Master's Thesis

High Spin Scattering Amplitudes

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Abstract

This thesis reviews and expands on the usage of scattering amplitude techniques to model gravitational collision events between rotating objects. In this formalism, classical angular momentum is encoded in an expansion in terms of the spin quantum number of the particles. By making use of the color-kinematics duality and the properties of Heavy-mass Effective Field Theories, we construct a fully covariant and spurious pole free form for the four- and five-point Compton amplitudes for a scattering process involving one massive spinning particle interacting with a number of massless gluons or gravitons. Using a bootstrap method where only gauge invariance and a correct factorization behavior are imposed, the solution is uniquely fixed when choosing a natural basis of functions to generate the *ansatz*. We also generalize this procedure to arbitrary number of particles. Lastly, we compare with several results in the literature and make appropriate contact term corrections to match the specific behavior for Kerr black hole collisions up to quartic order in spin.

Contents

1 Introduction

The need to understand gravitational physics has increased significantly in the last years after the first detection of gravitational waves caused by inspiraling astrophysical binary systems. These signals can provide new information about objects like black holes and neutron stars, and serve as a testing ground for Einstein's classical theory of gravity. Thus, numerous approaches have been developed in order to provide theoretical templates for these gravitational waveforms with the precision and scope required by present and future experiments. Remarkably, an avenue that has provided a great deal of novel results takes advantage of the modern techniques that have been developed over the last decades for the computation of perturbative scattering amplitudes involving elementary particles, prompted by a need to model collision processes in particle accelerators. In this formalism, super-heavy black holes are considered as point particles and their interaction is described by scattering amplitudes in Einstein's theory of gravity, where the perturbative expansion is performed in terms of Newton's constant G_N. This is known as the post-Minkowskian (PM) framework.
The fact that classical results emerge at all orders of the perturbative expansion was already observed long ago [1, 2]. Since then, several works [3–21] have contributed to extending the calculations for increasing orders in Newton's constant G_N, both in the elastic case and including radiation. In particular, fourth post-Minkowskian order (three-loop) computations have been pioneered by [22–24].
Since gravity is non-renormalizable, it has to be treated as an effective field theory (EFT), extracting non-local effects from the low energy regime and separating them from the ultraviolet contributions. This is of course not a problem, since we are mostly interested in so-called classical effects, which take place at large distances and correspond to the leading term of the amplitude when taking the limit $\hbar \to 0$. In this way, we can compute observables such as bending angles and waveforms, which are then compared to experimental data or alternative theoretical approaches such as the effective one-body formulation [25–28] or the worldline formalism [29–34].
Another avenue of interest is the addition of spin effects, as signals from interactions of Kerr black holes can present a higher intensity when compared to their Schwarzschild counterparts [35]. However, in contrast to the spinless case, significant complications arise even at low orders in the perturbative expansion. In particular, considering classically spinning bodies in the scattering amplitudes framework entails having to describe particles with arbitrarily high spin quantum number. As explained in [36], theories involving a massive particle with $s > 2$ coupled to gravity violate the unitarity principle at energies below the Planck scale. Thus, it is necessary to impose certain physical conditions in order to constrain the form of the amplitudes. In [37], the three-point amplitudes for a massive particle of spin s interacting with massless fields with $s = 1, 2$ were constructed in spinor-helicity

formalism by requiring minimal coupling, i.e. a high energy behaviour consistent with the dominant helicity configuration when $m \to 0$. It was later shown [38] that these amplitudes precisely described scattering processes corresponding to Kerr black holes, which seems to agree with the no hair theorem. In the original work, these three point amplitudes were extended to higher points via the Britto-Cachazo-Feng-Witten (BCFW) on-shell recursion. However, it was observed that for $s > 2$ there are spurious poles that obstruct the high energy behaviour, which was interpreted as the well known fact that massive higher spin particles cannot be elementary.

In subsequent works [39–45], the expression for the four point Compton amplitude was reworked to manifestly remove the spurious poles and obey additional physical constrains in order to be utilized in one-loop 2PM computations of classical observables. This is done by performing an additional expansion in terms of the ratio between the spin and the angular momentum, which explicitly represents the different induced multipoles in the amplitude. In particular, the expansion variable is usually chosen to be the rescaled intrinsic angular momentum expectation value $a = S/m$. In this framework, it was shown in [46] that the three and four-point amplitudes exponentiate for the precise helicity configuration that contributes to the classical limit. While this exponential still contains spurious poles, they can be cancelled by adding appropriate contact terms at each order in the spin expansion. Using this approach, a well behaved expression for the four point amplitude has been calculated and compared to General Relativity results up to fourth order in spin.

In spite of this, a closed formula for the high spin Compton amplitude is still to be obtained. Moreover, in most instances in the literature the amplitude is expressed in certain helicity configurations. Although it is true that for classical observable calculations only the opposite helicity piece contributes to the loop integral, a manifestly gauge invariant form of the amplitude is still desirable as to shed more light into the theories involving arbitrary spin fields. Lastly, in order to carry out computations at higher order in the PM expansion, higher point tree amplitudes are needed. Thus, the main goal is to devise a systematic procedure to obtain a well-behaved expression for these amplitudes.

In this work, we propose a bootstrap approach with which we are able to calculate the four and five-point tree amplitude for a massive spinning line interacting with two and three gravitons in a manifestly covariant, spurious pole free and gauge invariant form. Our framework is based on the Heavy-Mass Effective Field Theory (HEFT), which was greatly developed in [47–49] and used for gravitational computations in [21, 50, 51]. In this formulation, the massive momenta are parameterized as $k = m\, v$, and the amplitude is expanded in terms of the inverse mass $1/m$. As we will show here, this is completely equivalent to the soft graviton expansion ($q \to 0$) and, thus, the classical expansion $\hbar \to 0$.

Another feature which we make ample use of is the color-kinematics duality or, equivalently, the double copy. In [52–54], it was observed that the kinematic numerators

of gauge tree amplitudes could be chosen to obey the same Jacobi-like relations as the corresponding color factors (BCJ numerators). By substituting the latter for the former, one could obtain gravitational amplitudes as a double copy of gauge amplitudes, in a much simpler manner than using the Feynman rules for Einstein's theory. By combining this with the heavy-mass regime, remarkably compact expressions for the amplitudes are produced, which are also manifestly gauge invariant.
The procedure followed in this work is fairly simple: we firstly use the color-kinematics duality to determine the general structure that the amplitude involving a pair of massive spinning particles interacting with a number of massless fields must present. From this, we consider all the possible building blocks that this amplitude can be made of, ensuring that the expressions are manifestly gauge invariant and present the correct singularity structure. The final form for the amplitude is then fixed by imposing a consistent factorization behaviour into the Kerr three point amplitude presented in [37].
The thesis is organized as follows: in Section 2, we give a brief review of the color-kinematics duality, the double copy and the principles of generalized unitarity that are used to construct tree and loop gravity amplitudes. In Section 3, we clarify on the difference between the various representations of the BCJ numerators and introduce the HEFT amplitudes and their properties. In Section 4, we give a review of the current state and recent developments of high spin amplitudes. In Section 5, we construct the four and five-point Compton amplitude with massive spinning particles and propose a systematic procedure for an arbitrary number of points. In Section 6 and Section 7, we compare our results with those from the literature and add potential contact terms in order to match the Kerr black hole behaviour. Finally, in Section 8 we give some final remarks and discuss possible applications and questions for the future.

2 Aspects of scattering amplitudes

2.1 Color-kinematics duality and double copy

Although it can seem unexpected, a starting point in the construction of gravity amplitudes are non-abelian gauge theories such as Yang-Mills, where the interactions are controlled by a set of color tensors $\mathcal{C}$ like the structure constants f^{abc}. In most cases, these are not completely independent, but obey certain Jacobi relations:

$$f^{dae}f^{ebc} - f^{dbe}f^{eac} = f^{abe}f^{ecd}. \tag{2.1}$$

In the case of pure Y-M, the Lagrangian is given by:

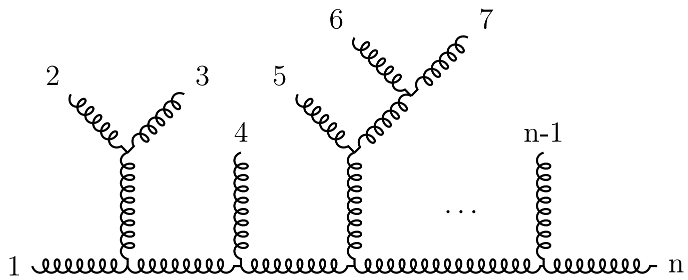
$$\mathcal{L}_{YM} = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu},\ F^a_{\mu\nu} = \delta_\mu A^a_\nu - \delta_\nu A^a_\mu + gf^{abc}A^b_\mu A^c\nu, \tag{2.2}$$

where A_μ^a are gauge vector fields which transform under the group $SU(N)$ ($N = 3$ for the standard theory of Yang-Mills), g is the coupling constant and $F_{\mu\nu}^a$ is the non-abelian field strength tensor. Using a Feynman diagram approach, we can express an n-point tree amplitude in this theory as:

$$\mathcal{A}_m^{\text{tree}} = -ig^{n-2} \sum_{i=1}^{(2n-5)!!} \frac{c_i N_i}{D_i}. \tag{2.3}$$

Here, the sum runs over all the possible cubic diagrams with n external legs. The c_i are the color factors, built as products of the structure constants associated to each vertex, and the D_i are propagator products corresponding to the diagram edges. Finally, the N_i contain all the remaining numerator information for each graph. While it is true that there can be graphs with four-gluon vertices, we can always incorporate their contribution into the cubic diagrams by multiplying and dividing by the missing propagator.

Since the structure constants obey (2.1), the color factors c_i are also going to be related to each other by Jacobi identities. Specifically, it can be shown that only $(n-2)!$ of them are independent, which means that we can construct a basis with which to express the remaining ones. To illustrate this, let's consider an arbitrary cubic diagram G like:



(2.4)

The corresponding color factor is:

$$c(G) = \tilde{f}^{a_1 bc} \tilde{f}^{ba_2a_3} \tilde{f}^{ca_4d} \tilde{f}^{deg} \tilde{f}^{ea_5h} \tilde{f}^{ha_6a_7} ... \tilde{f}^{za_{n-1}a_n}, \tag{2.5}$$

where we have rescaled the structure constants as $\tilde{f}^{abc} \equiv i\sqrt{2} f^{abc}$ to match the standard conventions (see e.g. [55]). By using a rank-2 tensor notation $(\tilde{f}^a)_{bc} \equiv \tilde{f}^{bac}$ and the Lie algebra relation:

$$\tilde{f}^{abc} \tilde{f}^c = [\tilde{f}^a, \tilde{f}^b], \tag{2.6}$$

we can express $c(G)$ as:

$$c(G) = ([\tilde{f}^{a_2}, \tilde{f}^{a_3}] \tilde{f}^{a_4} [\tilde{f}^{a_5}, [\tilde{f}^{a_6}, \tilde{f}^{a_7}]] ... \tilde{f}^{a_{n-1}})_{a_1 a_n}. \tag{2.7}$$

Following this, it is clear that any color factor can be written in terms of products of structure constants with legs 1 and m fixed:

$$c_i = \sum_{\sigma\in S_{n-2}} b_{i\sigma}(\tilde{f}^{a_{\sigma(2)}}\tilde{f}^{a_{\sigma(3)}}...\tilde{f}^{a_{\sigma(n-1)}})_{a_1 a_n},\ b_i = \{0, \pm 1\}, \tag{2.8}$$

which, in turn, means that any gauge tree amplitude can be expanded into this basis:

$$\mathcal{A}_n^{\text{tree}} = \sum_{\sigma\in S_{n-2}} A_n^{\text{tree}}(1,\sigma(2),...,\sigma(n-1),n)(\tilde{f}^{a_{\sigma(2)}}\tilde{f}^{a_{\sigma(3)}}...\tilde{f}^{a_{\sigma(n-1)}})_{a_1 a_n}. \tag{2.9}$$

The coefficients $A_n^{\text{tree}}(1,\sigma(2),...,\sigma(n-1),n)$ are called partial tree amplitudes. Only planar graphs with the corresponding ordering contribute to each A_m^{tree}, and they obey several useful relations [54]:

$$\begin{gathered}
\text{U(1) decoupling: } \sum_{i=2}^{n} A_n^{\text{tree}}(2,3,...,i,1,...,n) = 0,\\
\text{KK relations: } A_n^{\text{tree}}(1,\alpha,n,\beta) = (-1)^{|\beta|}\sum_{\sigma\in\alpha\sqcup\beta^T} A_n^{\text{tree}}(1,\sigma,n),\\
\text{BCJ relations: } \sum_{a=2}^{n-1}\left(\sum_{b=2}^{a} p_1\cdot p_b\right) A_n^{\text{tree}}(2,...,a-1,1,a,...,n-1,n),
\end{gathered} \tag{2.10}$$

where $\alpha\sqcup\beta^T$ indicates the set of all shuffles of α and the transpose of β (i.e. respecting the ordering within each subset). These identities reduce the number of independent partial amplitudes to $(n-3)!$, which means that (2.9) is still a redundant basis.

We now turn to study the color-kinematics duality for gauge amplitudes. In a general manner, it is defined as the fact that, for many gauge theories, it is possible to establish a one-to-one map between the Jacobi-like identities obeyed by the color factors of the different cubic diagrams and the relations of the kinematic numerators corresponding to those graphs. In other words, the kinematics numerators can be chosen so that they satisfy the same identities as the color factors. In this specific case, they are denoted as BCJ numerators.

This statement entails various interesting consequences. Firstly, by using (2.8), the partial amplitudes take the form:

$$A_n^{\text{tree}}(1,\alpha(2),...,\alpha(n-1),n) = -i\sum_{i\in planar} b_{i\alpha}\frac{N_i}{D_i}. \tag{2.11}$$

Now, if the kinematic numerators obey the same Jacobi identities as the color factors, then it is possible to reorganize the partial amplitude into an $(n-2)!$ KK basis:

$$A_n^{\text{tree}}(1,\alpha,n) = \sum_{\beta\in S_{n-2}} m(1\,\alpha\,n|1\,\beta\,n)N(1,\beta,n), \tag{2.12}$$

where $m(\alpha|\beta)$ is the so-called propagator matrix. Due to their convenience for generating gauge and gravity amplitudes, the systematic construction of expressions for the BCJ numerators $N(1,\beta,n)$ has been greatly developed over the last years. For example, in [56–60], various graphical algorithms for calculating the numerators were formulated from the so-called CHY formalism [61–63]. In this framework, field theory scattering amplitudes are written as integrals over the moduli space of punctured Riemann spheres constrained by a set of relations know as the scattering equations. If one tries to solve equation (2.12) for the numerators $N(1,\beta,n)$, it turns out that it is not possible, since the propagator matrix is singular. This is to be expected: we saw before that the partial amplitudes obey a set of relations that leave only $(n-3)!$ of them as independent objects, so any system of equations involving $(n-2)!$ amplitudes is bound to be degenerate. In other words, we can only solve for $(n-3)!$ numerators, while the rest contribute only as unfixed parameters manifesting a gauge freedom. We could for example make the following choice:

$$\begin{gathered} N(1,\sigma,n-1,n) \neq 0, \\ N(1,\sigma,n) = 0 \quad \text{if } \sigma(n-1) \neq n-1. \end{gathered} \tag{2.13}$$

We denote this as the KLT basis. In the next section, we will explore the properties of these numerators more thoroughly and establish a procedure to go from the KK to the KLT basis.

Another consequence of the color-kinematics duality is the fact that we can substitute the color factors of a gauge amplitude with another set of kinematic BCJ numerators, obtaining gravity amplitudes. This procedure is know as double copy, and it is a surprising feature, since there is no apparent connection between the different theories. Moreover, the expressions obtained from standard Lagrangian methods in gravity are, on the surface, significantly more complex than just a product of two gauge amplitudes, which hints at the fact that there is probably a much more efficient way of arranging the gravitational perturbative expansion.

In particular, it can be shown that the double copy of Yang-Mills theory corresponds to Einstein's theory of gravity coupled to an antisymmetric B-field and a scalar dilaton field ϕ, which can both be truncated into Einstein gravity by an adequate choice of symmetric traceless polarization tensors. By utilizing the different relations that the BCJ numerators inherit from the color-kinematics duality, we can express double copy gravitational amplitudes in several different ways:

$$M_n^{\text{tree}} = \sum_{i\in\text{graphs}} -i\left(\frac{\kappa}{2}\right)^{n-2} \frac{N_i\tilde{N}_i}{D_i} \tag{2.14}$$

$$= -i\left(\frac{\kappa}{2}\right)^{n-2} \sum_{\sigma\in S_{n-2}} N(1,\sigma,n)A_n^{\text{tree}}(1,\sigma,n) \tag{2.15}$$

$$= -i\left(\frac{\kappa}{2}\right)^{n-2} \sum_{\alpha,\beta\in S_{n-3}} A(1,\alpha,n-1,n)\mathcal{S}[\alpha|\beta]A(1,\beta,n,n-1). \tag{2.16}$$

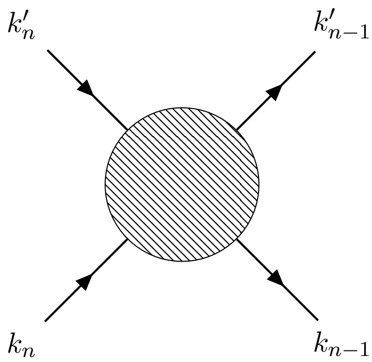
The expression in the last line is known as the KLT relation, which were first encountered in string theory [64]. The KLT kernel $\mathcal{S}[\alpha|\beta]$ is given by:

$$\mathcal{S}[\alpha|\beta] = \prod_{a=2}^{n-2}\left(2p_1\cdot p_{\alpha_a} + \sum_{b=2}^{a} 2p_{\alpha_a}\cdot p_{\alpha_b}\theta(\alpha_b,\alpha_a)_\beta\right), \tag{2.17}$$

where $\theta(\alpha_b,\alpha_a)_\beta = 1$ if α_b is before α_a with respect to β, and zero otherwise.
These double copy relations are remarkably convenient to obtain gravity amplitudes, since we only need to compute their much simpler gauge theory versions and then use the color-kinematics duality to square them. Then, once the gravity tree amplitude is generated, the next step is to perform loop calculations and extract the relevant classical observables.

2.2 Kinematics and notation

Moving forward, we will mainly analyze amplitudes pertaining to gravitational elastic scattering processes. Thus, we consider two incoming and two outgoing massive external particles, with momenta denoted by:



(2.18)

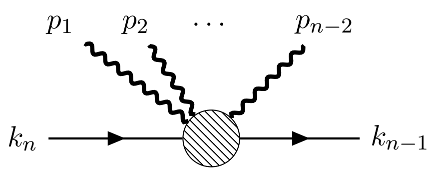
where $k_n^2 = k_{n-1}^2 = m_1$ and $k_n'^2 = k_{n-1}'^2 = m_2$. We have chosen this notation to reflect the fact that, most of the time, we are going to be working with the n-point tree amplitudes that are sewn together via generalized unitarity. We also define the transfer momentum as $q = k_n - k_{n-1} = k'_{n-1} - k'_n$ and the invariant Lorentz factor as:

$$y = \frac{k_n k'_n}{m_1 m_2}. \tag{2.19}$$

The on-shell conditions imply that:

$$k_n\cdot q = -k'_n\cdot q = \frac{q^2}{2}. \tag{2.20}$$

As we will see, these scattering processes are described by gravity loop amplitudes which, by the principle of generalized unitarity, can be constructed out of products of tree amplitudes where one massive line interacts with a number of gravitons:



(2.21)

In the cases where we consider tree amplitudes on their own, we will use two slightly different notations. For an n-point amplitude, the massless momenta of the gravitons are going to be represented by either l_i or p_i $(i = 1, ..., n-2)$, depending on whether we want to make the connection to loop integrals explicit or not.

2.3 Gravity loop integrals and classical observables

A key aspect of calculating loop gravity amplitudes is generalized unitarity, by which the non-analytic part of the amplitude can be expressed as a product of tree amplitudes integrated over the cut momenta [53]. This is especially convenient for long distance gravity scattering processes, since the exchanged gravitons can be taken on-shell and all the analytical pieces are discarded. Thus, the bulk of the important contributions are given by the multi-graviton cuts [65]:

$$i\mathcal{M}^{cut}_{L+1} = \hbar^{3L+1} \int (2\pi)^D \delta(q + l_1 + l_2 + ... + l_{n-2}) \prod_{i=1}^{n-2} \frac{i}{l_D^2} \frac{d^D l_i}{(2\pi\hbar)^D}$$
$$\frac{1}{(n-2)!} \sum_{h_i=\pm 2} M^{tree}_L(l_1^{h_2}, ..., l_{n-2}^{h_{n-2}}, -k_{n-1}, k_n) M^{tree}_R(k_2, -l_1^{h_2}, ..., -l_{n-2}^{h_{n-2}}, -k'_{n-1}, k'_n)^\dagger, \tag{2.22}$$

where the loop order is $L = n - 3$. Starting from 3PM (two loops), additional self-energy and vertex correction diagrams have to be added, which can be interpreted as radiation-reaction terms (although we won't concern ourselves with them in this work). Now, as we mentioned before, we are only interested in the classical contribution to the amplitude. This can be extracted by fixing the wavenumber of the gravitons $\bar{q} = q/\hbar$ and performing an expansion in terms of $\hbar$, keeping only the $1/\hbar$ term [15]. However, as we will see, expanding the tree amplitudes in the heavy-mass regime is turns out to be more convenient, since one does not have to subtract the hyperclassical terms that "feed down" from lower loop orders.

Classical observables are then computed by Fourier transforming the loop amplitude into impact parameter space (IPS):

$$\widetilde{\mathcal{M}}(b) = \frac{1}{4m_1 m_2 \sqrt{y^2 - 1}} \int \frac{d^{D-2}q}{(2\pi)^{D-2}} \; e^{-iq\cdot b} \mathcal{M}(q), \tag{2.23}$$

which, as shown in [20, 21], gives rise to an exponentiated expression for the S-matrix:

$$S = 1 + \widetilde{\mathcal{M}} = e^{i\delta}. \tag{2.24}$$

The eikonal phase δ can be expanded perturbatively to a certain loop order. In the HEFT, hyperclassical contributions factorize in IPS, and thus can be straightforwardly discarded (see Section 3.3). Finally, the scattering angle is given by the derivative of the real part of the eikonal phase with respect total angular momentum J:

$$\chi = -\frac{\partial(\mathrm{Re}\delta)}{\partial J}. \tag{2.25}$$

Having reviewed the fundamental aspects of the scattering amplitude methods in gravity, we now dedicate the next section to studying the HEFT and the construction of tree amplitudes and numerators in more detail.

3 KLT numerators and HEFT

3.1 KLT basis for BCJ numerators

Like we mentioned in previous sections, the colour-ordered tree-level amplitude for a massive line interacting with $(n-2)$ massless vector particles (gluons) can be written in terms of the so-called BCJ numerators in the $(n-2)!$-dimensional KK basis:

$$A(\alpha, \overline{n-1}, \overline{n}) = \sum_{\beta \in S_{n-2}} m(\alpha\, \overline{n-1}\, \overline{n} | \beta\, \overline{n-1}\, \overline{n}) N(\alpha, \overline{n-1}, \overline{n}), \tag{3.1}$$

where particles $\overline{n-1}, \overline{n}$ represent the massive line. The matrix $m(\alpha\, \overline{n-1}\, \overline{n} | \beta\, \overline{n-1}\, \overline{n})$ is known as the propagator matrix, and it is defined as:

$$m(\alpha\, \overline{n-1}\, \overline{n} | \beta\, \overline{n-1}\, \overline{n}) = \sum_{i \in \text{cubic graphs}} \frac{C_{i,\alpha} C_{i,\beta}}{D_i}, \tag{3.2}$$

where $C_{i,\alpha}$ is the coefficient relating the cubic graph color factor i to the color-factor of the half-ladder diagram associated with the permutation α, and D_i is the product of propagators of i. With this definition, it is easy to see that $m(\alpha\, \overline{n-1}\, \overline{n} | \beta\, \overline{n-1}\, \overline{n})$ corresponds to the amplitude for $(n-2)$ massless double-colored scalars interacting with a massive scalar line [63].

Now, although it appears that there are $(n-2)! \times (n-2)!$ independent components in the propagator matrix, this is actually not the case due to momentum conservation. One easy way to prove this is to use the CHY formalism, from which can deduce that only $(n-3)! \times (n-3)!$ of these double-colored amplitudes are independent.

This is because there are only $(n-3)!$ solutions for the scattering equations, and the amplitudes depend exclusively on the Riemann sphere punctures σ_a.
From this, it is clear that $m(\alpha\,\overline{n-1}\,\overline{n}|\beta\,\overline{n-1}\,\overline{n})$ must have a $(n-3)(n-3)!$ dimensional null space. To find this null space, we rewrite the fundamental BCJ relations for Y-M amplitudes using momentum conservation [66]:

$$0 = \sum_{a=2}^{n-1}\left(\sum_{b=a}^{n-1} 2p_1 \cdot p_b\right) A(2, ..., a-1, 1, a, ..., \overline{n-1}, \overline{n}). \tag{3.3}$$

To this, one has to add all possible permutations of $\{2, ..., n-2\}$, and we can also freely choose the particle that we move around. Considering this, the number of independent equations is precisely $(n-3)(n-3)!$
It is also clear that the BCJ equations can be expressed more generally as

$$0 = \sum_{\alpha \in S_{n-2}} d_i(\alpha, \overline{n-1}, \overline{n}) A(\alpha, \overline{n-1}, \overline{n}), \tag{3.4}$$

where we will call $d_i(\alpha, \overline{n-1}, \overline{n})$ the BCJ coefficients, and $1 \leq i \leq (n-3)(n-3)!$. If we now expand the color-ordered amplitudes in terms of the numerators, the relations become:

$$0 = \sum_{\alpha,\beta \in S_{n-2}} d_i(\alpha, \overline{n-1}, \overline{n}) m(\alpha\,\overline{n-1}\,\overline{n}|\beta\,\overline{n-1}\,\overline{n}) N(\beta, \overline{n-1}, \overline{n}). \tag{3.5}$$

Now, if we view the BCJ coefficients as an $(n-2)! \times (n-3)(n-3)!$ matrix (where the coefficients of each equation form a vector column), this can be written as:

$$d^T \times m \times N = 0 \tag{3.6}$$

where N represents a $(n-2)!$ dimensional vector with the BCJ numerators as components. However, the BCJ numerators in the KK basis are, in general, linearly independent with respect to the matrix product $d^T \times m$, which automatically implies:

$$d^T \times m = 0 \Rightarrow m \times d = 0. \tag{3.7}$$

Here, we have used the fact that the propagator matrix is symmetric, as seen in definition (3.2). In other words, the matrix of BCJ coefficients precisely spans the null space for $m(\alpha\,\overline{n-1}\,\overline{n}|\beta\,\overline{n-1}\,\overline{n})$.
Taking this into account, it is clear that any two sets of BCJ numerators in the KK basis are equivalent if they differ by a linear combination of vectors in the null space of the propagator matrix:

$$N'(\alpha, \overline{n-1}, \overline{n}) = N(\alpha, \overline{n-1}, \overline{n}) + \sum_i y_i d_i(\alpha, \overline{n-1}, \overline{n}), \quad y_i \in \mathbb{R}, \tag{3.8}$$

since, using equation (3.1), they will result in the same amplitude. Obviously, this means that we can choose a set of numerators where $N(\alpha, \overline{n-1}, \overline{n}) = 0$ if $\alpha(1) \neq 1$, i.e. only the numerators of the form $\mathcal{N}(1, \sigma, \overline{n-1}, \overline{n})$ with $\sigma \in S_{n-3}$ are non-zero. We call this specific set the KLT basis for the BCJ numerators, and denote them with a curly $\mathcal{N}$ to distinguish them from a general set. Of course, this saturates the linear dependence we exposed, and thus they have to be uniquely determined. In addition, these numerators must also be independently gauge invariant, i.e. they must vanish under a transformation $\varepsilon_i \to p_i$, with $i \in \{1, ..., n-2\}$. The reason for this is very simple: since, in terms of the KLT basis, the amplitude can be expressed as:

$$A(1, \alpha, \overline{n-1}, \overline{n}) = \sum_{\beta \in S_{n-3}} m(1\,\alpha\,\overline{n-1}\,\overline{n}|1\,\beta\,\overline{n-1}, \overline{n})\mathcal{N}(1, \beta, \overline{n-1}, \overline{n}), \tag{3.9}$$

and the matrix $m(1\,\alpha\,\overline{n-1}\,\overline{n}|1\,\beta\,\overline{n-1}, \overline{n})$ no longer has a non-trivial null space, one can invert it to express a particular numerator as a linear combination of color-ordered amplitudes:

$$\mathcal{N}(1, \beta, \overline{n-1}, \overline{n}) = \sum_{\alpha \in S_{n-3}} m^{-1}(1\,\beta\,\overline{n-1}\,\overline{n}|1\,\alpha\,\overline{n-1}, \overline{n})A(1, \beta, \overline{n-1}, \overline{n}). \tag{3.10}$$

Since the amplitudes $A(1, \beta, \overline{n-1}, \overline{n})$ have to be gauge invariant, it follows that the minimal basis numerators $\mathcal{N}(1, \beta, \overline{n-1}, \overline{n})$ also satisfy this condition. One interesting thing to note is that the inverse of the $(n-3)! \times (n-3)!$ propagator matrix can be shown to be equal to the KLT kernel $\mathcal{S}[\beta|\alpha]$ that relates gauge and gravity amplitudes via the double copy [63].
Expressing the KLT numerator in a minimal basis also implies that they are going to contain singularities in physical or unphysical poles (when considering color ordering). Indeed, if we insert (3.10) in the standard KLT relations [53], we see that amplitudes where a massive line is interacting with $(n-2)$ gravitons can be expressed as:

$$\begin{aligned} M(1, 2, ..., \overline{n-1}, \overline{n}) &= \sum_{\alpha, \beta \in S_{n-3}} A(1, \alpha, \overline{n-1}, \overline{n})\mathcal{S}[\alpha|\beta]A(1, \beta, \overline{n}, \overline{n-1}) \\ &= \sum_{\alpha \in S_{n-3}} A(1, \alpha, \overline{n-1}, \overline{n})\mathcal{N}(1, \alpha, \overline{n-1}, \overline{n}), \end{aligned} \tag{3.11}$$

where the elements of the KLT kernel are defined as in [67] to account for the permutation of legs $\overline{n-1}, \overline{n}$ in the second color-ordered amplitude. Now, gravity amplitudes are not ordered with respect to the massless particles, so they are going to contain terms with a non-zero residue in poles such as $p_1 \cdot k_{n-1} = 0$, $p_{12} \cdot k_{n-1} = 0$, and so on. However, these terms are not present in any of the color-ordered Yang-Mills amplitudes $A(1, \alpha, \overline{n-1}, \overline{n})$, and thus they must be present in the BCJ numerators $\mathcal{N}$. For example, at four points the numerator reads [68]:

$$\mathcal{N}(1, 2, \overline{3}, \overline{4}) = \frac{k_3 \cdot F_1 \cdot F_2 \cdot k_3}{(p_1 + k_3)^2 - m^2}, \tag{3.12}$$

where $F^{\mu\nu} = p^\mu \varepsilon^\nu - \varepsilon^\mu p\nu$ is the abelian strength tensor.

This example illustrates both the manifest gauge invariance in the massless legs and the presence of poles which don't respect the color ordering of the numerator.

Following the reasoning explained before, it is easy to derive with a systematic procedure to obtain the KLT numerators from any set of numerators in the KK basis. Like before, for the minimal basis we fix particles $\{1, \overline{n-1}, \overline{n}\}$. In that case, the first step would be to solve the matrix equation:

$$\sum_i d_i(\alpha, \overline{n-1}, \overline{n}) y_i = -N(\alpha, \overline{n-1}, \overline{n}), \quad \alpha \in S_{n-2} \text{ such that } \alpha(1) \neq 1. \tag{3.13}$$

Having obtained the vector $\mathbf{y}$, the KLT numerators can be written as:

$$\mathcal{N}(1, \sigma, \overline{n-1}, \overline{n}) = N(1, \sigma, \overline{n-1}, \overline{n}) + \sum_i d_i(1, \sigma, \overline{n-1}, \overline{n}) y_i. \tag{3.14}$$

Let's try to check this with the four point case. According to [60], a possible expression for the numerator in the KK basis is:

$$\begin{aligned}
N(1, 2, \overline{3}, \overline{4}) &= (\epsilon_1 \cdot k_4)(\epsilon_2 \cdot (p_1 + k_4)) - \frac{1}{2}(\epsilon_1 \cdot \epsilon_2)(p_1 \cdot k_4) \\
&= (\epsilon_1 \cdot (p_2 + k_3))(\epsilon_2 \cdot k_3) - \frac{1}{2}(\epsilon_1 \cdot \epsilon_2)(p_1 \cdot (p_2 + k_3)), \\
N(2, 1, \overline{3}, \overline{4}) &= (\epsilon_2 \cdot k_4)(\epsilon_1 \cdot (p_2 + k_4)) - \frac{1}{2}(\epsilon_1 \cdot \epsilon_2)(p_2 \cdot k_4) \\
&= (\epsilon_2 \cdot (p_1 + k_3)))(\epsilon_1 \cdot k_3) - \frac{1}{2}(\epsilon_1 \cdot \epsilon_2)(p_2 \cdot (p_1 + k_3)),
\end{aligned} \tag{3.15}$$

where we have used conservation of momentum and the on-shell conditions for the gluons $p_1^2 = p_2^2 = 0$. Next, we look at the only fundamental BCJ relation for 4 particles:

$$\begin{aligned}
&(p_2 \cdot k_3) A(1, 2, \overline{3}, \overline{4}) - (p_1 \cdot k_3) A(2, 1, \overline{3}, \overline{4}) = 0 \Rightarrow \\
&d(1, 2, \overline{3}, \overline{4}) = (p_2 \cdot k_3), \quad d(2, 1, \overline{3}, \overline{4}) = -(p_1 \cdot k_3).
\end{aligned} \tag{3.16}$$

The equation (3.13) would be given by:

$$d(2,1,\overline{3},\overline{4})y = -(p_1 \cdot k_3)y = -N(2,1,\overline{3},\overline{4}) \Rightarrow y = \frac{N(2,1,\overline{3},\overline{4})}{(p_1 \cdot k_3)}. \tag{3.17}$$

Finally, we can obtain our KLT numerator:

$$\begin{aligned}\mathcal{N}(1,2,\overline{3},\overline{4}) &= N(1,2,\overline{3},\overline{4}) + \frac{(p_2 \cdot k_3)}{(p_1 \cdot k_3)} N(2,1,\overline{3},\overline{4}) \\ &= 2\frac{(p_1 \cdot k_3)\left[(\epsilon_1 \cdot (p_2 + k_3)))(\epsilon_2 \cdot k_3) - \frac{1}{2}(\epsilon_1 \cdot \epsilon_2)(p_1 \cdot (p_2 + k_3))\right]}{(p_1 + k_3)^2 - m^2} \\ &+ 2\frac{(p_2 \cdot k_3)\left[(\epsilon_2 \cdot (p_1 + k_3)))(\epsilon_1 \cdot k_3) - \frac{1}{2}(\epsilon_1 \cdot \epsilon_2)(p_2 \cdot (p_1 + k_3))\right]}{(p_1 + k_3)^2 - m^2}.\end{aligned} \tag{3.18}$$

After using conservation of momentum and on-shell conditions and rearranging the terms, we get:

$$\begin{aligned}\mathcal{N}(1,2,\overline{3},\overline{4}) &= 2\frac{(k_3 \cdot p_1)(\epsilon_1 \cdot p_2)(\epsilon_2 \cdot k_3) - (k_3 \cdot \epsilon_1)(p_1 \cdot p_2)(\epsilon_2 \cdot k_3)}{(p_1 + k_3)^2 - m^2} \\ &- 2\frac{(k_3 \cdot p_1)(\epsilon_1 \cdot \epsilon_2)(p_2 \cdot k_3) + (k_3 \cdot \epsilon_1)(p_1 \cdot \epsilon_2)(p_2 \cdot k_3)}{(p_1 + k_3)^2 - m^2} = 2\frac{k_3 \cdot F_1 \cdot F_2 \cdot k_3}{(p_1 + k_3)^2 - m^2},\end{aligned} \tag{3.19}$$

which is indeed proportional to the minimal BCJ numerator (3.12). Also, when constructing the Y-M amplitude $A(1,2,\overline{3},\overline{4})$, the pole in the numerator is cancelled by the propagator matrix:

$$\begin{aligned}A(1,2,\overline{3},\overline{4}) &= 2m(1234|1234)\mathcal{N}(1,2,\overline{3},\overline{4}) \\ &= \left(\frac{1}{p_{12}^2} + \frac{1}{(p_2 + k_3)^2 - m^2}\right)\mathcal{N}(1,2,\overline{3},\overline{4}) = -2\frac{k_3 \cdot F_1 \cdot F_2 \cdot k_3}{p_{12}^2((p_2 + k_3)^2 - m^2)},\end{aligned} \tag{3.20}$$

correctly reflecting the color ordering in the amplitude.

3.2 Heavy Mass Effective Field Theory

When computing amplitudes for gravitational scattering processes, in most of the cases we are not interested in the full expression. Rather, we seek to extract the classical part of the amplitude, i.e. the leading contribution when $\hbar \to 0$. Since we normally work in natural units where $\hbar = c = 1$, we have to restore Planck's constant in the amplitude in order to reliably identify the classical contributions. As explained in [3, 50], this is done in the coupling constant (both QCD and gravity) by a factor of $\hbar^{-1/2}$ and the massless momenta as $q \to \hbar\overline{q}$. After that, one naively could take the leading term in the $1/\hbar$ expansion to obtain the classical piece. However, it turns out that loop level amplitudes contain terms that scale as a higher power in $1/\hbar$ than the tree level amplitude. For the amplitude to have a sensible classical limit, those terms have to cancel when computing physical observables (e.g. the boxed and crossed box contributions at one loop cancel when computing the classical potential [4]). Conventionally, one sets the $\hbar$ scaling so that the classical part of the amplitude behaves as $\hbar^{-1}$ [65].

From the discussion above, it follows that taking the classical limit corresponds to the massless particles (gravitons) being soft. In fact, it is possible to systematically extract the classical part of the amplitude by considering the multi-soft scaling of the graviton legs. In [69], the authors reorganize the amplitude so that it factorizes into pieces with definite scaling. In the following, we will review their derivation.

If we have a $n-2$ graviton amplitude $M(l_1, l_2, ..., k_{n-1}, k_n)$ (we slightly change the notation so that we can indicate sums of momenta within the expression for the amplitude), we can eliminate one of the l_i by using momentum conservation. Let's choose

$$\hat{l}_{n-2} = -\sum_{i=1}^{n-3} l_i + q, \tag{3.21}$$

where $q = k_{n-1} - k_n$ is the transfer momentum and we have denoted the massless momenta by l_i to indicate that we are working with loop amplitudes. Now, l_{n-2} is going to appear in a subset of the denominators in the amplitude, which means we can use the following relation:

$$\begin{aligned} &\frac{1}{(k_n + \hat{l}_{n-2} + l_{i_1} + ...) - m^2 + i\varepsilon} = \frac{1}{(k_n - l_{j_1} - ... - l_{j_k} + q)^2 - m^2 + i\varepsilon} = \\ &\hat{\delta}((k_n - l_{j_1} - ... - l_{j_k} + q)^2 - m^2) + \frac{1}{(k_n - l_{j_1} - ... - l_{j_k} + q)^2 - m^2 - i\varepsilon}, \end{aligned} \tag{3.22}$$

where $\hat{\delta}(x) = -2\pi i \delta(x)$ and we have used that:

$$\hat{\delta}(x) = \lim_{x \to 0^+} \frac{1}{x + i\varepsilon} - \frac{1}{x - i\varepsilon}. \tag{3.23}$$

These delta functions allow us to factorize parts of the amplitude:

$$\begin{aligned} &\mathrm{Res}\, M(l_1, l_2, ..., k_{n-1}, k_n)\Big|_{(k_n + \hat{l}_{n-2} + l_{i_1} + ...) - m^2 = 0} \to \\ &\to M(l_{i_1}, ..., \hat{l}_{n-2}, k_n, -k_n - l_{i_1} - ... - \hat{l}_{n-2}) \times M(l_{j_1}, ..., l_{j_k}, -k_{n-1}, k_n + l_{i_1} + \hat{l}_{n-2} + ...). \end{aligned} \tag{3.24}$$

If we perform this decomposition on all the propagators that contain the hatted leg, one can see that we will eventually obtain the following expansion for the amplitude:

$$\begin{aligned}M(l_1,...,\hat{l}_{n-2},k_{n-1},k_n) &= \sum_{r=0}^{n-3}\sum_{\{l_i\}}\left(\prod_{a=1}^{r}\hat{\delta}((k_n+\hat{l}_{n-2}+l_{i_1}+...+l_{i_{b_a}})^2-m^2\right)\\ &\times M^{(+)}(l_{i_1},...,l_{i_{b_1}},\hat{l}_{n-2},-k_n-l_{i_1}-...-l_{i_{b_1}}-\hat{l}_{n-2},k_n)\\ &\times M^{(+)}(...,l_{i_{b_2}},-k_n-l_{i_1}-...-l_{i_{b_1}}-...-l_{i_{b_2}}-\hat{l}_{n-2},k_n+l_{i_1}+...+l_{i_{b_1}}+\hat{l}_{n-2})\\ &\times...\times M^{(+)}(l_{j_1},...,l_{j_m},k_{n-1},k_n+l_{i_1}+...+l_{i_{b_1}}+...+l_{i_{b_r}}),\end{aligned} \tag{3.25}$$

where $k_a \geq a-1$, and the $(+)$ amplitudes are obtained by flipping the regulator in the propagators with the hatted leg:

$$\begin{aligned}(k_n+\hat{l}_{n-2}+l_{i_1}+...)^2-m^2+i\varepsilon &= -\left(2k_n\cdot(l_{j_1}+...+l_{j_k}-q)-(l_{j_1}+...+l_{j_k}-q)^2-i\varepsilon\right)\\ &\to -\left(2k_n\cdot(l_{j_1}+...+l_{j_k}-q)-(l_{j_1}+...+l_{j_k}-q)^2+i\varepsilon\right).\end{aligned} \tag{3.26}$$

The decomposition in (5.16) is useful because it allows for a uniform scaling when performing the multi-soft graviton limit. Indeed, the authors mention that one can observe that the $(+)$ amplitudes scale as $\mathcal{O}(|\vec{q}|^0)$. Since each delta function behaves as $\mathcal{O}(|\vec{q}|^{-1})$, we infer that the full amplitude have dominant terms starting from order $\mathcal{O}(|\vec{q}|^{-(n-3)})$. These terms appear solely because of the presence of the regulators $+i\varepsilon$, and are crucial to extract the classical part of the loop amplitudes. However, the fact that the $M^{(+)}$ scale uniformly in the soft limit is only an empirical observation. For example, at four points, the amplitude can be expressed as a sum of graphs with squared KK numerators:

$$M(l_1,l_2,k_3,k_4) = \frac{N(l_1,l_2,k_3,k_4)^2}{(k_4+l_1)^2-m^2+i\varepsilon}+\frac{N(l_2,l_1,k_3,k_4)^2}{(k_4+l_2)^2-m^2+i\varepsilon}+\frac{N([l_1,l_2],k_3,k_4)^2}{(l_1+l_2)^2+i\varepsilon}, \tag{3.27}$$

where $N([l_1,l_2],k_3,k_4)^2 \equiv N(l_1,l_2,k_3,k_4)-N(l_2,l_1,k_3,k_4)$. In the multi-soft limit $l_i \to |\vec{q}|\tilde{l}_i$, the only part of the numerator that scales as $\mathcal{O}(|\vec{q}|^0)$ reads (see Appendix B of [19]):

$$N(l_1,l_2,k_3,k_4) = 2(\epsilon_1\cdot k_4)(\epsilon_2\cdot k_4)+\mathcal{O}(|\vec{q}|). \tag{3.28}$$

Thus, the leading term of the amplitude in the soft limit is:

$$M(l_1,l_2,k_3,k_4) = 2\frac{(\epsilon_1\cdot k_4)^2(\epsilon_2\cdot k_4)^2}{|\vec{q}|}\left(\frac{1}{k_4\cdot\tilde{l}_1+i\varepsilon}+\frac{1}{k_4\cdot\tilde{l}_2+i\varepsilon}\right)+\mathcal{O}(|\vec{q}|^0). \tag{3.29}$$

If we use conservation of momentum and take only the leading term in the second propagator:

$$\frac{1}{k_4 \cdot l_2 + i\varepsilon} = \frac{1}{-k_4 \cdot (l_1 + q) + i\varepsilon} = \frac{-1}{|\vec{q}|(k_4 \cdot \tilde{l}_1 - i\varepsilon)} + \mathcal{O}(|\vec{q}|^0), \tag{3.30}$$

which means:

$$M(l_1, l_2, k_3, k_4) = 2\frac{(\epsilon_1 \cdot k_4)^2(\epsilon_2 \cdot k_4)^2}{|\vec{q}|}\hat{\delta}(k_4 \cdot \tilde{l}_1) + \mathcal{O}(|\vec{q}|^0). \tag{3.31}$$

This agrees with our previous discussion. However, if we consider $M^{(+)}$, which flips the regulator sign in e.g. the second propagator, we see that:

$$M^{(+)}(l_1, l_2, k_3, k_4) = 2\frac{(\epsilon_1 \cdot k_4)^2(\epsilon_2 \cdot k_4)^2}{|\vec{q}|}\left(\frac{1}{k_4 \cdot \tilde{l}_1 + i\varepsilon} - \frac{1}{k_4 \cdot \tilde{l}_1 + i\varepsilon}\right) + \mathcal{O}(|\vec{q}|^0) = \mathcal{O}(|\vec{q}|^0). \tag{3.32}$$

For the sake of completeness, here we will present an indirect proof for this scaling. For this, we firstly note that the $M^{(+)}$ amplitudes are computationally identical to tree amplitudes, i.e. we can treat them the same as if the regulator $\pm i\varepsilon$ wasn't there. This is because the only instance in which the presence of the regulator can lead to extra contributions is when two terms with the same pole structure are summed together. However, to get terms with the same denominator, momentum conservation has to be used in one of them, which flips the sign of its regulator. Flipping it back again (which is what we do when we consider $M^{(+)}$ amplitudes) results in two terms with the same regulator, which then can be ignored. Therefore, we can focus on proving the uniform scaling in tree amplitudes.

However, instead of performing the proof for the multi-soft limit of the amplitude, we are first going to show that this limit is completely equivalent to the so-called heavy mass approximation, and then prove our claim in this regime. The reason to do this (apart from the fact that it is easier to do) is that we will adopt the heavy mass limit for our results regarding amplitudes that involve massive classically spinning particles, which draws inspiration from previous work [21, 49, 68] using this approximation.

Heavy-mass Effective Field Theory (HEFT) is a regime in which the mass of the particle is taken to infinity, resulting in the leading term being the only one that contributes to the final amplitude. To quantify this, we redefine the massive momenta by

$$k^\mu_{n-1} = -mv^\mu - q^\mu, \qquad k^\mu_n = mv^\mu, \tag{3.33}$$

where v is the four-velocity of the heavy particle, satisfying $v^2 = 1$, and q is again the transfer momentum $q = p_1 + ... + p_{n-2}$. The heavy mass limit is then realized by taking $m \to \infty$ and keeping only the leading term in the $1/m$ expansion. Note that this effectively takes out one of the degrees of freedom in the kinematics, since the velocities of the both massive particles are the same up to a subdominant term. This limit also provides a useful relation:

$$v \cdot q = v \cdot p_{12...n-2} = \mathcal{O}(m^{-1}), \qquad p_{12...n-2} := p_1 + p_2 + ... + p_{n-2}, \tag{3.34}$$

which can be easily verified by using that:

$$k_{n-1}^2 = k_n^2 = m^2 \Rightarrow mv \cdot q = mv \cdot p_{12...n-2} = -q^2. \tag{3.35}$$

As we will see later, this significantly simplifies the computations of amplitudes and BCJ numerators.
For example, the four point amplitude (3.20) takes the following form in HEFT:

$$A(1,2,\overline{3},\overline{4}) = -2\frac{k_3 \cdot F_1 \cdot F_2 \cdot k_3}{p_{12}^2((p_2+k_3)^2 - m^2} \to A(1,2,v) = -\frac{m}{p_{12}^2}\frac{v \cdot F_1 \cdot F_2 \cdot v}{v \cdot p_1} \tag{3.36}$$

In [47], the authors use an approach based on a Hopf algebra to construct the minimal basis BCJ numerators. With this, they propose that the gauge and gravity amplitudes can be written as a sum over all the possible nested commutators of the massless legs:

$$\begin{aligned} A(1,2,...,n-2,v) &= \sum_{\Gamma \in \rho} \frac{\mathcal{N}(\Gamma, v)}{D_\Gamma}, \\ M(1,2,...,n-2,v) &= \sum_{\Gamma \in \tilde{\rho}} \frac{(\mathcal{N}(\Gamma, v))^2}{D_\Gamma}, \end{aligned} \tag{3.37}$$

where ρ ($\tilde{\rho}$) is the set of all possible (un)ordered nested commutators of particles $\{1, 2, ..., n-2\}$. For example, at five points:

$$\begin{aligned} \rho = \Big\{ &[[1,2],3] := (123) - (213) - (312) + (321), \\ &[1,[2,3]] := (123) - (132) - (231) + (321) \Big\}, \\ \tilde{\rho} = \Big\{ &[[1,2],3], [1,[2,3]], [[1,3],2] \Big\} \end{aligned} \tag{3.38}$$

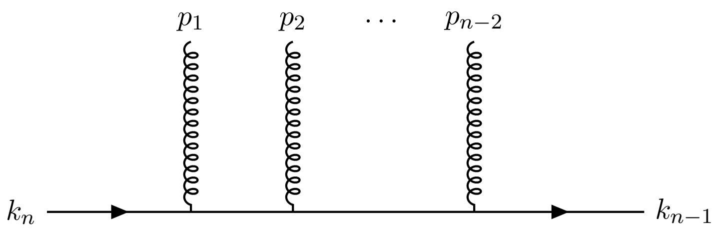
Meanwhile, D_Γ is defined as the products of propagators arising from the nested commutator Γ. Again, some examples at five points are:

$$\Gamma = [[1,2],3] \to D_\Gamma = p_{12}^2 p_{123}^2 \tag{3.39}$$

Now, the expression for the gravity amplitudes in the HEFT as written in (3.37) is not completely evident, since in principle it is not necessary that double copy translates directly in the heavy mass limit. However, one can show [49] that starting from the KLT relations (3.11) and taking $m \to \infty$, HEFT gravity amplitudes take the form shown before after using the color-kinematics duality and the definition of the propagator matrix in double-colored scalar amplitudes.

Lastly, it is easy to see that the Yang-Mills-scalar amplitudes $A(1, 2, ..., n-2, v)$ are of order $\mathcal{O}(m^1)$ independently of the number of particles (we will show that in the following), and thus gravity amplitudes $M(1, 2, ..., n-2, v)$ are going to behave universally as m^2. Note also that all the mass scaling is contained in the minimal BCJ numerators $\mathcal{N}(1, 2, ..., n-2, v)$, since the propagators in the amplitudes (3.37) are purely massless.

Before continuing to explore the properties of the amplitude in the heavy mass limit, we will prove the equivalence of this regime to the multi-soft graviton limit discussed before. For this, we will start by looking at Y-M amplitudes with two massive scalars. Luckily, it is remarkably easy to see the equivalence between taking the multi-soft limit $p_i \to |\vec{q}|\tilde{p}_i$, $|\vec{q}| \to 0$ and the heavy mass limit $k_{n-1} = -mv$, $m \to \infty$. Indeed, it is manifest at the level of Feynman diagrams. Using KK relations, we can reduce the number of independent BCJ numerators to those corresponding to the $(n-2)!$ set of DDM half-ladder diagrams:



(3.40)

Now, these diagrams include only vertices where a scalar line is coupled to a gluon. The first vertex of the diagram:

$$V_1 = (\epsilon_1 \cdot k_n) \tag{3.41}$$

is just order $\mathcal{O}(m)$ in the HEFT and $\mathcal{O}(|\vec{q}|^0)$ in the soft gluon limit. Meanwhile, we can group the rest of the terms in products of a massive propagator and a vertex, each of them giving:

$$\frac{(\epsilon_i \cdot (k_n + p_1 + ... + p_{i-1}))}{2(k_n \cdot (p_1 + ... + p_{i-1}))}. \tag{3.42}$$

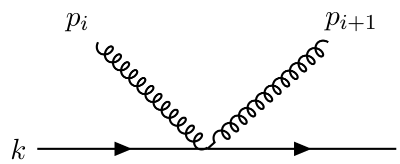
In the heavy mass limit, this expands as:

$$\frac{(\epsilon_i \cdot v)}{2(v \cdot (p_1 + .. + p_{i-1}))} + \frac{1}{m}\frac{(\epsilon_i \cdot (p_1 + ... + p_{i-1}))}{2(v \cdot (p_1 + .. + p_{i-1}))}. \tag{3.43}$$

This can be seen to be identical to the soft limit:

$$\frac{1}{|\vec{q}|}\frac{(\epsilon_i \cdot k_n)}{2(k_n \cdot (\tilde{p}_1 + .. + \tilde{p}_{i-1}))} + \frac{(\epsilon_i \cdot (\tilde{p}_1 + ... + \tilde{p}_{i-1}))}{2(k_n \cdot (\tilde{p}_1 + .. + \tilde{p}_{i-1}))}. \tag{3.44}$$

Something that we have to note is that these ladder diagrams have implicit "hidden" vertices where a scalar line connects to two gluons:

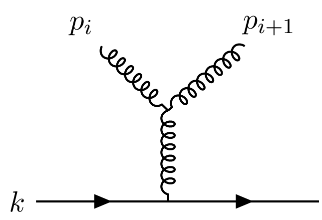


(3.45)

Here, $k = k_n + p_1 + ... + p_{i-1}$. Using the Feynman rules, we get:

$$\frac{\frac{1}{2}(k_n \cdot (p_1 + ... + p_{i-1} + p_i))}{2(k_n \cdot (p_1 + ... + p_{i-1}))}(\epsilon_i \cdot \epsilon_{i+1}), \tag{3.46}$$

where the factor in the numerators comes from the fact that we are dividing all the terms by the same propagator, so we have to cancel it for the two gluon vertex. Again, it is trivial to see that this scales in the same way for both limits.
Lastly, if we had a diagram with a massless line, like



(3.47)

we could factorize that part and use Jacobi identities on the numerators to express them as a combination of half-ladder diagrams. After some reduction, this would give:

$$\frac{(\epsilon_i \cdot k)(\epsilon_{i+1} \cdot p_i) - (\epsilon_{i+1} \cdot k)(\epsilon_i \cdot p_{i+1}) + \frac{1}{2}(\epsilon_i \cdot \epsilon_{i+1})(k_n \cdot (p_i - p_{i+1}))}{(p_i + p_{i+1})^2(2k_n \cdot (p_1 + ... + p_{i-1}))} \tag{3.48}$$

One can check that this again scales in the same way in the HEFT and the soft limit (it can be confusing that now the leading soft term is of order $\mathcal{O}(1/|\vec{q}|^2)$, but this is because the order in $1/|\vec{q}|$ increases with the number of propagators, and now we are considering two of them in this section of the diagram). In summary, one can see that the correspondence rule in an arbitrary Feynman diagram is $\mathcal{O}(m^n) \sim \mathcal{O}(|\vec{q}|^{2-g-n})$, where g is the number of external gluons (here n is an arbitrary integer). Moreover, it is straightforward to see that the leading term in the amplitudes that we consider

is of mass order m^1, regardless of the number or particles (it's really only the first vertex that contributes to the mass scaling, since in any other structure that we have analyzed, the vertex mass cancels with the massive propagator).
This means we have proven the equivalence for the Y-M amplitudes as a whole. Notice also that there aren't going to be cancellations between the diagrams, because these amplitudes are color-ordered.
Extending the correspondence to gravity is trivial, we only need the double copy. We know that gravity amplitudes can be expressed in the following form as a sum over all Feynman diagrams:

$$M(1,2,...,n-2,\overline{n-1},\overline{n}) = \sum_{i\in \text{cubic graphs}} \frac{N_i^2}{D_i}, \tag{3.49}$$

where the numerators N_Γ are borrowed from Yang Mills theory (i denotes an arbitrary Feynman diagram, not to be confused with the nested commutators Γ). Since we have proven the equivalent scaling of these quantities, it will hold for gravity amplitudes, too.
That being said, we have only showed that the amplitudes take the same form in the HEFT and the soft limit, but not how they scale exactly. Particularly, it has been observed that tree amplitudes suffer from cancellations in the first $(n-3)$ orders when taking the heavy or soft limits, and the first non-zero term always scales as $\mathcal{O}(m^2)$ or $\mathcal{O}(|\vec{q}|^0)$. To show that this is indeed the case for all amplitudes, we refer back to the fact that $M(1,2,...,n-2,v)$ can be written as:

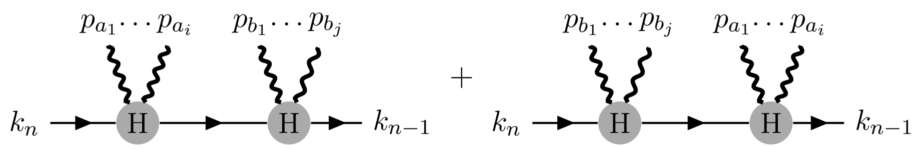
$$M(12...n-2,v) = \sum_{\Gamma\in\tilde{\rho}} \frac{(\mathcal{N}(\Gamma,v))^2}{D_\Gamma}, \tag{3.50}$$

i.e. the HEFT equivalence of the double copy. Now, all the propagators are massless, and Y-M amplitudes behave as m, which means that the HEFT numerators also scale as $\mathcal{O}(m)$, which means that the leading order in the amplitude will be $\mathcal{O}(m^2)$. Since, just accounting for the numerators, the correspondence is $\mathcal{O}(m^{n+g}) \sim \mathcal{O}(|\vec{q}|^{-n})$, the soft equivalent would be $\mathcal{O}(|\vec{q}|^0)$ when considering also the massless propagators.

3.3 Decomposition of the gravity amplitude in the heavy-mass limit

Before continuing on to compute the amplitudes for spinning massive particles, let's recall that, when taking the limit $m \to \infty$ (or, equivalently, the multisoft limit $|\vec{q}| \to 0$), the gravity n-point amplitude can be decomposed into a sum with products of s lower point amplitudes and delta functions imposing velocity cuts, which scale as $\mathcal{O}(m^{s+1})$ ($\mathcal{O}(|\vec{q}|^{1-s})$) and a n-point tree amplitude that scales as $\mathcal{O}(m^2)$ ($\mathcal{O}(|\vec{q}|^0)$). Although we already justified it in the soft limit, let us quickly repeat the argument from the perspective of HEFT. For this, we will closely follow the reasonings presented in section 4 of [21].

We start by noting that, since gravity amplitudes are not ordered, we have to sum over diagrams including all possible permutations of gravitons for a given set of massive and massless vertices. This means that we will encounter sums of diagrams such as:



(3.51)

where the grey blobs indicate a complete tree level amplitude. Now, both diagrams are identical except for the massive propagator separating the sets $\{a_1, ..., a_i\}$ and $\{b_1, ..., b_j\}$ of gravitons. Thus, the sum will be proportional to

$$\begin{aligned}&\frac{1}{(k_n + p_{a_1} + ... + p_{a_i})^2 - m^2 + i\varepsilon} + \frac{1}{(k_n + p_{b_1} + ... + p_{b_j})^2 - m^2 + i\varepsilon} \\ &= \frac{1}{(k_n + P_a)^2 - m^2 + i\varepsilon} + \frac{1}{(k_n + P_b)^2 - m^2 + i\varepsilon},\end{aligned} \tag{3.52}$$

were we have defined $P_a = p_{a_1} + ... + p_{a_i}$ and $P_b = p_{b_1} + ... + p_{b_j}$. In order to evaluate this sum, we can use that $P_b = q - P_a$, with q being the transfer momentum. However, since $k_n \cdot q = -q^2/2$, vector products like $k_n \cdot P_b$ won't have a uniform scaling in the transfer momentum (and thus, $\hbar$) when using momentum conservation. Therefore, we introduce the following variables:

$$\bar{k}_n = k_n + \frac{1}{2}q, \quad \bar{k}_{n-1} = k_{n-1} - \frac{1}{2}q. \tag{3.53}$$

This allows us to write a heavy-mass expansion like:

$$\begin{aligned}&\frac{1}{(k_n + P)^2 - m^2 + i\varepsilon} = \frac{1}{2k_n \cdot P + P^2 + i\varepsilon} \\ &= \frac{1}{2\bar{k}_n \cdot P + P^2 - q \cdot P + i\varepsilon} \approx \frac{1}{2\bar{k}_n \cdot P + i\varepsilon}\left(1 - \frac{P^2 - q \cdot P}{2\bar{k}_n \cdot P} + ...\right),\end{aligned} \tag{3.54}$$

where $\bar{k}_n = \bar{m}\bar{v}$. Since now $\bar{k}_n \cdot q = 0$, we can express the sum of the leading terms of the two propagators as:

$$\frac{1}{2\bar{k}_n \cdot P_a + i\varepsilon} + \frac{1}{2\bar{k}_n \cdot P_b + i\varepsilon} = \frac{1}{2\bar{k}_n \cdot P_a + i\varepsilon} + \frac{1}{-2\bar{k}_n \cdot P_a + i\varepsilon} = \hat{\delta}(\bar{k}_n \cdot P_a), \tag{3.55}$$

where again we have used (3.23). By recursively repeating this procedure, the HEFT amplitude can be written as:

$$\sum_{r=0}^{n-3} \sum_{\{P_a\}\in\mathcal{P}(r+1)} \left(\prod_{a=1}^{r} \hat{\delta}(\bar{m}\bar{v}\cdot P_a)\right) M(P_1,\bar{v})\cdots M(P_{r+1},\bar{v}), \tag{3.56}$$

where $\mathcal{P}(r+1)$ are all the possible partitions of the $(n-2)$ gravitons into $r+1$ non empty sets.

To illustrate this, consider the four point amplitude in gravity, which has an exact expression:

$$M(1,2,\bar{3},\bar{4}) = -\frac{(k_4\cdot F_1\cdot F_2\cdot k_4)(k_3\cdot F_1\cdot F_2\cdot k_3)}{p_{12}^2 k_4\cdot p_1 k_4\cdot p_2}, \tag{3.57}$$

which can be obtained by direct use of the KLT relations on (3.20). We first rewrite:

$$\frac{1}{k_4\cdot p_1 k_4\cdot p_2} = -\frac{2}{p_{12}^2}\left(\frac{1}{k_4\cdot p_1 + i\varepsilon} + \frac{1}{k_4\cdot p_2 + i\varepsilon}\right), \tag{3.58}$$

where we have used momentum conservation and reinstated the regulators $+i\varepsilon$. Changing to the barred massive momenta:

$$k_4 = \bar{k}_4 + \frac{q}{2} = \bar{m}\bar{v} - \frac{q}{2}, \tag{3.59}$$

leads to:

$$\frac{1}{k_4\cdot p_1 k_4\cdot p_2} = -\frac{2}{p_{12}^2\bar{m}}\left(\frac{1}{\bar{v}\cdot lp_1 - \frac{q\cdot p_1}{2\bar{m}} + i\varepsilon} + \frac{1}{\bar{v}\cdot p_2 - \frac{q\cdot p_2}{2\bar{m}} + i\varepsilon}\right). \tag{3.60}$$

Using that $\bar{v}\cdot p_2 = -\bar{v}\cdot p_1$ and expanding around $\frac{1}{\bar{m}}$:

$$\begin{aligned}\frac{1}{k_4\cdot p_1 k_4\cdot p_2} &= -\frac{2}{p_{12}^2\bar{m}}\left(\frac{1}{\bar{v}\cdot p_1 + i\varepsilon} - \frac{1}{\bar{v}\cdot l_1 - i\varepsilon}\right.\\ &\left.+\frac{q\cdot p_1}{2\bar{m}}\frac{1}{(\bar{v}\cdot p_1)^2 + i\varepsilon} + \frac{q\cdot p_2}{2\bar{m}}\frac{1}{(\bar{v}\cdot p_1)^2 - i\varepsilon} + \mathcal{O}\left(\frac{1}{\bar{m}^2}\right)\right).\end{aligned} \tag{3.61}$$

Here, we have rescaled the leading order in ε to get it out of the square. We can now see that the first two terms give a delta function and the two last, a principal value:

$$\begin{aligned}\frac{1}{k_4\cdot p_1 k_4\cdot p_2} &= -\frac{2}{p_{12}^2\bar{m}}\left(-2i\pi\delta(\bar{v}\cdot p_2) + \frac{p_{12}^2}{2\bar{m}}\mathrm{PV}\left(\frac{1}{(\bar{v}\cdot p_1)^2}\right) + \mathcal{O}\left(\frac{1}{\bar{m}^2}\right)\right)\\ &= -\frac{4i\pi\delta(\bar{v}\cdot p_1)}{p_{12}^2\bar{m}} + \frac{1}{\bar{m}^2}\mathrm{PV}\left(\frac{1}{(\bar{v}\cdot p_1)^2}\right) + \mathcal{O}\left(\frac{1}{\bar{m}^3}\right).\end{aligned} \tag{3.62}$$

This brings us to the following expression for the amplitude:

$$\begin{aligned}M(1,2,\bar{3},\bar{4}) = &-\frac{4i\pi\delta(\bar{v}\cdot p_2)(k_4\cdot F_1\cdot F_2\cdot k_4)(k_3\cdot F_1\cdot F_2\cdot k_3)}{(p_{12}^2)^2\bar{m}} \\ &+\frac{1}{\bar{m}^2}\frac{(k_4\cdot F_1\cdot F_2\cdot k_4)(k_3\cdot F_1\cdot F_2\cdot k_3)}{p_{12}^2(\bar{v}\cdot p_1)^2}+\mathcal{O}\left(\bar{m}\right). \end{aligned} \tag{3.63}$$

Now,

$$\begin{aligned}k_3\cdot F_1\cdot F_2\cdot k_3 = &\bar{m}^2\Big(\bar{v}\cdot F_1\cdot F_2\cdot\bar{v}-\frac{1}{2\bar{m}}q\cdot F_1\cdot F_2\cdot\bar{v} \\ &-\frac{1}{2\bar{m}}\bar{v}\cdot F_1\cdot F_2\cdot q+\frac{1}{4\bar{m}^2}q\cdot F_1\cdot F_2\cdot q\Big). \end{aligned} \tag{3.64}$$

One can see that, on the support of the delta function constrain, the second and third term vanish, while last one is going to contribute to at most order $\mathcal{O}(\bar{m})$ in the amplitude when squared. Thus, the leading and subleading terms for the amplitude are:

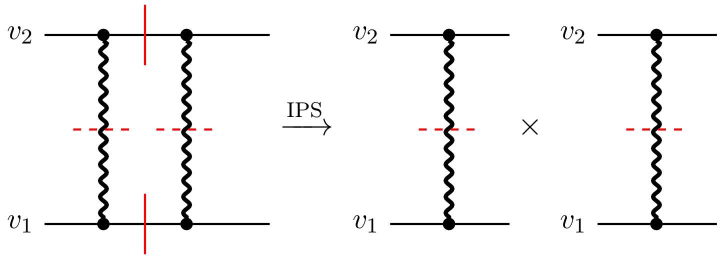
$$M(1,2,\bar{3},\bar{4}) = -i\pi\bar{m}^3(\bar{v}\cdot p_1)^2(\bar{v}\cdot p_2)^2\delta(\bar{v}\cdot p_1)+\frac{\bar{m}^2}{p_{12}^2}\left(\frac{\bar{v}\cdot F_1\cdot F_2\cdot\bar{v}}{\bar{v}\cdot p_1}\right)^2+\mathcal{O}(\bar{m}). \tag{3.65}$$

We see that this indeed corresponds to a product of two three point amplitudes on the support of a velocity cut delta function, of order $\mathcal{O}(\bar{m}^3)$, and a four point HEFT tree amplitude, which scales as $\mathcal{O}(\bar{m}^2)$.

Until now, we have been working in the HEFT formalism while using non-barred variables m and v. In contrast, it seems that our previous reasoning forces us to use $\bar{m}$ and $\bar{v}$ instead. However, although these redefined variables allow us to easily separate the amplitude in powers of the mass, each of the terms is not going to have a defined classical scaling, since:

$$\bar{m}^2 = m^2-\frac{q^2}{4}. \tag{3.66}$$

This can be seen as problematic, because it could be possible that hyperclassical terms "spill down" and contribute to the classical part of the amplitude. Nevertheless, it can be shown (see section 4.5 of [21]) that using the barred variables $\bar{m}$ and $\bar{v}$, hyperclassical diagrams (which are associated with two particle reducible diagrams) are convolutions integrals, which factorize into products of classical and quantum irreducible diagrams when Fourier transforming into impact parameter space. For example, the hyperclassical box diagram at one-loop is simply a product of two tree level amplitudes in IPS:



(3.67)

Because of the factorization of these diagrams, it is possible to express the S matrix in impact parameter space to some order in the coupling as a truncated exponential of the sum of the two particle irreducible diagrams to that particular order. For example, at two loops:

$$\tilde{S} = \exp\left(i\tilde{M}^{(0)} + i\tilde{M}^{(1)}_{2MPI} + i\tilde{M}^{(1,qu)}_{2MPI} + i\tilde{M}^{(2)}_{2MPI}\right) = \exp\left(\delta^{(0)}_{HEFT} + \delta^{(1)}_{HEFT} + \delta^{(2)}_{HEFT}\right). \tag{3.68}$$

Thus, the phase δ_{HEFT} contains only classical and quantum pieces. Since we don't care about the latter, the quantum contributions from $\bar{m}$ in the classical diagrams can be ignored, which means that we can just substitute the barred variables by the unbarred ones. We expect that this holds at every loop order and for the spinning case too, which is why we will exclusively use m and v from now on.

3.4 Factorization of the HEFT gravity amplitudes

One last subtlety we have to address with respect to the amplitudes in the heavy-mass limit is the fact that certain factorization channels get mixed with each other in gravitational amplitudes, giving rise to double poles which have to be taken into account when trying to evaluate the different cuts.
If we analyze the four point case one more time, we see that in color-ordered Yang-Mills theory the correspondence between the finite mass theory and the HEFT is given by:

$$A(12\bar{3}\bar{4}) = -\frac{k_3 \cdot F_1 \cdot F_2 \cdot k_3}{p_{12}^2(k_4 \cdot p_1)} \xrightarrow{m\to\infty} A(12, v) = -\frac{v \cdot F_1 \cdot F_2 \cdot v}{p_{12}^2(v \cdot p_1)} \tag{3.69}$$

As we can see, there is a one-to-one correspondence between the poles in the complete theory and the ones in the heavy-mass limit. That this extends to arbitrary points is also straightforward: the main reason is that the pole structure of the amplitude respects its color ordering, i.e. the only singularities present are of the form $k_n \cdot p_{12\ldots r}$, which don't get mixed when taking the limit $m \to \infty$. Even in the numerator structures, where there can be poles that don't correspond to physical factorization channels, they are never products of propagators that get mapped to the same object

in HEFT. This is because, due to the Hopf algebra rules [68], they always follow the structure $k_{n-1} \cdot p_{1\tau}$, where $\tau \in \{2, 3, ..., n-2\}$, and thus are never identical in the heavy-mass limit.
In gravity, however, the amplitudes are not ordered. Instead, one has to sum over all the possible permutations of massless particles when evaluating them using a Feynman diagram approach. As we saw before, this leads to the amplitude presenting terms where a velocity cut is imposed as a result of (3.23). On the other hand, the subleading contributions appear as a principal value of a squared propagator or, in other words, a double pole. This can also be seen by using the KLT formula (3.11), where the different orderings of the gauge amplitudes in the product is completely irrelevant in HEFT, leading to the same pole structure. Explicitly, at four points we see:

$$
\begin{aligned}
M(1,2,\bar{3},\bar{4}) &= -\frac{(k_4 \cdot F_1 \cdot F_2 \cdot k_4)(k_3 \cdot F_1 \cdot F_2 \cdot k_3)}{p_{12}^2 k_4 \cdot p_1 k_4 \cdot p_2} \\
&\xrightarrow{m\to\infty} M(12, v) = \frac{(v \cdot F_1 \cdot F_2 \cdot v)^2}{p_{12}^2 (v \cdot p_1)^2}.
\end{aligned}
\tag{3.70}
$$

The problem is evident: the two distinct massive poles $k_4 \cdot p_1$ and $k_4 \cdot p_2$ have been combined into the same HEFT pole $v \cdot p_1$. Thus, it is not clear what the interpretation of the factorization channels is anymore. This will become even more problematic when we analyze amplitudes involving spinning particles. As we will see, the Yang-Mills version of the amplitude presents a piece that doesn't contribute in the massive cut, i.e. it only contains the $1/p_{12}^2$ propagator. However, after performing the standard double copy with a scalar gauge amplitude, this term is going to acquire a $v \cdot p_1$ pole. We then have a term containing a double propagator $(v \cdot p_1)^2$ and another with a single copy. Obviously, the latter cannot contribute naively to the massive cut. This is because, if we trace back to the finite mass amplitude, it would mean that this piece has a residue in the $k_4 \cdot p_2$ pole, but not in the $k_4 \cdot p_1$ pole, breaking the symmetry under exchange of gravitons. Thus, we have to conclude that one cannot interpret massive HEFT poles as simple factorization channels. When we calculate the finite mass amplitude, we will see that this symmetry is indeed respected and that the simple pole piece is really a result of the spin flipping effect on the propagator that one has to take into account when considering the full theory.
In summary, factorization channels are not clearly defined in HEFT gravity amplitudes. If we want to compute any cut on an amplitude, we need to evaluate it first without taking the heavy-mass limit, and then collect the leading term when $m \to \infty$ of the resulting product of lower point amplitudes.

4 High spin amplitudes

In the previous sections, we have reviewed how the quantum scattering amplitudes approach is successfully used to compute classical observables in gravitational processes like black hole mergers. However, we have limited ourselves to the cases where the massive particles are scalar, which in the classical framework corresponds to having vanishing angular momentum (it is true that one can extend these methods to account for the massive particles having fundamental spin $s = \frac{1}{2}, 1, 2$, but that limits our analysis to quantum processes). If we wish to study scattering amplitudes where at least one of the massive objects has classical angular momentum, new complications arise due to the fact that we have to perform a power series expansion in the spin variable to describe all the possible values that the spin quantum number s can take. That this expansion indeed corresponds to a sum over fundamental spins can be seen by considering the amplitude in terms of the angular momentum operator J that is going to act on the spin states of the massive object. Generally, an amplitude $A_n^{h,s}$ describing a process where a massive spin s particle emits $n-2$ massless particles with helicity h (we will mostly be concerned with gluons and gravitons) can be written as a sum over operators transforming in the irreducible representations of the $SO(D-1)$ group up to dimension $2s+1$:

$$A_n^{h,s} = \mathcal{H}_n \times \sum_{r=0}^{s} \sum_{q=-r}^{r} \omega_{q,r}\, Q_q^{(r)}, \tag{4.1}$$

where $\mathcal{H}_n$ encodes all the scattering information of the process except for the spin state transition. In principle, the $\omega_{q,r}$ can be arbitrary coefficients of the operators Q, which act on the massive spinning states space (one has to interpret (4.1) as $\langle \varepsilon_n | A_n^{h,s} | \varepsilon_{n-1} \rangle$, where the $|\varepsilon\rangle$ transform in the spin s irreducible representation of the $SO(D-1)$ group). Now, by the Wigner-Eckart theorem, the matrix elements of the operators can be expressed as a product of a Clebsch-Gordan coefficient and a reduced matrix element that depends only on the spin r. Moreover, using the Cayley-Hamilton theorem we see that we can extend the sum to $r = \infty$, since for $r > s$, the fact that the largest space of states we are acting on corresponds to spin s implies that $Q^{(r)}$ can be written as a linear combination of lower representation operators. In other words, the sum effectively truncates at $r = s$. Incorporating the Clebsch-Gordan coefficients (and additional overall factors) into $\omega_{q,r}$, we are able to write:

$$A_n^{h,s} = \mathcal{H}_n \times \sum_{r=0}^{\infty} \omega_r Q^{(r)}, \tag{4.2}$$

where we note again that we are really reducing the matrix elements, not the operators themselves.

It is possible to translate this expression in terms of the generators of the $SO(D-1,1)$ group $J^{\mu\nu}$ [42]:

$$A_n^{h,s} = \mathcal{H}_n \times \sum_{j=0}^{\infty} \omega^{(2j)}_{\mu_1\cdots\mu_{2j}} J^{\mu_1\mu_2}\cdots J^{\mu_{2j-1}\mu_{2j}}. \tag{4.3}$$

Of course, since we are using tensor products of generators, which are generally not irreducible, each term in the multipole expansion is going to have lower spin contributions. Also, we can work only with the symmetric part of the operator products, since any commutator of the angular momentum operators can be reduced to lower powers of J via the $so(D-1,1)$ Lie algebra relations.
There is a certain ambiguity when promoting the multipoles to the $SO(D-1,1)$ group, since the massive spinning states on which the operators $J^{\mu\nu}$ are acting can transform in any of the representations $\left(s-\frac{n}{2},\frac{n}{2}\right)$ with $0 \le n \le 2s$. However, as it was shown in Appendix A of [70], a minimally coupled amplitude possesses *representation independence*, i.e. the massive spin states $\varepsilon_{1,2}$ can transform under any of the valid spin s representations of the $SO(D-1,1)$ group. As in [37], minimal coupling is defined as the interactions for which the high energy limit of the amplitudes is dominated by the opposite helicity configuration of the particles (thus matching with the massless case).
Moreover, we want to define a covariant spin vector associated with the $SO(D-1)$ Wigner rotations which, combined with a certain boost, add up to the set of Lorentz transformations that transform $k_{n-1} \to k_n$. Since these rotations are characterized by the time-like direction that they leave invariant, forming thus irreps of $SO(D-1)$ acting on the transverse space, we have to pick a certain reference direction. A possible choice [42] is to pick the average momentum of the incoming and outgoing massive particles $u = k/m = (k_{n-1}+k_n)/2m$. With this, boosts are realized as $K^\nu = u^\mu J^{\mu\nu}$, and the generators of Wigner rotations are given by $S^{\mu\nu} = J^{\mu\nu} - 2u^{[\mu}K^{\nu]}$. This automatically results in the so-called *Spin Supplementary Condition* (SSC) $u^\mu S^{\mu\nu} = 0$. Another common choice is to define the spin operator as the mass-rescaled Pauli-Lubanski pseudo-vector of the incoming particle [44, 70]:

$$S^\mu(p_1) = -\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}k_{n\nu}J_{\rho\sigma}. \tag{4.4}$$

It can be shown that any symmetrized product of Lorentz generators acting on the massive spinning states just results in:

$$\langle\varepsilon_n|(J^{\mu\nu})^{\odot n}|\varepsilon_{n-1}\rangle = (S^{\mu\nu})^n\langle\varepsilon_n|\varepsilon_{n-1}\rangle, \tag{4.5}$$

where the spin tensor is simply defined as:

$$S^{\mu\nu} = -\frac{1}{m}\epsilon^{\mu\nu\rho\sigma}k_\rho S_\sigma. \tag{4.6}$$

Lastly, since we are going to use it later, we also define the finite spin variable as:

$$a^\mu = \frac{S^\mu}{m} \tag{4.7}$$

In other words, we have shown how the amplitude describing a scattering process between a massive spinning line interacting with $n-2$ massless particles can be expressed as a power expansion in the spin variable S^μ (a^μ), which represents the action of Wigner rotations on the irreps for the massive particles.
Before continuing to review known results for high spin scattering, we will briefly discuss the parameters that we have to expand over in order to extract the classical limit of the amplitudes, since we ultimately are only interested in that contribution. As explained in [44, 70], we are working with three length scales, those being the Compton wavelength of the spinning body $\lambda_C \sim m^{-1}$ (equivalently we could use the de Broglie wavelength $\lambda_{dB} \sim k^{-1}$, since we are working in the regime where the velocity is of order $v \sim \mathcal{O}(1)$), the ring radius $|\vec{a}| \sim |\vec{S}|m^{-1}$ and the impact parameter of the scattering process $|\vec{b}| \sim |\vec{q}|^{-1}$. These scales obey the following hierarchy:

$$\lambda_C \ll |\vec{a}| \ll |\vec{b}|, \tag{4.8}$$

which obviously corresponds to the heavy-mass, finite size, large-distance regime. This gives us two main expansion parameters (dropping the 3-d vector notation):

$$\frac{\lambda_C}{a} \sim \frac{\hbar}{ma}, \qquad \frac{a}{b} \sim \hbar(q\cdot a), \tag{4.9}$$

where we also have to take into account the $\hbar$ scaling from the coupling constant $G \to \hbar G$ and the soft graviton momenta $l \to \hbar\tilde{l}$. However, since we already saw that the $m \to \infty$ limit is completely equivalent to the multi-soft limit $l \to 0$ and recovers the classical part of the amplitude, we can keep taking the leading term in the HEFT expansion if we are considering a finite ring radius. In the case where the object size is suppressed with respect to the impact parameter (i.e. low spinning objects), we can also take the spin expansion over $q \cdot a$.
Moreover, note that the first scale relation in (4.9) can also be written as:

$$\frac{\lambda_C}{a} \sim \frac{\hbar}{S} \sim s^{-1}, \tag{4.10}$$

where s is the fundamental spin of the particle. In other words, we expect classical spinning bodies to correspond to arbitrarily high spin fundamental particles.
One of the first advancements in the computation of arbitrary spin scattering amplitudes was made in [37], where the authors used massive spinor-helicity formalism to construct the minimal coupling three-point amplitude for a massive line (represented in the following by a field ϕ^s) with arbitrary spin interacting with a massless particle of helicity $h = \pm 1, \pm 2$.

In the case of gluons, it was given by:

$$A(1g^+, 2\phi^s, 3\bar{\phi}^s) = mx\frac{\langle\mathbf{23}\rangle^{2s}}{m^{2s}}, \qquad A(1g^-, 2\phi^s, 3\bar{\phi}^s) = \frac{m}{x}\frac{[\mathbf{23}]^{2s}}{m^{2s}}, \tag{4.11}$$

where $\langle\mathbf{23}\rangle$ and $[\mathbf{23}]$ are massive spinor products (see Appendix A for a brief introduction of massive spinor-helicity formalism), and x is defined in terms of the scalar amplitude:

$$A(1g^\pm, 2\phi, 3\bar{\phi}) = \frac{i}{\sqrt{2}}(k_2 - k_3)\cdot\varepsilon_1^\pm := mx^{\pm 1}. \tag{4.12}$$

For the case of gravitons, the amplitude is written as:

$$M(1h^+, 2\phi^s, 3\bar{\phi}^s) = im^2x^2\frac{\langle\mathbf{23}\rangle^{2s}}{m^{2s}}, \qquad M(1h^-, 2\phi^s, 3\bar{\phi}^s) = i\frac{m^2}{x^2}\frac{[\mathbf{23}]^{2s}}{m^{2s}}, \tag{4.13}$$

which, as pointed out in [36], is just the realization of the double copy using two amplitudes involving gluons with total massive spin s:

$$M(1h^\pm, 2\phi^s, 3\bar{\phi}^s) = iA(1g^\pm, 2\phi^{s_L}, 3\bar{\phi}^{s_L})A(1g^\pm, 2\phi^{s_R}, 3\bar{\phi}^{s_R}), \tag{4.14}$$

where $s_L + s_R = s$.

Despite this amplitude having a very compact form and being consistent with the massless helicity configurations in the high energy limit, we haven't argued why we should treat it as special compared to other possible amplitudes. However, it turns out that the minimally coupled expression is precisely the one that reproduces scattering processes involving Kerr black holes. In [38], the authors compared the impulse that a light particle experiments when moving through the electromagnetic or gravitational field sourced by a heavy spinning particle and the corresponding results obtained by using (4.11) and (4.13). Both types of background field are also related by a double copy [71]:

$$g^{\mu\nu} = \eta^{\mu\nu} + k^\mu k^\nu \phi(\vec{r}), \qquad A^{a\mu} = c^a k^\mu \phi(\vec{r}), \tag{4.15}$$

where k is a null vector depending on (r, θ) and $\phi(\vec{r})$ is a certain field common to both theories. Moreover, one can obtain obtain the Kerr metric (and its electromagnetic single copy, called $\sqrt{\text{Kerr}}$) by performing a shift in the coordinates $z \to z + ia$ [72]. Using this, it was shown that the impulse that a light particle experiences from a background field sourced by a heavy spinning particle matches the QFT result when using the minimally coupled amplitudes presented earlier. In other words, it seems like this minimal coupling does represent scattering processes involving Kerr black holes, which can also be seen as an agreement with the no hair theorem. This is going to be very important when we try to compute higher point arbitrary spin amplitudes since, by imposing unitarity conditions, we can guarantee that they factorize into the three point Kerr solution and thus behave as such when evaluating the cut kinematic configurations.

The authors of [37] also try to extend their general spin amplitudes to four points by using factorization behaviour consistency, i.e. imposing that the amplitude factorizes into the corresponding three point expressions when taking the residue in the various channels. The result in the case of spin 1 massless particles coupled to arbitrary spin massive particles is:

$$A(1g^+, 2g^-, 3\phi^s, 4\bar{\phi}^s) = \frac{\langle 2|k_4|1]^2}{(p_{14}^2 - m^2)(p_{24}^2 - m^2)} \left(\frac{\langle \mathbf{3}2\rangle[\mathbf{4}1] + \langle \mathbf{4}2\rangle[\mathbf{3}1]}{\langle 2|k_4|1]} \right)^{2S} . \tag{4.16}$$

Similarly, for the case of gravitons:

$$M(1h^+, 2h^-, 3\phi^s, 4\bar{\phi}^s) = \frac{\langle 2|k_4|1]^4}{(p_{14}^2 - m^2)(p_{24}^2 - m^2)p_{12}^2 M_{Pl}^2} \left(\frac{\langle \mathbf{3}2\rangle[\mathbf{4}1] + \langle \mathbf{4}2\rangle[\mathbf{3}1]}{\langle 2|k_4|1]} \right)^{2S} . \tag{4.17}$$

Upon superficial observation, we notice that these amplitudes have spurious poles for $S > 1$ and $S > 2$, respectively. This seems to indicate that there are some problems associated with particles of higher spin. Indeed, one can show that both (4.16) and (4.17) can be rewritten in a local form, but it contains inverse powers of m which obviously have are ill behaved in the high energy limit. Thus, one could interpret this as the fact that higher spin elementary particles cannot exist (clearly, composite particles with high spin can and do exist), except when considering the possibility of an infinite tower of internal particles with increasing spins, like the case of string theory.

As we explained before, these arbitrary spin amplitudes written in the spinor-helicity formalism must be equivalent to an expression of the form (4.3) in terms of the angular momentum operators $J^{\mu\nu}$. Indeed, it was shown in [39] that writing the angular momentum as a differential operator acting on spinor variables results in the following expression for the three point scattering amplitude between on massive line of spin s and one graviton:

$$M_3(1h^+, 2\phi^s, \bar{\phi}^s) = \frac{1}{m^{2s}}[3|^{2s}\hat{M}_3^+|2]^{2s}, \;\; M_3(1h^-, 2\phi^s, \bar{\phi}^s) = \frac{1}{m^{2s}}\langle 3|^{2s}\hat{M}_3^+|2\rangle^{2s}, \tag{4.18}$$

where the hatted amplitudes are given in terms of operators acting on the spin s massive states:

$$\hat{M}_3(1h^+, 2\phi^s, \bar{\phi}^s) = M_3^{(0)} \exp\left(i\frac{p_{1\mu}\varepsilon_\nu^+ J^{\mu\nu}}{k \cdot \varepsilon^+} \right),$$

$$\hat{M}_3(1h^-, 2\phi^s, \bar{\phi}^s) = M_3^{(0)} \exp\left(i\frac{p_{1\mu}\varepsilon_\nu^- J^{\mu\nu}}{k\cdot\varepsilon^-}\right). \tag{4.19}$$

Here, k refers to either the massive momentum k_2 or k_3 depending on which spinor state the exponential operator is acting on. Meanwhile, the notation e.g. $|2\rangle^{2s}$ simply reflects the fact that we can express a spin s state as a symmetrized tensor product of $2s$ spinor states.
This three point result was extended to four point (i.e. gravitational Compton amplitude) by generalizing the gravitational soft theorem to all orders in spin:

$$\hat{M}_4(1h^+, 2h^-, 3\phi^s, 4\bar{\phi}^s) = M_4^{(0)} \exp\left(i\frac{p_\mu\varepsilon_\nu J^{\mu\nu}}{k\cdot\varepsilon}\right), \tag{4.20}$$

where again k depends on the massive state the operator acts on and the massless momentum and polarization can be associated to either graviton by:

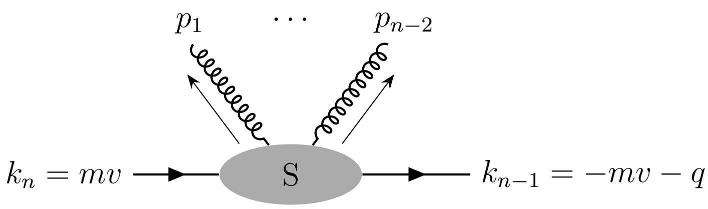
$$[4|^{2s} \exp\left(i\frac{p_{1\mu}\varepsilon_{1\nu}^+ J^{\mu\nu}}{k_1\cdot\varepsilon_1^+}\right)|3]^{2s} = \langle 4|^{2s} \exp\left(i\frac{p_{2\mu}\varepsilon_{2\nu}^- J^{\mu\nu}}{k_2\cdot\varepsilon_2^-}\right)|3\rangle^{2s}. \tag{4.21}$$

The identical helicity case can also be obtained with this methods, but since it doesn't contribute to the classical limit when calculating loop amplitudes [73], we won't focus on it for the time being. Now, one can write the action of the angular momentum operator in terms of the spin variable with the map $J^{\mu\nu} \to S^{\mu\nu}$ at three point (here, the spin tensor obeys the SSC $u_\mu S^{\mu\nu} = 0$). This allowed the authors in [39] to compute the scattering angle at 2PM (one loop) and check that it agreed with classical results in General Relativity up to third order in spin. Thus, we can use the amplitude (4.20) to check our results up to $\mathcal{O}(a^3)$.
Following this result, several advancements [40, 41, 44, 45, 70, 74, 75] have been made in the last couple of years to write the Compton amplitude in a more convenient form for classical computations (e.g. by trying to eliminate the spurious pole in the exponential) and perform loop calculations to extract observables describing spinning black hole scattering processes. In particular, in [74] a spurious pole free form of the Compton amplitude is achieved by adding contact terms to (4.20) while imposing a manifestly local amplitude and the so-called *black hole spin structure assumption*. The latter consists in expecting that the 2PM scattering amplitude depends on the quantity $(q\cdot a_i)(q\cdot a_j) - q^2(a_i\cdot a_j)$ $(j = 1, 2)$, a fact that has been observed to hold at order $\mathcal{O}(G^2)$. Using this, a unique form of the amplitude was obtained up to order $\mathcal{O}(a^4)$, which agreed with previous results (and so is also useful for us to compare our result).

5 SHEFT Compton Amplitudes

The objective of this project is to construct expressions for the amplitudes involving massive spinning particles that are manifestly covariant and gauge invariant and that contain only the classical information for the corresponding scattering process. Thus, we will extend the framework of the HEFT reviewed in previous sections to the case of spinning particles, generating only the leading term of the amplitude when expanding over $1/m$. We shall call this approach Spinning Heavy Mass Effective Field Theory (SHEFT).

In particular, we will focus first on calculating the amplitudes describing the interaction of one infinite mass and spin (where the ring radius $a = S/m$ remains finite) massive line interacting with $n-2$ massless gluons. This can be depicted as:



(5.1)

where, as we mentioned before, the four-velocity of the massive particle is normalized as $v \cdot v = 1$ and the heavy-mass limit provides the condition $v \cdot p_{12 \ldots n-2} = v \cdot q = 0$ to leading order in the mass. The corresponding gravity amplitude can then be calculated in a straightforward manner by use of the HEFT double copy [49].

In turn, the color-ordered gauge amplitude will be generated in terms of the minimal BCJ numerators $\mathcal{N}(1, 2, ..., n-2, v)$ as:

$$A_a(1, 2, ..., n-2, v) = \sum_{\Gamma \in \rho} \frac{\mathcal{N}_a(\Gamma, v)}{d_\Gamma}, \tag{5.2}$$

where the Γ are ordered nested commutators of the massless particles. Here, the subscript a indicates that the massive particle has classical spin, described by this vector.

Let's start looking at the three point amplitude, for which we can just apply the heavy-mass limit to the known result in minimal coupling:

$$A_a(1, v) = \mathcal{N}_a(1, v) = m(v \cdot \varepsilon_1) \exp\left(-i \frac{p_1 \cdot S \cdot \varepsilon_1}{m(v \cdot \varepsilon_1)}\right). \tag{5.3}$$

This expression still has a spurious pole in the exponent, but it actually is pretty straightforward to remove it in this covariant form. In order to do that, we first expand the exponential as a power series, separating the even and the odd terms:

$$\exp\left(-i\,\frac{p_1\cdot S\cdot\varepsilon_1}{m(v\cdot\varepsilon_1)}\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(i\,\frac{p_1\cdot S\cdot\varepsilon_1}{m(v\cdot\varepsilon_1)}\right)^n$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{p_1\cdot S\cdot\varepsilon_1}{m(v\cdot\varepsilon_1)}\right)^{2n}-i\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{p_1\cdot S\cdot\varepsilon_1}{m(v\cdot\varepsilon_1)}\right)^{2n+1}. \tag{5.4}$$

Next, we will transform every pair of dot products involving the $S^{\mu\nu}$ tensors into an expression with the spin variables a by using the fundamental identity:

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}=-\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}, \tag{5.5}$$

where the generalized Kronecker delta is given by:

$$\delta^{\mu_1\mu_2\ldots\mu_r}_{\nu_1\nu_2\ldots\nu_r}=\sum_{\sigma\in S_r}\pi(\sigma)\delta^{\mu_{\sigma(1)}}_{\nu_1}\delta^{\mu_{\sigma(2)}}_{\nu_2}\ldots\delta^{\mu_{\sigma(r)}}_{\nu_r}. \tag{5.6}$$

In particular, we have:

$$(p_1\cdot S\cdot\varepsilon_1)^2=-m^2(v\cdot\varepsilon_1)^2(p_1\cdot a)^2, \tag{5.7}$$

where we have used that $p_1^2=0$, $p_1\cdot\varepsilon_1=0$, $\varepsilon_1^2=0$ and the three point on-shell condition $v\cdot p_1=0$. Thus, it is easy to see that the pole is going to cancel completely once the spin tensors are expanded in pairs, resulting in the following:

$$\exp\left(-i\,\frac{p_1\cdot S\cdot\varepsilon_1}{m(v\cdot\varepsilon_1)}\right)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\,(p_1\cdot a)^{2n}-i\left(\frac{p_1\cdot S\cdot\varepsilon_1}{m(v\cdot\varepsilon_1)}\right)\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\,(p_1\cdot a)^{2n}$$
$$=\cosh(p_1\cdot a)-i\left(\frac{p_1\cdot S\cdot\varepsilon_1}{m(v\cdot\varepsilon_1)}\right)\frac{\sinh(p_1\cdot a)}{(p_1\cdot a)} \tag{5.8}$$

Of course, even if there seems to be a pole in $p_1\cdot a$ in the second term, the limit of the function $\sinh(x)/x$ when $x\to 0$ is well defined, so we don't need to worry about it. In order to avoid confusion and because it will be useful later, we will define the analytic function $G_1(x)$ as:

$$G_1(x):=\frac{\sinh(x)}{x}\;\forall x\neq 0,\qquad G_1(0)=\lim_{x\to 0}\frac{\sinh(x)}{x}=1. \tag{5.9}$$

Note: From now on, when the variables x_i appear as arguments of the G functions in the context of an amplitude, they will correspond to the kinematic objects $p_i\cdot a$. Inserting this result into the three point amplitude yields a spurious pole free, manifestly gauge invariant expression:

$$A_a(1,v)=m(v\cdot\varepsilon_1)\cosh(x_1)-i(p_1\cdot S\cdot\varepsilon_1)G_1(x_1). \tag{5.10}$$

This can be simplified even further by introducing the following vector:

$$w_1^\mu = m \cosh(x_1) v^\mu - i G_1(x_1)(p_1 \cdot S)^\mu, \tag{5.11}$$

which results in:

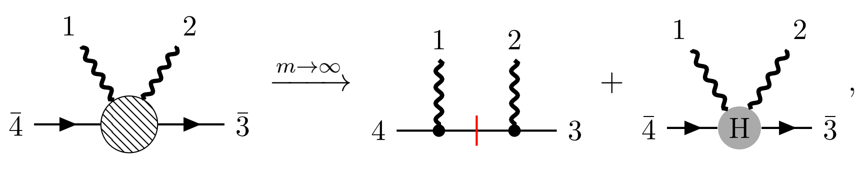
$$A_a(1, v) = w_1 \cdot \varepsilon_1. \tag{5.12}$$

Note that, written in this way, the amplitude is manifestly gauge invariant, since using both the on-shell condition $v \cdot p_1 = 0$ and the fact that $p_1 \cdot S \cdot p_1 = 0$ because of the antisymmetry of the spin tensor, we clearly see that $A_a(\varepsilon_1 \to p_1) = w_1 \cdot p_1 = 0$. The gravitational version of the amplitude can be easily obtain from its Yang-Mills counterpart by using the heavy-mass double copy. However, one thing to keep in mind is that we now have a freedom to choose the spin degree of the amplitudes we are using to perform the double copy. Indeed, if we consider a truncation of the series to order s, then every possible product of copies of order s_L and s_R will give the correct power counting as long as $s_L + s_R = s$. At three points, the resulting amplitude is independent on the values of $s_{L,R}$, which is straightforward to check when using the exponential form of the amplitude. However, as we will see, this doesn't necessarily hold at higher points. For the time being, we will follow the standard convention and use the double copy with a scalar and a spinning amplitude. Thus:

$$M_a(1, v) = A_a(1, v) A_0(1, v) = m(v \cdot \varepsilon_1)(w_1 \cdot \varepsilon_1). \tag{5.13}$$

5.1 Heavy-mass scaling

Before moving on to higher points, it is necessary to make the distinction of which part of the amplitude we are actually computing. As mentioned in previous sections, the HEFT amplitude in gravity can be decomposed into a sum of terms with definite scaling in the mass of the heavy particle. Discarding the quantum corrections, at n points there are $n-3$ superdominant contributions that scale as $\mathcal{O}(m^r)$ $(3 \leq r \leq n-1)$ and a classical contribution of order $\mathcal{O}(m^2)$. However, the former can be recursively expressed as products of lower point classical pieces on the support of a velocity cut delta function. Thus, the only new part of the amplitude is really the $\mathcal{O}(m^2)$, which we will designate as the HEFT tree amplitude. We expect the same to hold for arbitrary spin amplitudes. For example, at four points the complete amplitude (neglecting quantum corrections) is given by the sum of two diagrams:

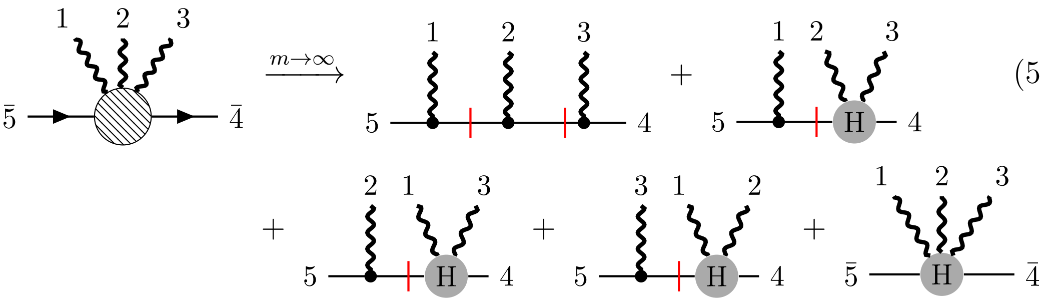


(5.14)

where the first one is of order $\mathcal{O}(m^3)$ and the second, of order $\mathcal{O}(m^2)$. In theory, we would need to calculate both of them to obtain the full gravity amplitude, but in practice, the first contribution is just:

$$\hat{\delta}(v \cdot p_1) M_a(1, v) M_a(2, v). \tag{5.15}$$

Since we already know the expression for the three point amplitude, the only thing we need to calculate is the four point HEFT tree amplitude. Thus, when we talk about the four point amplitude in the next sections, we really are referring to just this specific piece. Later, we will explicitly show this decomposition starting from the finite mass amplitude and taking $m \to \infty$.
The same reasoning holds at five points. Now the amplitude is divided into three different scalings:



(5.16)

Again, at this point the only unknown object is the five point HEFT amplitude (last term), which will be what we compute in the following.

5.2 Four point amplitude

Having made clear that we only need to calculate the tree part of the heavy-mass amplitudes, we proceed to construct the next simplest case, the four point. Using a minimal KLT basis, this means we can write:

$$A_a(12, v) = \frac{\mathcal{N}_a([1,2], v)}{p_{12}^2} = \frac{\mathcal{N}_a(12, v) - \mathcal{N}_a(21, v)}{p_{12}^2}. \tag{5.17}$$

Here, the numerator $\mathcal{N}(12, v)$ contains a piece that contributes to the massive pole $v \cdot p_1 = 0$. We can try to fix that part first by considering the cut condition:

$$\mathcal{N}_a(12, v)\Big|_{v \cdot p_1 = 0} \to \frac{1}{2v \cdot p_1}(p_1 \cdot p_2)\mathcal{N}_a(1, v)\mathcal{N}_a(2, v), \tag{5.18}$$

which induces the correct factorization behaviour for the amplitude. If we express the numerator as:

$$\mathcal{N}_a(12, v) = \frac{\mathcal{N}_a^{(v)}(12, v)}{2mv \cdot p_1} + \mathcal{N}'_a(12, v), \tag{5.19}$$

where $\mathcal{N}'_a$ contains no poles (i.e. we separate the analytical piece from the one that contributes to the massive cut), then we have the condition [49]:

$$\mathcal{N}_a^{(v)}(12, v)\Big|_{v\cdot p_1=0} \to (p_1 \cdot p_2)\mathcal{N}_a(1, v)\mathcal{N}_a(2, v) = (p_1 \cdot p_2)(w_1 \cdot \varepsilon_1)(w_2 \cdot \varepsilon_2). \tag{5.20}$$

Now, we want the BCJ numerator to be manifestly gauge invariant, which means we can express it in terms of the field strength tensors $F^{\mu\nu}$. With this in mind, there is a straightforward guess for the singular part of $\mathcal{N}_a(12, v)$:

$$\mathcal{N}_a^{(v)}(12, v) = -w_1 \cdot F_1 \cdot F_2 \cdot w_2. \tag{5.21}$$

For the sake of clarity, let's illustrate that this indeed factorizes correctly. Expanding the strength tensors:

$$\begin{aligned}\mathcal{N}_a^{(v)}(12, v) = &-(w_1 \cdot p_1)(\varepsilon_1 \cdot p_2)(\varepsilon_2 \cdot w_2) + (w_1 \cdot \varepsilon_1)(p_1 \cdot p_2)(\varepsilon_2 \cdot w_2)\\ &+(w_1 \cdot p_1)(\varepsilon_1 \cdot \varepsilon_2)(p_2 \cdot w_2) - (w_1 \cdot \varepsilon_1)(p_1 \cdot \varepsilon_2)(p_2 \cdot w_2).\end{aligned} \tag{5.22}$$

If we now take the cut $v \cdot p_1 = 0$, we see that $w_1 \cdot p_1 = w_2 \cdot p_2 = 0$ because of the HEFT on-shell condition $v \cdot p_{12} = 0$. Thus, only the second term remains and we get the result (5.20). In other words, our numerator can be written as:

$$\mathcal{N}_a(1, v) = -\frac{w_1 \cdot F_1 \cdot F_2 \cdot w_2}{2mv \cdot p_1} + \frac{1}{2}\mathcal{N}'_a(12, v). \tag{5.23}$$

We now turn to determine the analytical part of the numerator using the other cut condition on the massless pole $p_{12}^2 = 0$:

$$\mathcal{N}_a(12, v)\Big|_{p_{12}^2=0} \to \sum_{\lambda_i} \mathcal{N}_{\text{YM}}(12, i) \times \mathcal{N}_a(i, v), \tag{5.24}$$

where i indicates the internal massless particle that is being cut, and whose polarization states we have to sum over. Meanwhile, the three point Yang-Mills numerator can be written as:

$$\mathcal{N}_{\text{YM}}(12, i) = -\frac{\varepsilon_i \cdot F_1 \cdot F_2 \cdot \varepsilon_i}{\varepsilon_i \cdot p_1}, \tag{5.25}$$

as shown in [48]. In expanded form:

$$\mathcal{N}_{\text{YM}}(12, i) = (\varepsilon_i \cdot p_2)(\varepsilon_1 \cdot \varepsilon_2) - (\varepsilon_i \cdot \varepsilon_2)(\varepsilon_1 \cdot p_2) + (\varepsilon_i \cdot \varepsilon_1)(\varepsilon_2 \cdot p_1), \tag{5.26}$$

where we have used the three point on-shell conditions and the transversality of the polarization vectors. Using now the form of the three point SHEFT amplitude and the completeness relation:

$$\sum_{\lambda_i} \varepsilon_i^\mu \varepsilon_i^{\nu*} = -\eta^{\mu\nu}, \tag{5.27}$$

we get:

$$\mathcal{N}_a(12,v)\Big|_{p_{12}^2=0} \to (\varepsilon_1\cdot p_2)(w_{12}\cdot\varepsilon_2)-(\varepsilon_2\cdot p_1)(w_{12}\cdot\varepsilon_1)-(\varepsilon_1\cdot\varepsilon_2)(w_1\cdot p_2). \tag{5.28}$$

In contrast with the case with the massive pole, there is no clear way to propose an expression for the analytical part of the numerator $\mathcal{N}_a'(12,v)$. Not only does the cut condition show significantly less patterns to follow, but we also have to take into account the contribution from $\mathcal{N}_a^{(v)}(12,v)$ to the massless pole, since not all of the terms vanish when imposing $p_{12}^2=0$ (however, one can check that the pole $v\cdot p_1$ does disappear when expanding the strength tensors and taking the residue of the massless pole, as it should). Thus, we refrain from trying to guess how the rest of the numerator should look like, and instead take a bootstrap approach. In other words, we will make an *ansatz* with all the possible terms that $\mathcal{N}_a'$ could contain that satisfy our locality and gauge invariance conditions, and we will try to fix their coefficients by imposing the cut condition (5.28). Initially, the *ansatz* was constructed order by order in the spin variable expansion, but it quickly became clear that there was a structure that would allow for a closed form of the analytical piece of the BCJ numerator. In other words, it is possible to write $\mathcal{N}_a'(12,v)$ in terms of fundamental functions depending on the massless momenta and the spin variable a, as we will see. First, it is necessary to determine all the possible terms that could appear in the numerator. The expression (5.10) for the three point amplitude suggests that we should separate the even and odd part in the spin variable. Also, since the hyperbolic functions only depend on dot products of the massless momenta p_i and a, we assume that this is also the case at four points. Finally, every time there is a product of two spin tensors $S^{\mu\nu}$, we can use the identity (5.5) to express them in terms of a, which means that S will only appear once in each term of the odd part of the numerator. In other words, the *ansatz* will be composed a prefactor of monomials with dot products of massless momenta p_i, the velocity v, the field strength tensor F_i, the spin variable a and, at most, one instance of the spin tensor S. These monomials are multiplied by one or two hyperbolic functions depending on $p_i\cdot a$. Now, the monomials that can appear in the numerator are also constrained by the scaling of the amplitude. To start with, since we are dealing with the dominant contribution in the heavy-mass limit, the Yang-Mills numerator has to scale as $\mathcal{O}(m^1)$. We are considering terms with no massive pole $v\cdot p_1$, so there is one factor of v for each terms in the *ansatz*. For the odd part, this v is going to be hidden in the spin tensor S. Next, there has to be one factor of each field strength tensor F_1 and F_2 that contain the polarization information. Also, the spin scaling of the three point amplitude has to be preserved. This means that for each denominator containing a product $p_i\cdot a$ that appears in the hyperbolic part of the *ansatz* terms, there must be one power of the spin variable a (or the spin tensor S) in the monomials of the prefactor. We can state this condition in a more precise manner by introducing the concept of the *degree* of an object, which we define as its inverse scaling in λ when setting $a \to \lambda a$ and $\lambda \to (i\infty)$. Effectively, this measures the number of factors of $1/a$ that a certain object presents. For example, we would have:

$$\begin{gathered} \deg(p_1 \cdot a) = -1, \quad \deg(p_2 \cdot S \cdot F_1 \cdot a) = -2, \quad \deg(\cosh(x_1)) = 0, \\ \deg(G_1(x_1)) = 1, \quad \deg(G_1(x_1)G_1(x_2)) = 2. \end{gathered} \tag{5.29}$$

Our condition then translates to the fact that the total degree of each term in the ansatz has to be zero. Finally, the number of massless momenta p_i appearing in the prefactor monomials is determined by the energy scaling. Since the four point amplitude has dimensions of energy^0, and taking into account that:

$$\begin{gathered} [p_i^\mu] = \text{energy}^1, \quad [v^\mu] = \text{energy}^1, \quad [F_i^{\mu\nu}] = \text{energy}^1, \\ [a^\mu] = \text{energy}^{-1}, \quad [S^{\mu\nu}] = \text{energy}^0, \end{gathered} \tag{5.30}$$

we infer that $\mathcal{N}_a'(12, v)$ has to contain one less factor of p_i than factors of a and S combined (the argument is slightly different for the even and the odd part, but it leads to the same conclusion).

Regarding the hyperbolic functions that can appear, we would expect the following combinations:

$$\begin{gathered} \Big\{ \cosh(x_1)\cosh(x_2), \cosh(x_1)G_1(x_2), G_1(x_1)\cosh(x_2), \\ G_1(x_1)G_1(x_2), \cosh(x_{12}), G_1(x_{12}), \cosh(x_1 - x_2), G_1(x_1 - x_2) \Big\}. \end{gathered} \tag{5.31}$$

Now, we can discard the last two options because they don't appear in any of the cuts that we impose. Also, there is another special possibility that isn't being considered. It can be interpreted as an extension of the G_1 function to four points:

$$\begin{aligned} G_2(x_1; x_2) &:= \frac{1}{x_2}\left(G_1(x_{12}) - G_1(x_1)\cosh(x_2)\right) \\ &= \frac{1}{p_2 \cdot a}\left(\frac{\sinh(p_{12} \cdot a)}{p_{12} \cdot a} - \frac{\sinh(p_1 \cdot a)\cosh(p_2 \cdot a)}{p_1 \cdot a}\right). \end{aligned} \tag{5.32}$$

Although it may seems that this G_2 function contains some poles, it can easily be shown that this is not the case. Indeed, we know that the G_1 function is analytical in the whole complex plane, so the terms in parentheses are well-behaved. On the other hand, these terms vanish when we take $x_2 \to 0$, which means that it is going to be proportional to x_2 when expanding as a power series. This factor can then cancel with the denominator $1/x_2$, resulting in a fully polynomial expansion. In other words, G_2 is analytical in $\mathbb{C}^2$ [1].

[1] To be completely rigorous, we should have defined $G(x_1, 0)$ as the limit when $x_2 \to 0$, in the

It is important to consider this as a separate function, because the presence of the denominator allows us to incorporate additional factors of a to the monomials in front of the hyperbolic functions. In fact, it turns out that the G functions are natural building blocks for the analytical piece of the numerator. This is due to the fact that, when considering all the possible combinations of monomials and hyperbolic functions as explained above, the solution that fits the factorization conditions depends on a large number of free parameters, i.e. the form of the numerator isn't fixed only by gauge invariance and unitarity. However, if the set of terms is restricted to the one spanned by the functions G_1 and G_2, then the solution is uniquely determined once the factorization conditions are imposed. In order to show this, let's consider first which terms the *ansatz* is going to be made of. Taking into account the previously presented reasoning, we see that the only options are:

$$\begin{aligned}\text{even}: &(v\cdot X\cdot a)(p\cdot Y\cdot a)G_1(x_1)G_1(x_2),\ (v\cdot X\cdot p)(a\cdot Y\cdot a)G_1(x_1)G_1(x_2),\\ \text{odd}: &\operatorname{tr}(S\cdot X)G_1(x_{12}),\ (p\cdot X\cdot S\cdot Y\cdot a)G_2(x_1;x_2),\end{aligned}\tag{5.33}$$

where X, Y are arbitrary products of the field strength tensors $F_{1,2}$. Here, we have also used the fact that G_1 (G_2) is even (odd) under a transformation $a \to -a$ of the spin variable, which means that, for example, it is not possible to have terms like $(v\cdot X\cdot a)(p\cdot Y\cdot a)G_2(x_1;x_2)$ in the even part of the numerator, because this object is odd under reflections on a.

One could argue that there are some missing terms that could be included in the *ansatz*. For instance, upon further inspection of the function $G_2(x_1;x_2)$, we see that its Taylor expansion is still proportional to (x_1-x_2). In other words, it is possible to divide G_2 by this factor and consider prefactor monomials that contain three powers of the spin variable. However, we will see that the building blocks presented in (5.33) are enough to obtain a solution that satisfies the factorization conditions. Of course, the difference with any other solution has to be a pure contact term, since it needs to vanish when evaluating it on the cuts. This reflects the fact that we can start adding contact terms at cubic order in the spin expansion, making the solution no longer fully unique. The same conclusion can be reached when looking at the conditions for the terms in the *ansatz*: when considering a polynomial piece (contact) in the amplitude, we see that it needs to contain two less factors of p_i or v than factors of a to satisfy dimensionality constrains while maintaining a gauge invariant form. But, since it also needs to scale as v in order to be a leading term in the HEFT, we conclude that there need to be at least three power of the spin variable, i.e. the contact terms must start appearing at cubic order.

In order to fix the coefficients of the *ansatz*, a simple Mathematica program was used.

same way as we did for G_1

Generating all the possible terms with the conditions that we have stated before, we get the following expression for the proposal for $\mathcal{N}_a'$:

$$\begin{aligned}
&\mathcal{N}_a'(12,v) = G_1\left(x_{12}\right)\Big[c_1^{(1)}\operatorname{tr}\left(F_1\cdot F_2\cdot S\right)+c_2^{(1)}\operatorname{tr}\left(F_1\cdot S\cdot F_2\right)+c_3^{(1)}\operatorname{tr}\left(F_2\cdot S\cdot F_1\right)\Big]\\
&+G_2\left(x_1,x_2\right)\Big[c_1^{(2)}\left(a\cdot F_1\cdot F_2\cdot S\cdot p_1\right)+c_2^{(2)}\left(a\cdot F_1\cdot S\cdot F_2\cdot p_1\right)+c_3^{(2)}\left(a\cdot F_2\cdot F_1\cdot S\cdot p_1\right)\\
&+c_4^{(2)}\left(a\cdot F_1\cdot F_2\cdot S\cdot p_2\right)+c_5^{(2)}\left(a\cdot F_2\cdot F_1\cdot S\cdot p_2\right)+c_6^{(2)}\left(a\cdot F_2\cdot S\cdot F_1\cdot p_2\right)\Big]\\
&+m\,G_1\left(x_1\right)G_1\left(x_2\right)\Big[c_1^{(3)}\left(a\cdot v\right)\left(a\cdot F_1\cdot F_2\cdot p_1\right)+c_2^{(3)}\left(a\cdot F_1\cdot p_2\right)\left(a\cdot F_2\cdot v\right)\\
&+c_3^{(3)}\left(a\cdot v\right)\left(a\cdot F_2\cdot F_1\cdot p_2\right)+c_4^{(3)}\left(a\cdot F_2\cdot p_1\right)\left(a\cdot F_1\cdot v\right)+c_5^{(3)}\left(a\cdot p_1\right)\left(a\cdot F_1\cdot F_2\cdot v\right)\\
&+c_6^{(3)}\left(a\cdot p_1\right)\left(a\cdot F_2\cdot F_1\cdot v\right)+c_7^{(3)}\left(a\cdot p_2\right)\left(a\cdot F_1\cdot F_2\cdot v\right)+c_8^{(3)}\left(a\cdot p_2\right)\left(a\cdot F_2\cdot F_1\cdot v\right)\Big].
\end{aligned} \tag{5.34}$$

Adding that to the singular part of the numerator $\mathcal{N}_a^{(v)}$ and imposing the factorization condition (5.28), we obtain a system of 10 equations with 17 variables. We solve this by expanding out the field strength tensors and substituting each independent vector product by a numerical value repeatedly until there are enough independent linear equations to completely solve for the coefficients. Once this is done, the solution takes the form:

$$\begin{aligned}
&\mathcal{N}_a'(12,v) = G_1\left(x_{12}\right)\Big[c_1^{(1)}\operatorname{tr}\left(S\cdot F_1\cdot F_2\right)c_1^{(1)}\operatorname{tr}\left(S\cdot F_2\cdot F_1\right)+i\operatorname{tr}\left(S\cdot F_2\cdot F_1\right)\Big]\\
&+iG_2\left(x_1;x_2\right)\Big[\left(a\cdot F_1\cdot F_2\cdot S\cdot p_2\right)+\left(a\cdot F_2\cdot F_1\cdot S\cdot p_1\right)\Big]\\
&+m\,G_1\left(x_1\right)G_1\left(x_2\right)\Big[c_4^{(3)}\left(a\cdot F_1\cdot p_2\right)\left(a\cdot F_2\cdot v\right)+c_4^{(3)}\left(a\cdot F_2\cdot p_1\right)\left(a\cdot F_1\cdot v\right)\\
&-c_4^{(3)}\left(a\cdot p_2\right)\left(a\cdot F_1\cdot F_2\cdot v\right)-c_4^{(3)}\left(a\cdot p_1\right)\left(a\cdot F_2\cdot F_1\cdot v\right)-\left(a\cdot F_1\cdot p_2\right)\left(a\cdot F_2\cdot v\right)\\
&-\left(a\cdot p_1\right)\left(a\cdot F_2\cdot F_1\cdot v\right)\Big]
\end{aligned} \tag{5.35}$$

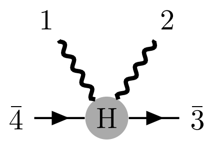
Now, it might seem like there is still some freedom left in the solution that we have obtained, since not all of the coefficients are fixed. However, this is just because some of the building blocks in the *ansatz* are not independent of each other. Indeed, it is easy to see that, because of the antisymmetry of the spin and field strength tensors, the traces that appear in the *ansatz* are odd under inversion, which means that $\operatorname{tr}\left(S\cdot F_1\cdot F_2\right)=\operatorname{tr}\left(S\cdot F_2\cdot F_1\right)$, making the $c(1,1)$ contribution cancel. On the other hand, we see that when expanding F_1 and F_2, the term multiplying $c(3,4)$ also vanishes. As a result, the form of the numerator is completely fixed:

$$\mathcal{N}'_a(12,v) = m\Big[(a\cdot p_1)(a\cdot F_2\cdot F_1\cdot v) - (a\cdot F_1\cdot p_2)(a\cdot F_2\cdot v)\Big]G_1\left(x_1\right)G_1\left(x_2\right)$$
$$+i\Big[\left(a\cdot F_1\cdot F_2\cdot S\cdot p_2 + a\cdot F_2\cdot F_1\cdot S\cdot p_1\right)G_2(x_1;x_2) + \mathrm{tr}\left(S\cdot F_2\cdot F_1\right)G_1\left(x_{12}\right)\Big] \tag{5.36}$$

In other words, we have shown that, after choosing a particular set of terms for the *ansatz*, which are spanned by the G_1 and G_2 functions, the factorization conditions together with gauge invariance provide a unique solution for the four point BCJ numerator involving a massive spinning line and two massless spin 1 particles. One can also see that the numerator is indeed antisymmetric under exchange of the massless particles, which is expected since we are working in the minimal KLT basis, i.e. there should only be one independent BCJ numerator. This property is not completely manifest in the expression for $\mathcal{N}_a(12,v)$, but can be checked by use of the relation:

$$\left(a\cdot F_1\cdot p_2\right)\left(a\cdot F_2\cdot v\right) + \left(a\cdot F_2\cdot p_1\right)\left(a\cdot F_1\cdot v\right)$$
$$-\left(a\cdot p_2\right)\left(a\cdot F_1\cdot F_2\cdot v\right) - \left(a\cdot p_1\right)\left(a\cdot F_2\cdot F_1\cdot v\right) = 0. \tag{5.37}$$

Finally, we can obtain an expression for the amplitude in the theory of gravity by performing the standard double copy in the SHEFT:



$$= M_a(12,v) = \frac{\mathcal{N}_0([1,2],v)\,\mathcal{N}_a([1,2],v)}{p_{12}^2}. \tag{5.38}$$

Now, one has to remember that this result only represents the factorizable part of the amplitude. In theory, we could add any set of contact terms as long as they satisfy gauge invariance and dimensionality/scaling constrains. Physically, this would just correspond to different types of spinning massive objects being scattered. In principle, we are interested in describing scattering processes involving Kerr black holes. Thus, we need to check that the amplitude that has been calculated indeed produces the correct expressions for the observables corresponding to such objects and, if not, modify it accordingly. This can be done by either comparing different forms of the amplitude (tree level Compton amplitude, one loop at 2PM, etc.) directly with the literature or by computing the observables themselves and contrast them with the ones obtained using classical General Relativity calculations. We will do that in later sections.

5.3 Finite mass four point amplitude

Before moving on to calculate the five point arbitrary spin amplitude, we will firstly take a look at four points in the case where the heavy-mass limit is not imposed. The regime is highly non trivial in comparison with the previous considerations, since we would have to take into account the spin flipping effects of the propagator. In short, since we are talking about high spin massive particles, which in general are going to be composite, we would expect that interacting with a spin 1 (gluon) or spin 2 (graviton) particle could change the intrinsic angular momentum quantum number (i.e. the eigenvalue of $\vec{S}^2$) by up to 1 or 2, respectively. Thus, the expression for the Feynman rules for the high spin propagators would need to include all the possible altered spin states. $\{s-1, s, s+1\}$ for the case of Yang-Mills and $\{s-2, s-1, s, s+1, s+2\}$ in the case of gravity. The reason for not having considered them before is that, as explained in [40], we expect these effects to be subleading in the heavy-mass expansion. Therefore, in principle we can naively assume that the four point amplitude factorizes directly into two three point high spin amplitudes when taking the cut in the massive factorization channel (see equation (5.18)).

Calculating the complete amplitude in the finite mass regime is out of the scope of this project, but we can compute the part that is going to have a non-zero contribution in the HEFT. This may not provide new physical insight to high spin scattering, but it will make the properties of the heavy-mass amplitude clearer, such as the factorization in the massive pole or its decomposition into definite scaling pieces.

As we mentioned in the beginning of this section, the three point amplitude in minimal coupling is well known to be:

$$A_a(1, \bar{2}, \bar{3}) = \mathcal{N}_a(1, \bar{2}, \bar{3}) = w_1^{(3)} \cdot \varepsilon_1 \Big|_{2s}, \tag{5.39}$$

where now:

$$(w_1^{(3)})^\mu := \cosh(x_1) k_3^\mu - iG_1(x_1)(p_1 \cdot S^{(3)})^\mu, \quad S_{\mu\nu}^{(3)} = \epsilon_{\mu\nu\rho\sigma} k_3^\rho a^\sigma, \tag{5.40}$$

and the $2s$ subscript indicates we are truncating the expansion in the spin variable at finite order $2s$. This is a consequence of no longer considering the regime where $m \to \infty$, which means that in order to keep $a = S/m$ finite, the expansion must stop at some finite order. Intuitively, one could justify this by arguing that a finite mass composite particle could not have arbitrarily high spin, since it would be made of a finite number of particles. However, since it won't affect our calculations, we will omit the subscript from now on.

At four points, we can write the expression for the exact amplitude as:

$$A_a(1, 2, \bar{3}, \bar{4}) = A_a^{(m)}(1, 2, \bar{3}, \bar{4}) + \mathcal{O}(m), \tag{5.41}$$

where $A_a^{(m)}(1, 2, \bar{3}, \bar{4})$ is the part of the amplitude contribution to the HEFT at order

$\mathcal{O}(m^2)$ or higher, and the rest are subleading terms that aren't going to appear when we take the limit $m \to \infty$. The next step is to assume that the color-kinematics duality still holds in an arbitrary spin theory, at least at the level of Yang-Mills. This allows us to use the minimal KLT basis to write the amplitude in terms of $(n-3)! = 1$ BCJ numerators:

$$A_a(1,2,\bar{3},\bar{4}) = \frac{\mathcal{N}_a([1,2],\bar{3},\bar{4})}{p_{12}^2}. \tag{5.42}$$

In order to construct the numerator $\mathcal{N}_a(1,2,\bar{3},\bar{4})$, we need to impose that the factorization condition (5.18) holds in the heavy-mass limit. With this in mind, a plausible expression would be:

$$\mathcal{N}_a(1,2,\bar{3},\bar{4}) = -\frac{w_1^{(4)}\cdot F_1\cdot F_2\cdot w_2^{(4)}}{2k_4\cdot p_1} + \frac{C}{2k_4\cdot p_1}\mathcal{N}_1'(1,2,\bar{3},\bar{4}) + \frac{1}{2}\mathcal{N}_2'(1,2,\bar{3},\bar{4}). \tag{5.43}$$

Indeed, we can easily see that the first term is going to simply factorize into a product of two three point numerators when we take the dominant part of the heavy-mass limit:

$$\begin{aligned} -w_1^{(4)}\cdot F_1\cdot F_2\cdot w_2^{(4)} =& -(w_1^{(4)}\cdot p_1)(\varepsilon_1\cdot p_2)(\varepsilon_2\cdot w_2^{(4)}) + (w_1^{(4)}\cdot\varepsilon_1)(p_1\cdot p_2)(\varepsilon_2\cdot w_2^{(4)}) \\ &+ (w_1^{(4)}\cdot p_1)(\varepsilon_1\cdot\varepsilon_2)(p_2\cdot w_2^{(4)}) - (w_1^{(4)}\cdot\varepsilon_1)(p_1\cdot\varepsilon_2)(p_2\cdot w_2^{(4)}) \\ \xrightarrow[\substack{k_4\cdot p_1\to 0\\ m\to\infty}]{}& (p_1\cdot p_2)(w_1^{(4)}\cdot\varepsilon_1)(\varepsilon_2\cdot w_2^{(4)}) = (p_1\cdot p_2)A_a(1,\bar{3},\bar{4})A_a(2,\bar{3},\bar{4}). \end{aligned} \tag{5.44}$$

Thus, in order for the numerator (5.43) to be well behaved in the HEFT, the second term needs to become polynomial in that regime. In other words, the pole $k_4\cdot p_1$ needs to cancel with the factor $C\mathcal{N}_1'(1,2,\bar{3},\bar{4})$. From this, we infer that the only choice is that $C = k_4\cdot p_2$, since this is the only object that will be proportional to the pole when we take $m\to\infty$ due to the on-shell condition $v\cdot p_{12} = 0$.
In other words, we have:

$$\mathcal{N}_a(1,2,\bar{3},\bar{4}) = -\frac{w_1^{(4)}\cdot F_1\cdot F_2\cdot w_2^{(4)}}{k_4\cdot p_1} + \frac{k_4\cdot p_2}{2k_4\cdot p_1}\mathcal{N}_1'(1,2,\bar{3},\bar{4}) + \frac{1}{2}\mathcal{N}_2'(1,2,\bar{3},\bar{4}). \tag{5.45}$$

In analogy with what we did in the HEFT, we can now fix the form for the rest of the numerator by imposing the correct factorization behaviour in the massless cut $p_{12}^2 = 0$. However, before this, we will also ensure that the amplitude satisfies the color-kinematics duality. This can be realized by making the Jacobi relations between numerators hold.

Using that $k_4 \cdot p_2 = -k_4 \cdot p_1 - p_1 \cdot p_2$, we can write the amplitude as:

$$A_a(1,2,\bar{3},\bar{4}) = -\frac{w_1^{(4)} \cdot F_1 \cdot F_2 \cdot w_2^{(4)}}{p_{12}^2(k_4 \cdot p_1)} + \frac{\mathcal{N}_2'(1,2,\bar{3},\bar{4}) - \mathcal{N}_1'(1,2,\bar{3},\bar{4})}{p_{12}^2} - \frac{\mathcal{N}_1'(1,2,\bar{3},\bar{4})}{2k_4 \cdot k_4}. \tag{5.46}$$

We will ignore the first term, since it can be shown that it satisfies the Jacobi relations on its own. Now, since a color-ordered amplitude can always be expressed as:

$$A(1,2,\bar{3},\bar{4}) = \frac{N(1,2,\bar{3},\bar{4})}{2k_4 \cdot p_1} + \frac{N([1,2],\bar{3},\bar{4})}{p_{12}^2}, \tag{5.47}$$

which means that our numerators must satisfy:

$$\begin{aligned} \mathcal{N}_1'(2,1,\bar{3},\bar{4}) - \mathcal{N}_1'(1,2,\bar{3},\bar{4}) &= \mathcal{N}_2'(1,2,\bar{3},\bar{4}) - \mathcal{N}_1'(1,2,\bar{3},\bar{4}) \\ \Rightarrow \mathcal{N}_2'(1,2,\bar{3},\bar{4}) &= \mathcal{N}_1'(2,1,\bar{3},\bar{4}). \end{aligned} \tag{5.48}$$

Repeating the argument for the amplitude $A_a(2,1,\bar{3},\bar{4})$, we obtain $\mathcal{N}_1'(1,2,\bar{3},\bar{4}) = \mathcal{N}_2'(2,1,\bar{3},\bar{4})$. It is straightforward to see that these conditions can be realized by:

$$\mathcal{N}_1'(1,2,\bar{3},\bar{4}) = -\mathcal{N}_2'(1,2,\bar{3},\bar{4}) := \frac{1}{2}\mathcal{N}_a'(1,2,\bar{3},\bar{4}). \tag{5.49}$$

We are now ready to impose the massless cut condition on the amplitude:

$$\begin{aligned} &\mathcal{N}_a(1,2,\bar{3},\bar{4})\Big|_{p_{12}^2=0} \to \sum_{\lambda_i} \mathcal{N}_{YM}(1,2,i) \times \mathcal{N}_a(i,\bar{3},\bar{4}) \\ &= (\varepsilon_1 \cdot p_2)(w_{12}^{(4)} \cdot \varepsilon_2) - (\varepsilon_2 \cdot p_1)(w_{12}^{(4)} \cdot \varepsilon_1) - (\varepsilon_1 \cdot \varepsilon_2)(w_{12}^{(4)} \cdot p_2). \end{aligned} \tag{5.50}$$

Expanding the term that contributes to the massive cut in the numerator, this translates explicitly to:

$$\begin{aligned} &- \mathcal{N}_a'(1,2,\bar{3},\bar{4})\Big|_{p_1 \cdot p_2 \to 0} \to (\varepsilon_1 \cdot p_2)(w_{12}^{(4)} \cdot \varepsilon_2) - (\varepsilon_2 \cdot p_1)(w_{12}^{(4)} \cdot \varepsilon_1) - (\varepsilon_1 \cdot \varepsilon_2) \\ &+ \Big[(w_1^{(4)} \cdot \varepsilon_1)(p_1 \cdot \varepsilon_2)\cosh(x_2) - \Big((\varepsilon_1 \cdot p_2)(\varepsilon_2 \cdot w_2^{(4)}) - (\varepsilon_1 \cdot \varepsilon_2)(p_2 \cdot w_2^{(4)})\Big)\cosh(x_1)\Big] \\ &= \Big[(a \cdot p_2)(a \cdot p_1)(\varepsilon_1 \cdot F_2 \cdot k_4) - (a \cdot p_2)(a \cdot p_1)(p_1 \cdot \varepsilon_2)(\varepsilon_1 \cdot k_4)\Big] G_1(x_1)G_1(x_2) \\ &+ i\Big[-(\varepsilon_1 \cdot F_2 \cdot S^{(4)} \cdot p_1) + (p_1 \cdot \varepsilon_2)(\varepsilon_1 \cdot S^{(4)} \cdot p_2)\Big] G_1(x_{12}) \\ &+ i\Big[(a \cdot p_1)(\varepsilon_1 \cdot F_2 \cdot S^{(4)} \cdot p_2) + (p_1 \cdot \varepsilon_2)(a \cdot p_2)(\varepsilon_1 \cdot S^{(4)} \cdot p_1)\Big] G_2(x_1; x_2). \end{aligned} \tag{5.51}$$

We can now make an appropriate *ansatz* for $\mathcal{N}_a'(1,2,\bar{3},\bar{4})$, just like we did in the heavy-mass limit, and solve for the cut condition. Choosing the same amount of building blocks as before, the solution is shown be unique, and it is given by:

$$\begin{aligned}
&\mathcal{N}'_a(1,2,\bar{3},\bar{4}) = \\
&= \Big[\frac{k_4\cdot p_2 - k_4\cdot p_1}{2}\,(a\cdot F_1\cdot F_2\cdot a) + (a\cdot F_1\cdot p_2)\,(a\cdot F_2\cdot k_4) - (a\cdot F_2\cdot p_1)(a\cdot F_1\cdot k_4)\Big] G_1(x_1)G_1(x_2) \\
&+ i\Big[(a\cdot F_1\cdot F_2\cdot S^{(4)}\cdot p_2) + (a\cdot F_2\cdot F_1\cdot S^{(4)}\cdot p_1)\Big] G_2(x_1;x_2) + i\,\mathrm{tr}(F_1\cdot F_2\cdot S^{(4)})G_1(x_{12}).
\end{aligned} \tag{5.52}$$

The full numerator is written as:

$$\mathcal{N}_a(1,2,\bar{3},\bar{4}) = -\frac{w_1^{(4)}\cdot F_1\cdot F_2\cdot w_2^{(4)}}{2k_4\cdot p_1} + \frac{k_4\cdot p_1 - k_4\cdot p_2}{2k_4\cdot p_1}\mathcal{N}'_a(1,2,\bar{3},\bar{4}). \tag{5.53}$$

Having obtained the piece of the amplitude in the finite mass regime that gives a non-zero contribution when taking the limit $m\to\infty$, let's check now that the corresponding gravity amplitude presents the correct factorization behaviour and decomposition into terms with definite mass scaling. Using the standard double copy:

$$\begin{aligned}
M_a(1,2,\bar{3},\bar{4}) &= -\frac{\mathcal{N}_a(1,2,\bar{3},\bar{4})\mathcal{N}_0(1,2,\bar{3},\bar{4})}{p_{12}^2} \qquad (5.54)\\
&= \frac{k_4\cdot F_1\cdot F_2\cdot k_4}{(k_4\cdot p_1)(k_4\cdot p_2)(p_1\cdot p_2)}\Big(w_1^{(4)}\cdot F_1\cdot F_2\cdot w_2^{(4)} + \frac{k_4\cdot p_1 - k_4\cdot p_2}{2}\mathcal{N}'_a(1,2,\bar{3},\bar{4})\Big).
\end{aligned}$$

If we take the massive cut $k_4\cdot p_1 = 0$ and use momentum conservation:

$$\begin{aligned}
M_a(1,2,\bar{3},\bar{4})\Big|_{k_4\cdot p_1=0} &\to \frac{1}{k_4\cdot p_1}\frac{(\varepsilon_1\cdot k_4)(p_1\cdot p_2)(\varepsilon_2\cdot k_4) + (\varepsilon_1\cdot k_4)(p_1\cdot p_2)(\varepsilon_2\cdot p_1)}{(p_1\cdot p_2)^2} \\
&\times\Big[(w_1^{(4)}\cdot\varepsilon_1)(p_1\cdot p_2)(\varepsilon_2\cdot w_2^{(4)}) + (w_1^{(4)}\cdot\varepsilon_1)(p_1\cdot p_2)(\varepsilon_2\cdot p_1)\cosh(x_2) \\
&\qquad -\frac{p_1\cdot p_2}{2}\mathcal{N}'_a(1,2,\bar{3},\bar{4})\Big|_{k_4\cdot p_1=0}\Big]
\end{aligned} \tag{5.55}$$

As we mentioned before, the last term involving $\mathcal{N}'_a(1,2,\bar{3},\bar{4})$ is a new contribution to the massive cut that results from the spin flipping effect of the propagator. Nevertheless, it can be seen to scale as $\mathcal{O}(m)$ when taking $k_4 = mv$, and thus is subleading in the $1/m$ expansion. This means that, in the heavy-mass limit, the amplitude correctly factorizes into a product of two three point HEFT amplitudes:

$$M_a(1,2,\bar{3},\bar{4})\Big|_{\substack{k_4\cdot p_1=0\\ m\to\infty}} \to \frac{m^3}{v\cdot p_1}(v\cdot\varepsilon_1)(w_1^{(4)}\cdot\varepsilon_1)(v\cdot\varepsilon_2)(w_2^{(4)}\cdot\varepsilon_2). \tag{5.56}$$

Lastly, we will decompose the amplitude into a series of terms with definite mass scaling and check that it is consistent with the previous results we obtained. Similarly to the scalar case, we write the propagator product in the denominator as:

$$\begin{aligned}\frac{1}{(k_4\cdot p_1+i\epsilon)(k_4\cdot p_2+i\epsilon)(p_1\cdot p_2)} &= -\frac{1}{(p_1\cdot p_2)^2}\left(\frac{1}{k_4\cdot p_1+i\epsilon}+\frac{1}{k_4\cdot p_2+i\epsilon}\right)\\ &\to \frac{\hat{\delta}(v\cdot p_1)}{m(p_1\cdot p_2)^2}-\frac{1}{m^2(p_1\cdot p_2)}\mathrm{PV}\left(\frac{1}{(v\cdot p_1)^2}\right).\end{aligned} \tag{5.57}$$

On the support of the delta function, the scalar numerator and the first terms of the spinning numerator simply factorize into two three point pieces. Meanwhile, the second term vanishes, since it is proportional to:

$$\frac{k_4\cdot p_1-k_4\cdot p_2}{2}=v\cdot p_1\to 0. \tag{5.58}$$

Meanwhile, by just changing to HEFT variables (without imposing any delta function velocity cut), we see that:

$$\begin{aligned}&\mathcal{N}_a'(1,2,\bar{3},\bar{4})\to\mathcal{N}_a'(12,v)=m\Big[-(v\cdot p_1)(a\cdot F_1\cdot F_2\cdot a)\\ &+(a\cdot F_1\cdot p_2)\,(a\cdot F_2\cdot v)-(a\cdot F_2\cdot p_1)(a\cdot F_1\cdot v)\Big]G_1(x_1)G_1(x_2)\\ &+i\Big[\Big((a\cdot F_1\cdot F_2\cdot S\cdot p_2)+(a\cdot F_2\cdot F_1\cdot S\cdot p_1)\Big)G_2(x_1;x_2)+\mathrm{tr}\,(S\cdot F_2\cdot F_1)\,G_1(x_{12})\Big],\end{aligned} \tag{5.59}$$

which exactly matches the entire part of the numerator (5.36) that we found in the previous section. As a result, we have:

$$\begin{aligned}&M_a(1,2,\bar{3},\bar{4})\to m^3(v\cdot\varepsilon_1)(w_1\cdot\varepsilon_1)(v\cdot\varepsilon_2)(w_2\cdot\varepsilon_2)\\ &-m^2\frac{v\cdot F_1\cdot F_2\cdot v}{v\cdot p_1}\left(\frac{w_1\cdot F_1\cdot F_2\cdot w_2}{v\cdot p_1}+\mathcal{N}_a'(12,v)\right)+\mathcal{O}(m)\\ &=m^3M_a(1,v)M_a(2,v)+m^2M_a(12,v),\end{aligned} \tag{5.60}$$

which correctly reflects the decomposition that we argued before, described by (5.16). However, we have to keep in mind that here we are really using the barred variables $\bar{m}$ and $\bar{v}$ implicitly, since we cannot write $v\cdot p_{12}=0$ otherwise. If one follows the calculations carefully, it is clear that we are neglecting some terms by use of this identity. For example, the numerator piece $\mathcal{N}_a'(1,2,\bar{3},\bar{4})$ on the support of the delta function would contribute to the $\mathcal{O}(m^2)$ piece of the amplitude, since the quantity $k_4\cdot(p_1-p_2)$ doesn't really vanish, but has a remainder $q^2/2$. Nevertheless, since this would be a contribution from the hyperclassical part of the amplitude in the classical part, we can safely neglect it due to the factorization of the amplitude in impact parameter space, as we argued previously. In other words, the expected heavy-mass decomposition in gravity still holds in the spinning case after taking into account all these considerations.

5.4 Five point amplitude

The next step will be to extend the result obtained previously to the five point case. Although the expressions get significantly more complex, the fundamental process remains the same: write the amplitude in terms of BCJ numerators using the minimal KLT basis and the HEFT, generate an *ansatz* by considering a sensible subset of building blocks that satisfy certain properties, and fix the coefficients by imposing gauge invariance and the correct factorization behaviour.
Using (3.37), the amplitude can be expressed in the $(n-3)!$ basis of BCJ numerators as a sum over nested commutators:

$$A_a(123,v) = \frac{\mathcal{N}_a([[1,2],3],v)}{p_{12}^2 p_{123}^2} + \frac{\mathcal{N}_a([1,[2,3]],v)}{p_{23}^2 p_{123}^2}. \tag{5.61}$$

At five points, the HEFT on-shell condition becomes $v \cdot p_{123} = 0$, so there are three independent massive poles. The factorization behaviour at each of them is given by:

$$\begin{aligned} \mathcal{N}_a(123,v)\Big|_{v\cdot p_{12}=0} &\to \frac{2}{3m\, v\cdot p_{12}}(p_{12}\cdot p_3)\mathcal{N}_a(12,v)\mathcal{N}_a(3,v), \\ \mathcal{N}_a(123,v)\Big|_{v\cdot p_{13}=0} &\to \frac{2}{3m\, v\cdot p_{13}}(p_1\cdot p_2)\mathcal{N}_a(13,v)\mathcal{N}_a(2,v), \\ \mathcal{N}_a(123,v)\Big|_{v\cdot p_{1}=0} &\to \frac{2}{3m\, v\cdot p_{1}}(p_1\cdot p_2)\mathcal{N}_a(1,v)\mathcal{N}_a(23,v). \end{aligned} \tag{5.62}$$

One might think that the cut $v \cdot p_{13} = 0$ is not to be taken here, since it isn't ordered in the context of the amplitude $A_a(123,v)$. However, since $\mathcal{N}_a(123,v)$ is a prenumerator, it doesn't have a particular ordering, and it is going to act as a building block for amplitudes involving other permutations of the massless particles, which is why this cut can be imposed.
Similarly to the four point case, the massive factorization conditions can be fixed by recasting the numerator as a sum with singular terms and an entire contribution:

$$\begin{aligned} \mathcal{N}_a(123,v) =& -\frac{(w_1\cdot F_1\cdot F_2\cdot w_2)\,(p_{12}\cdot F_3\cdot w_3)}{3m^2\,(v\cdot p_1)\,(v\cdot p_{12})} - \frac{(w_1\cdot F_1\cdot F_3\cdot w_3)\,(p_1\cdot F_2\cdot w_2)}{3m^2\,(v\cdot p_1)\,(v\cdot p_{13})} \\ &+ \frac{(w_1\cdot F_1\cdot F_2\cdot F_3\cdot w_3)\,\cosh(p_2\cdot a)}{3m\,(v\cdot p_1)} + \frac{\mathcal{N}'_a(12,v)\,(p_{12}\cdot F_3\cdot w_3)}{3m\, v\cdot p_{12}} \\ &+ \frac{\mathcal{N}'_a(13,v)\,(p_1\cdot F_2\cdot w_2)}{3m\, v\cdot p_{13}} - \frac{(w_1\cdot F_1\cdot p_2)\,\mathcal{N}'_a(23,v)}{3m\, v\cdot p_1} + \frac{1}{3}\mathcal{N}'_a(123,v). \end{aligned} \tag{5.63}$$

The logic behind the first three terms is that the arbitrary spin three point amplitude is identical to the scalar case but with the replacement $v \to w_i$. Thus, using the five point expression for the scalar amplitude obtained in [48] with this substitution should give an expression that factorizes correctly into at least part of the lower point numerators. However, at four points we saw that there is an entire part $\mathcal{N}'_a(ij, v)$ that appears when fixing the massless factorization condition, so we have to include it in the five point numerator. This is realized in the next three terms, which only have one pole. Lastly, the fully polynomial contribution in the numerator is given by $\mathcal{N}_a(123, v)$.

Let's check that this indeed gives the correct massive factorization behaviour. On the $v \cdot p_{12} = -v \cdot p_3 = 0$ pole, there are two contributions:

$$\begin{aligned}\mathcal{N}_a(123, v)\Big|_{v\cdot p_{12}=0} &= \frac{p_{12}\cdot p_3}{3m\, v\cdot p_{12}}\left[-\frac{w_1\cdot F_1\cdot F_2\cdot w_2}{m\, v\cdot p_1} + \mathcal{N}'_a(12, v)\right](\varepsilon_3\cdot w_3)\\ &= \frac{2}{3m\, v\cdot p_{12}}(p_{12}\cdot p_3)\mathcal{N}_a(12, v)\mathcal{N}_a(3, v),\end{aligned} \tag{5.64}$$

just as expected. The $v \cdot p_{13} = 0$ case is completely analogous, which leaves us with $v \cdot p_1 = v \cdot p_{23} = 0$:

$$\begin{aligned}&\mathcal{N}_a(123, v)\Big|_{v\cdot p_1=0} = \\ &= \frac{w_1\cdot\varepsilon_1}{3m\, v\cdot p_1}\left[\frac{(p_1\cdot F_2\cdot w_2)(p_2\cdot F_3\cdot w_3)}{v\cdot p_2} - (p_1\cdot F_2\cdot F_3\cdot w_3)\cosh(x_2) + (p_1\cdot p_2)\mathcal{N}'_a(23, v)\right].\end{aligned} \tag{5.65}$$

If we now use the fact that $(v \cdot p_2)\cosh(x_2) = w_2 \cdot p_2$ and we decompose the field strength tensors, the first two terms give the following:

$$\begin{aligned}&\frac{(p_1\cdot F_2\cdot w_2)(p_2\cdot F_3\cdot w_3)}{v\cdot p_2} - (p_1\cdot F_2\cdot F_3\cdot w_3)\cosh(x_2)\\ &= \frac{1}{v\cdot p_2}\big[(p_1\cdot F_2\cdot w_2)(p_2\cdot F_3\cdot w_3) - (w_2\cdot p_2)(p_1\cdot F_2\cdot F_3\cdot w_3)\big]\\ &= -\frac{p_1\cdot p_2}{v\cdot p_2}\big[(w_2\cdot p_2)(\varepsilon_2\cdot p_3)(\varepsilon_3\cdot_3) - (w_2\cdot\varepsilon_2)(p_2\cdot p_3)(\varepsilon_3\cdot w_3)\\ &- (w_2\cdot p_2)(\varepsilon_2\cdot\varepsilon_3)(p_3\cdot w_3) + (w_2\cdot\varepsilon_2)(p_2\cdot\varepsilon_3)(p_3\cdot w_3)\big] = -(p_1\cdot p_2)\frac{w_2\cdot F_2\cdot F_3\cdot w_3}{v\cdot p_2}\end{aligned} \tag{5.66}$$

Thus, we finally obtain:

$$\begin{aligned}\mathcal{N}_a(123, v)\Big|_{v\cdot p_1=0} &= \frac{p_1\cdot p_2}{3m\, v\cdot p_1}\left(-\frac{w_2\cdot F_2\cdot F_3\cdot w_3}{v\cdot p_2} + \mathcal{N}'_a(23, v)\right)(w_1\cdot\varepsilon_1)\\ &= \frac{2}{3m\, v\cdot p_1}(p_1\cdot p_2)\mathcal{N}_a(1, v)\mathcal{N}_a(23, v).\end{aligned} \tag{5.67}$$

Having checked that the numerator factorizes correctly on the massive poles, the only thing left is to fix the entire piece $\mathcal{N}'_a(123,v)$ using the massless factorization conditions. These are given by:

$$\begin{aligned} A_a(123,v)\Big|_{p_{123}^2=0} &\to \frac{1}{p_{123}^2}\sum_{\lambda_i} A_{YM}(123,i)\times A_a(i,v), \\ \mathcal{N}_a([[1,2],3],v)\Big|_{p_{12}^2=0} &\to \sum_{\lambda_i}\mathcal{N}_{YM}([1,2],i)\mathcal{N}_a([i,3],v), \\ \mathcal{N}_a([1,[2,3]],v)\Big|_{p_{23}^2=0} &\to \sum_{\lambda_i}\mathcal{N}_a([1,i],v)\mathcal{N}_{YM}([2,3],i). \end{aligned} \tag{5.68}$$

Note that we need to impose the second and third condition regardless of the massive factorization behaviour (5.62) since, although the cut e.g. $p_{12}^2=0$ is already encapsulated by $\mathcal{N}(12,v)$, it is not realized in the context of the five point numerator/amplitude. In other words, it is only fixing the double cut $v\cdot p_{12}=0$, $p_{12}^2=0$, but not the massless pole in its entirety. Moreover, it is clear that we need to use the numerators involving nested commutators, since the massless cuts are imposed at the level of the amplitude. Nevertheless, this is not a problem, since the prenumerator $\mathcal{N}_a(123,v)$ is crossing symmetric in the gluon legs, and thus other orderings like $\mathcal{N}_a(213,v)$ are simply obtained via the exchange $1\leftrightarrow 2$ in the expression for $\mathcal{N}_a(123,v)$.

In analogy to what we did at four points, we now set up an *ansatz* for $\mathcal{N}'_a(123,v)$ by considering all the possible terms that could appear in the entire part of the numerator. Again, we separate the even and odd parts in the spin expansion, where the former only contains dot products of p_i, v, a with the field strength tensors F_1, F_2, F_3 (for which there is one factor of each), while the latter also contains one spin tensor S. Since the amplitude has to scale as $\mathcal{O}(m)$, there can only appear one factor of v in the even part, and none in the odd part (it is included in S). Moreover, the five point amplitude has dimensions of energy^{-1}, so by taking into account the scaling of the objects v, F_i, a, S, we deduce that $\mathcal{N}'_a(123,v)$ has to contain one less power of the massless momenta p_i than factors of a and S.

Another important constrain for the *ansatz* is that each of the terms has to have total spin degree zero (recall the definition of degree from the previous section). At five points there are three massless particles, so we have a larger number of possible combinations for the hyperbolic functions. As in the four point case, we discard any terms depending on differences of the variables x_i, such as $G_1(x_1-x_2+x_3)$, because they don't appear in the lower point numerators when taking cuts. In addition, we restrict ourselves to products of G functions. Since now there is one more massless particle, we can extend their definition to include a new function with three arguments:

$$
\begin{aligned}
G_3(x_1; x_2, x_3) &:= \frac{1}{x_2 x_3}\big(G_2(x_{13}; x_2) - G_2(x_1; x_2)\cosh(x_3)\big) \\
&= \frac{1}{(p_2\cdot a)(p_3\cdot a)}\left(\frac{\sinh(p_{123}\cdot a)}{p_{123}\cdot a} - \frac{\sinh(p_{12}\cdot a)\cosh(p_3\cdot a)}{p_{12}\cdot a}\right. \\
&\quad \left. - \frac{\sinh(p_{13}\cdot a)\cosh(p_1\cdot a)}{p_{12}\cdot a} + \frac{\sinh(p_1\cdot a)\cosh(p_2\cdot a)\cosh(p_3\cdot a)}{p_1\cdot a}\right).
\end{aligned} \tag{5.69}
$$

Similarly to $G_2(x_1; x_2)$, one can show that $G_3(x_1; x_2, x_3)$ is indeed polynomial, i.e. the overall factor $1/(x_2 x_3)$ cancels out once we expand the numerator as a power series. We will save the proof for the general case when we extend the definition to an arbitrary number of arguments. Note also that G_3 is invariant under exchange of the last two arguments and odd under the transformation $a \to -a$, and that $\deg(G_3) = 3$. With this in mind, the possible hyperbolic functions that can appear in the *ansatz* are:

$$
\begin{aligned}
&\Big\{G_1(x_1)G_1(x_2)G_1(x_3), G_1(x_{12})G_1(x_3), G_1(x_{13})G_1(x_2), G_1(x_1)G_1(x_{23}), \\
&\quad G_1(x_{123}), G_2(x_1; x_2)G_1(x_3), G_2(x_1; x_3)G_1(x_2), G_2(x_2; x_3)G_1(x_1), \\
&G_2(x_{12}; x_3), G_2(x_{13}; x_2), G_2(x_{23}; x_1), G_3(x_1; x_2, x_3), G_3(x_2; x_1, x_3), G_3(x_3; x_1, x_2)\Big\}
\end{aligned} \tag{5.70}
$$

Considering all the restrictions presented above, we are now ready to determine the terms that can be included in $\mathcal{N}'_a(123, v)$:

$$
\begin{aligned}
\text{even} :\, & (v\cdot X\cdot a)\,(p\cdot Y\cdot a)\,G_1(x_i)\,G_1(x_{jk}), \\
& (v\cdot X\cdot a)\,(p\cdot Y\cdot a)\,(p\cdot Z\cdot a)\,G_1(x_i)\,G_2(x_j; x_k), \\
\text{odd} :\, & \mathrm{tr}(S\cdot X)\,G_1(x_{123}), (p\cdot X\cdot S\cdot Y\cdot a)\,G_2(x_{ij}; x_k), \\
& (p\cdot X\cdot S\cdot Y\cdot a)\,(p\cdot Z\cdot a)\,G_3(x_i; x_j, x_k), \\
& (p\cdot X\cdot S\cdot Y\cdot a)\,(p\cdot Z\cdot a)\,G_1(x_1)\,G_1(x_2)\,G_1(x_3),
\end{aligned} \tag{5.71}
$$

where X, Y, Z are just vector products of field strength tensors. We stress again that this doesn't exhaust all the possibilities for the *ansatz*. For example, as we mentioned before, $G_2(x_1; x_2)G_1(x_3)$ is still divisible by $(x_1 - x_2)$ while remaining a polynomial function, so we could consider monomial prefactors containing four spin variables. The same happens with the linear combination:

$$
\begin{aligned}
&G_3(x_1; x_2, x_3) - G_3(x_2; x_1, x_3) = \\
&= \frac{1}{630}(x_1 - x_2)\left(5x_1x_3^2 + 5x_2x_3^2 + 9x_1x_2x_3 - 126x_3 + 42x_1 + 42x_2 + \mathcal{O}(x^3)\right).
\end{aligned} \tag{5.72}
$$

The prefactor present when we take this linear combination means that we could include terms like:

$$\begin{aligned}&(p\cdot X\cdot a)(p\cdot Y\cdot a)(a\cdot Z\cdot a)\frac{G_3(x_i;x_j,x_k)-G_3(x_j;x_i,x_k)}{x_i-x_j},\\&(p\cdot X\cdot p)(a\cdot Y\cdot a)(a\cdot Z\cdot a)\frac{G_3(x_i;x_j,x_k)-G_3(x_j;x_i,x_k)}{x_i-x_j},\\&\dots\end{aligned}\tag{5.73}$$

However, as we will see, these would only add unnecessary freedom to the expression for $\mathcal{N}'_a(123,v)$. By considering just the terms depicted in (5.71), we obtain a unique solution for the numerator after imposing the massless factorization conditions.
The *ansatz* generated with these criteria is composed of 1548 terms. When solving the equations given by (5.68), the solution still has a fair amount of freedom (there are 621 free parameters left). However, upon expanding the field strength tensors and the hyperbolic functions, one can see that the terms corresponding to free parameters are really linear combinations of fixed terms. In other words, there is no real freedom in the solution, it is completely unique.
If we write the expression for the entire piece of the numerator as:

$$\mathcal{N}'_a(123,v)=\mathcal{N}'^{(even)}_a(123,v)+\mathcal{N}'^{(odd)}_a,\tag{5.74}$$

then the even and odd parts are given by:

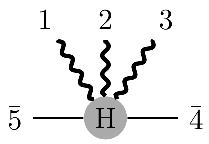
$$\begin{aligned}\mathcal{N}'^{(even)}_a&=m\Big[-\big((a\cdot F_1\cdot p_3)\,(a\cdot F_2\cdot p_1)\,(a\cdot F_3\cdot v)\big)\,G_1(x_3)\,G_2(x_1;x_2)\\&+\big((a\cdot F_1\cdot F_2\cdot p_3)\,(a\cdot F_3\cdot v)-(p_{12}\cdot a)(a\cdot F_3\cdot F_1\cdot F_2\cdot v)\big)\,G_1(x_3)\,G_1(x_{12})\\&-\big((p_1\cdot p_2)\,(a\cdot F_1\cdot v)\,(a\cdot F_3\cdot F_2\cdot a)+(p_{23}\cdot a)(a\cdot F_1\cdot p_2)(a\cdot F_3\cdot F_2\cdot v)\big)G_1(x_1)\,G_2(x_2;x_3)\\&+(a\cdot F_2\cdot p_1)(a\cdot F_3\cdot F_1\cdot v)G_1(x_1)\,G_1(x_2)\,\cosh(x_3)-(1\leftrightarrow 2)\Big]\\&+m\left((a\cdot F_1\cdot p_2)(a\cdot F_2\cdot F_3\cdot v)-(p_1\cdot a)(a\cdot F_2\cdot F_1\cdot F_3\cdot v)\right)G_1(x_1)\,G_1(x_2)\,\cosh(x_3)\,,\end{aligned}\tag{5.75}$$

$$\begin{aligned}\mathcal{N}'^{(odd)}_a&=i\Big[\mathrm{tr}(S\cdot F_3\cdot F_2\cdot F_1)G_1(x_{123})+G_3(x_1;x_2,x_3)\big((a\cdot F_2\cdot p_1)(a\cdot F_3\cdot F_1\cdot S\cdot p_1)\\&+(a\cdot F_2\cdot p_1)(a\cdot F_1\cdot F_3\cdot S\cdot p_3)-(p_1\cdot F_2\cdot S\cdot p_2)\,(a\cdot F_3\cdot F_1\cdot a)\big)\\&-(a\cdot F_3\cdot F_2\cdot[S,F_1]\cdot p_{12})\,G_2(x_{12};x_3)+(a\cdot F_2\cdot F_1\cdot[S,F_3]\cdot p_1)\,G_2(x_{13};x_2)-(1\leftrightarrow 2)\Big]\\&+i\Big[-(p_1\cdot a)(a\cdot F_2\cdot F_1\cdot F_3\cdot S\cdot p_3)\,G_3(x_1;x_2,x_3)-(a\cdot F_1\cdot F_2\cdot F_3\cdot S\cdot p_3)\,G_2(x_{12};x_3)\\&-(a\cdot F_2\cdot F_1\cdot F_3\cdot S\cdot p_3)\,G_2(x_{13};x_2)+(a\cdot F_1\cdot p_2)\,(a\cdot F_2\cdot F_3\cdot S\cdot p_3)\,G_1(x_1)\,G_1(x_2)\,G_1(x_3)\Big]\,.\end{aligned}\tag{5.76}$$

The appearance of functions like $\cosh(x_3)$ in the solution is nothing out of the ordinary: we have simply used the relation (5.32) to expand some of the G_2 functions, since doing that resulted in a more compact expression. In order to write $\mathcal{N}_a'(123, v)$ only in terms of the G functions, we would just have to revert this identity. Moreover, during the process of checking the dependence of the different terms, we came across the following interesting relation:

$$G_3(x_1; x_2, x_3) + G_3(x_2; x_1, x_3) + G_3(x_3; x_1, x_2) = -G_1(x_1)G_1(x_2)G_1(x_3), \tag{5.77}$$

which means that one of the functions $G_3(x_i; x_j, x_k)$ is not really independent, but can be expressed as a combination of the other G_3 and G_1 functions. This identity can actually be generalized to arbitrary number of points (see next section).
Finally, the five point gravity amplitude can be obtained by simple use of the standard double copy:



$$= M_a(123, v) = \frac{\mathcal{N}_0([[1,2],3], v)\,\mathcal{N}_a([[1,2],3], v)}{p_{12}^2\, p_{123}^2}$$
$$+ \frac{\mathcal{N}_0([1,[2,3]], v)\,\mathcal{N}_a([1,[2,3]], v)}{p_{23}^2\, p_{123}^2} + \frac{\mathcal{N}_0([[1,3],2], v)\,\mathcal{N}_a([[1,3],2], v)}{p_{13}^2\, p_{123}^2}, \tag{5.78}$$

As discussed in the four point case, it is possible that this amplitude doesn't describe processes involving Kerr black holes in particular. In order to make sure, one would have to perform 3PM (two loop) computations and compare with results obtained from classical calculations in GR. If there was a mismatch, the amplitude would have to be corrected by inserting additional contact terms. At five points, this is beyond the scope of this project, so we will leave it for future work.

5.5 Generalization to arbitrary number of points

The four and five point cases outline a clear methodology to construct an arbitrary point amplitude involving a massive spinning particle up to any order in spin in the heavy-mass limit. Firstly, we express it in terms of the BCJ numerators in the minimal $(n-3)!$ basis:

$$A_a(12...n-2, v) = \sum_{\Gamma \in \rho} \frac{\mathcal{N}_a(\Gamma, v)}{d_\Gamma}, \tag{5.79}$$

where $\Gamma \in \rho$ denotes the same set of nested ordered commutators as in (3.37) and d_Γ are the corresponding set of massless poles. As mentioned before, the prenumerator $\mathcal{N}_a$ is crossing symmetric, so we only need to compute one of them, e.g.
$\mathcal{N}_a(12...n-2, v)$.

To do this, we firstly fix the massive factorization conditions:

$$\mathcal{N}_a(12...n-2,v)\Big|_{v\cdot p_{1\tau}=0} \to \frac{|\tau||\omega|}{n-2}\frac{p_{1\tilde{\tau}}\cdot p_{\omega_1}}{m\,v\cdot p_{1\tau}}\mathcal{N}_a(1\tau,v)\mathcal{N}_a(\omega,v), \tag{5.80}$$

where τ and ω are two ordered disjoint subsets such that $\tau\cup\omega=\{1,2,...,n-2\}$ and $\tilde{\tau}\subset\tau$ is the subset of massless particles in τ which appear before the first element of ω in canonical ordering.

These factorization conditions can be satisfied by constructing a numerator piece $\mathcal{N}_a^{(v)}(12...n-2,v)$ with massive poles that accounts for all the possible products of lower point numerators that can be obtained on the different cuts. Although we don't have a completely streamlined process for the moment, the general procedure would consist in taking the scalar version of the numerator and making the replacements $v\cdot F_i \to w_i\cdot F_i$ while inserting factors of $\cosh(x_j)$ for every F_j that isn't dotted directly with a v. The result of this would satisfy the factorization conditions involving the terms that don't contain any lower point entire pieces $\mathcal{N}'_a(\omega,v)$. This can be seen inductively by noting that $v\cdot p_i=0 \Rightarrow w_i\cdot p_i=0$ and that $v\cdot p_i = w_i\cdot p_i$. As a consequence, a multilinear combination obtained by the replacement
$v\to v\cosh(x_i)+(p_i\cdot S)G_1(x_i)$ would have the same behaviour as the original, and thus would obey the same factorization behaviour. One should also add all the terms corresponding to the possible single or multiple cuts that contain entire lower point numerators pieces $\mathcal{N}'_a$. Again, a systematic way of doing this has still to be developed. For example, at six points the scalar numerator takes the form:

$$\begin{aligned}
\mathcal{N}_0(1234,v) &= \frac{(v\cdot F_1\cdot F_2\cdot v)(p_{12}\cdot F_3\cdot F_4\cdot v)}{(v\cdot p_1)(v\cdot p_{12})}\\
&-\frac{(v\cdot F_1\cdot F_2\cdot v)(p_{12}\cdot F_3\cdot v)(p_{123}\cdot F_4\cdot v)}{(v\cdot p_1(v\cdot p_{12})(v\cdot p_{123})}-\frac{(v\cdot F_1\cdot F_2\cdot v)(p_{12}\cdot F_3\cdot v)(p_{12}\cdot F_4\cdot v)}{(v\cdot p_1)(v\cdot p_{12})(v\cdot p_{124})}\\
&+\frac{(v\cdot F_1\cdot F_4\cdot v)(p_1\cdot F_2\cdot F_3\cdot v)}{(v\cdot p_1)(v\cdot p_{14})}+\frac{(v\cdot F_1\cdot F_3\cdot v)(p_1\cdot F_2\cdot F_4\cdot v)}{(v\cdot p_1)(v\cdot p_{13})}\\
&+\frac{(v\cdot F_1\cdot F_2\cdot F_3\cdot v)(p_{123}\cdot F_4\cdot v)}{(v\cdot p_1)(v\cdot p_{123})}+\frac{(v\cdot F_1\cdot F_2\cdot F_4\cdot v)(p_{12}\cdot F_3\cdot v)}{(v\cdot p_1)(v\cdot p_{124})}\\
&+\frac{(v\cdot F_1\cdot F_3\cdot F_4\cdot v)(p_1\cdot F_2\cdot v)}{(v\cdot p_1)(v\cdot p_{134})}-\frac{(v\cdot F_1\cdot F_3\cdot v)(p_1\cdot F_2\cdot v)(p_{132}\cdot F_4\cdot v)}{(v\cdot p_1)(v\cdot p_{13})(v\cdot p_{132})}\\
&-\frac{(v\cdot F_1\cdot F_3\cdot v)(p_1\cdot F_2\cdot v)(p_{13}\cdot F_4\cdot v)}{(v\cdot p_1)(v\cdot p_{13})(v\cdot p_{134})}-\frac{(v\cdot F_1\cdot F_4\cdot v)(p_1\cdot F_2\cdot v)(p_{12}\cdot F_3\cdot v)}{(v\cdot p_1)(v\cdot p_{14})(v\cdot p_{142})}\\
&-\frac{(v\cdot F_1\cdot F_4\cdot v)(p_1\cdot F_2\cdot v)(p_1\cdot F_3\cdot v)}{(v\cdot p_1)(v\cdot p_{14})(v\cdot p_{143})}-\frac{v\cdot F_1\cdot F_2\cdot F_3\cdot F_4\cdot v}{v\cdot p_1}.
\end{aligned} \tag{5.81}$$

Thus, the singular piece of the spinning version of this numerator would be:

$$
\begin{aligned}
\bar{\mathcal{N}}_a(1234,v) &= \mathcal{N}_0(1234,v)\Big|_{repl} + \frac{\mathcal{N}_a'(234,v)(p_2\cdot F_1\cdot w_1)}{m\,v\cdot p_1} \\
&+ \frac{\mathcal{N}_a'(34,v)(p_3\cdot F_2\cdot w_2)(p_2\cdot F_1\cdot w_1)}{m^2(v\cdot p_1)(v\cdot p_{12})} + \frac{\mathcal{N}_a'(24,v)(p_2\cdot F_3\cdot w_3)(p_2\cdot F_1\cdot w_1)}{m^2(v\cdot p_1)(v\cdot p_{13})} \\
&+ \frac{\mathcal{N}_a'(23,v)(p_{23}\cdot F_4\cdot w_4)(p_2\cdot F_1\cdot w_1)}{m^2(v\cdot p_1)(v\cdot p_{14})} + (\text{permutations}) \\
&+ \frac{(p_{12}\cdot p_3)\mathcal{N}_a'(12,v)\mathcal{N}_a'(34,v)}{m\,v\cdot p_{12}} + \frac{(w_3\cdot F_3\cdot F_4\cdot w_4)\mathcal{N}_a(12,v)}{m^2(v\cdot p_{12})(v\cdot p_{124})} \\
&+ \frac{(w_1\cdot F_1\cdot F_2\cdot w_2)\mathcal{N}_a(34,v)}{m^2(v\cdot p_{12})(v\cdot p_1)} + \frac{(p_{12}\cdot p_3)\mathcal{N}_a'(12,v)\mathcal{N}_a'(34,v)}{m\,v\cdot p_{12}} + (\text{permutations}),
\end{aligned}
\tag{5.82}
$$

where $\mathcal{N}_0(1234,v)\Big|_{repl}$ denotes the scalar version of the numerator with the replacements mentioned above.

The second step consists in computing the polynomial piece of the numerator $\mathcal{N}_a'(12...n-2,v)$, by imposing the massless factorization conditions on an *ansatz* containing a certain set of terms. From the results at four and five points, we expect that constructing an *ansatz* using only the G functions as building blocks is going to lead to a valid and potentially unique expression after solving the factorization equations.

For arbitrary number of points, we define the functions:

$$
G_r(x_1;x_2,...,x_r) := \frac{1}{x_r}\Big(G_{r-1}(x_1+x_r;x_2,\dots,x_{r-1}) - G_{r-1}(x_1;x_2,\dots,x_{r-1})\cosh(x_r)\Big),
\tag{5.83}
$$

with $G_1(x_1)$ given by (5.9). In appendix B, we will show that they can also be expressed as:

$$
\begin{aligned}
G_r(x_1;x_2,...,x_r) &= \frac{1}{x_2...x_r}\Big(G_1(x_{1...r}) - \sum_{i=2}^{r} G_1(x_{1...(i-1)(i+1)...r}\cosh(x_i) \\
&+ \sum_{i<j=2}^{r} G_1(x_{1...(i-1)(i+1)...(j-1)(j+1)...r})\cosh(x_i)\cosh(x_j) \\
&+ ... + (-1)^{n-2}\sum_{k=2}^{r} G_1(x_{1k})\cosh(x_2)...\cosh(x_{k-1})\cosh(x_{k+1})...\cosh(x_r) \\
&+ (-1)^{n-1}G_1(x_1)\cosh(x_2)...\cosh(x_r)\Big),
\end{aligned}
\tag{5.84}
$$

or, in a more compact form:

$$G_2(x_1;x_2,...x_r) = \frac{1}{x_2\cdots x_r}\times\left[\sum_{\substack{\rho_a\cup\rho_b=\{2,...,r\}\\ \rho_a\cap\rho_b=0}} G_1\big(x_{1\rho_a}\big)\prod_{i\in\rho_b}\big(-\cosh(x_i)\big)\right]. \tag{5.85}$$

From their definitions, it is clear that $\deg(G_r)=r$. It can also be proven that these specific functions are polynomial, i.e. they don't present any singularities when expanded as a Laurent series. Moreover, the following identity holds:

$$\sum_{i=1}^{n} G_n(x_i;x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) = \begin{cases}0 & n \text{ even}\\ (-1)^{\frac{n-1}{2}}G_1(x_1)\dots G_1(x_n) & n \text{ odd.}\end{cases} \tag{5.86}$$

In an n-point amplitude, we need the terms in $\mathcal{N}'_a(12...n-2,v)$ to contain hyperbolic functions depending on all the $n-2$ arguments x_i, since they will appear in lower point amplitudes when taking the cuts. Next, we need to take into consideration the different dimensionality conditions. In order to preserve gauge invariance and heavy-mass scaling, there needs to be one factor of each field strength tensor and one factor of v (included in the spin tensor S in the odd part of the numerator). Moreover, since the amplitude has dimensions of $(\text{energy})^{4-n}$ and $\mathcal{N}'_a(12...n-2,v)$ is divided by $n-3$ massless propagators, the number of factors of p_i has to be one less than the number of factors of a and S combined.
With all of this in mind, we see that the *ansatz* has to be composed of the following building blocks:

$$\begin{aligned}\text{even: }&\big\{(v\cdot X\cdot a)\mathcal{F}(p^{r-1},a^{r-1}),(v\cdot X\cdot p)\mathcal{F}(p^{r-2},a^{r})\big\}\times\prod_{\tau}G_{d_\tau}(x_\tau),\\ \text{odd:}&\big\{(p\cdot X\cdot S\cdot Y\cdot p)\mathcal{F}(p^{r-3},a^{r-1}),(p\cdot X\cdot S\cdot Y\cdot a)\mathcal{F}(p^{r-2},a^{r-2}),\\ &(a\cdot X\cdot S\cdot Y\cdot a)\mathcal{F}(p^{r-1},a^{r-3}),\mathrm{tr}(X\cdot S)\mathcal{F}(p^{r-1},a^{r-1})\big\}\times\prod_{\tau}G_{d_\tau}(x_\tau),\end{aligned} \tag{5.87}$$

where X,Y are arbitrary vector products of field strength tensors, $\mathcal{F}$ are products of F_i with the number of factors of p and a indicated in the arguments (which can include traces), and τ are the elements of a certain partition of the massless particles $\{1,2,...,n-2\}$ with $\sum_\tau d_\tau = r$ and such that the complete building blocks have the correct parity under $a\to -a$. We stress again that we are missing several possible terms involving e.g. linear combinations of the G functions, which would be divisible by some function of the variables x_i. However, the results at four and five points suggest that such an *ansatz* will lead to a unique solution or $\mathcal{N}'_a(12...n-2,v)$ once we impose the massless factorization conditions:

(5.88)

Moreover, although the expression for the entire piece of the numerator does not have a clear structure yet and has to be bootstrapped using an *ansatz*, we expect to eventually construct a set of rules to compute the expression for the arbitrary spin amplitude, potentially based on the Hopf algebra formalism developed in [48, 49, 68] for the scalar case. We leave this for future work.

6 Comparison with the literature up to quadratic order in spin

Although we have computed a manifestly gauge invariant form of the high spin amplitude to all orders in the angular momentum, this is not the first time that the four point amplitude has been calculated. There are multiple works in the literature [39, 41, 44, 74, 76] that provide an expression for the Compton amplitude to a particular order in spin. In most cases, they are obtained using the spinor-helicity formalism, since the most interesting part of the amplitude corresponds to the massless particles being of opposite helicity. This is because the same helicity contribution is actually subleading in the classical limit [74], so it can be neglected for Post-Minkowskian computations in gravity. This makes the process of obtaining the amplitude slightly simpler, due to the fact that many terms that would contribute in the covariant form vanish in spinor-helicity by adequately choosing the reference momenta for polarization vectors.
In recent years, the gravity spinning Compton amplitude has been calculated and compared with classical results in General Relativity up to fourth order in spin, mainly geodesic computations using the Mathisson-Papapetrou-Dixon equations [77–79] and black hole perturbation theory (BHPT) results via the Teukolsky equations[80]. For the first case, it is necessary to perform a 2PM loop integral and from this compute the corresponding observables that are being compared. In our case, we will firstly check our amplitude up to quadratic order in spin using the covariant expression provided in Appendix E of [41]. After a relabelling of the particles' momenta and a rescaling of the coupling constants to fit our conventions, it reads:

$$M_a(1,2,\bar{3},\bar{4}) = \frac{\langle\omega^{(0)}\rangle}{8p_{12}^2(k_4\cdot p_1)^2}\Big[\langle\omega^{(0)}\rangle + i\langle\omega^{(1)\mu\nu}\rangle\epsilon_{\mu\nu\rho\sigma}k_4^\rho a^\sigma + \langle\omega^{(2)}_{\alpha\beta}\rangle a^\alpha a^\beta\Big], \tag{6.1}$$

where the classical limit of the multipole coefficients are given by:

$$
\begin{aligned}
&\langle \omega^{(0)} \rangle = 2k_4 \cdot F_1 \cdot F_2 \cdot k_4, \\
&\langle \omega^{(1)\mu\nu} \rangle = k_4 \cdot F_1 \cdot p_2 F_2^{\mu\nu} + k_4 \cdot F_2 \cdot p_1 F_1^{\mu\nu} + \frac{k_4 \cdot (p_1 - p_2)}{2} [F^{\mu}_{1,\rho} F_2^{\rho\nu} - F^{\mu}_{2,\rho} F_1^{\rho\nu}], \\
&\langle \omega^{(2)\alpha\beta} \rangle = \left[k_4 \cdot F_1 \cdot F_2 \cdot k_4 q_\mu q_\nu \mathcal{P}^{\mu\nu\alpha\beta} - 2p_1 \cdot p_2 m^2 \left(\mathcal{P}^{\mu\nu\alpha\beta} + \frac{\eta^{\mu\nu}\eta^{\alpha\beta}}{2} \right) F_1^{(\mu|\delta} F_2^{\gamma|\nu)} \eta_{\gamma\delta} \right], \\
&\mathcal{P}^{\mu\nu\alpha\beta} = \frac{\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}}{2} - \eta^{\mu\nu}\eta^{\alpha\beta}.
\end{aligned}
\tag{6.2}
$$

Taking the heavy-mass limit $k_4 = mv$ and grouping everything together, we get the following:

$$
\begin{aligned}
M_a(12, v) = m^2 \frac{v \cdot F_1 \cdot F_2 \cdot v}{4p_{12}^2 (v \cdot p_1)^2} \Big[& 2(v \cdot F_1 \cdot F_2 \cdot v) + i\big((v \cdot F_1 \cdot p_2)\mathrm{tr}(F_2 \cdot S) \\
& + (v \cdot F_2 \cdot p_1)\mathrm{tr}(F_1 \cdot S) + 2(v \cdot p_1)\mathrm{tr}(F_1 \cdot F_2 \cdot S)\big) \\
& + (v \cdot F_1 \cdot F_2 \cdot v)\big((p_{12} \cdot a)^2 - p_{12}^2 a^2\big)\big) - \frac{a^2}{2}\mathrm{tr}(F_1 \cdot F_2) \Big].
\end{aligned}
\tag{6.3}
$$

Now, if we were to expand our four point gravity amplitude (5.38) to second order in spin, we would obtain something somewhat different to this last equation. However, as we saw when determining the form of the entire part of the BCJ numerator, this is just because of the ambiguity that exists when writing in a manifestly gauge invariant form, i.e. in terms of field strength tensors. After expanding using $F_i^{\mu\nu} = p_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu p_i^\nu$ and (5.5), the two expression are shown to be identical. Since the latter has been compared against classical results and corresponds to scattering processes of Kerr black holes, we can confidently say that our amplitude describes the intended phenomena up to second order in spin. However, as we will see, checking the $\mathcal{O}(a^3)$ and $\mathcal{O}(a^4)$ is not as straightforward, since no covariant form is presented in the literature, and it turns out that our proposed amplitude is missing some contact terms in order to match the appropriate Kerr behaviour.

7 Cubic order and beyond

When trying to analyze the validity of our expression for the Compton amplitude at order $\mathcal{O}(a^3)$ and higher, there is no reference in the literature that provides an explicit formula in a covariant formalism. Rather, it is generally calculated using an helicity-dependent framework where the kinematic information of the massless particles is presented in the form of Weyl spinors (sometimes also the massive ones). This is known as spinor-helicity formalism. Here, we only focus on the results that concern the arbitrary spin Compton amplitude, but in Appendix A we provide a brief introduction of this specific formulation for scattering amplitudes, in case the reader is not familiar with it.

For the purpose of checking our gravitational Compton amplitude, we will use the expression found in [74], where the authors start from the opposite helicity amplitude introduced in [37] and add a set of contact terms order by order in the spin variable with the intent of eliminating the spurious poles that appear as result of the ill-behaved high energy regime and satisfying the spin structure assumption mentioned in section 4. However, as it turns out, such contact terms are not needed until fifth order in spin and higher; since these contributions have not been contrasted with classical computations in General Relativity, we will not concern ourselves with them. Up to $\mathcal{O}(a^4)$, the gravitational Compton amplitude for the $(-+)$ helicity configuration is given by:

$$M_{lit}^{-+}(1,2,\bar{3},\bar{4}) = -\frac{4y^4}{p_{12}^2(p_4\cdot p_2)(p_4\cdot p_1)}\exp\left((p_2-p_1)\cdot a+(w\cdot a)\frac{(p_4\cdot p_1)-(p_4\cdot p_2)}{y}\right)\Bigg|_{a^4}, \tag{7.1}$$

where $y=[2|p_4|1\rangle$, $w^\mu=[2|\overline{\sigma}^\mu|1\rangle/2$ and the subscript in the exponential indicates that it is to be truncated at fourth order in the spin variable. Since the classical limit has already been taken in this formula, we can safely write the massive momenta in terms of the velocity v and apply the HEFT on-shell condition (3.34). This yields:

$$M_{lit}^{-+}(12,v) = 4m^2\frac{y^4}{p_{12}^2(v\cdot p_1)^2}\exp\left((p_2-p_1)\cdot a+2(w\cdot a)\frac{(v\cdot p_1)}{y}\right)\Bigg|_{a^4}, \tag{7.2}$$

where now $y=[2|v|1\rangle$. The next step is to convert our expression for the gravitational amplitude to spinor-helicity formalism, using $\varepsilon_1=\varepsilon_1^-,\varepsilon_2=\varepsilon_2^+$. This is relatively straightforward, we only have to use:

$$\varepsilon_{1\mu}^- = -\frac{[2|\gamma_\mu|1\rangle}{\sqrt{2}[21]},\ \varepsilon_{2\mu}^+ = \frac{\langle 1|\gamma_\mu|2]}{\sqrt{2}\langle 12\rangle}, p_{1\mu}=[1|\gamma_\mu|1\rangle,\ p_{2\mu}=[2|\gamma_\mu|1\rangle, \tag{7.3}$$

which leads to the following relations:

$$\begin{gathered}
\varepsilon_1^-\cdot p_2=\varepsilon_2^+\cdot p_1=0,\quad \varepsilon_1^-\cdot\varepsilon_2^+=0,\quad p_{12}^2=\langle 12\rangle[21],\\
\varepsilon_1^-\cdot v=-\frac{y}{\sqrt{2}[21]},\quad \varepsilon_2^+\cdot v=\frac{y}{\sqrt{2}\langle 12\rangle},\quad \varepsilon_1^-\cdot S\cdot\varepsilon_2^+=0,\\
p_1\cdot S\cdot\varepsilon_2^+=-\frac{\langle 12\rangle}{[21]}p_1\cdot S\cdot\varepsilon_1^-,\quad p_2\cdot S\cdot\varepsilon_1^-=-\frac{[21]}{\langle 12\rangle}p_2\cdot S\cdot\varepsilon_2^+.
\end{gathered} \tag{7.4}$$

In addition, we also have to substitute the product $p \cdot S \cdot \varepsilon$ by its helicity dependent form, which is [39]:

$$p \cdot S \cdot \varepsilon^{\pm} = (\pm) i \big[(v \cdot p)(\varepsilon^{\pm} \cdot a) - (v \cdot \varepsilon^{\pm})(p \cdot a) \big]. \tag{7.5}$$

This allows us to find the last undetermined product $p_1 \cdot S \cdot p_2$ by expanding the rank two tensor S in a Gram basis (e.g. $\{v, a, p_1, \varepsilon_1\}$) and using the previous expressions. After expanding the field strength tensors, performing a Taylor series on the spin variable up to fourth order and substituting the covariant vector products by the helicity dependent expressions, our amplitude takes the following form:

$$\begin{aligned}
M_a^{-+}(12,v) = &-\frac{y^3\,(a\cdot p_1)^{\,3}(w\cdot a)}{3p_{12}^2(v\cdot p_1)} + \frac{y^3\,(a\cdot p_2)^{\,3}(w\cdot a)}{3p_{12}^2(v\cdot p_1)} - \frac{y^3\,(a\cdot p_1)^{\,2}(w\cdot a)}{p_{12}^2(v\cdot p_1)}\\
&- \frac{y^3(a\cdot p_1)\,(a\cdot p_2)^{\,2}(w\cdot a)}{3p_{12}^2(v\cdot p_1)} - \frac{y^3\,(a\cdot p_2)^{\,2}(w\cdot a)}{p_{12}^2(v\cdot p_1)} - \frac{2y^3(a\cdot p_1)(w\cdot a)}{p_{12}^2(v\cdot p_1)}\\
&+ \frac{y^3\,(a\cdot p_1)^{\,2}(a\cdot p_2)(w\cdot a)}{3p_{12}^2(v\cdot p_1)} + \frac{2y^3(a\cdot p_1)(a\cdot p_2)(w\cdot a)}{3p_{12}^2(v\cdot p_1)} + \frac{2y^3(a\cdot p_2)(w\cdot a)}{p_{12}^2 p_1\cdot v_1} - \frac{2y^3(w\cdot a)}{p_{12}^2(v\cdot p_1)}\\
&+ \frac{y^4\,(a\cdot p_1)^{\,4}}{24p_{12}^2\,(v\cdot p_1)^{\,2}} + \frac{y^4\,(a\cdot p_2)^{\,4}}{24p_{12}^2\,(v\cdot p_1)^{\,2}} + \frac{y^4\,(a\cdot p_1)^{\,3}}{6p_{12}^2\,(v\cdot p_1)^{\,2}} - \frac{y^4(a\cdot p_1)\,(a\cdot p_2)^{\,3}}{6p_{12}^2\,(v\cdot p_1)^{\,2}}\\
&- \frac{y^4\,(a\cdot p_2)^{\,3}}{6p_{12}^2\,(v\cdot p_1)^{\,2}} + \frac{y^4\,(a\cdot p_1)^{\,2}}{2p_{12}^2\,(v\cdot p_1)^{\,2}} + \frac{y^4\,(a\cdot p_1)^{\,2}\,(a\cdot p_2)^{\,2}}{4p_{12}^2\,(p_1\cdot v_1)^{\,2}} + \frac{y^4(a\cdot p_1)\,(a\cdot p_2)^{\,2}}{2p_{12}^2\,(v\cdot p_1)^{\,2}}\\
&+ \frac{y^4\,(a\cdot p_2)^{\,2}}{2p_{12}^2\,(v\cdot p_1)^{\,2}} + \frac{y^4(a\cdot p_1)}{p_{12}^2\,(p_1\cdot v_1)^{\,2}} - \frac{y^4\,(a\cdot p_1)^{\,3}(a\cdot p_2)}{6p_{12}^2\,(v\cdot p_1)^{\,2}} - \frac{y^4\,(a\cdot p_1)^{\,2}(a\cdot p_2)}{2p_{12}^2\,(v\cdot p_1)^{\,2}}\\
&- \frac{y^4(a\cdot p_1)(a\cdot p_2)}{p_{12}^2\,(v\cdot p_1)^{\,2}} - \frac{y^4(a\cdot p_2)}{p_{12}^2\,(p_1\cdot v_1)^{\,2}} + \frac{y^2\,(a\cdot p_1)^{\,2}(w\cdot a)^2}{3p_{12}^2} + \frac{y^2\,(a\cdot p_2)^{\,2}(w\cdot a)^2}{3p_{12}^2} +\\
&\frac{2y^2(a\cdot p_1)(w\cdot a)^2}{3p_{12}^2} - \frac{2y^2(a\cdot p_2)(w\cdot a)^2}{3p_{12}^2} + \frac{2y^2(w\cdot a)^2}{p_{12}^2} + \frac{y^4}{p_{12}^2\,(p_1\cdot v_1)^{\,2}}
\end{aligned} \tag{7.6}$$

Grouping everything together, we find:

$$\begin{aligned}
M_a^{-+}(12,v) = &M_{lit}^{-+}(12,v) - \frac{4y^2(a\cdot p_1)(w\cdot a)^2}{3p_{12}^2} + \frac{4y^2(a\cdot p_2)(w\cdot a)^2}{3p_{12}^2}\\
&+ \frac{4y(w\cdot a)^3 v\cdot p_1}{3p_{12}^2} - \frac{2(w\cdot a)^4\,(v\cdot p_1)^{\,2}}{3p_{12}^2} - \frac{4y^3(a\cdot p_1)(a\cdot p_2) w\cdot a}{3p_{12}^2(v\cdot p_1)} - \frac{2y^2\,(a\cdot p_1)^{\,2}(w\cdot a)^2}{3p_{12}^2}\\
&- \frac{2y^3\,(a\cdot p_1)^{\,2}(a\cdot p_2) w\cdot a}{3p_{12}^2(v\cdot p_1)} + \frac{4y(a\cdot p_1)(w\cdot a)^3 v\cdot p_1}{3p_{12}^2} - \frac{4y(a\cdot p_2)(w\cdot a)^3 v\cdot p_1}{3p_{12}^2}\\
&+ \frac{2y^2(a\cdot p_1)(a\cdot p_2)(w\cdot a)^2}{p_{12}^2} - \frac{2y^2\,(a\cdot p_2)^{\,2}(w\cdot a)^2}{3p_{12}^2} + \frac{2y^3(a\cdot p_1)\,(a\cdot p_2)^{\,2} w\cdot a}{3p_{12}^2(v\cdot p_1)}.
\end{aligned} \tag{7.7}$$

In other words, the expression for our amplitude doesn't completely agree with the

results obtained by comparing to the classical computations for Kerr black holes. However, we can see that our previous claims remain consistent: the difference between both Compton amplitudes starts at cubic order in the spin variable, which means that $M_a(12, v)$ correctly describes Kerr black hole scattering up to quadratic order. Moreover, none of the terms that constitute the difference presents a massive double pole $(v \cdot p_1)^2$ in the denominator. This can be interpreted as the fact that the disagreement doesn't stem from QCD theory, but rather from a subleading effect in the HEFT expansion that manifests itself in the gravity amplitude after summing over all the possible graviton orderings. Indeed, despite the fact that the remainder contains massless poles $1/p_{12}^2$, they are only a byproduct of the spinor-helicity objects y and w, and would disappear once transformed back into a covariant expression in terms of the polarization vectors ε. This means that all the terms in the difference would contain at most one single massive pole $1/(v \cdot p_1)$, and thus cannot originate from the double copy of the gluon amplitudes. We conclude that these terms must represent some kind of subleading spin flipping effect in the complete high spin theory, altering the factorization behaviour in the massive poles. In order to determine them from first principles, we would have to devise an expression for the massive propagator to all orders in spin, something that is beyond the scope of this project and will be left for future research.
However, we are still able to add a finite number of pseudo-contact terms (which contain a single massive propagator) to correct the Compton amplitude order by order. This can be done by just considering all the possible gauge invariant objects with a single massive propagator that are cubic or quartic in the spin variable, and then imposing the total amplitude to be equal to M_{lit}^{-+} (resp. M_{lit}^{+-}) after particularizing to this helicity configuration.
This procedure yields the following fixing terms:

$$\begin{aligned}
M_a^{cont}(12, v) = &-\frac{2i(a \cdot F_2 \cdot v)(a \cdot F_1 \cdot S \cdot F_1 \cdot F_2 \cdot v)}{3(v \cdot p_1)} \\
&+ \frac{(a \cdot p_2)(a \cdot F_1 \cdot v)(a \cdot F_2 \cdot v)(a \cdot F_1 \cdot F_2 \cdot v)}{3(v \cdot p_1)} - \frac{(a \cdot p_1)(a \cdot F_1 \cdot v)(a \cdot F_2 \cdot v)(a \cdot F_1 \cdot F_2 \cdot v)}{3(v \cdot p_1)} \\
&+ \frac{1}{3}(a \cdot F_1 \cdot F_2 \cdot a)(a \cdot F_1 \cdot v)(a \cdot F_2 \cdot v). \qquad (7.8)
\end{aligned}$$

In other words, the amplitude can be deformed to match the Kerr black hole behaviour with just one additional term at cubic order in the spin variable and three additional terms at quartic order. We note that this doesn't provide a complete picture, because only one helicity configuration has been considered when comparing the amplitudes, but it is enough to reflect the behavior obtained in classical scattering computations. This approach could be repeated *ad infinitum* for every order in the spin variable using General Relativity calculations. However, as we mentioned before, there should be a way to establish a complete theory for massive spinning particles by accounting for the spin flipping effects at every order. We expect this theory to describe Kerr black hole scattering processes when taking the classical limit in the amplitudes, reflecting the correspondence between these objects and minimally coupled theories at an arbitrary number of points.

8 Conclusions and future research

In this work, we have reviewed and build upon the current research on gravitational scattering amplitudes involving massive spinning particles that correspond to classical objects with non-zero angular momentum such as Kerr black holes. This amplitudes framework could both help to create accurate templates for gravitational wave signals generated by collision processes of astrophysical binaries and to further our understanding of quantum field theories describing particles with arbitrary spin quantum number. In this endeavour, tools such as the color-kinematics duality, the double copy or the Heavy Mass Effective Field Theory are remarkably helpful to simplify the computation of said amplitudes, and reflect deeper relations between seemingly disconnected field theories that have yet to be unveiled.

In particular, we have shown that the HEFT framework is completely equivalent to the soft graviton expansion of the amplitude, which makes it a convenient way to extract its classical contributions due to the straightforward decomposition into definite mass scaling pieces when taking $m \to \infty$. However, one important downside is that the factorization behavior of the gravity amplitudes is no longer apparent, since the poles corresponding to two mirrored sets of gravitons get mapped into the same object in the heavy-mass limit. Thus, one has to be careful when taking massive cuts directly on the gravitational amplitudes.

Next, we have constructed the four- and five-point tree level amplitude for a process involving one massive particle with arbitrary spin in both Yang-Mills theory and Einstein gravity. Starting from the three point amplitude presented in [37], which was shown to describe Kerr black hole scattering in the classical limit, we have boostrapped the higher point expression by simply imposing gauge invariance, locality and a correct factorization behavior. Although this didn't completely restrict the solution, we observed that a particular family of functions arose naturally when fixing the factorization in the massless poles. By limiting the quantities appearing in the amplitude to these special functions, we were able to uniquely determine its form to all orders in spin in a manifestly covariant and gauge invariant way. We also presented an algorithm that generalizes the procedure of obtaining the amplitude for an arbitrary number of points.

The four point gravitational Compton amplitude obtained in this work was compared to other results obtained in the literature and shown to be identical up to quadratic order in the spin variable, which means that it exactly describes scattering processes involving Kerr black holes up to $\mathcal{O}(a^2)$. At cubic and higher orders, finite differences were detected with respect to other expressions in the kinematic configuration where the gravitons have opposite helicities, which could nevertheless be fixed by adding a small number of contact terms to the amplitude. In addition, these discrepancies could also be fixed by considering an alternative double copy to the standard procedure, where the two gauge amplitudes correspond to a spinning and a scalar copy, respectively. By using a suitable linear combination of spin order truncations for each copy of the Yang-Mills amplitudes, it is possible that the result can be matched to the literature at all orders in spin without needing to add any counterterms.

In addition to this, there are several directions in which one could continue the research in this topic. For instance, the tree amplitudes obtained in this work can be then used to calculate loop integrals and extract classical observables to all orders in spin up to 3PM for scattering processes involving spinning black holes. This would have a set of additional complications with respect to the non-rotating case due to the appearance of higher and higher powers of graviton momenta, which could make the integral badly divergent in the high energy regime. Thus, one would have to use integration by regions or equivalent techniques in order to obtain a sensible result.

Another open question is how to generalize this framework to the finite mass regime, i.e. obtain a covariant and well behaved form of the amplitude that accounts for all the spin flipping effects that occur for subleading orders in the classical expansion. Although the expressions in [37] are valid at any energy scale, they are still helicity dependent and present unwanted spurious poles for $s \geq 2$. Moreover, it doesn't consider the case where the spin quantum number of the particles change as a result of their interaction, which is entirely possible both in an elementary particle picture and in the case with composite particles where the angular momenta can be added into a direct sum of representations with different spin. Computing a well behaved expression for the finite mass case could aid in the development of a complete theory for high spin particles. In order to do this, one would need to fully understand the factorization behavior of the amplitudes and draw a correspondence between fundamental particle representations and the expansion into spin multipoles. Eventually, it could be possible to establish a well understood Lagrangian formalism that describes these processes. We leave all these exciting questions for future research.

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A Appendix: Introduction to Spinor-Helicity

Here, we will provide a brief review of the spinor-helicity formalism and its applications to scattering amplitudes. For a more thorough analysis, see e.g. [55].
In order to eliminate the redundancy that comes with using bi-fundamental objects such as polarization tensors in amplitudes, it is convenient to replace all the kinematic information for quantities that transform under the lowest possible representation of the Lorentz group, the spinor representation. In order to do that, one writes a massless momentum p as:

$$p_{\alpha\dot{\alpha}} = \sigma^{\mu}_{\alpha\dot{\alpha}} p_{\mu} = \lambda_{\alpha}\tilde{\lambda}_{\dot{\alpha}}, \tag{A.1}$$

where we are able to express the momentum as a product of two Weyl spinors $\lambda, \tilde{\lambda}$ because the determinant of the matrix vanishes due to the massless condition, $\det p_{\alpha\dot{\alpha}} = 0$, and thus it is of rank 1. For real momenta, we have $\tilde{\lambda}_{\dot{\alpha}} = (\lambda_{\alpha})^*$. Aside from manifestly satisfying the massless on-shell condition while being completely unconstrained, the 2-d spinors also transform straightforwardly under both the Lorentz and the little group (in contrast to other quantities like Lorentz vectors). For massless particles, the little group is just $U(1)$, and it is reflected in the spinors as the fact that the momentum $p_{\alpha\dot{\alpha}}$ is invariant under the transformation:

$$\lambda \to w\lambda, \quad \tilde{\lambda} \to w^{-1}\tilde{\lambda}, \tag{A.2}$$

where w is just a complex number in the case of general complex momenta. It is also common to express the spinors in terms of the angle and bracket notation:

$$|i\rangle \equiv \lambda_i^{\alpha}, \quad |i] \equiv \tilde{\lambda}_{i\dot{\alpha}}, \tag{A.3}$$

such that spinor products are written as:

$$\langle ij\rangle \equiv \epsilon_{\alpha\beta}\lambda_i^{\alpha}\lambda_j^{\beta}, \quad [ij] \equiv \epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_i^{\dot{\alpha}}\tilde{\lambda}_j^{\dot{\beta}}, \tag{A.4}$$

where $\epsilon_{\alpha\beta}$ ($\epsilon_{\dot{\alpha}\dot{\beta}}$) is the two-dimensional (anti-)fundamental Levi-Civita symbol, which makes these spinor products Lorentz invariant.

With these conventions, we can express vector products between massless momenta as:

$$s_{ij} = 2p_i \cdot p_j = \langle ij\rangle[ji]. \tag{A.5}$$

Although less apparent, polarization vectors can also be written in a direct manner in terms of spinors:

$$\varepsilon_i^{\mu+} = \frac{\langle q|\gamma^\mu|i]}{\sqrt{2}\langle qi\rangle},\ \varepsilon_i^{\mu-} = -\frac{[q|\gamma^\mu|i\rangle}{\sqrt{2}\langle qi\rangle}, \tag{A.6}$$

where $q \neq p_i$ is a reference momentum that can be chosen arbitrarily, and which reflects the gauge invariance of ϵ_i under transformations of the form $\epsilon^\mu \to \epsilon^\mu + xp^\mu$. Although the final amplitudes won't be dependent on the reference momenta, this freedom is actually very convenient to simplify the computations if one chooses an adequate set of q vectors such that certain products automatically vanish. For example, if the massless spin-one particles $1, 2$ have helicity $h_1 = +1$, $h_2 = -1$, then we can choose $q_1 = p_2$, $q_2 = p_1$ so that:

$$\varepsilon_1^+ \cdot p_2 = \varepsilon_2^- \cdot p_1 = \varepsilon_1^+ \varepsilon_2^- = 0. \tag{A.7}$$

Thus, as seen in Section **??**, the amplitudes take a very simple form once we particularize to each helicity configuration. Of course, this entails giving up on a gauge invariant form of the amplitude and dealing with some other problems like the appearance of spurious poles and a non-trivial factorization behaviour, but it is undeniable that the expression for the amplitudes greatly simplifies when using the spinor-helicity formalism.

In [37], the authors also introduce a straightforward generalization of the framework to massive particles. Since now the matrix $p_{\alpha\dot{\alpha}}$ has rank 2, we need to sum over two rank 1 matrices, which means that we need to introduce two sets of Weyl spinors:

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha^I \tilde{\lambda}_{\dot{\alpha}I}, \quad I = 1, 2. \tag{A.8}$$

This new index I also reflects the transformation properties under the little group, which in the massive case is $SU(2)$. More generally, for complex momenta we have:

$$\lambda^I \to W_J^I \lambda^J, \quad \tilde{\lambda}_I \to (W^{-1})_J^I \tilde{\lambda}^J, \tag{A.9}$$

where $W \in SL(2, \mathbb{C})$. In order to not overcrowd the expressions, the little group indices of massive particles are hidden inside bold spinors:

$$|i^J\rangle \to |\mathbf{i}\rangle, \quad |i^J] \to |\mathbf{i}]. \tag{A.10}$$

Aside from taking these indices into account, the way of operating with these spinors remains the same as in the massless case.

B Appendix: Properties of the G functions

We will now present the proof that the functions:

$$G_n(x_1;...,x_n) = \frac{1}{x_n}\Big(G_{n-1}(x_1+x_n;x_2,...,x_{n-1}) - G_{n-1}(x_1;x_2,...,x_{n-1})\cosh(x_n)\Big),$$
$$G_1(x_1) := \frac{\sinh(x_1)}{x_1}, \tag{B.1}$$

are free of divergences at the origin, i.e. when expanding around $x_1 = x_2 = ... = x_n = 0$, the result is a polynomial. We will do so by induction. At $n = 1$, the function expands as:

$$G_1(x_1) = 1 - \frac{1}{6}x_1^2 + \frac{1}{120}x_1^4 + \mathcal{O}(x_1^6), \tag{B.2}$$

which is clearly non-singular. For $n = 2$:

$$G_2(x_1;x_2) = \frac{1}{x_2}\Big(G_1(x_1+x_2) - G_1(x_1)\cosh(x_2)\Big). \tag{B.3}$$

Because G_1 is an analytical function, the term in parenthesis also is. Moreover, since it vanishes when $x_2 = 0$, it means that its Taylor series must be proportional to x_2. In other words, when expanding around the origin, the denominator $1/x_2$ is going to be cancelled by this overall factor, making G_2 also analytical.
If we now assume that the statement is true for $n-1$, and we look at the recursive definition (B.1), we see that the term in parenthesis in G_n is analytical and it vanishes when $x_n = 0$. Thus, it is proportional to x_n, cancelling the denominator when expanding as a power series and ensuring that G_n as a whole is analytical.
Next, we will prove that the explicit expression for the G functions is given by:

$$\begin{aligned}
G_n(x_1;...,x_n) = {} & \frac{1}{x_2...x_n}\Big(G_1(x_1+...+x_n) \\
& - \sum_{i=2}^{n} G_1(x_1+...+x_{i-1}+x_{i+1}+...+x_n)\cosh(x_i) \\
& + \sum_{i<j=2}^{n} G_1(x_1+...+x_{i-1}+x_{i+1}+...+x_{j-1}+x_{j+1}+...+x_n)\cosh(x_i)\cosh(x_j) \\
& + ... + (-1)^{n-1}G_1(x_1)\cosh(x_2)...\cosh(x_n)\Big).
\end{aligned} \tag{B.4}$$

The case $n = 2$ is trivial. Let's assume it holds for $n-1$. First, let's rewrite the previous expression as:

$$G_n(x_1; x_2, ..., x_n) = \frac{1}{x_2 ... x_n} \sum_{r=0}^{n-1} (-1)^r \sum_{i_1<i_2<...<i_r=2}^{n} G_1\left(\sum_{j=1}^{n} x_j - \sum_{k=1}^{r} x_{i_k}\right) \prod_{k=1}^{r} \cosh(x_{i_k}) \tag{B.5}$$

Using the recursive definition, we see that:

$$\begin{aligned}
&G_n(x_1; x_2, ..., x_n) = \frac{1}{x_n}\Big(G_{n-1}(x_1 + x_n; x_2, ..., x_{n-1}) - G_{n-1}(x_1; x_2, ..., x_{n-1}) \cosh(x_n)\Big) \\
&= \frac{1}{x_2 ... x_n}\left[G_1\left(\sum_{j=1}^{n} x_j\right) + \sum_{r=1}^{n-2} (-1)^r \sum_{i_1<i_2<...<i_r=2}^{n-1} G_1\left(\sum_{j=1}^{n} x_j - \sum_{k=1}^{r} x_{i_k}\right) \prod_{k=1}^{r} \cosh(x_{i_k})\right. \\
&- \sum_{r=0}^{n-3} (-1)^r \sum_{i_1<i_2<...<i_r=2}^{n-1} G_1\left(\sum_{j=1}^{n} x_j - \sum_{k=1}^{r} x_{i_k}\right) \cosh(x_n) \prod_{k=1}^{r} \cosh(x_{i_k}) \\
&\left. - (-1)^{n-2} G_1(x_1) \cosh(x_2) ... \cosh(x_n)\right].
\end{aligned} \tag{B.6}$$

Now, by a change of the summation variable $r \to r-1$, the term in the third line becomes:

$$\sum_{r=1}^{n-2} (-1)^r \sum_{\substack{i_1<i_2<...<i_{r-1}=2 \\ i_r=n}}^{n} G_1\left(\sum_{j=1}^{n} x_j - \sum_{k=1}^{r} x_{i_k}\right) \prod_{k=1}^{r} \cosh(x_{i_k}), \tag{B.7}$$

which means that, if we sum the terms in the second and third line, we obtain:

$$\begin{aligned}
&G_n(x_1; x_2, ..., x_n) = \frac{1}{x_n}\Big(G_{n-1}(x_1 + x_n; x_2, ..., x_{n-1}) - G_{n-1}(x_1; x_2, ..., x_{n-1}) \cosh(x_n)\Big) \\
&= \frac{1}{x_2 ... x_n}\left[G_1\left(\sum_{j=1}^{n} x_j\right) + \sum_{r=1}^{n-2} (-1)^r \sum_{i_1<i_2<...<i_r=2}^{n} G_1\left(\sum_{j=1}^{n} x_j - \sum_{k=1}^{r} x_{i_k}\right) \prod_{k=1}^{r} \cosh(x_{i_k})\right. \\
&\qquad \left. - (-1)^{n-2} G_1(x_1) \cosh(x_2) ... \cosh(x_n)\right] = \\
&= \frac{1}{x_2 ... x_n} \sum_{r=0}^{n-1} (-1)^r \sum_{i_1<i_2<...<i_r=2}^{n} G_1\left(\sum_{j=1}^{n} x_j - \sum_{k=1}^{r} x_{i_k}\right) \prod_{k=1}^{r} \cosh(x_{i_k}),
\end{aligned} \tag{B.8}$$

just as we wanted to prove.

Another interesting property that these functions obey is:

$$\sum_{l=1}^{n} G_n(x_l; x_1, ..., x_n) = \begin{cases} 0 & n \text{ even} \\ (-1)^{\frac{n-1}{2}} G_1(x_1)..G_1(x_n) & n \text{ odd.} \end{cases} \tag{B.9}$$

To prove this relation, we will firstly define a similar kind of function:

$$\begin{aligned} F_n(x_1;x_2, ..., x_n) &= \sinh(x_1 + ... + x_n) - \sum_{i=2}^{n} \sinh(x_1 + ... + x_{i-1} + x_{i+1} + ... + x_n)\cosh(x_i) \\ &+ \sum_{i<j=2}^{n} \sinh(x_1 + ... + x_{i-1} + x_{i+1} + ... + x_{j-1} + x_{j+1} + ... + x_n)\cosh(x_i)\cosh(x_j) \\ &+ ... + (-1)^{n-1}\sinh(x_1)\cosh(x_2)...\cosh(x_n) = \\ &= \sum_{r=0}^{n-1}(-1)^r \sum_{i_1<i_2<...<i_r=2}^{n} \sinh\left(\sum_{j=1}^{n} x_j - \sum_{k=1}^{r} x_{i_k}\right)\prod_{k=1}^{r}\cosh(x_{i_k}), \end{aligned} \tag{B.10}$$

and we will prove the following:

$$F_n(x_1; x_2, ..., x_n) = \begin{cases} (-1)^{\frac{n}{2}-1}\cosh(x_1)\sinh(x_2)...\sinh(x_n) & n \text{ even} \\ (-1)^{\frac{n-1}{2}}\sinh(x_1)...\sinh(x_n) & n \text{ odd.} \end{cases} \tag{B.11}$$

This can be done again by induction. For example, if $n = 2$:

$$F_2(x_1; x_2) = \sinh(x_1 + x_2) - \sinh(x_1)\cosh(x_2) = \cosh(x_2)\sinh(x_1). \tag{B.12}$$

Now, assume that n is even. Then, if the statement is true for $n - 1$, we have:

$$\begin{aligned} F_{n-1}(x_1 + x_n; x_2, ..., x_{n-1}) =& \sum_{r=0}^{n-2}(-1)^r \sum_{i_1<i_2<...<i_r=2}^{n-1} \sinh\left(\sum_{j=1}^{n} x_j - \sum_{k=1}^{r} x_{i_k}\right)\prod_{k=1}^{r}\cosh(x_{i_k}) \\ =& (-1)^{\frac{n}{2}-1}\sinh(x_1 + x_n)\sinh(x_2)...\sinh(x_{n-1}) \end{aligned} \tag{B.13}$$

The remaining terms are:

$$\begin{aligned} &F_n(x_1, ..., x_n) - F_{n-1}(x_1 + x_n, ..., x_{n-1}) = \\ &= \sum_{r=1}^{n-1}(-1)^r \sum_{\substack{i_1<i_2<...<i_{r-1}=2 \\ i_r=n}}^{n} \sinh\left(\sum_{j=1}^{n} x_j - \sum_{k=1}^{r} x_{i_k}\right)\prod_{k=1}^{r}\cosh(x_{i_k}) \\ &= -\cosh(x_n)\sum_{r=0}^{n-2}(-1)^r \sum_{i_1<i_2<...<i_r=2}^{n-1} G_1\left(\sum_{j=1}^{n} x_j - \sum_{k=1}^{r} x_{i_k}\right)\prod_{k=1}^{r}\cosh(x_{i_k}) \\ &= -F_{n-1}(x_1, ..., x_{n-1})\cosh(x_n) = -(-1)^{\frac{n}{2}-1}\sinh(x_1)\sinh(x_2)...\sinh(x_{n-1})\cosh(x_n). \end{aligned} \tag{B.14}$$

Summing both contributions, one indeed obtains the correct expression for F_n. The exact same reasoning can be followed for the case where n is odd.
Now, we return to the G_n functions. Define:

$$\tilde{G}_n(x_1; x_2, ..., x_n) = (x_2...x_n) \times G_n(x_1; x_2, ..., x_n). \tag{B.15}$$

It is straightforward to show that:

$$\begin{aligned}
&\sum_{l=1}^{n} x_l \tilde{G}_n(x_l; x_1, ..., x_n) = \sinh(x_1 + ... + x_n) \\
&- \sum_{j=1}^{n} \sinh(x_1 + ... + x_{j-1} + x_{j+1} + ... + x_n) \cosh(x_j) \\
&+ \sum_{k<j=1}^{n} \sinh(x_1 + ... + x_{j-1} + x_{j+1} + ... + x_{k-1} + x_{k+1} + ... + x_n) \cosh(x_j) \cosh(x_k) \\
&+ ... + (-1)^{n-1} \sum_{i=1}^{n} \sinh(x_i) \cosh(x_1) ... \cosh(x_n).
\end{aligned} \tag{B.16}$$

Extracting from this expression the terms that contain $\cosh(x_n)$ and separating them from the rest:

$$\begin{aligned}
&\sum_{i=1}^{n} x_i \tilde{G}_n(x_i; x_1, ..., x_n) = -\sinh(x_1 + ... + x_{n-1}) \cosh(x_n) \\
&+ \sum_{j=1}^{n-1} \sinh(x_1 + ... + x_{j-1} + x_{j+1} + ... + x_{n-1}) \cosh(x_j) \cosh(x_n) \\
&+ ... + (-1)^{n-1} \sum_{i=1}^{n-1} \sinh(x_i) \cosh(x_1) ... \cosh(x_{n-1}) \cosh(x_n) \\
&+ \sinh(x_n + x_1 + ... + x_{n-1}) - \sum_{j=1}^{n} \sinh(x_n + x_1 + ... + x_{j-1} + x_{j+1} + ... + x_{n-1}) \cosh(x_j) \\
&+ \sum_{k<j=1}^{n-1} \sinh(x_n + x_1 + ... + x_{j-1} + x_{j+1} + ... + x_{k-1} + x_{k+1} + ... + x_{n-1}) \cosh(x_j) \cosh(x_k) \\
&+ ... + (-1)^{n-1} \sinh(x_n) \cosh(x_1) ... \cosh(x_{n-1}).
\end{aligned} \tag{B.17}$$

The first two lines of this expression are just the $n-1$ version of the sum over the G functions, while the rest form the previously defined F function. In other words:

$$\sum_{i=1}^{n} x_i \tilde{G}_n(x_i; x_1, ..., x_n) = -\cosh(x_n) \sum_{i=1}^{n-1} x_i \tilde{G}_{n-1}(x_i; x_1, ...x_{n-1}) + F_n(x_n; x_1, ..., x_{n-1}). \tag{B.18}$$

This gives us the perfect setup to apply induction again. For n even, if (B.9) is true for $n-1$, then:

$$\begin{aligned}\sum_{i=1}^{n} x_i \tilde{G}_n(x_i; x_1, ..., x_n) = &- (-1)^{\frac{n}{2}-1} \cosh(x_n) \sinh(x_1) ... \sinh(x_{n-1}) \\ &+ (-1)^{\frac{n}{2}-1} \cosh(x_n) \sinh(x_1) ... \sinh(x_{n-1}) = 0.\end{aligned} \tag{B.19}$$

Meanwhile, for n odd:

$$\sum_{i=1}^{n} x_i \tilde{G}_n(x_i; x_1, ..., x_n) = F_n(x_n; x_1, ..., x_{n-1}) = (-1)^{\frac{n-1}{2}} \sinh(x_n) \sinh(x_1) ... \sinh(x_{n-1}). \tag{B.20}$$

The only thing left to do is to prove the first step of induction, $n = 2$:

$$\begin{aligned}&x_1 \tilde{G}_2(x_1; x_2) + x_2 \tilde{G}_2(x_2; x_1) \\ &= x_1 G_1(x_1 + x_2) - x_1 G_1(x_1) \cosh(x_2) + x_2 G_1(x_1 + x_2) - x_2 G_1(x_2) \cosh(x_1) \\ &= \sinh(x_1 + x_2) - \sinh(x_1) \cosh(x_2) - \cosh(x_2) \sinh(x_1) = 0.\end{aligned} \tag{B.21}$$

And with this, we have proven equation (B.9) in its totality.

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