in the spin and becomes relevant exactly once we get to the level that is non-linear in gravity and cubic in the spins, i.e. at this sector, as was already explicitly noted in [11]. We will address in detail the particular contributions coming from this new feature in section 4 below after we have done the essential computation, which requires only the leading approximation to the linear momentum, similar to what was considered in all past PN computations with spin, in the following section.

3 The essential computation

In this section we carry out the perturbative expansion of the effective action in terms of Feynman graphs, and provide the value of each diagram, under the leading approximation of the linear momentum. At the NLO level, i.e. up to the G^2 order, with spins all of the three relevant topologies are realized even when the beneficial KK decomposition of the field is used, as discussed in [6, 7, 11, 18]. As shown in figures 2-4 below (drawn using Jaxodraw [40, 41] based on [42]) there is a total of $49 = 10 + 15 + 24$ graphs making up the sector, distributed among the relevant topologies of one- and two-graviton exchanges and cubic self-interaction, respectively. As shown in table 1 about half of the total graphs require a one-loop evaluation (the highest loop in this sector). We note that as we go into the nonlinear regime of the sector, the options for the make up of the interaction become more intricate.

At the one-graviton exchange level we only have two kinds of interaction contributing, similar to the LO in [10], namely either an octupole-monopole or a quadrupole-dipole interaction. As noted in [10] there are nice analogies among these interactions according to the parity of the multipole moments involved. Following these analogies the relevant graphs of one-graviton exchange are easily constructed. Yet, once we proceed to the level nonlinear in the gravitons further types of interactions emerge. In particular, there are also interactions involving the various multipoles on two different points of the same worldline, which add up to interactions that are cubic in the spin, such as a spin and a spin-induced

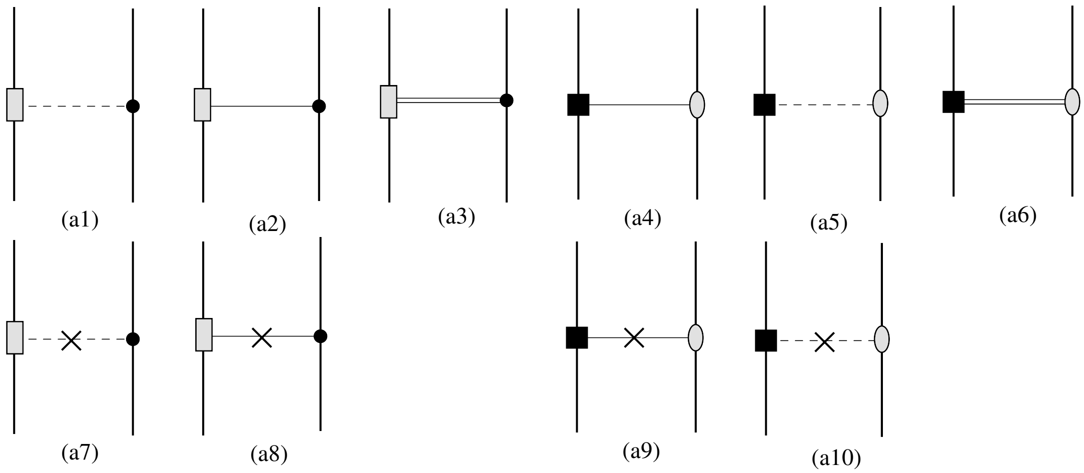


Figure 2. The one-graviton exchange Feynman graphs, which contribute to the NLO cubic-in-spin interaction at the 4.5PN order for maximally rotating compact objects. These graphs should be included together with their mirror images, i.e. with the worldline labels $1 \leftrightarrow 2$ exchanged. At the one-graviton exchange level we only have two kinds of interactions contributing, similar to the LO in [10], namely either a quadrupole-dipole or an octupole-monopole one. As noted in [10] there are nice analogies among these interactions according to the parity of the multipole moments involved. Following these analogies the relevant graphs here are easily constructed. Notice that we have here the four graphs that appeared at the LO with the quadratic time insertions on the propagators at graphs (a7)-(a10), and a new octupole coupling involving the tensor component of the KK fields at graph (a3).

quadrupole, or two spin dipoles on the same worldline, as can already be seen as of the NLO spin-squared sector [11, 13].

We note that all the graphs in this sector should be included together with their mirror images, i.e. with the worldline labels $1 \leftrightarrow 2$ exchanged. For more specific details on the generation of the Feynman graphs, and their evaluation, including the conventions and notations used here, we refer the reader to [18] and references therein.

3.1 One-graviton exchange

As can be seen in figure 2 we have 10 graphs of one-graviton exchange in this sector, the majority of which already involve time derivatives to be applied. Consistent with former works by one of the authors we keep all of the higher order time derivative terms that emerge in the evaluations of the graphs, and they will be treated properly via redefinitions of the position and the rotational variables as shown in [43]). Notice that we have here the 4 graphs that appeared at the LO with the quadratic time insertions on the propagators at graphs 1(a7)-(a10), and a new octupole coupling involving the tensor component of the KK fields at graph 1(a3).

The graphs in figure 2 are evaluated as follows:

$$\text{Fig. 2(a1)} = -C_{1(BS^3)}\frac{G}{r^4}\frac{m_2}{m_1^2}\Big[\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\big(2\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}+\vec{v}_2\cdot\vec{n}\big(S_1^2-5\big(S_1\cdot\vec{n}\big)^2\big)-\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}\big)$$

$$
\begin{aligned}
&+ \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \Big(\big(S_1^2 - 5\big(\vec{S}_1 \cdot \vec{n}\big)^2 \big) \vec{v}_1 \cdot \vec{v}_2 + \vec{S}_1 \cdot \vec{v}_2 \big(\vec{S}_1 \cdot \vec{v}_2 - 5 \vec{S}_1 \cdot \vec{n}\ \vec{v}_2 \cdot \vec{n} \big) \\
&- \vec{S}_1 \cdot \vec{v}_2 \big(\vec{S}_1 \cdot \vec{v}_1 - 5 \vec{S}_1 \cdot \vec{n}\ \vec{v}_1 \cdot \vec{n} \big) \Big) + \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \Big(\frac{1}{2} S_1^2 \big(v_1^2 + v_2^2 \big) \\
&- \vec{S}_1 \cdot \vec{v}_1 \big(\vec{S}_1 \cdot \vec{v}_2 - 5 \vec{S}_1 \cdot \vec{n}\ \vec{v}_2 \cdot \vec{n} \big) - \frac{5}{2} \big(\vec{S}_1 \cdot \vec{n} \big)^2 \big(v_1^2 + v_2^2 \big) \Big) \\
&- \frac{1}{2} \vec{v}_1 \cdot \vec{v}_2 \times \vec{n}\ \vec{S}_1 \cdot \vec{v}_1 \big(S_1^2 - 5 \big(\vec{S}_1 \cdot \vec{n} \big)^2 \big) \Big] \\
&- \frac{1}{3} C_{1(BS^3)} \frac{G}{r^3} \frac{m_2}{m_1^2} \Big[\vec{S}_1 \cdot \vec{v}_1 \times \vec{a}_2 \big(S_1^2 - 3 \big(\vec{S}_1 \cdot \vec{n} \big)^2 \big) - 3 \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \vec{S}_1 \cdot \vec{a}_2\ \vec{S}_1 \cdot \vec{n}
\end{aligned} \tag{3.1}
$$

$$
+ 3 \vec{S}_1 \cdot \vec{a}_2 \times \vec{n} \vec{S}_1 \cdot \vec{v}_1\ \vec{S}_1 \cdot \vec{n} \Big], \tag{3.2}
$$

$$
\begin{aligned}
\text{Fig. 2(a2)} = &\frac{1}{2} C_{1(BS^3)} \frac{G}{r^4} \frac{m_2}{m_1^2} \Big[\vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \Big(S_1^2 \big(v_1^2 + 3 v_2^2 \big) - 2 \vec{S}_1 \cdot \vec{n} \big(\vec{S}_1 \cdot \vec{v}_2 - 5\ \vec{S}_1 \cdot \vec{n}\ \vec{v}_2 \cdot \vec{n} \big) \\
&- 5 (\vec{S}_1 \cdot \vec{n})^2 \big(v_1^2 + 3 v_2^2 \big) \Big) - 2 \vec{S}_1 \cdot \vec{v}_1 \times \vec{v}_2\ \vec{S}_1 \cdot \vec{v}_1\ \vec{S}_1 \cdot \vec{n} \Big],
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
\text{Fig. 2(a3)} = &- C_{1(BS^3)} \frac{G}{r^4} \frac{m_2}{m_1^2} \Big[\big(\vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\ v_2^2 - \vec{S}_1 \cdot \vec{v}_2 \times \vec{n}\ \vec{v}_1 \cdot \vec{v}_2 \big) \big(S_1^2 - 5 (\vec{S}_1 \cdot \vec{n})^2 \big) \Big] \\
&+ C_{1(BS^3)} \frac{G}{r^3} \frac{m_2}{m_1^2} \Big[\vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \big(\vec{S}_1 \cdot \vec{v}_2 \dot{\vec{S}}_1 \cdot \vec{n} + \dot{\vec{S}}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \big) \\
&+ \dot{\vec{S}}_1 \cdot \vec{v}_2 \times \vec{n}\ \vec{S}_1 \cdot \vec{v}_2\ \vec{S}_1 \cdot \vec{n} \Big],
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
\text{Fig. 2(a4)} = &\frac{3}{2} C_{1(ES^2)} \frac{G}{r^4} \frac{1}{m_1} \Big[2 \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \Big(\vec{S}_1 \cdot \vec{v}_1\ \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \cdot \vec{n} \big(3 v_1^2 + v_2^2 \big) \Big) \\
&+ 2 \vec{S}_2 \cdot \vec{v}_1 \times \vec{v}_2 \Big(2 S_1^2\ \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \cdot \vec{v}_1\ \vec{S}_1 \cdot \vec{n} \Big) \\
&- \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} \Big(S_1^2 \big(5 v_1^2 + v_2^2 - 10 \big(\vec{v}_1 \cdot \vec{n} \big)^2 \big) - 2 \vec{S}_1 \cdot \vec{v}_1 \big(\vec{S}_1 \cdot \vec{v}_1 - 5 \vec{S}_1 \cdot \vec{n}\ \vec{v}_1 \cdot \vec{n} \big) \\
&- 5 \big(\vec{S}_1 \cdot \vec{n} \big)^2 \big(3 v_1^2 + v_2^2 \big) \Big) \Big] \\
&+ C_{1(ES^2)} \frac{G}{r^3} \frac{1}{m_1} \Big[2\ \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2\ \big(\vec{S}_1 \cdot \vec{a}_1 + \dot{\vec{S}}_1 \cdot \vec{v}_1 \big) + 2 \dot{\vec{S}}_1 \cdot \vec{S}_2 \times \vec{v}_2\ \vec{S}_1 \cdot \vec{v}_1 \\
&+ 4 \vec{S}_2 \cdot \vec{v}_1 \times \vec{v}_2\ \dot{\vec{S}}_1 \cdot \vec{S}_1\ + 2 \vec{S}_2 \cdot \vec{v}_2 \times \vec{a}_1\ S_1^2\ + \vec{S}_2 \cdot \vec{v}_2 \times \vec{a}_2 \big(S_1^2 - 3 \big(\vec{S}_1 \cdot \vec{n} \big)^2 \big) \\
&- 6 \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} \Big(S_1^2\ \vec{a}_1 \cdot \vec{n} - 2 \dot{\vec{S}}_1 \cdot \vec{S}_1\ \vec{v}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{v}_1\ \dot{\vec{S}}_1 \cdot \vec{n} + \dot{\vec{S}}_1 \cdot \vec{v}_1\ \vec{S}_1 \cdot \vec{n} \\
&+ \vec{S}_1 \cdot \vec{a}_1\ \vec{S}_1 \cdot \vec{n} \Big) \Big] \\
&- 4 C_{1(ES^2)} \frac{G}{r^2} \frac{1}{m_1} \vec{S}_2 \cdot \vec{v}_2 \times \vec{n} \Big(\dot{S}_1^2 + \ddot{\vec{S}}_1 \cdot \vec{S}_1 \Big),
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
\text{Fig. 2(a5)} = &- \frac{3}{2} C_{1(ES^2)} \frac{G}{r^4} \frac{1}{m_1} \Big[2 \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 \big(\vec{S}_1 \cdot \vec{v}_1\ \vec{v}_1 \cdot \vec{n} - \vec{S}_1 \cdot \vec{n}\, v_1^2 \big) \\
&- 6 \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2\ \vec{S}_1 \cdot \vec{n}\ \vec{v}_1 \cdot \vec{v}_2 + \vec{S}_2 \cdot \vec{v}_1 \times \vec{v}_2 \Big(S_1^2 \vec{v}_2 \cdot \vec{n} + 2 \vec{S}_1 \cdot \vec{v}_2\ \vec{S}_1 \cdot \vec{n} \\
&- 5 \big(\vec{S}_1 \cdot \vec{n} \big)^2 \vec{v}_2 \cdot \vec{n} \Big) - \vec{S}_2 \cdot \vec{v}_1 \times \vec{n} \Big(S_1^2\ \big(3 v_1^2 - 10 \big(\vec{v}_1 \cdot \vec{n} \big)^2 \big) - 2 \big(\vec{S}_1 \cdot \vec{v}_1 \big)^2 \\
&+ 10 \vec{S}_1 \cdot \vec{v}_1\ \vec{S}_1 \cdot \vec{n}\ \vec{v}_1 \cdot \vec{n} - 5 \big(\vec{S}_1 \cdot \vec{n} \big)^2\ \vec{v}_1^2 \Big)
\end{aligned}
$$

$$
\begin{aligned}
& - 3\vec{S}_2\cdot\vec{v}_2\times\vec{n}\,\left(S_1^2 - 5\big(\vec{S}_1\cdot\vec{n}\big)^2\right)\vec{v}_1\cdot\vec{v}_2\,\Big] \\
& - 3C_{1(ES^2)}\frac{G}{r^3}\frac{1}{m_1}\,\Big[\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\dot{\vec{S}}_1\cdot\vec{n} + \dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\Big] \\
& - \frac{1}{2}C_{1(ES^2)}\frac{G}{r^3}\frac{1}{m_1}\Big[\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\big(4\,\dot{\vec{S}}_1\cdot\vec{v}_2 - 3\dot{\vec{S}}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\big) \\
& + \dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\big(4\,\vec{S}_1\cdot\vec{v}_2 - 3\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\big) - 3\vec{S}_1\cdot\vec{S}_2\times\vec{n}\Big(\vec{S}_1\cdot\vec{v}_1\,\vec{a}_1\cdot\vec{n} \\
& + \dot{\vec{S}}_1\cdot\vec{v}_1\,\vec{v}_1\cdot\vec{n} + \vec{S}_1\cdot\vec{a}_1\,\vec{v}_1\cdot\vec{n} - 2\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{a}_1 - \,\dot{\vec{S}}_1\cdot\vec{n}\,v_1^2\Big) \\
& - 3\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\big(\vec{S}_1\cdot\vec{v}_1\,\vec{v}_1\cdot\vec{n} - \vec{S}_1\cdot\vec{n}\,v_1^2\big) \\
& + 4\big(\vec{S}_2\cdot\vec{v}_1\times\vec{a}_2 + \dot{\vec{S}}_2\cdot\vec{v}_1\times\vec{v}_2\big)\Big(S_1^2 - 3\big(\vec{S}_1\cdot\vec{n}\big)^2\Big) \\
& - 3\vec{S}_2\cdot\vec{v}_1\times\vec{n}\Big(2S_1^2\,\vec{a}_1\cdot\vec{n} + 4\dot{\vec{S}}_1\cdot\vec{S}_1\,\vec{v}_1\cdot\vec{n} - \vec{S}_1\cdot\vec{v}_1\,\dot{\vec{S}}_1\cdot\vec{n} - \dot{\vec{S}}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{n} \\
& - \vec{S}_1\cdot\vec{a}_1\,\vec{S}_1\cdot\vec{n}\Big) - 3\vec{S}_2\cdot\vec{a}_1\times\vec{n}\big(2S_1^2\,\vec{v}_1\cdot\vec{n} - \vec{S}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{n}\big) \\
& + 3\vec{S}_2\cdot\vec{v}_2\times\vec{n}\big(8\dot{\vec{S}}_1\cdot\vec{S}_1\,\vec{v}_2\cdot\vec{n} - 3\vec{S}_1\cdot\vec{v}_2\dot{\vec{S}}_1\cdot\vec{n} - 3\dot{\vec{S}}_1\cdot\vec{v}_2\,\vec{S}_1\cdot\vec{n}\big)\Big] \\
& + 2C_{1(ES^2)}\frac{G}{r^2}\frac{1}{m_1}\Big[\big(\vec{S}_1\cdot\vec{S}_2\times\vec{a}_2 + \vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{v}_2\big)\dot{\vec{S}}_1\cdot\vec{n} \\
& + \big(\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2\times\vec{v}_2 + \dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{a}_2\big)\vec{S}_1\cdot\vec{n} - 2\big(\dot{\vec{S}}_2\cdot\vec{v}_2\times\vec{n} + \vec{S}_2\cdot\vec{a}_2\times\vec{n}\big)\dot{\vec{S}}_1\cdot\vec{S}_1\Big],
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
\text{Fig. 2(a6)} = & -3C_{1(ES^2)}\frac{G}{r^4}\frac{1}{m_1}\Big[2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{v}_2 - 2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}\,v_1^2 \\
& + \left(\vec{S}_2\cdot\vec{v}_1\times\vec{n}\,\vec{v}_1\cdot\vec{v}_2 - \vec{S}_2\cdot\vec{v}_2\times\vec{n}\,v_1^2\right)\left(S_1^2 - 5\big(\vec{S}_1\cdot\vec{n}\big)^2\right)\Big] \\
& + C_{1(ES^2)}\frac{G}{m_1 r^3}\Big[\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\dot{\vec{S}}_1\cdot\vec{v}_2 + \dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_1\vec{S}_1\cdot\vec{v}_2 + 2\vec{S}_1\cdot\vec{S}_2\times\vec{a}_1\vec{S}_1\cdot\vec{v}_2 \\
& - 2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\big(\dot{\vec{S}}_1\cdot\vec{v}_1 + \vec{S}_1\cdot\vec{a}_1\big) - 2\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{v}_1 \\
& + 3\vec{S}_1\cdot\vec{S}_2\times\vec{n}\Big(\vec{S}_1\cdot\vec{n}\,\vec{a}_1\cdot\vec{v}_2 - \vec{S}_1\cdot\vec{v}_2\,\vec{a}_1\cdot\vec{n} + \dot{\vec{S}}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{v}_2\Big) + 3\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{v}_2 \\
& - 6\vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\dot{\vec{S}}_1\cdot\vec{S}_1 + 2\vec{S}_2\cdot\vec{v}_2\times\vec{a}_1 S_1^2 + 3\vec{S}_2\cdot\vec{v}_1\times\vec{n}\Big(2\dot{\vec{S}}_1\cdot\vec{S}_1\,\vec{v}_2\cdot\vec{n} \\
& - \vec{S}_1\cdot\vec{v}_2\,\dot{\vec{S}}_1\cdot\vec{n} - \dot{\vec{S}}_1\cdot\vec{v}_2\,\vec{S}_1\cdot\vec{n}\Big) - 3\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big(S_1^2\,\vec{a}_1\cdot\vec{n} + 4\dot{\vec{S}}_1\cdot\vec{S}_1\,\vec{v}_1\cdot\vec{n} \\
& - 2\vec{S}_1\cdot\vec{v}_1\,\dot{\vec{S}}_1\cdot\vec{n} - 2\dot{\vec{S}}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{n} - 2\vec{S}_1\cdot\vec{n}\,\vec{S}_1\cdot\vec{a}_1\Big) \\
& + 3\vec{S}_2\cdot\vec{a}_1\times\vec{n}\big(S_1^2\vec{v}_2\cdot\vec{n} - \vec{S}_1\cdot\vec{v}_2\,\vec{S}_1\cdot\vec{n}\big)\Big] \\
& - C_{1(ES^2)}\frac{G}{r^2}\frac{1}{m_1}\Big[\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\ddot{\vec{S}}_1\cdot\vec{v}_2 + 2\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\,\dot{\vec{S}}_1\cdot\vec{v}_2 + \ddot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{v}_2 \\
& - 2\vec{S}_2\cdot\vec{v}_2\times\vec{n}\,\big(\dot{S}_1^2 + \ddot{\vec{S}}_1\cdot\vec{S}_1\big)\Big],
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
\text{Fig. 2(a7)} = & \frac{1}{2}C_{1(BS^3)}\frac{G}{r^4}\frac{m_2}{m_1^2}\Big[\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\Big(S_1^2\vec{v}_2\cdot\vec{n} + 2\vec{S}_1\cdot\vec{v}_2\,\vec{S}_1\cdot\vec{n} - 5\big(\vec{S}_1\cdot\vec{n}\big)^2\vec{v}_2\cdot\vec{n}\Big) \\
& - \vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big(S_1^2\big(\vec{v}_1\cdot\vec{v}_2 - 5\vec{v}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\big) + 2\vec{S}_1\cdot\vec{v}_1\,\vec{S}_1\cdot\vec{v}_2
\end{aligned}
$$

$$
\begin{aligned}
&- 10\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n} - 10\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\\
&- 5\big(\vec{S}_1\cdot\vec{n}\big)^2\big(\vec{v}_1\cdot\vec{v}_2 - 7\vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\Big)\Big]\\
&+ \frac{1}{6}C_{1(BS^3)}\frac{G}{r^3}\frac{m_2}{m_1^2}\Big[\vec{S}_1\cdot\vec{v}_1\times\vec{a}_2\big(S_1^2 - 3\big(\vec{S}_1\cdot\vec{n}\big)^2\big)\\
&- 6\vec{S}_1\cdot\vec{v}_2\times\vec{n}\big(\dot{\vec{S}}_1\cdot\vec{S}_1\vec{v}_2\cdot\vec{n} + \dot{\vec{S}}_1\cdot\vec{v}_2\vec{S}_1\cdot\vec{n} + \vec{S}_1\cdot\vec{v}_2\dot{\vec{S}}_1\cdot\vec{n} - 5\dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\\
&+ 3\vec{S}_1\cdot\vec{a}_2\times\vec{n}\big(S_1^2\vec{v}_1\cdot\vec{n} + 2\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n} - 5\big(\vec{S}_1\cdot\vec{n}\big)^2\vec{v}_1\cdot\vec{n}\big)\\
&- 3\dot{\vec{S}}_1\cdot\vec{v}_2\times\vec{n}\big(S_1^2\vec{v}_2\cdot\vec{n} + 2\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n} - 5\big(\vec{S}_1\cdot\vec{n}\big)^2\vec{v}_2\cdot\vec{n}\big)\Big]\\
&- \frac{1}{6}C_{1(BS^3)}\frac{G}{r^2}\frac{m_2}{m_1^2}\Big[2\vec{S}_1\cdot\vec{a}_2\times\vec{n}\big(\dot{\vec{S}}_1\cdot\vec{S}_1 - 3\dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\big)\\
&+ \dot{\vec{S}}_1\cdot\vec{a}_2\times\vec{n}\big(S_1^2 - 3(\vec{S}_1\cdot\vec{n})^2\big)\Big],
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
\text{Fig. 2(a8)} = {}& \frac{1}{2}C_{1(BS^3)}\frac{G}{r^4}\frac{m_2}{m_1^2}\Big[\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\Big(S_1^2\vec{v}_1\cdot\vec{n} + 2\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}\ - 5(\vec{S}_1\cdot\vec{n})^2\vec{v}_1\cdot\vec{n}\Big)\\
&+ \vec{S}_1\cdot\vec{v}_1\times\vec{n}\Big(S_1^2\big(\vec{v}_1\cdot\vec{v}_2 - 5\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big) + 2\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{v}_2\\
&- 10\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n} - 10\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\\
&- 5(\vec{S}_1\cdot\vec{n})^2\big(\vec{v}_1\cdot\vec{v}_2 - 7\vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\Big)\Big]\\
&- \frac{1}{6}C_{1(BS^3)}\frac{G}{r^3}\frac{m_2}{m_1^2}\Big[2\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\big(\dot{\vec{S}}_1\cdot\vec{S}_1 - 3\dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\big)\\
&- \big(\vec{S}_1\cdot\vec{v}_2\times\vec{a}_1 - \dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{v}_2\big)\big(S_1^2 - 3\big(\vec{S}_1\cdot\vec{n}\big)^2\big) - 6\vec{S}_1\cdot\vec{v}_1\times\vec{n}\big(\dot{\vec{S}}_1\cdot\vec{S}_1\ \vec{v}_2\cdot\vec{n}\\
&+ \dot{\vec{S}}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n} + \vec{S}_1\cdot\vec{v}_2\ \dot{\vec{S}}_1\cdot\vec{n} - 5\dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\\
&- 3\big(\vec{S}_1\cdot\vec{a}_1\times\vec{n} + \dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{n}\big)\big(S_1^2\ \vec{v}_2\cdot\vec{n} + 2\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n} - 5(\vec{S}_1\cdot\vec{n})^2\vec{v}_2\cdot\vec{n}\big)\Big],
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
\text{Fig. 2(a9)} = {}& -\frac{3}{2}C_{1(ES^2)}\frac{G}{r^4}\frac{1}{m_1}\Big[2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\Big(\vec{S}_1\cdot\vec{v}_1\ \vec{v}_2\cdot\vec{n}\ + \vec{S}_1\cdot\vec{v}_2\ \vec{v}_1\cdot\vec{n}\\
&+ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{v}_2\ - 5\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\ \Big) - \vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\Big(S_1^2\vec{v}_2\cdot\vec{n} + 2\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}\\
&- 5\big(\vec{S}_1\cdot\vec{n}\big)^2\vec{v}_2\cdot\vec{n}\Big) - \vec{S}_2\cdot\vec{v}_2\times\vec{n}\ \Big(S_1^2\big(\vec{v}_1\cdot\vec{v}_2\ - 5\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big) - 2\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{v}_2\\
&+ 10\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n} + 10\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\Big)\\
&+ 5\big(\vec{S}_1\cdot\vec{n}\big)^2\big(\vec{v}_1\cdot\vec{v}_2\ - 7\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\Big)\Big]\\
&+ \frac{1}{2}C_{1(ES^2)}\frac{G}{r^3}\frac{1}{m_1}\Big[2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\big(\dot{\vec{S}}_1\cdot\vec{v}_2 - 3\dot{\vec{S}}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\\
&- 2\Big(\vec{S}_1\cdot\vec{S}_2\times\vec{a}_2 + \vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{v}_2\Big)\Big(\vec{S}_1\cdot\vec{v}_1 - 3\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\Big)\\
&+ 2\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\big(\vec{S}_1\cdot\vec{v}_2 - 3\vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\\
&- \Big(\vec{S}_2\cdot\vec{v}_1\times\vec{a}_2 + \dot{\vec{S}}_2\cdot\vec{v}_1\times\vec{v}_2\Big)\Big(S_1^2 + 3\big(\vec{S}_1\cdot\vec{n}\big)^2\Big) + 6\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big(\dot{\vec{S}}_1\cdot\vec{S}_1\ \vec{v}_2\cdot\vec{n}\\
&- \dot{\vec{S}}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n} - \vec{S}_1\cdot\vec{v}_2\ \dot{\vec{S}}_1\cdot\vec{n} + 5\dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\Big)
\end{aligned}
$$

$$- 3\Big(\dot{\vec{S}}_2 \cdot \vec{v}_2 \times \vec{n} + \vec{S}_2 \cdot \vec{a}_2 \times \vec{n}\Big)\Big(S_1^2 \ \vec{v}_1 \cdot \vec{n} - 2\vec{S}_1 \cdot \vec{v}_1 \ \vec{S}_1 \cdot \vec{n} + 5\big(\vec{S}_1 \cdot \vec{n}\big)^2 \ \vec{v}_1 \cdot \vec{n}\Big)\Big]$$

$$- C_{1(ES^2)} \frac{G}{r^2} \frac{1}{m_1} \Big[\vec{S}_1 \cdot \vec{S}_2 \times \vec{a}_2 \ \dot{\vec{S}}_1 \cdot \vec{n} + \vec{S}_1 \cdot \dot{\vec{S}}_2 \times \vec{v}_2 \ \dot{\vec{S}}_1 \cdot \vec{n} + \dot{\vec{S}}_1 \cdot \vec{S}_2 \times \vec{a}_2 \ \vec{S}_1 \cdot \vec{n}$$

$$+ \dot{\vec{S}}_1 \cdot \dot{\vec{S}}_2 \times \vec{v}_2 \ \vec{S}_1 \cdot \vec{n} - \Big(\dot{\vec{S}}_2 \cdot \vec{v}_2 \times \vec{n} + \vec{S}_2 \cdot \vec{a}_2 \times \vec{n}\Big)\Big(\dot{\vec{S}}_1 \cdot \vec{S}_1 + 3\dot{\vec{S}}_1 \cdot \vec{n} \ \vec{S}_1 \cdot \vec{n}\Big)\Big], \tag{3.10}$$

$$\text{Fig. 2(a10)} = \frac{3}{2} C_{1(ES^2)} \frac{G}{r^4} \frac{1}{m_1} \Big[2\vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 \Big(\vec{S}_1 \cdot \vec{v}_1 \ \vec{v}_2 \cdot \vec{n} + \vec{S}_1 \cdot \vec{v}_2 \ \vec{v}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{v}_2$$

$$- 5\vec{S}_1 \cdot \vec{n} \ \vec{v}_1 \cdot \vec{n} \ \vec{v}_2 \cdot \vec{n}\Big) - \vec{S}_2 \cdot \vec{v}_1 \times \vec{v}_2 \Big(S_1^2 \vec{v}_1 \cdot \vec{n} - 2\vec{S}_1 \cdot \vec{v}_1 \ \vec{S}_1 \cdot \vec{n}$$

$$+ 5\big(\vec{S}_1 \cdot \vec{n}\big)^2 \vec{v}_1 \cdot \vec{n}\Big) - \vec{S}_2 \cdot \vec{v}_1 \times \vec{n} \Big(S_1^2 \big(\vec{v}_1 \cdot \vec{v}_2 - 5 \ \vec{v}_1 \cdot \vec{n} \ \vec{v}_2 \cdot \vec{n}\big)$$

$$- 2\vec{S}_1 \cdot \vec{v}_1 \ \vec{S}_1 \cdot \vec{v}_2 + 10\vec{S}_1 \cdot \vec{v}_1 \ \vec{S}_1 \cdot \vec{n} \vec{v}_2 \cdot \vec{n} + 10\vec{S}_1 \cdot \vec{v}_2 \ \vec{S}_1 \cdot \vec{n} \vec{v}_1 \cdot \vec{n}$$

$$+ 5\big(\vec{S}_1 \cdot \vec{n}\big)^2 \big(\vec{v}_1 \cdot \vec{v}_2 - 7\vec{v}_1 \cdot \vec{n} \ \vec{v}_2 \cdot \vec{n}\big)\Big)\Big]$$

$$- \frac{1}{2} C_{1(ES^2)} \frac{G}{r^3} \frac{1}{m_1} \Big[\vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 \big(\dot{\vec{S}}_1 \cdot \vec{v}_2 - 3\dot{\vec{S}}_1 \cdot \vec{n} \ \vec{v}_2 \cdot \vec{n}\big)$$

$$+ 2\vec{S}_1 \cdot \vec{S}_2 \times \vec{a}_1 \big(\vec{S}_1 \cdot \vec{v}_2 - 3\vec{S}_1 \cdot \vec{n} \ \vec{v}_2 \cdot \vec{n}\big) + \dot{\vec{S}}_1 \cdot \vec{S}_2 \times \vec{v}_1 \big(\vec{S}_1 \cdot \vec{v}_2 - 3\vec{S}_1 \cdot \vec{n} \ \vec{v}_2 \cdot \vec{n}\big)$$

$$- 2\vec{S}_1 \cdot \dot{\vec{S}}_2 \times \vec{v}_1 \Big(\vec{S}_1 \cdot \vec{v}_1 - 3\vec{S}_1 \cdot \vec{n} \ \vec{v}_1 \cdot \vec{n}\Big) - \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \big(\dot{\vec{S}}_1 \cdot \vec{v}_1 - 3\dot{\vec{S}}_1 \cdot \vec{n} \ \vec{v}_1 \cdot \vec{n}\big)$$

$$- \dot{\vec{S}}_1 \cdot \vec{S}_2 \times \vec{v}_2 \Big(\vec{S}_1 \cdot \vec{v}_1 - 3\vec{S}_1 \cdot \vec{n} \ \vec{v}_1 \cdot \vec{n}\Big) + 3\vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \Big(\dot{\vec{S}}_1 \cdot \vec{v}_1 \ \vec{v}_2 \cdot \vec{n}$$

$$+ \dot{\vec{S}}_1 \cdot \vec{v}_2 \ \vec{v}_1 \cdot \vec{n} + \dot{\vec{S}}_1 \cdot \vec{n} \ \vec{v}_1 \cdot \vec{v}_2 - 5\dot{\vec{S}}_1 \cdot \vec{n} \ \vec{v}_1 \cdot \vec{n} \ \vec{v}_2 \cdot \vec{n}\Big)$$

$$+ 3\dot{\vec{S}}_1 \cdot \vec{S}_2 \times \vec{n} \Big(\vec{S}_1 \cdot \vec{v}_1 \ \vec{v}_2 \cdot \vec{n} + \vec{S}_1 \cdot \vec{v}_2 \ \vec{v}_1 \cdot \vec{n} + \vec{S}_1 \cdot \vec{n} \ \vec{v}_1 \cdot \vec{v}_2$$

$$- 5\vec{S}_1 \cdot \vec{n} \ \vec{v}_1 \cdot \vec{n} \ \vec{v}_2 \cdot \vec{n}\Big) - 2\vec{S}_2 \cdot \vec{v}_1 \times \vec{v}_2 \big(\dot{\vec{S}}_1 \cdot \vec{S}_1 + 3\dot{\vec{S}}_1 \cdot \vec{n} \ \vec{S}_1 \cdot \vec{n}\big)$$

$$- \vec{S}_2 \cdot \vec{a}_1 \times \vec{v}_2 \Big(S_1^2 + 3\big(\vec{S}_1 \cdot \vec{n}\big)^2\Big) + 6\vec{S}_2 \cdot \vec{v}_1 \times \vec{n} \Big(\dot{\vec{S}}_1 \cdot \vec{S}_1 \ \vec{v}_2 \cdot \vec{n} - \vec{S}_1 \cdot \vec{v}_2 \dot{\vec{S}}_1 \cdot \vec{n}$$

$$- \dot{\vec{S}}_1 \cdot \vec{v}_2 \ \vec{S}_1 \cdot \vec{n} + 5\dot{\vec{S}}_1 \cdot \vec{n} \ \vec{S}_1 \cdot \vec{n} \ \vec{v}_2 \cdot \vec{n}\Big) + 3\vec{S}_2 \cdot \vec{a}_1 \times \vec{n} \Big(S_1^2 \vec{v}_2 \cdot \vec{n} - 2\vec{S}_1 \cdot \vec{v}_2 \ \vec{S}_1 \cdot \vec{n}$$

$$+ 5\big(\vec{S}_1 \cdot \vec{n}\big)^2 \vec{v}_2 \cdot \vec{n}\Big) - 3\dot{\vec{S}}_2 \cdot \vec{v}_1 \times \vec{n} \big(S_1^2 \ \vec{v}_1 \cdot \vec{n} - 2\vec{S}_1 \cdot \vec{v}_1 \ \vec{S}_1 \cdot \vec{n} + 5(\vec{S}_1 \cdot \vec{n})^2 \vec{v}_1 \cdot \vec{n}\big)\Big]$$

$$+ \frac{1}{2} C_{1(ES^2)} \frac{G}{r^2} \frac{1}{m_1} \Big[\vec{S}_1 \cdot \dot{\vec{S}}_2 \times \vec{v}_1 \ \dot{\vec{S}}_1 \cdot \vec{n} + \dot{\vec{S}}_1 \cdot \dot{\vec{S}}_2 \times \vec{v}_1 \ \vec{S}_1 \cdot \vec{n}$$

$$+ \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2 \ \ddot{\vec{S}}_1 \cdot \vec{n} + 2\dot{\vec{S}}_1 \cdot \vec{S}_2 \times \vec{v}_2 \ \dot{\vec{S}}_1 \cdot \vec{n} + \ddot{\vec{S}}_1 \cdot \vec{S}_2 \times \vec{v}_2 \ \vec{S}_1 \cdot \vec{n}$$

$$+ \vec{S}_1 \cdot \vec{S}_2 \times \vec{n} \big(\ddot{\vec{S}}_1 \cdot \vec{v}_2 - 3\ddot{\vec{S}}_1 \cdot \vec{n} \ \vec{v}_2 \cdot \vec{n}\big) - \vec{S}_1 \cdot \dot{\vec{S}}_2 \times \vec{n} \Big(\dot{\vec{S}}_1 \cdot \vec{v}_1 - 3\dot{\vec{S}}_1 \cdot \vec{n} \ \vec{v}_1 \cdot \vec{n}\Big)$$

$$+ 2\dot{\vec{S}}_1 \cdot \vec{S}_2 \times \vec{n} \big(\dot{\vec{S}}_1 \cdot \vec{v}_2 - 3\dot{\vec{S}}_1 \cdot \vec{n} \ \vec{v}_2 \cdot \vec{n}\big) - \dot{\vec{S}}_1 \cdot \dot{\vec{S}}_2 \times \vec{n} \Big(\vec{S}_1 \cdot \vec{v}_1 - 3\vec{S}_1 \cdot \vec{n} \ \vec{v}_1 \cdot \vec{n}\Big)$$

$$+ \ddot{\vec{S}}_1 \cdot \vec{S}_2 \times \vec{n} \big(\vec{S}_1 \cdot \vec{v}_2 - 3\vec{S}_1 \cdot \vec{n} \ \vec{v}_2 \cdot \vec{n}\big) + 2\vec{S}_1 \cdot \dot{\vec{S}}_2 \times \vec{a}_1 \ \vec{S}_1 \cdot \vec{n}$$

$$- 2 \ \dot{\vec{S}}_2 \cdot \vec{v}_1 \times \vec{n} \Big(\dot{\vec{S}}_1 \cdot \vec{S}_1 + 3\dot{\vec{S}}_1 \cdot \vec{n} \ \vec{S}_1 \cdot \vec{n}\Big) - \dot{\vec{S}}_2 \cdot \vec{a}_1 \times \vec{n} \Big(S_1^2 + 3\big(\vec{S}_1 \cdot \vec{n}\big)^2\Big)\Big]$$

$$- \frac{1}{2} C_{1(ES^2)} \frac{G}{r} \frac{1}{m_1} \Big[\vec{S}_1 \cdot \dot{\vec{S}}_2 \times \vec{n} \ \ddot{\vec{S}}_1 \cdot \vec{n} + 2\dot{\vec{S}}_1 \cdot \dot{\vec{S}}_2 \times \vec{n} \ \dot{\vec{S}}_1 \cdot \vec{n} + \ddot{\vec{S}}_1 \cdot \dot{\vec{S}}_2 \times \vec{n} \ \vec{S}_1 \cdot \vec{n}\Big]. \tag{3.11}$$

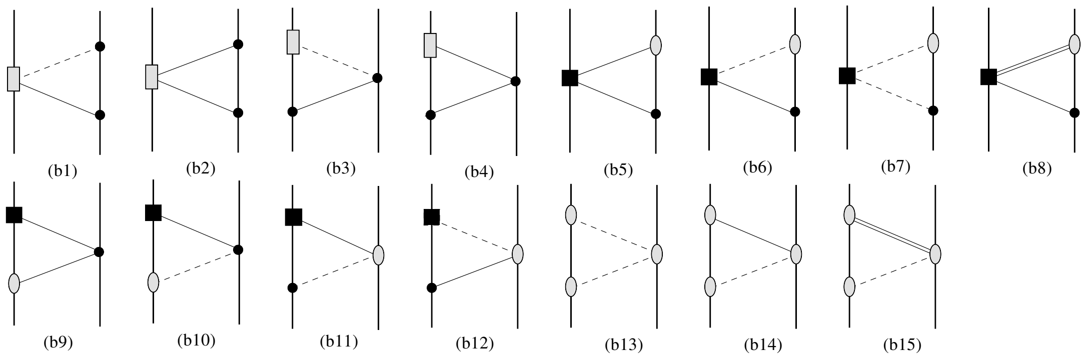


Figure 3. The two-graviton exchange Feynman graphs, which contribute to the NLO cubic-in-spin interaction at the 4.5PN order for maximally rotating compact objects. These graphs should be included together with their mirror images, i.e. with the worldline labels $1 \leftrightarrow 2$ exchanged. These graphs include all relevant interactions among the spin-induced quadrupole, octupole, and the mass and spin, in particular here at the nonlinear level there are also interactions involving the various multipoles on two different points of the same worldline, which add up to interactions that are cubic in the spin, such as a spin dipole and a spin-induced quadrupole or two spin dipoles on the same worldline as can already be seen as of the NLO spin-squared sector [11, 13]. Consequently notice that there are nonlinearities originating from gravitons sourced strictly from minimal coupling to the worldline as shown in graphs (b13)-(b15). We also have here two new two-graviton–octupole couplings in graphs (b1), (b2).

Note that almost all these graphs contain higher order time derivatives terms, notably second order time derivatives, where graph 1(a10) even contains third order ones.

Further notice that the value of graph 1(a5) is unique in that it also contains time derivatives of the spin, which appeared already in graph 2(a) of the LO in [10], but eventually did not contribute at the LO. At this order, as we will see here in section 5 these terms actually contribute.

3.2 Two-graviton exchange

As can be seen in figure 3 we have 15 graphs of two-graviton exchange in this sector. Here the majority of the graphs do not involve time derivatives. We have here two new two-graviton–octupole couplings in graphs 1(b1), 1(b2), and on the other hand we have here nonlinearities originating from gravitons sourced strictly from minimal coupling to the worldline as in graphs 1(b13)-1(b15).

The graphs in figure 3 are evaluated as follows:

$$\text{Fig. 3(b1)} = C_{1(BS^3)} \frac{G^2}{r^5} \frac{m_2^2}{m_1^2} \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \left[9S_1^2 - 50(\vec{S}_1 \cdot \vec{n})^2\right], \tag{3.12}$$

$$\text{Fig. 3(b2)} = -\frac{1}{3} C_{1(BS^3)} \frac{G^2}{r^5} \frac{m_2^2}{m_1^2} \vec{S}_1 \cdot \vec{v}_1 \times \vec{n} \left[11S_1^2 - 54(\vec{S}_1 \cdot \vec{n})^2\right], \tag{3.13}$$

$$\text{Fig. 3(b3)} = C_{1(BS^3)} \frac{G^2}{r^5} \frac{m_2}{m_1} \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \left[S_1^2 - 5(\vec{S}_1 \cdot \vec{n})^2\right], \tag{3.14}$$

$$\text{Fig. 3(b4)} = -C_{1(BS^3)}\frac{G^2}{r^5}\frac{m_2}{m_1}\,\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left[S_1^2-5(\vec{S}_1\cdot\vec{n})^2\right], \tag{3.15}$$

$$\text{Fig. 3(b5)} = 8C_{1(ES^2)}\frac{G^2}{r^5}\frac{m_2}{m_1}\,\left[3\,\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}+\vec{S}_2\cdot\vec{v}_2\times\vec{n}\left[2S_1^2-9(\vec{S}_1\cdot\vec{n})^2\right]\right], \tag{3.16}$$

$$\begin{aligned}\text{Fig. 3(b6)} = {}& C_{1(ES^2)}\frac{G^2}{r^5}\frac{m_2}{m_1}\Big[-23\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n}+13\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}\\ &-\vec{S}_2\cdot\vec{v}_1\times\vec{n}\big(31S_1^2-66\big(\vec{S}_1\cdot\vec{n}\big)^2\big)-\vec{S}_1\cdot\vec{S}_2\times\vec{n}\left(10\vec{S}_1\cdot\vec{v}_1-51\vec{S}_1\cdot\vec{n}\,\vec{v}_1\cdot\vec{n}\right)\\ &+\vec{S}_1\cdot\vec{S}_2\times\vec{n}\left(11\vec{S}_1\cdot\vec{v}_2-54\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right)\Big]\\ &-13C_{1(ES^2)}\frac{G^2}{r^4}\frac{m_2}{m_1}\,\left[\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\;\vec{S}_1\cdot\vec{n}\right],\end{aligned} \tag{3.17}$$

$$\begin{aligned}\text{Fig. 3(b7)} = {}& 2C_{1(ES^2)}\frac{G^2}{r^5}\frac{m_2}{m_1}\Big[2\,\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{S}_2\times\vec{n}\left(\vec{S}_1\cdot\vec{v}_2-3\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right)\\ &+\vec{S}_2\cdot\vec{v}_2\times\vec{n}\left(2S_1^2-3(\vec{S}_1\cdot\vec{n})^2\right)\Big],\end{aligned} \tag{3.18}$$

$$\begin{aligned}\text{Fig. 3(b8)} = {}& -C_{1(ES^2)}\frac{G^2}{r^5}\frac{m_2}{m_1}\Big[2\,\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}+3\vec{S}_1\cdot\vec{S}_2\times\vec{n}\left(\vec{S}_1\cdot\vec{v}_2-2\vec{S}_1\cdot\vec{n}\,\vec{v}_2\cdot\vec{n}\right)\\ &+\vec{S}_2\cdot\vec{v}_2\times\vec{n}\left(5S_1^2-12(\vec{S}_1\cdot\vec{n})^2\right)\Big],\end{aligned} \tag{3.19}$$

$$\text{Fig. 3(b9)} = -C_{1(ES^2)}\frac{G^2}{r^5}\frac{m_2}{m_1}\,\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left[S_1^2-3(\vec{S}_1\cdot\vec{n})^2\right], \tag{3.20}$$

$$\text{Fig. 3(b10)} = C_{1(ES^2)}\frac{G^2}{r^5}\frac{m_2}{m_1}\,\vec{S}_1\cdot\vec{v}_2\times\vec{n}\left[S_1^2-3(\vec{S}_1\cdot\vec{n})^2\right], \tag{3.21}$$

$$\text{Fig. 3(b11)} = -4C_{1(ES^2)}\frac{G^2}{r^5}\,\vec{S}_2\cdot\vec{v}_1\times\vec{n}\left[S_1^2-3(\vec{S}_1\cdot\vec{n})^2\right], \tag{3.22}$$

$$\begin{aligned}\text{Fig. 3(b12)} = {}& -12C_{1(ES^2)}\frac{G^2}{r^5}\,\left[2\,\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\,\vec{S}_1\cdot\vec{n}+\vec{S}_2\cdot\vec{v}_1\times\vec{n}\left(S_1^2-5(\vec{S}_1\cdot\vec{n})^2\right)\right]\\ &+12C_{1(ES^2)}\frac{G^2}{r^4}\,\left[\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\dot{\vec{S}}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\right],\end{aligned} \tag{3.23}$$

$$\begin{aligned}\text{Fig. 3(b13)} = {}& 2\frac{G^2}{r^5}\,\left[\vec{S}_1\cdot\vec{v}_2\times\vec{n}\left(\vec{S}_1\cdot\vec{S}_2-3\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\right)-\vec{S}_1\cdot\vec{S}_2\;\vec{S}_1\cdot\vec{v}_1\times\vec{n}\right]\\ &+2\frac{G^2}{r^4}\left[\dot{\vec{S}}_1\cdot\vec{n}\;\vec{S}_1\cdot\vec{S}_2\times\vec{n}-\vec{S}_1\cdot\vec{n}\;\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}-\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{S}_2\right],\end{aligned} \tag{3.24}$$

$$\text{Fig. 3(b14)} = -8\frac{G^2}{r^5}\,\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left[\vec{S}_1\cdot\vec{S}_2-3\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\right], \tag{3.25}$$

$$\begin{aligned}\text{Fig. 3(b15)} = {}& -\frac{G^2}{r^5}\Big[2\,\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{v}_1-\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left(5\,\vec{S}_1\cdot\vec{S}_2-9\vec{S}_1\cdot\vec{n}\,\vec{S}_2\cdot\vec{n}\right)\\ &+3\,\vec{S}_2\cdot\vec{v}_1\times\vec{n}\left(S_1^2-(\vec{S}_1\cdot\vec{n})^2\right)\Big].\end{aligned} \tag{3.26}$$

3.3 Cubic self-interaction

As can be seen in figure 4 we have 24 graphs of cubic self-interaction in this sector, 6 of which contain time-dependent self-interaction, similar to what we have in the odd parity spin-orbit sector [7, 11, 12]. Similar to the nonlinear graphs of two-graviton exchange, these

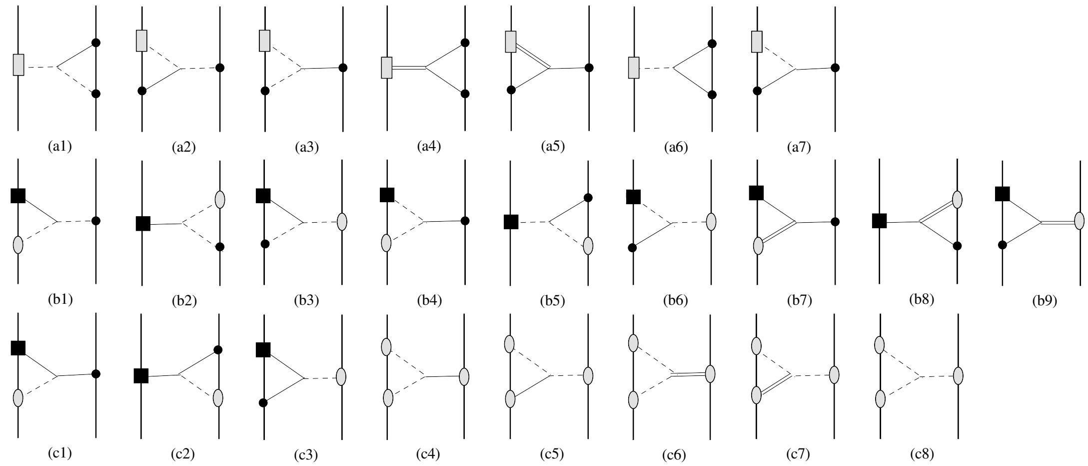


Figure 4. The Feynman graphs at one-loop level, i.e. with cubic self-gravitational interaction, which contribute to the NLO cubic-in-spin interaction at the 4.5PN order for maximally rotating compact objects. These graphs should be included together with their mirror images, i.e. with the worldline labels $1 \leftrightarrow 2$ exchanged. Similar to the nonlinear graphs of two-graviton exchange, these graphs include all relevant interactions among the spin-induced quadrupole, octupole, and the mass and spin, and we have here nonlinearities originating from gravitons sourced strictly from minimal coupling to the worldline as shown in graphs (c4)-(c8). We also have here cubic vertices containing time derivatives, similar to what we have in the NLO odd parity spin-orbit sector [7, 11, 12].

graphs include all relevant interactions among the spin-induced quadrupole, octupole, and the mass and spin, and we have here nonlinearities originating from gravitons sourced strictly from minimal coupling to the worldline as shown in graphs (c4)-(c8). This sector required using tensor one-loop integrals of up to order 5.

The graphs in figure 4 are evaluated as follows:

$$\text{Fig. 4(a1)} = -\frac{16}{3}C_{1(BS^3)}\frac{G^2}{r^5}\frac{m_2^2}{m_1^2}\,\vec{S}_1\cdot\vec{v}_2\times\vec{n}\left[S_1^2-6(\vec{S}_1\cdot\vec{n})^2\right], \tag{3.27}$$

$$\text{Fig. 4(a2)} = -\frac{3}{2}C_{1(BS^3)}\frac{G^2}{r^5}\frac{m_2}{m_1}\,\vec{S}_1\cdot\vec{v}_2\times\vec{n}\left[S_1^2-5(\vec{S}_1\cdot\vec{n})^2\right], \tag{3.28}$$

$$\text{Fig. 4(a3)} = \frac{3}{2}C_{1(BS^3)}\frac{G^2}{r^5}\frac{m_2}{m_1}\,\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left[S_1^2-5(\vec{S}_1\cdot\vec{n})^2\right], \tag{3.29}$$

$$\text{Fig. 4(a4)} = -\frac{1}{3}C_{1(BS^3)}\frac{G^2}{r^5}\frac{m_2^2}{m_1^2}\,\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left[S_1^2-6(\vec{S}_1\cdot\vec{n})^2\right], \tag{3.30}$$

$$\text{Fig. 4(a5)} = -\frac{1}{8}C_{1(BS^3)}\frac{G^2}{r^5}\frac{m_2}{m_1}\,\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left[S_1^2-5(\vec{S}_1\cdot\vec{n})^2)\right], \tag{3.31}$$

$$\text{Fig. 4(a6)} = \frac{1}{3}C_{1(BS^3)}\frac{G^2}{r^5}\frac{m_2^2}{m_1^2}\,\vec{S}_1\cdot\vec{v}_2\times\vec{n}\left[S_1^2-6(\vec{S}_1\cdot\vec{n})^2\right], \tag{3.32}$$

$$\text{Fig. 4(a7)} = \frac{1}{8}C_{1(BS^3)}\frac{G^2}{r^5}\frac{m_2}{m_1}\,\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left[S_1^2-5(\vec{S}_1\cdot\vec{n})^2\right], \tag{3.33}$$

$$\text{Fig. 4(b1)} = -\frac{1}{2}C_{1(ES^2)}\frac{G^2}{r^5}\frac{m_2}{m_1}\,\vec{S}_1\cdot\vec{v}_2\times\vec{n}\left[S_1^2+3(\vec{S}_1\cdot\vec{n})^2\right], \tag{3.34}$$

$$\text{Fig. 4(b2)} = -8C_{1(ES^2)}\frac{G^2}{r^5}\frac{m_2}{m_1}\left[\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\ \vec{S}_1\cdot\vec{n}+\ \vec{S}_2\cdot\vec{v}_2\times\vec{n}\big(S_1^2-3(\vec{S}_1\cdot\vec{n})^2\big)\right], \tag{3.35}$$

$$\begin{aligned}\text{Fig. 4(b3)} = {} & 4C_{1(ES^2)}\frac{G^2}{r^5}\Big[4\ \vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\ \vec{S}_1\cdot\vec{n}+\ \vec{S}_1\cdot\vec{S}_2\times\vec{n}\big(\vec{S}_1\cdot\vec{v}_1-6\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\big)\\ & +\vec{S}_2\cdot\vec{v}_1\times\vec{n}\big(2S_1^2-9(\vec{S}_1\cdot\vec{n})^2\big)\Big],\end{aligned} \tag{3.36}$$

$$\begin{aligned}\text{Fig. 4(b4)} = {} & \frac{1}{2}C_{1(ES^2)}\frac{G^2}{r^5}\frac{m_2}{m_1}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left[S_1^2+3(\vec{S}_1\cdot\vec{n})^2\right]\\ & -2C_{1(ES^2)}\frac{G^2}{r^4}\frac{m_2}{m_1}\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{n}\ \vec{S}_1\cdot\vec{n},\end{aligned} \tag{3.37}$$

$$\begin{aligned}\text{Fig. 4(b5)} = {} & 8C_{1(ES^2)}\frac{G^2}{r^5}\frac{m_2}{m_1}\Big[\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\ \vec{S}_1\cdot\vec{n}+\vec{S}_2\cdot\vec{v}_1\times\vec{n}\big(S_1^2-3(\vec{S}_1\cdot\vec{n})^2\big)\Big]\\ & -4C_{1(ES^2)}\frac{G^2}{r^4}\frac{m_2}{m_1}\left[\vec{S}_1\cdot\vec{S}_2\times\vec{n}\ \dot{\vec{S}}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\ \vec{S}_1\cdot\vec{n}\right],\end{aligned} \tag{3.38}$$

$$\begin{aligned}\text{Fig. 4(b6)} = {} & 4C_{1(ES^2)}\frac{G^2}{r^5}\Big[2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\ \vec{S}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{S}_2\times\vec{n}\big(\vec{S}_1\cdot\vec{v}_1-6\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\big)\\ & +\vec{S}_2\cdot\vec{v}_1\times\vec{n}\big(2S_1^2-9(\vec{S}_1\cdot\vec{n})^2\big)\Big]\\ & -12C_{1(ES^2)}\frac{G^2}{r^4}\left[\vec{S}_1\cdot\vec{S}_2\times\vec{n}\ \dot{\vec{S}}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\ \vec{S}_1\cdot\vec{n}\right],\end{aligned} \tag{3.39}$$

$$\text{Fig. 4(b7)} = -\frac{3}{8}C_{1(ES^2)}\frac{G^2}{r^5}\frac{m_2}{m_1}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\left[S_1^2-5(\vec{S}_1\cdot\vec{n})^2\right], \tag{3.40}$$

$$\text{Fig. 4(b8)} = 2C_{1(ES^2)}\frac{G^2}{r^5}\frac{m_2}{m_1}\left[\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}+\vec{S}_2\cdot\vec{v}_2\times\vec{n}\big(S_1^2-3(\vec{S}_1\cdot\vec{n})^2\big)\right], \tag{3.41}$$

$$\begin{aligned}\text{Fig. 4(b9)} = {} & \frac{1}{4}C_{1(ES^2)}\frac{G^2}{r^5}\Big[4\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\ \vec{S}_1\cdot\vec{n}-2\vec{S}_1\cdot\vec{S}_2\times\vec{n}\big(\vec{S}_1\cdot\vec{v}_2-3\vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\\ & +3\vec{S}_2\cdot\vec{v}_2\times\vec{n}\big(S_1^2-5(\vec{S}_1\cdot\vec{n})^2\big)\Big],\end{aligned} \tag{3.42}$$

$$\begin{aligned}\text{Fig. 4(c1)} = {} & \frac{3}{8}C_{1(ES^2)}\frac{G^2}{r^5}\frac{m_2}{m_1}\Big[\vec{S}_1\cdot\vec{v}_1\times\vec{n}\big(S_1^2-5(\vec{S}_1\cdot\vec{n})^2\big)\Big]\\ & -C_{1(ES^2)}\frac{G^2}{r^4}\frac{m_2}{m_1}\left[\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{n}\ \vec{S}_1\cdot\vec{n}\right],\end{aligned} \tag{3.43}$$

$$\text{Fig. 4(c2)} = -2C_{1(ES^2)}\frac{G^2}{r^5}\frac{m_2}{m_1}\left[\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\,\vec{S}_1\cdot\vec{n}+\vec{S}_2\cdot\vec{v}_2\times\vec{n}\big(S_1^2-3(\vec{S}_1\cdot\vec{n})^2\big)\right], \tag{3.44}$$

$$\begin{aligned}\text{Fig. 4(c3)} = {} & -\frac{1}{4}C_{1(ES^2)}\frac{G^2}{r^5}\Big[4\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\ \vec{S}_1\cdot\vec{n}-2\vec{S}_1\cdot\vec{S}_2\times\vec{n}\big(\vec{S}_1\cdot\vec{v}_1-3\vec{S}_1\cdot\vec{n}\vec{v}_1\cdot\vec{n}\big)\\ & +3\ \vec{S}_2\cdot\vec{v}_1\times\vec{n}\big(S_1^2-5(\vec{S}_1\cdot\vec{n})^2\big)\Big]\\ & +C_{1(ES^2)}\frac{G^2}{r^4}\left[\vec{S}_1\cdot\vec{S}_2\times\vec{n}\ \dot{\vec{S}}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\ \vec{S}_1\cdot\vec{n}\right],\end{aligned} \tag{3.45}$$

$$\text{Fig. 4(c4)} = 4\frac{G^2}{r^5}\left[\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\ \vec{S}_1\cdot\vec{n}-3\vec{S}_2\cdot\vec{v}_2\times\vec{n}\ (\vec{S}_1\cdot\vec{n})^2\right], \tag{3.46}$$

$$\text{Fig. 4(c5)} = 4\frac{G^2}{r^5}\left[\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\ \vec{S}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\ \vec{S}_1\cdot\vec{S}_2\right], \tag{3.47}$$

$$
\text{Fig. 4(c6)} = -\frac{1}{2}\frac{G^2}{r^5}\Big[\, 15\vec{S}_1\cdot\vec{S}_2\times\vec{n}\big(\vec{S}_1\cdot\vec{v}_2-\vec{S}_1\cdot\vec{n}\;\vec{v}_2\cdot\vec{n}\big) - \vec{S}_1\cdot\vec{v}_2\times\vec{n}\big(14\vec{S}_1\cdot\vec{S}_2-12\vec{S}_1\cdot\vec{n}\;\vec{S}_2\cdot\vec{n}\big)+\frac{1}{2}\vec{S}_2\cdot\vec{v}_2\times\vec{n}\big(29S_1^2-33(\vec{S}_1\cdot\vec{n})^2\big)\Big], \tag{3.48}
$$

$$
\text{Fig. 4(c7)} = -\frac{G^2}{r^5}\Big[4\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\;\vec{S}_1\cdot\vec{n}+3\vec{S}_1\cdot\vec{S}_2\times\vec{n}\big(\vec{S}_1\cdot\vec{v}_1-4\vec{S}_1\cdot\vec{n}\;\vec{v}_1\cdot\vec{n}\big) + \vec{S}_2\cdot\vec{v}_1\times\vec{n}\big(S_1^2-6(\vec{S}_1\cdot\vec{n})^2\big)\Big], \tag{3.49}
$$

$$
\begin{aligned}
\text{Fig. 4(c8)} = \frac{1}{4}\frac{G^2}{r^5}\Big[\, & 4\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\;\vec{S}_1\cdot\vec{n}+8\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\;\vec{S}_1\cdot\vec{n} \\
& + 6\vec{S}_1\cdot\vec{S}_2\times\vec{n}\big(\vec{S}_1\cdot\vec{v}_1-5\vec{S}_1\cdot\vec{n}\;\vec{v}_1\cdot\vec{n}\big)+8\vec{S}_1\cdot\vec{S}_2\times\vec{n}\big(\vec{S}_1\cdot\vec{v}_2-6\vec{S}_1\cdot\vec{n}\;\vec{v}_2\cdot\vec{n}\big) \\
& + 8\vec{S}_1\cdot\vec{v}_1\times\vec{n}\big(\vec{S}_1\cdot\vec{S}_2-3\vec{S}_1\cdot\vec{n}\;\vec{S}_2\cdot\vec{n}\big)+\vec{S}_2\cdot\vec{v}_1\times\vec{n}\big(5S_1^2-9(\vec{S}_1\cdot\vec{n})^2\big)\Big] \\
& + \frac{G^2}{r^4}\Big[\vec{S}_1\cdot\vec{S}_2\times\vec{n}\;\dot{\vec{S}}_1\cdot\vec{n}-2\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\;\vec{S}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\;\vec{S}_1\cdot\vec{n}\Big].
\end{aligned} \tag{3.50}
$$

4 New features from gauge of rotational DOFs

The formulation of the EFT of a spinning gravitating particle in [11] consisted of an action initially taken in the covariant gauge as formulated by Tulczyjew in [44]. The latter presented the spin supplementary condition (SSC) given by $S_{\mu\nu}p^\nu = 0$, which as noted in [11], corresponds to the choice $e_0^\mu = p^\mu/\sqrt{p^2}$ for the timelike component of the worldline tetrad. This gauge is distinguished among possible covariant gauges as the only gauge of rotational DOFs for which the existence and uniqueness of a corresponding "center" for the spinning particle were proven rigorously in General Relativity [45, 46].

For this reason the formulation in [11] was made in terms of the linear momentum p^μ rather than the four-velocity u^μ for example, as in general the former is given by

$$
p_\mu = -\frac{\partial L}{\partial u^\mu} = m\frac{u_\mu}{\sqrt{u}^2}+\mathcal{O}(RS^2). \tag{4.1}
$$

Therefore the difference between using p_μ and u_μ would show up, as was pointed out in [11], as of the NLO of the sector cubic in the spins, namely the sector that we are studying in this work.

Let us then find how these new feature transpires in this sector. Since we are working to cubic order in the spin in this sector, we should take into account in the linear momentum beyond the leading term, which was the only one required for lower order spin sectors thus far, only the first correction, that is we now consider also

$$
\Delta p_\kappa \equiv p_\kappa - \bar{p}_\kappa \simeq \frac{C_{ES^2}}{2m}S^\mu S^\nu\left(\frac{2}{u}R_{\mu\alpha\nu\kappa}u^\alpha-\frac{1}{u^3}R_{\mu\alpha\nu\beta}u^\alpha u^\beta u_\kappa\right), \tag{4.2}
$$

where we have denoted the leading approximation to the linear momentum as $\bar{p}_\kappa \equiv \frac{m}{u}u_\kappa$. Let us also note that due to eq. (4.8) of [11] at this order the spin vectors and the spin vectors can be used interchangeably.

Hence, the part that is linear in the spin in the action of the spinning particle actually gives rise to a new type of worldline-graviton couplings that are cubic in the spin, due to its dependence in the linear momentum. We recall that the relevant part of the Lagrangian is given as follows [11]:

$$L_S = -\frac{1}{2}\hat{S}_{ab}\hat{\Omega}^{ab}_{\text{flat}} - \frac{1}{2}\hat{S}_{ab}\omega_\mu^{ab}u^\mu - \frac{\hat{S}_{ab}p^b}{p^2}\frac{Dp^a}{D\lambda}, \tag{4.3}$$

where the hatted DOFs represent the generic rotational DOFs. Therefore the new contributions arise from substituting in the gauge, which we choose here as the canonical gauge formulated in [11] as

$$\hat{\Lambda}_{[0]}^{\ a} = \delta^a_0, \quad \hat{S}^{ab}\left(p_b + p\delta_{0b}\right) = 0, \tag{4.4}$$

into the linear-in-spin couplings, as well as from the extra term that enters from minimal coupling, appearing last in eq. (4.3), which was found in [11] to be related with the gauge of the rotational DOFs, and stands for the Thomas precession as was noted in section 2. Let us stress again that the subtlety here is not about switching from the covariant gauge, but rather about advancing from using in the gauge u_ν to p_ν as the basic covariant gauge, which is necessary as of this nonlinear order in gravity and cubic order in spins.

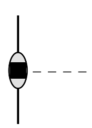
Working out explicitly this part of the action in terms of the local spin variable in the canonical gauge similarly to the derivations in [11], and keeping only terms that lead to new cubic-in-spin couplings, we obtain here the following contribution:

$$L_{S\to S^3} = \omega_\mu^{ij}u^\mu\frac{\hat{S}_{ik}p^k p_j}{p\left(p+p^0\right)} - \omega_\mu^{0i}u^\mu\frac{\hat{S}_{ij}p^j}{p} + \frac{\hat{S}_{ij}p^i\dot{p}^j}{p\left(p+p^0\right)}. \tag{4.5}$$

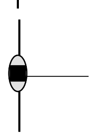
where in principle all the indices here are in the locally flat frame. In order to obtain the new cubic-in-spin couplings we only need to substitute in the correction to the linear momentum from eq. (4.2) to linear order, keeping in mind that all the contributions at the zeroth order are taken into account in section 3 above, and section 5 below.

At this point it becomes clear that the first two terms in eq. (4.5) give rise to new two-graviton couplings, and that the last term gives rise to new one-graviton couplings containing higher order time derivatives.

The resulting new Feynman rules for the one-graviton couplings are then:



$$= \int dt\,\left[\frac{C_{\text{ES}^2}}{4m^2}S_iS_j\epsilon_{klm}\Big[\Big(2S_ma^k + \dot{S}_mv^k\Big)\Big(A_{l,ij} - A_{j,il}\Big)\Big]\right], \tag{4.6}$$



$$= \int dt\,\Bigg[-\frac{C_{\text{ES}^2}}{2m^2}S_iS_j\epsilon_{klm}\Big[2S_ma^k\Big(2\big(\phi_{,ij}v^l - \phi_{,il}v^j\big) + \delta_{ij}\left(\partial_t\phi_{,l} + \phi_{,ln}v^n\right)\Big) - \dot{S}_mv^k\Big(2\phi_{,il}v^j - \delta_{ij}\left(\partial_t\phi_{,l} + \phi_{,ln}v^n\right) + \delta_{il}\left(\partial_t\phi_{,j} + \phi_{,jn}v^n\right)\Big)\Big]\Bigg], \tag{4.7}$$

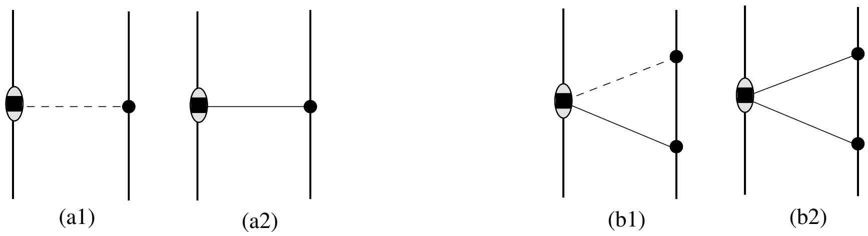
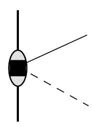
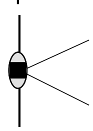


Figure 5. The extra one- and two-graviton exchange Feynman graphs, which appear at the NLO cubic-in-spin interaction at the 4.5PN order for maximally rotating compact objects. These graphs should be included together with their mirror images, i.e. with the worldline labels $1 \leftrightarrow 2$ exchanged. These graphs contain a new type of worldline-graviton couplings, which we dub as the "composite" octupole ones, and obviously yield similar graphs to the corresponding graphs with the "elementary" spin-induced octupole in figure 2 (a1),(a2) and in figure 3 (b1),(b2).

where a black square mounted on an oval blob represents this new type of "composite" cubic-in-spin worldline couplings. Notice that all these rules contain accelerations and even time derivatives of spins similar to the acceleration terms that appear first in the rules for the spin-orbit sector [11].

For the new two-graviton couplings we get the following rules:


$$= \int dt\, \left[\frac{C_{\rm ES^2}}{2m^2} S_i S_j \epsilon_{klm} S_m \phi_{,k} \left(A_{l,ij} - A_{j,il}\right)\right], \tag{4.8}$$


$$= \int dt\, \left[-\frac{C_{\rm ES^2}}{m^2} S_i S_j \epsilon_{klm} S_m \phi_{,k} \Big(2\big(\phi_{,ij} v^l - \phi_{,il} v^j\big) + \delta_{ij}\left(\partial_t \phi_{,l} + \phi_{,ln} v^n\right)\Big)\right]. \tag{4.9}$$

Note that the mass ratio together with the Wilson coefficients in these rules for the new cubic-in-spin couplings indicate that these are truly new couplings that cannot be absorbed in the existing "elementary" octupole operator.

These new couplings give rise to four additional graphs shown in figure 5, similar to those in figure 2 (a1), (a2), and in figure 3 (b1), (b2). The graphs in figure 5 are evaluated as follows:

$$\begin{aligned}
\text{Fig. 5(a1)} = -C_{1(ES^2)} \frac{G}{r^3} \frac{m_2}{m_1^2} \Bigg[& 2\vec{S}_1 \cdot \vec{v}_2 \times \vec{a}_1 \left(S_1^2 - 3\left(\vec{S}_1 \cdot \vec{n}\right)^2\right) - 6\vec{S}_1 \cdot \vec{a}_1 \times \vec{n} \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \\
& - \dot{\vec{S}}_1 \cdot \vec{S}_1 \times \vec{v}_1 \vec{S}_1 \cdot \vec{v}_2 - \dot{\vec{S}}_1 \cdot \vec{v}_1 \times \vec{v}_2 \left(S_1^2 - 3\left(\vec{S}_1 \cdot \vec{n}\right)^2\right) \\
& - 3\dot{\vec{S}}_1 \cdot \vec{v}_1 \times \vec{n} \vec{S}_1 \cdot \vec{v}_2 \vec{S}_1 \cdot \vec{n} \Bigg],
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
\text{Fig. 5(a2)} = \frac{1}{2} C_{1(ES^2)} \frac{G}{r^3} \frac{m_2}{m_1^2} \Bigg[& 6\vec{S}_1 \cdot \vec{v}_1 \times \vec{a}_1 \left(S_1^2 - 2\left(\vec{S}_1 \cdot \vec{n}\right)^2\right) - 2\vec{S}_1 \cdot \vec{v}_2 \times \vec{a}_1 S_1^2 \\
& + 6\vec{S}_1 \cdot \vec{a}_1 \times \vec{n} \left(S_1^2 \big(\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}\big) - 2\vec{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{n}\right)
\end{aligned}$$

$$- \dot{\vec{S}}_1 \cdot \vec{S}_1 \times \vec{v}_1 \left(3\vec{S}_1 \cdot \vec{v}_1 - \vec{S}_1 \cdot \vec{v}_2 - 3\vec{S}_1 \cdot \vec{n}\,(\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n})\right)$$

$$+ \dot{\vec{S}}_1 \cdot \vec{v}_1 \times \vec{v}_2\, S_1^2 + 3\dot{\vec{S}}_1 \cdot \vec{v}_1 \times \vec{n} \left(S_1^2\,(\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}) - 2\vec{S}_1 \cdot \vec{v}_1 \vec{S}_1 \cdot \vec{n}\right)\Bigg], \tag{4.11}$$

$$\text{Fig. 5(b1)} = -2C_{1(ES^2)} \frac{G^2}{r^5} \frac{m_2^2}{m_1^2}\, \vec{S}_1 \cdot \vec{v}_2 \times \vec{n} \left[S_1^2 - 3(\vec{S}_1 \cdot \vec{n})^2\right], \tag{4.12}$$

$$\text{Fig. 5(b2)} = C_{1(ES^2)} \frac{G^2}{r^5} \frac{m_2^2}{m_1^2} \left[3\vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\big(S_1^2 - 2(\vec{S}_1 \cdot \vec{n})^2\big) - \vec{S}_1 \cdot \vec{v}_2 \times \vec{n}\, S_1^2\right]. \tag{4.13}$$

5 The cubic-in-spin action at the next-to-leading order

Let us then put together all the results from sections 3 and 4 to get the final effective action for this sector. This summation includes the values presented above plus similar results under the exchange of particle labels $1 \leftrightarrow 2$, where $\vec{n} \to -\vec{n}$. Next, we apply the 4 vectors identity for 3 dimensions presented in eq. (3.14) of [10], to further simplify and compress the results. As was already noted these results contain higher order time derivatives of both the velocity and the spin, which would be treated rigorously at the level of the action, following the procedure shown in [43], by making variable redefinitions which will remove the higher order terms (in complete analogy to the removal of redundant/on-shell operators by field redefinitions in effective theories as was pointed out by one of the authors in [43]).

The final result of these steps is then given as follows:

$$L^{\text{NLO}}_{\text{S}^3} = L^{\text{NLO}}_{\text{S}_1^2\text{S}_2} + L^{\text{NLO}}_{\text{S}_1^3} + (1 \leftrightarrow 2), \tag{5.1}$$

where we have:

$$\begin{aligned} L^{\text{NLO}}_{\text{S}_1^2\text{S}_2} = &+ \frac{G^2}{r^5} L_{(1)} + C_{1(ES^2)} \frac{G}{r^4} \frac{1}{m_1} L_{(2)} + C_{1(ES^2)} \frac{G^2}{r^5} L_{(3)} + C_{1(ES^2)} \frac{G^2 m_2}{r^5 m_1} L_{(4)} \\ &+ \frac{G^2}{r^4} L_{(5)} + C_{1(ES^2)} \frac{G}{r^3} \frac{1}{m_1} L_{(6)} + C_{1(ES^2)} \frac{G^2}{r^4} L_{(7)} + C_{1(ES^2)} \frac{G^2 m_2}{r^4 m_1} L_{(8)} \\ &+ C_{1(ES^2)} \frac{G}{r^2} \frac{1}{m_1} L_{(9)} + C_{1(ES^2)} \frac{G}{r} \frac{1}{m_1} L_{(10)}, \end{aligned} \tag{5.2}$$

with the following pieces:

$$\begin{aligned} L_{(1)} = &+ \vec{S}_1 \cdot \vec{S}_2 \times \vec{n}\Big(- 5\vec{S}_1 \cdot \vec{v}_1 + \vec{S}_1 \cdot \vec{v}_2 + \frac{9}{2}\, \vec{S}_1 \cdot \vec{n}\; \vec{v}_1 \cdot \vec{n} - \frac{9}{2}\, \vec{S}_1 \cdot \vec{n}\; \vec{v}_2 \cdot \vec{n}\Big) \\ &+ \vec{S}_1 \cdot \vec{v}_1 \times \vec{n}\; \big(\frac{5}{2}\, \vec{S}_1 \cdot \vec{S}_2 + 9\, \vec{S}_1 \cdot \vec{n}\; \vec{S}_2 \cdot \vec{n}\big) - \frac{3}{4} \vec{S}_2 \cdot \vec{v}_2 \times \vec{n}\big(S_1^2 + 5\big(\vec{S}_1 \cdot \vec{n}\big)^2\big) \\ &+ \vec{S}_1 \cdot \vec{v}_2 \times \vec{n}\big(\frac{5}{2} \vec{S}_1 \cdot \vec{S}_2 - 12\, \vec{S}_1 \cdot \vec{n}\; \vec{S}_2 \cdot \vec{n}\big) - \frac{1}{4}\, \vec{S}_2 \cdot \vec{v}_1 \times \vec{n}\; \big(17\, S_1^2 - 27\big(\vec{S}_1 \cdot \vec{n}\big)^2\big) \\ &+ \frac{1}{2}\big(5\vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 - \vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_2\big)\; \vec{S}_1 \cdot \vec{n}, \end{aligned} \tag{5.3}$$

$$L_{(2)} = + 3\vec{S}_1 \cdot \vec{S}_2 \times \vec{v}_1 \Big(\vec{S}_1 \cdot \vec{v}_1\; \big(- \vec{v}_1 \cdot \vec{n} + \vec{v}_2 \cdot \vec{n}\big) + \vec{S}_1 \cdot \vec{v}_2\; \big(\vec{v}_1 \cdot \vec{n} - \vec{v}_2 \cdot \vec{n}\big)\Big)$$

$$
\begin{aligned}
&+3\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\Big(\vec{S}_1\cdot\vec{v}_1\ \vec{v}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{v}_2\ \vec{v}_1\cdot\vec{n}\Big)+\frac{3}{2}\ \vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\ \vec{v}_1\cdot\vec{n}\big(3S_1^2-5\big(\vec{S}_1\cdot\vec{n}\big)^2\big)\\
&-3\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\ \vec{S}_1\cdot\vec{S}_2+\vec{S}_2\cdot\vec{v}_1\times\vec{n}\Big(\frac{15}{2}S_1^2\ \big(v_1^2-\vec{v}_1\cdot\vec{v}_2-2(\vec{v}_1\cdot\vec{n})^2-\vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\\
&+3\vec{S}_1\cdot\vec{v}_1\big(-\vec{S}_1\cdot\vec{v}_1+\vec{S}_1\cdot\vec{v}_2+5\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}-5\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)-15\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\\
&+\frac{15}{2}\big(\vec{S}_1\cdot\vec{n}\big)^2\ \big(-v_1^2+\vec{v}_1\cdot\vec{v}_2+7\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\Big)\\
&+\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big(\frac{3}{2}\ S_1^2\big(-5v_1^2+8\vec{v}_1\cdot\vec{v}_2-3v_2^2+10(\vec{v}_1\cdot\vec{n})^2+5\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\\
&+3\vec{S}_1\cdot\vec{v}_1\big(\vec{S}_1\cdot\vec{v}_1-\vec{S}_1\cdot\vec{v}_2-5\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}+5\vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)+15\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\\
&+\frac{15}{2}\big(\vec{S}_1\cdot\vec{n}\big)^2\big(v_1^2-2\vec{v}_1\cdot\vec{v}_2+v_2^2-7\vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\Big)\\
&+3\vec{S}_1\cdot\vec{S}_2\times\vec{n}\ \Big(\vec{S}_1\cdot\vec{v}_1\big(\ v_1^2-\vec{v}_1\cdot\vec{v}_2-5\vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\\
&+\vec{S}_1\cdot\vec{v}_2\big(-v_1^2-v_2^2+2\vec{v}_1\cdot\vec{v}_2+5\vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\Big)-3\vec{S}_1\cdot\vec{v}_1\times\vec{n}\ \vec{S}_1\cdot\vec{S}_2\ \big(v_1^2-\vec{v}_1\cdot\vec{v}_2\\
&-5\vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)+3\vec{S}_1\cdot\vec{v}_2\times\vec{n}\ \vec{S}_1\cdot\vec{S}_2\ \big(v_1^2-2\vec{v}_1\cdot\vec{v}_2+v_2^2-5\vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big),
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
L_{(3)}=&+\frac{1}{2}\ \vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\ \vec{S}_1\cdot\vec{n}-\frac{1}{2}\ \vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\ \vec{S}_1\cdot\vec{n}+\frac{3}{2}\vec{S}_1\cdot\vec{S}_2\big(\vec{S}_1\cdot\vec{v}_1\times\vec{n}-\vec{S}_1\cdot\vec{v}_2\times\vec{n}\big)\\
&+\ \vec{S}_1\cdot\vec{S}_2\times\vec{n}\big(-\vec{S}_1\cdot\vec{v}_1+\vec{S}_1\cdot\vec{v}_2-\frac{3}{2}\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}+\frac{3}{2}\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\\
&-\frac{3}{4}\ \big(\vec{S}_2\cdot\vec{v}_1\times\vec{n}-\vec{S}_2\cdot\vec{v}_2\times\vec{n}\big)\big(3S_1^2-5(\vec{S}_1\cdot\vec{n})^2\big),
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
L_{(4)}=&-10\vec{S}_1\cdot\vec{n}\big(\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1-\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\ \big)+21\vec{S}_1\cdot\vec{S}_2\big(\vec{S}_1\cdot\vec{v}_1\times\vec{n}-\vec{S}_1\cdot\vec{v}_2\times\vec{n}\big)\\
&-2\Big(\vec{S}_2\cdot\vec{v}_1\times\vec{n}-\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big)\big(14\ S_1^2-21\big(\vec{S}_1\cdot\vec{n}\big)^2\big)\\
&+\vec{S}_1\cdot\vec{S}_2\times\vec{n}\big(-31\vec{S}_1\cdot\vec{v}_1+31\vec{S}_1\cdot\vec{v}_2+63\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}-66\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big),
\end{aligned}
\tag{5.6}
$$

$$
L_{(5)}=\ +3\vec{S}_1\cdot\vec{S}_2\times\vec{n}\ \dot{\vec{S}}_1\cdot\vec{n}-\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\ \vec{S}_1\cdot\vec{n}-2\ \vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\ \vec{S}_1\cdot\vec{n}+2\vec{S}_1\cdot\dot{\vec{S}}_1\times\vec{S}_2,
\tag{5.7}
$$

$$
\begin{aligned}
L_{(6)}=&\ +\frac{1}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\Big(\dot{\vec{S}}_1\cdot\vec{v}_2-2\vec{S}_1\cdot\vec{a}_2+3\vec{S}_1\cdot\vec{n}\ \vec{a}_1\cdot\vec{n}+3\ \dot{\vec{S}}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}-3\ \dot{\vec{S}}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\Big)\\
&+\frac{1}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\Big(\dot{\vec{S}}_1\cdot\vec{v}_1+2\vec{S}_1\cdot\vec{a}_1-2\ \dot{\vec{S}}_1\cdot\vec{v}_2-3\ \dot{\vec{S}}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}-6\ \vec{S}_1\cdot\vec{n}\ \vec{a}_1\cdot\vec{n}\Big)\\
&+\frac{3}{2}\ \vec{S}_1\cdot\vec{S}_2\times\vec{a}_1\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}+\frac{1}{2}\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_1\Big(\vec{S}_1\cdot\vec{v}_2+3\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}-3\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\Big)\\
&+\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{v}_1\big(\vec{S}_1\cdot\vec{v}_1-\vec{S}_1\cdot\vec{v}_2\big)\\
&+\frac{1}{2}\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\Big(\vec{S}_1\cdot\vec{v}_1-3\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}-2\ \vec{S}_1\cdot\vec{v}_2\Big)\\
&-\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\ \vec{S}_1\cdot\dot{\vec{S}}_2-\vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\big(\dot{\vec{S}}_1\cdot\vec{S}_1-3\ \dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\big)\\
&+\big(\vec{S}_1\cdot\vec{v}_2\times\vec{a}_1-\vec{S}_1\cdot\vec{v}_1\times\vec{a}_2\big)\vec{S}_1\cdot\vec{S}_2+\big(\vec{S}_2\cdot\vec{v}_2\times\vec{a}_2+\frac{1}{2}\vec{S}_2\cdot\vec{v}_2\times\vec{a}_1\big)\big(S_1^2-3(\vec{S}_1\cdot\vec{n})^2\big)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3}{2}\big(\vec{S}_2\cdot\vec{v}_1\times\vec{a}_2+\dot{\vec{S}}_2\cdot\vec{v}_1\times\vec{v}_2\big)\big(S_1^2-3(\vec{S}_1\cdot\vec{n})^2\big)\\
&+\frac{3}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{n}\Big(2\ \vec{S}_1\cdot\vec{n}\ \vec{a}_1\cdot\vec{v}_2+\ \dot{\vec{S}}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{v}_2-\dot{\vec{S}}_1\cdot\vec{v}_2\ \vec{v}_2\cdot\vec{n}\\
&-\dot{\vec{S}}_1\cdot\vec{v}_2\ \vec{v}_1\cdot\vec{n}+2\vec{S}_1\cdot\vec{a}_1\ \vec{v}_2\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{v}_1\ \vec{v}_2\cdot\vec{n}\\
&+2\vec{S}_1\cdot\vec{a}_2\ \vec{v}_1\cdot\vec{n}-2\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{a}_1-\dot{\vec{S}}_1\cdot\vec{n}\ v_1^2+5\dot{\vec{S}}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\Big)\\
&+\frac{3}{2}\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\Big(\vec{S}_1\cdot\vec{v}_1\ \vec{v}_2\cdot\vec{n}-\vec{S}_1\cdot\vec{v}_2\ \vec{v}_2\cdot\vec{n}-\vec{S}_1\cdot\vec{v}_2\ \vec{v}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{n}\ v_1^2\\
&+\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{v}_2+5\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\Big)\\
&-3\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\big(\vec{S}_1\cdot\vec{v}_1-\vec{S}_1\cdot\vec{v}_2\big)\ \vec{v}_1\cdot\vec{n}\\
&+\frac{3}{2}\vec{S}_2\cdot\vec{v}_1\times\vec{n}\Big(S_1^2\ \vec{a}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{S}_1\ (2\vec{v}_1\cdot\vec{n}+6\vec{v}_2\cdot\vec{n})-\vec{S}_1\cdot\vec{v}_1\ \dot{\vec{S}}_1\cdot\vec{n}-\dot{\vec{S}}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}\\
&-\vec{S}_1\cdot\vec{a}_1\ \vec{S}_1\cdot\vec{n}-10\ \dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\Big)\\
&+\frac{3}{2}\vec{S}_2\cdot\vec{a}_1\times\vec{n}\Big(S_1^2\ \vec{v}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}+3S_1^2\ \vec{v}_2\cdot\vec{n}-5\ (\vec{S}_1\cdot\vec{n})^2\vec{v}_2\cdot\vec{n}\Big)\\
&+\frac{3}{2}\vec{S}_1\cdot\vec{v}_1\times\vec{n}\big(2\vec{S}_1\cdot\vec{S}_2\ \vec{a}_1\cdot\vec{n}+2\vec{S}_1\cdot\dot{\vec{S}}_2\ \vec{v}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{S}_2\ \vec{v}_1\cdot\vec{n}-2\dot{\vec{S}}_1\cdot\vec{S}_2\ \vec{v}_2\cdot\vec{n}\big)\\
&-\frac{3}{2}\vec{S}_1\cdot\vec{v}_2\times\vec{n}\big(2\vec{S}_1\cdot\vec{S}_2\ \vec{a}_1\cdot\vec{n}+2\vec{S}_1\cdot\dot{\vec{S}}_2\ \vec{v}_1\cdot\vec{n}-\dot{\vec{S}}_1\cdot\vec{S}_2\ \vec{v}_2\cdot\vec{n}\big)\\
&+\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big(\frac{3}{2}(\dot{\vec{S}}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{v}_2\ \dot{\vec{S}}_1\cdot\vec{n})-3\vec{v}_2\cdot\vec{n}\ \big(4\ \dot{\vec{S}}_1\cdot\vec{S}_1-5\ \dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\big)\Big)\\
&-\frac{3}{2}\dot{\vec{S}}_2\cdot\vec{v}_1\times\vec{n}\big(S_1^2\vec{v}_1\cdot\vec{n}+2\ \vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}-5(\vec{S}_1\cdot\vec{n})^2\vec{v}_1\cdot\vec{n}\big)\\
&+\big(\dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{n}+\vec{S}_1\cdot\vec{a}_1\times\vec{n}\big)\big(\frac{3}{2}\ \vec{v}_1\cdot\vec{n}-3\vec{v}_2\cdot\vec{n}\big)\vec{S}_1\cdot\vec{S}_2\\
&+\frac{3}{2}\dot{\vec{S}}_1\cdot\vec{v}_2\times\vec{n}\ \vec{v}_2\ \ \cdot\vec{n}\ \vec{S}_1\cdot\vec{S}_2-3\vec{S}_1\cdot\vec{a}_2\times\vec{n}\ \vec{v}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{S}_2\\
&+\frac{3}{2}\big(\vec{S}_2\cdot\vec{a}_2\times\vec{n}+\dot{\vec{S}}_2\cdot\vec{v}_2\times\vec{n}\big)\big(S_1^2\ \vec{v}_1\cdot\vec{n}+2\ \vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}-5(\vec{S}_1\cdot\vec{n})^2\vec{v}_1\cdot\vec{n}\big),
\end{aligned}
\tag{5.8}
$$

$$
L_{(7)}=\ +\vec{S}_1\cdot\vec{S}_2\times\vec{n}\ \dot{\vec{S}}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\ \vec{S}_1\cdot\vec{n},
\tag{5.9}
$$

$$
L_{(8)}=\ -4\vec{S}_1\cdot\vec{S}_2\times\vec{n}\ \dot{\vec{S}}_1\cdot\vec{n}-13\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\ \vec{S}_1\cdot\vec{n}-4\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\ \vec{S}_1\cdot\vec{n},
\tag{5.10}
$$

$$
\begin{aligned}
L_{(9)}=&-\frac{1}{2}\ \vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{v}_1\ \dot{\vec{S}}_1\cdot\vec{n}-\frac{1}{2}\ \dot{\vec{S}}_1\cdot\dot{\vec{S}}_2\times\vec{v}_1\ \vec{S}_1\cdot\vec{n}-\frac{1}{2}\ddot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\ \vec{S}_1\cdot\vec{n}\\
&-\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\ \dot{\vec{S}}_1\cdot\vec{n}-\frac{1}{2}\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\ \ddot{\vec{S}}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{v}_2\times\vec{n}\ \ddot{\vec{S}}_1\cdot\vec{S}_2\\
&-\Big(\dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{n}+\vec{S}_1\cdot\vec{a}_1\times\vec{n}\Big)\vec{S}_1\cdot\dot{\vec{S}}_2\\
&-\Big(\ddot{\vec{S}}_1\cdot\vec{v}_2\times\vec{n}+\dot{\vec{S}}_1\cdot\vec{a}_2\times\vec{n}\Big)\vec{S}_1\cdot\vec{S}_2-\dot{\vec{S}}_1\cdot\vec{v}_2\times\vec{n}\big(2\dot{\vec{S}}_1\cdot\vec{S}_2+\vec{S}_1\cdot\dot{\vec{S}}_2\big)\\
&-\Big(\vec{S}_1\cdot\vec{v}_1\times\vec{n}+\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big)\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2-\vec{S}_1\cdot\vec{a}_2\times\vec{n}\ \dot{\vec{S}}_1\cdot\vec{S}_2
\end{aligned}
$$

$$
\begin{aligned}
&-\Big(\vec{S}_2\cdot\vec{a}_2\times\vec{n}+\dot{\vec{S}}_2\cdot\vec{v}_2\times\vec{n}-\dot{\vec{S}}_2\cdot\vec{v}_1\times\vec{n}\Big)\big(\dot{\vec{S}}_1\cdot\vec{S}_1-3\dot{\vec{S}}_1\cdot\vec{n}\;\vec{S}_1\cdot\vec{n}\big)\\
&+\vec{S}_1\cdot\vec{S}_2\times\vec{n}\Big(\dot{\vec{S}}_1\cdot\vec{a}_1+\frac{1}{2}\ddot{\vec{S}}_1\cdot\vec{v}_2-\frac{3}{2}\ddot{\vec{S}}_1\cdot\vec{n}\;\vec{v}_2\cdot\vec{n}\Big)\\
&+\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\Big(\frac{1}{2}\dot{\vec{S}}_1\cdot\vec{v}_1+\vec{S}_1\cdot\vec{a}_1+\dot{\vec{S}}_1\cdot\vec{v}_2+\frac{3}{2}\dot{\vec{S}}_1\cdot\vec{n}\;\vec{v}_1\cdot\vec{n}\Big)\\
&+\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\Big(\vec{S}_1\cdot\vec{a}_2+\dot{\vec{S}}_1\cdot\vec{v}_2-3\dot{\vec{S}}_1\cdot\vec{n}\;\vec{v}_2\cdot\vec{n}\Big)\\
&+\frac{1}{2}\ddot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\Big(\vec{S}_1\cdot\vec{v}_2-3\vec{S}_1\cdot\vec{n}\;\vec{v}_2\cdot\vec{n}\Big)\\
&+\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2\times\vec{n}\Big(\frac{1}{2}\vec{S}_1\cdot\vec{v}_1+\vec{S}_1\cdot\vec{v}_2+\frac{3}{2}\vec{S}_1\cdot\vec{n}\;\vec{v}_1\cdot\vec{n}\Big), \qquad (5.11)
\end{aligned}
$$

$$
L_{(10)} = -\frac{1}{2}\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\;\ddot{\vec{S}}_1\cdot\vec{n}-\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2\times\vec{n}\;\dot{\vec{S}}_1\cdot\vec{n}-\frac{1}{2}\ddot{\vec{S}}_1\cdot\dot{\vec{S}}_2\times\vec{n}\;\vec{S}_1\cdot\vec{n}, \qquad (5.12)
$$

and also:

$$
\begin{aligned}
L^{\rm NLO}_{\rm S_1^3} = {}& C_{1(ES^2)}\frac{G^2m_2}{r^5m_1}L_{[1]}+C_{1(ES^2)}\frac{G^2m_2^2}{r^5m_1^2}L_{[2]}+C_{1(BS^3)}\frac{Gm_2}{r^4m_1^2}L_{[3]}\\
&+C_{1(BS^3)}\frac{G^2m_2}{r^5m_1}L_{[4]}+C_{1(BS^3)}\frac{G^2m_2^2}{r^5m_1^2}L_{[5]}+C_{1(ES^2)}\frac{Gm_2}{r^3m_1^2}L_{[6]}\\
&+C_{1(ES^2)}\frac{G^2m_2}{r^4m_1}L_{[7]}+C_{1(BS^3)}\frac{Gm_2}{r^3m_1^2}L_{[8]}+C_{1(BS^3)}\frac{Gm_2}{r^2m_1^2}L_{[9]}, \qquad (5.13)
\end{aligned}
$$

with the pieces:

$$
L_{[1]} = \frac{1}{2}(-\vec{S}_1\cdot\vec{v}_1\times\vec{n}+\vec{S}_1\cdot\vec{v}_2\times\vec{n})\big(S_1^2-9\big(\vec{S}_1\cdot\vec{n}\big)^2\big), \qquad (5.14)
$$

$$
L_{[2]} = +\vec{S}_1\cdot\vec{v}_2\times\vec{n}\left(5S_1^2-6(\vec{S}_1\cdot\vec{n})^2\right), \qquad (5.15)
$$

$$
\begin{aligned}
L_{[3]} = {}& +\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\Big(-3\vec{S}_1\cdot\vec{v}_1\;\vec{S}_1\cdot\vec{n}+\;\vec{S}_1\cdot\vec{v}_2\;\vec{S}_1\cdot\vec{n}-\frac{1}{2}\big(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big)\big(S_1^2-5\big(\vec{S}_1\cdot\vec{n}\big)^2\big)\Big)\\
&+\vec{S}_1\cdot\vec{v}_1\times\vec{n}\Big(\;\frac{1}{2}\;S_1^2\big(v_1^2-\vec{v}_1\cdot\vec{v}_2+v_2^2-5\;\vec{v}_1\cdot\vec{n}\;\vec{v}_2\cdot\vec{n}\big)\\
&+\vec{S}_1\cdot\vec{v}_1\big(-\vec{S}_1\cdot\vec{v}_1+\frac{1}{2}\vec{S}_1\cdot\vec{v}_2+5\;\vec{S}_1\cdot\vec{n}\;\left(\vec{v}_1\cdot\vec{n}-\frac{1}{2}\vec{v}_2\cdot\vec{n}\right)\big)-5\vec{S}_1\cdot\vec{v}_2\;\vec{S}_1\cdot\vec{n}\;\vec{v}_1\cdot\vec{n}\\
&-\frac{5}{2}(\vec{S}_1\cdot\vec{n})^2\big(v_1^2-\;\vec{v}_1\cdot\vec{v}_2+\;v_2^2-7\;\vec{v}_1\cdot\vec{n}\;\vec{v}_2\cdot\vec{n}\big)\Big)\\
&+\vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big(-\frac{1}{2}S_1^2\big(v_1^2-\vec{v}_1\cdot\vec{v}_2+v_2^2-5\;\vec{v}_1\cdot\vec{n}\;\vec{v}_2\cdot\vec{n}\big)\\
&+\vec{S}_1\cdot\vec{v}_1\big(\frac{3}{2}\vec{S}_1\cdot\vec{v}_1-\vec{S}_1\cdot\vec{v}_2-5\vec{S}_1\cdot\vec{n}\;\left(\frac{3}{2}\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\right)\big)+5\vec{S}_1\cdot\vec{v}_2\;\vec{S}_1\cdot\vec{n}\;\vec{v}_1\cdot\vec{n}\\
&+\frac{5}{2}(\vec{S}_1\cdot\vec{n})^2\big(v_1^2-\;\vec{v}_1\cdot\vec{v}_2+\;v_2^2-7\;\vec{v}_1\cdot\vec{n}\;\vec{v}_2\cdot\vec{n}\big)\Big),
\end{aligned}
\qquad (5.16)
$$

$$L_{[4]} = +\frac{1}{2}\big(\vec{S}_1\cdot\vec{v}_1\times\vec{n}-\vec{S}_1\cdot\vec{v}_2\times\vec{n}\big)\big(S_1^2-5\big(\vec{S}_1\cdot\vec{n}\big)^2\big), \tag{5.17}$$

$$L_{[5]} = -4\,\big(\vec{S}_1\cdot\vec{v}_1\times\vec{n}-\vec{S}_1\cdot\vec{v}_2\times\vec{n}\big)\big(S_1^2-5\big(\vec{S}_1\cdot\vec{n}\big)^2\big), \tag{5.18}$$

$$\begin{aligned}L_{[6]} = &+2\vec{S}_1\cdot\vec{v}_1\times\vec{a}_1\left(S_1^2-3\left(\vec{S}_1\cdot\vec{n}\right)^2\right)-6\vec{S}_1\cdot\vec{v}_2\times\vec{a}_1\left(S_1^2-2\left(\vec{S}_1\cdot\vec{n}\right)^2\right)\\
&-6\vec{S}_1\cdot\vec{a}_1\times\vec{n}\left(S_1^2\vec{v}_2\cdot\vec{n}+\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{n}-2\vec{S}_1\cdot\vec{v}_2\vec{S}_1\cdot\vec{n}\right)\\
&-\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{v}_1\left(2\vec{S}_1\cdot\vec{v}_1-3\vec{S}_1\cdot\vec{v}_2-3\vec{S}_1\cdot\vec{n}\left(\vec{v}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\right)\right)\\
&+3\dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{v}_2\left(S_1^2-2\left(\vec{S}_1\cdot\vec{n}\right)^2\right)\\
&-3\dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{n}\left(S_1^2\vec{v}_2\cdot\vec{n}-\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{n}-2\vec{S}_1\cdot\vec{v}_2\vec{S}_1\cdot\vec{n}\right),\end{aligned} \tag{5.19}$$

$$L_{[7]} = -3\,\dot{\vec{S}}_1\cdot\vec{S}_1\times\vec{n}\;\vec{S}_1\cdot\vec{n}, \tag{5.20}$$

$$\begin{aligned}L_{[8]} =&\frac{1}{6}\Big(2\;\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\big(\dot{\vec{S}}_1\cdot\vec{S}_1-3\;\dot{\vec{S}}_1\cdot\vec{n}\;\vec{S}_1\cdot\vec{n}\big)+\dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{v}_2\big(S_1^2-3(\vec{S}_1\cdot\vec{n})^2\big)\\
&+\vec{S}_1\cdot\vec{a}_1\times\vec{v}_2\big(S_1^2-3(\vec{S}_1\cdot\vec{n})^2\big)+\vec{S}_1\cdot\vec{v}_1\times\vec{a}_2\big(S_1^2-3(\vec{S}_1\cdot\vec{n})^2\big)\\
&-6\;\vec{S}_1\cdot\vec{v}_1\times\vec{n}\;\Big(\vec{S}_1\cdot\vec{v}_1\;\dot{\vec{S}}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{v}_1\;\vec{S}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{a}_1\;\vec{S}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big(\dot{\vec{S}}_1\cdot\vec{S}_1-5\;\dot{\vec{S}}_1\cdot\vec{n}\;\vec{S}_1\cdot\vec{n}\big)\Big)\\
&-3\;\dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{n}\;\Big(2\vec{S}_1\cdot\vec{v}_1\;\vec{S}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big(S_1^2-5(\vec{S}_1\cdot\vec{n})^2\big)\Big)\\
&-3\;\vec{S}_1\cdot\vec{a}_1\times\vec{n}\;\Big(2\vec{S}_1\cdot\vec{v}_1\;\vec{S}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big(S_1^2-5(\vec{S}_1\cdot\vec{n})^2\big)\Big)\\
&+6\;\vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big(\vec{S}_1\cdot\vec{v}_1\;\dot{\vec{S}}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{v}_1\;\vec{S}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{a}_1\;\vec{S}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big(\dot{\vec{S}}_1\cdot\vec{S}_1-5\;\dot{\vec{S}}_1\cdot\vec{n}\;\vec{S}_1\cdot\vec{n}\big)\Big)\\
&+3\;\dot{\vec{S}}_1\cdot\vec{v}_2\times\vec{n}\Big(2\vec{S}_1\cdot\vec{v}_1\;\vec{S}_1\cdot\vec{n}-\vec{v}_2\cdot\vec{n}\big(S_1^2-5(\vec{S}_1\cdot\vec{n})^2\big)\Big)\\
&+3\;\vec{S}_1\cdot\vec{a}_2\times\vec{n}\Big(2\vec{S}_1\cdot\vec{v}_1\;\vec{S}_1\cdot\vec{n}+\vec{v}_1\cdot\vec{n}\big(S_1^2-5(\vec{S}_1\cdot\vec{n})^2\big)\Big)\Big),\end{aligned} \tag{5.21}$$

$$L_{[9]} = -\frac{1}{3}\;\vec{S}_1\cdot\vec{a}_2\times\vec{n}\big(\dot{\vec{S}}_1\cdot\vec{S}_1-3\;\dot{\vec{S}}_1\cdot\vec{n}\;\vec{S}_1\cdot\vec{n}\big)-\frac{1}{6}\dot{\vec{S}}_1\cdot\vec{a}_2\times\vec{n}\big(S_1^2-3\big(\vec{S}_1\cdot\vec{n}\big)^2\big). \tag{5.22}$$

As can be seen in the result above we have grouped together terms according to their mass ratios and Wilson coefficients, and the total number/order of their higher order time derivatives. At this stage this result is rather bulky, but it is easy to see that after the reduction of the action to an ordinary action by the removal of higher order time derivative terms, we will only be left with such pieces as the first 4 ones in $L^{\rm NLO}_{S_1^2S_2}$ and the first 5 ones in $L^{\rm NLO}_{S_1^3}$, which becomes significantly more compact.

However, before we will proceed to handle via redefinitions the higher order time derivatives appearing in the cubic-in-spin sector at this order, we need to also take into account all the contributions to the action in this sector at this order, which originate from lower order redefinitions of the variables made at lower order sectors in order to remove higher order time derivatives there, as was shown in detail in section 6 of [11]. First, for example we recall that we have kinematic contributions as noted in eq. (5.28) of [11], that are linear in the spin but have no field coupling. Those are required here to NLO as follows:

$$L_{\rm kin} = -\vec{S}\cdot\vec{\Omega} - \frac{1}{2}\left(1+\frac{3}{4}v^2\right)\epsilon_{ijk}S_k v^j a^i, \tag{5.23}$$

where $S_{ij} = \epsilon_{ijk}S_k$, and $\Omega_{ij} = \epsilon_{ijk}\Omega_k$. At LO, e.g., we define the following shift of the positions, $\Delta\vec{y}_I$, according to

$$\vec{y}_1 \to \vec{y}_1 + \frac{1}{2m_1}\vec{S}_1\times\vec{v}_1, \tag{5.24}$$

and similarly for particle 2 with $1 \leftrightarrow 2$ to remove the leading accelerations.

Note that as of the NLO linear-in-spin level higher order time derivatives of spin also appear, where it was shown how to generically treat these in section 5 of [43]. Yet, since the leading spin redefinition is of higher PN order, terms quadratic in the leading redefinition contribute only at the next-to-NNLO (NNNLO) level. Therefore, here it is sufficient to consider the redefinition of the spins to linear order.

To recap, let us list the additional contributions coming from lower order variable redefinitions that we will have. From position shifts in lower order sectors we will have:

1. The LO (1.5PN) position shift in eq. (5.24) implemented to linear order on the NLO quadratic-in-spin (spin1-spin2 + spin-squared) sectors.

2. The above LO position shift implemented to quadratic order on the Newtonian and LO spin-orbit sectors.

3. The above LO position shift to cubic order implemented on the Newtonian sector.

4. The NLO position shift at 2.5PN order in eq. (6.20) of [11] implemented to linear order on the LO quadratic-in-spin sectors.

5. The NLO position shifts at 3PN order in eqs. (6.30), (6.43) of [11] implemented to linear order on the *shifted* LO spin-orbit sector.

The leading redefinition of spin (of 2PN order) in eq. (6.21) of [11] will not contribute to our sector. From spin redefinitions, i.e. rotations of the spin, we will have then:

1. The spin redefinitions at 2.5PN order in eqs. (6.31), (6.44) of [11] implemented to linear order on the LO quadratic-in-spin sectors.

2. The spin redefinitions at 3PN order, which were required at the LO cubic-in-spin sector [10], implemented to linear order on the LO spin-orbit sector.

In a forthcoming publication we will present the full details of these redefinitions and the contributions which add up to get the reduced effective action.

6 Conclusions

In this work we derived for the first time the complete NLO cubic-in-spin PN effective action for the interaction of generic compact binaries via the EFT formulation for gravitating spinning objects in [11] and its extension to the leading sector, where gravitational non-linearities are considered at an order in the spins that is beyond quadratic. This sector, which enters at the 4.5PN order for rapidly rotating compact objects, completes finite size effects up to this order, and is the first sector completed beyond the current state of the art for generic compact binary dynamics at the 4PN order. Once again we see that the EFT of gravitating spinning objects has enabled pushing the state of the art in PN Gravity. Yet the analysis in this work indicates that going beyond this sector into the intriguing gray area of table 1 may become impossibly intricate.

We have seen that at this order in spins with nonlinearities in gravity we have to take into account additional terms, which arise from a new type of worldline couplings, due to the fact that at this order the Tulczyjew gauge, which involves the linear momentum, can no longer be approximated only in terms of the four-velocity, as the latter approximation differs from the linear momentum by an order $\mathcal{O}(RS^2)$. The correction gives rise to new "composite" couplings from the gauge of rotational DOFs. It is interesting to consider whether these new couplings have an insightful physical interpretation.

As we noted in section 1 one of the main motivations for us to tackle this sector was also to see what happens when we go to a sector at order higher than quadratic in the spins and nonlinear in gravity, which corresponds to a gravitational Compton scattering with quantum spins of $s \geq 3/2$, and to possibly also get an insight on the non-uniqueness of fixing its amplitude from factorization when spins of $s \geq 5/2$ are involved [32]. From [11] and the analysis in section 4 we can see that going to an order quintic in spins, or in the quantum case to $s = 5/2$, exactly corresponds to where the correction to p_μ in eq. (4.2) has to be taken into account at quadratic order. We will discuss this interesting connection between the classical and the quantum levels at a forthcoming publication. A general observation that we can clearly make already is that even-parity sectors in l, see table 1, are easier to handle than odd ones. In the quantum context this corresponds to the greater ease of dealing with bosons compared to fermions.

Unless all the additional terms from section 4 conspire to cancel out eventually, we obtain an effective action that differs from that with the gauge used in lower spin sectors, involving only the four-velocity. Yet, it could be that when computing the consequent observable quantities, such as the binding energy, or the EOMs, one finds that this difference does not show up, and the two gauges are physically equivalent. In a forthcoming publication we will present the resulting Hamiltonian, EOMs, and finally gauge invariant quantities, such as the binding energy, and get an answer for this question.

At the moment it is not clear whether computations carried out within an amplitudes framework can capture all the classical effects derived in this paper. The generic results in this work can serve to streamline such a framework, as that which was initiated in [35, 36], or provide a crosscheck for the conjectured result for the scattering angle at one-loop level in the restricted case of black holes with aligned spins in [39].

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Chapter 4

Conclusions

In this thesis we reviewed of the EFT formulation to compute the interactions in a coalescent binary system of compact objects, which has gained special attention through the years, as a way to test theories of gravity, and especially after the first gravitational waves were detected at 2015 by the Advanced LIGO detector, [1], and presented new high precision computations in classical gravity, [15], which was the cubic-in-spin interaction that enters at the 4.5PN order. This sector revealed a group of 49 Feynman diagrams at the leading term of the four-momentum, plus 4 graphs that arise from a new type of worldline coupling. The latter appear because of the dependence in the spin variable of the four-momentum that enters at the 4.5 PN order, as can be seen in the equation below and is thoroughly computed in [15]:

$$p^\mu = -\frac{\partial L}{\partial u^\mu} = m\frac{u^\mu}{\sqrt{u^2}} + \mathcal{O}(S^2) \tag{4.1}$$

At this order we see that the linear momentum can no longer be approximated as the four-velocity u^μ, as it was usually done in the previously computed spin sectors. This contribution will reveal a new type of "composite" coupling to the worldline and therefore new graphs that contribute in the interaction, [15]. These new coupling terms have not yet been fully understood in their meaning in terms of physical interpretation and leaves open interesting research topic to follow.

An alternative expansion that is being used to compute scattering amplitudes and classical gravitational quantities of a coalescent binary is the post-Minkowskian (PM) approximation, in which the gravitational field is still taken to be weak, but the velocities are not the small parameters anymore, and can in fact be completely relativistic, [32]. The work in [32] also uses an effective field theory formulation, in which the operators are expanded in powers of the transfer momentum divided by the mass of the matter field, and enables one to extract classical contributions for the loop amplitudes of large mass scalar and fermions weakly coupled to gravity in the PM approximation.

In general, a parallel construction of classical and quantum formulations to the interaction of classical gravitational objects has been gaining more and more attention and it is a promising research topic, bringing the discussion of the two-body problem to a scattering amplitudes framework.

For instance, an interesting outcome of the computations from the work developed in this thesis is the connection to a quantum formalism, to compute the scattering amplitudes that was first approached in [33], based on new spinor-helicity formalism for massive spins introduced in [34]. In this work, the classical spin effects with the spin to the lth order corresponds to the amplitudes with a quantum spin of $s = \frac{l}{2}$. This means that, as of the one-loop sector and larger than quadratic classical spin order i.e the gray area in the table in figure 1.2, the Compton scattering amplitude, which is the scattering amplitude involving two massive scalar and two massless gravitons, with a quantum spin $s \geqslant \frac{3}{2}$ is already required in the quantum formalism to capture the binary system interactions. However, there is a related issue that a Compton scattering amplitude with $s \geqslant \frac{5}{2}$ cannot be uniquely fixed due to the difficulty in formulating a perturbative UV completion of gravity with higher spins, [34]. Curiously, the gray section in the table in figure 1.2 also corresponds to where the four-momentum is no longer independent of the spin. This could suggest some kind of connection between the classical and quantum formulations i.e the gray area in the table in figure 1.2 and higher order spins in the Compton scattering, to be better understood in the future. Could these classical computations within an effective particle perspective bring us some insight into the non-uniqueness of the Compton scattering for spins $s > 2$?

Other follow ups of the work done through this thesis are, straightforwardly, the extension of the high precision computation through the diagonal of the table in figure 1.2 i.e pushing to higher loop orders and spin orders. Some recent work pushing the frontier has already been done in [17] and [18], as well as in [6]. One can also seek to obtain the Hamiltonian, the equations of motion and possibly binding energies starting from the results of the computations accomplished during this thesis. That would be a possible good way to get an answer on whether the different gauges applied at this sector and at lower orders of the spin sectors, that included only the leading term in the four-momentum i.e the four-velocity term, will turn out to be equivalent or not, at the level of gauge invariant quantities.

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