Hydrodynamics and Magnetohydrodynamics as an effective field theory

Master thesis

Written by *Mads Sørensen*

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Supervised by

Troels Harmark & Jay Armas

University of Copenhagen

UNIVERSITY OF COPENHAGEN

NAME OF INSTITUTE:	Copenhagen University
NAME OF DEPARTMENT:	Copenhagen University: Niels Bohr Institute
AUTHOR(S):	Mads Sørensen
EMAIL:	wfm599@alumni.ku.dk
TITLE AND SUBTITLE:	Hydrodynamics and Magnetohydrodynamics as an effective field theory -
SUPERVISOR(S):	Troels Harmark & Jay Armas
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NAME ______________________

SIGNATURE ______________________

DATE ______________________

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Abstract

We review relativistic hydrodynamics and magnetohydrodynamics as effective field theories to a first-order in the most general frame. This allows to study both frames' linear stability and causality by applying constraints on some of the involved transport coefficients. This shows that a set of frames satisfies both stability and causality under certain constraints. Furthermore, the Eckart and Landau-Lifshitz frame is commented on, and the latter is considered in more details for magnetohydrodynamics. We find the linear system for small perturbations and comment on the linear stability and causality criteria. This modern perspective gives sensible physics without introducing new degrees of freedom as were done in the Israel-Stewart theory. The constraints for the general frame give criteria of what frames to choose to ensure sensible relativistic theories. This can lead to a better understanding of dissipative effects and include them in astrophysical settings, where it has been common practice to only consider the ideal cases.

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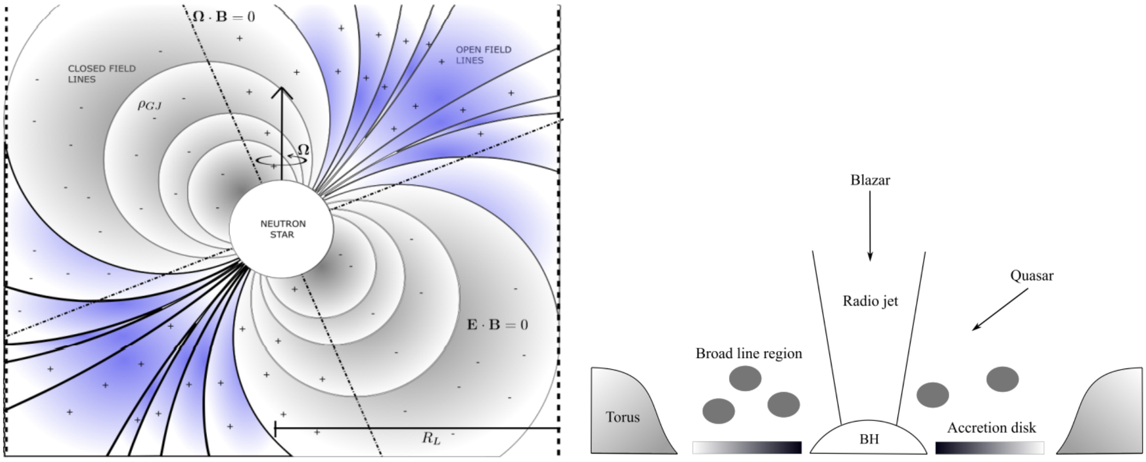
Contents

1 Introduction

Relativistic hydrodynamics and magnetohydrodynamics (MHD) offer a toy model description of fluids and plasmas in many astrophysical settings: Shocks from short Gamma Ray Burst (GRB) [1] and Neutron stars [2]. Relativistic Reaction fronts produced by white dwarfs (WD) merging [3]. Toy models of jets propagating from pulsars and quasars [4][5]. In astrophysics, shocks are described as a non-continuous front that travels faster than the local sound speed in the medium. They occur in the interstellar medium (ISM) and are produced in high-energy settings such as GRB, binary merges and supernovas [6]. Due to the shock fronts, particles in the ISM will be accelerated, and energy is released in the form of radiation. The acceleration of particles due to shock fronts in GRB is believed to produce relativistic bulk velocities. The current understanding of GRB is described by a fireball model: There is an inner engine, which can be a black hole, surrounded by a local medium. Jets can propagate due to matter being injected into the engine. This leads to relativistic jets that create an internal shock front is known as the glow and emit low to high gamma radiations. When the shock wave reaches the local ISM, it is called the external shock or afterglow and emits high-energy gamma-rays and X-rays [1]. In order to understand these mechanics, it is necessary to understand the inner engine. The propagation of jets has been studied for both pulsars and Active Galactic Nucleus (AGN). Pulsars are fast rotating neutron stars that have jets propagating from their poles. Neutron stars are believed to be remnants of massive stars that have undergone core collapse and are extremely dense with a mass of $\sim 1.4\ \mathrm{M}_\odot$ and a radius of ~ 10 km, making them one of the densest objects in the Universe, and have been found to have a large angular momentum Ω. A force-free approximation describes a model of the magnetosphere of a pulsar: Here, the Lorentz force is assumed to be vanishing due to a screening of the electric field. The screening occurs since electron-positron plasma is created due to the presence of a large magnetic field $B_0 \sim 10^8$ T. The electric field occurs when the density of the plasma is equal to or larger than the Goldreich-Julian (GJ) density ρ_{GJ}. In this model, it is assumed that no dissipative effects are taken into account, and the plasma follows the ideal MHD description. Then the magnetic field lines are frozen into the plasma, and two regions of the magnetosphere: A region of closed magnetic field lines where the plasma reaches the GJ density and is co-rotating with the neutron star. Then, since the particles can not travel faster than the speed of light, all magnetic field crossing a light cylinder is open. The model of the pulsar's magnetosphere is illustrated in figure (1a). A similar approach has been developed by black holes surrounded by accretion disks or envelopes. These objects are known as quasars and/or blazers, but are both believed to be AGN with jets, and the intensity of their radio wave depends on the observation angle, which is illustrated in figure (1b). Relativistic hydrodynamics and MHD continue to be actively studied theories. The first successful description of relativistic dissipative hydrodynamics was done by Eckart [7], and later developed further by Landau and Lifshitz [8]. In both descriptions, stability and causality were not satisfied, and their description led to inconsistent physics. This has been solved by introducing extra degrees of freedom, which modifies the theory of hydrodynamics. This was first done in the Israel-Stewart theory [9][10]. This allows for a full description of relativistic hydrodynamics with dissipative effects that are stable and causal. However, introducing the new degrees of freedom makes the theory more complex. Similar relativistic hydrodynamics have been studied from a kinetic theory, by finding moments of the relativistic Boltzmann equation [11][12]. Hydrodynamics has recently been treated as an effective field theory known as the Bemfica-Disconzi-Noronha-Kovtun (BDNK) theory. This modern perspective of relativistic hydrodynamics allows defining conserved quantities fully governed by three degrees of freedom T, μ and u^μ. They can locally be taught the temperature, chemical potential and fluid velocity at equilibrium. Simultaneously, the hydrodynamics equation in the BDNK theory is stable and satisfies causality. Similarly this has recently been done for relativistic MHD in describing plasma, where the equations of interest are governed by four degrees of freedom with the fourth field corresponding to the magnetic fields. In both the fluids and plasmas description, the theories are both stable and causal.



(a) (b)

Figure 1: Left figure: Illustration of the magnetosphere of a pulsar. At the centre the Neutron star, where the grey area's are the closed field lines, and blue areas are the open field lines. The dotted lines correspond to a light cylinder, and magnetic field lines crossing these must remain open. The fine dashed lines correspond to a change in sign for the GJ density. Right figure: An illustration of the upper half of an AGN, where the radio wave intensity depends on the observation angle. Both figures are original from [13]

In this project, we will review the findings of the BDNK formulation/theory for both hydrodynamics and MHD to a first-order: First, a definition of fluids will be given, and hydrodynamics in the non-relativistic case to allow us to draw parallels with the BDNK theory. Afterwards, the description of the BDNK theory is given, where we study the stability and causality criteria for a general frame [1] . The same approach will be done for MHD where first a classical description will be considered and then the corresponding BDNK theory. Here the Alfvén and magnetosonic modes will be derived in the Landau frame, and the results found in [14] for the general MHD frame will be presented and commented on. For this project, we assume that the reader is familiar with special relativity. Furthermore, we will make use of the Einstein summation convention, where we use the standard notation in which Greek letters μ, ν etc., takes the values (0,1,2,3) and Latin letters

[1]The word frame in this context refer to of how one defines T, μ and u^μ for hydrodynamics, and T, μ , u^μ and h^μ in MHD.

i, j denote the spatial components with values $(1, 2, 3)$, unless else is stated. Furthermore, the metric will have a Lorentzian signature $(-, +, +, +)$ which divides the four-dimensional space-time into spatial coordinates and a time coordinate such that we are able to describe three-dimensional values still. We will for most computation consider a flat background, such that the general metric $g_{\mu\nu} = \eta_{\mu\nu}$ is the Minkowski metric in standard coordinates $\eta_{\mu\nu} = (-1, 1, 1, 1)$. Finally, we set the speed of light $c = 1$, in the relativistic descriptions.

2 Physics description of fluids

Hydrodynamics is a toy model that describes flows of quantities conserved under collisions in fluids. In order to understand the underlying physics behind fluids, this section will go through their definition and briefly consider the underlying kinetic theory, which gives fundamental insight into hydrodynamics. For a system of N particles that can deform under pressure and motion, i.e., a system that is not a rigid body, suppose that the N particles can still interact through some coupling, e.g., through collisions. For a system where the number of particles is large, it becomes an impossible task to determine the equations of motions, and statistical physics is a more variable approach. If the system has a scale length L, and a mean free path ℓ determining how long a particle can travel before it undergoes collision, and satisfies

$$\frac{\ell}{L} \ll 1, \tag{2.1}$$

then a fluid description of the system is possible [3]. Any system with a large N which is not a rigid body and satisfies eq. (2.1) can be described as fluids, e.g., gases and liquids. A system that satisfies the criteria for fluids will have a collection of particles with a position x^i and a velocity u^i, and will be contained inside a phase volume d^3xd^3u. If the volume contains a high number of particles but is small enough to ensure homogeneity, then all the particles in that collection have the same velocity. Therefore, each phase volume can be viewed as one point in the fluids and can be expected to follow a distribution function which is dependent on the position and velocity at a time t

$$f = f(t, x^i, u^i).$$

It is standard to refer to such a phase volume as the fluid element, and this term will be maintained throughout this project. Without loss of generality, only mono-atomic fluids are considered here. The number of particles in each fluid element can be determined by

$$fd^3xd^3u = f(t, x^i, u^i)d^3xd^3u.$$

Then, considering the change of the distribution function by an infinitesimal time dt allows us to derive an expression of the conservation of the distribution function over time, called the Boltzmann equation.

2.1 Boltzmann equation

Suppose first that in the fluids, no collisions occur such that all particles centred around (x^i, u^i) at t, will at an infinitesimal time dt be centred around $(x^i + u^i dt, u^i + a^i dt)$. The distribution function is invariant between t and dt

$$f(t + dt, x^i + u^i dt, u^i + a^i dt)d^3x'd^3u' = f(t, x^i, u^i)d^3xd^3u,$$

where the prime denotes the coordinates at dt. This corresponds to a coordinates transformation

$$x'^i = x^i + u^i dt$$
$$u'^i = u^i + a^i dt$$

where a^i is the acceleration due to external forces that acts on all the particles. Suppose now that the force leading to the acceleration is independent on the velocity, then the fluid elements under a coordinates transformation corresponds to writing $d^3x'd^3u' = \mathbf{J}d^3xd^3u$, where $\mathbf{J}$ is the Jacobian, and to a second order in dt satisfy

$$\mathbf{J}(x^i, u^j) = \begin{pmatrix} \frac{\partial x'^i}{\partial x^j} & \frac{\partial x'^i}{\partial u^j} \\ \frac{\partial u'^i}{\partial x^j} & \frac{\partial u'^i}{\partial u^j} \end{pmatrix} = 1 + \mathcal{O}(dt^2),$$

such that

$$d^3x'd^3u' = d^3xd^3u + \mathcal{O}(dt^2).$$

If collisions are taken into consideration, particles at the point (x^i, u^i) might not move to the point $(x^i + u^i dt, u^i + a^i dt)$ and particles that were not originally contained in (x^i, u^i) can enter into the point $(x^i + u^i dt, u^i + a^i dt)$. Thus, a correction for collisions needs to be taken into consideration: let $\Gamma(f)$ correspond to such a collision term. Then the correction is given by

$$f(t + dt, x^i + u^i dt, u^i + a^i dt)d^3x'd^3u' = f(t, x^i, u^i)d^3xd^3u + \Gamma(f)d^3xd^3udt. \tag{2.2}$$

Taking an expansion around dt to a first-order corresponds to writing

$$\left(f + \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial t}dt + \frac{\partial f}{\partial u^i}\frac{\partial u^i}{\partial t}dt\right)d^3xd^3u = fd^3xd^3u + \Gamma(f)dtd^3xd^3u.$$

Where the first term of left hand side (LHS) and the right hand side (RHS) cancels out, and the relations then reads

$$\partial_t f + u^i \partial_i f + a^i \frac{\partial f}{\partial u^i} = \Gamma(f). \tag{2.3}$$

This is the Boltzmann equation, and the RHS describes the collisions of the particles between the time t and $t + dt$, and is known as the collisions integral [3]. In order to retrieve the complete Boltzmann equation, an expression for the collision term needs to be derived. Assume that all collisions are elastic and there are only collisions between two particles. In its centre of mass during the velocity change u^i and $u^i + a^i dt$, with each particle having a different velocity u_1^i and u_2^i. The distribution function will give the number of the particles in each respective fluid element

$$f_1 d^3u_1 = f(t, x^i, u_1^i)d^3u_1 \ , \quad f_2 d^3u_2 = f(t, x^i, u_2^i)d^3u_2.$$

After the collisions, the particles will have the velocities $u_1'^i$ and $u_2'^i$, and by neglecting any inelastic collisions, the relative velocity will be conserved

$$u_R^i \equiv |u_1^i - u_2^i| = {u'_R}^i \equiv |u_1'^i - {u'_2}^i|.$$

At the point of collision, the particles scatter through a solid angle. Defining the total cross-section $\sigma(\Omega)$ by taking the integral over the whole solid angle $d\Omega$. The flux of particles that scatters out from the volume $d^3u_1d^3u_2$ of f_1 into the solid angle given by $|u_1^i - u_2^i|f_1f_2\sigma(\Omega)d\Omega$. Dividing with d^3u and integrating over the solid angle $d\Omega$ and d^3u_2 is given by

$$\Gamma(f_1) = \int\int f_1f_2u_R^i d^3u\sigma(\Omega)d\Omega.$$

The same argument holds for particles being scattered into f_1, where the distribution function is dependent on the primed velocities $f_1' = f'(t, x^i, u_1'^i)$, such that

$$\Gamma(f_1') = \int\int f_1f_2u_R'^i d^3u\sigma'(\Omega)d\Omega.$$

Since the velocity is invariant under these collisions, then so is the solid angle. Subtracting these two expressions, and writing $d\Omega = \sin\theta d\theta d\phi$, with θ, ϕ being the scattering angle and the azimuthal angle respectively, the collision integral becomes

$$\Gamma(f) = \int d^3u_2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta |u_1^i - u_2^i|(f_1f_2 - f_1'f_2')\sigma(\Omega). \tag{2.4}$$

If the collision integral vanishes, the Boltzmann equation reduces to a differential equation and describes how the distribution function is conserved. Before considering solutions, we will here, without proof, describe an essential consequence of the Boltzmann equation. The collision integral leads to the H-theorem, which states that if a system is out of equilibrium, the collisions will bring it back to an equilibrium state. This happens when $f_1f_2 - f_2'f_2 = 0$, and the distribution function for a mono-atomic system is proportional to [3]

$$f \propto \exp(-u). \tag{2.5}$$

Finally, the Boltzmann equation is difficult to solve due to the collision integral, but it is sufficient to simplify the problem by considering invariant quantities under collisions.

2.2 Moment's equation

For a mono-atomic fluid in an elastic collision, the conserved quantities are the mass m, the momentum mu^i and the kinetic energy $(1/2)mu^2$. The fluids consist of N particles. By averaging over the whole velocity space, the number density is given by

$$n = \int d^3 u f(t, x^i, u^i), \tag{2.6}$$

while the average velocity v^i, and the internal energy per number density ϵ as [3]

$$v^i = \frac{1}{n} \int d^3 v u f(t, x^i, u^i) \tag{2.7a}$$

$$\epsilon = \frac{1}{2n} \int w^2 d^3 u, \tag{2.7b}$$

where $w^i = u^i - v^i$ and describes random motion in the fluids. Note that the average of w^i, satisfy $\langle w^i \rangle = 0$ while in general $\langle w^2 \rangle$ and $\langle w^i w^j \rangle$ does not. The average velocity v^i describes the velocity flow and will be referred as the fluid velocity and describes the global dynamic of fluids. For a static fluid, the fluid velocity vanishes, but a stationary fluid can still have local motion denoted by u^i. Thus the local velocity u^i is independent of time, while the fluid velocity is not per definition. Suppose a force acting on the particles in the fluids is uniform, such that $\langle a^i \rangle = a^i$, and that the force is independent of the velocity. For mono-atomic fluids, $m = \langle m \rangle$, and multiplying the mass with the Boltzmann equation and integrating over the whole velocity space gives

$$\begin{aligned} \int d^3 u m \left(\partial_t f + u^i \partial_i f + a^i \frac{\partial f}{\partial u^i} \right) &= \partial_t \left(\int m f d^3 u \right) + \partial_i \left(\int u^i m f d^3 u \right) \\ &= \partial_t (mn) + \partial_i (m n v^i) = 0, \end{aligned} \tag{2.8}$$

where the first equality holds due to the product rule. The third term on the LHS goes to zero since the distribution function consists of finite particles and according to eq. (2.5); it goes towards zero when the velocity goes towards infinity. Eq.(2.8) describes the conservation of mass in fluids; thus, no particles are created or destroyed. With the same assumption, the conservation of momentum becomes

$$\begin{aligned} \int d^3 u m u^i \left(\partial_t f + u^i \partial_i f + a^i \frac{\partial f}{\partial u^i} \right) &= \partial_t \left(\int m u^i f d^3 u \right) + \partial_i \left(\int u^i m u^i f d^3 u \right) - m \int a^i f d^3 u \\ &= \partial_t (m n u^i) + \partial_j (m n \langle u^j u^i \rangle) - m n a^i = 0. \end{aligned} \tag{2.9}$$

Using the relation

$$\langle u^i u^j \rangle = \langle (u^i - v^i)(u^j - v^j) \rangle + \langle u^i \rangle v^j + v^i \langle u^j \rangle - v^i v^j = \langle w^i w^j \rangle + v^i v^j,$$

and defining

$$\Pi^{ij} = nmv^i v^j + P^{ij} \text{ , with } P^{ij} = nm\langle w^i w^j \rangle, \tag{2.10}$$

simplifies eq.(2.9), and reduces to

$$\partial_t(nm) + \partial_i \Pi^{ij} = 0. \tag{2.11}$$

Note that eq.(2.10) is the momentum flux tensor and describes the flux through an area orthogonal to the i-th direction of the j-th components, which is precisely just the spatial component of the energy-stress tensor. The term P^{ij} is the pressure tensor and is still dependent on the distribution function. The expression of eq. (2.11) for $a^i = 0$ describes the conservation of the linear momentum through the momentum flux tensor. Lastly, multiplying the Boltzmann equation with the internal energy

$$\begin{aligned}\int d^3u \frac{1}{2}mu^2 \left(\partial_t f + u^i \partial_i f + a^i \frac{\partial f}{\partial u^i} \right) &= \partial_t \left(\frac{1}{2}mu^2 d^3u \right) + \partial_i \left(\int \frac{1}{2}mu^2 d^3u \right) - \int \frac{1}{2}ma^i \frac{\partial u^2}{\partial u^i} d^3u \\ &= \partial_t(mn\langle u^2 \rangle) + \partial_i(mn\langle u^2 u^i \rangle) - nma^i u_i,\end{aligned} \tag{2.12}$$

and for the average brackets, the following relations holds

$$\begin{aligned}\langle u^2 \rangle &= \langle u^i u_i \rangle = \langle w^i w_i \rangle + v^i v_i = \epsilon + v^2 \\ \langle u^2 v^i \rangle &= \langle (w^2 + v^2 + 2w^j v_k)(w^i + v^i) \rangle = \langle w^2 w^i \rangle + \langle w^2 \rangle v^i + v^2 v^i + 2\langle w^j w^i \rangle v_j.\end{aligned}$$

Substituting this into eq.(2.12) and using the definition of the thermal flux [3]:

$$q^i = \frac{1}{2}nm\langle w^i w^2 \rangle \tag{2.13}$$

and describes the flow of thermal energy. With eq.(2.7b) and the pressure tensor gives

$$\partial_t \left(mn\epsilon + \frac{1}{2}mnv^2 \right) + \partial_i \left(\left(nm\epsilon + \frac{nmv^2}{2} \right) \right) + \partial_i q^i + \partial_i (P^{ij} v_j) - nma^i v_i. \tag{2.14}$$

Similar to the momentum equation, setting $a^i = 0$ describe the energy conservation through the heat flux and pressure tensor. Defining now the density and the internal kinetic energy per unit volume

$$\rho \equiv nm, \tag{2.15}$$

$$\varepsilon \equiv \rho\epsilon, \tag{2.16}$$

then eq.(2.8),(2.11) and (2.14) can be written together as

$$\partial_t(\rho) + \partial_i(\rho v^i) = 0 \tag{2.17a}$$

$$\partial_t(\rho v^i) + \partial_j \Pi^{ij} = \rho a^i \tag{2.17b}$$

$$\partial_t\left(\varepsilon + \frac{\rho v^2}{2}\right) + \partial_i\left(\left(\varepsilon + \frac{\rho v^2}{2}\right)v^i\right) = -\partial_i q^i - \partial_i(P^{ij}v_j) + \rho a^i v_i. \tag{2.17c}$$

These equations are known as the Boltzmann moment's equation, and setting $a^i = 0$ describes macroscopic quantities conserved through collisions. The thermal flux in eq. (2.13 and the pressure tensor in eq. (2.10) are still dependent on the distribution function, even if such a function was known to satisfy eq. (2.17) the physical implication, if any, is not clear to us.

Nevertheless, while the exact value of the distribution function may not be obtainable, it is possible to consider a derivation expansion, known as the Chapman-Enskog expansion. The computation of such expansion is involved and is out of this project's scope. Therefore, we will only present the idea behind such an expansion and comment on the results. For an exact computation, we refer the reader to [15]. The general idea is to introduce a variable τ that is of the order of ℓ/L. Then by making an expansion for small τ, the Chapman-Enskog expansion reads [15]

$$f = f_{(0)} + \tau f_{(1)} + \mathcal{O}(\tau^2). \tag{2.18}$$

The first term designates the zeroth-order approximation, and the collision integral vanishes such that the Boltzmann equation is automatically conserved. Since the integral collision equals zero, the H-theorem states that the Maxwell distribution satisfies the Boltzmann equation. For the first-order, the collision integral does not vanish and is thus only conserved by collision invariant quantities. Furthermore, the expansion further implies that the distribution function also is expanded in a power series of the gradient. The thermal flux and pressure tensor satisfies [16]

$$P^{ij} = P^{ij}_{(0)} + P^{ij}_{(1)} = p\delta^{ij} - \eta\sigma^{ij} - \zeta\delta^{ij}\partial_k v^k \tag{2.19a}$$

$$q^i = -\kappa\partial_i T, \tag{2.19b}$$

where δ^{ij} is the Kronecker delta, and σ^{ij} is the shear tensor given by

$$\sigma^{ij} = \eta\bigg(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta^{ij}\partial_k v^k\bigg). \tag{2.19c}$$

The first term in the last equality in eq. (2.19a) is the zeroth-order contribution and corresponds to isotropic pressure, with P^{ij} given only by diagonal elements. The two last terms correspond to the first-order contribution and describes dissipation effects. The thermal flux has no contribution in the zeroth-order, and no heat exchange occurs. The coefficients p, η, κ and ζ are the isotropic pressure, shear viscosity, bulk viscosity, and thermal conductivity, respectively. Here η, κ and ζ are transport coefficients and describe how rapidly fluids out of equilibrium will return to their equilibrium state, and the Chapman-Enskog expansion determines their exact value. Thus, when writing the hydrodynamic equations, these transport coefficients can not be evaluated by hydrodynamics but must instead be found from underlying physics theory. This will be a fundamental premise when considering the BDNK theory. Before writing the hydrodynamics equations, the implication of thermodynamics should be considered.

2.3 Thermodynamic equilibrium

Thermodynamic quantities, such as pressure, density, internal energy etc., play a vital role in describing fluids. For a system isolated from its environment, the first law of thermodynamics states that the internal energy E can vary due to heat exchange, work done to the system, and the creation/destruction of particles. The first law of thermodynamics reads

[3]

$$dE = TdS - pdV + \mu dN, \tag{2.20}$$

where T is the temperature of the system, S the entropy, p the pressure, V the volume, μ the chemical potential and N the particle number. A thermodynamical system isolated from its environment is said to be in equilibrium if its properties do not change. In a system in a global thermodynamic equilibrium (GTE), properties such as the temperature, pressure and chemical potential are uniform throughout the system. At the same time, the entropy, internal energy and volume depend on the system's size, and are known as extensive properties [17]. These properties can be made independent by dividing it by the system's volume or the density. Setting ε as the internal energy per unit volume, $s = \rho S$ as the entropy per unit volume, with the specific volume $V = 1/\rho$. For a mono-atomic fluid $dN = 0$, the first law reads

$$d\varepsilon = \rho T ds + \frac{w}{\rho} d\rho. \tag{2.21}$$

Alternatively, with a constant $R \in \mathbb{R}$, the internal energy satisfies

$$RE(S, V, N) = E(RS, RV, RN),$$

differentiating with respect to R and setting $R = 1$, leads to $E = TS - pV + \mu N$. Writing this in unit volumes reads

$$\varepsilon + p = Ts + \rho\mu \ , \ dp = sdT + \rho d\mu \ , \ d\varepsilon = Tds + \mu d\rho. \tag{2.22}$$

The relationship between the pressure and thermodynamics variables is known as the equation of state. This dictates that the pressure is a function satisfy $p = p(T, \mu)$, and according to the chain rule this corresponds to writing

$$dp(T, \mu) = \frac{\partial p}{\partial T} dT + \frac{\partial p}{\partial \mu} d\mu \implies s = \frac{\partial p}{\partial T} \ \ , \ \ \rho = \frac{\partial p}{\partial \mu}.$$

The two last relations are implied by comparing with eq. (2.22). Due to the relation of the entropy and density, it can further be seen that [14]

$$\left(\frac{\partial s}{\partial \mu}\right)_T = \left(\frac{\partial \rho}{\partial T}\right)_\mu. \tag{2.23}$$

The equation of state plays an essential role in describing a model of the universe: diluted gas is usually considered, and an equation of state is chosen to describe the expansion of a universe made up of specific types of matter [18]. Lastly, the second law of thermodynamics states that the entropy of a closed system must increase. This corresponds to stating that thermodynamic processes are irreversible in the direction they occur [3]. The second law directly impacts hydrodynamics in terms of an entropy current. Nevertheless, the GTE is not satisfied for fluids, which are not necessarily closed systems and can still interact with their environment. The first law of thermodynamics can not be applied for fluids in the global scheme. Instead a local thermodynamic equilibrium *LTE* is adopted, such that for a system not satisfying GTE it can be made up of sub systems that locally satisfies thermodynamic equilibrium. The fluid elements have local values of temperature, internal energy etc., and fluids is a continuum of these local values. Therefore, the thermodynamic quantities are dependent on a time t and its position x^i in the system, and can be seen as scalar fields of the fluids [3], satisfying the relations in eq. (2.22) as

$$dp(t,x^i) = sdT(t,x^i) + \rho d\mu(t,x^i)\ , \quad d\varepsilon = Tds(t,x^i) + \mu d\rho(t,x^i). \tag{2.24}$$

For the variation of the entropy ds and the density $d\rho$ the chain rule applies

$$ds(t,x^i) = \frac{\partial s}{\partial T}dT(t,x^i) + \frac{\partial s}{\partial \mu}d\mu(t,x^i)\ , \quad d\rho(t,x^i) = \frac{\partial \rho}{\partial T}dT(t,x^i) + \frac{\partial \rho}{\partial \mu}d\mu(t,x^i). \tag{2.25}$$

Later, perturbation around the thermodynamic equilibrium will be considered, and with the definition of the scalar fields, it will be possible to define amplitudes for these perturbations.

3 Non-relativistic hydrodynamics

From the discussion above, we summarise the findings and the assumption that allows to write the hydrodynamic equations:

- Fluids have a scale length much larger than the mean free path of the particles. The inequality in eq. (2.1) ensures that collisions occur, and according to the H-theorem, fluids out of equilibrium will eventually return to their equilibrium state.

- Hydrodynamics describes only conserved quantities, which are the system's mass, momentum and total energy, and in deriving the moment's equation, only elastic collisions occur.

- Hydrodynamics is described by an expansion of ℓ/L or equivalent of the gradient ∂_i. The zeroth-order approximation corresponds to having a stress tensor with only diagonal elements. The first-order approximation is a correction to the zeroth-order and has a non-vanishing heat flux, and a pressure tensor has non-diagonal components.

The zeroth-order of hydrodynamics, are also called ideal or perfect fluids [16]. By setting $P^{ij} = p\delta^{ij}$ and $q^i = 0$, in the moments equation eqs. (2.17) the ideal hydrodynamics equations become

$$\partial_t \rho + \partial_i(\rho v^i) = 0 \tag{3.1a}$$

$$\partial_t(\rho v^i) + \partial_j \Pi^{ij}_{(0)} = \rho a^i \tag{3.1b}$$

$$\partial_t \left(\varepsilon + \frac{\rho v^2}{2} \right) + \partial_i \left(\left(\varepsilon + p + \frac{\rho v^2}{2} \right) v^i \right) = \rho a^j v_j \tag{3.1c}$$

where $\Pi^{ij}_{(0)}$ reads

$$\Pi^{ij}_{(0)} = p\delta^{ij} + \rho v^i v^j.$$

These equations govern the ideal hydrodynamics, where no terms contain the transport coefficients η, ζ and κ. The first equation describes the conservation of matter, known as the continuity equation, and states that no particles can be created or destroyed. For the relativistic case, this corresponds to the conservation of a four-current related to the symmetries of the first unitary group $U(1)$. The second and third equations are the conservation of momentum and the total energy. For the relativistic case, this corresponds to the conservation of the energy-momentum tensor. Ideal hydrodynamics consists of one vector field $v^i(t, x^i)$, three scalar fields $\rho(t, x^i), p(t, x^i)$ and $\varepsilon(t, x^i)$ that correspond to the local fluid velocity, density, pressure and internal energy respectively. The eqs. (3.1) describe monoatomic fluids, and for n multi-component fluids, a summation is needed over the conserved quantities, e.g., $\rho = \sum_{i=1}^{n} \rho_n$ [8].

Suppose now that no external force is present, then by multiplying the momentum equation with the velocity v_j, and using the continuity equation, leads to

$$\frac{1}{2}\partial_t(\rho v^2) + \frac{1}{2}\partial_i(\rho v^2 v^i) - v^j\partial_j(p\delta^{ij}).$$

Subtracting this with eq.(3.1c), reduces the energy conservation to

$$\partial_t\varepsilon + \partial_i(\varepsilon v^i) + p\partial_i v^i = 0. \tag{3.2}$$

Inserting eq.(2.21) into the eq.(3.2), and using the continuity equation gives

$$\partial_t s + \partial_i(s v^i) = 0. \tag{3.3}$$

Here su^μ is the entropy current, and is conserved for ideal hydrodynamics. For fluids with constant density $\rho \in \mathbb{R}$, they are said to be incompressible. In this case the continuity equation implies that the fluid velocity is divergenceless.

The ideal hydrodynamics is not a realistic picture of fluids since the only dynamic occurs due to pressure. Therefore, substituting eqs. (2.19) with the first-order correction into the moments equation, gives

$$\partial_t\rho + \partial_i(\rho v^i) = 0 \tag{3.4a}$$

$$\partial_t(\rho v^i) + \partial_j\Pi^{ij} = \rho a^i \tag{3.4b}$$

$$\partial_t\left(\varepsilon + \frac{\rho v^2}{2}\right) + \partial_i\left(\left(\varepsilon + p + \frac{\rho v^2}{2}\right)v^i\right) + j_\varepsilon = \rho a^j v_j, \tag{3.4c}$$

where for compactness

$$j_\varepsilon \equiv -\partial_i(\eta\sigma^{ij}) - \partial_i(\zeta\delta^{ij}\partial_k v^k) - \partial_i(\kappa\partial_i T),$$

and

$$\Pi_{ij} = p\delta_{ij} + \rho v_i v_j - \eta\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_k v^k\right) - \zeta\delta_{ij}\partial_k v^k.$$

The particles move due to collisions and small inhomogeneities in the temperature, density, velocity, etc., which is due to the first-order correction. Thus, the first-order approximation gives a closer description of actual fluids. The first equality is unchanged from the zeroth-order, and the second equality is known as the Navier-stokes equation and describes the motion of viscous fluids [3]. The third is the heat-conduction equation which describes the heat/energy distribution in a given fluid over time. Again, the transport coefficients η, ζ and κ exact values can not be determined by the hydrodynamic equations but are found from the Boltzmann equation. Determining the exact values of η, ζ and κ requires specific assumptions, and dependent on their derivation can lead to different hydrodynamics [16]. This is explicitly seen by considering fluids with $\kappa = 0$, $\eta \neq 0$ and $\zeta \neq 0$, describing viscous fluids with no heat exchange, while $\kappa \neq 0$ and $\eta = \zeta = 0$ describe fluids with heat exchange but no viscous effects, i.e., the pressure is isotropic. This will be expanded on further when considering the BDNK theory.

3.1 Hydrodynamics fluctuations

Suppose a given vector field v^i and two scalar fields ρ and ε satisfy the hydrodynamic equations at an equilibrium configuration. Suppose the fields undergo small disturbances, such that the initial flow undergoes small infinitesimal fluctuations. These small fluctuations give rise to amplitude, e.g., the velocity v^i will have the corresponding amplitude δv^i. Writing the velocity in power of the amplitude such that

$$v^i = v_0^i + \delta v^i + \mathcal{O}(\delta^2). \tag{3.5}$$

The discussion in the previous section allow us to consider this expansion to a first-order for hydrodynamics, such that only terms linear to these amplitudes are retained. Furthermore, only linear equations will be considered, meaning that the disturbance is non-finite, while non-linear systems consider finite amplitudes. If the amplitude vanishes, then the zeroth power of the amplitude series in eq. (3.5) must satisfy the fluid velocity in equilibrium. Since the fluctuations are small, then $v^i \gg \delta v^i$. Suppose now that the fluids is in a 3-dimensional space, with the coordinates $x^i = (x, y, z)$. We can analyse the small fluctuations of the velocity field in terms of plane waves govern by wave vector k^i, with a given wave number

$k = \sqrt{k^i k_i}$. Let $\delta v^i_k(t)$ be the disturbance associated with the wave number k, the amplitude $\delta v^i(t, x^i)$ can be expanded by considering an infinite homogeneous background by

$$\delta v^i(t, x^i) = \int_{-\infty}^{\infty} d\mathbf{k} \delta v^i_k(t) \exp(ik^i x_i). \tag{3.6}$$

This expansion is general for any amplitude and will also be valid for the perturbation of the thermodynamic fields. Studying the hydrodynamic equations under these small fluctuations gives rise to two modes.

3.2 Hydrodynamics modes

The small fluctuations of the relevant fields for hydrodynamics correspond to an amplitude $\delta \sim \exp(ik^i x_i)$. To see their effect on the hydrodynamic equations, suppose that the fluid velocity $v^i = 0$ at equilibrium, and that $\rho, \varepsilon, p \in \mathbb{R}$. This setup is a solution to the hydrodynamic equation. The fields out of thermal equilibrium reads

$$\rho = \rho + \delta\rho \ , \quad \varepsilon = \varepsilon + \delta\varepsilon \ , \quad \delta p = p + \delta p \ , \quad v^i \to \delta v^i, \tag{3.7}$$

where again, only terms linear in the perturbation will be retained. Using the equation of state $p = p(\varepsilon, \rho)$ for the perturbation corresponds to writing

$$\partial_i \delta p = \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \partial_i \delta\varepsilon + \left(\frac{\partial p}{\partial \rho}\right)_\varepsilon \partial_i \delta\rho.$$

Inserting now eq. (3.7) into the ideal hydrodynamic equations, gives

$$\partial_t \delta\rho + \rho \partial_i \delta v^i = 0 \tag{3.8a}$$

$$\rho \partial_t \delta v^i + \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \partial_i \delta\varepsilon + \left(\frac{\partial p}{\partial \rho}\right)_\varepsilon \partial_i \delta\rho = 0 \tag{3.8b}$$

$$\partial_t \delta\varepsilon + w \partial_i \delta v^i = 0 \tag{3.8c}$$

where $w = \varepsilon + p$ is the enthalpy. Then decompose the velocity perturbation into components that are parallel and perpendicular with respect to the wave vector k^i, and write $\delta v^i = \delta v^i_\parallel + \delta v^i_\perp$. Defining $\delta v^i_\parallel = (k^i k^j \delta v_j)/k^2$. Taking the time derivative of eq.(3.8b) while using the relation of eq. (3.8a) and (3.8c) the perturbed momentum equation becomes

$$\rho\partial_t^2\delta v^i + \left(w\left(\frac{\partial p}{\partial\varepsilon}\right)_\rho + \rho\left(\frac{\partial p}{\partial\rho}\right)_\varepsilon\right)\partial_i^2\delta v^i = 0.$$

Using now the relation of the amplitude in eq. (3.6) and defining the sound speed $v_s^2 \equiv (\partial p/\partial\rho)_\varepsilon + (\partial p/\partial_\varepsilon)_\rho(w/\rho)$, gives

$$\partial_t^2\delta v^i + k^2 v_s^2\delta v^i = 0.$$

This satisfies the wave equation and describes a hydrodynamic mode that propagates with the sound speed. By inserting the decomposition of the fluid velocity, then the only contribution is from the longitudinal components since $k^i\delta v_\perp^i = 0$. The transverse velocity component is thus time-independent, and the thermodynamic fields decouples. Therefore, there are two modes in hydrodynamics: one that corresponds to the longitudinal velocities called the sound modes, and the other that corresponds to the transverse velocities called the shear modes [19]. For the sound channel the perturbation of the thermodynamic fields are present, and are not for the shear modes. Furthermore, from eqs. (3.8), considering the sound modes, that the following relations holds

$$\left|\frac{\delta\rho}{\rho}\right| = \left|\frac{\delta v^i}{v_s}\right|, \quad \left|\frac{\delta\varepsilon}{w}\right| = \left|\frac{\delta v^i}{v_s}\right|.$$

These relations require that the velocity perturbation must be much smaller than the sound speed v_s. Since it is required that hydrodynamics satisfies a mean free path much smaller than the characteristic length according to eq.(2.1), which corresponds to stating that [16]

$$|\partial_i\rho| \ll |\rho/\ell|,$$

and from the above relations corresponds to having $|\delta v^i| \ll |v_s|$.

3.3 Linear analysis

The hydrodynamic equations satisfy the wave equation and define the sound modes for the longitudinal velocities. While the wave equation is a solution to the hydrodynamic equations, we will introduce the idea of linear analysis, which will become important for more involved solutions. Since the longitudinal velocity components satisfy the wave equation, then $\delta v_k(t)$ are given by

$$\delta v_k^i(t) = \delta v_k^i \exp(\sigma_k t) \implies \delta v^i = \delta v^i \exp(ik^i x_i - \sigma_k) \tag{3.9}$$

where σ_k will generally be complex. To see this explicitly, consider eqs. (3.8) written again for convenience

$$\begin{aligned}\partial_t \delta\rho &= -ik\rho\delta v^i \\ \partial_t \delta v^i &= -i\frac{a}{\rho}k\delta\varepsilon - i\frac{b}{\rho}k\delta\rho \\ \partial_t \varepsilon &= -iwk\delta v^i\end{aligned}$$

where for compactness $a = (\partial p/\partial\varepsilon)_\rho$ and $b = (\partial p/\partial\rho)_\varepsilon$. By defining a vector

$$\delta^a = (\delta\rho, \delta v^i, \delta\varepsilon)$$

the linear system can be expressed in terms of a matrix M_{ab}, and written as

$$\partial_t \delta_a = M_{ab}\delta^b, \tag{3.10}$$

which is a linear differential equation that has the solution satisfying the relation eq.(3.9). Suppose that the eigenfrequnecy of the system is given by σ_k such that

$$M_{ab} - \sigma I = \begin{pmatrix} -\sigma_k & -i\rho k & 0 \\ -i\frac{b}{\rho}k & -\sigma_k & -i\frac{a}{\rho}k \\ 0 & -iwk & -\sigma_k \end{pmatrix}, \tag{3.11}$$

and taking the determinant of the matrix and solving for σ_k to find that

$$\sigma_k = \pm i\sqrt{\frac{b\rho + aw}{\rho}}k = \pm i v_s k = \pm i\omega.$$

This corresponds to the result of the wave equation considered in the previous section. Nevertheless, linear analysis is essential when studying fluctuations of hydrodynamics and will be the primary approach for more complicated systems. Furthermore, note that $\omega = v_s k$

has the unit of frequency, and thus the dispersion relation gives the eigenfrequency of the modes. For the first-order approximation, an additional term of order $\mathcal{O}(k^2)$ would be part of the dispersion relation $\omega = \pm v_s k + i\omega_2 k^2$. The fluctuations will be considered to be proportional to $\exp(i(k^i x_i - \omega t))$, and we will make use of the linear analysis in studying dispersion relations. For relativistic hydrodynamics, many fundamental ideas of the classical description translate directly over, which will also be the case for the BDNK theory.

4 Linear stability theory

Before realising the relativistic hydrodynamics, it is necessary to study the stability of the system. Lets Suppose that the hydrodynamics variables are the density ρ, the internal energy ε and the fluid velocity v^i. Assuming that they satisfy the hydrodynamic equations, such that they describe an initial flow: if this flow is disturbed by small fluctuations, will the fluctuations grow in amplitude or slowly return to equilibrium? For the latter, the system is stable, but if the amplitude grows such that fluids never return to their initial flow, the system is unstable [20]. For illustrative purposes, consider a ball with a mass m. If the ball is at the top of a hill, any disturbances will send it down the hill, and it departs from its initial state. If the ball is at the bottom of a potential well, it will return to its initial state under certain conditions. For example, if the disturbance is small, the ball will not gain enough kinetic energy to leave the potential well and return to its initial state. This example is illustrated in figure (4). Suppose the disturbance makes the ball oscillate with a frequency $\omega > 0$. Let $x(t)$ be the position for the ball at a time t, such that the equation of motion reads

$$\partial_t^2 x(t) = \pm\omega x(t),$$

where the plus sign is for the stable system, and the minus is for the unstable system. The solution for the stable case is given by $x(t) = x_0 \exp(-i\omega t)$, while the unstable case $x(t) \sim x_0 \exp(\omega t)$ and thus grows exponential.

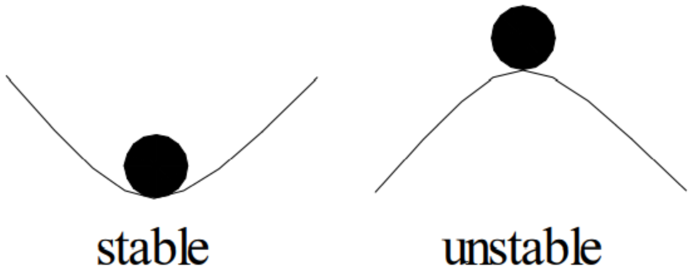


Figure 2: Illustration of a ball with a mass m of a stable system in the potential well (left) and the unstable case on top of the hill (right). Figure adapted from [21]

The case studies for the hydrodynamics case correspond to stating that the amplitudes can grow exponentially. From the linear analysis, the eigenfrequency was given by ω, and the amplitudes $\sim \exp(ik^i x_i - i\omega t)$. The dispersion relations have a real and imaginary part, and thus to avoid the amplitudes to grow exponential, it is required that

$$\mathrm{Im}\omega(\mathbf{k}) \leq 0, \tag{4.1}$$

for the system to be stable. However, the stability requirement above is for the dispersion relation only, and thus only ensures stability for small k. To find the stability for arbitrary k, it is necessary to imply stability of the linear system's determinant. This can be done using the Routh-Hurwitz criterion, that states that for any polynomial with real coefficients describing a closed system, the stability is determined by a Routh array or a Hurwitz matrix [22][23]. For the Hurwitz matrix, consider any given polynomial with coefficients that are real

$$f(x) \equiv a_0 x^4 + a_1 x^3 + a_2 x^2 + a_4,$$

with $a_0 > 0$, the leading minors of the Hurwitz determinant must all be positive. The

Hurwitz Matrix are given by [23]

$$H(f) = \begin{pmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{pmatrix}.$$

The polynomial is then said to be stable if all the leading minors are positive, where the first minors dictate that $a_1 > 0$. For studying the linear stability of hydrodynamics fluctuations, the dispersion relation must satisfy eq. (4.1), while for arbitrary k it is necessary to impose the Routh-Hurwitz criterion.

5 Relativistic hydrodynamics

When writing hydrodynamics from now on, it will refer to relativistic hydrodynamics. Before considering the BDNK theory, we will present general concepts of velocity in a relativistic theory and briefly comment on the relativistic Boltzmann equation. The theory of relativity describes a space-time given by a smooth manifold with a Lorentzian metric. The line-element is given by [24]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

which is then usually maximised to give the proper time $d\tau = -ds$. Given a set of coordinates x^μ where $\mu = 0, 1, 2, 3,$ with its respective proper time τ allows to define the four-velocity as [24]

$$u^\mu = \frac{dx^\mu}{d\tau}. \tag{5.1}$$

By introducing the Lorentz Factor $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, and using the definition of the spatial velocity $\beta^i = \frac{dx}{dt}$, the time and spatial components are given by $u^0 = \gamma$ and $u^i = \gamma\beta^i$, with the speed of light $c = 1$. The components of the four-velocity reads

$$u^\mu = \gamma(1, \beta^i). \tag{5.2}$$

The four-velocity is the fundamental degree of freedom in relativistic hydrodynamics. For a static fluid, i.e. when there is no relative motion between the fluid elements, the spatial velocity vanishes. For an arbitrary smooth space-time manifold embedded with a general metric $g^{\mu\nu}$, then at any given point, a tangent space is spanned, and the space-time is considered locally flat. That is, locally, the effects of any gravitational potential can be neglected, and the metric can locally be given by the Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$. For hydrodynamics settings, for small fluid elements, the metric should reduce to the Minkowski metric $\eta_{\mu\nu}$. Thus locally, Einstein's Equivalence principle is satisfied for any fluid element [25]. The fluid velocity u^μ is normalised accordingly

$$\eta_{\mu\nu} u^\mu u^\nu = u^\mu u_\mu = -1. \tag{5.3}$$

The normalisation of the fluid velocity, is the first constraint for hydrodynamics, which will be taken into account together with the constraints of the shear and sound modes. Furthermore, the fluid velocity u^μ is per definition a vector, and thus must transform accordingly under a Lorentz-transformation on a general space-time

$$u'^\mu = \Lambda^\mu{}_\nu u^\nu. \tag{5.4}$$

For an arbitrary space-time, the direction derivative ∂_i can be generalised by introducing the covariant derivative for any vector field, and the covariant derivative of the fluid velocity reads [24]

$$\nabla_\mu u_\nu = \partial_\mu u_\nu - \Gamma^\rho{}_{\mu\nu} u_\rho.$$

Here $\Gamma^\rho{}_{\mu\nu}$ is the Christoffel symbol and is dependent on the metric, and for a Minkowski metric with cartesian coordinates, all the Christoffel symbols are zero. Thus, the covariant derivative reduces to the directional derivative. The four accelerations in a general space-time is given by

$$a^\mu = u^\nu \nabla_\nu u^\mu. \tag{5.5}$$

The normalisation criteria in eq.(5.3) implies that the four accelerations must be orthogonal to the four-velocity such that $u_\mu a^\nu = 0$. Thus, any fluid element has a four-velocity that is tangent to its worldlines, while the acceleration will be orthogonal to its worldlines, which is illustrated in Figure (3).

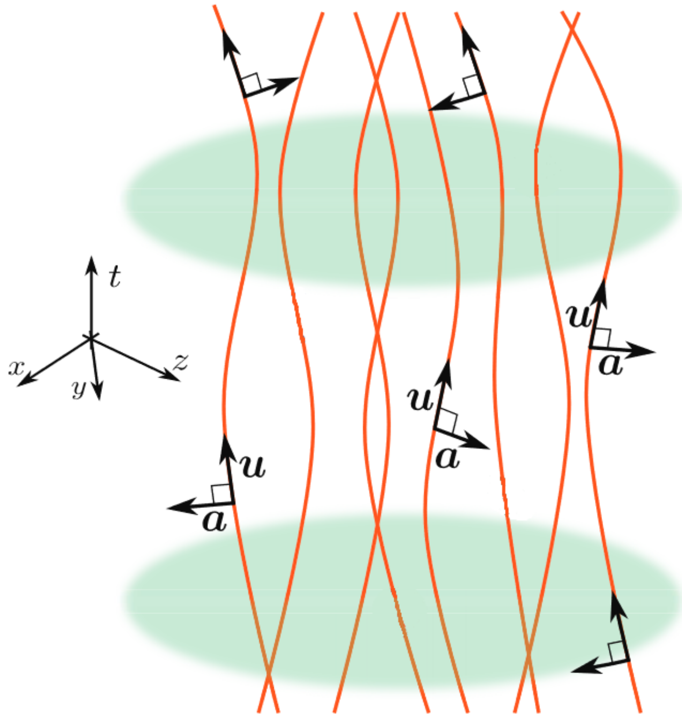


Figure 3: The worldlines of fluid elements with a given fluid velocity, and respective four acceleration. Figure is adapted from [3]

The covariant derivative of the four-velocity $\nabla_\mu u^\nu$ gives rise to a two-rank tensor. It is often useful to split such a tensor in a parallel and perpendicular direction to a given vector, allowing us to study such properties of the directional components. First, consider any general two-rank tensor $A^{\mu\nu}$. It is always possible to write it in terms of symmetric and antisymmetric parts

$$A^{\mu\nu} = A^{(\mu\nu)} + A^{[\mu\nu]} = \frac{1}{2}(A^{\mu\nu} + A^{\nu\mu}) + \frac{1}{2}(A^{\mu\nu} - A^{\nu\mu}),$$

where (...) and [...] denote symmetric and antisymmetric over the indices, respectively. Given the four-velocity and a generic metric $g^{\mu\nu}$ a projection tensor $\Delta^{\mu\nu}$ can be defined that projects any tensor on the hypersurface orthogonal to the four-velocity, given by [3]

$$\Delta^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}. \tag{5.6}$$

The projection tensor satisfies $u_\mu \Delta^{\mu\nu} = 0$, $\Delta^{\mu\nu}\Delta^\rho{}_\nu = \Delta^{\mu\rho}$ and $\Delta^\rho{}_\nu \nabla_\mu u^\nu = \nabla_\mu u^\rho$, and its trace $\Delta^\mu{}_\mu = 3$. The space of the fluid is spanned by u^μ and $\Delta^{\mu\nu}$. Any two rank tensor dependent on the fluid velocity, can be decomposed in parts: A term that is parallel, orthogonal and transverse to the fluid velocity, which further can be decomposed in symmetric traceless and antisymmetric parts. The tensor $\nabla_\mu u_\nu$ decomposes as

$$\nabla_\mu u_\nu = a u_\mu u_\nu + b\Delta_{\mu\nu} + 2C_{(\mu} u_{\nu)} + \sigma_{\mu\nu} + \omega_{\mu\nu}. \tag{5.7}$$

The scalars a, b are parallel and orthogonal respectively. The vector C_μ is transverse, $\sigma_{\mu\nu}$ is the symmetric traceless and $\omega_{\mu\nu}$ is the skew-symmetric part. The scalars, vectors and tensors are given by:

$$\begin{gathered}
a = u^\lambda u^\rho \nabla_\lambda u_\rho \quad , \quad b = \frac{1}{3}\Delta^{\lambda\rho}\nabla_\lambda u_\rho \\
\sigma_{\mu\nu} = \Delta^\lambda{}_\mu \Delta^\rho{}_\nu (\nabla_\rho u_\lambda + \nabla_\lambda u_\rho - \frac{2}{3} g_{\rho\lambda}\nabla_\sigma u^\sigma) \\
\omega_{\mu\nu} = \Delta^\lambda{}_\mu \Delta^\rho{}_\nu \nabla_{[\rho} u_{\nu]} \ , \ C_\mu = -\Delta^\rho{}_\mu u^\lambda \nabla_\lambda u_\rho,
\end{gathered}$$

where the fraction in the relations of b is present due to the trace of the projection tensor. The scalar a vanishes since the four-velocity is orthogonal to the acceleration, the decomposition reduces to

$$\nabla_\mu u_\nu = \omega_{\mu\nu} + \sigma_{\mu\nu} + b\Delta_{\mu\nu} + C_{(\mu} u_{\nu)}, \tag{5.8}$$

where $\omega_{\mu\nu}$ is called the kinematic vorticity tensor and $\sigma_{\mu\nu}$ is the shear tensor [3]. Both are transverse to the four-velocity, and the shear tensor is traceless, and thus satisify

$$\omega_{\mu\nu}\Delta^{\mu\nu} = \sigma_{\mu\nu}\Delta^{\mu\nu} = 0.$$

The physical description of these quantities, can be illustrated by taken the action and defining the vorticity four-vector $\omega^\mu = (1/2)\varepsilon^{\mu\nu\alpha\beta}\omega_{\alpha\beta}u_\nu$, where $\varepsilon^{\mu\nu\alpha\beta}$ is the levi-civita symbol. Taking the action of the vorticity vector ω_μ gives rise to a rotation around a fixed axis, while the action of the shear tensor describe distortion of the fluid. Furthermore, the scalar b contains $\Delta^{\lambda\rho}\nabla_\lambda u_\rho = \nabla_\lambda u^\lambda$, which describes the expansion of the fluid. All three cases is illustrated in Figure (4).

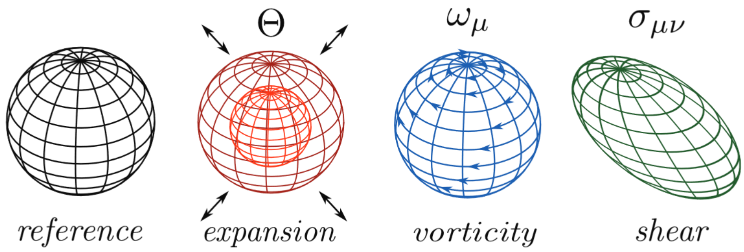


Figure 4: Illustration of the different composition terms of $\nabla_\mu u_\nu$. First figure to the left is a reference, second the expansion where $\Theta = \nabla_\lambda u^\lambda$. Third rotation around the axis intersecting the poles and lastly the shear of the sphere due to shear tensor $\sigma_{\mu\nu}$. Original figure [3]

The introduction of the four-velocity, gives rise to a four-momentum, let m be the rest mass of any sort of particle, then the four-momentum is given accordingly

$$p^\mu = mu^\mu.$$

The time component is the energy, and the spatial component is the momentum, and are normalised such that $p_\mu p^\mu = -m^2$. Lastly, the four acceleration also introduce the four-force given by

$$\mathcal{F}^\mu = ma^\mu. \tag{5.9}$$

The four-velocity u^μ, the four-acceleration a^μ, the momentum p^μ and the four-force $\mathcal{F}^\mu$ gives all the quantities to derive a relativistic version of the Boltzmann equation. Its derivation is more involved but similar to the classical description. The difference in the relativistic case is that the phase space d^3xd^3u must now be Lorentz invariant under infinitesimal time dt. The derivation will not be considered here, but finding the relativistic moment's equation gives conserved quantities similar to the classical description. The conserved quantities are the stress-energy momentum tensor $T^{\mu\nu}$, and the four current J^μ. That is, with respect to the covariant derivative [3]

$$\nabla_\mu T^{\mu\nu} = 0 \quad , \quad \nabla_\mu J^\mu = 0 \tag{5.10}$$

where the four-current represents the conservation of electric charge, and the energy-momentum tensor is the conservation of linear momentum and energy. The energy-momentum tensor is symmetric, and the physical interpretation of the components is given by

- T^{00} is the energy density
- T^{i0} the momentum density
- T^{ii} the isotropic pressure
- T^{ij} with $j \neq i$ is the shear stress .

The kinetic theory allows to derive expressions of the zeroth and first-order approximation of the energy-momentum tensor and the four-current. However, the kinetic description will not be considered, and alternatively a description of hydrodynamics as an effective field theory (EFT).

6 BDNK-Theory

The description of hydrodynamics from the kinetic theory requires a solution to the relativistic Boltzmann equation, which can be rather involved. However, describing hydrodynamics as an EFT allows for simplification in writing out the equations governing hydrodynamics. For any EFT the theory is characterised by an effective action, that in turns can be constrained by the following points [26]

- *Degrees of freedom*: The system needs to be characterised by variables or fields corresponding to degrees of freedom for the system.
- *Symmetries*: Constrain on an effective action given by certain symmetries, i.e., as a gauge, global symmetries etc., such that they correspond to constraints on the system's dynamics.
- *Expansion*: Physics phenomena act differently for a wide range of energy, length and time scales. For example, particles in microscopic quantities do not behave similarly

to macroscopic quantities. Thus, while an effective action may have an infinite amount of terms, they can be collected in a power expansion, such that zeroth-order term is collected together.

From the classical description, it was evident that the hydrodynamic equation describes conserved quantities, that are characterised by two thermodynamics fields and one vector field. The three points above must be satisfied to develop hydrodynamics as an EFT. Firstly, the conservation of the energy-momentum tensor and the four-current comes from the results of a kinetic theory. However, how do we establish these conservations from an EFT perspective? This is done by studying the symmetries of the action and using Noether's theorem, which states that: For any symmetry of the Lagrangian, there is a conserved quantity [16]. Translations, rotations and boosts give the symmetries of space-time. First, defining the energy-momentum tensor for a material system is achieved by considering the variation of the action S with a background of the metric $g_{\mu\nu}$, then a variation of the action with respect to the metric corresponds to writing $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ and the energy-momentum tensor may be written as [24]

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}(x)}. \tag{6.1}$$

From this definition since the metric $g_{\mu\nu}$ is symmetric over its indicies, then so is $T^{\mu\nu}$. The general action S under a translation gives the energy-momentum tensor, that according to the Neother theorem, is conserved. For direct proof of this, we refer to [24]. Furthermore, let A_μ be a vector potential, then with similar argumentation for the energy-momentum tensor, a four-current can be written by considering variation of the vector potential $A_\mu \to A_\mu + \delta A_\mu$, such that the four-current can be written as

$$J^{\mu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta A_{\mu}(x)}. \tag{6.2}$$

This corresponds to a global $U(1)$ symmetry [16]. Thus, since the energy-momentum tensor emits translation symmetries, and the four-current emits a $U(1)$ symmetry, they are both conserved and satisfy eqs. (5.10). The equation governing hydrodynamics can be expressed with six fields, while they are parametrised by two thermodynamics fields and a vector field. For this purpose, it is common practice to use μ, T and u^μ which are the chemical potential, temperature and fluid velocity. However, we note that these fields are auxiliary fields, and when defining equilibrium for the BDNK theory, the fields can be seen as a local fluid velocity, temperature and chemical potential [27]. Lastly, the thermodynamical fields can be choosen to be ε, p, ρ etc., it is merely a choice and they related through the first law of thermodynamics and the equation of state.

With this brief discussion, the first two points of forming an effective field theory has been satisfied. Firstly the degrees of freedom are given by the fields T, μ and u^μ, and the symmetries of the action are given by the energy-momentum tensor and the four-currents. Lastly, the expansion is a derivative expansion, according to the Chapman-Enskog expansion, and thus the energy-momentum tensor and the four-current can be expanded as

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} \quad , \quad J^\mu = J^\mu_{(0)} + J^\mu_{(1)}. \tag{6.3}$$

The zeroth-order is in the power of $\mathcal{O}(1)$ and the first-order approximation is in the power of $\mathcal{O}(\partial)$. As final note for this section, the four-current will be considered to be the charge density, such that parity symmetry is satisfied. Lastly, the symmetries of the boost and rotation, are not necessary taken into account, but their symmetries are satisfied by the 3-rank tensor $M^{\mu\nu\lambda} = x^\mu T^{\mu\lambda} - x^\nu T^{\mu\lambda}$, which is also conserved [16]. Thus there are more conserved quantities that could be taken into account. However, the hydrodynamic equations in eq. (5.10) are dependent on six variables and its linear system is solvable, and thus the last symmetries is not considered in this project.

6.1 General Frame for hydrodynamics

The definition of the energy-momentum tensor dictates the values of the different components. The time-time component give the energy density, while the diagonal elements are the pressure. From the classical description, the zeroth-order correspond to having isotropic pressure. Therefore, decomposing the energy-momentum tensor $T^{\mu\nu}$ and current vector J^μ similar to the decomposition of the fluid velocity in eq. (5.7). The general tensor and vector can be decomposed into terms containing constitution relations, which can be thought of as a relation between the physical values, such as the internal energy, pressure and the current density contained in the four-current. Thus, for a comoving frame $u^\mu = (1, 0, 0, 0)$ let us define $\mathcal{E}$ as the components of T^{00}, $\mathcal{P}$ for T^{ii}, q^μ for T^{0i} and $t^{\mu\nu}$ for T^{ij}, while for the four-current $\mathcal{N}$ for J^0, and j^μ for J^i. Furthermore, from eq. (6.3) and according to the Chapman-Enskog expansion, it corresponds to writing these constitution scalars, vectors and tensor as

$$
\begin{aligned}
\mathcal{E} &= \mathcal{E}_{(0)} + \varepsilon\mathcal{E}_{(1)} = \varepsilon + \delta\varepsilon \\
\mathcal{P} &= \mathcal{P}_{(0)} + \pi\mathcal{P}_{(1)} = p + \delta\pi \\
\mathcal{N} &= \mathcal{N}_{(0)} + \tau\mathcal{N}_{(1)} = \rho + \delta\tau \\
q^\mu &= q^\mu_{(0)} + rq^\mu_{(1)} = q^\mu_{(1)} \\
j^\mu &= j^\mu_{(0)} + lj^\mu_{(0)} = j^\mu_{(1)} \\
t^{\mu\nu} &= t^{\mu\nu}_{(0)} + \eta t^{\mu\nu}_{(1)} = t^{\mu\nu}_{(1)}.
\end{aligned}
$$

The vectors and tensor does not contribute to the zeroth-order approximation, since the energy-momentum tensor is expected only to have diagonal elements, and the four-current equals the current density. The variables $\varepsilon, \pi, \tau, l, r$ and η are the dummy variables, which now can be considered transport coefficients and have been integrated into the first-order term. It is convenient to write up the zeroth-order and first order of the conserved quantities separately. Thus, the zeroth-order decomposition gives

$$T^{\mu\nu}_{(0)} = (\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu} \tag{6.4a}$$

$$J^\mu_{(0)} = \rho u^\mu, \tag{6.4b}$$

and the first-order are given by

$$T^{\mu\nu}_{(1)} = \delta\varepsilon u^\mu u^\nu + \delta\pi\Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + t^{\mu\nu} \tag{6.5a}$$

$$J^\mu_{(1)} = \delta\tau u^\mu + j^\mu. \tag{6.5b}$$

Note that only at equilibrium can ε and p be taught of as the local energy and the pressure, while out of equilibrium they are auxiliary parameters. It is necessary to chose an equilibrium configuration, which is considered in the next section. Nevertheless, the constitution relation are given by

$$\begin{aligned}
&\delta\varepsilon = u_\mu u_\nu T^{\mu\nu}_{(1)} \ , \delta\pi = \frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu}_{(1)} \ , \ q^\mu = -\Delta^\mu{}_\nu u_\rho T^{\nu\rho} \\
&t^{\mu\nu} = \frac{1}{2}\left(\Delta^\nu{}_\lambda\Delta^\nu{}_\rho + \Delta^\nu{}_\lambda\Delta^\nu{}_\rho - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\lambda\rho}\right)T^{\lambda\rho} \\
&\delta\tau = -u_\mu J^\mu_{(1)} \ , \ j^\mu = \Delta^\mu{}_\rho J^\rho .
\end{aligned} \tag{6.6}$$

The fraction in the relation of $\delta\pi$ comes from the trace of the projection tensor. For the zeroth-order the scalars follow same definition, by replacing $T^{\mu\nu}_{(1)}$ with $T^{\mu\nu}_{(0)}$. The first-order approximation is of an order of $\mathcal{O}(\partial)$, and its constitution relation can be written in terms of the derivation of the field that parameterise them, i.e. T, μ and u^μ. Thus, for the constitution relations, the scalars can be written in terms of the scalar fields $u^\mu\nabla_\mu T$, $u^\mu\nabla_\mu\mu$ and $\nabla_\mu u^\mu$, and for the vectors $\Delta^{\mu\nu}\nabla_\mu T$, $\Delta^{\mu\nu}\nabla_\mu\mu$ and $u^\mu\nabla_\mu u^\nu$. For the tensor $t^{\mu\nu}$ can be written up in terms of the shear tensor $\sigma^{\mu\nu}$ defined in eq. (5.7). Note that the derivation terms that define the constitution relation are merely a choice, and it is common to write the term of $\nabla_\mu(\mu)$ as $\nabla_\mu(\mu/T)$ [28]. The constitution relations in a general expansion then read

$$\begin{aligned}
&\delta\varepsilon = \varepsilon_1\frac{1}{T}u^\lambda\nabla_\lambda T + \varepsilon_2\nabla_\lambda u^\lambda + \varepsilon_3 u^\lambda\nabla_\lambda\left(\frac{\mu}{T}\right) \\
&\delta\pi = \pi_1\frac{1}{T}u^\lambda\nabla_\lambda T + \pi_2\nabla_\lambda u^\lambda + \pi_3 u^\lambda\nabla_\lambda\left(\frac{\mu}{T}\right) \\
&\delta\tau = \tau_1\frac{1}{T}u^\lambda\nabla_\lambda T + \tau_2\nabla_\lambda u^\lambda + \tau_3 u^\lambda\nabla_\lambda\left(\frac{\mu}{T}\right) \\
&q^\mu = r_1 u^\lambda\nabla_\lambda u^\mu + \frac{r_2}{T}\Delta^{\mu\lambda}\nabla_\lambda T + r_3\Delta^{\mu\lambda}\left(\frac{\mu}{T}\right) \\
&j^\mu = l_1 u^\lambda\nabla_\lambda u^\mu + \frac{l_2}{T}\Delta^{\mu\lambda}\nabla_\lambda T + l_3\Delta^{\mu\lambda}\nabla_\lambda\left(\frac{\mu}{T}\right) \\
&t^{\mu\nu} = -\eta\sigma^{\mu\nu}.
\end{aligned} \tag{6.7}$$

The minus sign in $t^{\mu\nu}$ is merely a choice and will be shown later to state that $\eta > 0$. Nevertheless, the constitution relations have sixteen transport coefficients: ε_i, π_i, τ_i, r_i, l_i and η. However, since T, μ and u^μ are auxiliary parameters and out of equilibrium they have no microscopic definition. In order to understand this, a definition of what is meant with equilibrium is considered in the following section.

6.2 Equilibrium constraints

From non-relativistic hydrodynamics, equilibrium corresponds to values of the thermodynamics fields and fluid velocity that satisfy the conservation equations. Describing hydrodynamics as an EFT should also reflect on this definition, such that specific values for the fields T, μ and u^μ satisfy the conservation equation in eqs. (5.10). The equilibrium configuration for the BDNK theory is also a choice, and it is conventional to choose a time-independent equilibrium. This corresponds to having a time-like vector K^μ corresponding to a killing vector [29]-[30]. For K^μ to be a killing vector, it must satisfy the killing equation such that [31]

$$\mathcal{L}_K g_{\mu\nu} = 0, \tag{6.8}$$

where $\mathcal{L}$ is the lie-derivative. Moreover, it states the direction of the symmetries of the space-time. The killing equation implies that the general metric is time-independent $\partial_t g_{\mu\nu} = 0$. Studies of such an equilibrium configuration have been done in [29] - [30], and show that the first-order approximation must vanish. For a time-like killing vector, the energy-momentum tensor is still invariant under translation, and is Lie dragged along K, which corresponds to writing

$$\mathcal{L}_K T^{\mu\nu} = 0.$$

For this to be satisfied $\mathcal{L}_K$ must vanish for the thermodynamic variables. Suppose that T_0, μ_0 and u_0^μ satisfy a time-independent equilibrium; then according to the killing equation, it corresponds to writing [28]

$$\begin{gathered} u^\mu \nabla_\mu T = u^\mu \nabla_\mu \mu = 0 \ , \quad T u^\mu \nabla_\mu u^\rho + \Delta^{\rho\nu} \nabla_\nu T = 0 \\ T \Delta^{\mu\nu} \nabla_\nu \left(\frac{\mu}{T} \right) = 0 \ , \quad \nabla_\mu u^\mu = 0 \ , \quad \sigma^{\mu\nu} = 0. \end{gathered} \tag{6.9}$$

By inserting the eqs. (6.9) into the first order constitution relations eqs. (6.7) the first order should vanish. The constitution relations of the first-order does not contribute, and all goes to zero, except for the vectors, which states

$$q^\mu = r_1 u^\lambda \nabla_\lambda u^\mu + \frac{r_2}{T}\Delta^{\mu\lambda}\nabla_\lambda T$$
$$j^\mu = l_1 u^\lambda \nabla_\lambda u^\mu + \frac{l_2}{T}\Delta^{\mu\lambda}\nabla_\lambda T.$$

However, at this equilibrium they must be equal to zero and comparing with eqs. (6.9) it vanishes if $r_1 = r_2$ and $l_1 = l_2$. The zeroth-order approximation gives the equilibrium solution according to eqs. (6.4), that ε and p can be seen as the local internal energy and the local pressure, respectively, and thus satisfy eqs. (2.22). The zeroth-order approximation is conserved and leads to some interesting consequences. For classical hydrodynamics, it was found that the internal energy, mass and momentum are all conserved. The conservation of the energy-momentum tensor contracted with the fluid velocity u_μ leads to a conservation of the energy density, while contracted along the projection tensor is the conservation of the momentum. The four-current govern the conservation of the charge density, and the ideal hydrodynamics satisfy

$$\nabla_\mu J^\mu_{(0)} = u^\mu \nabla_\mu \rho + \rho \nabla_\mu u^\mu = 0 \tag{6.10a}$$
$$\Delta^\rho{}_\nu \nabla_\mu T^{\mu\nu}_{(0)} = (\varepsilon + p) u^\mu \Delta^\rho{}_\nu \nabla_\mu u^\nu + \Delta^{\mu\rho}\nabla_\mu p = 0 \tag{6.10b}$$
$$-u_\nu \nabla_\mu T^\mu_{(0)} = u^\mu \nabla_\mu(\varepsilon) + (\varepsilon + p)\nabla_\mu u^\mu = 0, \tag{6.10c}$$

where it has been used that $u_\mu u^\mu = -1$, $\Delta^{\mu\nu}u_\nu = 0$ and $\Delta^\rho{}_\nu \Delta^{\mu\nu} = \Delta^{\rho\mu}$, and it is understood that $\varepsilon, p, \rho > 0$. Furthermore, substituting the relations of eq.(2.22) into the energy conservation and using the conservation of the charge density gives

$$\begin{gathered}\nabla_\mu(\epsilon u^\mu) + p\nabla_\mu(u^\mu) = T\nabla_\mu(s u^\mu) + \mu(u^\mu \nabla_\mu \rho + \rho \nabla_\mu u^\mu) = 0 \\ \implies \nabla_\mu(s u^\mu) = 0.\end{gathered} \tag{6.11}$$

This states that the entropy is conserved along the fluid velocity and defines an entropy current $S^\mu = s u^\mu$ that is conserved for ideal fluids. Note that for hydrodynamics to a first-order, the entropy current is no longer conserved, and must follow the second law of thermodynamics, by imposing that $\nabla_\mu S^\mu$ must increase [16]. As final notes for this section, outside of equilibrium, the auxiliary parameters T, μ and u^μ no longer have a microscopic definition and thus can be seen just as two scalar fields and a vector field, respectively. Thus, in equilibrium, the thermodynamic relations in eqs.(2.22) are satisfied automatically, while at a first order correction they are not. However, corresponding relations have been found to a higher-order but does no longer represent local thermodynamics relations [29]-[30]. As a consequence, out of equilibrium the value of the auxiliary fields can be chosen freely, as long as they agree in equilibrium. In the following section, this corresponds to transforming T, μ and u^μ. Where in choosing this equilibrium configuration corresponds to a choice of frame where the zeroth-order constitution relations are the local temperature T, chemical potential μ and fluid velocity u^μ. This corresponds to the discussion of LTE in the classical description. Studying the field transformation, implies that not all the transport coefficients in eqs.(6.7) are genuine since we will see they are not invariant under such transformation.

6.3 Field transformation

The thermodynamics scalar fields at the equilibrium are the temperature T and the chemical potential μ. However, outside of equilibrium, the fields have no microscopic definition, and the small perturbation of the fields does not define any microscopic quantity. It is thus always possible to redefine the fields as long as they agree in equilibrium. By introducing small perturbations $\delta T, \delta\mu$ and δu^μ that is in order of $\mathcal{O}(\partial)$. The fields transform as

$$T \to T + \delta T \ , \ \mu \to \mu + \delta\mu \ , \ u^\mu \to u^\mu + \delta u^\mu. \tag{6.12}$$

To ensure that the normalisation in eq. (5.3) is satisfied for any perturbation consider

$$u^\mu u_\mu = (u^\mu + \delta u^\mu)(u_\mu + \delta u_\mu) = u_\mu u^\mu + u^\mu \delta u_\mu + u_\mu \delta u^\mu + \delta u_\mu \delta u^\mu,$$

the last term is of higher-order and can be neglected, while the second and third term must satisfy

$$u_\mu \delta u^\mu = 0, \tag{6.13}$$

to ensure $u^\mu u_\mu = -1$. This is not guaranteed to be satisfied at higher-order hydrodynamics, but is satisfied at first-order [28]. These fields' transformations correspond to transforming the energy-momentum tensor and four-current as

$$T^{\mu\nu} \to T^{\mu\nu}_{(0)} + \delta T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + \delta T^{\mu\nu}_{(1)} \ , \ J^{\mu} \to J^{\mu}_{(0)} + \delta J^{\mu}_{(0)} + J^{\mu}_{(1)} + \delta J^{\mu}_{(1)}.$$

The terms $\delta T^{\mu\nu}_{(1)}$ and $\delta J^{\mu}_{(1)}$ is of second-order, and thus neglected under these transformations since only first-order terms are of interest. Since any higher-orders can be neglected then it corresponds to stating that

$$\delta u_\mu \delta T^{\mu\nu}_{(0)} = 0 \ , \quad \delta u_\mu T^{\mu\nu}_{(1)} = 0 \ , \quad u_\mu \delta J^{\mu}_{(0)} = 0 \ , \quad \delta u_\mu J^{\mu}_{(1)} = 0$$

and due to eq. (6.13) then

$$\delta u_\mu T^{\mu\nu}_{(0)} = \delta u_\nu ((\varepsilon + p) u^\mu u^\nu + p g^{\mu\nu}) = p \delta u^\nu.$$

Transforming the constitution relations in eqs. (6.7), for $\mathcal{E}'$ transform as

$$\begin{aligned}
\mathcal{E}' &= u'_\mu u'_\nu T'^{\mu\nu} \\
&= (u_\mu + \delta u_\mu)(u_\nu + \delta u_\nu)(T^{\mu\nu} + \delta T^{\mu\nu}_{(0)}) \\
&= (u_\mu u_\nu + u_\mu \delta u_\nu + u_\nu \delta u_\mu + \delta u_\mu \delta u_\mu)(T^{\mu\nu} + \delta T^{\mu\nu}_{(0)}) \\
&= \mathcal{E} + (u_\mu \delta u_\nu + u_\nu \delta u_\mu) T^{\mu\nu}_{(0)} \\
&= \mathcal{E} + p(u_\mu \delta u^\mu + u_\nu \delta u^\nu) \\
&= \mathcal{E}.
\end{aligned}$$

The energy-momentum tensor and the four-current are invariant under these transformations. To ensure this, it is enough to impose that the remaining constitution relations transform as

$$\mathcal{P}' = \mathcal{P} \ , \ \mathcal{N}' = \mathcal{N} \ , \ t^{\mu'\nu'} = t^{\mu\nu} \tag{6.14a}$$

$$q^{\mu'} = q^\mu - (\varepsilon + p)\delta u^\mu \ , \ j^{\mu'} = j^\mu - \rho \delta u^\mu. \tag{6.14b}$$

Their transformation are written out explicitly in Appendix A. From the definition of the constitution relations of the scalars, the following relations hold

$$\begin{aligned}
\varepsilon' + \delta\varepsilon' &= \varepsilon + \delta\varepsilon \\
p' + \delta\pi' &= p + \delta\pi \\
\rho' + \delta\tau' &= \rho + \delta\tau,
\end{aligned}$$

using now that $\varepsilon = \varepsilon(T,\mu)$ with the relations in eq. (2.22), the constitution scalars satisfy

$$\begin{aligned}
\delta\varepsilon' &= \varepsilon + \delta\varepsilon - \varepsilon' = \delta\varepsilon - \left(\frac{\partial\varepsilon}{\partial T}\right)_\mu \delta T - \left(\frac{\partial\varepsilon}{\partial\mu}\right)_T \delta\mu \\
\delta\pi' &= p + \delta\pi - p' = \delta\pi - \left(\frac{\partial p}{\partial T}\right)_\mu \delta T - \left(\frac{\partial p}{\partial\mu}\right)_T \delta\mu \\
\delta\tau' &= \rho + \delta\tau - \rho' = \delta\pi - \left(\frac{\partial\rho}{\partial T}\right)_\mu \delta T - \left(\frac{\partial\rho}{\partial\mu}\right)_T \delta\mu.
\end{aligned} \tag{6.15}$$

The perturbation of the fields is of first-order. Therefore, they can be written generally as an expansion as done for the constitution relations. Defining a_i, b_i and c_i that are function of T, μ, the perturbation can be written as

$$\begin{aligned}
\delta T &= a_1 \frac{1}{T} u^\mu \nabla_\mu T + a_2 \nabla_\mu u^\mu + a_3 u^\mu \nabla_\mu \left(\frac{\mu}{T}\right) \\
\delta\mu &= b_1 \frac{1}{T} u^\mu \nabla_\mu T + b_2 \nabla_\mu u^\mu + b_3 u^\mu \nabla_\mu \left(\frac{\mu}{T}\right) \\
\delta u^\mu &= c_1 u^\nu \nabla_\nu u^\mu + \frac{c_2}{T} \Delta^{\mu\nu} \nabla_\nu T + c_3 \Delta^{\mu\nu} \nabla_\mu \left(\frac{\mu}{T}\right).
\end{aligned}$$

This is similar to

$$\begin{aligned}
\delta\varepsilon' &= \delta\varepsilon - \left(\frac{\partial\varepsilon}{\partial T}\right)_\mu \delta T - \left(\frac{\partial\varepsilon}{\partial\mu}\right)_T \delta\mu \\
&= \varepsilon_1' \frac{1}{T} u^\mu \nabla_\mu T + \varepsilon_2' \nabla_\mu u^\mu + \varepsilon_3' u^\mu \nabla_\mu \left(\frac{\mu}{T}\right).
\end{aligned}$$

For $\delta\epsilon'$ in eqs. (6.15) corresponds to writing

$$\varepsilon_i' = \varepsilon_i - \left(\frac{\partial\varepsilon}{\partial T}\right)_\mu a_i - \left(\frac{\partial\varepsilon}{\partial\mu}\right)_T b_i,$$

and thus the transformation of the constitution relations correspond to transforming the transport coefficient as

$$\delta\varepsilon \to \delta\varepsilon' \implies \varepsilon_i' \to \varepsilon_i - \left(\frac{\partial\varepsilon}{\partial T}\right)_\mu a_i - \left(\frac{\partial\varepsilon}{\partial\mu}\right)_T b_i \tag{6.16a}$$

$$\delta\pi \to \delta\pi' \implies \pi_i' \to \pi_i - \left(\frac{\partial p}{\partial T}\right)_\mu a_i - \left(\frac{\partial p}{\partial\mu}\right)_T b_i \tag{6.16b}$$

$$\delta\tau \to \delta\tau' \implies \tau_i' \to \tau_i - \left(\frac{\partial\rho}{\partial T}\right)_\mu a_i - \left(\frac{\partial\rho}{\partial\mu}\right)_T b_i. \tag{6.16c}$$

For the constitution vector q^μ, we can write

$$
\begin{aligned}
q'^\mu &= q^\mu - (\varepsilon + p)\delta u^\mu \\
&= (r_1 - (\varepsilon+p)c_1)u^\nu\nabla_\nu u^\mu + (r_2 - (\varepsilon+p)c_2)\frac{1}{T}\Delta^{\mu\nu}\nabla_\nu T + (r_3 - (\varepsilon+p)c_3)\Delta^{\mu\nu}\nabla_\mu\left(\frac{\mu}{T}\right),
\end{aligned}
$$

and similar for j^μ. Thus the transport coefficient related to the constitution vectors transform as

$$
\begin{aligned}
r_i' &\to r_i - (\varepsilon+p)c_i \\
l_i &\to l_i - \rho c_i.
\end{aligned}
\tag{6.16d}
$$

The constitution tensor transforms as

$$
t^{\mu\nu} \to t'^{\mu\nu} \implies \eta \to \eta. \tag{6.16e}
$$

The transport coefficient η is invariant under these transformations, but $\varepsilon_i, \pi_i, \tau_i, r_i$ and l_i are not. For the transport coefficients to be genuine they must be invariant under such field transformation in order for the physics to remain the same. A combination of the none genuine transport coefficients are invariant, consider

$$
\begin{aligned}
\left(\frac{\partial p}{\partial\varepsilon}\right)_\rho \varepsilon_i' &= \left(\frac{\partial p}{\partial\varepsilon}\right)_\rho\left(\varepsilon_i - \left(\frac{\partial\varepsilon}{\partial T}\right)_\mu a_i - \left(\frac{\partial\varepsilon}{\partial\mu}\right)_T b_i\right) \\
\left(\frac{\partial p}{\partial\rho}\right)_\rho \tau_i' &= \left(\frac{\partial p}{\partial\rho}\right)_\rho\left(\tau_i - \left(\frac{\partial\rho}{\partial T}\right)_\mu a_i - \left(\frac{\partial\rho}{\partial\mu}\right)_T b_i\right),
\end{aligned}
$$

and using eq. (2.22) we find that

$$
\begin{aligned}
-\left(\left(\frac{\partial p}{\partial\varepsilon}\right)_\rho\left(\frac{\partial\varepsilon}{\partial T}\right)_\mu + \left(\frac{\partial p}{\partial\rho}\right)_\varepsilon\left(\frac{\partial p}{\partial T}\right)_\mu\right)a_i &= -s a_i \\
-\left(\left(\frac{\partial p}{\partial\varepsilon}\right)_\rho\left(\frac{\partial\varepsilon}{\partial\mu}\right)_T + \left(\frac{\partial p}{\partial\rho}\right)_\varepsilon\left(\frac{\partial p}{\partial T}\right)_\mu\right)b_i &= -\rho b_i \\
\left(\frac{\partial p}{\partial T}\right)_\mu a_i = s a_i \ , \quad \left(\frac{\partial p}{\partial\mu}\right)_T b_i &= \rho b_i.
\end{aligned}
$$

Then there are three genuine transport coefficients that are given by η and by the following relations

$$
f_i = \pi_i - \left(\frac{\partial p}{\partial\varepsilon}\right)_\rho \varepsilon_i - \left(\frac{\partial p}{\partial\rho}\right)_\varepsilon \tau_i \ , \quad \ell_i = l_i - \frac{\rho}{\varepsilon+p} r_i, \tag{6.17}
$$

and from the constraints of equilibrium configuration it implies that $\ell_1 = \ell_2$. The two relations in eq. (6.17) are written explicitly in Appendix B. The discussion above shows that it is possible to choose T, μ and u^μ arbitrarily by the transformation of the transport coefficient by choosing a value for a_i, b_i and c_i. Such a choice is referred to as choosing a hydrodynamics frame [16]. By choosing values for a_i, b_i and c_i, new differential equations will govern hydrodynamics, and can lead to instabilities that are not present in the general frame. To clarify the consequences of such a choice, the well studied frames, Eckart and Landau frame will be considered in the following section.

7 Landau and Eckart frame

Eckart achieved the first formulation of first-order hydrodynamics by assuming that the four-current did not contribute at a first-order hydrodynamics [7]. Landau and Lifshits later revised this description to allow hydrodynamics with a first-order four-current [8]. In both cases, hydrodynamics was parameterised by two thermodynamics fields and the fluid velocity. However, the Eckart frame and the Landau-Lifshitz (Landau) frame are unstable. It was then proposed to add extra fields to ensure both stability and causality are satisfied by the Israel-Stewart formulation [3]. This gives the two frames extra degrees of freedom, and the description of hydrodynamics becomes more involved. The previous section showed that the auxiliary parameters under transformation lead to a new frame. Thus, the Landau and Eckart frames can be derived from the general frame.

7.1 Landau frame in hydrodynamics

Suppose that the energy flux does not contribute in the rest frame at the first-order approximation, from the definition of the energy-momentum tensor that corresponds to stating that $T^{0i}_{(1)}$ is equal to zero, and the time component for the four-current, likewise, vanishes. The criteria for the Landau frame is thus given by [3]

$$u_\mu T^{\mu\nu}_{(1)} = 0 \ , \quad u_\mu J^\mu_{(1)} = 0. \tag{7.1}$$

A frame can always be chosen such that the Landau conditions are satisfied. From eqs.(6.16) it corresponds to choosing a_i, b_i and c_i appropriately such that $\delta\varepsilon = \delta\tau = q^\mu = 0$. This can be achieved by setting $\varepsilon_i = \pi_i = r_i = a_i = b_i = c_i = 0$, or by using eqs.(6.16) and choosing

$$\begin{aligned} a_i &= \frac{\varepsilon_i}{\alpha} - \frac{\xi(\tau_i\alpha - \varepsilon_i\chi)}{\alpha(\lambda\alpha - \chi\xi)} \\ b_i &= \frac{\tau_i\alpha - \epsilon_i\chi}{\lambda\alpha - \chi\xi} \\ c_i &= \frac{r_i}{\varepsilon + p}, \end{aligned}$$

where

$$\alpha \equiv \left(\frac{\partial\varepsilon}{\partial T}\right)_\mu, \quad \xi \equiv \left(\frac{\partial\varepsilon}{\partial\mu}\right)_T, \quad \chi \equiv \left(\frac{\partial\rho}{\partial T}\right)_\mu, \lambda \equiv \left(\frac{\partial\rho}{\partial\mu}\right)_T.$$

Then the invariant transport coefficients are given by $f_i = \pi_i, \ell_i = l_i$ and $\eta = \eta$. The energy-momentum tensor and four-current, reduce to

$$\begin{aligned} T^{\mu\nu} &= (\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu} + \left(\frac{f_1}{T}u^\mu\nabla_\mu T + f_2\nabla_\lambda u^\lambda + f_3 u^\lambda\nabla_\lambda\left(\frac{\mu}{T}\right)\right)\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} \\ J^\mu &= \rho u^\mu + l_1 u^\lambda\nabla_\lambda u^\mu + \frac{l_2}{T}\Delta^{\mu\lambda}\nabla_\lambda T + l_3\Delta^{\mu\lambda}\nabla_\lambda\left(\frac{\mu}{T}\right). \end{aligned}$$

For the landau frame the sixteen transport coefficients reduces to six, namely $f_{1,2,3}, \ell_{1,2,3}$ and η, where $\ell_1 = \ell_2$. The Landau frame can be simplified further by evaluating the energy-momentum tensor and the four-current in the hydrodynamic equations at equilibrium. The energy-momentum tensor and the four-current can be evaluated on-shell, by imposing the zeroth-order approximation is equal to zero in eqs.(6.10). This gives two on-shell relations for the scalars $u^\mu\nabla_\mu T, u^\mu\nabla_\mu\mu$ and $\nabla_\mu u^\mu$, and one on-shell relation of the vectors $u^\mu\nabla_\mu u^\nu, \Delta^{\mu\nu}\nabla_\nu T$ and $\Delta^{\mu\nu}\nabla_\nu T$. Solving for $u^\mu\nabla_\mu\mu$, $u^\mu\nabla_\mu T$ and $u^\mu\nabla_\mu u^\nu$ in eqs. (6.10) gives the following relations

$$\begin{aligned} u^\mu\nabla_\mu\mu &= \frac{-s\frac{\partial s}{\partial\mu} + \rho\frac{\partial s}{\partial T}}{\left(\frac{\partial s}{\partial\mu}\right)^2 - \frac{\partial s}{\partial T}\frac{\partial\rho}{\partial\mu}}\nabla_\mu u^\mu \\ u^\mu\nabla_\mu T &= \frac{-\rho\frac{\partial s}{\partial\mu} + s\frac{\partial\rho}{\partial\mu}}{\left(\frac{\partial s}{\partial\mu}\right)^2 - \frac{\partial s}{\partial T}\frac{\partial\rho}{\partial\mu}}\nabla_\mu u^\mu \\ u^\mu\nabla_\mu u^\nu &= -\frac{1}{Ts + \mu\rho}(s\nabla_\mu T + \rho\nabla_\mu\mu). \end{aligned} \tag{7.2}$$

Inserting these relations into the energy-momentum tensor and four-current for the Landau frame, they reduce to

$$\begin{aligned} T^{\mu\nu} &= (\varepsilon + p)u^\mu u^\nu - \zeta \nabla_\lambda u^\lambda \Delta^{\mu\nu} - \eta\sigma^{\mu\nu} \\ J^\mu &= \rho u^\mu + \chi_T \Delta^{\mu\nu}\nabla_\nu T - T\sigma\Delta^{\mu\nu}\nabla_\nu\left(\frac{\mu}{T}\right), \end{aligned} \tag{7.3}$$

where ζ, χ_T and σ are given by

$$\begin{aligned} \zeta &= -f_2 + \frac{1}{\frac{\partial s}{\partial T}\frac{\partial \rho}{\partial \mu} - \left(\frac{\partial s}{\partial \mu}\right)^2}\left(\left(s\frac{\partial \rho}{\partial \mu} - \rho\frac{\partial s}{\partial \mu}\right)f_1 + \left(\rho\frac{\partial s}{\partial T} - s\frac{\partial s}{\partial \mu} + \frac{\mu}{T}\left(\rho\frac{\partial s}{\partial \mu} - s\frac{\partial \rho}{\partial \mu}\right)\right)f_3\right) \\ \sigma &= \frac{\rho}{Ts + \mu\rho}\ell_1 - \frac{1}{T}\ell_3 \\ \chi_T &= \frac{1}{T}(\ell_2 - \ell_1) \end{aligned}$$

where ζ and σ are the Bulk viscosity and the charge conductivity, and the transport coefficient χ_T becomes zero, due to the constraints from equilibrium. Thus by choosing a frame such that $\delta\varepsilon = \delta\tau = q^\mu = 0$ and with $\chi_T = 0$ leaves three transport coefficients η, ζ and σ. By setting $\chi_T = 0$, the same frame proposed by Landau and Lifshitz is achieved [8].
The general frame is a collection of frames according to the transformation of the transport coefficients. The same applies to the Landau frame: Firstly, using the on-shell constraints to eliminate the scalar fields $u^\mu\nabla_\mu T$, $u^\mu\nabla_\mu(\mu/T)$ and the vector field $u^\mu\nabla_\mu u^\nu$ leads to one frame that is evaluated in hydrodynamics. If instead we chose to retain $u^\mu\nabla_\mu T$, would have lead to a different hydrodynamics frame, and would in return impose different physics, and different stability criteria. The problem is that the choice becomes rather arbitrary, and there are no constraints to dictate a "correct way" to eliminate fields.

7.2 Eckart frame in hydrodynamics

To emphasise that transformation of the transport coefficient leads to different hydrodynamics. While generally focusing on the Landau and the general frame in the rest of the project, the Eckart frame will briefly be introduced here. Suppose that the first-order the four-current vanishes. This corresponds to state that there is no charge flow [3]. From the definition of the energy-momentum tensor and the four-current, stating that the charge flow vanishes in the rest frame corresponds to writing

$$u_\mu u_\nu T^{\mu\nu}_{(1)} = 0 \ , \quad J^\mu_{(1)} = 0. \tag{7.4}$$

This corresponds to setting $\delta\varepsilon = \delta\tau = 0$ and $j^\nu = 0$, which can be achieved by setting $\varepsilon_i = \tau_i = l_i = a_i = b_i = c_i = 0$, or by choosing

$$\begin{aligned} a_i &= \frac{\varepsilon_i}{\alpha} - \frac{\beta(\tau_i\alpha - \varepsilon_i\chi)}{\alpha(\lambda\alpha - \chi\beta)} \\ b_i &= \frac{\tau_i\alpha - \epsilon_i\chi}{\lambda\alpha - \chi\beta} \\ c_i &= \frac{l_i}{\rho}. \end{aligned}$$

Then the invariant quantities becomes $\ell_i = -\frac{\rho}{\varepsilon+p} r_i$ and $f_i = \pi_i$. The energy-momentum tensor and the four-current to a first-order reads

$$\begin{aligned} T^{\mu\nu} &= (\varepsilon + p) + pg^{\mu\nu} + \delta\pi\Delta^{\mu\nu} + 2q^{(\mu}u^{\nu)} - \eta\sigma^{\mu\nu} \\ J^\mu &= \rho u^\mu \end{aligned}$$

where

$$\begin{aligned} q^\mu &= -\frac{\varepsilon + p}{\rho}\left(\ell_1 u^\lambda \nabla_\lambda u^\mu + \frac{\ell_2}{T}\Delta^{\mu\lambda}\nabla_\lambda T + \ell_3 \Delta^{\mu\lambda}\nabla_\lambda\left(\frac{\mu}{T}\right)\right) \\ \delta\pi &= \frac{f_1}{T} u^\lambda \nabla_\lambda T + f_2 \nabla_\lambda u^\lambda + f_3 u^\lambda \nabla_\lambda \left(\frac{\mu}{T}\right). \end{aligned}$$

Similar to the Landau frame the on-shell relations can be imposed, retaining on the scalar field $\nabla_\mu u^\mu$ leads to the same $\delta\pi = -\zeta\nabla_\mu u^\mu$ as for the Landau frame. However, solving for $\Delta^{\mu\lambda}\nabla_\lambda(\mu/T)$ or $u^\lambda\nabla_\lambda u^\mu$ respectively gives [28]

$$\begin{aligned} q^\mu &= -\kappa\left(T u^\lambda \nabla_\lambda u^\mu + \Delta^{\mu\lambda}\nabla_\lambda T\right) + \chi_T \Delta^{\mu\lambda}\nabla_\lambda T \\ q^\mu &= \frac{\varepsilon + p}{\rho}\sigma T \Delta^{\mu\lambda}\nabla_\lambda(\mu/T) - \frac{\varepsilon + p}{\rho}\chi_T \Delta^{\mu\lambda}\nabla_\lambda T \end{aligned}$$

Where σ and χ_T is given by the relation found for the Landau-frame, while

$$\kappa \equiv (\varepsilon + p)^2\sigma/(\rho^2 T)$$

Thus, imposing the on-shell relation, different hydrodynamic frames are manifested. That each have different instabilities, note that choosing to eliminate $\Delta^{\mu\lambda}\nabla_\lambda(\mu/T)$ and recalling that $\ell_1 = \ell_2$, the Newtonian limit can be taken to give the classical description of the first-order hydrodynamics [16][3]. Nevertheless, in the discussion of Landau and Eckart's frames can be arbitrarily chosen, but it is not guaranteed that they will lead to sensible physics, i.e. stable frames. From the discussion of hydrodynamics as a field theory, it seems more sensible that frames should be chosen from constraints in the general frame, such that stability and causality are guaranteed. Lastly, the Landau frame can also be derived from a kinetic theory and possibly the Eckart frame. However, this has recently been achieved for the general frame, where the effective field theory description are taken into account; such derivation can be found in [32].

8 Covariant entropy current

Thermodynamics plays a vital role in hydrodynamics, which is evident from the constraints in equilibrium and the hydrodynamic equations themselves. Therefore, it is useful also to consider the second law of thermodynamics, stating that the entropy for a closed system will always increase. For the ideal hydrodynamics, it was found that the entropy is conserved in the form of an entropy current $\nabla_\mu(su^\mu)$. Which corresponds to writing

$$u_\nu T^{\mu\nu}_{(0)} + \mu J^\mu_{(0)} = -T\nabla_\mu(su^\mu), \tag{8.1}$$

according with eq.(6.10). The entropy current to a first-order can be written as

$$S^\mu = S^\mu_{(0)} + S^\mu_{(1)},$$

with the second term being the correction term. From eq. (8.1) it corresponds to writing

$$S^\mu = su^\mu - \frac{1}{T}u_\nu T^{\mu\nu}_{(1)} - \frac{\mu}{T}J^\mu_{(1)}. \tag{8.2}$$

The above expression is known as the canonical entropy and has been considered in [28] and studied in more detail in [33]. It is also a full covariant version of the relation $Ts = p+\varepsilon-\mu\rho$

[16]. For simplicity, consider the case of uncharged fluids, such that $\mu = 0$ and $J^\mu = 0$. The entropy then reduces to

$$S^\mu = su^\mu - \frac{1}{T}u_\nu T^{\mu\nu}_{(1)}. \tag{8.3}$$

Taking the derivative of the entropy fully, that is including all order of derivative such that

$$\nabla_\mu S^\mu = \nabla_\mu(su^\mu) - \frac{1}{T}u_\nu \nabla_\mu T^{\mu\nu}_{(1)} - T^{\mu\nu}_{(1)}\nabla_\mu\left(u_\mu \frac{1}{T}\right). \tag{8.4}$$

The energy-momentum tensor is conserved, which implies a relation between the zeroth and first-order, which for uncharged fluids corresponds to writing $\nabla_\mu(su^\mu) = (1/T)u_\nu \nabla_\mu T^{\mu\nu}_{(1)}$. Using this relation, the derivative of the entropy current reads

$$\nabla_\mu S^\mu = -T^{\mu\nu}_{(1)}\nabla_\mu\left(u_\nu \frac{1}{T}\right). \tag{8.5}$$

Since the energy-momentum tensor is symmetric, then the antisymmetric parts of $\nabla_\mu(u_\nu/T)$ can be neglected, and it is convenient to define

$$\chi_{\mu\nu} = \nabla_{(\mu}\left(u_{\nu)}\frac{1}{T}\right).$$

This is a two-rank symmetric tensor and can be decomposed similarly of the energy-momentum tensor and is given by

$$\chi_{\mu\nu} = \mathcal{A}u_\mu u_\nu + \mathcal{B}\Delta_{\mu\nu} + 2\mathcal{Q}_{(\mu}u_{\nu)} + \mathcal{T}^{\mu\nu},$$

where its constitution relations read

$$\begin{aligned} \mathcal{A} = -\frac{1}{T}u^\lambda \nabla_\lambda T \ , \quad \mathcal{B} = \frac{1}{3T}\Delta^{\lambda\sigma}\nabla_\lambda u_\sigma \\ \mathcal{Q}_\mu = -\frac{1}{2T}\Delta_\mu{}^\rho\left(\frac{1}{T}\nabla_\rho T + u^\lambda \nabla_\lambda u_\rho\right) \ , \quad \mathcal{T}^{\mu\nu} = \frac{1}{2T}\sigma^{\mu\nu}. \end{aligned} \tag{8.6}$$

Using these relations, together with the definition of $T^{\mu\nu}_{(1)}$ given in eq. (6.5) the entropy current is given by

$$\nabla_\mu S^\mu = -\mathcal{A}\delta\varepsilon - 3\mathcal{B}\delta\pi + 2q^\mu \mathcal{Q}_\mu + \frac{1}{2T}\sigma^{\mu\nu}\sigma_{\mu\nu}. \tag{8.7}$$

The exact expression is derived in Appendix C. The coefficient 3 in the second term arises due to $\Delta^{\mu\nu}\Delta_{\mu\nu} = g^{\mu\nu}\Delta_{\mu\nu} = \Delta^\nu{}_\nu = 3$ since $\Delta^{\mu\nu}u_\nu = 0$, similar since both q^μ and $\mathcal{Q}_\mu$ is transverse with the fluid velocity, then $(2q^{(\mu}u^{\nu)})(2\mathcal{Q}_{(\mu}u_{\nu)}) = 2q^{\mu}\mathcal{Q}_{\mu}$. The expression in eq. (8.7) corresponds to all orders of the derivation, and not all terms contribute to a first-order hydrodynamics. This is why higher-order terms must be eliminated. To ensure this, the entropy must be evaluated on-shell. For uncharged fluids the thermodynamic relations read

$$\varepsilon + p = Ts \ , \quad dp = sdT \ , \quad d\varepsilon = TcdT \text{ where } \quad c = \frac{\partial s}{\partial T}. \tag{8.8}$$

Using these relation with the on-shell, the derivative of the entropy current reads

$$T\nabla_{\mu}S^{\mu} = (v_s^2(\varepsilon_2 + \pi_1) - \pi_2 - v_s^4\varepsilon_1)(\nabla_{\mu}u^{\mu})^2 + \frac{\eta}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}, \tag{8.9}$$

and the derivation is written explicitly in Appendix C. The first term is along the fluid velocity, and the second term is transverse to u^{μ}. For this reason the terms can be evaluated separately. Then up to $\mathcal{O}(\partial^2)$, and demanding that the covariant entropy current satisfy $\nabla_{\mu}S^{\mu} \geq 0$ then the following inequalities are satisfied

$$\eta \geq 0 \ , \quad v_s^2(\varepsilon_2 + \pi_1) - \pi_2 - v_s^4\varepsilon_1 \geq 0. \tag{8.10}$$

The second inequality corresponds to the bulk viscosity for uncharged fluids [28], and thus $\zeta \geq 0$. These constraints ensure that an increasing entropy current is present for first-order hydrodynamics, which agrees with the H-theorem described in the derivation of the Boltzmann equation. The BDNK theory imposes constraints from a time-independent equilibrium and entropy. In the following section, continuing with uncharged fluids, we find small perturbation around equilibrium to study the linear stability of the general frame.

9 Modes for the general frame

The hydrodynamics equations in equilibrium are characterised by the two scalar fields temperature $T(t, x^i)$, the chemical potential $\mu(t, x^i)$ and the vector field $u^{\mu}(t, x^i)$. First consider a unboosted frame, such that $\beta^i = 0$. The hydrodynamics frame is assumed to be for uncharged fluids, corresponding to the flow being one-dimensional. There are two modes similar to the classical description for the relativistic modes: the shear and sound modes.

The auxiliary parameters can be transformed by corresponding to small perturbations that give an amplitude. Here, again, it will be considered that the amplitudes are plan waves, such that the amplitudes are proportional to the exponential of the wave vector given by

$$k^\mu = (\omega, \kappa \sin\theta, \kappa\cos\theta, 0). \tag{9.1}$$

The transformation of the auxiliary parameters must agree in equilibrium. For compactness, it is convenient to define $\Delta = \delta \exp(ik^\mu x_\mu)$ such that the fields transform as

$$T \to T + \Delta T \ , \quad u^\mu \to u^\mu + \Delta u^\mu.$$

Furthermore, the covariant derivative, in this case, corresponds to $\nabla_\mu \to ik_\mu$, and for clarification a flat-background is considered such that $g_{\mu\nu} = \eta_{\mu\nu}$, with $\eta_{\mu\nu} = (-1, 1, 1, 1)$. The perturbation of the fluid velocity Δu^μ is spanned by u^μ and the wave vector k^μ. Defining ΔU_1 and ΔU_2 as the basis for the vector, then $\Delta u^\mu = \Delta U_1 u^\mu + \Delta U_2 k^\mu$. Recalling that a consequence of the normalisation in eq. (5.3), the perturbation satisfies $u_\mu \Delta u^\mu = 0$. Solving for ΔU_1 under this constrain implies $\Delta U_1 = -\omega\Delta U_2$. Thus, Δu^μ is written as

$$\Delta u^\mu = (k^\mu - \omega u^\mu)\Delta U_2, \tag{9.2}$$

and satisfies the following relations

$$k_\mu \Delta u^\mu = \kappa^2 \Delta U_2 \ , \quad \Delta^\rho{}_\mu \Delta u^\mu = \Delta^\rho{}_\mu k^\mu \Delta U_2. \tag{9.3}$$

Lastly, for the shear mode, the fluid velocity is transverse with the wave vector such that $k_\mu \Delta u^\mu = 0$, while for the sound channel, the relation in eq. (9.3) holds. The equations of interest is the energy and momentum conservation. For convenience, we write the zeroth-order again as

$$\Delta^\rho{}_\nu \nabla_\mu T^{\mu\nu}_{(0)} = (\varepsilon + p) u^\mu \nabla_\mu u^\rho + \Delta^{\mu\rho}\nabla_\mu p = 0$$
$$u_\nu \nabla_\mu T^\mu = -u^\mu \nabla_\mu(\varepsilon) - (\varepsilon + p)\nabla_\mu u^\mu = 0,$$

and the corresponding equation for the first-order are

$$u_\nu \nabla_\mu T^{\mu\nu}_{(1)} = -u^\mu \nabla_\mu \delta\varepsilon - \nabla_\mu q^\mu \tag{9.4a}$$
$$\Delta^\rho{}_\nu \nabla_\mu T^{\mu\nu}_{(1)} = \Delta^{\mu\rho}\nabla_\mu \delta\pi + \Delta^\rho{}_\nu u^\mu \nabla_\mu q^\nu - \eta \Delta^\rho{}_\nu \nabla_\mu \sigma^{\mu\nu}, \tag{9.4b}$$

where each derivation of the constitution relations are explicitly given by

$$
\begin{aligned}
\nabla_\mu \delta\varepsilon &= -\frac{\varepsilon_1}{T} u^\lambda \nabla_\mu \nabla_\lambda \Delta T - \varepsilon_2 \nabla_\mu \nabla_\lambda \Delta u^\lambda \\
\nabla_\mu \delta\pi &= \frac{\pi_1}{T} u^\lambda \nabla_\mu \nabla_\lambda \Delta T - \pi_2 \nabla_\mu \nabla_\lambda \Delta u^\lambda \\
\nabla_\mu q^\nu &= r_1 \bigg(u^\lambda \nabla_\mu \nabla_\lambda \Delta u^\nu + \frac{1}{T} \Delta^{\nu\lambda} \nabla_\mu \nabla_\lambda \Delta T \bigg) \\
\nabla_\mu q^\mu &= r_1 \bigg(u^\lambda \nabla_\mu \nabla_\lambda \Delta u^\mu + \frac{1}{T} \Delta^{\mu\lambda} \nabla_\mu \nabla_\lambda \Delta T \bigg) \\
\nabla_\mu \sigma^{\mu\nu} &= \Delta^{\mu\sigma} \Delta^{\nu\lambda} \bigg(\nabla_\mu \nabla_\lambda \Delta u_\sigma + \nabla_\mu \nabla_\sigma \Delta u_\lambda - \frac{2}{3} g_{\sigma\lambda} \nabla_\mu \nabla_\alpha \Delta u^\alpha \bigg).
\end{aligned}
\tag{9.5}
$$

The first expression in eqs. (9.4) is the first-order correction of conservation of energy, while the second equation is the first-order correction of the momentum conservation. Furthermore, in finding the derivation of the constitution relations, all terms that are of order $\mathcal{O}(\partial^3)$ is neglected, since the amplitudes are of $\mathcal{O}(\partial)$ per definition, then $\nabla_\mu u^\lambda \nabla_\lambda T = \nabla_\mu \Delta u^\lambda \nabla_\lambda \Delta T$ is of higher-order, and does not contribute in a first-order hydrodynamics.

9.1 Shear modes in the general frame

For the shear channel, the thermodynamics fields decouples, which in this case implies that $\Delta T = 0$. Furthermore, the constitution scalars all vanish for the shear modes, since for uncharged fluids they are parameterised by $\nabla_\mu u^\mu$ and $u^\mu \nabla_\mu T$. Likewise, the term of $\nabla_\mu q^\mu$ also equals zero which can be read of from eqs. (9.5). The only contributing equations for the shear modes are the momentum conservation, and for zeroth-order reads

$$\Delta^\rho{}_\nu \nabla_\mu T^{\mu\nu}_{(0)} = -iw\omega \Delta^\rho{}_\nu k^\nu \Delta u_\perp,$$

and the first-order

$$\Delta^\rho{}_\nu \nabla_\mu T^{\mu\nu}_{(1)} = r_1 \omega^2 \Delta^\rho{}_\nu k^\nu \Delta u_\perp - \eta \kappa^2 \Delta^{\rho\lambda} k_\lambda \Delta u_\perp.$$

Adding the zeroth and first-order contribution together, and multiplying with $i \exp(ik^i x_i)$ gives us the following equation

$$(\varepsilon + p)\omega - i(r_1 \omega^2 - \eta \kappa^2) = 0. \tag{9.6}$$

The Routh-Hurwitz criteria states that r_1 and η must be positive. An inequality between r_1 and η appears for the boosted frame.

9.2 Sound modes in the general frame

For the sound channel, the perturbation of T is non-vanishing. Thus, the energy conservation contributes to the sound modes. Using the relation in eq. (9.3), with $\Delta T \neq 0$, the zeroth-order contribution gives

$$\begin{aligned} u_\nu \nabla_\mu T^{\mu\nu}_{(0)} &= i\omega \Delta T - iwk_\mu \Delta u^\mu = 0 \\ \Delta^\rho{}_\nu \nabla_\mu T^{\mu\nu}_{(0)} &= -iw\omega \Delta^\rho{}_\nu k^\nu \Delta U_2 + is\Delta^{\rho\mu} k_\mu \Delta T = 0, \end{aligned} \tag{9.7}$$

and the first-order correction are given by

$$\begin{aligned} u_\nu \nabla_\mu T^{\mu\nu}_{(1)} &= -\varepsilon_1 \frac{1}{T} u^\lambda u^\mu \nabla_\mu \nabla_\lambda \Delta T - \varepsilon_2 u^\mu \nabla_\mu \nabla_\lambda \Delta u^\lambda - \frac{r_1}{T}\left(u^\lambda \nabla_\mu \nabla_\lambda \Delta u^\lambda + \frac{1}{T}\Delta^{\nu\lambda}\nabla_\mu\nabla_\lambda \Delta T\right) \\ &= \frac{\varepsilon_1}{T}\omega^2 \Delta T - \varepsilon_2 \omega \kappa^2 \Delta U_2 - r_1\left(\frac{1}{T}\kappa^2 \Delta T - \omega\kappa^2 \Delta U_2\right) \\ &= \exp(ik^\mu x_\mu)\left(\frac{1}{T}(\varepsilon_1\omega^2 + r_1\kappa^2)\delta T - \kappa^2(\varepsilon_2 + r_1)\omega \delta U_2\right) \\ \Delta^\rho{}_\nu \nabla_\mu T^{\mu\nu}_{(1)} &= \frac{\pi_1}{T}\Delta^{\mu\rho} u^\lambda_\mu \nabla_\lambda \Delta T - \pi_2 \Delta^{\mu\rho}\nabla_\mu\nabla_\lambda \Delta u^\lambda + r_1\left(\Delta^\rho{}_\nu u^\lambda \nabla_\mu \nabla_\lambda u^\nu + \frac{1}{T}\Delta^{\rho\lambda}\nabla_\mu\nabla_\lambda \Delta T\right) \\ &- \eta \Delta^{\mu\sigma}\Delta^{\rho\lambda}\left(\nabla_\mu\nabla_\lambda \Delta u_\sigma + \nabla_\mu \nabla_\sigma \Delta u_\lambda - \frac{2}{3} g_{\sigma\lambda}\nabla_\mu\nabla_\alpha \Delta u^\alpha\right) \\ &= \frac{\pi_1}{T}\omega \Delta^{\rho\mu} k_\mu \Delta T - \pi_2 \kappa^2 \Delta^{\rho\mu} k_\mu \Delta U_2 - r_1 \omega^2 \Delta^\rho{}_\nu k^\nu \Delta U_2 + \frac{r_1}{T}\omega \Delta^{\sigma\rho} k_\sigma \Delta T + \frac{4}{3}\eta\kappa^2 \Delta^{\rho\lambda} k_\lambda \Delta U_2 \\ &= \exp(ik^\mu x_\mu)\left((\pi_1 + r_1)\frac{1}{T}\omega \Delta^{\rho\mu} k_\mu \delta T + \Delta^{\rho\mu} k_\mu \left(\frac{4}{3}\eta - \pi_2\right)\kappa^2 \delta U_2 - r_1 \omega^2 \Delta^\rho{}_\nu k^\nu \delta U_2\right). \end{aligned} \tag{9.8}$$

For uncharged fluids π_2 can be given in terms of the bulk viscosity according to eq. (8.10) as

$$\pi_2 = v_s^2(\pi_1 - v_s^2 \varepsilon_1) + v_s^2 \varepsilon_2 - \zeta, \tag{9.9}$$

and allow us to write the system in terms of ζ. Adding the zeroth and first-order contribution together, for the energy and momentum conservation respectively, and multiplying with

$-i\exp(ik^\mu x_\mu)$ leads to the following equations:

$$\frac{1}{T}(w\omega - iv_s^2(\varepsilon_1\omega^2 + r_1\kappa^2))\delta T - \kappa^2 v_s^2(w - i\omega(\varepsilon_2 + r_1))\delta U_2 = 0$$
$$\frac{1}{T}(w - i(\pi_1 + r_1)\omega)\delta T - \left(\omega w + i\kappa^2\left(\frac{4}{3}\eta + \zeta - v_s^2(\pi_1 - v_s^2\varepsilon_2) - v_s^2\varepsilon_1\right) + ir_1\omega^2\right)\delta U_2 = 0.$$

This correspond to solving the linear system $M_{ij}\delta^i = 0$ with the matrix given by

$$M_{ij} = \begin{pmatrix} \frac{1}{T}(w\omega - iv_s^2(\varepsilon_1\omega^2 + r_1\kappa^2)) & \kappa^2 v_s^2(w - i\omega(\varepsilon_2 + r_1)) \\ \frac{1}{T}(w - i(\pi_1 + r_1)\omega) & -\omega w - i\kappa^2\left(\frac{4}{3}\eta + \zeta - v_s^2(\pi_1 - v_s^2\varepsilon_2) - v_s^2\varepsilon_1\right) + ir_1\omega^2 \end{pmatrix}$$

where $w = (\varepsilon + p)$ is the enthalpy. Defining $D_V = \frac{4}{3}\eta + \zeta$ as the longitudinal kinematic viscosity [28], the determinant of the linear system gives

$$\begin{aligned} &v_s^2\varepsilon_1 r_1\omega^4 + iw(v_s^2\varepsilon_1 + r_1)\omega^3 - i\kappa^2 w(v_s^2\varepsilon_1 + D_V + v_s^2 r_1)\omega \\ &-(w^2 + v_s^2\kappa^2(v_s^4\varepsilon_1^2 + D_V\varepsilon_1 + (\varepsilon_1 + \pi_1)(r_1 - v_s^2\varepsilon_1) + \varepsilon_2\pi_1))\omega^2 \\ &+\kappa^2 v_s^2(w^2 + \kappa^2 r_1(v_s^2(\varepsilon_2 + \pi_1 - v_s^2\varepsilon_1) - D_V)) = 0. \end{aligned} \tag{9.10}$$

For the linearized system of the sound channel, there are two gapped modes and two sound waves. The two gapped modes, are given by $\omega = \omega_0 + \mathcal{O}(\partial)$, such tat for $\kappa \to 0$ $\omega \neq 0$. The gapped modes are thus found by letting $\kappa \to 0$, and solving for ω in power of κ, the zeroth order of this dispersion gives

$$\omega = -i\frac{w}{v_s^2\varepsilon_1}\ , \quad \omega = -i\frac{w}{r_1}. \tag{9.11}$$

For the gapless mode, the dispersion relation must vanish when $\kappa \to 0$, and for small κ reads $\omega = v_s\kappa + i\omega_2\kappa^2$, and is given by

$$\omega_{sound} = \pm v_s\kappa - \frac{i}{2}\frac{D_V}{w}\kappa^2. \tag{9.12}$$

This first term corresponds to the dispersion relation for zeroth-order hydrodynamics, and the classical description gives a similar dispersion relation in zeroth-order. For the frame to be stable in the sound channel for small κ, it is enough to demand that eq. (4.1) is satisfied, such that $\mathrm{Im}\omega \leq 0$. Thus from the gapped and the gapless modes, the involved transport coefficient is positive

$$D_V > 0\ , \quad \varepsilon_1 > 0\ , \quad r_1 > 0. \tag{9.13}$$

For the stable hydrodynamics frame, it is also required to check for arbitrary κ, here the Routh-Hurwitz critera is imposed. The determinant that corresponds to the system's characteristic equation can be written as a polynomial with the variables $\Delta = i\omega$. According to eq. (9.10) this can readily be achived, the first term containing ω^4 corresponds to writing $(i\omega)^4 = \Delta$, etc. The determinant can be re-written in powers of Δ as

$$A_0^{(\kappa)}\Delta^4 + A_1^{(\kappa)}\Delta^3 + A_2^{(\kappa)}\Delta^2 + A_3^{(\kappa)}\Delta + A_4^{(\kappa)} = 0 \tag{9.14}$$

where the coefficients are given by

$$\begin{aligned}
A_0^{(\kappa)} &= v_s^2\varepsilon_1 r_1 \\
A_1^{(\kappa)} &= w(v_s^2\varepsilon_1 + r_1) \\
A_2^{(\kappa)} &= w^2 + v_s^2\kappa^2(v_s^4\varepsilon_1^2 + D_V\varepsilon_1 + (\varepsilon_1+\pi_1)(r_1 - v_s^2\varepsilon_1) + \varepsilon_2\pi_1) \\
A_3^{(\kappa)} &= \kappa^2 w(v_s^2\varepsilon_1 + D_V + v_s^2 r_1) \\
A_4^{(\kappa)} &= \kappa^2 v_s^2(w^2 + \kappa^2 r_1(v_s^2(\varepsilon_2 + \pi_1 - v_s^2\varepsilon_1) - D_V)).
\end{aligned} \tag{9.15}$$

The polynomial corresponds to the following Hurwitz Matrix

$$H^{ij}_{(sound)} = \begin{pmatrix} A_1^{(\kappa)} & A_3^{(\kappa)} & 0 & 0 \\ A_0^{(\kappa)} & A_2^{(\kappa)} & A_4^{(\kappa)} & 0 \\ 0 & A_1^{(\kappa)} & A_3^{(\kappa)} & 0 \\ 0 & A_0^{(\kappa)} & A_2^{(\kappa)} & A_4^{(\kappa)} \end{pmatrix},$$

and for stability to be satisfied, it is required that the leading order minors $\Delta_1, \Delta_2, \Delta_3$ and Δ_4 are all positive, note that the determinant of the whole Matrix, can be written as $\Delta_4 = A_4^{(\kappa)}\Delta_3$, thus for arbitrary κ the stability requirements are given by $A_0^{(\kappa)} > 0$, $A_1^{(\kappa)} > 0$, $A_4^{(\kappa)} > 0$ and

$$A_1^{(\kappa)}A_2^{(\kappa)} - A_0^{(\kappa)}A_3^{(\kappa)} > 0 \;, \quad \left(A_1^{(\kappa)}A_2^{(\kappa)} - A_0^{(\kappa)}A_3^{(\kappa)}\right)A_3^{(\kappa)} - \left(A_1^{(\kappa)}\right)^2 A_4^{(\kappa)} > 0. \tag{9.16}$$

The first inequality $A_0^{(\kappa)} > 0$ implies that $\varepsilon_1 r_1 \geq 0$, which is in agreement with eq. (9.13). The second inequality states that $v_s^2\varepsilon_1 + r_1 > 0$, which holds if the speed of sound is real and positive, for $\varepsilon_1, r_1 > 0$. The first inequality in eq. (9.16) is automatically satisfied if the second equality is as well. They give stability requirements for ε_1, ε_2, π_1 and r_1 in a non-linear way. The last inequality $A_4^{(\kappa)} > 0$ corresponds to stating that

$$\varepsilon_2 + \pi_1 > v_s^2 \varepsilon_1 + \frac{1}{v_s^2} D_V. \tag{9.17}$$

For a frame with $\beta^i = 0$, and demanding that the constraints are stable are illustrated in figure (5)(left). Lastly, by studying the determinant at large κ, such that for short wavelengths $\kappa \to \infty$, given by a linear dispersion $\omega = W\kappa$, with W determined by

$$-r_1(D_V + v_s^2(v_s^2\varepsilon_1 - \varepsilon_2 - \pi_1)) - (D_V\varepsilon_1 - vs^4\varepsilon_1^2 - r_1\varepsilon_2 - (r_1 + \varepsilon_2)\pi_1 + v_s^2\varepsilon_1(\varepsilon_2 + \pi_1))W^2 + r_1\varepsilon_1 W^4 = 0,$$

here $0 < v_s < 1$. The above expression leads to other stability and causality constraints. For causality to be satisfied it is enough to require that [28]

$$0 < \lim_{\mathbf{k}\to\infty} \frac{\text{Re }\omega(\mathbf{k})}{\mathbf{k}} < 1. \tag{9.18}$$

Therefore, for $0 < W < 1$ causality is satisfied. The stability and causality constraints from the dispersion relation $\omega = W\kappa$ are illustrated in figure (5) (right). The general frame is stable at spatial velocity $\beta^i = 0$. To find stability for frames with non-vanishing spatial velocity, it is enough to boost the frame, which follows in the next section.

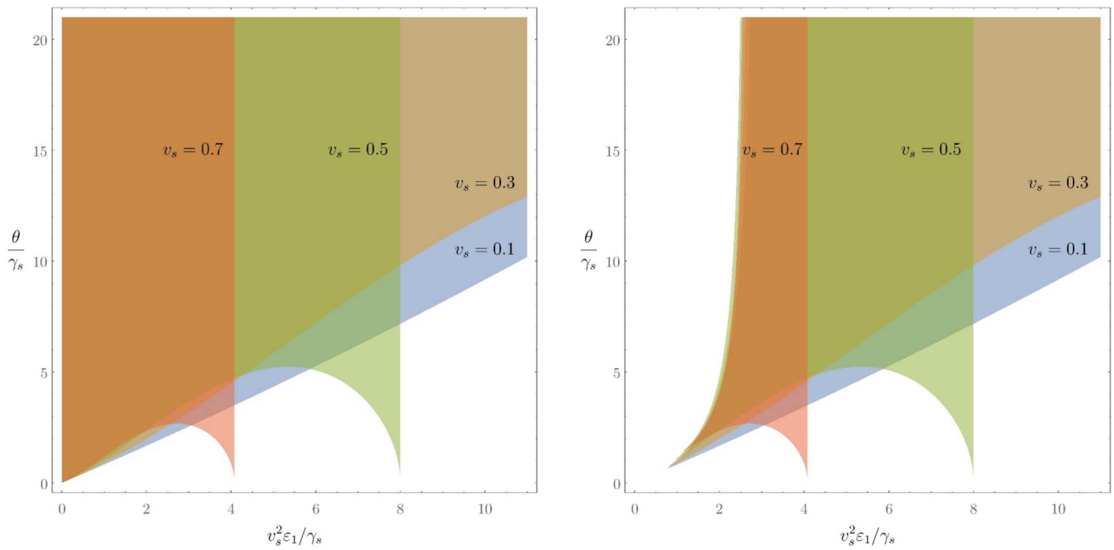


Figure 5: Illustration of different stability regions for different sound speeds. Here dimensionless quantities have been defined as $\tilde{\varepsilon}_1 = v_s^2\varepsilon_1/D_V$, where $D_V = \gamma_s$ on the axis. The transport coefficients $\varepsilon_2 = 0$ and $\pi/D_V = 3/v_s^2$. Left: Stability region for arbitrary κ. Right: Stability region for small wavelengths, with causality criteria. Original figure from [28]

10 Modes in the boosted frame

The general frame in a co-moving frame satisfies $u^\mu = (1, 0, 0, 0)$, and corresponds to stating $\beta^i = 0$ according with eq. (5.2). To study hydrodynamics with spatial velocity the velocity can be boosted according to eq. (5.4), written here again

$$u^\mu = \Lambda^\mu{}_{\nu'} u^{\nu'}.$$

However, it is more useful to boost the frequency ω and the wave vector k^μ. For an arbitrary wave vector in the comoving frame k^μ, analogously for the fluid velocity can be written as [14]

$$k^\mu = (\omega, k^i).$$

The hydrodynamics modes correspond to longitudinal and transverse velocities to the wave vector. Instead, consider the spatial component of the wave vector in terms that are transverse and longitudinal to the spatial velocity, such that $k^i = k^i_\perp + k^i_\parallel$. For this case, the boost only occurs for $k^i_\parallel$, and the spatial components of the wave-vector transform as

$$k^i = k^{i'}_\perp + \gamma(r^{i'}_\parallel - \beta\omega'). \tag{10.1}$$

The perpendicular component can be re-written in terms of $k^{i'}$ and $k^{i'}_\parallel$ as $k^{i'}_\perp = k^{i'} - k^{i'}_\parallel$, furthermore, the dot product of the velocity and $k^{i'}_\perp$ satisfy $k^i_\perp\beta_i = k\beta cos(\phi = 0)$, with ϕ being the angle between k^i and β^i. The perpendicular component can then be written in terms of k^i as

$$k^i_\parallel = \frac{k^i\beta_i}{\beta^2}\beta^j. \tag{10.2}$$

The β^2 can be given in terms of the gamma factor, according to

$$\beta^2 = \frac{\gamma - 1}{\gamma},$$

Substituting these two relations into eq. (10.1), gives

$$k^i = k^{i'} + \gamma\left(\frac{\gamma}{1+\gamma}\beta^j k_j - \omega\right)\beta^i. \tag{10.3}$$

Thus, to find the modes in a boosted frame, it is sufficient to make the following substitutions

$$\omega \to \gamma(\omega' - \beta^i k_i') \ , \quad k^i \to k^{i'} + \gamma\left(\frac{\gamma}{1+\gamma}\beta^i k_i' - \omega'\right)\beta^i, \tag{10.4a}$$

where the squared of the wave vector $k^i k_i = k^2$ are given by

$$k^2 \to k^i k_i = k'^2 + 2\gamma\left(\frac{\gamma}{1+\gamma}\beta^i k_i' - \omega'\right)\beta^i k_i' + \gamma^2\left(\frac{\gamma}{1+\gamma}\beta^i k_i' - \omega'\right)^2\beta^2. \tag{10.4b}$$

For arbitrary ϕ and non-vanishing spatial velocities $\beta^i \neq 0$, the dispersion relations can be found with the above transformations.

10.1 Boosted shear channel

For the boosted frames, without loss of generality the wave vector can again be given by $k^\mu = (\omega, \kappa\sin\theta, \kappa\cos\theta, 0)$, such that $k^2 = \kappa^2$, and κ^2 transforms according to (10.4b).

Nevertheless, the transformation in eq. (10.4a) and eq. (10.4b) can be substituted into the determinant of the shear channel in eq. (9.6) and gives

$$(r_1-\beta^2)(\omega')^2+\left(\frac{iw}{\gamma}-2(r_1-\eta)k^{i'}\beta_i\right)\omega'-\frac{iw}{\gamma}k^{i'}\beta_i-\frac{\kappa^2\eta}{\gamma^2}+(k^{i'}\beta_{i'})^2(r_1-\eta)=0 \quad (10.5)$$

In the boosted frame, the shear channel emits two gapped modes, with the dispersion relation $\omega'=\omega_0+i\omega_1\beta^i k_i$, where ω_0 are found by letting $k\to 0$, and solving for ω_0 gives

$$\omega'=\frac{i(\varepsilon+p)\sqrt{1-\beta^2}}{\eta\beta^2-r_1}, \quad (10.6)$$

while the gapless modes follow the dispersion relation $\omega=c(k^{i'})+i\omega_2(k^{i'})^2$, and for small k^i the gapless modes are given by

$$\omega'=\beta^i k_i'-\frac{i\eta}{(\varepsilon+p)}\sqrt{1-\beta^2}((k^{i'})^2-(\beta^i k_i')^2), \quad (10.7)$$

which corresponds to a shear wave, that propagates perpendicular to the longitudinal waves [28]. From the two relations, the same stability constraints are obtained, but now also stating that

$$r_1>\eta>0 \quad (10.8)$$

according to the gapped modes. The Landau frame is unstable, since $r_1=0$ for non-vanishing η, and do not satisfy the stability criteria of the shear channel for constant β^i. The same criteria for arbitrary k^i can be found with the Routh-hurwitz criteria, and the boosted determinant of the Alfvén channel can be written in terms of $\Delta=i\omega'$, taking only the real parts gives

$$A^{\kappa}_{(0)}\Delta^2+A^{\kappa}_{(1)}\Delta+A^{\kappa}_{(2)}, \quad (10.9)$$

where

$$A^{(\kappa)}_{(0)}=r_1-\beta^2\eta\ ,\quad A^{\kappa}_{(1)}=2\beta^i k_i'(\eta-r_1)$$
$$A^{(\kappa)}_{(2)}=(r_1-\eta)(\kappa^i\beta_i)^2-(1-\beta^2)\eta(k^{i'})^2.$$

The Routh-Hurwitz criteria then implies that $A^{\kappa}_{(0)}>0$, which corresponds to stating that $r_1>\eta$, and similar for the remaining minors. The causality criteria is found in a similar manner as for the sound wave in the non-boosted frame. Thus, for large k^i, the modes follow a linear dispersion of $c_{shear}(\phi)$ which for a boosted frame along x^1 can be shown to be given by [28]

$$(r_1 - \beta^2\eta)c_{shear}^2 - 2\beta\cos\phi(r_1 - \eta)c_{shear} + \beta^2(r_1\cos^2\phi + \eta\sin^2\phi) - \eta = 0, \tag{10.10}$$

where $\beta < 1$, and for $r_1 > \eta$ the c_{shear} must satisfy

$$|c_{shear}| < \frac{1 + \beta\sqrt{\frac{r_1}{\eta}}}{\beta + \sqrt{\frac{r_1}{\eta}}} < 1, \tag{10.11}$$

for the solution of c_{shear} to be real, and thus ensure that the causality criteria holds. With the above discussion, the eigenfrequency at small κ emits the two gapped and gapless modes, while for large κ follow a linear dispersion relation. To see this explicitly, it is convenient to write the determinant in eq. (10.5) in dimensionless quantities, by writing ω and k^i in units of $(\varepsilon + p)/\eta$, such that

$$\tilde{\omega} = \frac{\eta}{\varepsilon + p}\omega\ , \quad \tilde{k}^i = \frac{\eta}{\varepsilon + p}k^i.$$

The dimensionless determinant reduces to

$$-\tilde{k}^2 + \gamma\tilde{\omega}(i + \gamma B\tilde{\omega}) + \gamma\tilde{k}^i\beta_i(\tilde{k}^i\beta_i\gamma A - i - 2\gamma A)$$

where we have defined

$$A = \frac{r_1}{\eta} - 1\ , \quad B = \frac{r_1}{\eta} - \beta^2$$

Solving for $\tilde{\omega}$ gives the dispersion relation for arbitrary $\tilde{k}^i$, and are illustrated in figure (6)

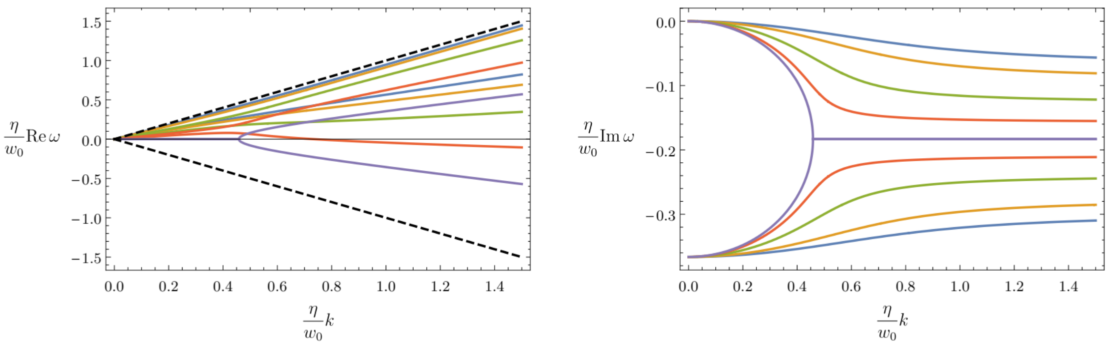


Figure 6: The real and imaginary part of the dispersion relations boosted in the x-direction, for arbitrary $k = \kappa$. Here $w_0 = \varepsilon + p$, $r_1/\eta = 2$, and $0 \leq \phi \leq 1$, where the blue lines corresponds to $\phi = 0$ and purple lines corresponds to $\phi = \pi/2$. In both figures the gapped and gapless modes appear for small $\tilde{\kappa}$, and follow a linear dispersion relation for large $\tilde{\kappa}$. The left figure: here the dashed lines are the light-cone, and all the curves are contained between $\pm\kappa$. The causality criteria is satisfied according to eq. (9.18). The right figure: shows that all the curves are at the lower imaginary plane, and thus satisfies stability. Original figure from [28]

10.2 Boosted sound channel

The sound channel emits two gapped and gapless modes for $\beta^i = 0$, and are found from the determinant in eq. (9.10). Unfortunately, the determinant in the boosted frame is too long to be shown in this project. Nevertheless, it is found by substituting eq. (10.4a) and eq. (10.4b). The gapless modes can then be found for small k^i, and satisfy the dispersion relation $\omega' = c_\pm\kappa + i\omega_2'$, where

$$c_\pm = \frac{1 - v_s^2}{1 - v_s^2\beta^2}\beta\cos(\phi) \pm \frac{v_s}{1 - v_s^2\beta^2}\sqrt{(1-\beta^2)(1 - v_s^2\beta^2 - \beta^2(1 - v_s^2)\cos^2\phi)}. \tag{10.12}$$

This corresponds to a relativistic addition of the phase velocities for arbitrary angles ϕ, and for $\phi = 0$ eq. (10.12) reduces to

$$c_\pm(\phi = 0) = \frac{\beta \pm v_s}{1 \pm v_s\beta}. \tag{10.13}$$

From eq. (10.12) the stability and causality requires that $0 < c_\pm < 1$, and setting $\beta = 0$, implies that in frame with vanishing spatial velocity, the speed of sound is real and must be large than zero, but smaller than the speed of light. Unfortunately, because of lack of time, we could not derive the gapless modes to power of $\mathcal{O}(\partial)$ and, likewise for the gapped modes. However, the gapless modes was found in [28], and their findings will be presented here, which leads to some interesting constraints of stability and causality for the sound channel. For an arbitrary angle ϕ the dispersion relation can be found to be given by

$$\omega' = c_\pm(\phi)\kappa - \frac{i}{2}\Gamma_s(\phi)\kappa^2, \tag{10.14}$$

where

$$\Gamma_s = \frac{D_V}{\varepsilon + p}\frac{c_\pm - \beta\cos\phi}{\sqrt{1-\beta^2}}\frac{1+\beta^2 c_\pm - 2c_\pm\beta\cos\phi - \beta^2\sin^2\phi}{c_\pm(1-\beta^2 v_s^2) - \beta(1-v_s^2)\cos\phi}, \tag{10.15}$$

and it is understood that $c_\pm$ is dependent on ϕ. The coefficient $\Gamma_s(\phi)$ is the damping coefficient and is always positive for different β. The damping coefficient are obtained by considering the linear dispersion relation for when $k^i \to \infty$. Furthermore, the gapped modes also change, and it is found that a necessary criteria for stability is given by [28]

$$v_s^2\varepsilon_1 + r_1 > \frac{D_V}{1-v_s^2}. \tag{10.16}$$

The constraints in figure (5) still holds for the frame with $\beta = 0$, since that they were found for large k^i. As was found for the boosted Alfvén channel, it is possible to define dimensionless quantities

$$\tilde{\varepsilon}_1 = \frac{v_s}{D_V}\varepsilon_1\ ,\quad \tilde{\varepsilon}_2 = \varepsilon_2 D_V^{-1}\ ,\quad r_1 = r_1 D_V^{-1}\ ,\quad \tilde{\pi}_1 = \pi_1 D_V^{-1},$$

and finding the dispersion relation for arbitrary $\tilde{\kappa}$ which is illustrated in figure (7).The discussion of the perturbation of the general frame leads to sets of frames that satisfy stability and causality simultaneously. It also illustrates that frame transformations should be chosen according to the constraints of the transport coefficients to ensure stability and thus sensible physics. The same approach can be applied to fluids that have the property of electrical conductivity, known as plasmas. While the approach is the same, the plasmas description becomes more involved, which will be considered in the rest of this project, and concludes the hydrodynamics description as an EFT theory. Nevertheless, some comments will be directed towards hydrodynamics in the discussion.

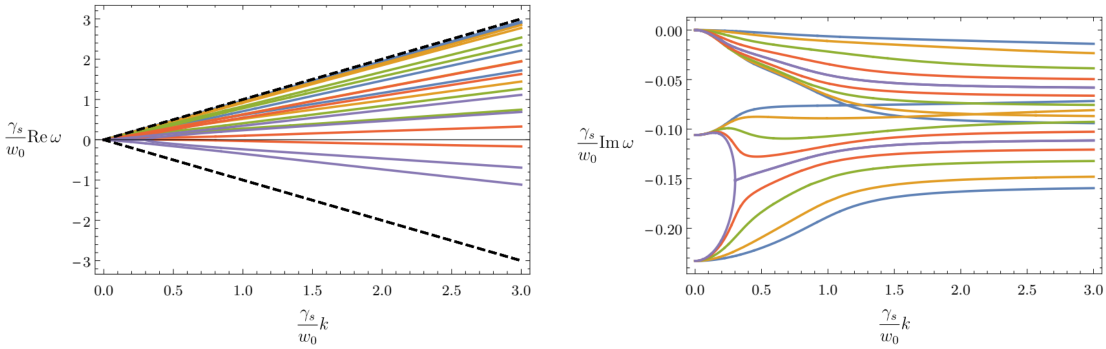


Figure 7: Illustration of the dispersion relation for arbitrary $\tilde{\kappa}$ in the sound channel. In both figure $\tilde{\varepsilon}_1 = 3$, $\tilde{\pi}_1 = 3v_s^{-2}$, $\tilde{r}_1 = 4$, $\tilde{\varepsilon}_2 = 0$ and $1 \leq \phi \leq 0$, where the blue curves corresponds to $\phi = 0$ and purple $\phi = \pi/2$. The left figure: Shows that all curves are inside $\pm\kappa$, and thus causaility is satisfied. The left figure: shows that all imaginary parts are in the lower half-plane and satisfies stability. Furthermore, the gapped modes can be seen for the curves between $\sim [-0.9, -0.24]$. Both figures shows that for large $\tilde{\kappa}$ the dispersion relation is linear. Original figure from [28]

11 Physics description of plasma

Plasmas have properties that differ from a liquid, gas and solid. For that reason, it is known as the fourth state of matter. For most descriptions, plasma has a high temperature. However, a description of cold plasma has also been developed [21]. Magnetohydrodynamics (MHD) offers a toy model for plasmas, and the necessary approximation will be reviewed here. Plasmas are fluids that have the property of electrical conduction and possible a magnetic field due to the movement of the electric charge. The dynamics of plasma corresponds to a coupling of matter and electrodynamic fields, and such dynamics are governed by the

Maxwell's law in matter, given by [34]

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\
\partial_t B + \nabla \times \mathbf{E} &= 0 \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{B} &= \mu_0(\mathbf{J} + \varepsilon_0 \partial_t \mathbf{E}).
\end{aligned}
\tag{11.1}
$$

Here $\mathbf{E}, \mathbf{B}$ are the electric and magnetic field respectively, while $\mathbf{J}, \rho, \mu_0, \varepsilon_0$ are the current density, charge density, vacuum permeability and vacuum permittivity. Maxwell's equation in matter offers a dynamic of plasmas, together with the Lorentz force given by

$$
\mathbf{a} = \frac{q}{m}(\mathbf{E} + \mathbf{u} \times \mathbf{B}), \tag{11.2}
$$

with $\mathbf{a}$, $\mathbf{u}$, m and q being the acceleration, local velocity, m the particle's mass and q the charge of the particles. For a system containing N particles that are charged, a summation over m and q is needed, together with ρ in the divergence of the electric field. The dynamic of the system can be determined accordingly: given an electric and magnetic field, the velocity $\mathbf{u}$ and position $\mathbf{x}$ are given by the Lorentzian force. In contrast, if a position and velocity are known, Maxwell's equation gives the electrodynamic fields. However, such a description becomes impossible for a system of N particles. Instead, it is of more interest to integrate the Lorentz force into the Boltzmann equation by taking the average overall acceleration due to the Lorentzian force. The Boltzmann equation for a monoatomic plasma now reads

$$
\partial_t f + u^i \partial_i f + \frac{q}{m} \frac{\partial}{\partial \mathbf{u}} \cdot ((\mathbf{E} + \mathbf{u} \times \mathbf{B}) f) = \Gamma(f). \tag{11.3}
$$

For the description of plasma according to MHD, it is necessary to find the moment's equation. This was done for non-relativistic hydrodynamics, by averaging over collision invariant quantities. However, the collision term does not apply here: The present of electric and magnetic field leads to long range interactions, and thus random collisions can not bring the system back to an equilibrium state [19]. Thus, to ensure that MHD is govern by the same moment's equation as fluids, with an external force corresponding to the

Lorentz force. It is necessary to make the following key assumptions: 1) The fluid velocities are non-relativistic, 2) the velocities of the positive and negative charges are locked together, such that the plasma is described by a single velocity $v^i(t, x^i)$, corresponding to the fluid velocity. 3) plasmas are strongly collisional such that the particle motion corresponds to the fluid motion. 4) The conductivity σ_e is large, such that in the rest frame a magnetic field is produced even for small electric field. 5) The plasma is quasi-neutral, and the electric field is divergencesless [21]. The last assumption occurs due to Debye shielding which will be discussed in the following section.

11.1 Debye shielding

Suppose that the plasma is quasi-neutral, and ρ_e, ρ_n corresponds to the density for negative and positive charged particles, respectively. Quasi-neutral then means that $\rho_e \simeq \rho_n$, and the divergence of the electric field is given by

$$\nabla \cdot \mathbf{E} = \frac{q}{\varepsilon_0}[\rho_e - \rho_n].$$

Assuming that the inductive electric fields are negligible, such that $E \sim -\nabla_i \phi$, then the above relation corresponds to the Poisson equation. Suppose now that a positive test charge q_t is placed inside the plasma at the centre $r = 0$. This leads to inhomogeneity in the plasma since the positive charge repels all the positive charges and attracts all the negative charges so that at $r = 0$, the electron density increases. For a stationary test charge, the Poisson's equation in terms of the number densities n_e, n_n, then reads [21]

$$\nabla^2 \phi(r) = -\frac{q}{\varepsilon_0}(n_e(r) - n_n(r)) - \frac{q_t}{\varepsilon_0}\delta(r), \tag{11.4}$$

where $\delta(r)$ is the Kronecker delta function. If the temperature is spatially uniform and the plasma remains in thermal equilibrium after the insertion of the test charge. Then if the test particle is surrounded by electrons, then n_e is given by the Boltzmann relation, that reads [21]

$$n_e = n_0 \exp\left(-\frac{q\phi}{k_B T_e}\right),$$

where n_0, T_e, k_b is the equilibrium electron number density, Boltzmann constant, and the electron temperature respectively. The potential ϕ for a single test particle, is expected to be infinitesimal small, so away from origin $q\phi \ll k_B T_e$, and expanding the energy density simplifies to

$$n_e = n_0 \left(1 - \frac{q\phi}{k_B T} \right).$$

Substituting this into the Poisson equation and defining $D = 1 + (x/\lambda_{De}^2)$, with

$$\lambda_{De} = \sqrt{\frac{\varepsilon_0 k_B T_e}{n_0 q^2}}, \tag{11.5}$$

being the Debye length for electrons [21]. The Poisson equation can then be solved by a Fourier transformation

$$\phi(r) = \frac{q_t}{8\pi^3 \varepsilon_0} \int d^3x \frac{\exp(ix^i r_i)}{x^2 D},$$

and the solution then reads

$$\phi(r) = \frac{q_t}{4\pi\varepsilon_0 r} \exp\left(- \frac{r}{\lambda_{De}} \right). \tag{11.6}$$

For $r \ll \lambda_{De}$, the potential describes a test charge in vacuum [34], where for $r \gg \lambda_{De}$, the potential goes towards zero, and the electric field is completely shielded. Thus for the non-relativistic case, plasmas satisfy:

- The scale length $L >> \lambda_{De}$ such that outside of the corresponding fluid element for plasmas, the electric field is screened, and the divergence of the electric field vanishes.
- The screening can only occur if enough particles are present in the fluid elements, which can be shown to be satisfied when the number of particles satisfies $N = (4\pi/3)n_0\lambda_{De}^3 \gg 1$, which corresponds to weakly coupled plasma.[21].

Furthermore, for the non-relativistic case, the velocities are assumed to be small, and thus the displacement field can be neglected. Thus going forward, when referring to Maxwell's equations, we mean

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0 \tag{11.7a}$$

$$\nabla \mathbf{B} = 0 \tag{11.7b}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{11.7c}$$

The discussion above allows to derive the conservation of equation governing the MHD, and can be found by inserting the Lorentz force into the hydrodynamics equation. Their derivation will be considered in the following section, and we will briefly comment on the consequences of the coupling of the electrodynamics fields.

12 Non relativistic MHD

For MHD the plasmas are considered as a system with N particles that satisfies the dynamics of fluids, coupled with with Maxwell's equations. Thus, deriving the conservation of MHD the Lorentz force have to be substituted in for the force term in eqs. (3.1) for the ideal case, and eqs. (3.4) for the first-order correction. For this reason the continuity equation stays unchanged, and particles can not be created or destroyed in MHD. The only two terms that changes are the momentum equation and the energy-conservation. The acceleration term in the momentum equation reads $\rho\mathbf{a}$, and due to Debye shielding the electric flux vanishes $\rho\mathbf{E} = 0$. Using the definition of the current density $\mathbf{J} = qn\mathbf{v}$, the "flux" of the Lorentz force reads

$$\rho\mathbf{a} = \mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B} - \frac{1}{2\mu_0}\nabla B^2. \tag{12.1}$$

For the second equality eq. (11.7c) have been used to substitute for $\mathbf{J}$, together with general identities and eq. (11.7b). The last term in the second equality corresponds to pressure produced by the magnetic field, and the momentum flux for a plasma can be defined by

$$\tilde{\mathbf{\Pi}} = \mathbf{\Pi}_{(0)} + \mathbf{\Pi}_{(1)} + \frac{1}{2\mu_0}\nabla B^2, \tag{12.2}$$

where $\Pi_{(0)}$ and $\Pi_{(1)}$ correspond to the zeroth and first-order found for hydrodynamics. The term $(\mathbf{B}\cdot\nabla)\mathbf{B}$ corresponds to a magnetic tension, and applies a force to the curved magnetic fields, and is for that reason often referred to as a restoring force [19]. To see this explicitly, it is convenient to consider the normalisation unit vector of $\mathbf{B}$ defined by

$$\mathbf{h} = \frac{\mathbf{B}}{B}. \tag{12.3}$$

Substituting this into eq. (12.1) gives

$$\mathbf{J} \times \mathbf{B} = \frac{B^2}{\mu_0}(\mathbf{h} \cdot \nabla)\mathbf{h} - \nabla\left(\frac{B^2}{2\mu_0}\right),$$

where $(\mathbf{h} \cdot \nabla)\mathbf{h}$ points towards the centre for a curved magnetic field, and go towards zero as the magnetic lines gets straighten out. Lastly, for the non-relativistic case the velocities are much smaller than the speed of light. Thus, the Ohm's law is under a Galilean transformation in the non-rest frame, given by [34]

$$E + \mathbf{v} \times B = \eta_e \mu_0 \mathbf{J}. \tag{12.4}$$

Here η_e is the resistivity, and the subscript e is denoted to not confuse it with shear viscosity η. One of the key assumption for MHD, was that the conductivity was large to ensure the production of a magnetic field. In the ideal case, this corresponds to having a plasma that is a perfect conductor, and thus $\eta_e \to 0$. Setting the RHS of eq. (12.4) to zero implies that the time derivative of the magnetic field satisfies

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{12.5}$$

which is known as the induction equation [21]. The continuity equation, momentum conservation and induction equation together with the relation $\nabla \cdot \mathbf{B} = 0$ gives eight equations, where there are eight unknown variables $\rho, p, \mathbf{v}$ and $\mathbf{B}$, making the system solvable. For the ideal hydrodynamics it was shown that the total energy was conserved, this also applies for MHD. To see this, consider the acceleration term of eq. (3.1c): $\rho \mathbf{a} \cdot \mathbf{v}$. The acceleration are given by the Lorentz force, and the RHS of eq. (3.1c) reads

$$\begin{aligned}
\mathbf{v} \cdot (\mathbf{J} \times \mathbf{B}) &= -\mathbf{J} \cdot \mathbf{E} \\
&= \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \cdot \mathbf{E} \\
&= -\frac{1}{2\mu_0}\frac{\partial}{\partial t}B^2 - \frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}).
\end{aligned}$$

The first equality holds for any odd permutations, and due to Ohm's law stating $\mathbf{E} = -v \times \mathbf{B}$. The remaining equality's are given by standard identities of the cross product. The last term is the Poynting vector $\mathbf{S}$, and the energy conservation can be written as

$$\frac{\partial}{\partial t}\left(\varepsilon + \frac{\rho v^2}{2} + \frac{B^2}{2\mu_0}\right) + \nabla \cdot \left(\left(w + \frac{\rho v^2}{2}\right)\mathbf{v} + \mathbf{S}\right) = 0. \tag{12.6}$$

Note that the electromagnetic energy corresponds to multiplying the induction law with $\frac{1}{\mu_0}\mathbf{B}$, that is

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right) = -\nabla \cdot \mathbf{S}.$$

Using this expression, together with the continuity and momentum equation, leads to a conservation of the entropy current

$$\frac{\partial s}{\partial t} + \nabla \cdot (s\mathbf{v}) = 0, \tag{12.7}$$

or equivalently, the euler equation

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot (\varepsilon \mathbf{v}) - \nabla p = 0. \tag{12.8}$$

Thus for the ideal MHD, the entropy current is conserved. The equations governing the zeroth-order MHD are then given by

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \\ \frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot \tilde{\Pi}_{(0)} = \frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B} \\ \frac{\partial}{\partial t}\left(\varepsilon + \frac{\rho v^2}{2} + \frac{B^2}{2\mu_0}\right) + \nabla \cdot \left(\left(w + \frac{\rho v^2}{2}\right)\mathbf{v} + \mathbf{S}\right) = 0, \end{aligned} \tag{12.9}$$

together with the divergence of the magnetic field and an equation of state. A final remark is necessary before writing the first-order correction of MHD. For the zeroth-order MHD the plasma follows a frozen-in flux, which can either be stated as the magnetic field lines must pass through the fluid elements, or the magnetic fields move with the plasma. This means that the magnetic field lines can not change, and are "frozen" into the plasma.

The first-order approximation requires that eq. (12.4) is non vanishing, and the curl of $\eta_e \mu_0 \mathbf{J}$ can be written as

$$\nabla \times (\eta \mu_0 \mathbf{J}) = \eta \nabla^2 \mathbf{B},$$

and the first order correction reads

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\
\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \tilde{\mathbf{\Pi}} &= \frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B} \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \mu_0 \mathbf{J}) \\
\frac{\partial}{\partial t}\left(\varepsilon + \frac{\rho v^2}{2} + \frac{B^2}{2\mu_0}\right) + \nabla \cdot \mathbf{j}_\varepsilon &= 0,
\end{aligned}$$

where for compactness, we have defined

$$\mathbf{j}_\varepsilon \equiv \left(w + \frac{\rho v^2}{2}\right)\mathbf{v} - \eta \overline{\sigma} - \zeta \nabla (\nabla \cdot \mathbf{v}) - \kappa \nabla T + \frac{1}{\mu}\mathbf{S} - \eta \mu_0 \mathbf{J}^2,$$

together with the divergence of the magnetic field and an equation of state. For Hydrodynamics two modes where found, and two corresponding modes appear for MHD.

12.1 Magnetohydrodynamics modes

The zeroth-order approximation emits two modes, corresponding to the shear and sound modes of hydrodynamics. For simplicity, the zeroth-order will be considered with the following assumptions: at equilibrium, the velocity is zero, the magnetic field is constant, and the plasma is homogeneous such that the entropy is constant. Then only the continuity, momentum and induction equation are present. The field out of equilibrium is then

$$\rho \to \rho + \delta\rho \ , \quad p \to p + \delta p \ , \quad v^i \to \delta v^i \ , \quad B \to B + \delta B. \tag{12.10}$$

The amplitude is described as a plane wave; similar to hydrodynamics, only linear terms in the gradients are of interest. For an isotropic flow, the pressure satisfies [19]

$$\delta p = v_s^2 \delta\rho. \tag{12.11}$$

Furthermore, recall that $\partial_i \to ik_i$ and $\partial_t \to -i\omega$, the perturbation of eq. (12.9) in components are then given by

$$
\begin{aligned}
-i\omega\delta\rho + \rho i k_i \delta v^i &= 0 \\
-\omega\rho\delta v^i + ik^i v_s \delta\rho + \frac{i}{\mu_0} k^i B_0^j \delta B_j - \frac{i}{\mu_0} k_j B_0^j \delta B^i &= 0 \\
-i\omega B^i - i(k^j \delta v_j) B_0^i - i(k_j B_0^j)\delta v^i &= 0 \\
&.
\end{aligned}
$$

Solving for $\delta\rho$ and δB in the continuity equation and induction equation and substituting into the momentum equation then gives

$$
\begin{aligned}
-\omega^2 \delta v^i + ik^i v_s k_i \delta v^i + &\frac{i}{\mu_0 \rho} B^j ((k_n \delta v^n) B_j - (k_n B^n)\delta v_j) k^i \\
&- \frac{i}{\mu_0 \rho} k_j B^j ((k_n \delta v^n) B^i - (k_n B^n)\delta v^i) = 0
\end{aligned}
$$

where the momentum equation have been divided with ρ and multiplied with ω. It is convenient to consider the unit vector of the magnetic field given in eq. (12.3) writing $B^i = h^i B$, and defining

$$
v_A^2 \equiv \frac{B^2}{\mu_0 \rho}, \tag{12.12}
$$

the momentum equation reduces to

$$
\omega^2 \delta v^i - ((v_s^2 + v_A^2) k^j v_j - v_A^2 (k^j h_j)(h^n \delta v_n)) k^i + c_A^2 (k^j h_j)((k^n h_n)\delta v^i - h^i (k^n \delta v_n)) = 0. \tag{12.13}
$$

The linear system can now be written up in terms of $\delta v^i = (\delta v^1, \delta v^2, \delta v^3)$ corresponding to the x,y and z components of δv^i, furthermore, without loss of generality suppose that h^i and k^i are given by

$$
h^i = (0, 0, 1)\ , \quad k^i = (\kappa \sin\theta, 0, \kappa\cos\theta).
$$

The linear system can then be written as $M_{ab}\delta v^a = 0$, with the matrix given by

$$
M_{ab} = \begin{pmatrix} \omega^2 - \kappa^2 (v_A^2 + v_s^2 \sin^2\theta) & 0 & -\kappa^2 v_s^2 \sin\theta\cos\theta \\ 0 & \omega^2 - \kappa^2 v_A^2 \cos^2\theta & 0 \\ -\kappa^2 c_s^2 \sin\theta\cos\theta & 0 & \omega^2 - \kappa^2 v_s^2 \cos^2\theta \end{pmatrix}. \tag{12.14}
$$

Recall that for the shear channel the perturbation of the velocity is perpendicular to the wave vector, that is $\delta v^i k_i = 0$, this corresponds to consider the y-component of the fluid velocity, which immediately can be read as

$$\omega^2 - \kappa^2 v_A^2 \cos^2\theta = 0, \tag{12.15}$$

which gives the following dispersion relations

$$\omega = \pm\kappa v_A \cos\theta. \tag{12.16}$$

Thus, v_A corresponds to a velocity along the magnetic field lines, which is seen by eq. (12.12). For $B^2 \gg \mu_0\rho$ the velocity increases, while for $B^2 \ll \mu_0\rho$ the velocity goes towards zero [20]. The dispersion relation reaches its highest value when $\theta = 0$ and can not propagates perpendicular to the magnetic field since $\theta = \pi/2$ implies $\omega = 0$. The shear mode for MHD is referred to as the Alfvén mode, and this will be maintained throughout the rest of the project.

Similar to hydrodynamics, longitudinal modes emits for MHD. These modes are known as the magnetosonic modes [19], and corresponds to velocities parallel to the wave vector k^i. For this particular case, it corresponds to x and z component of δv^i and the Matrix for this linear system can be written as

$$M_{ab} = \begin{pmatrix} \omega^2 - \kappa^2(v_A^2 + v_s^2\sin^2\theta) & -\kappa^2 v_s^2 \sin\theta\cos\theta \\ -\kappa^2 c_s^2 \sin\theta\cos\theta & \omega^2 - \kappa^2 v_s^2\cos^2\theta \end{pmatrix}, \tag{12.17}$$

with its respective determinate reads

$$\omega^4 - (v_A^2 + v_s^2)\kappa^2\omega^2 + v_A^2 v_s^2 \kappa^2 \cos^2\theta = 0. \tag{12.18}$$

Solving this gives the following dispersion relation

$$\omega = \pm\kappa c_\pm \tag{12.19}$$

where $\pm$ defines the direction, and

$$c_\pm = \sqrt{\frac{v_A^2 + v_s^2 \pm \sqrt{(v_A^2 + v_s^2)^2 - 4v_s^2 c_A^2\cos^2\theta}}{2}}. \tag{12.20}$$

Thus, in the magnetosonic channel, there are two-speed modes, a slow c_- and a fast c_+. Furthermore, for the case of $v_s > v_A$, the gas pressure is dominating, while in the reverse case, the plasma is magnetically dominated. Both the sound speed and the Alfvén speed are positive and real, and thus both the fast and slow magnetosonic speed is real and positive. Lastly, the magnetosonic waves reduces to the sound waves in the case of $v_A \to 0$, which corresponds to stating that the magnetic field vanish. Thus, MHD and hydrodynamics are closely related, but the introduction of the magnetic field can lead to more complex physics. This will also be shown to be the case when describing MHD as an EFT, while it is similar to the hydrodynamics approach, it is more involved with the introduction of the extra field **B**. Before discussing MHD as an EFT, we will first consider some consequences of the relativistic Maxwell's equations.

13 Relativistic Maxwell's equations

Maxwell's equations give dynamics to the electromagnetic fields; however, as stated in the description of plasma, such description becomes too complex. Going onwards, when stating MHD, it refers to relativistic MHD, and likewise for plasmas. Nevertheless, to describe MHD, which corresponds to a coupling of the electromagnetic fields to fluids, the introduction of Maxwell's equation is necessary. Given a potential A_μ that is related to the electric and magnetic fields, the strength fields tensor is given by [24]

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \tag{13.1}$$

where $F^{\mu\nu}$ is fully antisymmetric, and its time-spatial components is related to the electric field, and spatial-spatial components give the magnetic field. The Maxwell equations are then govern by

$$\nabla_{[\rho} F_{\mu\nu]} = 0 \ , \quad \nabla_\nu F^{\mu\nu} = J^\mu. \tag{13.2}$$

The first term is the Bianchi identity and gives dynamics to the magnetic fields, while the second term gives dynamics to the electric fields. For MHD, the full dynamic is given by a matter and electromagnetic section, such that the four-current can be written as

$J^\mu = J^\mu_{(em)} + J^\mu_{(matter)}$. Since the $F^{\mu\nu}$ is anti-symmetric over its indices, then it follows that $\nabla_\mu \nabla_\nu F^{\mu\nu} = 0$, which imposes that the four-current J^μ is conserved. Therefore,

$$\nabla_\mu J^\mu = 0. \tag{13.3}$$

Similar the energy-momentum tensor can be written as

$$T^{\mu\nu} = T^{\mu\nu}_{(em)} + T^{\mu\nu}_{(matter)},$$

where

$$T^{\mu\nu}_{(em)} = F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu} F_{\rho\lambda} F^{\rho\lambda}.$$

While coupling the matter to the electromagnetic fields is straightforward, a direct calculation can be difficult. For that reason, a common assumption is that $T^{\mu\nu}_{(em)} \gg T^{\mu\nu}_{(matter)}$ for many astrophysical settings. For example, in studying a zero-force approximation for pulsars and black holes [4][5]. The assumption is sound for astrophysical settings with large $|B|$, and while the Maxwell equations are not Lorentz invariant themselves, it is the case for relations between the magnetic and electric fields. For example

$$F^2 = F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}(B^2 - E^2), \tag{13.4}$$

which is invariant, and in order to satisfy causality it correspond to stating that $B^2 > E^2$ [5]. The relation $T^{\mu\nu}_{(em)} \gg T^{\mu\nu}_{(matter)}$ corresponds to stating that the plasma is weakly coupled, however, many astrophysical settings have a coupling constant $\Gamma \sim 10^{12}$, and corresponds to the ratio of potential to kinetic energy [35]. Thus, for many astrophysical settings the assumption of weak coupling is inconsistent. These assumption can be neglected by describing MHD as an EFT, and will be considered in the following section.

14 Magnetohydrodynmics as an EFT

The description of MHD as an EFT follows hydrodynamics analogously, with the exception of the coupling of an extra field. As such, the degrees of freedom follow T, μ and u^μ, which in equilibrium corresponds to a local temperature, chemical potential and fluid velocity.

Another degree of freedom at equilibrium correspond to the magnetic field, and as in the non-relativistic case, it is enough to consider the unit vector h^μ. Similarly, the expansion of MHD will be given as an expansion of the covariant derivative. For hydrodynamics, the conserved quantities of interest were $T^{\mu\nu}$ and a four-current associated with $U(1)$ symmetries. However, the four-current J^μ is not the generalized global symmetry for electrodynamics. The global symmetry is instead given as a two-form current [36]

$$J^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}, \tag{14.1}$$

which should be treated similar to the four-current J^μ, and is associated with a two-form potential $b_{\mu\nu}$. For the 2-form potential, a 3-form strength field can be written analogously to $F^{\mu\nu}$

$$H_{\mu\nu\rho} = \partial_\mu b_{\nu\rho} + \partial_\nu b_{\rho\mu} + \partial_\rho b_{\mu\nu}. \tag{14.2}$$

Then from the Bianchi identity, the 2-form current is conserved, such that

$$\nabla_\mu J^{\mu\nu} = 0. \tag{14.3}$$

This corresponds to the state that the magnetic field is divergenceless, and as such, the 2-form current can be viewed as a string corresponding to the magnetic field lines. It was explained in [36] that instead of considering magnetic field lines, the dual of $F^{\mu\nu}$ corresponds to electric flux lines. However, such quantity is not conserved in the electrically charged matter because the electric field lines end on charges, and thus electrodynamics only have one conserved 2-form current. The absence of a magnetic charge allows for different symmetries, and they are better suited for studying plasma. If the metric is the background of $T^{\mu\nu}$ and the 2-form potential $b_{\mu\nu}$ external source of $J^{\mu\nu}$, the conserved quantities read [14]

$$\nabla_\mu T^{\mu\nu} = H^\nu{}_{\rho\sigma} J^{\rho\sigma} \ , \quad \nabla_\mu J^{\mu\nu} = 0. \tag{14.4}$$

Therefore, the electric field is not necessary for describing MHD as an effective field theory. Without loss of generality, it is enough to consider the unit vector of the magnetic field h^μ corresponding to eq. (12.3) and this unit vector will be referred to as the magnetic field.

The magnetic field satisfies

$$h^\mu h_\mu = 1 \ , \quad u^\mu h_\mu = 0, \tag{14.5}$$

where the first equality counts for any unit vectors, while the second relation holds due to the magnetic field lines having no time direction. Lastly, the chemical potential in eq. (2.22) now corresponds to the chemical potential for the charge J^{0i} [36]. Before considering the expansion of these quantities, note that per definition, the 2-form current is antisymmetric over its indices, while the energy-momentum tensor remains symmetric. Finally the RHS term in the conservation of $T^{\mu\nu}$ corresponds to a work done by an external source, and vanishes for $b_{\mu\nu} = 0$ [36].

14.1 MHD: General frame

This section follows analogously the BDNK theory for hydrodynamics. An expression of the expansion of the conserved quantities will be written together with the constitution relations. Then constraints from the covariant entropy and the equilibrium state will be reviewed, and small fluctuations around this equilibrium will be studied. Unfortunately, the fluctuations of the general frame were not realised in this project, due to lack of time. However, the stability criteria were found in [14], and we will present their findings and comment on the results. Instead of finding the dispersion relation for the general frame, the Landau frame will instead be considered. It offers simpler algebra but follows analogously the derivation of the dispersion relation in the general frame, which allows us to compare the dispersion relation and show inconsistencies with the Landau Frame. Nevertheless, the energy-momentum tensor and the 2-form current can be written in power of $\mathcal{O}(\partial)$

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} \ , \quad J^{\mu\nu} = J^{\mu\nu}_{(0)} + J^{\mu\nu}_{(1)}. \tag{14.6}$$

With the introduction of the magnetic field, the decomposition will have terms transverse and along h^μ. For this reason the projection tensor now reads

$$\Delta^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu} - h^\mu h^\nu, \tag{14.7}$$

and satisfies

$$\Delta^{\mu}{}_{\nu} = 2 \ , \quad \Delta^{\mu\nu}\Delta^{\rho}{}_{\nu} = \Delta^{\mu\rho} \ , \quad \Delta^{\mu\nu}u_{\mu} = \Delta^{\mu\nu}h_{\mu} = 0. \tag{14.8}$$

The zeroth orders decompose as

$$T^{\mu\nu}_{(0)} = (\varepsilon + p)u^{\mu}u^{\nu} + pg^{\mu\nu} - \mu\rho h^{\mu}h^{\nu} \ , \quad J^{\mu\nu}_{(0)} = 2\rho u^{[\mu}h^{\nu]}, \tag{14.9}$$

where the transport coefficients ε, p, ρ and μ in equilibrium correspond to a local energy density, pressure, charge density and chemical potential. This is satisfied due to some equilibrium states studied similar to the hydrodynamics, and we will later comment on this. Nevertheless, the first-order can be decomposed in terms that are transverse, longitudinal to h^{μ} and u^{μ}. For the energy-momentum tensor an additional term is symmetric and traceless, and for the 2-form current a term that is antrisymmetric. They both can be written as

$$\begin{aligned} T^{\mu\nu}_{(1)} &= \delta\varepsilon u^{\mu}u^{\nu} + \delta\pi\Delta^{\mu\nu} + \delta\xi h^{\mu}h^{\nu} + 2\delta\chi h^{(\mu}u^{\nu)} + 2\ell^{(\mu}h^{\nu)} + 2k^{(\mu}u^{\nu)} + t^{\mu\nu} \\ J^{\mu\nu}_{(1)} &= 2\delta\tau u^{[\mu}h^{\nu]} + 2m^{[\mu}h^{\nu]} + 2n^{[\mu}u^{\nu]} + s^{\mu\nu}. \end{aligned} \tag{14.10}$$

There are five constitution scalars $\delta\varepsilon, \delta\pi, \delta\xi, \delta\tau$ and $\delta\chi$, where the latter is a scalar of mixed terms. The constitution vectors $\ell^{\mu}, k^{\mu}, m^{\mu}$ and n^{μ} are transverse to both h^{μ} and u^{ν}. Lastly the constitution tensors $t^{\mu\nu}, s^{\mu\nu}$ are both transverse, where the first is traceless symmetric and the last one is antisymmetric. The constitution relation is per definition in order of $\mathcal{O}(\partial)$, and can be expanded in terms of the degrees of freedom T, μ, u^{μ} and h^{μ}, and it possible to define them as [14]

$$
\begin{aligned}
\delta\varepsilon &= -\varepsilon_1 \Delta^{\mu\nu}\nabla_\mu u_\nu - \varepsilon_2 h^\mu h^\nu \nabla_\mu u_\nu - \varepsilon_3 u^\mu \nabla_\mu T - \varepsilon_4 u^\mu \nabla_\mu \left(\frac{\mu}{T}\right) \\
\delta\pi &= -\pi_1 \Delta^{\mu\nu}\nabla_\mu u_\nu - \pi_2 h^\mu h^\nu \nabla_\mu u_\nu - \pi_3 u^\mu \nabla_\mu T - \pi_4 u^\mu \nabla_\mu \left(\frac{\mu}{T}\right) \\
\delta\xi &= -\xi_1 \Delta^{\mu\nu}\nabla_\mu u_\nu - \xi_2 h^\mu h^\nu \nabla_\mu u_\nu - \xi_3 u^\mu \nabla_\mu T - \xi_4 u^\mu \nabla_\mu \left(\frac{\mu}{T}\right) \\
\delta\tau &= -\tau_1 \Delta^{\mu\nu}\nabla_\mu u_\nu - \tau_2 h^\mu h^\nu \nabla_\mu u_\nu - \tau_3 u^\mu \nabla_\mu T - \tau_4 u^\mu \nabla_\mu \left(\frac{\mu}{T}\right) \\
\delta\chi &= -T\chi_1 u^\mu h^\nu \delta_B g_{\mu\nu} - T\chi_1 \nabla_\mu (T\rho h^\mu) \\
\ell^\mu &= -T\ell_1 \Delta^{\mu\sigma} h^\nu \delta_B g_{\nu\sigma} - T\ell_2 \Delta^{\mu\sigma} u^\nu \delta_B b_{\sigma\nu} \\
k^\mu &= -Tk_1 \Delta^{\mu\sigma} h^\nu \delta_B b_{\nu\sigma} - Tk_2 \Delta^{\mu\sigma} u^\nu \delta_B u_{\sigma\nu} \\
n^\mu &= -Tn_1 \Delta^{\mu\sigma} h^\nu \delta_B g_{\nu\sigma} - Tn_2 \Delta^{\mu\sigma} u^\nu \delta_B b_{\sigma\nu} \\
m^\mu &= -Tm_1 \Delta^{\mu\sigma} h^\nu \delta_B b_{\nu\sigma} - Tm_2 \Delta^{\mu\sigma} u^\nu \delta_B g_{\sigma\nu} \\
t^{\mu\nu} &= -T\eta_\perp \left(\Delta^{\mu\rho}\Delta^{\nu\sigma} - \frac{1}{2}\Delta^{\mu\nu}\Delta^{\rho\sigma}\right)\delta_B g_{\rho\sigma} \\
s^{\mu\nu} &= -Tr_\| \Delta^{\mu\rho}\Delta^{\nu\sigma}\delta_B b_{\rho\sigma},
\end{aligned}
\tag{14.11a}
$$

where

$$
\delta_B g_{\mu\nu} = 2\nabla_{(\mu}\left(u_{\nu)}\frac{1}{T}\right)\ , \quad \delta_B b_{\mu\nu} = 2\nabla_{[\mu}\left(h_{\nu]}\frac{\mu}{T}\right) + \frac{1}{T}u^\lambda H_\lambda \mu\nu.
$$

There are 28 unidentified transport coefficients that depends on T and μ. The general frame can be transformed into arbitrary frames, by considering small fluctuations out of the equilibrium for the fields. Similar to hydrodynamics, it corresponds to stating that not all transport coefficients are genuine.

14.2 MHD: Frame transformation

The auxiliary parameters T and μ, together with the fluid velocity and the magnetic field h^μ outside of equilibrium have no microscopic definition, and just as in hydrodynamics they transform as

$$
T \to T + \delta T\ , \quad \mu \to \mu + \delta\mu\ , \quad u^\mu \to u^\mu + \delta u^\mu\ , \quad h^\mu \to h^\mu + \delta h^\mu, \tag{14.12}
$$

where the terms δ are of order $\mathcal{O}(\partial)$, and due to the normalisation of the fluid velocity and the magnetic field, the following constraints must be satisfied

$$u_\mu \delta u^\mu = 0\ ,\quad h_\mu \delta h^\mu = 0\ ,\quad u_\mu \delta h^\mu + h_\mu \delta u^\mu = 0. \tag{14.13}$$

The difference in the field transformations in MHD from hydrodynamics, is that the fields should not only be invariant, but respect parity and charge conjugation symmetry. By considering vectors $\alpha_\mu,\ \gamma_\mu$ and a scalar $\tilde{\beta}$ that are dependent on μ, T and satisfy CPT symmetries [2] , the two vectors δu^μ and δh^μ can be decomposed as

$$\delta u^\mu = \alpha_\nu \Delta^{\mu\nu} + \tilde{\beta} h^\mu\ ,\quad \delta h^\mu = \gamma_\nu \Delta^{\mu\nu} + \tilde{\beta} u^\mu.$$

Then generally the scalars $\tilde{\beta},\ \delta T$ and $\delta\mu$ in the transformed frame can be treated the same as the constitution relation, and can be written as an expansion

$$\begin{aligned}
\delta T &= -a_1 \Delta^{\mu\nu} \nabla_\mu u_\nu - a_2 h^\mu h^\nu \nabla_\mu u_\nu - a_3 u^\mu \nabla_\mu T - a_4 u^\mu \nabla_\mu \left(\frac{\mu}{T}\right) \\
\delta\mu &= -b_1 \Delta^{\mu\nu} \nabla_\mu u_\nu - b_2 h^\mu h^\nu \nabla_\mu u_\nu - b_3 u^\mu \nabla_\mu T - b_4 u^\mu \nabla_\mu \left(\frac{\mu}{T}\right) \\
\tilde{\beta} &= -T \tilde{\beta}_1 u^\mu h^\nu \delta_B g_{\mu\nu} - \tilde{\beta}_2 \nabla_\mu (T\rho h^\mu) \\
\gamma^\mu &= -T\gamma_1 \Delta^{\mu\nu} h^\sigma \delta_B g_{\nu\sigma} - T\gamma_2 \Delta^{\mu\nu} u^\sigma \delta_B b_{\nu\sigma} \\
\alpha^\mu &= -Tc_1 \Delta^{\mu\nu} h^\lambda \delta_B b_{\nu\lambda} - Tc_2 \Delta^{\mu\nu} u^\lambda \delta_B g_{\nu\lambda}.
\end{aligned} \tag{14.14}$$

By inserting eq.(14.2) and (14.14) into the constitution relations eq.(14.11), it can be shown that the transformation corresponds to transforming the transport coefficients. Here $\delta\varepsilon$, $\delta\pi$ and $\delta\tau$ transform similar to eq. (6.16) with a sign change

$$\begin{aligned}
\varepsilon_i &\to \varepsilon_i + \frac{\partial\varepsilon}{\partial T} a_i + \frac{\partial\varepsilon}{\partial\mu} b_i \\
\pi_i &\to \pi_i + \frac{\partial p}{\partial T} a_i + \frac{\partial p}{\partial\mu} b_i \\
\tau_i &\to \tau_i + \frac{\partial\rho}{\partial T} a_i + \frac{\partial\rho}{\partial\mu} b_i
\end{aligned} \tag{14.15a}$$

[2] CPT stands for charge conjugation (C), parity transformation (P) and time reversal (T), where C corresponds to symmetries with particles respective anti particles. P is a sign flip in the spatial coordinates and T: $t \to -t$.

where $i = 1, 2, 3, 4$ and the constitution vectors m_i and k_i correspond to the vectors in eq. (6.16) such that

$$\begin{aligned} k_i &\to k_i + (\varepsilon + p)c_i \\ m_i &\to m_i + \rho c_i, \end{aligned} \tag{14.15b}$$

where $i = 1, 2$. The remaining transport coefficients can be shown to satisfy the following transformations [14]

$$\begin{aligned} \tilde{\xi}_i &\to \xi_i + \left(\frac{\partial p}{\partial T} - \mu\frac{\partial \rho}{\partial T}\right)a_i - \mu\frac{\partial \rho}{\partial \mu}b_i \\ \chi_i &\to \chi_i + Ts\tilde{\beta}_i \\ \ell_i &\to \ell_i - \mu\rho\gamma_i \\ n_i &\to n_i - \rho\gamma_i \\ \eta_\perp &\to \eta_\perp \\ r_\parallel &\to r_\parallel \end{aligned} \tag{14.15c}$$

where $i = 1, 2, 3, 4$ for $\tilde{\xi}_i$, and $i = 1, 2$ for the remaining ones. The transport coefficients are not invariant under the transformation in eq. (14.15), except $\eta_\perp$ and $r_\parallel$. However, similar to hydrodynamics it can be found that a combination of the transport coefficients are invariant, and can shown to by given by [14]

$$\begin{aligned} f_i &\equiv \pi_i - \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \varepsilon_i - \left(\frac{\partial p}{\partial \rho}\right)_\varepsilon \tau_i \\ \tilde{\ell}_i &\equiv \ell_i - \mu n_i \\ \tilde{m}_i &\equiv m_i - \frac{\rho}{\varepsilon + p}k_i \\ \tilde{\xi} &\equiv \xi_i - \left(\frac{\partial}{\partial \varepsilon}(p - \mu\rho)\right)_\rho \varepsilon_i - \left(\frac{\partial}{\partial \rho}(p - \mu\rho)\right)_\varepsilon \tau_i \\ \eta_\perp &\to \eta_\perp \ , \quad r_\parallel \to r_\parallel . \end{aligned} \tag{14.16}$$

Here ε_i, π_i and τ_i have $i = 1, 2, 3, 4$ and n_i, ℓ_i, k_i and m_i have $i = 1, 2$. The transport coefficients f_i, $\tilde{\ell}_i$ and $\eta_\perp$ corresponds to those from hydrodynamics, while the remaining are additional for MHD. There are in total 14 genuine transport coefficients, and choosing a_i,

b_i, c_i, ℓ_i, n_i etc., corresponds to changing the frame in MHD. For the stability and causality criteria found in eq. (4.1 and eq. (9.18) to be satisfied, it requires certain constraints of the transport coefficients that must be found in the two normal modes for MHD. Furthermore, in hydrodynamics constraints of the transport coefficients were found by considering the equilibrium settings and by demanding that $\nabla_\mu S^\mu \geq 0$ at all orders. These constraints were found in [14], and will be reviewed in the following section.

14.3 MHD: Equilibrium and entropy constraints

For the rest of this project, the background will be assumed to be flat, and no external force is present such that

$$g_{\mu\nu} = \eta_{\mu\nu} \ , \quad b_{\mu\nu} = 0.$$

The last equality implies that the 3-form $H_{\mu\nu\rho} = 0$. The equilibrium configuration is found in a similar way from hydrodynamics, with an additional killing field ζ^μ is chosen and aligns with the magnetic field h^μ, such equilibrium configurations have been studied in [37][38] [39], and it implies that [14].

$$\delta_B g_{\mu\nu} = 0 \ , \quad \delta_B g_{\mu\nu} = 0 \ , \quad \nabla_\mu (T\rho h^\mu) = 0. \tag{14.17}$$

For such configuration, the first-order vanish and in equilibrium the following equalities hold

$$(\varepsilon + p) > 0 \ , \ \mu\rho > 0 \ , \quad s > 0 \ , \quad T > 0, \tag{14.18}$$

together with extra constraints for a MHD with a non zero spatial velocity

$$\left(\frac{\partial\rho}{\partial\mu}\right)_T \leq 0 \ , \ T\left(\frac{\partial\varepsilon}{\partial T}\right)_\mu + \mu\left(\frac{\partial\varepsilon}{\partial\mu}\right)_T \geq 0 \ , \ \left(\frac{\partial\varepsilon}{\partial T}\right)_\mu \left(\frac{\partial\rho}{\partial\mu}\right)_T - \left(\frac{\partial\rho}{\partial T}\right)_\mu \left(\frac{\partial\varepsilon}{\partial\mu}\right)_T \geq 0 \tag{14.19}$$

For compactness we will define the variables

$$c \equiv \left(\frac{\partial s}{\partial T}\right)_\mu \ , \quad \chi \equiv \left(\frac{\partial\rho}{\partial\mu}\right) \text{ and } \ \lambda = \left(\frac{\partial s}{\partial\mu}\right)_T \equiv \left(\frac{\partial\rho}{\partial T}\right)_\mu . \tag{14.20}$$

Thus, at equilibrium, the transport coefficient in the zeroth-order can be read as the energy density, pressure, chemical potential and charge density. The zeroth orders can be contracted by the fluid velocity, magnetic field and the projection tensor and for $T^{\mu\nu}_{(0)}$ reads

$$\begin{aligned}
u_\nu \nabla_\mu T^{\mu\nu}_{(0)} &= -u^\mu \nabla_\mu \varepsilon - (\varepsilon + p)\nabla_\mu u^\mu - \mu\rho u_\nu h^\mu \nabla_\mu h^\nu = 0 \\
h_\nu \nabla_\mu T^{\mu\nu}_{(0)} &= (\varepsilon + p) h_\nu u^\mu \nabla_\mu u^\nu + h^\mu \nabla_\mu p - h^\mu \nabla_\mu(\mu\rho) - \mu\rho \nabla_\mu h^\mu = 0 \\
\Delta^\rho{}_\nu \nabla_\mu T^{\mu\nu} &= (\varepsilon + p)\Delta^\rho{}_\nu u^\mu \nabla_\mu u^\nu + \Delta^{\rho\mu}\nabla_\mu p - \mu\rho \Delta^\rho{}_\nu h^\mu \nabla_\mu h^\nu = 0,
\end{aligned} \tag{14.21a}$$

while the 2-form current reads

$$\begin{aligned}
u_\nu J^{\mu\nu}_{(0)} &= h^\mu \nabla_\mu \rho - \rho u^\mu u_\nu \nabla_\mu h^\nu + \rho \nabla_\mu h^\mu \\
h_\nu J^{\mu\nu}_{(0)} &= u^\mu \nabla_\mu \rho + \rho \nabla_\mu u^\mu - \rho h^\mu h_\nu \nabla_\mu u^\nu = 0 \\
\Delta^\rho{}_\nu \nabla_\mu J^{\mu\nu}_{(0)} &= \Delta^\rho{}_\nu u^\mu \nabla_\mu h^\nu - \Delta^\rho{}_\nu h^\mu \nabla_\mu = 0
\end{aligned} \tag{14.21b}$$

For an ideal MHD, they define a general coupling of the magnetic field to charged matter, the conservation of $T^{\mu\nu}_{(0)}$ along the fluid velocity corresponds to the conservation of the energy density in the plasma, while the contraction of h_ν and $\Delta^\rho{}_\nu$ corresponds to the momentum conservation longitudinal and transverse to the background field. Similarly, for the 2-current, $u_\mu \nabla_\mu J^{\mu\nu}$ the continuity equation is retained, while the two remaining describes the dynamic of the magnetic field. Where the transverse part is Lie dragged by the fluid along the velocity and corresponds to the frozen-in theorem. Lastly, the entropy for the zeroth-order is also conserved, which can be seen by considering

$$u_\nu \nabla_\mu T^{\mu\nu}_{(0)} + h_\nu J^{\mu\nu}_{(0)} = -T\nabla_\mu (s u^\mu) = 0. \tag{14.22}$$

The entropy current was conserved through the induction equation for the classical description. Thus, $h_\nu \nabla_\mu J^{\mu\nu}_{(0)}$ can be seen as a general form of that equation. Finally, the term $\mu\rho$ is related to the magnetic tension and can be seen as the magnetic pressure since that $T^{33} = \mu\rho$, which should correspond to an isotropic pressure term from the magnetic fields, a more detailed argumentation for such interpretation is given in [36].

Finally, the remaining constraints are retained from the covariant entropy, which now reads

$$S^{\mu\nu} = s u^\mu - \frac{1}{T} u_\mu T^{\mu\nu}_{(1)} - \frac{\mu}{T} J^{\mu\nu}_{(1)}. \tag{14.23}$$

The covariant derivative can then be evaluated at all orders and reads

$$\nabla_\mu S^\mu = -T^{\mu\nu}_{(1)} \delta_B g_{\mu\nu} - J^{\mu\nu}_{(1)} \delta_B b_{\mu\nu} \tag{14.24}$$

Writing out the 2-current and the energy-momentum tensor, and imposing the on-shell relation while neglecting any terms of higher order than $\mathcal{O}(\partial^2)$ give constraints for the transport coefficients. Such work has been carried out in [14], and can shown to be given by

$$r_\perp, r_\parallel, \zeta_\perp, \zeta_\parallel, \eta_\perp, \eta_\parallel \geq 0 \ , \quad \zeta_\perp\zeta_\parallel \geq \frac{1}{4}(\zeta_\times + \zeta'_\times)^2, \tag{14.25}$$

where

$$\begin{aligned}
\eta_\parallel &= \tilde{\ell}_1 - \mu\tilde{\ell}_2 \\
r_\perp &= \tilde{m}_1 - \frac{\rho}{\varepsilon+p}\tilde{m}_2 \\
\zeta_\perp &= \tilde{f}_1 - T\left(\frac{p}{\partial\varepsilon}\right)_\rho \tilde{f}_3 - \frac{1}{T}\left(\frac{\partial p}{\partial\rho}\right)_\varepsilon \tilde{f}_4 \\
\zeta_\times &= \tilde{f}_2 - T\left(\frac{\partial(p-\mu\rho)}{\partial\varepsilon}\right)_\rho \tilde{f}_3 - \frac{1}{T}\left(\left(\frac{\partial(p-\mu\rho)}{\partial\varepsilon}\right)_\varepsilon + \mu\right)\tilde{f}_4 \\
\zeta'_\times &= \tilde{\chi}_1 - T\left(\frac{\partial p}{\partial\varepsilon}\right)_\rho \tilde{\xi}_3 - \frac{1}{T}\left(\frac{\partial p}{\partial\varepsilon}\right)_\rho \tilde{\xi}_4 \\
\zeta_\parallel &= \tilde{\xi}_2 - T\left(\frac{\partial(p-\mu\rho)}{\partial\varepsilon}\right)_\rho \tilde{\xi}_3 - \frac{1}{T}\left(\left(\frac{\partial(p-\mu\rho)}{\partial\varepsilon}\right)_\varepsilon + \mu\right)\tilde{\xi}_4.
\end{aligned} \tag{14.26}$$

Studying the underlying linear response theory allows finding relations between these transport coefficients, and such computation shows that $\zeta_\times = \zeta'_\times$ [14] [3]

15 Landau frame in MHD

The general frame at first-order is determined by eqs. (14.10) and simplifies a lot when considering the Landau-frame. By imposing a frame change that satisfies the Landau conditions in eq. (7.1) implies that at first-order MHD

$$\begin{aligned}
u_\nu T^{\mu\nu}_{(1)} &= -\delta\varepsilon u^\mu - \delta\chi h^\mu - k^\mu = 0 \\
u_\nu J^{\mu\nu}_{(1)} &= \delta\tau h^\mu - n^\nu = 0,
\end{aligned}$$

which corresponds to choosing a frame such that the constitution relations $\delta\varepsilon = \delta\chi = k^\mu = 0$ and $\delta\tau = m^\mu = 0$. Again, this is achieved by transforming the fields by either demanding

[3]The exact method corresponds to finding a dissipation matrix, which is done by finding Kubo formulae. Then Onsager's relations state that these dissipation matrix are symmetric over its indices. In doing so, the relation $\zeta_\times = \zeta'_\times$ is found. See [14]

that $\varepsilon_i = \chi_i = k_i = \tau_i = n_i = 0$, which can either be achieved by setting them to zero, or by setting relevant relations in eqs. (14.15) equal to zero, and then solve for a_i, b_i, c_i, $\tilde{\beta}_i$ and γ_i, which give

$$\begin{aligned}
a_i &= \frac{\varepsilon_i}{\frac{\partial\varepsilon}{\partial T}} + \frac{\frac{\partial\varepsilon}{\partial\mu}}{\frac{\partial\varepsilon}{\partial T}} \frac{\frac{\partial\varepsilon}{\partial T}\tau_i - \frac{\partial\rho}{\partial T}\varepsilon_i}{\frac{\partial\rho}{\partial\mu}\frac{\partial\varepsilon}{\partial T} - \frac{\partial\rho}{\partial T}\frac{\partial\varepsilon}{\partial\mu}} \ , \quad b_i = \frac{\frac{\partial\varepsilon}{\partial T}\tau_i - \frac{\partial\rho}{\partial T}\varepsilon_i}{\frac{\partial\rho}{\partial\mu}\frac{\partial\varepsilon}{\partial T} - \frac{\partial\rho}{\partial T}\frac{\partial\varepsilon}{\partial\mu}} \\
c_i &= -\frac{k_i}{\varepsilon + p} \ , \quad \tilde{\beta}_i = -\frac{\xi_i}{Ts} \ , \quad \gamma_i = \frac{n_i}{\rho}.
\end{aligned}$$

These relations further impose that the invariant transport coefficient now satisfy

$$f_i = \pi_i \ , \quad \tilde{\ell}_i = \ell_i \ , \quad \tilde{m}_i = m_i \ , \ \tilde{\xi} = \xi_i \tag{15.1}$$

In using the transformations for the general frame, and eliminating the constitution relations such that the Landau conditions are satisfied, the conserved quantities reads

$$\begin{aligned}
T^{\mu\nu}_{(1)} &= \delta\pi\Delta^{\mu\nu} + \delta\xi h^\mu h^\nu + 2\ell^{(\mu}h^{\nu)} + t^{\mu\nu} \\
J^{\mu\nu}_{(1)} &= 2m^{[\mu}h^{\nu]} + s^{\mu\nu}.
\end{aligned}$$

Following now the example for the Landau frame in hydrodynamics, by imposing on-shell. The constitution relations can be shown to be given by ([14])

$$\begin{aligned}
\delta\pi &= -\zeta_\perp\Delta^{\mu\nu}\nabla_\mu u_\nu - \zeta_\times h^\mu h^\nu\nabla_\mu u_\nu \\
\delta\xi &= -\zeta_\times\Delta^{\mu\nu}\nabla_\mu u_\nu - \zeta_\parallel h^\mu h^\nu\nabla_\mu u_\nu \\
\ell^\mu &= -2\eta_\parallel\Delta^{\mu\sigma}h^\nu\nabla_{(\sigma}u_{\nu)} \\
m^\mu &= -2r_\perp\Delta^{\mu\beta}h^\nu\left(T\nabla_{[\beta}\left(h_{\nu]}\frac{\mu}{T}\right) + u_\sigma H^\sigma{}_{\beta\nu}\right) \\
t^{\mu\nu} &= -2\eta_\perp\left(\Delta^{\mu\rho}\Delta^{\nu\sigma} - \frac{1}{2}\Delta^{\mu\nu}\Delta^{\rho\sigma}\right)\nabla_{(\rho}u_{\sigma)} \\
s^{\mu\nu} &= 2r_\parallel\Delta^{\mu\rho}\Delta^{\nu\sigma}(\mu\nabla_{[\rho}h_{\sigma]} + u_\lambda h^\lambda{}_{\rho\sigma}),
\end{aligned}$$

where the transport coefficients $\zeta_\perp$, $\zeta_\times$, $\zeta_\parallel$, $\eta_\parallel$, $r_\parallel$, $\eta_\perp$ and $r_\perp$ are given by eq. (14.26) with eq. (15.1). These constitution relations define the Landau frame, and it is possible to investigate the Alfvén and Magnetosonic channel, which will be considered next.

15.1 Alfvén channel in the Landau frame

The Alfvén channel describes the shear modes in MHD, and similar for hydrodynamics the perturbation of the fluid velocity is perpendicular to the wave vector. Similar to the classical description, h^μ can be considered being along the z-component such that $h^\mu = (0,0,0,1)$. Then, under a general field transformation according to eq. (14.12), δh^μ can be split into components parallel and perpendicular to k^μ. Then since the thermodynamics fields decouples in the Alfvén channel the following relations holds

$$k_\mu \delta u^\mu = k_\mu \delta h^\nu = 0 \ , \quad h_\mu \delta u^\mu = u_\mu \delta h^\mu = 0 \ , \quad \delta T = \delta \mu = 0. \tag{15.2}$$

The perturbations of the fields are proportional to $\exp(ik^\mu x_\mu)$, where without loss of generality set

$$k^\mu = (\omega, \kappa \sin\theta, 0, \kappa\cos\theta). \tag{15.3}$$

We then define $\Delta \equiv \delta \exp(ik^\mu x_\mu)$, and the field transformation reads

$$T \to T + \Delta T \ , \quad \mu \to \mu + \Delta\mu \ , \quad u^\mu \to u^\mu + \Delta u^\mu \ , \quad h^\mu \to h^\mu + \Delta h^\mu. \tag{15.4}$$

We write in details the different contractions of u_ν, h_ν and $\Delta^\rho{}_\nu$ in Appendix D, and will here just comment on the results, and only write the equations that do not automatically vanish due to the relations in eqs. (15.2). Recall that for $g_{\mu\nu} = \eta_{\mu\nu}$, the covariant derivative $\nabla_\mu \to \partial_\mu$, and this implies that $\nabla_\mu \to ik_\mu$. The zeroth-order MHD contracted with the fluid velocity u_μ and h_μ does not contribute in the Alfvén channel since $k_\mu \Delta u^\mu$ and $k_\mu \Delta h^\mu$ equals zero. Thus for the zeroth-order approximation, the following relations holds

$$u_\nu \nabla_\mu T^{\mu\nu}_{(0)} = 0 \ , \quad u_\nu \nabla_\mu J^{\mu\nu}_{(0)} = 0 \ , \quad h_\nu \nabla_\mu T^{\mu\nu}_{(0)} = 0 \ , \quad h_\nu \nabla_\mu J^{\mu\nu}_{(0)} = 0.$$

The only contributing equations in the Alfvén channel are the momentum equations. Thus, contracting with the projection tensor $\Delta^\rho{}_\nu$ gives

$$\begin{aligned}
\Delta^\rho{}_\nu \nabla_\mu T^{\mu\nu}_{(0)} &= \Delta^\rho{}_\nu \nabla_\mu((\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu} - \mu\rho h^\mu h^\nu) \\
&= (\varepsilon + p)\Delta^\rho{}_\nu \nabla_\mu(u^\mu u^\nu) - \Delta^\rho{}_\nu \mu\rho \nabla_\mu (h^\mu h^\nu) \\
&= (\varepsilon + p)u^\mu \Delta^\rho{}_\nu \nabla_\mu u^\nu - \mu\rho h^\mu \Delta^\rho{}_\nu \nabla_\mu h^\nu \\
&= i(\varepsilon + p)u^\mu k_\mu \Delta^\rho{}_\nu \Delta u^\nu - i\mu\rho h^\mu k_\mu \Delta^\rho{}_\nu \Delta h^\nu \\
&= i\exp(ik^\mu x_\mu)(-(\varepsilon + p)\omega \Delta^\rho{}_\nu \delta u^\nu - \mu\rho\kappa\cos\theta \Delta^\rho{}_\nu \delta h^\nu) \\
\Delta^\rho{}_\nu \nabla_\mu J^{\mu\nu}_{(0)} &= \Delta^\rho{}_\nu \nabla_\mu (2\rho u^{[u} h^{\nu]}) \\
&= \rho \Delta^\rho{}_\nu \nabla_\mu (u^\mu h^\nu) - \rho \Delta^\rho{}_\nu \nabla_\mu (u^\nu h^\mu) \\
&= \rho u^\mu \Delta^\rho{}_\nu \nabla_\mu h^\nu - \rho h^\mu \Delta^\rho{}_\nu \nabla_\mu u^\nu \\
&= i\rho(u^\mu k_\mu \Delta h^\rho - \rho h^\mu k_\mu \Delta u^\rho) \\
&= i\rho\exp(ik^\mu x_\mu)(-\omega \Delta^\rho{}_\nu \delta h^\nu - \kappa\cos\theta \Delta^\rho{}_\nu \delta u^\nu).
\end{aligned}$$

There are two equations in the linear system, which is evident from the restraints of eqs. (15.2) since that in the Alfvén Channel, the perturbations of the magnetic field and the four velocity are only non-zero in its y-component i.e., $\delta u^\mu = (0, 0, \delta u^2, 0)$ and $\delta h^\mu = (0, 0, \delta h^2, 0)$. The eigenfrequency ω of the zeroth-order approximation for the Landau frame in the Alfvén channel is determined by the linear system $M_{ij}\delta^i = 0$ with $\delta^i = (\delta u^2, \delta h^2)$, and the Matrix is given by

$$M_{ij} = \begin{pmatrix} (p+\varepsilon)\omega & \kappa\mu\rho\cos\theta \\ \kappa\rho\cos\theta & -\rho\omega \end{pmatrix},$$

while the determinant of the Matrix of zeroth-order is given by

$$(p + \varepsilon)\omega^2 + \kappa^2 \mu\rho\cos^2\theta = 0. \tag{15.5}$$

Dividing by $\varepsilon + p$ and defining $\mathcal{V}_A \equiv \frac{\mu\rho}{\varepsilon + p}$ the zeroth-order approximation then satisfy

$$\omega = \mathcal{V}_A \kappa \cos\theta, \tag{15.6}$$

where $\mathcal{V}_A$ is the Alfvén velocity, and for the relativistic case, it corresponds to the tension of the strings. This is expected from the classical description that the fluid elements propagates along the magnetic field lines, i.e the strings. The first-order approximation can be obtained similarly. It is written explicitly in Appendix D. First consider the quantities $\delta\pi$ and $\delta\xi$ in the Alfvén channel. They consist of two terms that are $\sim \Delta^{\mu\nu}\nabla_\mu u_\nu$ and $\sim h^\mu h^\nu \nabla_\mu u_\nu$. Both of these terms must vanish in the Alfvén channel, since $\nabla_\mu \to ik_\mu$ and the wave vector does not have any y-components, by using that $\Delta^{\mu\nu}\nabla_\mu u_\nu = \nabla_\mu u^\mu$ then

$$\Delta^{\mu\nu}\nabla_\mu u_\nu = i\Delta^{\mu\nu}k_\mu \Delta u_\mu = ik_\mu \Delta u^\mu = 0,$$

while the other term vanishes due to $h^\mu \Delta u_\mu = 0$. For the Alfvén channel the Bulk viscosity is not included, and all the the constitution scalars can be neglected. Thus by contracting the projection tensor $\Delta^\rho{}_\nu$ for the first-order approximation we obtain

$$\begin{aligned}
\Delta^\rho{}_\nu T^{\mu\nu}_{(1)} &= \Delta^\rho{}_\nu \nabla_\mu(\delta\pi\Delta^{\mu\nu} + \delta\tau h^\mu h^\nu + 2\ell^{(\mu}h^{\nu)} + t^{\mu\nu}) \\
&= -\Delta^\rho{}_\nu h^\mu \nabla_\mu \ell^\nu + \Delta^\rho{}_\nu \nabla_\mu t^{\mu\nu} \\
&= \eta_\parallel \Delta^\rho{}_\nu \Delta^{\nu\sigma} h^\mu h^\lambda \nabla_\mu(\nabla_\sigma u_\lambda - \nabla_\lambda u_\sigma) - \eta_\perp \Delta^\rho{}_\nu(\Delta^{\mu\lambda}\Delta^{\nu\sigma} - \frac{1}{2}\Delta^{\mu\nu}\Delta^{\lambda\sigma})\nabla_\mu(\nabla_\lambda u_\sigma - \nabla_\sigma u_\lambda) \\
&= -\eta_\parallel \Delta^{\rho\sigma} h^\mu k_\mu h^\lambda k_\lambda \Delta u_\sigma + \eta_\perp \Delta^{\mu\lambda}\Delta^{\rho\sigma}(k_\mu k_\lambda \Delta u_\sigma - k_\mu k_\sigma \Delta u_\lambda) \\
&= \kappa^2(\eta_\parallel \cos^2\theta + \eta_\perp \sin^2\theta)\Delta^{\rho\sigma}\Delta u_\sigma \\
&= \exp(ik^\mu x_\mu)\kappa^2(\eta_\parallel \cos^2\theta + \eta_\perp \sin^2\theta)\Delta^{\rho\sigma}\delta u_\sigma \\
\Delta^\rho{}_\nu \nabla_\mu J^{\mu\nu}_{(1)} &= \Delta^\rho{}_\nu \nabla_\mu(2m^{[\mu}h^{\nu]} + s^{\mu\nu}) \\
&= -\Delta^\rho{}_\nu h^\mu \nabla_\mu m^\nu + \Delta^\rho{}_\nu \nabla_\mu s^{\mu\nu} \\
&= \mu r_\perp \Delta^\rho{}_\nu \Delta^{\nu\sigma} h^\mu h^\lambda \nabla_\mu(\nabla_\sigma h_\lambda - \nabla_\lambda h_\sigma) - r_\parallel \Delta^\rho{}_\nu \Delta^{\nu\sigma}\Delta^{\mu\lambda}\mu\nabla_\mu(\nabla_\lambda h_\sigma - \nabla_\sigma h_\lambda) \\
&= -\mu r_\perp \Delta^{\mu\rho} k_\mu h^\lambda k_\lambda \Delta h_\sigma - \mu r_\parallel \Delta^{\mu\lambda}\Delta^{\rho\sigma} k_\mu k_\lambda \Delta h_\sigma \\
&= \kappa^2\mu(r_\perp \cos^2\theta + r_\parallel \sin^2\theta)\Delta^{\sigma\rho}\Delta u_\sigma \\
&= \exp(ik^\mu x_\mu)\kappa^2\mu(r_\perp \cos^2\theta + r_\parallel \sin^2\theta)\Delta^{\sigma\rho}\delta u_\sigma,
\end{aligned}$$

where the relation in eqs.(15.2) haven been used. It is convenient to define

$$\mathcal{T}_r \equiv r_\perp \cos^2\theta + r_\parallel \sin^2\theta\ , \quad \mathcal{T}_\eta \equiv \eta_\parallel \cos^2\theta + \eta_\perp \sin^2\theta. \tag{15.7}$$

Then, adding the contributing equation with the first-order approximation and multiplying with $i\exp(-ik^\mu x_\mu)$ gives the linear system

$$\omega(\varepsilon+p)\delta u^2 + \mu\rho\kappa\cos\theta\delta h^2 + i\kappa^2\mathcal{T}_\eta\delta u^2 = 0 \tag{15.8}$$

$$\omega\delta h^2 + \rho\kappa\cos\theta\delta u^2 + i\mu\kappa^2\mathcal{T}_r\delta h^2 = 0, \tag{15.9}$$

that can be written in Matrix form as

$$M_{ij} = \begin{pmatrix} (\varepsilon+p)\omega + i\kappa^2\mathcal{T}_\eta & \mu\rho\kappa\cos\theta \\ \rho\kappa\cos\theta & \rho\omega + i\mu\kappa^2\mathcal{T}_r \end{pmatrix}, \tag{15.10}$$

with its respective determinant

$$\rho(p+\varepsilon)\omega^2 + i(\mu(p+\varepsilon)\mathcal{T}_r - \rho\mathcal{T}_\eta)\kappa^2\omega - \kappa^2(\mu\kappa^2\mathcal{T}_\eta\mathcal{T}_r + \mu\rho^2\cos^2\theta) = 0. \tag{15.11}$$

Solving for ω gives the dispersion relation to $\mathcal{O}(\kappa^2)$ reads

$$\omega = \pm\mathcal{V}_A\kappa\cos^2\theta - \frac{i}{2}\left(\frac{1}{\varepsilon+p}\mathcal{T}_\eta + \frac{\mu}{\rho}\mathcal{T}_r\right)\kappa^2 \tag{15.12}$$

Letting the viscosity and resistivity going towards zero gives the zeroth order dispersion relation as expected. It is also possible to find two diffusive modes: however, it was found in [40] that these modes are not physical. The reason for this, is that they are first found by letting $\theta \to \pi/2$ and then $\kappa \to 0$. But, the Alfvén waves can not propagate perpendicular to the magnetic field lines, and $\theta \to \pi/2$ is unphysical.

15.2 Landau: Alfvén channel in boosted frame

The Alfvén channel has two gapped and gapless modes for a spatial velocity $\beta^i \neq 0$. This can be realised by substituting eq. (10.4a) and (10.4b) together with $h^\mu \to h^{\mu'}$. To see this, consider $h^\mu = (h_t, h^i)$: they transform similar to k^μ, such that

$$h_t = \gamma(h_t' - \beta^i h_i')\ , \quad h^i = h^{i'} + \gamma\left(\frac{\gamma}{1+\gamma}\beta^i h_i' - h_t'\right)\beta^i. \tag{15.13}$$

However, $h_t = 0$ and the magnetic field and fluid velocity satisfy $h^\mu u_\mu = 0$. For frames with non-vanishing β^i it corresponds to stating $\beta^i h_i' = 0$. The first equality then reads $h_t' = 0$

and the second equality reduces to $h^i \to h^{i'}$. The determinant in the boosted frame then reads

$$\gamma^2 w((\omega')^2 - (\beta^i k_i')^2) - k^2\mu\rho^2\cos^2\theta - i\rho\gamma k^2\beta^i k_i'\mathcal{T}_\eta - k^2\mu\mathcal{T}_\eta(i\gamma w\beta^i k_i' + k^2\mathcal{T}_\eta) = 0, \quad (15.14)$$

where $k^2 = \kappa^2$ and should be read according to eq. (10.4b). The gapped modes follow a dispersion relation $\omega_G' = \omega_0 + \omega_1\beta^i k_i$, where ω_0 is found by letting $\kappa \to 0$, in which case the determinant reduces to

$$\rho w - \beta^2\mu\rho^2\cos^2\theta + i\beta^2\gamma\omega'(\rho\mathcal{T}_\eta + \mathcal{T}_r(w + i\beta^2\gamma\omega'\mathcal{T}_\eta)) = 0. \quad (15.15)$$

Setting $\omega' = \omega_0$, and solving for ω_0 gives

$$\omega_0 = \frac{i}{2}\frac{\sqrt{1-\beta^2}(w\mu\mathcal{T}_r + \rho\mathcal{T}_\eta \pm \mathcal{D})}{\beta^2\mu\mathcal{T}_r\mathcal{T}_\eta}, \quad (15.16)$$

where $\mathcal{D}$ is defined as

$$\mathcal{D}^2 \equiv (w\mu\mathcal{T}_r - \rho\mathcal{T}_\eta)^2 + 4\mathcal{T}_\eta\mathcal{T}_r\beta^2\mu^2\cos^2\theta.$$

We were unable to find a compact way in writing the first-order term for ω_G'. However, by letting $\theta \to \pi/2$, then $\mathcal{T}_r = r_\parallel$ and $\mathcal{T}_\eta = \eta_\perp$: the two gapped modes reduce to

$$\begin{aligned}\omega_G^- &= i\frac{\sqrt{1-\beta^2}}{\beta^2}\frac{p+\varepsilon}{\eta_\perp} + \frac{(2-\beta^2)}{\beta^2}\beta^i k_i' \\ \omega_G^+ &= i\frac{\sqrt{1-\beta^2}}{\beta^2}\frac{\rho}{\mu r_\parallel} + \frac{2-\beta^2}{\beta^2}\beta^i k_i',\end{aligned} \quad (15.17)$$

Note that this is inconsistent with the stability criteria in eq. (4.1), since that for Im $\omega < 0$, it requires that $\eta_\perp$ and $r_\parallel$ to be negative, which is inconsistent with the inequality from the entropy current. Furthermore, these modes do not allow one to return to an unboosted frame since both terms go towards infinity when $\beta^i \to 0$. Which is most likely due to a bad choice of frame.

15.3 Magnetosonic channel in the Landau frame

The magnosonic channel consists of the longitudinal components of the fluid velocity u^μ and the magnetic field h^μ. The perturbations δu^μ and δh^μ are spanned by the equilibrium values u^μ, h^μ and the wave vector k^μ such that

$$\delta u^\mu = Au^\mu + Bk^\mu + Ch^\mu \ , \quad \delta h^\mu = Dk^\mu + Fu^\mu + Gh^\mu .$$

Using now that $u_\mu \delta u^\mu = 0$, and $h_\mu \delta h^\mu = 0$, and setting $B = \delta U_1$, $C = \delta U_2$, $D = \delta H_1$ and $F = \delta H_2$ the above relations reads

$$\delta u^\mu = (k^\mu - \omega u^\mu)\delta U_1 + h^\mu \delta U_2 \ , \quad \delta h^\mu = (k^\mu - \kappa \cos\theta h^\mu)\delta H_1 + u^\mu \delta H_2 .$$

Furthermore the normalisation in eq. (14.13) impose that $u_\mu \delta h^\mu + h_\mu \delta u^\mu = 0$, which implies

$$\delta H_2 = \kappa \cos\theta \delta U_1 - \omega \delta H_1 .$$

The amplitudes for the sound channel can then be written as

$$\begin{aligned} \delta u^\mu &= (k^\mu - \omega u^\mu)\delta U_1 + h^\mu \delta U_2 \\ \delta h^\mu &= (k^\mu - \kappa \cos\theta h^\mu - \omega u^\mu)\delta H_1 + u^\mu(\kappa \cos\theta \delta U_1 + \delta U_2), \end{aligned} \tag{15.18}$$

the magnetosonic channel is parameterised by $\delta U_1, \delta U_2, \delta H_1, \delta T$ and $\delta\mu$ that satisfies the following relations

$$\begin{gathered} \Delta^\rho{}_\mu \delta u^\mu = \Delta^\rho{}_\mu k^\mu \delta U_1 \ , \quad k_\mu \delta u^\mu = \kappa^2 \delta U_1 + \kappa \cos\theta \delta U_2 \\ \Delta^\rho{}_\mu \delta h^\mu = \Delta^\rho{}_\mu k^\mu \delta H_1 \ , \quad k_\mu \delta h^\mu = \kappa^2 \sin^2\theta \delta H_1 - \omega(\cos\theta \delta U_1 + \delta U_2) \\ h_\mu \delta u^\mu = \kappa \cos\theta \delta U_1 + \delta U_2 \ , \quad u_\mu \delta h^\mu = -\kappa \cos\theta \delta U_1 - \delta U_2 . \end{gathered} \tag{15.19}$$

For the sound channel all the constitution relations are non-vanishing, except for the anti-symmetric part of the 2-form current, $s^{\mu\nu}$. This is explicitly seen by

$$\begin{aligned} s^{\mu\nu} &= -2r_\| \Delta^{\mu\rho}\Delta^{\nu\sigma}\mu \nabla_{[\rho} h_{\sigma]} \\ &= -i2r_\| \Delta^{\mu\rho}\Delta^{\nu\sigma}(k_\mu \Delta h_\sigma - k_\sigma \Delta h_\rho) \\ &= -i2r_\| \Delta^{\mu\rho}\Delta^{\nu\sigma}(k_\mu k_\sigma \Delta H_1 - k_\sigma k_\rho \Delta H_1) \\ &= 0. \end{aligned}$$

The linear system is then found by contracting $\nabla_\mu T^{\mu\nu}$ and $\nabla_\mu J^{\mu\nu}$ with the fluid velocity u_ν, the magnetic field h_ν and the projection tensor $\Delta^\rho{}_\nu$. For the magnetosonic channel, it is convenient to consider relation of the entropy current. Thus, the linear system consist of the following contractions

$$\begin{aligned} u_\nu \nabla T^{\mu\nu} + \mu h_\nu \nabla_\mu J^{\mu\nu} &= 0 \\ h_\nu \nabla_\mu T^{\mu\nu} &= 0 \\ u_\nu \nabla_\mu J^{\mu\nu} &= 0 \\ \Delta^\rho{}_\nu \nabla_\mu T^{\mu\nu} &= 0 \\ \Delta^\rho{}_\nu \nabla_\mu J^{\mu\nu} &= 0. \end{aligned}$$

Where the first expression in the zeroth-approximation corresponds to considering $T\nabla_\mu(su^\mu) = 0$. We write the calculation of these equations explicitly in Appendix E and contracted equations reads

$$\begin{aligned} T(s\delta U_1\kappa^2 - (c\delta T + \delta\mu\lambda)\omega + s\delta U_2\kappa\cos\theta) - i\mu r_\perp(\delta\mu - \frac{\mu}{T}\delta T - \mu\kappa\cos\theta\delta H_1) &= 0 \\ \omega T s\delta U_1 - \kappa\cos\theta(\mathcal{A} - \mu\rho\kappa^2\sin^2\theta\delta H_1) - \kappa^2(\cos^2\theta(\delta U_1 + \kappa\cos\theta)\zeta_\parallel + \sin^2\theta\mathcal{B}) &= 0 \\ (\delta T\lambda + \delta\mu\chi)\cos\theta + \delta H_1\kappa\rho\sin^2\theta &= 0 \\ (s\delta T + \rho\delta\mu) - (\varepsilon + p)\omega\delta U_1 - \mu\rho\kappa\cos\theta\delta H_1 - i\kappa\mathcal{C}\delta U_1 - 2\eta_\parallel\cos\theta(\cos\theta\delta U_1 + \delta U_2) &= 0 \\ \rho(\omega\delta H_1 + \kappa\cos\theta\delta U_1) + i\frac{r_\perp}{T}\kappa\cos\theta(T\delta\mu - \mu\delta T - \mu\kappa\cos\theta\delta H_1) &= 0, \end{aligned} \tag{15.20}$$

where

$$\begin{aligned} \mathcal{A} &= (s - \mu\lambda)\delta T - \omega T s\delta U_1 + \mu\chi\delta\mu \\ \mathcal{B} &= \delta U_2\eta_\parallel + \delta U_1\kappa\cos\theta(\zeta_\times + 2\eta_\parallel) \\ \mathcal{C} &= \zeta_\times\cos^2\theta + \sin^2\theta(\zeta_\perp + \eta_\perp). \end{aligned}$$

The Matrix is to long to be shown in this section, but for clarification, its components can be found in Appendix (F). Alternatively it can be read directly from eq.(15.20). Nevertheless, finding the determinant of the linear system allows to set up an expansion in powers of κ, to find the following dispersion relation

$$\omega = \pm v_\pm\kappa + i\tau\kappa^2, \tag{15.21}$$

where

$$v_{\pm} = \frac{1}{2}((\mathcal{V}_A^2 + \mathcal{V}_0^2)\cos^2\theta + \mathcal{V}_s^2\sin^2\theta) \pm \frac{1}{2}\sqrt{((\mathcal{V}_A^2 - \mathcal{V}_0^2)\cos^2\theta + \mathcal{V}_s^2\sin^2\theta)^2 + 4\mathcal{V}^4\cos^2\theta\sin^2\theta} \tag{15.22}$$

with

$$\mathcal{V}_0^2 \equiv \frac{s\chi}{T(c\chi - \lambda^2)}\ ,\quad \mathcal{V}_s^2 \equiv \frac{s^2\chi + c\rho^2 - 2s\rho\lambda}{(\varepsilon + p)(c\chi - \lambda^2)}\ ,\quad \mathcal{V}^4 = \frac{s(\rho\lambda - s\chi)^2}{T(c\chi - \lambda^2)^2(\varepsilon + p)}, \tag{15.23}$$

The results correspond to the findings in the classical description, and it can be read that there are a slow v_- and a fast v_+ mode. For $\theta = 0$, the fast mode $c_+ \to \mathcal{V}_0$ while for the slow mode $c_- \to \mathcal{V}_A$. For $\theta = \pi/2$, $c_+ \to \mathcal{V}_s$ and $c_- \to 0$. For the classical description at $\theta \to \pi/2$ implied that the fast mode was equal to the sound speed. For this reason $\mathcal{V}_s$ can be seen as a sort of speed of sound for the magnetosonic channel for the fast mode.

The first order-term τ, are as well to long to be presented, but it simplifies greatly for specific angles and different modes

$$\begin{aligned}
\tau(\mathcal{V}_0, \theta = 0) &= \frac{\zeta_\parallel}{2sT} \\
\tau(\mathcal{V}_A, \theta = 0) &= \frac{1}{2}\left(\frac{\mu}{\rho}r_\perp + \frac{\eta_\parallel}{\varepsilon + p}\right) \\
\tau(\mathcal{V}_s, \theta = \pi/2) &= -\frac{1}{2}\frac{(\rho(cT + \lambda\mu) - s(T\lambda + \mu\chi))^2}{T^2(\lambda^2 - c\chi)(c\rho^2 + s^2\chi - 2s\lambda\rho)} + \frac{\zeta_\perp + \eta_\perp}{\varepsilon + p} \\
\tau(0, \theta = \pi/2) &= \frac{1}{2}\left(\frac{\eta_\parallel}{sT} + \frac{r_\perp(\varepsilon + p)^2}{T^2(s^2\chi + \rho^2 c - 2\rho s\lambda)}\right),
\end{aligned} \tag{15.24}$$

here when writing $\tau(\mathcal{V}_0)$ it corresponds to setting all the other modes to zero. Finally two diffusive modes can again be found by first setting $\theta = \pi/2$, and then $\kappa \to 0$. However, due to their unphysical nature they are left out of the computation. The procedure that have been provided is analogous to the general frame, which has been done in [14] and for the purpose of the discussion, their findings will be presented and commented.

16 MHD modes in the General frame

For the Landau frame, the two normal modes were found, the Alfvén and magnetosonic channel. The same procedure can be considered for the general frame determined by eq.(14.10), with its respective constitution relations. The general frame consists of a set of stable and causal frames, as was also evident in the case of hydrodynamics. However, the general frame for MHD is more involved than the Landau frame. Due to the lack of time, we were unfortunately unable to derive the normal modes of the general frame, but we will present the findings for the Alfvén channel in [14] and comment on the different modes. To compare the Landau frame from the general frame, it is only necessary to consider the first-order relations, since the zeroth-order is independent of the transport coefficients. The equation of interest in the general frame is given by

$$\begin{aligned}
u_\nu \nabla_\mu T^{\mu\nu}_{(1)} + \mu h_\nu \nabla_\mu J^{\mu}_{(1)} &= -u^\mu \nabla_\mu \delta\varepsilon - h^\mu \nabla_\mu \delta\chi - \nabla_\mu k^\mu + \mu u^\mu \nabla_\mu \delta\tau + \mu \nabla_\mu m^\mu \\
h_\nu \nabla_\mu T^{\mu\nu}_{(1)} &= h^\mu \nabla_\mu \delta\xi - u^\mu \nabla_\mu \delta\chi + \nabla_\mu \ell^\mu \\
u_\nu \nabla_\mu J^{\mu\nu}_{(1)} &= h^\mu \nabla_\mu \delta\tau - \nabla_\mu n^\mu \\
\Delta^\rho{}_\nu \nabla_\mu T^{\mu\nu}_{(1)} &= \Delta^{\rho\mu} \nabla_\mu \delta\pi + \Delta^\rho{}_\nu h^\mu \nabla_\mu \ell^\nu + \Delta^\rho{}_\nu u^\mu \nabla_\mu k^\nu + \Delta^\rho{}_\nu \nabla_\mu t^{\mu\nu} \\
\Delta^\rho{}_\nu \nabla_\mu J^{\mu\nu}_{(1)} &= -\Delta^\rho{}_\nu h^\mu \nabla_\mu m^\nu - \Delta^\rho{}_\nu u^\mu \nabla_\mu n^\nu + \Delta^\rho{}_\nu \nabla_\mu s^{\mu\nu}.
\end{aligned} \tag{16.1}$$

Comparing these equations with the Landau frame, the difference lies in the vanishing four additional terms containing the constitution relation. The work carried out in [14] a flat background with no external source was considered. For the Alfvén channel the only contributing equations are

$$\begin{aligned}
\Delta^\rho{}_\nu \nabla_\mu T^{\mu\nu}_{(1)} &= \Delta^\rho{}_\nu h^\mu \nabla_\mu \ell^\nu + \Delta^\rho{}_\nu u^\mu \nabla_\mu k^\nu + \Delta^\rho{}_\nu \nabla_\mu t^{\mu\nu} \\
\Delta^\rho{}_\nu \nabla_\mu J^{\mu\nu}_{(1)} &= -\Delta^\rho{}_\nu h^\mu \nabla_\mu m^\nu - \Delta^\rho{}_\nu u^\mu \nabla_\mu n^\nu + \Delta^\rho{}_\nu \nabla_\mu s^{\mu\nu}.
\end{aligned} \tag{16.2}$$

By writing the derivation and solving the linear system, the Alfvén mode emits two gapless and two gapped modes. The gapless modes are in the same form as eq. (15.12) where $\eta_\parallel, \eta_\perp, r_\parallel$ and $r_\perp$ now defined by eq. (14.26). The gapless modes are [14]

$$\omega = i\frac{\rho}{\mu n_2}\ , \quad \omega = i\frac{(\varepsilon + p)}{k_2}. \tag{16.3}$$

They are not present in the Landau frame for $\beta^i = 0$, and in the boosted frame they would go towards infinity when the spatial velocity going towards zero. The stability criteria then requires that

$$n_2 < 0\ , \quad k_2 < 0. \tag{16.4}$$

While in the Landau frame, one set $n_2 = k_2 = 0$. Thus it is clear that the Landau Frame is not stable in the Alfvén Channel. Furthermore, gapped modes also appear in the magnetosonic channel, and here the stability criteria also requires that $k_2 < 0$, and the Landau frame is also unstable in the magnetosonic channel. It was shown by [14] that the general frame emits stable and causal frames.

17 Discussion

In this thesis, we have briefly reviewed the classical description of fluids and plasmas to draw parallels with the relativistic case. The BDNK theory has been applied to hydrodynamics and MHD by writing them as a gradient expansion in a flat background. In order to study the linear stability and causality of hydrodynamics, a general frame has been studied. The transport coefficients of this frame have been constrained by an equilibrium configuration and a covariant version of the second law of thermodynamics. We have then studied the linear stability and causality for the general frame in hydrodynamics by considering the small perturbations out of equilibrium that correspond to transforming the involved fields. This gives a linear system, where the two normal modes (shear and sound modes), have been found for small wave numbers. The linear stability and causality criteria of these two modes have then been reviewed, and the determinant allows us to study the linear stability and causality for arbitrary wavelengths. For magnetohydrodynamics, a similar approach was taken. Namely, first, a definition of plasma was reviewed, and the classical description of the conservation equation governing MHD. Then the relativistic MHD was described as an EFT, following analogously to the BDNK theory for hydrodynamics. An equilibrium configuration allowed to describe physics interpretation of the conservation equation to a zeroth-order. Lastly, the linear system was found in the Landau frame and was shown to be inconsistent and not to satisfy stability and causality as expected. The results from [14] were then reviewed, showing that the general frame for MHD is stable and causal. The Eckart and Landau frame lead to non-sensible physics since both frames' stability and causality criteria are not satisfied. The BDNK theory shows that a general frame can be derived, confirmed by a kinetic theory as well in [32]. Without applying the Israel-Stewart theory and introducing extra degrees of freedom. The theories are guided by symmetries on the effective action and give conserved quantities up to a first-order approximation that agrees with the classical description. Namely, for hydrodynamics, the governing equation is given by a continuity equation, momentum equation and energy equation. Similar to MHD, with the extra equation due to coupling of the electromagnetic fields. The constraints imposed by the transport coefficients from the normal modes ensure causality and stability and give rise to a set of frames. The Eckart and Landau frame can be derived directly, and the constraints show that it is inconsistent. This agrees with previous studies in hydrodynamics [9]. To our knowledge, the general frame for MHD has not been derived from the relativistic Boltzmann equation. Nevertheless, the BDNK theory gives sensible physics: Firstly, as already stated, the general frame is stable and causal. Secondly, the contractions of the conserved quantities corresponds to those in the classical description, and both for hydrodynamics and MHD the entropy current is conserved at zeroth-order.

Furthermore, for the description of MHD, no assumption of the coupling constant or the separation of the energy-momentum tensor is needed. Thus, it gives a more complete theory since the coupling of matter and electromagnetic fields are retained. However, it should be noted that hydrodynamics and MHD remain toy models, and their prediction power may not be sufficient enough to predict physics phenomena in astrophysical settings. Rather or not this is true is unclear to us. The future outlook for hydrodynamics and MHD, is a bit exciting. We hope that the BDNK theory allows for a better understanding of the physical effects of dissipative hydrodynamics and MHD in various situations, for example, in shocks. Furthermore, studying the stability and causality criteria for an arbitrary metric could be interesting and likewise, with an external background source. Lastly, we would find it interesting to apply the force-free approximation for the general frame, to first study the linear stability and causality criteria, and hopefully apply a stable MHD theory to the magnetosphere for pulsars. This could give new insights and a better understanding of how pulsars work.

From the above discussion, we conclude that both hydrodynamics and MHD, in their general frames, give a complete toy model of fluids and plasmas. The constraints imposed by the general frame should be used as guidelines of what frames to choose. Finally, we think this modern approach can lead to a better understanding of various astrophysical settings.

A Field transformations for hydrodynamics

For the field transformations, only orders of $\mathcal{O}(\partial)$ is of interest. Therefore, $\delta u_\mu \delta u_\nu$ and $\delta u_\nu T^{\mu\nu}_{(1)}$ are neglected. The transformation of $\mathcal{P}$ is invariant up to a first-order, accordingly

$$\begin{aligned}
\mathcal{P} &= \frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu} \\
&= \frac{1}{3}\big((u_\mu + \delta u_\mu)(u_\nu + \delta u_\nu) + g_{\mu\nu}\big)(T^{\mu\nu} + \delta T^{\mu\nu}_{(0)}) \\
&= \frac{1}{3}(u_\mu u_\nu + g_{\mu\nu})T^{\mu\nu} + \frac{1}{3}(u_\mu \delta u_\nu T^{\mu\nu} + u_\nu \delta u_\mu T^{\mu\nu} + u_\mu u_\nu \delta T^{\mu\nu}_{(0)} + g_{\mu\nu}\delta T^{\mu\nu}_{(0)}) \\
&= \frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu} + \frac{1}{3}(u_\mu \delta u_\nu + u_\nu \delta u_\mu)T^{\mu\nu}_{(0)} \\
&= \frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu} = \mathcal{P}.
\end{aligned}$$

The terms in the second parenthesis cancel out, this is specifically seen by

$$\begin{aligned}
u_\mu \delta u_\nu T^{\mu\nu} &= u_\mu \delta u_\nu T^{\mu\nu}_{(0)} + u_\mu \delta u_\nu T^{\mu\nu}_{(1)} \\
&= u_\mu \delta u_\nu (\varepsilon + p) u^\mu u^\nu \\
&= 0 \\
u_\mu u_\nu \delta T^{\mu\nu}_{(0)} &= (\varepsilon + p) u_\mu u_\nu (\delta u^\mu u^\nu + u^\mu \delta u^\nu) \\
&= -(\varepsilon + p)(u_\mu \delta u^\mu + u_\nu \delta u^\nu) = 0 \\
g_{\mu\nu}\delta T^{\mu\nu}_{(0)} &= (\varepsilon + p) g_{\mu\nu}(\delta u^\mu u^\nu + u^\mu \delta u^\nu) \\
&= (\varepsilon + p)(u_\mu \delta u^\mu + u_\nu \delta u^\nu) = 0.
\end{aligned}$$

Similar $\mathcal{N}$ is invariant under these transformations

$$\begin{aligned}
\mathcal{N}' &= -u'_\mu J^{\mu\prime} \\
&= -(u_\mu + \delta u_\mu)(J^\mu + \delta J^\mu_{(0)}) \\
&= -u_\mu J^\mu - u_\mu \delta J^\mu_{(0)} \\
&= -u_\mu J^\mu \\
&= \mathcal{N},
\end{aligned}$$

where again $\delta u_\mu \delta J^\mu_{(0)}$ is of higher order and

$$u_\mu \delta J^\mu_{(0)} = \rho u_\mu \delta u^\mu = 0.$$

The transformation of the constitution vector q^μ becomes

$$\begin{aligned}
q^{\mu'} &= -\Delta^\mu{}_\nu u'_\rho T^{\nu\rho'} \\
&= -\Delta^\mu{}_\nu (u_\rho + \delta u_\rho)(T^{\nu\rho} + \delta T^{\nu\rho}_{(0)}) \\
&= -\Delta^\mu{}_\nu u_\rho T^{\nu\rho} - \Delta^\mu{}_\nu u_\rho \Delta T^{\nu\rho}_{(0)} \\
&= -\Delta^\mu{}_\nu u_\rho T^{\nu\rho} - \Delta^\mu{}_\nu u_\rho (\varepsilon + p)(\delta u^\nu u^\rho + u^\nu \delta u^\rho) \\
&= -\Delta^\mu{}_\nu u_\rho T^{\nu\rho} + (\varepsilon + p)\delta u^\mu \\
&= q^\mu + (\varepsilon + p)\delta u^\mu.
\end{aligned}$$

Here we have used that

$$\Delta^\mu{}_\nu \delta^\nu = (u^\mu u_\nu + g^\mu{}_\nu)\delta u^\nu = g^\mu{}_\nu \delta u^\nu = \delta u^\mu.$$

Similar for $j^{\mu'}$

$$\begin{aligned}
j^{\mu'} &= \Delta^\mu{}_\nu J^\nu \\
&= \Delta^\mu{}_\nu (J^\mu + \delta J^\mu_{(0)}) \\
&= \Delta^\mu{}_\nu J^\mu + \rho \Delta^\mu{}_\nu \delta u^\nu \\
&= j^\mu + \rho \delta u^\mu.
\end{aligned}$$

Lastly, the constitution tensor the energy-momentum tensor is replaced with $T^{\mu\nu} + \delta T^{\mu\nu}_{(0)}$, thus to show that $t^{\mu\nu'} = t^{\mu\nu}$, it is sufficient to consider

$$\begin{aligned}
\Delta^\mu{}_\lambda \Delta^\nu{}_\rho \delta T^{\lambda\rho}_{(0)} &= \Delta^\mu{}_\lambda \Delta^\nu{}_\rho \delta(\varepsilon + p)(\delta u^\lambda u^\rho + u^\lambda \delta u^\rho) = 0 \\
\Delta_{\lambda\rho} \delta T^{\lambda\rho}_{(0)} &= 0.
\end{aligned}$$

Thus, $t^{\mu\nu'} = t^{\mu\nu}$ under these transformations, since $T^{\mu\nu}$ and J^μ are invariant under these transport coefficient. It is enough to demand that the constitution vectors transform as eq. (6.14b).

B Hydrodynamics: Invariant transport coefficients

Consider first the invariant transport coefficients

$$f_i = \pi_i - \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \varepsilon_i - \left(\frac{\partial p}{\partial \rho}\right)_\varepsilon \tau_i \ , \quad \ell_i = l_i - \frac{\rho}{\varepsilon + p} r_i.$$

Using now the transformation relations in eqs. (6.16), then ℓ_i' reads

$$\begin{aligned}
\ell_i' &= l_i' - \frac{\rho}{\varepsilon+p} r_i' \\
&= l_i - \rho c_i - \frac{\rho}{\varepsilon+p} r_i + \frac{\rho}{\varepsilon+p}(\varepsilon+p)c_i \\
&= \ell_i.
\end{aligned}$$

For f_i' using again eqs. (6.16) reads

$$\begin{aligned}
f_i' &= \pi_i' - \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \varepsilon_i' - \left(\frac{\partial p}{\partial \rho}\right)_\varepsilon \tau_i' \\
&= \pi_i - \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \varepsilon_i - \left(\frac{\partial p}{\partial \rho}\right)_\varepsilon \tau_i + \mathcal{Y} a_i + \mathcal{X} b_i,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{Y} &= \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \left(\frac{\partial \varepsilon}{\partial T}\right)_\mu + \left(\frac{\partial p}{\partial \rho}\right)_\varepsilon \left(\frac{\partial \rho}{\partial T}\right)_\mu - \left(\frac{\partial p}{\partial T}\right)_\mu \\
\mathcal{X} &= \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \left(\frac{\partial \varepsilon}{\partial \mu}\right)_T + \left(\frac{\partial p}{\partial \rho}\right)_\varepsilon \left(\frac{\partial \rho}{\partial \mu}\right)_T - \left(\frac{\partial p}{\partial \mu}\right)_T.
\end{aligned}$$

For $f_i' = f_i$, then $\mathcal{Y} = \mathcal{X} = 0$ must be satisfied. From eq. (2.22) the following relations holds

$$\frac{\partial \varepsilon}{\partial \mu} = T\lambda + \mu\chi \ , \quad \frac{\partial \varepsilon}{\partial T} = Tc + \mu\lambda$$
$$\left(\frac{\partial p}{\partial T}\right)_\mu = s \ , \quad \left(\frac{\partial p}{\partial \mu}\right)_T = \rho,$$

where

$$\lambda = \frac{\partial s}{\partial \mu} = \frac{\partial \rho}{\partial T} \ , \quad \chi = \frac{\partial \rho}{\partial \mu} \ , \quad c = \frac{s}{\partial T}.$$

First consider $d\rho = 0$, then

$$d\rho = \chi d\mu + \lambda dT \implies dT = -\frac{\chi}{\lambda} d\mu,$$

then $(\partial p/\partial \varepsilon)_\rho$ reads

$$\begin{aligned}\left(\frac{\partial p}{\partial \varepsilon}\right)_\rho &= \frac{sdT+\rho d\mu}{T(cdT+\lambda d\mu)}\\ &= \frac{-s\frac{\chi}{\lambda}+\rho}{T(-c\frac{\chi}{\lambda}+\lambda)}\\ &= \frac{s\chi-\rho\lambda}{T(c\chi-\lambda^2)}\end{aligned}$$

and for $d\varepsilon = 0$ it reads

$$d\varepsilon = (Tc+\mu\lambda)dT + (\lambda T+\mu\chi)d\mu = 0 \implies dT = -\frac{T\lambda+\mu\chi}{cT+\mu\lambda}d\mu.$$

Then $(\partial p/\partial \rho)_\varepsilon$ reads

$$\begin{aligned}\left(\frac{\partial p}{\partial \varepsilon}\right)_\varepsilon &= \frac{sdT+\rho d\mu}{\chi d\mu+\lambda dT}\\ &= \frac{\lambda(sT-\rho\mu)+s\chi\mu-\rho Tc}{T(\lambda^2-\chi c)}.\end{aligned}$$

Substituting these relations into $\mathcal{Y}$ and $\mathcal{X}$ gives

$$\begin{aligned}\mathcal{Y} &= \frac{s\chi+\rho\lambda}{T(\chi c-\lambda^2)}(Tc+\mu\lambda) + \frac{\lambda(sT-\rho\mu)+\chi s\mu-\rho Tc}{T(\lambda^2-\chi c)}\lambda - s = 0\\ \mathcal{X} &= \frac{s\chi-\rho\lambda}{T(\chi c-\lambda^2)}(T\lambda+\mu\chi) + \frac{\lambda(sT-\rho\mu)+\chi s\mu-\rho Tc}{T(\lambda^2-\chi c)}\chi - \rho = 0.\end{aligned}$$

Thus $f'_i = f_i$, is invariant under these transformations.

C Covariant entropy current for uncharged fluids

The entropy current to a first-order hydrodynamics satisfies eq. (8.7, written again here as

$$\nabla_\mu S^\mu = -\mathcal{A}\delta\varepsilon - 3\mathcal{B}\delta\pi + 2q^\mu \mathcal{Q}_\mu + \frac{1}{2T}\sigma^{\mu\nu}\sigma_{\mu\nu}.$$

For simplicity consider each term separately. By using eq. (6.7), the first term reads

$$\begin{aligned}-\mathcal{A}\delta\varepsilon &= \left(-\frac{1}{T^2}u^\lambda\nabla_\lambda T\right)\left(\frac{\varepsilon_1}{T}u^\lambda\nabla_\lambda T + \varepsilon_2\nabla_\lambda u^\lambda\right)\\ &= -\frac{1}{T^3}\varepsilon_1\big(u^\lambda\nabla_\lambda T\big)^2 - \frac{1}{T^2}\varepsilon_2 u^\lambda\nabla_\lambda T\nabla_\sigma u^\sigma,\end{aligned}$$

and the second term

$$
\begin{aligned}
-3\mathcal{B}\delta\pi &= -3\left(\frac{1}{3T}\nabla_\lambda u^\lambda\right)\left(\frac{\pi_1}{T}u^\lambda\nabla_\lambda T + \pi_2\nabla_\lambda u^\lambda\right) \\
&= -\frac{1}{T^2}\pi_1\nabla_\lambda u^\lambda u^\sigma\nabla_\sigma T - \frac{1}{T}\pi_2\big(\nabla_\lambda u^\lambda\big)^2.
\end{aligned}
$$

The third term containing the constitution vector q^μ reads

$$
\begin{aligned}
2q^\mu\mathcal{Q}_\mu &= 2\left(r_1 u^\lambda\nabla_\lambda u^\mu + \frac{r_1}{T}\Delta^{\mu\lambda}\nabla_\lambda T\right)\left(-\frac{1}{2T^2}\Delta_\mu{}^\rho\nabla_\rho T - \frac{1}{2T}\Delta_\mu{}^\rho u^\lambda\nabla_\lambda u_\rho\right) \\
&= -\frac{r_1}{T^2}(u^\lambda\nabla_\lambda u^\mu)(\Delta_\mu{}^\rho\nabla_\rho T) - \frac{r_1}{T}\big(u^\lambda\nabla_\lambda u^\mu u^\sigma\nabla_\sigma u_\mu\big) - \frac{r_1}{T^3}\Delta^{\mu\rho}\Delta_\mu{}^\rho\nabla_\lambda T\nabla_\rho T - \frac{r_1}{T^2}\Delta^{\rho\lambda}\nabla_\lambda T u^\sigma\nabla_\sigma u_\rho \\
&= -\frac{2r_1}{T^2}(u^\lambda\nabla_\lambda u^\mu)(\Delta_\mu{}^\rho\nabla_\rho T) - \frac{r_1}{T}\big(u^\lambda\nabla_\lambda u^\mu u^\sigma\nabla_\sigma u_\mu\big) - \frac{r_1}{T^3}\Delta^{\mu\rho}\Delta_\mu{}^\rho\nabla_\lambda T\nabla_\rho T,
\end{aligned}
$$

and the last term remain the same. The on-shell relations needs to be taken into account, and since only uncharged-fluids are considered, then

$$
dp = sdT\ ,\quad d\varepsilon = Tds = TcdT.
$$

The energy conservation then reads

$$
-u^\mu\nabla_\mu\varepsilon - (\varepsilon+p)\nabla_\mu u^\mu = -Tcu^\mu\nabla_\mu T - Ts\nabla_\mu u^\mu = 0,
$$

solving for $u^\mu\nabla_\mu T$, and using that $v_s^2 = s/cT$ gives

$$
u^\mu\nabla_\mu T = -Tv_s^2\nabla_\mu u^\mu.
$$

The momentum equation reads

$$
(\varepsilon+p)u^\mu\nabla_\mu u^\rho + \Delta^{\mu\rho}\nabla_\mu p = Tsu^\mu\nabla_\mu u^\rho + s\Delta^{\mu\rho}\nabla_\mu T = 0,
$$

and solving for $\Delta^{\mu\rho}\nabla_\mu T$ gives

$$
\Delta^{\mu\rho}\nabla_\mu T = -Tu^\mu\nabla_\mu u^\rho.
$$

Inserting these on-shell relations into each term gives

$$
\begin{aligned}
-\mathcal{A}\delta\varepsilon &= -\frac{1}{T}v_s^4\varepsilon_1(\nabla_\mu u^\mu)^2 + \frac{1}{T}\varepsilon_2 v_s^2(\nabla_\mu u^\mu)^2 \\
-3\mathcal{B}\delta\pi &= \frac{1}{T}\pi_1 v_s^2\big(\nabla_\mu u^\mu\big)^2 - \frac{1}{T}\pi_2\big(\nabla_\mu u^\mu\big)^2.
\end{aligned}
$$

The vector term cancels out, and to see this explicitly we write

$$2q^\mu \mathcal{Q}_\mu = \frac{2}{T} r_1 \big(u^\lambda \nabla_\lambda u^\mu u^\sigma \nabla_\sigma u_\mu\big) - \frac{r_1}{T}\big(u^\lambda \nabla_\lambda u^\mu u^\sigma \nabla_\sigma u_\mu\big) - \frac{1}{T} r_1 \big(u^\lambda \nabla_\lambda u^\mu u^\sigma \nabla_\sigma u_\mu\big) = 0.$$

Note, that it only vanishes due to the equilibrium settings $r_1 = r_2$, and might not vanish at higher orders. Setting it all together gives

$$\nabla_\mu S^\mu = \frac{1}{T}(v_s^2(\varepsilon_2 + \pi_1) - \pi_2 - v_s^4 \varepsilon_1)\big(\nabla_\mu u^\mu\big)^2 + \frac{\eta}{2T}\sigma_{\mu\nu}\sigma^{\mu\nu}.$$

Multiplying with T on both sides, agrees with eq. (8.9).

D Derivation of Alfvén channel in the Landau Frame

The only contributing equations in the Alfvén channel are those contracted by the projection tensor. To see explicitly that the other terms vanishes, they are written out in this appendix. First recall that $\delta T = \delta\mu = 0$ and $\nabla_\mu u^\mu = ik_\mu \Delta u^\mu = ik_\mu h^\mu = 0$. The contraction of the fluid velocity for both conserved quantities gives

$$\begin{aligned}
u_\nu \nabla_\mu T^{\mu\nu}_{(0)} &= u_\nu \nabla_\mu((\varepsilon + p)u^\mu u^\nu + p g^{\mu\nu} - \mu\rho h^\mu h^\nu) \\
&= (\varepsilon + p)u_\nu \nabla_\mu (u^\mu u^\nu) - \mu\rho u_\nu \nabla_\mu (h^\mu h^\nu) \\
&= (\varepsilon + p)u_\nu u^\nu \nabla_\mu u^\mu + (\varepsilon + p)u_\nu u^\mu \nabla_\mu u^\nu - \mu\rho u_\nu h^\mu \nabla_\mu h^\nu \\
&= -i(\varepsilon + p)k_\mu \Delta u^\mu - i\mu\rho h^\mu k_\mu u_\nu \Delta h^\nu = 0 \\
u_\nu \nabla_\mu J^{\mu\nu}_{(0)} &= u_\nu \nabla_\mu (2\rho u^{[u} h^{\nu]}) \\
&= \rho u_\nu \nabla_\mu (u^\mu h^\nu) - \rho u_\nu \nabla_\mu (u^\nu h^\mu) \\
&= \rho u_\nu u^\mu \nabla_\mu h^\nu - \rho u_\nu h^\mu \nabla_\mu u^\nu - \rho u_\nu u^\nu \nabla_\mu h^\mu \\
&= \rho u_\nu u^\mu \nabla_\mu h^\nu - \rho u_\nu h^\mu \nabla_\mu u^\nu + \rho \nabla_\mu h^\mu \\
&= i\rho u_\nu u^\mu k_\mu \Delta h^\nu - \rho u_\nu h^\mu k_\mu \Delta u^\nu + \rho k_\mu \Delta h^\mu = 0,
\end{aligned}$$

where it has been used that $u_\mu \Delta h^\mu = h_\mu \Delta u^\mu$, and recall that due to the normalisation of u^μ and h^μ, then $h_\mu \Delta h^\mu = u_\mu \Delta u^\mu = 0$. Similar for the contractions by h_ν

$$\begin{aligned}
h_\nu \nabla_\mu T^{\mu\nu}_{(0)} &= h_\nu \nabla_\mu ((\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu} - \mu\rho h^\mu h^\nu) \\
&= (\varepsilon + p)h_\nu \nabla_\mu (u^\mu u^\nu) - \mu\rho h_\nu \nabla_\mu (h^\mu h^\nu) \\
&= (\varepsilon + p)h_\nu u^\mu \nabla_\mu u^\nu - \mu\rho \nabla_\mu h^\mu - \mu\rho h_\nu h^\mu \nabla_\mu h^\nu \\
&= i(\varepsilon + p)h_\nu u^\mu k_\mu \Delta u^\nu - i\mu\rho k_\mu \Delta h^\mu - i\mu\rho h_\nu h^\mu k_\mu \Delta h^\nu = 0 \\
h_\nu \nabla_\mu J^{\mu\nu}_{(0)} &= h_\nu \nabla_\mu (2\rho u^{[u} h^{\nu]}) \\
&= \rho h_\nu \nabla_\mu (u^\mu h^\nu) - \rho h_\nu \nabla_\mu (u^\nu h^\mu) \\
&= \rho \nabla_\mu u^\mu + \rho h_\nu u^\mu \nabla_\mu h^\nu - \rho h_\nu h^\mu \nabla_\mu u^\nu \\
&= i\rho k_\mu \Delta u^\mu + i\rho h_\nu u^\mu k_\mu \Delta h^\nu - i\rho h_\nu h^\mu_\mu \Delta u^\nu = 0.
\end{aligned}$$

Similar at first-order the contraction of the fluid velocity vanishes:

$$\begin{aligned}
u_\nu \nabla_\mu T^{\mu\nu}_{(1)} &= u_\nu \nabla_\mu (\delta f \Delta^{\mu\nu} + \delta\tau h^\mu h^\nu + 2\ell^{(\mu} h^{\nu)} + t^{\mu\nu}) \\
&= u_\nu \delta\tau \nabla_\mu (h^\mu h^\nu) + u_\nu \nabla_\mu (\ell^\mu h^\nu) + u_\nu \nabla_\mu (\ell^\nu h^\mu) \\
&= \delta\tau u_\nu h^\mu \nabla_\mu h^\nu + u_\nu \ell^\mu \nabla_\mu h^\nu + u_\nu h^\mu \nabla_\mu \ell^\nu \\
&= \delta\tau u_\nu h^\mu \nabla_\mu h^\nu + u_\nu \ell^\mu \nabla_\mu h^\nu - 2\eta_\parallel \Delta^{\mu\sigma} u_\nu h^\mu \nabla_\mu (h^\rho \nabla_{(\sigma} u_{\rho)})) \\
&= i\delta\tau u_\nu h^\mu k_\mu \Delta h^\nu + i u_\nu \ell^\mu k_\mu \Delta h^\nu = 0 \\
u_\nu \nabla_\mu J^{\mu\nu}_{(1)} &= u_\nu \nabla_\mu (2m^{[\mu} h^{\nu]} + s^{\mu\nu}) \\
&= u_\nu \nabla_\mu (m^\mu h^\nu) - u_\nu \nabla_\mu (m^\nu h^\mu) \\
&= u_\nu m^\mu \nabla_\mu h^\nu - u_\nu h^\mu \nabla_\mu m^\nu \\
&= u_\nu m^\mu \nabla_\mu h^\nu + 2r_\perp \Delta^{\nu\beta} u_\nu h^\mu \nabla_\mu (h^\rho T \nabla_{[\beta} (h_{\rho]} \mu / T)) \\
&= i u_\nu m^\mu k_\mu \Delta h^\nu = 0,
\end{aligned}$$

and by contracting h_ν

$$
\begin{aligned}
h_\nu \nabla_\mu T^{\mu\nu}_{(1)} &= h_\nu \nabla_\mu (\delta f \Delta^{\mu\nu} + \delta\tau h^\mu h^\nu + 2\ell^{(\mu} h^{\nu)} + t^{\mu\nu}) \\
&= h_\nu \nabla_\mu (\delta\tau) h^\mu h^\nu + \delta\tau h_\nu \nabla_\mu (h^\mu h^\nu) + h_\nu \nabla_\mu (\ell^\mu h^\nu) + h_\nu \nabla (\ell^\nu h^\mu) \\
&= h^\mu \nabla_\mu \delta\tau + \delta\tau \nabla_\mu h^\mu + \delta\tau h_\nu h^\mu \nabla_\mu h^\nu + \nabla_\mu \ell^\mu + h_\nu \ell^\mu \nabla_\mu h^\nu + h_\nu h^\mu \nabla_\mu \ell^\nu \\
&= h^\mu \nabla_\mu (-\zeta_x \Delta^{\rho\lambda} \nabla_\rho u_\lambda - \zeta_\| h^\rho h^\lambda \nabla_\rho u_\lambda) - 2\eta_\| \Delta^{\mu\sigma} \nabla_\mu (h^\rho \nabla_{(\sigma} u_{\rho)}) \\
&= h^\mu \nabla_\mu (-i\zeta_\times \Delta^{\rho\lambda} k_\rho \Delta u_\lambda - i\eta_\| h^\rho h^\lambda k_\rho \Delta u_\lambda) - \eta_\| \Delta^{\mu\sigma} \nabla_\mu (i h^\rho k_\sigma \Delta u_\rho - i h^\rho k_\rho \Delta u_\sigma) \\
&= -i\zeta_\times \Delta^{\rho\lambda} k_\rho h^\mu \nabla_\mu \Delta u_\lambda + i\eta_\| \Delta^{\mu\sigma} h^\rho \kappa_\rho \nabla_\mu \Delta u_\sigma \\
&= -i\zeta_\times (u^\rho u^\lambda + g^{\rho\lambda} - h^\rho h^\lambda) k_\rho h^\mu \nabla_\mu \Delta u_\lambda + i\eta_\| (u^\mu u^\sigma + g^{\mu\sigma} - h^\mu h^\sigma) h^\rho k_\rho \nabla_\mu \Delta u_\sigma \\
&= -i\zeta_\times k^\lambda h^\mu k_\mu \Delta u_\lambda + i\eta_\| h^\rho k_\rho k^\sigma \Delta u_\sigma = 0 \\
h_\nu \nabla_\mu J^{\mu\nu}_{(1)} &= h_\nu \nabla_\mu (2m^{[\mu} h^{\nu]} + s^{\mu\nu}) \\
&= h_\nu \nabla_\mu (m^\mu h^\nu) - h_\nu \nabla_\mu (m^\nu h^\mu) \\
&= \nabla_\mu m^\mu + h_\nu m^\mu \nabla_\mu h^\nu - h_\nu h^\mu \nabla_\mu m^\nu \\
&= -2r_\perp \Delta^{\mu\beta} \nabla_\mu (h^\rho T \nabla_{[\beta} (h_\rho \mu / T)) + i h_\nu m^\mu k_\mu \Delta h^\nu + 2r_\perp \Delta^{\nu\beta} h_\nu h^\mu \nabla_\mu (h^\rho T \nabla_{[\beta} (h_{\rho]} \mu / T)) \\
&= -2r_\perp \mu \Delta^{\mu\beta} \nabla_\mu (h^\rho \nabla_\beta h_\rho - h^\rho \nabla_\rho h_\beta) \\
&= -2r_\perp \mu \Delta^{\mu\beta} \nabla_\mu (i h^\rho k_\beta \Delta h_\rho - h^\rho k_\rho \Delta h_\beta) \\
&= 2r_\perp \mu \Delta^{\mu\beta} h^\rho k_\rho k_\rho k^\beta \Delta h_\beta = 0.
\end{aligned}
$$

Thus, for the first-order only the projection by $\Delta^\rho{}_\nu$ contributes, the zeroth and first-order can be added together to give the full linear system for the Alfvén channel.

E Derivation of the magnetosonic channel in the Landau Frame

The first equation we consider are $\nabla_\mu(su^\mu) = 0$ for the zeroth approximation, which corresponds to considering $u_\nu \nabla_\mu T^{\mu\nu}_{(0)} + \mu h_\nu \nabla_\mu J^{\mu\nu}_{(0)} = 0$ we find

$$\begin{aligned}
\nabla_\mu(su^\mu) &= u^\mu \nabla_\mu s + s\nabla_\mu u^\mu && \text{(E.1)}\\
&= \lambda u^\mu \nabla_\mu \mu + cu^\mu \nabla_\mu T + s\nabla_\mu u^\mu && \text{(E.2)}\\
&= i(-\lambda\omega\Delta\mu - c\omega\Delta T + s\kappa^2\Delta U_1 + s\kappa\cos\theta\Delta U_2) && \text{(E.3)}\\
&= i\exp(ik^\mu x_\mu)(-\lambda\omega\delta\mu - c\omega\delta T + s\kappa^2\delta U_1 + s\kappa\cos\theta\delta U_2) && \text{(E.4)}
\end{aligned}$$

We next consider the contraction of $u_\nu \nabla_\mu J^{\mu\nu}_{(0)}$, where we find that

$$\begin{aligned}
u_\nu \nabla_\mu J^{\mu\nu}_{(0)} &= u_\nu \nabla_\mu (2\rho u^{[\mu} h^{\nu]})\\
&= (u^\mu h^\nu - u^\nu h^\mu) u_\nu \nabla_\mu \rho + \rho u_\nu \nabla_\mu (u^\mu h^\nu - u^\nu h^\mu)\\
&= h^\mu(\chi\nabla_\mu \mu + \lambda\nabla_\mu T) + i\rho u^\mu k_\mu u_\nu \Delta h^\nu + i\rho k_\mu \Delta h^\mu\\
&= i\kappa\cos\theta(\chi\Delta\mu + \lambda\Delta T) + i\omega\rho(\kappa\cos\theta\Delta U_1 - \Delta U_2) + i\rho(\kappa^2\sin^2\theta\Delta H_1 - \omega(\kappa\cos\theta\Delta U_1 + \Delta U_2))\\
&= i\exp(ik^\mu x_\mu)(\kappa\cos\theta(\chi\delta\mu + \lambda\delta\mu) + \rho\kappa^2\sin^2\theta\delta H_1)
\end{aligned} \tag{E.5}$$

Likewise, for the contraction of $h_\nu \nabla_\mu T^{\mu\nu}_{(0)}$ we find that

$$\begin{aligned}
h_\nu \nabla_\mu T^{\mu\nu}_{(0)} &= h_\nu \nabla_\mu((\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu} - \mu\rho h^\mu h^\nu)\\
&= (\varepsilon + p)h_\nu u^\mu \nabla_\mu u^\mu + h^\mu(s\nabla_\mu T + \rho\nabla_\mu \mu) - \rho h^\mu \nabla_\mu \mu - \mu h^\mu(\chi\nabla_\mu \mu + \lambda\nabla_\mu T) - \mu\rho\nabla_\mu h^\mu\\
&= i(\varepsilon + p)u^\mu k_\mu h_\nu \Delta u^\nu + ih^\mu k_\mu(s\Delta T + \rho\Delta\mu) - i\rho h^\mu k_\mu \Delta\mu - \mu h^\mu k_\mu(\Delta\mu + \lambda\Delta T) - i\mu\rho k_\mu \Delta h^\mu\\
&= -i\omega Ts\Delta U_2 + i\kappa\cos\theta((s - \mu\lambda)\Delta T - \omega Ts\Delta U_1 - \mu\chi\Delta\mu) - i\mu\rho\kappa^2\sin^2\theta\Delta H_1\\
&= i\exp(ik^\mu x_\mu)(-\omega Ts\delta U_1 + \kappa\cos\theta((s - \mu\lambda)\delta T - \omega Ts\delta U_1 - \mu\chi\delta\mu) - \mu\rho\kappa^2\sin^2\theta\delta H_1)
\end{aligned} \tag{E.6}$$

While for the contraction with the projection tensor, for both the conserved quantities in the zeroth-order we explicitly find

$$
\begin{aligned}
\Delta^{\rho}{}_{\nu}\nabla_{\mu}T^{\mu\nu}_{(0)} &= \Delta^{\rho}{}_{\nu}\nabla_{\mu}((\varepsilon+p)u^{\mu}u^{\nu}+pg^{\mu\nu}-\mu\rho h^{\mu}h^{\nu}) \\
&= (\varepsilon+p)\Delta^{\rho}{}_{\nu}u^{\mu}\nabla_{\mu}u^{\nu}+\Delta^{\mu\rho}(s\nabla_{\mu}T+\rho\nabla_{\mu}\mu)-\mu\rho\Delta^{\rho}{}_{\nu}h^{\mu}\nabla_{\mu}h^{\nu} \\
&= -i(\varepsilon+p)\omega\Delta^{\rho}{}_{\nu}k^{\nu}\Delta U_1+i\Delta^{\rho\mu}k_{\mu}(s\Delta T+\rho\Delta\mu)-i\mu\rho\kappa\cos\theta\Delta^{\rho}{}_{\nu}k^{\nu}\Delta H_1 \\
&= -i\exp(ik^{\mu}x_{\mu})(\Delta^{\rho\mu}k_{\mu}(s\delta T+\rho\delta\mu)-(\varepsilon+p)\omega\Delta^{\rho}{}_{\nu}k^{\nu}\delta U_1-\mu\rho\kappa\cos\theta\Delta^{\rho}{}_{\nu}k^{\nu}\delta H_1) \\
\Delta^{\rho}{}_{\nu}\nabla_{\mu}J^{\mu\nu}_{(0)} &= \Delta^{\rho}{}_{\nu}\nabla_{\mu}(2\rho u^{[\mu}h^{\nu]}) \\
&= \rho\Delta^{\rho}{}_{\nu}u^{\mu}\nabla_{\mu}h^{\nu}-\rho\Delta^{\rho}{}_{\nu}h^{\mu}\nabla_{\mu}u^{\nu} \\
&= -i\rho\omega\Delta^{\rho}{}_{\nu}k^{\nu}\Delta H_1-i\rho\kappa\cos\theta\Delta^{\rho}{}_{\nu}k^{\nu}\Delta U_1 \\
&= -i\exp(ik^{\mu}x_{\mu})\rho\Delta^{\rho}{}_{\nu}k^{\nu}(\omega\Delta H_1+\kappa\cos\theta\delta U_1)
\end{aligned}
\tag{E.7}
$$

Next we consider the first-order approximation, in the same order as for the zeroth-approximation. We can start by considering $u_{\nu}\nabla_{\mu}T^{\mu\nu}_{(0)}+\mu h_{\nu}\nabla_{\mu}J^{\mu\nu}_{(0)}=0$. We note here again that for the sound channel $s^{\mu\nu}=0$.

$$
\begin{aligned}
u_{\nu}\nabla_{\mu}T^{\mu\nu}_{(1)}+\mu h_{\nu}\nabla_{\mu}J^{\mu\nu}_{(1)} &= u_{\nu}\nabla_{\mu}(\delta\pi\Delta^{\mu\nu}+\delta\xi h^{\mu}h^{\nu}+2\ell^{(\mu}h^{\nu)}+t^{\mu\nu})+\mu h_{\nu}\nabla_{\mu}(2m^{[\mu}h^{\nu]}) \\
&= \mu\nabla_{\mu}m^{\mu} \\
&= -2\mu r_{\perp}\Delta^{\mu\sigma}h^{\lambda}T\nabla_{\mu}\nabla_{[\sigma}\left(h_{\lambda]}\frac{\mu}{T}\right) \\
&= \mu r_{\perp}\Delta^{\mu\sigma}(k_{\mu}k_{\sigma}\Delta\mu-\frac{\mu}{T}k_{\mu}k_{\sigma}\Delta T-\mu\kappa\cos\theta k_{\mu}\Delta h_{\sigma}) \\
&= \mu r_{\perp}\Delta^{\mu\sigma}k_{\mu}k_{\sigma}(\Delta\mu-\frac{\mu}{T}\Delta T-\kappa\cos\theta\Delta H_1) \\
&= \exp(ik^{\mu}x_{\mu})(\mu r_{\perp}\kappa^2\sin^2\theta(\delta\mu-\frac{\mu}{T}\delta T-\mu\kappa\cos\theta\delta H_1))
\end{aligned}
\tag{E.8}
$$

Next the conservation of the current contracted with the fluid velocity

$$u_{\nu}\nabla_{\mu}J^{\mu\nu}_{(1)}=u_{\nu}\nabla_{\mu}(2m^{[\mu}h^{\nu]})=u_{\nu}\nabla_{\mu}(m^{\mu}h^{\nu}-m^{\nu}h^{\mu})=0 \tag{E.9}$$

This complete vanish since that $u_\nu h^\mu = 0$, and m^ν is transverse to the fluid velocity, while

$$\begin{aligned}
h_\nu \nabla_\mu T^{\mu\nu}_{(1)} &= h_\nu \nabla_\mu (\delta\pi \Delta^{\mu\nu} + \delta\xi h^\mu h^\nu + 2\ell^{(\mu} h^{\nu)} + t^{\mu\nu}) \\
&= h^\mu \nabla_\mu \delta\pi - \nabla_\mu \ell^\mu \\
&= -\zeta_\times h^\mu \Delta^{\rho\sigma} \nabla_\mu \nabla_\rho u_\sigma - \zeta_\| h^\mu h^\rho h^\sigma \nabla_\mu \nabla_\rho u_\sigma + 2\eta_\| \Delta^{\mu\sigma} h^\lambda \nabla_\mu \nabla_{(\sigma} u_{\lambda)} \\
&= -\zeta_\times \kappa \cos\theta \Delta^{\rho\sigma} k_\rho k_\sigma \Delta U_1 - \zeta_\| \kappa^2 \cos^2\theta h^\sigma \Delta u_\sigma - \eta_\| \kappa^2 \sin^2\theta h^\lambda \Delta u_\lambda - \eta_\| \kappa \cos\theta \Delta^{\rho\sigma} k_\sigma k_\rho \Delta U_1 \\
&= \kappa^2 \sin^2\theta (-\kappa \cos\theta \Delta U_1 (\zeta_\times + 2\eta_\|) - \eta_\| \Delta U_2) - \zeta_\| \kappa^2 \cos^2\theta (\kappa \cos\theta \Delta U_1 - \Delta U_2) \\
&= \exp(ik^\mu x_\mu) \kappa^2 (\cos^2\theta (\delta U_1 + \kappa \cos\theta) \zeta_\| + \sin^2\theta (\delta U_2 \eta_\| + \delta U_1 \kappa \cos\theta (\zeta_\times + 2\eta_\|))).
\end{aligned} \tag{E.10}$$

Lastly we have the contraction of the projection tensor

$$\begin{aligned}
\Delta^{\rho}{}_{\nu}\nabla_{\mu}T^{\mu\nu}_{(1)} &= \Delta^{\rho}{}_{\nu}\nabla_{\mu}(\delta\pi\Delta^{\mu\nu} + \delta\xi h^{\mu}h^{\nu} + 2\ell^{(\mu}h^{\nu)} + t^{\mu\nu}) \\
&= \Delta^{\mu\rho}\nabla_{\mu}\delta\pi + \Delta^{\rho}{}_{\nu}h^{\mu}\nabla_{\mu}\ell^{\nu} + \Delta^{\rho}{}_{\nu}\nabla_{\mu}t^{\mu\nu} \\
&= -\zeta_{\perp}\Delta^{\mu\rho}\Delta^{\sigma\lambda}\nabla_{\mu}\nabla_{(\sigma}u_{\lambda)} - \zeta_{\times}\Delta^{\mu\rho}h^{\sigma}h^{\lambda}\nabla_{\mu}\nabla_{(\sigma}u_{\lambda)} - 2\eta_{\|}\Delta^{\rho}{}_{\nu}\Delta^{\nu\sigma}h^{\mu}h^{\lambda}\nabla_{\mu}\nabla_{(\sigma}u_{\lambda)} \\
&- 2\eta_{\perp}\Delta^{\rho}{}_{\nu}(\Delta^{\mu\lambda}\Delta^{\nu\sigma} - \frac{1}{2}\Delta^{\mu\nu}\Delta^{\lambda\sigma})\nabla_{\mu}\nabla_{(\lambda}u_{\sigma)} \\
&= \zeta_{\perp}\Delta^{\mu\rho}\Delta\sigma\lambda k_{\mu}k_{\sigma}\Delta u_{\lambda} + \zeta_{\times}\Delta^{\mu\rho}h^{\sigma}h^{\lambda}k_{\mu}k_{\sigma}\Delta u_{\lambda} + \eta_{\|}h^{\mu}k_{\mu}\Delta^{\rho\sigma}k_{\sigma}h^{\lambda}\Delta u_{\lambda} \\
&+ \eta_{\|}h^{\mu}k_{\mu}\Delta^{\rho\sigma}h^{\lambda}k_{\lambda}\Delta u_{\sigma} + \eta_{\perp}\Delta^{\mu\lambda}k_{\mu}\Delta^{\rho\sigma}k_{\lambda}\Delta u_{\sigma} + \eta_{\perp}\Delta^{\mu\lambda}k_{\mu}\Delta^{\rho\sigma}k_{\sigma}\Delta u_{\lambda} \\
&- \frac{1}{2}\eta_{\perp}\Delta^{\mu\rho}k_{\mu}\Delta^{\lambda\sigma}k_{\lambda}\Delta u_{\sigma} - \frac{1}{2}\eta_{\perp}\Delta^{\mu\rho}k_{\mu}\Delta^{\lambda\sigma}k_{\sigma}\Delta u_{\lambda} \\
&= \zeta_{\perp}\Delta^{\mu\rho}k_{\mu}\kappa^{2}\sin^{2}\theta\Delta U_{1} + \zeta_{\times}\Delta^{\mu\rho}k_{\mu}\kappa\cos\theta(\kappa\cos\theta\Delta u_{1} + \Delta U_{2}) \\
&+ \eta_{\|}\Delta^{\rho\sigma}k_{\sigma}\kappa\cos\theta(\kappa\cos\theta\Delta U_{1} + \Delta U_{2}) + \eta_{\|}\Delta^{\rho\sigma}k_{\sigma}\kappa^{2}\cos^{2}\theta\Delta U_{1} + 2\eta_{\perp}\Delta^{\rho\sigma}K_{\sigma}\kappa^{2}\sin^{2}\theta\Delta U_{1} \\
&- \eta_{\perp}\Delta^{\mu\rho}k_{\mu}\kappa^{2}\sin^{2}\theta\Delta U_{1} \\
&= \exp(ik^{\mu}x_{\mu})\Delta^{\mu\rho}k_{\mu}\kappa(\zeta_{\times}\cos\theta\delta U_{1} + \kappa(\zeta_{\times}\cos^{2}\theta + \sin^{2}\theta(\zeta_{\perp} + \eta_{\perp}))\delta U_{1} \\
&+ \eta_{\|}\cos\theta(2\kappa\cos\theta\delta U_{1} + \delta U_{2})) \\
\Delta^{\rho}{}_{\nu}\nabla_{\mu}J^{\mu\nu}_{(1)} &= \Delta^{\rho}{}_{\nu}\nabla_{\mu}(2m^{[\mu}h^{\nu]}) \\
&= \Delta^{\rho}{}_{\nu}h^{\mu}\nabla_{\mu}m^{\nu} \\
&= 2r_{\perp}\Delta^{\rho}{}_{\nu}\Delta^{\nu\sigma}h^{\lambda}h^{\mu}T\nabla_{\mu}\nabla_{[\sigma}\left(h_{\lambda]}\frac{\mu}{T}\right) \\
&= -r_{\perp}\Delta^{\rho\sigma}h^{\lambda}h^{\mu}T\nabla_{\mu}\left(\nabla_{\sigma}\left(h_{\lambda}\frac{\mu}{T}\right) - \nabla_{\lambda}\left(h_{\sigma}\frac{\mu}{T}\right)\right) \\
&= r_{\perp}\Delta^{\rho\sigma}h^{\mu}\left(\nabla_{\mu}\nabla_{\sigma}\mu - \frac{\mu}{T}\nabla_{\mu}\nabla_{\sigma}T - \mu h^{\lambda}\nabla_{\mu}\nabla_{\lambda}h_{\sigma}\right) \\
&= -r_{\perp}\Delta^{\rho\sigma}h^{\mu}k_{\mu}\left(k_{\sigma}\Delta\mu - \frac{\mu}{T}k_{\sigma}\Delta T - \mu h^{\lambda}k_{\lambda}\Delta h_{\sigma}\right) \\
&= -\frac{1}{T}r_{\perp}\Delta^{\rho\sigma}\kappa\cos\theta(T\Delta\mu - \mu\Delta T - \mu\kappa\cos\theta\Delta H_{1}) \\
&= -\exp(ik^{\mu}x_{\mu})r_{\perp}\Delta^{\rho\sigma}k_{\sigma}\kappa\cos\theta(T\delta\mu - \mu\delta T - \mu\kappa\cos\theta\delta H_{1})
\end{aligned} \tag{E.11}$$

F Magnetosonic matrix

The magnetosonic modes are determined by the linear system

$$M_{ab}\delta^a = 0,$$

where $\delta = (\delta U_1, \delta U_2, \delta H_1, \delta\mu, \delta T)$, where $a = 1, 2, 3, 4$. The Matrix can be written as

$$M_{ab} = \begin{pmatrix} \kappa^2\rho & \kappa\rho\cos\theta & 0 & M_{14} & M_{15} \\ M_{21} & M_{22} & 0 & 0 & -s\kappa\cos\theta \\ M_{31} & M_{32} & -\frac{1}{\varepsilon+p}\kappa\mu\cos\theta & \frac{\rho}{p+\varepsilon} & \frac{s}{p+\varepsilon} \\ 0 & 0 & \kappa^2\rho\sin^2\theta & \kappa\chi\cos\theta & \kappa\lambda\cos\theta \\ -\kappa\rho\cos\theta & 0 & -\rho\omega - i\kappa^2\mu\cos^2\theta r_\perp & i\kappa\cos\theta r_\perp & -\frac{i}{\mu}\kappa\mu\cos\theta r_\perp, \end{pmatrix}$$

where

$$\begin{aligned}
M_{14} &= -\frac{1}{sT}\Big(T\lambda\rho\omega - i\kappa^2\mu(\mu\chi\cos^2\theta + \rho\sin^2\theta)r_\perp\Big) \\
M_{15} &= -\frac{1}{sT^2}\Big(cT^2\rho\omega - i\kappa^2\mu^2(T\lambda\cos^2\theta - \rho\sin^2\theta)r_\perp\Big) \\
M_{21} &= \kappa\cos\theta\Big(sT\omega + i\kappa^2(\cos^2\theta\zeta_\| + \sin^2\theta(\zeta_\times + 2\eta_\|))\Big) \\
M_{22} &= sT\omega + i\kappa^2(\cos^2\theta\zeta_\| + \sin^2\theta\eta_\|) \\
M_{31} &= -\frac{1}{\varepsilon+p}\Big((\varepsilon+p)\omega + i\kappa^2(\sin^2\theta(\zeta_\perp + \eta_\perp) + \cos^2\theta(\zeta_\times + 2\eta_\|))\Big) \\
M_{32} &= -\frac{i}{\varepsilon+p}\kappa\cos\theta(\zeta_\times + \eta_\|)
\end{aligned}$$

The Matrix can be read directly from eq. (15.20), and its determinant can be written directly from the Matrix.

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