Wilson lines in AdS/dCFT

The holographic particle-interface potential

Master thesis

Written by *Khalil Idiab*

July 22, 2020

Supervised by
Charlotte Fløe Kristjansen

University of Copenhagen

UNIVERSITY OF
COPENHAGEN

Name of Institute:	The Niels Bohr Institute
Author:	Khalil Idiab
Email:	ljw251@alumni.ku.dk
Title and subtitle:	Wilson lines in AdS/dCFT - The holographic particle-interface potential
Supervisor:	Charlotte Fløe Kristjansen
Handed in:	23.07.2020
Defended:	12.08.2020

Abstract

The project computes the expectation value of Wilson lines to one-loop order for two holographic defect conformal field theories. Supersymmetry is completely broken for both setups, but only one of which is described by an integrable boundary state. The known computation for the supersymmetric dCFT with $so(3)$ R-symmetry is briefly reviewed and its results are reproduced. After finding the Wilson line expectation value for all three setups we solve their gravity dual. First treating the brane configuration for the setups and then computing the string action dual to the Wilson lines. We find perfect agreement for the non-local observable in the double scaling limit for all three setups as predicted by the holographic dictionary.

Acknowledgments

I, K.I., would like to thank my lovely supervisor Charlotte Kristjansen for offering and suggesting the project. I am sincerely appreciative of her endless patience and I feel deeply honoured to have been welcomed into her kind-hearted group. In this regard I would like to thank Sara Bonansea, Matthias Volk and Rasmus Nielsen for illuminating discussions and helpful remarks. I would also like to thank the Niels Bohr Institute for a well-structured physics programme.

Contents

1 Introduction

The idea that a string theory is dual to a large N Yang-Mills theory dates all the way back to 't Hooft in 1974 [1]. It was, however, as late as 1998 that Maldacena found a precise example of such a duality, the famous AdS/CFT correspondence [2]. Ever since then, has duality and holography been among the most hot topics in theoretical physics. A duality means that two theories with distinct interpretation, are equivalent at some fundamental level (a map exists). Holography is a duality between two theories of different spacetime dimension.

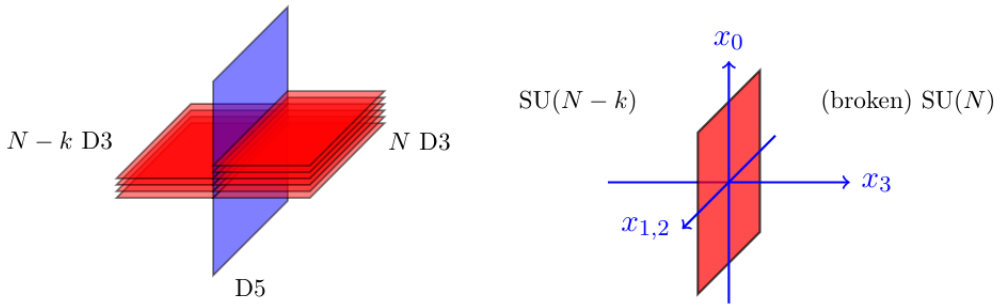
Historically supersymmetry has been important for holography. The common holographic dualities have a large amount of symmetry (supersymmetry and conformal symmetry) and are dual to a third, integrable, system. One could suspect that the only reason that their holographic dictionary works, is that the symmetry is sufficient to uniquely specify the observables on both sides. The goal of the project is to test if it is necessary for holography to have supersymmetry and integrability, or that our notion of holography is more fundamental.

We intend on achieving this by computing an observable on both sides of the proposed AdS/dCFT duality. Our observable of choice is the infinite Wilson line treated in [3]. The idea is to perform a check of the correspondence for two setups with completely broken supersymmetry where only one of which is described by an integrable boundary state. A description and a perturbative framework for the two setups are laid out in [4, 5]. The hope is to deepen the understanding of the interplay between holography and supersymmetry and between holography and integrability. The organization of the following sections is as follows; section 2 will introduce the general setup and review some previous work, section 3 and 4 will delve into the computation on the field theory side and string theory side respectively. In the last section we conclude and summarize the results.

2 The setup and previous work

2.1 Introduction to the setup

In the original AdS/CFT setup the starting point is a stack of N D3 branes [2]. For AdS/dCFT setups we similarly start with a stack of N D3 branes, but in this case we also have a D5 or D7 probe brane that the D3 branes end on [6]. On the other side of the probe brane there is fewer D3 branes, say $N-k$, see Figure 1. As usual, the stack of N D3 branes corresponds to a gauge theory with gauge group $\mathrm{SU}(N)$. Since the stack ends at the probe brane and only $N-k$ branes continue on the other side, there will be a $\mathrm{SU}(N-k)$ gauge group on the other side with a interface between them. We place the coordinate system on the field theory side



(a) (b)

Figure 1: (a) depicts the brane configuration on the string theory side. (b) depicts the corresponding dCFT.

such that x_3 is perpendicular to the interface defect and positive x_3 is the SU(N) side. In particular, we will be interested in three cases, one with a D5 probe brane and two with a D7 probe brane with different geometries. The observable that we will be comparing is the particle-interface potential for positive x_3, as computed by the supersymmetric Wilson line and its holographic dual being the action of a string suspended between the probe brane and the AdS boundary.

2.2 Characterization of the three setups

The three brane setups of interest are summarized in Table 1. For the D3-D7 setups supersymmetry is completely broken. The D3-D5 setup and the second D3-D7 setup is described by an integrable boundary state.

Brane setup	D3-D5	D3-D7 (I)	D3-D7 (II)
Supersymmetry	1/2-BPS	None	None
Brane geometry	AdS$_4\times$ S^2	AdS$_4\times$ S^2 $\times$ S^2	AdS$_4\times$ S^4
Magnetic flux on spheres	k	k_1, k_2	$\frac{(n+1)(n+2)(n+3)}{6}$
D.s. parameter	$\frac{\lambda}{\pi^2 k^2}$	$\frac{\lambda}{\pi^2(k_1^2+k_2^2)}$	$\frac{\lambda}{\pi^2 n^2}$
Boundary state	Integrable	Non-integrable	Integrable

Table 1: A brief overview of the three brane configurations that we consider.

For the D3-D5 setup, there is missing k D3 branes on one side of the probe brane, see Figure 1. For the first D3-D7 brane setup there is $N - k_1 k_2$ on one side and for the other D3-D7 setup there is $N - d_G$ with $d_G = (n+1)(n+2)(n+3)/6$. The theories will be comparable to their dual dCFT only in a certain double scaling limit. As is the case in the usual AdS/CFT we first take $N \to \infty$, but when we take $\lambda \to \infty$ we do it in such a way that the double scaling parameter stays finite but small. Refer to Table 1 for the different setups respective double scaling parameters. Observables on both sides of the duality will be expressed as a perturbative expansion in the double scaling parameter, and since this parameter can be taken small, they can be compared order by order.

Now we will look at the dCFT dual to the three setups. Only $x_3 > 0$ will be considered. The action here is the regular $\mathcal{N} = 4$ super Yang-Mills

$$S_{\mathcal{N}=4} = \frac{2}{g_{\mathrm{YM}}^2} \int d^4x \, \mathrm{tr} \bigg[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \phi_i D^\mu \phi_i + \frac{i}{2} \bar{\Psi} \Gamma^\mu D_\mu \Psi + \frac{1}{2} \bar{\Psi} \tilde{\Gamma}^i [\phi_i, \Psi] + \frac{1}{4} [\phi_i, \phi_j][\phi_i, \phi_j] \bigg], \tag{2.1}$$

where the field strength tensor and covariant derivative are defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \tag{2.2}$$

$$D_\mu = \partial_\mu - i[A_\mu, \cdot\ ]. \tag{2.3}$$

The equations of motion for the scalar fields becomes

$$\nabla^2 \phi_i^{\mathrm{cl}}(x) = \left[\phi_j^{\mathrm{cl}}(x), \left[\phi_j^{\mathrm{cl}}(x), \phi_i^{\mathrm{cl}}(x)\right]\right]. \tag{2.4}$$

To match the symmetries of the D3-D5 brane configuration we require that the scalar fields have SO(3) symmetry, this matches the S^2 part of the brane geometry $AdS_4 \times S^2$. A particular solution to the equations of motion for the scalar fields with the relevant symmetry is [7]

$$\phi_i^{\mathrm{cl}}(x) = -\frac{1}{x_3} t_i^k \oplus 0^{(N-k)}, \qquad i = 1, 2, 3, \tag{2.5}$$

$$\phi_i^{\mathrm{cl}}(x) = 0^{(N)}, \qquad i = 4, 5, 6, \tag{2.6}$$

where t_i^k is the k-dimensional irreducible representation of $so(3)$, it is padded with zeros to make it a N by N matrix. To see this is a solution to the equations of motion we may simply plug it in and use the defining commutation relations that the generators of so(3) satisfy. We will do it explicitly for this case

$$-\nabla^2 \frac{1}{x_3} t_i^k = \left[\phi_j^{\mathrm{cl}}(x), \left[\phi_j^{\mathrm{cl}}(x), -\frac{1}{x_3} t_i^k\right]\right], \tag{2.7}$$

notice that (4.87) is trivially satisfied for the $N-k$ block and for $i=4,5,6$.

$$-\frac{2}{x_3^3}t_i^k = \sum_{j=1}^{3}\left[-\frac{1}{x_3}t_j^k, \left[-\frac{1}{x_3}t_j^k, -\frac{1}{x_3}t_i^k\right]\right], \tag{2.8}$$

$$-\frac{2}{x_3^3}t_i^k = -\frac{1}{x_3^3}\sum_{j=1}^{3}\left[t_j^k, \left[t_j^k, t_i^k\right]\right]. \tag{2.9}$$

The Lie brackets are now replaced with the structure constants and the Einstein summation convention will be employed

$$\frac{2}{x_3^3}t_i^k = \frac{1}{x_3^3}\left[t_j^k, i f_{jil} t_l^k\right] = -\frac{1}{x_3^3} f_{jil} f_{jlm} t_m^k. \tag{2.10}$$

For so(3) the structure constants are $f_{ijl} = \epsilon_{ijl}$ where ϵ denotes the totally antisymmetric symbol. The contraction of the antisymmetric symbols becomes $\epsilon_{jil}\epsilon_{jlm} = -2\delta_{im}$ and (2.10) is thus satisfied and we have confirmed that (4.36), (2.6) is an appropriate solution to the equations of motion. Suitable solutions for the D3-D7 setups have also been found, for the first setup with SO(3)×SO(3) symmetry we use [8]

$$\phi_i^{\text{cl (I)}}(x) = -\frac{1}{x_3}\left(t_i^{k_1}\otimes 1^{k_2}\right)\oplus 0^{(N-k_1k_2)} \qquad \text{for} \qquad i=1,2,3, \tag{2.11}$$

$$\phi_i^{\text{cl (I)}}(x) = -\frac{1}{x_3}\left(1^{k_1}\otimes t_{i-3}^{k_2}\right)\oplus 0^{(N-k_1k_2)} \qquad \text{for} \qquad i=4,5,6. \tag{2.12}$$

t_i^k again denotes the generators of so(3) in the k-dimensional irreducible representation. For the second setup with SO(5) symmetry we use the following solution [9]

$$\phi_i^{\text{cl (II)}}(x) = \frac{G_{i6}^{d_G}}{\sqrt{2}x_3}\oplus 0^{(N-d_G)} \qquad \text{for} \qquad i=1,...,5; \qquad \phi_6^{\text{cl (II)}}(x) = 0^{(N)}. \tag{2.13}$$

Here $G_{i6}^{d_G}$ are the generators of so(5) in the $d_G = (n+1)(n+2)(n+2)/6$ dimensional irreducible representation. This concludes the basic characterization of the three setups, quantum fluctuations around the classical solutions will be dealt with later.

2.3 The particle-interface potential

2.3.1 Gauge theory side

In order to check that the theories are dual, we compute and compare an observable on both sides. As previously mentioned, our observable of choice is the constant

x_3 Wilson line and its dual string. In this section we focus on the D3-D5 setup as treated in [3]. We let the path be parameterized by

$$x^\mu(\gamma(\lambda)) = (\lambda, 0, 0, z) \tag{2.14}$$

and compute the associated supersymmetric Wilson integral

$$W = \lim_{T\to\infty} \mathrm{tr}\left[\mathrm{Pexp}\int_{-T/2}^{T/2} \mathcal{A}(t)dt\right], \tag{2.15}$$

with

$$\mathcal{A} = iA_0 - \sin\chi\phi_3 - \cos\chi\phi_6. \tag{2.16}$$

The potential V as seen by the particle is then related by $\langle W\rangle = e^{-TV}$. Note that the origin of this potential is due to the fact that we had to find non-vanishing vacuum solutions for the scalar fields in order to match the symmetries of the holographic dual. We can now find the classical potential by plugging in the classical vacuum expectation values for the D3-D5 setup as given in (4.36) and (2.6).

$$\langle W\rangle_{\mathrm{tree}} = \lim_{T\to\infty} \mathrm{tr}\left[\mathrm{Pexp}\int_{-T/2}^{T/2} dt\sin\chi\frac{1}{z}t_3^k\right], \tag{2.17}$$

$$\langle W\rangle_{\mathrm{tree}} = \lim_{T\to\infty} \mathrm{tr}\exp\left[\frac{T}{z}\sin\chi t_3^k\right]. \tag{2.18}$$

In the large T limit, only the largest eigenvalue of t_3^k will contribute, following the conventions of [3] the largest eigenvalue is $\eta = (k-1)/2$,

$$\langle W\rangle_{\mathrm{tree}} = \lim_{T\to\infty} \exp\left[\frac{T}{2z}\sin\chi(k-1)\right], \tag{2.19}$$

the potential can now be identified as

$$V_{\mathrm{tree}}(x) = -\frac{k-1}{2x_3}\sin\chi. \tag{2.20}$$

In order to compare to the string theory side we will have to take the correct double scaling limit, which includes the large k limit, in this case we find

$$V_{\mathrm{tree}}(x) = -\frac{k}{2x_3}\sin\chi. \tag{2.21}$$

Of course this is not the full story, since there will be quantum fluctuations around the vacuum expectation values. The above result is simply the classical result and is therefore denoted tree, as it amounts to the tree diagram in the Feynman expansion.

2.3.2 String theory side

The Wilson line corresponds to the area of the string suspended from the AdS boundary and perpendicularly attached to the probe brane. In order to perform this computation one first has to find where the brane is sitting. Once this is found by extremizing the Dirac-Born-Infeld action, the precise boundary conditions can be written. The boundary conditions were found in [10] and summarized in Table 2. Λ is the angle between the probe brane and the D3 branes, for the D3-D5 setup

	$y(\sigma_0,\tau) = 0$	$y(\sigma_1,\tau) = \frac{1}{\Lambda}x_3(\sigma_1,\tau)$
BC:	$x_3(\sigma_0,\tau) = z$	$y'(\sigma_1,\tau) = -\Lambda x_3'(\sigma_1,\tau)$
	$\psi(\sigma_0,\tau) = \frac{\pi}{2}-\chi$	$\psi(\sigma_1,\tau) = 0$

Table 2: Summary of the boundary conditions for the string.

[10] found this to be $\Lambda = \pi k/\alpha'$. The boundary conditions have two parameters, z and χ. z is the initial distance to the probe brane and is the same z as in the path parameterization for the Wilson line. χ is a parameter that controls the suspension in the S^5 part of the space, see Figure 2. One simply proceeds by solving the equations of motion given the boundary conditions and plugging it back into the Nambu-Goto or Polyakov action. The action is then related to the potential by $\langle W\rangle = e^S$ where $\langle W\rangle$ is the Wilson line and S is the action. The action and the associated potential was found to be [10]

$$V(x) = -\frac{k}{2x_3}\sin\chi\left[1+\frac{\lambda\sin\chi}{8\pi^2k^2\cos^3\chi}(\pi-2\chi-\sin 2\chi)+O\left(\frac{\lambda^2}{\pi^4k^4}\right)\right]. \quad (2.22)$$

The gauge theory side was computed in [3] and found agreement to at least one-loop order. The details of the computation on both the string theory and gauge theory side will be carried out in the following sections for the D3-D7 brane setups.

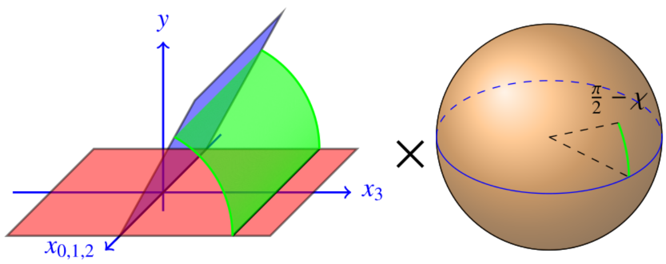


Figure 2: The green represents the string worldsheet in AdS_5 and S^5. The red is the AdS boundary and the blue is the probe brane.

3 Gauge theory computation

3.1 Dealing with quantum corrections

The goal is to compute the Wilson line

$$\langle W\rangle = \lim_{T\to\infty}\left\langle \mathrm{tr}\left[\mathrm{Pexp}\int_{-T/2}^{T/2}\mathcal{A}(t)dt\right]\right\rangle, \tag{3.1}$$

with

$$\mathcal{A}^{(\mathrm{I})}(t) = iA_0(t) - \phi_3^{(\mathrm{I})}(t)\sin(\chi) - \phi_6^{(\mathrm{I})}(t)\cos(\chi), \tag{3.2}$$

$$\mathcal{A}^{(\mathrm{II})}(t) = iA_0(t) - \phi_5^{(\mathrm{II})}(t)\sin(\chi) - \phi_6^{(\mathrm{II})}(t)\cos(\chi), \tag{3.3}$$

for the two D3-D7 setups respectively. In order to deal with the quantum fluctuations, we do the usual trick of reexpanding all fields around their classical vacuum such as $\phi = \phi^{\mathrm{cl}} + \tilde{\phi}$. For convenience, we define

$$U(\alpha,\beta) = \mathrm{Pexp}\int_\alpha^\beta \mathcal{A}(t)dt, \tag{3.4}$$

the gauge invariant infinite Wilson line is then given by

$$W = \lim_{T\to\infty}\mathrm{tr}\,U\left(-\frac{T}{2},\frac{T}{2}\right). \tag{3.5}$$

The expansion of the fields

$$\mathcal{A} = \mathcal{A}^{\mathrm{cl}} + \tilde{\mathcal{A}} \tag{3.6}$$

is now plugged into (3.4)

$$U(\alpha,\beta) = \mathrm{Pexp}\left[\int_\alpha^\beta dt\,\Big(\mathcal{A}^{\mathrm{cl}}(t) + \tilde{\mathcal{A}}(t)\Big)\right], \tag{3.7}$$

$$= \mathrm{P}\left[\exp\int_\alpha^\beta dt\mathcal{A}^{\mathrm{cl}}(t)\right]\left[\exp\int_\alpha^\beta dt\tilde{\mathcal{A}}(t)\right], \tag{3.8}$$

since it is under the path ordering operator, everything commutes and we are able to split the exponential. We define the first exponential

$$U^{\mathrm{cl}}(\alpha,\beta) = \mathrm{Pexp}\left[\int_\alpha^\beta dt\mathcal{A}^{\mathrm{cl}}(t)\right]. \tag{3.9}$$

The second exponential is now expanded, it turns out that up to second order is necessary for the one-loop result,

$$\mathrm{Pexp}\int_{\alpha}^{\beta} dt\tilde{\mathcal{A}}(t) = 1 + \mathrm{P}\int_{\alpha}^{\beta} dt\tilde{\mathcal{A}}(t) + \frac{1}{2}\mathrm{P}\left(\int_{\alpha}^{\beta} dt\tilde{\mathcal{A}}(t)\right)^2 + O\left(\tilde{\mathcal{A}}^3\right). \tag{3.10}$$

The result from (3.10) and the definition from (3.9) can now be used to rewrite (3.8)

$$U(\alpha,\beta) = \mathrm{P}U^{\mathrm{cl}}(\alpha,\beta)\left[1 + \int_{\alpha}^{\beta} dt\tilde{\mathcal{A}}(t) + \frac{1}{2}\left(\int_{\alpha}^{\beta} dt\tilde{\mathcal{A}}(t)\right)^2\right] + O\left(\tilde{\mathcal{A}}^3\right). \tag{3.11}$$

The final step is to put everything in correct path order and remove the path ordering operator

$$\begin{aligned} U(\alpha,\beta) = {} & U^{\mathrm{cl}}(\alpha,\beta) + \int_{\alpha}^{\beta} dt U^{\mathrm{cl}}(\alpha,t)\tilde{\mathcal{A}}(t)U^{\mathrm{cl}}(t,\beta) \\ & + \int_{\alpha}^{\beta} dt \int_{t}^{\beta} dt' U^{\mathrm{cl}}(\alpha,t)\tilde{\mathcal{A}}(t)U^{\mathrm{cl}}(t,t')\tilde{\mathcal{A}}(t')U^{\mathrm{cl}}(t',\beta) + O(\tilde{\mathcal{A}}^3). \end{aligned} \tag{3.12}$$

To find the Wilson line we are interested in, we will need take the trace and expectation value. This means we will be needing to find the one-loop corrections to the vacuum expectation values $\langle\tilde{\mathcal{A}}(t)\rangle$ and the propagators $\langle\tilde{\mathcal{A}}(t)\tilde{\mathcal{A}}(t')\rangle$. The corresponding diagram to each term in (3.12) is illustrated in Figure 3. The following subsections will be devoted to treating each of the terms. It will be convenient to define

$$U_{\mathrm{lol}}(\alpha,\beta) = \int_{\alpha}^{\beta} dt U^{\mathrm{cl}}(\alpha,t)\tilde{\mathcal{A}}(t)U^{\mathrm{cl}}(t,\beta), \tag{3.13}$$

$$U_{\mathrm{tad}}(\alpha,\beta) = \int_{\alpha}^{\beta} dt \int_{t}^{\beta} dt' U^{\mathrm{cl}}(\alpha,t)\tilde{\mathcal{A}}(t)U^{\mathrm{cl}}(t,t')\tilde{\mathcal{A}}(t')U^{\mathrm{cl}}(t',\beta), \tag{3.14}$$

and

$$\langle W\rangle_{\mathrm{tree}} = \lim_{T\to\infty}\left\langle \mathrm{tr}\, U^{\mathrm{cl}}\left(-\frac{T}{2},\frac{T}{2}\right)\right\rangle, \tag{3.15}$$

$$\langle W\rangle_{\mathrm{lol}} = \lim_{T\to\infty}\left\langle \mathrm{tr}\, U_{\mathrm{lol}}\left(-\frac{T}{2},\frac{T}{2}\right)\right\rangle, \tag{3.16}$$

$$\langle W\rangle_{\mathrm{tad}} = \lim_{T\to\infty}\left\langle \mathrm{tr}\, U_{\mathrm{tad}}\left(-\frac{T}{2},\frac{T}{2}\right)\right\rangle, \tag{3.17}$$

such that

$$\langle W\rangle = \langle W\rangle_{\mathrm{tree}} + \langle W\rangle_{\mathrm{lol}} + \langle W\rangle_{\mathrm{tad}} + O(2-\mathrm{loop}). \tag{3.18}$$

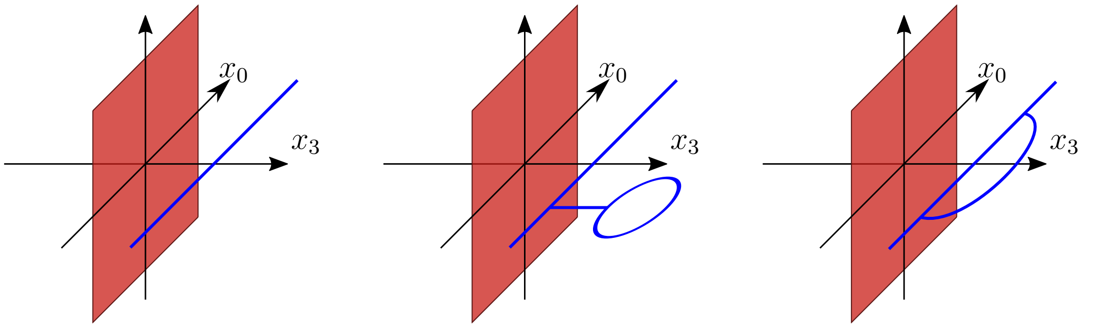


Figure 3: Diagrams at tree level and one-loop order. The middle diagram will be called the lollipop and the third diagram will be called the tadpole.

3.1.1 One-point functions and propagators

Finding the propagators $\langle \tilde{\mathcal{A}}(t)\tilde{\mathcal{A}}(t')\rangle$ and the expectation value of the correction $\langle \tilde{\mathcal{A}}(t)\rangle$ is a rather lengthy and complicated procedure. Only the results will be stated here. The procedure starts with the action. The action consists of the usual $\mathcal{N} = 4$ super Yang Mills action in the bulk and the action of the fields living on the defect. At one loop order we only need to consider the action of the bulk as interactions with the defect fields will be of higher order. To proceed we perform the well-known trick of substituting $\phi \to \phi^{\text{cl}} + \tilde{\phi}$ in the action (2.1) and treating $\tilde{\phi}$ as the fundamental fields. Since it is the action of an interacting field theory, many terms will be generated where ϕ^{cl} will play the role of mass coefficients and interaction coupling constants. The gauge freedom will have to be fixed as well before doing field theoretic computation. After fixing the gauge the resulting gauge-fixed action was found in [6]

$$S_{\mathcal{N}=4} + S_{\text{gh}} = S_{\text{kin}} + S_{\text{m,b}} + S_{\text{m,f}} + S_{\text{cubic}} + S_{\text{quartic}}, \tag{3.19}$$

where S_{kin} are the kinetic terms, $S_{\text{m,b}}$ are the mass terns for the bosons, $S_{\text{m,f}}$ are the mass terms for the fermions, S_{cubic} are the cubic interactions and S_{quartic} are the quartic interactions. In particular, they found

$$\begin{aligned} S_{\text{m,b}} = \frac{1}{g^2_{\text{YM}}} \int d^4x \,\text{tr} \Big[&[\phi^{\text{cl}}_i, \phi^{\text{cl}}_j][\tilde{\phi}_i, \tilde{\phi}_j] + [\phi^{\text{cl}}_i, \tilde{\phi}_j][\phi^{\text{cl}}{}_i, \tilde{\phi}_j] + [\phi^{\text{cl}}_i, \tilde{\phi}_j][\tilde{\phi}_i, \phi^{\text{cl}}{}_j] \\ &+ [\phi^{\text{cl}}_i, \tilde{\phi}_i][\phi^{\text{cl}}{}_j, \tilde{\phi}_j] + [A_\mu, \phi^{\text{cl}}_i][A^\mu, \phi^{\text{cl}}_i] + 2i[A^\mu, \tilde{\phi}_i]\partial_\mu \phi^{\text{cl}}_i \Big]. \end{aligned} \tag{3.20}$$

Two difficulties are apparent with this, first is that it mixes fields and secondly is that ϕ^{cl}, and hence mass coefficients, are spacetime dependent. The spacetime dependence means that the propagators will be related to the AdS propagators.

And to deal with the mixing one has to diagonalize the mass matrix and find the resulting spectrum. Then one can go back to finding the propagators of the original fields by writing them as a sum of the mass eigenstates.

Before proceeding, we write out the two-point function we are interested in

$$\langle \tilde{\mathcal{A}}(t)\tilde{\mathcal{A}}(t')\rangle = \big\langle (i\tilde{A}_0(t) - \tilde{\phi}_a(t)\sin\chi - \tilde{\phi}_6(t)\cos\chi) \\ (i\tilde{A}_0(t') - \tilde{\phi}_a(t')\sin\chi - \tilde{\phi}_6(t')\cos\chi)\big\rangle, \tag{3.21}$$

where $\tilde{\phi}_a = \tilde{\phi}_3$ for the D3-D5 and the first D3-D7 setup and $\tilde{\phi}_a = \tilde{\phi}_5$ for the second D3-D7 setup. Common for all three setups, there is no propagation between $\tilde{A}_0$ and a different field

$$\langle \tilde{\mathcal{A}}(t)\tilde{\mathcal{A}}(t')\rangle = -\big\langle \tilde{A}_0(t)\tilde{A}_0(t')\big\rangle + \big\langle \tilde{\phi}_a(t)\tilde{\phi}_a(t')\big\rangle \sin^2\chi \tag{3.22}$$

$$+ \big\langle \tilde{\phi}_6(t)\tilde{\phi}_6(t')\big\rangle \cos^2\chi + 2\big\langle \tilde{\phi}_a(t)\tilde{\phi}_6(t')\big\rangle \sin\chi\cos\chi, \tag{3.23}$$

A useful way to write it is

$$\langle \tilde{\mathcal{A}}(t)\tilde{\mathcal{A}}(t')\rangle = \Big(\big\langle \tilde{\phi}_a(t)\tilde{\phi}_a(t')\big\rangle - \big\langle \tilde{A}_0(t)\tilde{A}_0(t')\big\rangle\Big)\sin^2\chi \tag{3.24}$$

$$+ \Big(\big\langle \tilde{\phi}_6(t)\tilde{\phi}_6(t')\big\rangle - \big\langle \tilde{A}_0(t)\tilde{A}_0(t')\big\rangle\Big)\cos^2\chi \tag{3.25}$$

$$+ 2\big\langle \tilde{\phi}_a(t)\tilde{\phi}_6(t')\big\rangle \sin\chi\cos\chi. \tag{3.26}$$

As will be argued later, it is only the off diagonal block that will end up contributing. The propagators for the D3-D5 setup was found in [6]. For the $k \times (N-k)$ and $(N-k)\times k$ blocks they found

$$\big\langle [\tilde{\phi}_3(t)]_{a\mu}[\tilde{\phi}_3(t')]_{\mu b}\big\rangle - \big\langle [\tilde{A}_0(t)]_{a\mu}[\tilde{A}_0(t')]_{\mu b}\big\rangle = \delta_{ab}(N-k)\bigg(\frac{k+1}{2k}K^{m^2=\frac{(k-2)^2-1}{4}} \\ + \frac{k-1}{2k}K^{m^2=\frac{(k+2)^2-1}{4}} - K^{m^2=\frac{k^2-1}{4}}\bigg), \tag{3.27}$$

$$\big\langle [\tilde{\phi}_6(t)]_{a\mu}[\tilde{\phi}_6(t')]_{\mu b}\big\rangle - \big\langle [\tilde{A}_0(t)]_{a\mu}[\tilde{A}_0(t')]_{\mu b}\big\rangle = 0, \tag{3.28}$$

$$\big\langle [\tilde{\phi}_3(t)]_{a\mu}[\tilde{\phi}_6(t')]_{\mu b}\rangle\big\rangle = 0, \tag{3.29}$$

where $K^{m^2} = K^{m^2}(t,t')$ is the solution to

$$\left(-\partial_\mu\partial^\mu + \frac{m^2}{x_3^2}\right)K(t,t') = \frac{g_{\mathrm{YM}}^2}{2}\delta(t-t'). \tag{3.30}$$

For the first D3-D7 setup the propagators were found in [4]

$$\left\langle[\tilde{\phi}_3(t)]_{a\mu}[\tilde{\phi}_3(t')]_{\mu b}\right\rangle = (N-k_1k_2)\left(\delta_{ab}K^{\phi,(1)}_{\text{sing}} - [t_3^{k_1}t_3^{k_1}\otimes 1_{k_2}]_{ab}K^{\phi,(1)}_{\text{sym}}\right), \tag{3.31}$$

$$\left\langle[\tilde{\phi}_6(t)]_{a\mu}[\tilde{\phi}_6(t')]_{\mu b}\right\rangle = (N-k_1k_2)\left(\delta_{ab}K^{\phi,(2)}_{\text{sing}} - [t_3^{k_2}t_3^{k_2}\otimes 1_{k_1}]_{ab}K^{\phi,(2)}_{\text{sym}}\right), \tag{3.32}$$

$$\left\langle[\tilde{A}_0(t)]_{a\mu}[\tilde{A}_0(t')]_{\mu b}\right\rangle = \delta_{ab}(N-k_1k_2)K^{m^2=\frac{1}{4}(k_1^2+k_2^2-2)}, \tag{3.33}$$

$$\left\langle[\tilde{\phi}_3(t)]_{a\mu}[\tilde{\phi}_6(t')]_{\mu b}\rangle\right\rangle = (N-k_1k_2)[t_3^{k_1}\otimes t_3^{k_2}]_{ab}\left(\frac{K^{m_-^2}}{N_-}+\frac{K^{m_+^2}}{N_+}-\frac{K^{m_0^2}}{N_0}\right), \tag{3.34}$$

for the $k_1k_2\times(N-k_1k_2)$ and $(N-k_1k_2)\times k_1k_2$ blocks, where

$$K^{\phi,(1)}_{\text{sing}} = \frac{k_1+1}{2k_1}K^{m^2_{(1),+}} + \frac{k_1-1}{2k_1}K^{m^2_{(1),-}}, \tag{3.35}$$

$$K^{\phi,(2)}_{\text{sing}} = \frac{k_2+1}{2k_2}K^{m^2_{(2),+}} + \frac{k_2-1}{2k_2}K^{m^2_{(2),-}}, \tag{3.36}$$

$$K^{\phi,(1)}_{\text{sym}} = \frac{2K^{m^2_{(1),+}}}{k_1^2+k_1} + \frac{2K^{m^2_{(1),-}}}{k_1^2-k_1} - \frac{k_2^2-1}{k_1^2-1}\frac{K^{m_0^2}}{N_0} - \frac{K^{m_-^2}}{N_-} - \frac{K^{m_+^2}}{N_+}, \tag{3.37}$$

$$K^{\phi,(2)}_{\text{sym}} = \frac{2K^{m^2_{(2),+}}}{k_2^2+k_2} + \frac{2K^{m^2_{(2),-}}}{k_2^2-k_2} - \frac{k_1^2-1}{k_2^2-1}\frac{K^{m_0^2}}{N_0} - \frac{K^{m_-^2}}{N_-} - \frac{K^{m_+^2}}{N_+}, \tag{3.38}$$

with

$$\lambda_\pm = -\frac{1}{2}\pm\frac{1}{2}\sqrt{k_1^2+k_2^2-1}, \tag{3.39}$$

$$N_\pm = \lambda_\mp\left(\lambda_\mp-\lambda_\pm\right), \tag{3.40}$$

$$N_0 = -\lambda_+\lambda_-, \tag{3.41}$$

and

$$m^2_{(1),+} = \frac{1}{4}\left(k_1^2-4k_1+k_2^2+2\right), \tag{3.42}$$

$$m^2_{(1),-} = \frac{1}{4}\left(k_1^2+4k_1+k_2^2+2\right), \tag{3.43}$$

$$m^2_{(2),+} = \frac{1}{4}\left(k_2^2-4k_2+k_1^2+2\right), \tag{3.44}$$

$$m^2_{(2),-} = \frac{1}{4}\left(k_2^2+4k_2+k_1^2+2\right), \tag{3.45}$$

$$m_0^2 = \frac{1}{4}\left(k_1^2+k_2^2+6\right), \tag{3.46}$$

$$m_+^2 = \frac{1}{4}\left(k_1^2+k_2^2-2\right)-2\lambda_+, \tag{3.47}$$

$$m_-^2 = \frac{1}{4}\left(k_1^2+k_2^2-2\right)-2\lambda_-. \tag{3.48}$$

For the second D3-D7 setup the propagators in the off-diagonal block were found in [5]

$$\Big\langle[\tilde{\phi}_5(t)]_{a\mu}[\tilde{\phi}_5(t')]_{\mu b}\Big\rangle = (N-d_G)\bigg(\delta_{ab}f^{\rm sing} + 4[G_{56}G_{56}]_{ab}f^{\rm prod}\bigg), \tag{3.49}$$

$$\Big\langle[\tilde{\phi}_6(t)]_{a\mu}[\tilde{\phi}_6(t')]_{\mu b}\Big\rangle = \delta_{ab}(N-d_G)K^{m^2=\frac{1}{8}n(n+4)}, \tag{3.50}$$

$$\Big\langle[\tilde{A}_0(t)]_{a\mu}[\tilde{A}_0(t')]_{\mu b}\Big\rangle = \delta_{ab}(N-d_G)K^{m^2=\frac{1}{8}n(n+4)}, \tag{3.51}$$

$$\Big\langle[\tilde{\phi}_5(t)]_{a\mu}[\tilde{\phi}_6(t')]_{\mu b}\Big\rangle\rangle = 0, \tag{3.52}$$

where

$$f^{\rm sing} = \frac{n}{2(n+2)}K^{m^2_{-+}} + \frac{n+4}{2n+4}K^{m^2_{++}}, \tag{3.53}$$

$$f^{\rm prod} = -\frac{K^{m^2-+}}{2n(n+2)} - \frac{K^{m^2_{++}}}{2(n+2)(n+4)} + \frac{K^{m^2_{-}}}{4N_+} + \frac{K^{m^2_{+}}}{4N_-}, \tag{3.54}$$

$$N_\pm = \frac{n}{2}(n+4) + 1 \pm \sqrt{\frac{n}{2}(n+4)+1}, \tag{3.55}$$

with

$$m^2_{++} = \frac{1}{8}n^2, \tag{3.56}$$

$$m^2_{-+} = \frac{1}{8}(n+4)^2, \tag{3.57}$$

$$m^2_{+} = \frac{1}{8}\left(n^2+4n+8+4\sqrt{2n^2+8n+4}\right), \tag{3.58}$$

$$m^2_{-} = \frac{1}{8}\left(n^2+4n+8-4\sqrt{2n^2+8n+4}\right). \tag{3.59}$$

We notice now that for all three setups the two-point function has the following form

$$\langle[\tilde{\mathcal{A}}(t)]_{d\mu}[\tilde{\mathcal{A}}(t')]_{\mu c}\rangle = \sum_n D^n_{dc}\sum_i \lambda_{i,n}K^{m^2_{i,n}}(t,t') \tag{3.60}$$

where D is a diagonal prefactor. That is everything we will be needing from the propagators. As for the one-point function, the same references that found the propagators were able to use them to find the one-loop corrections to the classical solutions. For they D3-D5 setup they found [6]

$$\langle\tilde{\mathcal{A}}\rangle = 0, \tag{3.61}$$

which is due to the supersymmetry and thus does not vanish for the D3-D7 setups. For the first D3-D7 setup they found [4]

$$\langle \tilde{\mathcal{A}} \rangle = \frac{2\lambda}{4\pi^2(k_1^2+k_2^2)^3}\left(k_2^4 \sin\chi\phi_3^{\mathrm{cl}} + k_1^4 \cos\chi\phi_6^{\mathrm{cl}}\right). \tag{3.62}$$

For the second D3-D7 setup they found [5]

$$\langle \tilde{\mathcal{A}} \rangle = \frac{\lambda}{4\pi^2 n^2}\sin\chi\phi_5^{\mathrm{cl}}, \tag{3.63}$$

at one-loop level.

3.2 The tree

The tree level contribution $\langle W\rangle_{\mathrm{tree}}$ as given by (3.15) is computed for the two D3-D7 setups

$$\langle W\rangle_{\mathrm{tree}} = \lim_{T\to\infty}\left\langle \mathrm{tr}\,\mathrm{Pexp}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} dt\mathcal{A}^{\mathrm{cl}}(t)\right]\right\rangle, \tag{3.64}$$

$$\langle W\rangle_{\mathrm{tree}} = \lim_{T\to\infty}\mathrm{tr}\exp\int_{-\frac{T}{2}}^{\frac{T}{2}} dt\mathcal{A}^{\mathrm{cl}}(t). \tag{3.65}$$

For both setups $\mathcal{A}$ is time independent

$$\langle W\rangle_{\mathrm{tree}} = \lim_{T\to\infty}\mathrm{tr}\exp\left(T\mathcal{A}^{\mathrm{cl}}\right), \tag{3.66}$$

$$= \lim_{T\to\infty}\left[\exp\left(T\mathcal{A}^{\mathrm{cl}}\right)\right]_{ii} \tag{3.67}$$

In the large T limit, only the fastest growing exponential will contribute. The result depends on the largest eigenvalue η/z and its multiplicity μ

$$\langle W\rangle_{\mathrm{tree}} = \mu\exp\left(T\frac{\eta}{z}\right). \tag{3.68}$$

For the first setup the largest eigenvalue of $\mathcal{A}^{(\mathrm{I})}$ is $\eta^{(\mathrm{I})}/z$ where $2\eta^{(\mathrm{I})} = (k_1-1)\sin\chi + (k_2-1)\cos\chi$ with multiplicity $\mu^{(\mathrm{I})}=1$. For the second setup we need the largest eigenvalue of G_{56} and its multiplicity, this was found to be $\eta^{(\mathrm{II})} = \frac{n}{\sqrt{8}}\sin\chi$ and $\mu^{(\mathrm{II})} = n+1$ in [11]. The tree level results can finally be written

$$\langle W\rangle_{\mathrm{tree}}^{(\mathrm{I})} = \mu^{(\mathrm{I})}\exp\left(T\frac{(k_1-1)\sin\chi + (k_2-1)\cos\chi}{2z}\right), \tag{3.69}$$

$$\langle W\rangle_{\mathrm{tree}}^{(\mathrm{II})} = \mu^{(\mathrm{II})}\exp\left(T\frac{n}{\sqrt{8}z}\right). \tag{3.70}$$

Since the Wilson line is related to the particle interface potential by $\langle W\rangle = e^{-TV}$ we can identify the tree-level potentials

$$V^{(\mathrm{I})}_{\mathrm{tree}}(x) = -\lim_{T\to\infty}\frac{1}{T}\langle W\rangle^{(\mathrm{I})}_{\mathrm{tree}} = -\frac{k_1\sin\chi + k_2\cos\chi}{2x_3}, \tag{3.71}$$

$$V^{(\mathrm{II})}_{\mathrm{tree}}(x) = -\lim_{T\to\infty}\frac{1}{T}\langle W\rangle^{(\mathrm{II})}_{\mathrm{tree}} = -\frac{n\sin\chi}{\sqrt{8}x_3}, \tag{3.72}$$

having taken the large k_1, k_2 limit as implied by the double scaling limit, such that it is comparable to the holographic dual.

3.3 The lollipop

The focus of this subsection is the evaluation of the lollipop contribution (3.16)

$$\langle W\rangle_{\mathrm{lol}} = \lim_{T\to\infty}\left\langle \mathrm{tr}\int_{-T/2}^{T/2} dt\, U^{\mathrm{cl}}\left(-\tfrac{T}{2}, t\right)\tilde{\mathcal{A}}(t)U^{\mathrm{cl}}\left(t, \tfrac{T}{2}\right)\right\rangle. \tag{3.73}$$

The trace allows us to cycle the U^{cl} and combine their exponentials

$$\langle W\rangle_{\mathrm{lol}} = \lim_{T\to\infty}\left\langle \mathrm{tr}\int_{-T/2}^{T/2} dt\, U^{\mathrm{cl}}\left(-\tfrac{T}{2}, \tfrac{T}{2}\right)\tilde{\mathcal{A}}(t)\right\rangle. \tag{3.74}$$

We use that the expectation values are time independent and plug in the result for U^{cl}

$$\langle W\rangle_{\mathrm{lol}} = \lim_{T\to\infty} T\,\mathrm{tr}\, e^{T\mathcal{A}^{\mathrm{cl}}}\left\langle\tilde{\mathcal{A}}\right\rangle_{1-\mathrm{loop}}. \tag{3.75}$$

We will need 1-loop corrections to the vacuum expectation value. In the large T limit only the components multiplying the fastest growing exponential will contribute. In our conventions this will be the first component in both cases

$$\langle W\rangle_{\mathrm{lol}} = \lim_{T\to\infty} T\mu e^{T\frac{\eta}{z}}\left\langle\left[\tilde{\mathcal{A}}\right]_{11}\right\rangle_{1-\mathrm{loop}}. \tag{3.76}$$

Given the one-loop corrections (3.61),(3.62) and (3.63), we find

$$\langle W\rangle^{(\mathrm{D5})}_{\mathrm{lol}} = 0, \tag{3.77}$$

$$\langle W\rangle^{(\mathrm{I})}_{\mathrm{lol}} = -\mu^{(\mathrm{I})}\frac{\lambda T e^{T\eta^{(\mathrm{I})}/z}}{4\pi^2 z\left(k_1^2+k_2^2\right)^3}\left(k_1k_2^4\sin(\chi) + k_2k_1^4\cos(\chi)\right), \tag{3.78}$$

$$\langle W\rangle^{(\mathrm{II})}_{\mathrm{lol}} = -\mu^{(\mathrm{II})}\frac{\lambda T e^{T\eta^{(\mathrm{II})}/z}}{4\sqrt{8}\pi^2 zn}\sin(\chi), \tag{3.79}$$

having again taken the double scaling limit in (3.77). For the D3-D5 setup the one-loop corrections to the vacuum expectation values are vanishing due to supersymmetry, that means the lollipop contribution vanishes for the D3-D5 setup

$$\langle W\rangle_{\mathrm{lol}}^{(\mathrm{D5})} = 0. \tag{3.80}$$

3.4 And the tadpole

As in [3], the tadpole term (3.17) is the least straight forward term to compute. However, the same techniques can be employed with just minor complications. To proceed the analysis we write out all matrix indices explicitly

$$U_{\mathrm{tad}}(\alpha,\beta) = \int_\alpha^\beta dt \int_t^\beta dt' [U^{\mathrm{cl}}(\alpha,t)]_{ab}[\tilde{\mathcal{A}}(t)]_{bc}[U^{\mathrm{cl}}(t,t')]_{cd}[\tilde{\mathcal{A}}(t')]_{de}[U^{\mathrm{cl}}(t',\beta)]_{ef}. \tag{3.81}$$

They are all $N\times N$ matrices and have the same block structure.

$$\left[\begin{array}{c|c} p\times p & p\times(N-p) \\ \hline (N-p)\times p & (N-p)\times(N-p) \end{array}\right] \tag{3.82}$$

For the D3-D5 $p=k$, for the first D3-D7 setup $p=k_1k_2$ and for the second D3-D7 setup $p=d_G$. We decompose the indices such that latin indices run from 1 to p and greek indices run from $p+1$ to N and take the trace

$$\begin{aligned}
\operatorname{tr} U_{\mathrm{tad}}(\alpha,\beta) = &\int_\alpha^\beta dt \int_t^\beta dt' [1]_{\mu\nu}[\tilde{\mathcal{A}}(t)]_{\nu\rho}[1]_{\rho\sigma}[\tilde{\mathcal{A}}(t')]_{\sigma\gamma}[1]_{\gamma\mu} \\
&+ \int_\alpha^\beta dt \int_t^\beta dt' [1]_{\mu\nu}[\tilde{\mathcal{A}}(t)]_{\nu c}[e^{(t'-t)\mathcal{A}^{\mathrm{cl}}}]_{cd}[\tilde{\mathcal{A}}(t')]_{d\gamma}[1]_{\gamma\mu} \\
&+ \int_\alpha^\beta dt \int_t^\beta dt' [e^{(t-\alpha)\mathcal{A}^{\mathrm{cl}}}]_{ab}[\tilde{\mathcal{A}}(t)]_{b\rho}[1]_{\rho\sigma}[\tilde{\mathcal{A}}(t')]_{\sigma e}[e^{(\beta-t')\mathcal{A}^{\mathrm{cl}}}]_{ea} \\
&+ \int_\alpha^\beta dt \int_t^\beta dt' [e^{(t-\alpha)\mathcal{A}^{\mathrm{cl}}}]_{ab}[\tilde{\mathcal{A}}(t)]_{bc}[e^{(t'-t)\mathcal{A}^{\mathrm{cl}}}]_{cd}[\tilde{\mathcal{A}}(t')]_{de}[e^{(\beta-t')\mathcal{A}^{\mathrm{cl}}}]_{ea},
\end{aligned} \tag{3.83}$$

notice that $U^{\rm cl}(\alpha,\beta)$ is $e^{(\beta-\alpha)\mathcal{A}^{\rm cl}}$ in the $p\times p$ block and 1 elsewhere. The exponentials are combined and the expectation value is taken

$$\begin{aligned}\langle \operatorname{tr} U_{\rm tad}(\alpha,\beta)\rangle =& \int_\alpha^\beta dt \int_t^\beta dt' \langle [\tilde{\mathcal{A}}(t)]_{\mu\rho}[\tilde{\mathcal{A}}(t')]_{\rho\mu}\rangle \\ &+ \int_\alpha^\beta dt \int_t^\beta dt' [e^{(t'-t)\mathcal{A}^{\rm cl}}]_{cd}\langle [\tilde{\mathcal{A}}(t')]_{d\mu}[\tilde{\mathcal{A}}(t)]_{\mu c}\rangle \\ &+ \int_\alpha^\beta dt \int_t^\beta dt' [e^{(\beta-\alpha+t-t')\mathcal{A}^{\rm cl}}]_{eb}\langle [\tilde{\mathcal{A}}(t)]_{b\rho}[\tilde{\mathcal{A}}(t')]_{\rho e}\rangle \\ &+ \int_\alpha^\beta dt \int_t^\beta dt' [e^{(\beta-\alpha+t-t')\mathcal{A}^{\rm cl}}]_{eb}[e^{(t'-t)\mathcal{A}^{\rm cl}}]_{cd}\langle [\tilde{\mathcal{A}}(t)]_{bc}[\tilde{\mathcal{A}}(t')]_{de}\rangle .\end{aligned} \tag{3.84}$$

We briefly remind the reader that the two-point function can be expanded as

$$\langle \tilde{\mathcal{A}}(t)\tilde{\mathcal{A}}(t')\rangle = \Big(\big\langle \tilde{\phi}_a(t)\tilde{\phi}_a(t')\big\rangle - \big\langle \tilde{A}_0(t)\tilde{A}_0(t')\big\rangle\Big)\sin^2\chi \tag{3.85}$$

$$+\Big(\big\langle \tilde{\phi}_6(t)\tilde{\phi}_6(t')\big\rangle - \big\langle \tilde{A}_0(t)\tilde{A}_0(t')\big\rangle\Big)\cos^2\chi \tag{3.86}$$

$$+ 2\big\langle \tilde{\phi}_a(t)\tilde{\phi}_6(t')\big\rangle \sin\chi\cos\chi, \tag{3.87}$$

where $\tilde{\phi}_a = \tilde{\phi}_3$ for the D3-D5 and the first D3-D7 setup and $\tilde{\phi}_a = \tilde{\phi}_5$ for the second D3-D7 setup. Looking back at the first term of (3.84), we may conclude it vanishes since in the $(N-p)\times(N-p)$ block the fields have the same usual $\mathcal{N}=4$ super Yang-Mills propagators. That means we are left with the last three terms. Notice that the second and third term contains $\langle [\tilde{\mathcal{A}}(t')]_{d\mu}[\tilde{\mathcal{A}}(t)]_{\mu c}\rangle$ and for all setups we have that

$$\langle [\tilde{\mathcal{A}}(t')]_{a\alpha}[\tilde{\mathcal{A}}(t)]_{\beta b}\rangle \propto \delta_{\alpha\beta}. \tag{3.88}$$

This means the contraction over the Kronecker delta will give a factor $\delta_{\mu\mu} = (N-p)$, and since the final term of (3.84) is just a contraction over the $p\times p$ block, it will be independent of N. We conclude that, if the second or third term does not vanish, the final term will not contribute in the large N limit. (3.84) is now reduced to

$$\begin{aligned}\langle \operatorname{tr} U_{\rm tad}(\alpha,\beta)\rangle =& \int_\alpha^\beta dt \int_t^\beta dt' [e^{(t'-t)\mathcal{A}^{\rm cl}}]_{cd}\langle [\tilde{\mathcal{A}}(t')]_{d\mu}[\tilde{\mathcal{A}}(t)]_{\mu c}\rangle \\ &+ \int_\alpha^\beta dt \int_t^\beta dt' [e^{(\beta-\alpha+t-t')\mathcal{A}^{\rm cl}}]_{eb}\langle [\tilde{\mathcal{A}}(t)]_{b\rho}[\tilde{\mathcal{A}}(t')]_{\rho e}\rangle .\end{aligned} \tag{3.89}$$

We use the definition from (3.17) to find the Wilson line contribution

$$\langle W\rangle_{\rm tad} = \lim_{T\to\infty}\int_{-T/2}^{T/2} d\alpha \int_\alpha^{T/2} d\beta \Big[e^{-(\alpha-\beta)\mathcal{A}^{\rm cl}} + e^{(\alpha-\beta+T)\mathcal{A}^{\rm cl}}\Big]_{cd}\langle [\tilde{\mathcal{A}}]_{d\mu}(\alpha)[\tilde{\mathcal{A}}]_{\mu c}(\beta)\rangle, \tag{3.90}$$

having relabeled the integration variables. As previously mentioned, the propagator has the following form in all three setups

$$\langle [\tilde{\mathcal{A}}]_{d\mu}(\alpha)[\tilde{\mathcal{A}}]_{\mu c}(\beta)\rangle = \sum_n D^n_{dc} \sum_i \lambda_{i,n} K^{m^2_{i,n}}(\alpha,\beta), \tag{3.91}$$

where D is a diagonal prefactor and $K^{m^2_{i,n}}$ is the spacetime part of the propagator given in (3.92) below. We use the following representation of the propagator

$$K^{m_i^2}(\alpha,\beta) = \frac{g^2_{YM} z}{4\pi^2}\int_0^\infty dr\, r \frac{\sin(\delta r)}{\delta} I_{\nu_i}(rz)K_{\nu_i}(rz), \tag{3.92}$$

$$\nu_i = \sqrt{m_i^2 + \frac{1}{4}}, \tag{3.93}$$

having defined $\delta = \beta - \alpha$. We will have to perform integrals of the form

$$\langle W\rangle_{\text{tad}} = \lim_{T\to\infty}\int_{-T/2}^{T/2} d\alpha \int_\alpha^{T/2} d\beta \\ \left[e^{-(\alpha-\beta)\mathcal{A}^{\text{cl}}} + e^{(\alpha-\beta+T)\mathcal{A}^{\text{cl}}}\right]_{cd} \sum_n D^n_{dc}\sum_i \lambda_{i,n} K^{m^2_{i,n}}(\alpha,\beta). \tag{3.94}$$

We plug in the representation (3.92) and following [3], we proceed by changing variables $\alpha = \delta - T/2$, rescaling $r \to r/z$ and doing the β integration,

$$\langle W\rangle_{\text{tad}} = \frac{g^2_{YM} z}{4\pi^2}\lim_{T\to\infty}\int_0^T d\delta \int_{-T/2+\delta}^{T/2} d\beta \int_0^\infty dr\, r\frac{\sin(\delta r)}{\delta} \\ \left[e^{-(\alpha-\beta)\mathcal{A}^{\text{cl}}} + e^{(\alpha-\beta+T)\mathcal{A}^{\text{cl}}}\right]_{cd} \sum_n D^n_{dc}\sum_i \lambda_{i,n} I_{\nu_i}(rz)K_{\nu_i}(rz), \tag{3.95}$$

$$= \frac{g^2_{YM}}{4\pi^2 z}\lim_{T\to\infty}\int_0^T d\delta\,(T-\delta)\int_0^\infty dr\, r\frac{\sin(\delta r/z)}{\delta} \\ \left[e^{\delta\mathcal{A}^{\text{cl}}} + e^{(T-\delta)\mathcal{A}^{\text{cl}}}\right]_{cd} \sum_n D^n_{dc}\sum_i \lambda_{i,n} I_{\nu_{i,n}}(r)K_{\nu_{i,n}}(r). \tag{3.96}$$

The strategy is now to use integration by parts on the r integral, differentiating $\sin(\delta r/z)$ in order to cancel the $\frac{1}{\delta}$ and then perform the δ integration

$$\langle W\rangle_{\text{tad}} = \frac{g^2_{YM}}{4\pi^2 z^2}\lim_{T\to\infty}\int_0^T d\delta\,(T-\delta)\left[e^{\delta\mathcal{A}^{\text{cl}}} + e^{(T-\delta)\mathcal{A}^{\text{cl}}}\right]_{cd} \tag{3.97} \\ \sum_n D^n_{dc}\int_0^\infty dr\cos(\delta r/z)\int_r^\infty dr' r' \sum_i \lambda_{i,n} I_{\nu_{i,n}}(r')K_{\nu_{i,n}}(r').$$

This particular antiderivative was chosen because it makes the boundary term vanish at infinity, whilst the $\sin(\delta r/z)$ part makes the boundary term vanish at $r = 0$. We can now perform the δ integration. In the large T limit we have

$$\int_0^T d\delta\,(T-\delta)\left[e^{\delta\mathcal{A}^{\rm cl}} + e^{(T-\delta)\mathcal{A}^{\rm cl}}\right]_{cd} \sum_n D^n_{dc}\cos(\delta r/z) = \mu e^{\eta T/z} T z\,\frac{\eta}{\eta^2+r^2}\sum_n D^n_{1,1}, \tag{3.98}$$

since for our setups the largest eigenvalue of D coincides with the largest eigenvalue of $\mathcal{A}^{\rm cl}$ in the first entry. We plug the result back into (3.97)

$$\langle W\rangle_{\rm tad} = \mu\frac{g^2_{YM}Te^{\eta T/z}}{4\pi^2 z}\int_0^\infty dr\,\frac{\eta}{\eta^2+r^2}$$
$$\sum_n D^n_{1,1}\int_r^\infty dr'\,r'\sum_i \lambda_{i,n}I_{\nu_{i,n}}(r')K_{\nu_{i,n}}(r'). \tag{3.99}$$

It is here and in the following implicit that T is large. We will now perform the r' integration in the double scaling limit, we write it as

$$\int_r^\infty r'\sum_i\lambda_i I_{\nu_i}(r')K_{\nu_i}(r')dr' = -\sum_i\lambda_i\int_0^r r'I_{\nu_i}(r')K_{\nu_i}(r')dr'$$
$$+\sum_i\lambda_i\int_0^\infty r'I_{\nu_i}(r')K_{\nu_i}(r')dr', \tag{3.100}$$

and for convenience we define the functions F and A

$$F_{\nu_{i,n}}(r) = \int_0^r dr'\,r'I_{\nu_{i,n}}(r')K_{\nu_{i,n}}(r'), \tag{3.101}$$

$$A(r) = \int_r^\infty dr'\,r'\sum_i\lambda_{i,n}I_{\nu_{i,n}}(r')K_{\nu_{i,n}}(r') \tag{3.102}$$

$$= -\sum_i\lambda_{i,n}F_{\nu_{i,n}}(r) + \lim_{r'\to\infty}\sum_i\lambda_{i,n}F_{\nu_{i,n}}(r'). \tag{3.103}$$

This definition allows us to write (3.99) as

$$\langle W\rangle_{\rm tad} = \mu\frac{g^2_{YM}Te^{\eta T/z}}{4\pi^2 z}\int_0^\infty dr\,\frac{\eta}{\eta^2+r^2}\sum_n D^n_{1,1}A(r). \tag{3.104}$$

By doing the integral from (3.82) we find $F_{\nu_{i,n}}(r)$, the result is

$$F_{\nu_{i,n}}(r) = \frac{1}{2}r^2\left(I_{\nu_{i,n}+1}(r)K_{\nu_{i,n}-1}(r) + I_{\nu_{i,n}}(r)K_{\nu_{i,n}}(r)\right), \tag{3.105}$$

$$= -\frac{\nu_{i,n}}{2} + \frac{1}{2}(r^2+\nu^2_{i,n})I_{\nu_{i,n}}(r)K_{\nu_{i,n}}(r) - \frac{1}{2}r^2I'_{\nu_{i,n}}(r)K'_{\nu_{i,n}}(r). \tag{3.106}$$

The double scaling limit implies large $\nu_{i,n}$, we use the behaviour of the Bessel functions at large order and finite argument [12] and find

$$F_{\nu_{i,n}}(r) = -\frac{\nu_{i,n}}{2} + \frac{1}{2}(r^2+\nu_{i,n}^2)\frac{1}{2}\left(\nu_{i,n}^2+r^2\right)^{-1/2} + \frac{1}{2}r^2\frac{(\nu_{i,n}^2+r^2)^{1/2}}{2r^2} + O\left(\nu_{i,n}^{-1}\right), \tag{3.107}$$

$$F_{\nu_{i,n}}(r) = -\frac{\nu_{i,n}}{2} + \frac{1}{2}\left(\nu_{i,n}^2+r^2\right)^{1/2} + O\left(\nu_{i,n}^{-1}\right). \tag{3.108}$$

Given (3.103) we find $A(r)$

$$\begin{aligned} A(r) =& -\sum_i \lambda_{i,n}\left(-\frac{\nu_{i,n}}{2} + \frac{1}{2}\left(\nu_{i,n}^2+r^2\right)^{1/2}\right) \\ &+ \lim_{r'\to\infty}\sum_i \lambda_{i,n}\left(-\frac{\nu_{i,n}}{2} + \frac{1}{2}\left(\nu_{i,n}^2+r'^2\right)^{1/2}\right) + O\left(\nu_{i,n}^{-1}\right), \end{aligned} \tag{3.109}$$

$$= -\frac{1}{2}\sum_i \lambda_{i,n}\left(\nu_{i,n}^2+r^2\right)^{1/2} + \frac{1}{2}\lim_{r'\to\infty}\sum_i \lambda_{i,n}\left(\nu_{i,n}^2+r'^2\right)^{1/2} + O\left(\nu_{i,n}^{-1}\right) \tag{3.110}$$

The second term is Taylor expanded for large r'

$$\left(\nu_{i,n}^2+r'^2\right)^{1/2} = r'\left(1+\frac{\nu_{i,n}^2}{r'^2}\right)^{1/2} = r' + \frac{\nu_{i,n}^2}{2r'} + O\left(r'^{-3}\right), \tag{3.111}$$

$$A(r) = -\frac{1}{2}\sum_i \lambda_{i,n}\left(\nu_{i,n}^2+r^2\right)^{1/2} + \frac{1}{2}\lim_{r'\to\infty} r'\sum_i \lambda_{i,n} + O\left(\nu_{i,n}^{-1}\right). \tag{3.112}$$

We note that $A(r)$ is divergent unless $\sum_i \lambda_{i,n} = 0$, properly bunching our terms we can satisfy this condition for the three setups. In this case we find

$$A(r) = -\frac{1}{2}\sum_i \lambda_{i,n}\left(\nu_{i,n}^2+r^2\right)^{1/2} + O\left(\nu_{i,n}^{-1}\right). \tag{3.113}$$

This result is now plugged into (3.104) and the final integral is performed

$$\langle W\rangle_{\text{tad}} = -\mu\frac{g_{YM}^2 T e^{\eta T/z}}{8\pi^2 z}\sum_n D_{1,1}^n \int_0^\infty dr\,\frac{\eta}{\eta^2+r^2}\sum_i \lambda_{i,n}\left(\nu_{i,n}^2+r^2\right)^{1/2} \tag{3.114}$$

$$= -\mu\frac{g_{YM}^2 T e^{\eta T/z}}{16\pi^2 z}\sum_n D_{1,1}^n$$

$$\sum_i \lambda_{i,n}\left[2\sqrt{\nu_{i,n}^2-\eta^2}\ \text{arccot}\left(\frac{\eta}{\sqrt{\nu_{i,n}^2-\eta^2}}\right) - \eta\log\left(\nu_{i,n}^2\right)\right], \tag{3.115}$$

where we again used $\sum_i \lambda_{i,n} = 0$ in the second line. We are finally ready to plug in coefficients. For the D3-D5 setup we find

$$\langle W\rangle^{(\mathrm{D5})}_{\mathrm{tad}} = -\frac{\lambda T e^{\frac{T\eta^{(\mathrm{D5})}}{z}}}{16\pi^2 z\, k}\frac{\sin^2(\chi)}{\cos^3(\chi)}\left(2\chi - \pi + \sin(2\chi)\right). \tag{3.116}$$

For the first D3-D7 setup we let $k_2 = k_1 \tan(\psi_0)$ and take the large k_1 limit

$$\langle W\rangle^{(\mathrm{I})}_{\mathrm{tad}} = -\mu^{(\mathrm{I})}\frac{\lambda T e^{\frac{T\eta^{(\mathrm{I})}}{z}}\cos(\psi_0)}{4\pi^2 z\, k_1}\frac{\sin^2(\psi_0+\chi)}{4\cos^3(\psi_0+\chi)}\left(2\psi_0 + 2\chi - \pi + \sin(2\psi_0+2\chi)\right), \tag{3.117}$$

notice that $\cos(\psi_0)/k_1 = \left(k_1^2 + k_2^2\right)^{-1/2}$ gives the combination appearing in the double scaling parameter. For the second D3-D7 setup in the large n limit we find

$$\langle W\rangle^{(\mathrm{II})}_{\mathrm{tad}} = -\mu^{(\mathrm{II})}\frac{\lambda T e^{\frac{T\eta^{(\mathrm{II})}}{z}}}{4\sqrt{8}\pi^2 z n}\frac{\sin^2(\chi)}{\cos^3(\chi)}\left(2\chi - \pi + \sin(2\chi)\right). \tag{3.118}$$

3.5 The full one-loop potential

The full one-loop result is now obtained by adding the lollipop and the tadpole contribution

$$\langle W\rangle_{1-\mathrm{loop}} = \langle W\rangle_{\mathrm{lol}} + \langle W\rangle_{\mathrm{tad}} \tag{3.119}$$

For the D3-D5 setup the lollipop term vanishes

$$\langle W\rangle^{(\mathrm{D5})}_{1-\mathrm{loop}} = \langle W\rangle^{(\mathrm{D5})}_{\mathrm{tad}}. \tag{3.120}$$

For the first D3-D7 setup we find

$$\begin{aligned}\langle W\rangle^{(\mathrm{I})}_{1-\mathrm{loop}} = &-\mu^{(\mathrm{I})}\frac{\lambda T e^{\frac{T\eta^{(\mathrm{I})}}{z}}\cos(\psi_0)}{4\pi^2 z k_1}\Bigg[\cos(\chi)\sin(\psi_0)\cos^4(\psi_0)\\ &+\sin(\chi)\cos(\psi_0)\sin^4(\psi_0)\\ &+\frac{\sin^2(\psi_0+\chi)}{4\cos^3(\psi_0+\chi)}\left(2\psi_0+2\chi-\pi+\sin(2\psi_0+2\chi)\right)\Bigg],\end{aligned} \tag{3.121}$$

having also expressed the lollipop contribution in terms of $\psi_0 = \arctan(k_2/k_1)$. For the second setup we have

$$\langle W\rangle^{(\mathrm{II})}_{1-\mathrm{loop}} = -\mu^{(\mathrm{II})}\frac{\lambda T}{4\pi^2 n}\frac{e^{\frac{T\eta^{(\mathrm{II})}}{z}}}{\sqrt{8}z}\left[\sin(\chi) - \frac{\sin^2(\chi)}{\cos^3(\chi)}\left(\pi - 2\chi - \sin(2\chi)\right)\right]. \tag{3.122}$$

We now have to find the corresponding correction to the particle-interface potential. The potential and Wilson line are related by $\langle W\rangle = e^{-TV}$, we expand the potential as $V = V_{\text{tree}} + \tilde{V}$

$$\langle W\rangle = e^{-T(V_{\text{tree}}+\tilde{V})} = e^{-TV_{\text{tree}}}e^{-T\tilde{V}}, \tag{3.123}$$

$$= e^{-TV_{\text{tree}}}(1 - T\tilde{V} + O(\tilde{V}^2)), \tag{3.124}$$

For one-loop level we identify

$$\langle W\rangle_{\text{tree}} = e^{-TV_{\text{tree}}}, \tag{3.125}$$

$$\langle W\rangle_{1-\text{loop}} = -TV_{1-\text{loop}}e^{-TV_{\text{tree}}}, \tag{3.126}$$

the correction to the potential is thus

$$V_{1-\text{loop}} = -\lim_{T\to\infty}\frac{1}{T}\frac{\langle W\rangle_{1-\text{loop}}}{\langle W\rangle_{\text{tree}}}, \tag{3.127}$$

which concludes the gauge theory computation with the following results:

$$V^{(\text{D5})}_{1-\text{loop}} = V^{(\text{D5})}_{\text{tree}}\frac{\lambda}{8\pi^2k^2}\frac{\sin(\chi)}{\cos^3(\chi)}\left(\pi - 2\chi - \sin(2\chi)\right), \tag{3.128}$$

$$V^{(\text{I})}_{1-\text{loop}} = V^{(\text{I})}_{\text{tree}}\left(\frac{\lambda}{\pi^2\left(k_1^2+k_2^2\right)}\right)\frac{1}{2\sin(\psi_0+\chi)}$$
$$\left[\frac{\sin^2(\psi_0+\chi)}{4\cos^3(\psi_0+\chi)}\left(\pi - 2\psi_0 - 2\chi - \sin(2\psi_0+2\chi)\right)\right.$$
$$\left. - \cos(\chi)\sin(\psi_0)\cos^4(\psi_0) - \sin(\chi)\cos(\psi_0)\sin^4(\psi_0)\right], \tag{3.129}$$

$$V^{(\text{II})}_{1-\text{loop}} = V^{(\text{II})}_{\text{tree}}\left(\frac{\lambda}{\pi^2n^2}\right)\left[\frac{\sin(\chi)}{4\cos^3(\chi)}\left(\pi - 2\chi - \sin(2\chi)\right) - \frac{1}{4}\right]. \tag{3.130}$$

4 String theory computation

4.1 Solving the brane geometry

The dual to the Wilson line is the string that is suspended from the boundary and perpendicularly attached to the probe brane. For this reason, we must start with a treatment of the probe brane to find the exact position. The background is $AdS_5 \times S^5$ and in Table 3 is the orientation of the branes summarized.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	○	○	○	○	×	×	×	×	×	×
D5	○	○	○	×	○	○	○	×	×	×
D7	○	○	○	×	○	○	○	○	○	×

Table 3: The table shows the directions the branes are extended in. ○ denotes that the brane is extended in the coordinate.

We adopt the conventions of [10], including units where the radius of the AdS_5 is 1. The metric of the background in this coordinate system is given by

$$ds^2 = r^2(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{dr^2}{r^2} + d\psi^2 + \cos^2\psi \left(d\theta^2 + \sin^2\theta d\phi^2\right) + \sin^2\psi \left(d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\phi}^2\right), \tag{4.1}$$

where the latin coordinates are the coordinates of AdS_5 and the greek are coordinates of S^5. In these coordinates, the boundary of AdS_5 is at $r \to \infty$. The Ramond-Ramond 4-form is given by

$$C^{(4)} = r^4 dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 + \frac{c(\psi)}{2} d\cos\theta \wedge d\phi \wedge d\cos\tilde{\theta} \wedge d\tilde{\phi}, \tag{4.2}$$

here the gauge $C^{(4)}_{0123} = r^4$ is chosen.

4.1.1 The D5 brane

For the D3-D5 setup the D5 brane has geometry $AdS_4 \times S^2$ and there is k units of magnetic charge on the S^2. We may write the world volume gauge flux as

$$\mathcal{F} = \frac{k}{2}\, d\cos\theta \wedge d\phi. \tag{4.3}$$

The action of the D5 brane is given by

$$S_{D5} = S_{DBI} + S_{WZ}, \tag{4.4}$$

$$S_{DBI} = -\frac{T_5}{g_s} \int d^6\sigma \sqrt{-\det(G + 2\pi\alpha'\mathcal{F})}, \tag{4.5}$$

$$S_{WZ} = \frac{2\pi\alpha' T_5}{2g_s} \int C^{(4)} \wedge \mathcal{F}, \tag{4.6}$$

$$T_p = \frac{1}{(2\pi)^p \alpha'^{(p+1)/2} g_s}. \tag{4.7}$$

Here G is the induced metric and is defined by the pull-back of the metric

$$G_{\mu\nu} = \frac{\partial x^M}{\partial \xi^\mu}\frac{\partial x^N}{\partial \xi^\nu} g_{MN}. \tag{4.8}$$

We would like to compute the induced metric, we start start by writing the $AdS_5 \times S^5$ metric as a matrix

$$g_{MN} = \begin{bmatrix} -r^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r^{-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos^2\psi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos^2\psi\sin^2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin^2\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin^2\psi\sin^2\tilde{\theta} \end{bmatrix}. \tag{4.9}$$

The pull-back is now computed with the ansatz $r = r(x_3)$

$$G_{\mu\nu} = \begin{bmatrix} -r^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2 + r^{-2}\left(\frac{\partial r}{\partial x_3}\right)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos^2\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos^2\psi\sin^2\theta \end{bmatrix}. \tag{4.10}$$

To find the combination appearing in the Dirac Born Infeld action, we have to add $2\pi\alpha'\mathcal{F}$ it is convenient to define $\kappa = \pi\alpha' k$ this allows us to write

$$(G + 2\pi\alpha'\mathcal{F})_{\mu\nu} = \begin{bmatrix} -r^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2 + r^{-2}\left(\frac{\partial r}{\partial x_3}\right)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos^2\psi & \kappa\sin\theta \\ 0 & 0 & 0 & 0 & -\kappa\sin\theta & \cos^2\psi\sin^2\theta \end{bmatrix}. \tag{4.11}$$

The determinant can now readily be computed

$$\sqrt{-\det(G + 2\pi\alpha'\mathcal{F})} = r^4\sin\theta\sqrt{1 + r^{-4}\left(\frac{\partial r}{\partial x_3}\right)^2}\sqrt{\kappa^2 + \cos^4\psi}. \tag{4.12}$$

This gives us the DBI action

$$S_{DBI} = -\frac{T_5}{4g_s}\int d^6\sigma r^4 \sin\theta \sqrt{1 + r^{-4}\left(\frac{\partial r}{\partial x_3}\right)^2}\sqrt{\kappa^2 + \cos^4\psi}, \tag{4.13}$$

$$S_{DBI} = -\frac{\pi T_5 V}{g_s}\int dx_3 r^4 \sqrt{1 + r^{-4}\left(\frac{\partial r}{\partial x_3}\right)^2}\sqrt{\kappa^2 + \cos^4\psi} \tag{4.14}$$

where we have defined $\int dx_0 dx_1 dx_2 = V$. We will now do the Wess-Zumino part of the action, we multiply out the relevant part

$$S_{WZ} = \frac{2\pi\alpha' T_5}{2g_s}\int r^4 dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge \frac{\kappa}{4\pi\alpha'} d\cos\theta \wedge d\phi, \tag{4.15}$$

$$= \frac{\pi T_5}{g_s} V \int dx_3 r^4 \kappa, \tag{4.16}$$

where we have used the same definition of V. The full action is now

$$S_{D5} = -\frac{\pi T_5 V}{g_s}\sqrt{\kappa^2 + \cos^4\psi}\int dx_3 r^4 \sqrt{1 + r^{-4}\left(\frac{\partial r}{\partial x_3}\right)^2} - \frac{\kappa r^4}{\sqrt{\kappa^2 + \cos^4\psi}}. \tag{4.17}$$

The equations of motion are found using the Euler-Lagrange equations

$$\frac{\partial}{\partial x_3}\left[\left(1 + \frac{1}{r^4}\left(\frac{\partial r}{\partial x_3}\right)^2\right)^{-1/2}\frac{\partial r}{\partial x_3}\right] = 4r^3\sqrt{1 + \frac{1}{r^4}\left(\frac{\partial r}{\partial x_3}\right)^2}$$
$$- 2r^{-1}\left(1 + \frac{1}{r^4}\left(\frac{\partial r}{\partial x_3}\right)^2\right)^{-1/2}\left(\frac{\partial r}{\partial x_3}\right)^2 - \frac{4\kappa r^3}{\sqrt{\kappa^2 + \cos^4\psi}}, \tag{4.18}$$

$$\frac{\partial}{\partial\psi}\sqrt{\kappa^2 + \cos^4\psi} = 0. \tag{4.19}$$

Let us start with (4.19)

$$\frac{\partial}{\partial\psi}\sqrt{\kappa^2 + \cos^4\psi} = -\frac{4\cos^3\psi\sin\psi}{2\sqrt{\kappa^2 + \cos^4\psi}} = 0. \tag{4.20}$$

This means that the D5 brane sits at a particular value $\psi = \psi_1 = n\pi/2$, $n \in \mathbb{Z}$ for the equation of motion (4.20) to be satisfied. The relevant solution for us will be $\psi = 0$. To deal with (4.18) we plug in the ansatz

$$r(x_3) = \frac{\Lambda}{x_3}, \tag{4.21}$$

$$\frac{\partial}{\partial x_3}\left[\left(1+\frac{x_3^4}{\Lambda^4}\frac{\Lambda^2}{x_3^4}\right)^{-1/2}\left(-\frac{\Lambda}{x_3^2}\right)\right]=4\frac{\Lambda^3}{x_3^3}\sqrt{1+\frac{x_3^4}{\Lambda^4}\frac{\Lambda^2}{x_3^4}}$$
$$-2\frac{x_3}{\Lambda}\left(1+\frac{x_3^4}{\Lambda^4}\frac{\Lambda^2}{x_3^4}\right)^{-1/2}\frac{\Lambda^2}{x_3^4}-\frac{\Lambda^3}{x_3^3}\frac{8\kappa}{\sqrt{\frac{\kappa^2}{4}+\cos^4\psi}}, \tag{4.22}$$

$$\left(1+\frac{1}{\Lambda^2}\right)^{-1/2}\left(\frac{2\Lambda}{x_3^3}\right)=4\frac{\Lambda^3}{x_3^3}\sqrt{1+\frac{1}{\Lambda^2}}-2\frac{x_3}{\Lambda}\left(1+\frac{1}{\Lambda^2}\right)^{-1/2}\frac{\Lambda^2}{x_3^4}$$
$$-\frac{\Lambda^3}{x_3^3}\frac{4\kappa}{\sqrt{\kappa^2+\cos^4\psi}}, \tag{4.23}$$

notice that all x_3 cancels and the ansatz was successful. We solve for Λ

$$2\Lambda=4\Lambda^3\left(1+\frac{1}{\Lambda^2}\right)-2\Lambda-\Lambda^3\frac{4\kappa}{\sqrt{\kappa^2+\cos^4\psi}}\left(1+\frac{1}{\Lambda^2}\right)^{1/2}, \tag{4.24}$$

$$0=4\Lambda-\frac{4\kappa}{\sqrt{\kappa^2+\cos^4\psi}}\left(\Lambda^2+1\right)^{1/2}, \tag{4.25}$$

$$\Lambda^2+1=\frac{\Lambda^2}{\kappa^2}\left(\kappa^2+\cos^4\psi\right), \tag{4.26}$$

$$\Lambda^2=-\frac{\kappa^2}{\kappa^2-(\kappa^2+\cos^4\psi)}, \tag{4.27}$$

$$\Lambda=\pm\frac{\kappa}{\cos^2\psi}. \tag{4.28}$$

It is important to note, that for large flux Λ will also be large. This will be key for the perturbative computations to come and this will also be the case for the other setups. We are finally ready to write down the the boundary conditions of the string dual to the Wilson line. Let σ and τ be worldsheet parameters on the string and let $\sigma=\sigma_0$ and $\sigma=\sigma_1$ denote the two endpoints. The string should start at the AdS_5 boundary which is at $r=\infty$ and to match the Wilson line at $x_3=z$ the string should also start at $x_3=z$. This gives us the first two boundary conditions

$$r(\sigma_0,\tau)=\infty, \tag{4.29}$$

$$x_3(\sigma_0,\tau)=z. \tag{4.30}$$

The other end of the string has to be perpendicularly attached to the probe brane. This means for x_3 and r we will have the following Dirichlet and Neumann boundary conditions

$$r(\sigma_1,\tau) = \frac{\Lambda}{x_3(\sigma_1,\tau)}, \tag{4.31}$$

$$x_3'(\sigma_1,\tau) = \frac{r'(\sigma_1,\tau)}{\Lambda r^2(\sigma_1,\tau)}, \tag{4.32}$$

having arbitrarily chosen the positive Λ solution. For x_1 and x_2 the brane is filling that subspace, that means the string cannot have any suspension in those coordinates and still be perpendicular to the brane. This leaves us only the S^5 to consider. As previously mentioned, the brane sits at $\psi = 0$, which is the north pole. From any point on the S^5 the shortest perpendicular path will only be controlled by one parameter being the initial ψ coordinate. We introduce χ as a parameter that controls the suspension along ψ.

$$\psi(\sigma_0,\tau) = \frac{\pi}{2} - \chi, \tag{4.33}$$

$$\psi(\sigma_1,\tau) = 0, \tag{4.34}$$

where χ is defined as such to make it match the χ parameter on the gauge theory side. All the boundary conditions for the setup is summarized in Table 4.

	$r(\sigma_0,\tau) \;= \infty$	$r(\sigma_1,\tau) \;= \frac{\Lambda}{x_3(\sigma_1,\tau)}$
D3-D5 BC:	$x_3(\sigma_0,\tau) = z$	$x_3'(\sigma_1,\tau) = \frac{r'(\sigma_1,\tau)}{\Lambda r^2(\sigma_1,\tau)}$
	$\psi(\sigma_0,\tau) \;= \frac{\pi}{2} - \chi$	$\psi(\sigma_1,\tau) \;= 0$

Table 4: Boundary conditions for the fundamental string for the D3-D5 setup.

4.1.2 The first D7 brane

For the first D3-D7 setup the D7 brane has geometry $AdS_4 \times S^2 \times S^2$ and there is k_1 and k_2 units of magnetic charge on the two S^2's respectively. We may write the world volume gauge flux as

$$\mathcal{F} = \frac{k_1}{2} d\cos\theta \wedge d\phi + \frac{k_2}{2} d\cos\tilde{\theta} \wedge d\tilde{\phi}. \tag{4.35}$$

The action of the D7 brane is given by

$$S_{D7} = -\frac{T_7}{g_s}\int d^8\sigma \sqrt{-\det(G + 2\pi\alpha'\mathcal{F})} + \frac{(2\pi\alpha')^2 T_7}{2g_s}\int C^{(4)} \wedge \mathcal{F} \wedge \mathcal{F}, \tag{4.36}$$

$$T_p = \frac{1}{(2\pi)^p \alpha'^{(p+1)/2} g_s}. \tag{4.37}$$

Here G is the pull-back of the metric

$$G_{\mu\nu} = \frac{\partial x^M}{\partial \xi^\mu}\frac{\partial x^N}{\partial \xi^\nu} g_{MN}. \tag{4.38}$$

We would like to compute the induced metric, we start start by writing the $AdS_5 \times S^5$ metric as a matrix

$$g_{MN} = \begin{bmatrix} -r^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r^{-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos^2\psi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos^2\psi\sin^2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin^2\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin^2\psi\sin^2\tilde{\theta} \end{bmatrix}. \tag{4.39}$$

The pull-back is now computed with the ansatz $r = r(x_3)$

$$G_{\mu\nu} = \begin{bmatrix} -r^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2 + r^{-2}\left(\frac{\partial r}{\partial x_3}\right)^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos^2\psi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos^2\psi\sin^2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin^2\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin^2\psi\sin^2\tilde{\theta} \end{bmatrix}. \tag{4.40}$$

To find the combination appearing in the Dirac Born Infeld action, we have to add $2\pi\alpha'\mathcal{F}$ it is convenient to define $(f_1, f_2) = 2\pi\alpha'(k_1, k_2)$ this allows us to write

$$(G + 2\pi\alpha'\mathcal{F})_{\mu\nu} = \begin{bmatrix} -r^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2 + r^{-2}\left(\frac{\partial r}{\partial x_3}\right)^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos^2\psi & \frac{f_1}{2}\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{f_1}{2}\sin\theta & \cos^2\psi\sin^2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin^2\psi & \frac{f_2}{2}\sin\tilde{\theta} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{f_2}{2}\sin\tilde{\theta} & \sin^2\psi\sin^2\tilde{\theta} \end{bmatrix}. \tag{4.41}$$

The determinant can now readily be computed

$$\sqrt{-\det(G + 2\pi\alpha'\mathcal{F})} = \frac{r^2}{4}\sin\theta\sin\tilde{\theta}\sqrt{r^4 + \left(\frac{\partial r}{\partial x_3}\right)^2} \sqrt{f_1^2 f_2^2 + 4f_1\sin^4\psi + 4f_2\cos^4\psi + 16\sin^4\psi\cos^4\psi}. \tag{4.42}$$

This gives us the DBI action

$$S_{DBI} = -\frac{T_7}{4g_s}\int d^8\sigma r^4 \sin\theta\sin\tilde{\theta}\sqrt{1 + \frac{1}{r^4}\left(\frac{\partial r}{\partial x_3}\right)^2} \sqrt{f_1^2 f_2^2 + 4f_1\sin^4\psi + 4f_2\cos^4\psi + 16\sin^4\psi\cos^4\psi}, \tag{4.43}$$

$$S_{DBI} = -\frac{4\pi^2 T_7 V}{g_s}\int dx_3 r^4 \sqrt{1 + \frac{1}{r^4}\left(\frac{\partial r}{\partial x_3}\right)^2} \sqrt{f_1^2 f_2^2 + 4f_1\sin^4\psi + 4f_2\cos^4\psi + 16\sin^4\psi\cos^4\psi} \tag{4.44}$$

where we have defined $\int dx_0 dx_1 dx_2 = V$. We will now focus on the Wess-Zumino part of the action

$$S_{WZ} = \frac{(2\pi\alpha')^2 T_7}{2g_s}\int C^{(4)} \wedge \mathcal{F} \wedge \mathcal{F}. \tag{4.45}$$

We multiply out the relevant part

$$S_{WZ} = 2\frac{(2\pi\alpha')^2 T_7}{2g_s}\int r^4 dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge \frac{\sqrt{\lambda}}{4\pi} f_1 d\cos\theta \wedge d\phi \wedge \frac{\sqrt{\lambda}}{4\pi} f_2 d\cos\tilde{\theta} \wedge d\tilde{\phi}, \tag{4.46}$$

$$= \frac{(2\pi)^2 T_7}{g_s} V \int dx_3 r^4 f_1 f_2, \tag{4.47}$$

where we have used the same definition of V and the fact that we have taken the radius of curvature to be $1 = \alpha'\sqrt{\lambda}$. The full action is now

$$S_{D7} = -\frac{4\pi^2 T_7 V}{g_s}\int dx_3 r^4 \sqrt{1 + \frac{1}{r^4}\left(\frac{\partial r}{\partial x_3}\right)^2}\kappa - r^4 f_1 f_2, \tag{4.48}$$

where we have defined

$$\kappa = \sqrt{f_1^2 f_2^2 + 4 f_1 \sin^4\psi + 4 f_2 \cos^4\psi + 16\sin^4\psi\cos^4\psi}. \tag{4.49}$$

The equations of motion are found using the Euler-Lagrange equations

$$\frac{\partial}{\partial x_3}\left[\kappa\left(1 + \frac{1}{r^4}\left(\frac{\partial r}{\partial x_3}\right)^2\right)^{-1/2}\frac{\partial r}{\partial x_3}\right] = 4\kappa r^3\sqrt{1 + \frac{1}{r^4}\left(\frac{\partial r}{\partial x_3}\right)^2}$$
$$- 2\kappa r^{-1}\left(1 + \frac{1}{r^4}\left(\frac{\partial r}{\partial x_3}\right)^2\right)^{-1/2}\left(\frac{\partial r}{\partial x_3}\right)^2 - 4r^3 f_1 f_2, \tag{4.50}$$

$$\frac{\partial}{\partial\psi}\kappa = 0. \tag{4.51}$$

Let us start with (4.51)

$$\frac{\partial}{\partial\psi}\kappa = \frac{1}{2\kappa}\left(16 f_1^2\sin^3\psi\cos\psi - 16 f_2^2\sin\psi\cos^3\psi + 64\sin^3\psi\cos^5\psi - 64\sin^5\psi\cos^3\psi\right), \tag{4.52}$$

$$f_1^2\sin^2\psi - f_2^2\cos^2\psi + 4\sin^2\psi\cos^4\psi - 4\sin^4\psi\cos^2\psi = 0. \tag{4.53}$$

This means that the D7 brane sits at a particular value $\psi = \psi_1$ for the equation of motion (4.53) to be satisfied. To deal with (4.50) we plug in the ansatz

$$r(x_3) = \frac{\Lambda}{x_3} \tag{4.54}$$

$$\frac{\partial}{\partial x_3}\left[\left(1+\frac{x_3^4}{\Lambda^4}\frac{\Lambda^2}{x_3^4}\right)^{-1/2}\left(-\frac{\Lambda}{x_3^2}\right)\right] = 4\frac{\Lambda^3}{x_3^3}\sqrt{1+\frac{x_3^4}{\Lambda^4}\frac{\Lambda^2}{x_3^4}}$$
$$-2\frac{x_3}{\Lambda}\left(1+\frac{x_3^4}{\Lambda^4}\frac{\Lambda^2}{x_3^4}\right)^{-1/2}\frac{\Lambda^2}{x_3^4} - \frac{4\Lambda^3}{\kappa x_3^3}f_1 f_2, \tag{4.55}$$

$$\left(1+\frac{1}{\Lambda^2}\right)^{-1/2}\left(\frac{2\Lambda}{x_3^3}\right) = 4\frac{\Lambda^3}{x_3^3}\sqrt{1+\frac{1}{\Lambda^2}} - 2\frac{x_3}{\Lambda}\left(1+\frac{1}{\Lambda^2}\right)^{-1/2}\frac{\Lambda^2}{x_3^4} - \frac{4\Lambda^3}{\kappa x_3^3}f_1 f_2, \tag{4.56}$$

also for this setup does all the x_3 cancel and the ansatz is successful. We solve for Λ

$$2\Lambda = 4\Lambda^3\left(1+\frac{1}{\Lambda^2}\right) - 2\Lambda - \frac{4\Lambda^3}{\kappa}f_1 f_2\left(1+\frac{1}{\Lambda^2}\right)^{1/2}, \tag{4.57}$$

$$0 = 4\Lambda^2 - \frac{4\Lambda^2}{\kappa}f_1 f_2\left(1+\frac{1}{\Lambda^2}\right)^{1/2}, \tag{4.58}$$

$$\frac{1}{\Lambda^2} = \frac{\kappa^2}{f_1^2 f_2^2} - 1, \tag{4.59}$$

$$\Lambda = \pm\sqrt{\frac{f_1^2 f_2^2}{\kappa^2 - f_1^2 f_2^2}}. \tag{4.60}$$

For the boundary conditions, the same considerations for the AdS_5 part of the geometry will be valid. For the S^5 part of the geometry the brane is sitting at $\psi = \psi_1$ where ψ_1 is the solution to (4.53). That means the string must end at this value of ψ. As for the other coordinates on S^5 the brane is covering them, that means the string cannot have suspension in these coordinates. The boundary conditions will again have a parameter χ that controls the initial ψ coordinates similar to the D3-D5 setup. Table 5 shows a summary of the boundary conditions for this D3-D7 setup.

	$r(\sigma_0,\tau) = \infty$	$r(\sigma_1,\tau) = \frac{\Lambda}{x_3(\sigma_1,\tau)}$
D3-D7$^{(I)}$ BC:	$x_3(\sigma_0,\tau) = z$	$x_3'(\sigma_1,\tau) = \frac{r'(\sigma_1,\tau)}{\Lambda r^2(\sigma_1,\tau)}$
	$\psi(\sigma_0,\tau) = \frac{\pi}{2} - \chi$	$\psi(\sigma_1,\tau) = \psi_1$

Table 5: Boundary conditions for the string for the first D3-D7 setup.

4.1.3 The second D7 brane

The discussion in this subsection is largely borrowed from [9, 13]. In this setup we have $d_G = \frac{1}{6}(n+1)(n+2)(n+3)$ D3 branes ending on $N_7 = n+2$ of D7 branes with geometry $AdS_4 \times S^4$ sitting at $\psi = 0$. Similar to previous setups, we have d_G units of flux on the S^4. But in this case the gauge field is a non-abelian gauge field with $U(N_7)$ symmetry. The second Chern class on the S^4 in this setup is given by

$$d_G = \frac{1}{8\pi^2} \int_{S^4} \mathrm{tr}\, F \wedge F. \tag{4.61}$$

For this setup it is convenient to use coordinates on $AdS_5 \times S^5$ such that the metric takes the form

$$ds^2 = r^2(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{dr^2}{r^2} + d\psi^2 + \cos^2\psi d\Omega_{S^4}^4, \tag{4.62}$$

where $d\Omega_{S^4}^4$ is the metric of S^4. The action for the D7 branes is given by

$$S_{D7} = S_{DBI} + S_{WZ}, \tag{4.63}$$

$$S_{DBI} = \frac{T_7}{g_s} \mathrm{STr} \int d\sigma^8 \sqrt{-\det(G + 2\pi\alpha' F)}, \tag{4.64}$$

$$S_{WZ} = \frac{T_7}{g_s} \mathrm{STr} \int d\sigma^8 \frac{(2\pi\alpha')^2}{2} C^{(4)} \wedge F \wedge F. \tag{4.65}$$

where the STr prescription involves taking symmetric averages over all orderings of non-abelian fields, in our case being F. For the Wess-Zumino term we find

$$S_{WZ} = \frac{T_7 V}{g_s} \int dx_3 r^4 \frac{(2\pi\alpha')^2}{2} d_G, \tag{4.66}$$

having again defined $V = \int dx_0 dx_1 dx_2$. The Dirac Born Infeld part was treated carefully in [8, 13]

$$S_{DBI} = \frac{T_7}{g_s} \int d\sigma^8 \sqrt{g_{AdS}} \sqrt{g_S} \left(N_7 + \frac{(2\pi\alpha')^2}{8} \mathrm{tr}\, \epsilon_{abcd} F^{ab} F^{cd} \right), \tag{4.67}$$

$$S_{DBI} = \frac{T_7 V N_7}{3 g_s} \int dx_3 r^4 \sqrt{1 + \frac{1}{r^4} \left(\frac{\partial r}{\partial x_3} \right)^2} \left(1 + \frac{6\pi^2 \alpha'^2}{N_7} d_G \right). \tag{4.68}$$

The complete action is now

$$S_{D7} = \frac{T_7 V N_7}{3 g_s} \int dx_3 r^4 \sqrt{1 + \frac{1}{r^4} \left(\frac{\partial r}{\partial x_3} \right)^2} (1 + Q) + r^4 Q, \tag{4.69}$$

$$Q = \frac{6\pi^2 \alpha'^2}{N_7} d_G. \tag{4.70}$$

The equation of motion is now found

$$\frac{\partial}{\partial x_3}\left[(1+Q)\left(1+\frac{1}{r^4}\left(\frac{\partial r}{\partial x_3}\right)^2\right)^{-1/2}\frac{\partial r}{\partial x_3}\right] = (1+Q)4r^3\sqrt{1+\frac{1}{r^4}\left(\frac{\partial r}{\partial x_3}\right)^2}$$
$$-2(1+Q)r^{-1}\left(1+\frac{1}{r^4}\left(\frac{\partial r}{\partial x_3}\right)^2\right)^{-1/2}\left(\frac{\partial r}{\partial x_3}\right)^2 + 4r^3Q. \tag{4.71}$$

Similar to the other setups this can be solved with the same ansatz $r = \frac{\Lambda}{x_3}$,

$$\left(1+\frac{1}{\Lambda^2}\right)^{-1/2}\left(\frac{2\Lambda}{x_3^3}\right) = 4\frac{\Lambda^3}{x_3^3}\sqrt{1+\frac{1}{\Lambda^2}} - 2\frac{x_3}{\Lambda}\left(1+\frac{1}{\Lambda^2}\right)^{-1/2}\frac{\Lambda^2}{x_3^4} + \frac{4\Lambda^3}{(1+Q)x_3^3}Q, \tag{4.72}$$

$$0 = \Lambda^2\sqrt{1+\frac{1}{\Lambda^2}} - \left(1+\frac{1}{\Lambda^2}\right)^{-1/2} + \frac{\Lambda^2}{(1+Q)}Q, \tag{4.73}$$

$$\frac{Q^2}{(1+Q)^2}\left(1+\frac{1}{\Lambda^2}\right) = 1, \tag{4.74}$$

$$\Lambda^2 = \frac{1}{\frac{(1+Q)^2}{Q^2}-1}, \tag{4.75}$$

$$\Lambda = \pm\frac{Q}{\sqrt{1+2Q}}. \tag{4.76}$$

The boundary conditions are similar to the first D3-D7 setup just with a different Λ and for this case we have $\psi_1 = 0$. The complete set of boundary conditions are summarized in Table 6.

D3-D7$^{(\mathrm{II})}$ BC:	$r(\sigma_0,\tau) = \infty$	$r(\sigma_1,\tau) = \frac{\Lambda}{x_3(\sigma_1,\tau)}$
	$x_3(\sigma_0,\tau) = z$	$x_3'(\sigma_1,\tau) = \frac{r'(\sigma_1,\tau)}{\Lambda r^2(\sigma_1,\tau)}$
	$\psi(\sigma_0,\tau) = \frac{\pi}{2}-\chi$	$\psi(\sigma_1,\tau) = 0$

Table 6: Boundary conditions for the string for the first D3-D7 setup.

4.2 The string action

Now that we have the precise boundary conditions for the string, we can look to find the least action. We start with the Polyakov action and the Virasoro constraints

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \left(-\dot{X}^M \dot{X}_M + X'^M X'_M\right), \tag{4.77}$$

$$\dot{X}^M \dot{X}_M + X'^M X'_M = 0, \tag{4.78}$$

$$\dot{X}^M X'_M = 0, \tag{4.79}$$

here dot denotes derivation with respect to τ and prime denotes derivation with respect to σ where τ and σ are worldsheet parameters. We use the following ansatz

$$x_0 = x_0(\tau), \qquad x_3 = x_3(\sigma), \qquad r = r(\sigma), \qquad \psi = \psi(\sigma). \tag{4.80}$$

The action and constraints becomes

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \frac{1}{r^2} r'^2 + r^2 \left(\dot{x}_0^2 + x_3'^2\right) + \psi'^2, \tag{4.81}$$

$$r^2 \dot{x}_0^2 = r^2 x_3'^2 + \frac{1}{r^2} r'^2 + \psi'^2, \tag{4.82}$$

The equations of motion are found using the Euler-Lagrange equations

$$\ddot{x}_0 = 0, \tag{4.83}$$

$$\psi'' = 0, \tag{4.84}$$

$$\left(r^2 x_3'\right)' = 0, \tag{4.85}$$

$$\left(\frac{2r'}{r^2}\right)' = 2r\left(\dot{x}_0^2 + x_3'^2\right) - 2\frac{1}{r^3} r'^2. \tag{4.86}$$

The first three equations are solved by

$$x_0 = z_1 \tau + z_0, \tag{4.87}$$

$$\psi' = m, \tag{4.88}$$

$$r^2 x_3' = c. \tag{4.89}$$

We choose the gauge for the time parameterization to simply have

$$x_0 = \tau. \tag{4.90}$$

m and c will have to be determined given the boundary conditions. We start with m as it is most straight forward. We let the worldsheet parameter σ run from 0 to π, this allows us to write the solution to (4.89) as

$$\psi(\sigma) = (\psi_1 - \psi_0) \frac{\sigma}{\pi} + \psi_0, \tag{4.91}$$

where ψ_1 and ψ_0 are simply the boundary conditions for the ψ coordinate. We thus have

$$m = \frac{\psi_1 - \psi_0}{\pi}. \tag{4.92}$$

As previously alluded to, c will be more complicated to link to the boundary conditions. To initiate our analysis, we plug in (4.88), (4.89) and (4.90) into the Virasoro constraint (4.82)

$$r^2 = \frac{c^2}{r^2} + \frac{1}{r^2} r'^2 + m^2, \tag{4.93}$$

$$r' = \sqrt{r^4 - m^2 r^2 - c^2}, \tag{4.94}$$

this will be a useful result throughout. Ideally, we would like to relate c to the initial value $x_3(\sigma_0, \tau) = z$, however we will have to settle for a relation between $r(\sigma_1, \tau) = r_1$ and c. To find this we simply consider (4.94) at σ_1 and use the boundary condition $x_3'(\sigma_1, \tau) = \frac{r'(\sigma_1,\tau)}{\Lambda r^2(\sigma_1,\tau)}$

$$\Lambda r_1^2 x_3'(\sigma_1, \tau) = \sqrt{r_1^4 - m^2 r_1^2 - c^2}. \tag{4.95}$$

We can now use (4.89) and solve for c

$$\Lambda c = \sqrt{r_1^4 - m^2 r_1^2 - c^2}, \tag{4.96}$$

$$\Lambda^2 c^2 = r_1^4 - m^2 r_1^2 - c^2, \tag{4.97}$$

$$c^2 = \frac{r_1^4 - m^2 r_1^2}{\Lambda^2 + 1}. \tag{4.98}$$

This gives us an equation for c, but we still need to find r_1, we integrate the Virasoro constraint (4.94) from $\sigma = 0$ to $\sigma = \pi$

$$\frac{1}{\sqrt{r^4 - m^2 r^2 - c^2}} r' = 1, \tag{4.99}$$

$$\int_\infty^{r_1} \frac{dr}{\sqrt{r^4 - m^2 r^2 - c^2}} = \pi, \tag{4.100}$$

this can be inverted to find r_1, notice that it does not depend on z. The equation of motion (4.89) with the boundary condition $x_3'(\sigma_1, \tau) = \frac{r'(\sigma_1,\tau)}{\Lambda r^2(\sigma_1,\tau)}$ and the Virasoro constraint (4.94) is sufficient to uniquely specify r_1. Now that we know how to find m and c given the boundary conditions, we can go back to the action (4.81)

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \frac{1}{r^2} r'^2 + r^2 \left(\dot{x}_0^2 + x_3'^2 \right) + \psi'^2, \tag{4.101}$$

$$= \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma r^2, \tag{4.102}$$

having used the Virasoro constraint (4.82) and $x_0 = \tau$ in the second line. We let $T = \int d\tau$

$$S = \frac{T\sqrt{\lambda}}{4\pi} \int_{\infty}^{r_1} \frac{r^2}{r'} dr. \tag{4.103}$$

The action is divergent in the $r = \infty$ boundary, this is expected since we had a Neumann boundary condition, the correct mapping to the Wilson line is taking the Legendre transform of the action [14]. Practically, this simply amounts to subtracting the linear divergence

$$S = \frac{T\sqrt{\lambda}}{4\pi} \int_{1/\epsilon}^{r_1} \frac{r^2}{r'} dr. \tag{4.104}$$

We use the Virasoro constraint (4.94) to replace r'

$$S = \frac{T\sqrt{\lambda}}{4\pi} \int_{1/\epsilon}^{r_1} \frac{r^2}{\sqrt{r^4 - m^2 r^2 - c^2}} dr, \tag{4.105}$$

notice the following identity

$$\frac{d}{dr}\left[\frac{1}{r}\sqrt{r^4 - m^2 r^2 - c^2}\right] = \frac{r^2}{\sqrt{r^4 - m^2 r^2 - c^2}} + \frac{c^2}{r^2\sqrt{r^4 - m^2 r^2 - c^2}}. \tag{4.106}$$

Since the first term on the right hand side matches the integrand of (4.105), we can use the identity to rewrite the action

$$S = \frac{T\sqrt{\lambda}}{4\pi}\left[\frac{1}{r}\sqrt{r^4 - m^2 r^2 - c^2}\Bigg|_{r=1/\epsilon}^{r=r_1} - \int_{1/\epsilon}^{r_1} \frac{c^2 dr}{r^2\sqrt{r^4 - m^2 r^2 - c^2}}\right]. \tag{4.107}$$

For the first term we have

$$\frac{1}{r}\sqrt{r^4 - m^2 r^2 - c^2}\Bigg|_{r=1/\epsilon}^{r=r_1} = \frac{1}{r_1}\sqrt{r_1^4 - m^2 r_1^2 - c^2} - \epsilon\sqrt{\epsilon^{-4} - m^2\epsilon^{-2} - c^2}, \tag{4.108}$$

for small epsilon the second term becomes a simple pole

$$\epsilon\sqrt{\epsilon^{-4} - m^2\epsilon^{-2} - c^2} = \frac{1}{\epsilon}\sqrt{1 - m^2\epsilon^2 - c^2\epsilon^4} = \frac{1}{\epsilon} + O(\epsilon), \tag{4.109}$$

which is exactly what we need to remove in order to match the Wilson line. The first term on the right hand side of (4.108) can be rewritten using (4.96)

$$\frac{1}{r_1}\sqrt{r_1^4 - m^2 r_1^2 - c^2} = \frac{\Lambda c}{r_1}. \tag{4.110}$$

The action finally becomes

$$S = \frac{T\sqrt{\lambda}}{4\pi} \left[\frac{\Lambda c}{r_1} - \int_{\infty}^{r_1} \frac{c^2 dr}{r^2\sqrt{r^4 - m^2 r^2 - c^2}} \right]. \tag{4.111}$$

Before initiating the perturbative expansion in order to compare to the gauge theory side, it is highly convenient to start with an expression where the proper z dependence is explicit. To treat the z dependence we integrate (4.89)

$$\frac{dz}{d\sigma} = \frac{c}{r^2}, \tag{4.112}$$

$$z_1 - z = \int_0^{\pi} \frac{c}{r^2} d\sigma \tag{4.113}$$

where $x_3(\sigma_1, \tau) = z_1$. We use the Virasoro constraint (4.94)

$$z_1 - z = c \int_{\infty}^{r_1} \frac{dr}{r^2\sqrt{r_1^4 - m^2 r_1^2 - c^2}}, \tag{4.114}$$

we can now use the boundary condition $x_3(\sigma_1, \tau) = \Lambda / r(\sigma_1, \tau)$

$$z = \frac{\Lambda}{r_1} - c \int_{\infty}^{r_1} \frac{dr}{r^2\sqrt{r_1^4 - m^2 r_1^2 - c^2}}. \tag{4.115}$$

We can finally use this result to rewrite the action to show the correct z dependence explicitly

$$S = \frac{T\sqrt{\lambda}}{4\pi z} c \left[\frac{\Lambda}{r_1} - c \int_{\infty}^{r_1} \frac{dr}{r^2\sqrt{r^4 - m^2 r^2 - c^2}} \right]^2. \tag{4.116}$$

4.2.1 Expanding the potential perturbatively

The particle-interface potential is given by $V = S/T$, to compare the result to the gauge theory result we have to expand (4.116) in the double scaling parameter. First we find r_1 by inverting (4.100) together with (4.98)

$$\int_{\infty}^{r_1} \frac{dr}{\sqrt{r^4 - m^2 r^2 - c^2}} = \pi, \tag{4.117}$$

$$\int_{\infty}^{r_1} \frac{dr}{\sqrt{r^4 - m^2 r^2 - \frac{r_1^4 - m^2 r_1^2}{\Lambda^2 + 1}}} = \pi, \tag{4.118}$$

as previously remarked, in the double scaling limit Λ becomes large for all three setups. It is useful to define the small parameter

$$a = \frac{1}{\Lambda^2 + 1}. \tag{4.119}$$

We are now looking for a solution of r_1 with the following form

$$r_1 = \sum_n c_n a^n. \tag{4.120}$$

To proceed we expand (4.118) in a power series using Taylors theorem

$$\sum_n \frac{a^n}{n!} \frac{d^n}{da^n} \int_\infty^{r_1(a)} \frac{dr}{\sqrt{r^4 - m^2 r^2 - a(r_1^4(a) - m^2 r_1^2(a))}} \bigg|_{a=0} = \pi, \tag{4.121}$$

Since this should be valid for all values of a, we have the following set of equations

$$\frac{d^n}{da^n} \left[\int_\infty^{r_1(a)} \frac{dr}{\sqrt{r^4 - m^2 r^2 - a(r_1^4(a) - m^2 r_1^2(a))}} - \pi \right]_{a=0} = 0. \tag{4.122}$$

Luckily, we are able to solve the first equation $n = 0$

$$\int_\infty^{c_0} \frac{dr}{\sqrt{r^4 - m^2 r^2}} - \pi = 0, \tag{4.123}$$

$$\frac{\operatorname{arccsc}\left(\frac{c_0}{m}\right)}{m} = \pi, \tag{4.124}$$

this gives us the unperturbed result

$$c_0 = \frac{m}{\sin(m\pi)}. \tag{4.125}$$

We will need the first correction to this, to find this we have to consider the first derivative

$$\frac{d}{da} \int_\infty^{r_1(a)} \frac{dr}{\sqrt{r^4 - m^2 r^2 - a(r_1^4(a) - m^2 r_1^2(a))}} \bigg|_{a=0} = 0. \tag{4.126}$$

Since the integrand and limits are continuously differentiable we can invoke the Leibniz integral rule, it will be useful to give the integrand a name

$$f(r, a) = \frac{1}{\sqrt{r^4 - m^2 r^2 - a(r_1^4(a) - m^2 r_1^2(a))}}, \tag{4.127}$$

$$\frac{dr_1}{da} f(r_1, a) \bigg|_{a=0} + \int_\infty^{r_1(a)} \frac{\partial}{\partial a} f(r, a) dr \bigg|_{a=0} = 0. \tag{4.128}$$

The first term simply becomes

$$\frac{dr_1}{da} f(r_1, a) \bigg|_{a=0} = \frac{c_1}{\sqrt{c_0^4 - m^2 c_0^2}}. \tag{4.129}$$

For the second term we have

$$\int_{\infty}^{r_1(a)} \frac{\partial}{\partial a} f(r,a) dr \bigg|_{a=0} = \int_{\infty}^{r_1} \frac{r_1^4 - m^2 r_1^2}{2\left(r^4 - m^2 r^2 - a(r_1^4 - m^2 r_1^2)\right)^{3/2}} dr \bigg|_{a=0}, \quad (4.130)$$

$$= \frac{c_0^4 - m^2 c_0^2}{2} \int_{\infty}^{c_0} \frac{1}{\left(r^4 - m^2 r^2\right)^{3/2}} dr, \quad (4.131)$$

$$= \frac{\sqrt{c_0^2 - m^2}}{4m^4}(m^2 - 3c_0^2) + \frac{3(c_0^2 - m^2)}{4m^5} \operatorname{arccsc}\left(\frac{c_0}{m}\right). \quad (4.132)$$

We can now write an equation for c_1

$$\frac{c_1}{\sqrt{c_0^4 - m^2 c_0^2}} + \frac{\sqrt{c_0^2 - m^2}}{4m^4}(m^2 - 3c_0^2) + \frac{3(c_0^2 - m^2)}{4m^5} \operatorname{arccsc}\left(\frac{c_0}{m}\right) = 0, \quad (4.133)$$

plugging in our solution for $c_0 = m/\sin(m)$ we find c_1 to be

$$c_1 = \frac{m}{8} \frac{\cos^2(m\pi)}{\sin^5(m\pi)} \left(5 + \cos(2m\pi) - 6m\pi \cot(m\pi)\right). \quad (4.134)$$

This is all we will need to match the result to the one-loop order on the gauge theory side

$$r_1 = \frac{m}{\sin(m\pi)} + a \frac{m}{8} \frac{\cos^2(m\pi)}{\sin^5(m\pi)} \left(5 + \cos(2m\pi) - 6m\pi \cot(m\pi)\right) + O(a^2). \quad (4.135)$$

It is now straight forward to find c^2 by plugging in r_1 into (4.98) and reexpanding

$$c^2 = a\left(r_1^4 - m^2 r_1^2\right), \quad (4.136)$$

$$= a \frac{m^4 \cos^2(m\pi)}{\sin^4(m\pi)} + a^2 \frac{m^4}{4} \frac{\cos^2(m\pi)}{\sin^8(m\pi)} \left(5 + \cos(2m\pi) - 6m\pi \cot(m\pi)\right)\left(2 - \sin^2(m\pi)\right) + O(a^3). \quad (4.137)$$

To find an expansion for the action we need to expand

$$\int_{\infty}^{r_1} \frac{dr}{r^2 \sqrt{r^4 - m^2 r^2 - c^2}}, \quad (4.138)$$

at the lowest order this becomes

$$\int_{\infty}^{\frac{m}{\sin(m\pi)}} \frac{dr}{r^2 \sqrt{r^4 - m^2 r^2}} = -\frac{\sin(2m\pi)}{4m^3} - \frac{\pi}{2m^2}. \quad (4.139)$$

Notice that in (4.116) the first term on the right hand side has factor Λ which is large and the second term has factor c which is small, that means this expansion of the integral is sufficient

$$\int_{\infty}^{r_1} \frac{dr}{r^2\sqrt{r^4 - m^2 r^2 - c^2}} = -\frac{\sin(2m\pi)}{4m^3} - \frac{\pi}{2m^2} + O(a). \tag{4.140}$$

We use the above results to write an expansion of the action in (4.116)

$$S = \frac{T\sqrt{\lambda}}{4\pi z} c\left[\frac{\Lambda}{r_1} - c\int_{\infty}^{r_1} \frac{dr}{r^2\sqrt{r^4 - m^2 r^2 - c^2}}\right]^2, \tag{4.141}$$

$$= \frac{T\sqrt{\lambda}}{4\pi z}\left[\Lambda\cos(m\pi) + \frac{1}{4\Lambda}\frac{\cos^2(m\pi)}{\sin^2(m\pi)}\left(7\cos(m\pi) + \frac{m\pi}{\sin(m\pi)}\right) + O(\Lambda^{-2})\right], \tag{4.142}$$

having used that $a = \frac{1}{\Lambda^2+1}$ and reexpanded for large Λ. To find the potential we need to find m for the three setups and expand Λ in the double scaling parameter. For m we have already found the relation (4.92)

$$m = \frac{\psi_1 - \psi_0}{\pi}, \tag{4.143}$$

for the D3-D5 setup and the second D3-D7 setup we have $\psi_1 = 0$ and $\psi_0 = \pi/2 - \chi$ this means we have

$$m^{(\mathrm{D5})} = m^{(\mathrm{II})} = \frac{\chi}{\pi} - \frac{1}{2}. \tag{4.144}$$

For the first D3-D7 setup the ψ_1 value satisfies (4.53)

$$4\pi^2\frac{k_1^2}{\lambda}\sin^2\psi_1 - 4\pi^2\frac{k_2^2}{\lambda}\cos^2\psi_1 + 4\sin^2\psi_1\cos^4\psi_1 - 4\sin^4\psi_1\cos^2\psi_1 = 0, \tag{4.145}$$

where we are interested in the behaviour in the double scaling limit. Let

$$\psi_1 = \sum_n \phi_n \left(\frac{\lambda}{4\pi^2(k_1^2 + k_2^2)}\right)^n \tag{4.146}$$

We multiply (4.145) through by the double scaling parameter

$$\frac{k_1^2}{(k_1^2 + k_2^2)}\tan^2\psi_1 - \frac{k_2^2}{(k_1^2 + k_2^2)} = \frac{4\lambda}{4\pi^2(k_1^2 + k_2^2)}\left(\sin^4\psi_1\cos^2\psi_1 - \sin^2\psi_1\cos^4\psi_1\right), \tag{4.147}$$

at lowest order in the double scaling parameter we find

$$\frac{k_1^2}{(k_1^2+k_2^2)}\tan^2\phi_0 = \frac{k_2^2}{(k_1^2+k_2^2)}, \tag{4.148}$$

$$\phi_0 = \arctan\left(\frac{k_2}{k_1}\right). \tag{4.149}$$

Equation (4.147) at first order in the double scaling parameter is

$$\frac{2k_1^2}{(k_1^2+k_2^2)}\frac{\tan\phi_0}{\sin^2\phi_0}\phi_1 = 4\sin^4\phi_0\cos^2\phi_0 - 4\sin^2\phi_0\cos^4\phi_0, \tag{4.150}$$

using our solution for ϕ_0 we find

$$\phi_1 = \frac{2k_1^3k_2(k_1+k_2)(k_2-k_1)}{(k_1^2+k_2^2)^3}. \tag{4.151}$$

This is all we need for the one-loop result

$$\psi_1 = \arctan\left(\frac{k_2}{k_1}\right) + \frac{\lambda}{4\pi^2(k_1^2+k_2^2)}\frac{2k_1^3k_2(k_1+k_2)(k_2-k_1)}{(k_1^2+k_2^2)^3} + O\left(\left(\tfrac{\lambda}{4\pi^2(k_1^2+k_2^2)}\right)^2\right). \tag{4.152}$$

We conclude that for the first D3-D7 setup m is

$$m^{(\mathrm{I})} = \frac{\psi_1}{\pi} + \frac{\chi}{\pi} - \frac{1}{2}, \tag{4.153}$$

with the ψ_1 given in (4.152). Only the angles Λ remains to be expanded, for the D3-D5 case there are no corrections

$$\Lambda^{(\mathrm{D5})} = \frac{\pi k}{\sqrt{\lambda}}. \tag{4.154}$$

For the first D3-D7 setup we found

$$\Lambda^{(\mathrm{I})} = \frac{\frac{4\pi^2k_1k_2}{\lambda}}{\sqrt{\left(\frac{4\pi^2k_1^2}{\lambda}+4\cos^4\psi_1\right)\left(\frac{4\pi^2k_2^2}{\lambda}+4\sin^4\psi_1\right)-\frac{4^2\pi^4k_1^2k_2^2}{\lambda^2}}}. \tag{4.155}$$

This can be expanded in the double scaling parameter

$$\Lambda^{(\mathrm{I})} = \frac{\pi(k_1^2+k_2^2)^{1/2}}{\sqrt{\lambda}}\left[1 - \frac{\lambda}{\pi^2(k_1^2+k_2^2)}\frac{k_1^2k_2^2}{2(k_1^2+k_2^2)^2} + O\left(\left(\tfrac{\lambda}{\pi^2(k_1^2+k_2^2)}\right)^2\right)\right]. \tag{4.156}$$

For the second D3-D7 setup we had

$$\Lambda^{(\mathrm{II})} = \frac{\frac{\pi^2(n+1)(n+3)}{\lambda}}{\sqrt{1+\frac{2\pi^2(n+1)(n+3)}{\lambda}}}. \tag{4.157}$$

This can easily be expanded in the double scaling parameter

$$\Lambda^{(\mathrm{II})} = \frac{\pi n}{\sqrt{2\lambda}}\left[1-\frac{\lambda}{4\pi^2 n^2}+O\left(\left(\frac{\lambda}{\pi^2 n^2}\right)^2\right)\right]. \tag{4.158}$$

We can finally plug these Λ and m into the action (4.142) and divide by T and reexpand to identify the string theory prediction of the potentials. We first state the result for the D3-D5 setup

$$V^{(\mathrm{D5})}(x) = -\frac{k\sin(\chi)}{2x_3}\left[1+\frac{\lambda}{\pi^2 k^2}\frac{\sin(\chi)}{8\cos^3(\chi)}\left(\pi-2\chi-\sin(2\chi)\right)+\mathcal{O}\left(\frac{\lambda^2}{\pi^4 k^4}\right)\right]. \tag{4.159}$$

For the first D3-D7 setup we found

$$\begin{aligned} V^{(\mathrm{I})}(x) = -\frac{k_1\sin(\chi)+k_2\cos(\chi)}{2x_3}\Bigg[1&+\frac{\lambda}{\pi^2\left(k_1^2+k_2^2\right)}\frac{1}{2\sin(\psi_0+\chi)} \\ &\Bigg[\frac{\sin^2(\psi_0+\chi)}{4\cos^3(\psi_0+\chi)}\left(\pi-2\psi_0-2\chi-\sin(2\psi_0+2\chi)\right) \\ &-\cos(\chi)\sin(\psi_0)\cos^4(\psi_0)-\sin(\chi)\cos(\psi_0)\sin^4(\psi_0)\Bigg]+\mathcal{O}\left(\frac{\lambda^2}{\pi^4(k_1^2+k_2^2)^2}\right)\Bigg], \end{aligned} \tag{4.160}$$

here we used the same definition as on the gauge theory side $\tan\psi_0 = k_2/k_1$. Finally, for the second D3-D7 setup we found

$$V^{(\mathrm{II})}(x) = -\frac{n\sin(\chi)}{2\sqrt{2}x_3}\left[1+\frac{\lambda}{\pi^2 n^2}\left(\frac{\sin(\chi)}{4\cos^3(\chi)}\left(\pi-2\chi-\sin(2\chi)\right)-\frac{1}{4}\right)+\mathcal{O}\left(\frac{\lambda^2}{\pi^4 n^4}\right)\right]. \tag{4.161}$$

5 Conclusion and outlook

Comparing the string theory prediction (4.159), (4.160) and (4.161) with the gauge theory results (3.128), (3.129) and (3.130), we find perfect agreement for all three cases. We may thus conclude the project as a positive test of the robustness of the AdS/dCFT correspondence. The non-local observable matches to at least one-loop order even for setups with completely broken supersymmetry and boundary integrability. This means that these holographic dualities are independent of these other notions. One could have imagined that in the D3-D5 setup the string only matches the Wilson line because they are entirely constrained by symmetries, however with the two D3-D7 setups the symmetries is weakened sufficiently to require a perturbative calculation.

It is interesting to compare the computation on both sides. In particular, a main difference in the character of the computations for the supersymmetric and non-supersymmetric setups on the gauge theory side, is that for the supersymmetric setup the one-loop correction to the one-point function is vanishing. This correction results in an extra term in the potential at one-loop level. On the string theory side, albeit the computations look deceptively similar for the three setups, this extra correction gets slipped in as a correction to the D7 brane angle Λ.

Another interesting comparison between the sides is the growth of complexity order by order. One could just as easily have gone to second loop order on the string side as it simply amounts to Taylor expanding to one higher order. But on for the gauge theory side going to two-loop order is completely unfeasible at the current stage. The first thing one has to change is using the two-loop corrections to the one-point function, and already when trying to compute this one encounters difficulties in the form of integrals that are yet to be solved.

For the special case $\pi m = \psi_1 - \psi_0 = 0$ on the string theory side, one can scale out r_1 and find the potential exactly for the two integrable setups. As a proof of concept we let $m = 0$ in (4.118) and perform the substitution $r = r_1 v$

$$\int_{\infty}^{r_1} \frac{dr}{\sqrt{r^4 - \frac{r_1^4}{\Lambda^2+1}}} = \pi, \tag{5.1}$$

$$r_1 = \frac{1}{\pi} \int_{\infty}^{1} \frac{dv}{\sqrt{v^4 - \frac{1}{\Lambda^2+1}}}. \tag{5.2}$$

We can use this to find $c^2 = r_1^4/\sqrt{\Lambda^2+1}$ and plug the results into the equation for the action (4.116). We use the same substitution on the other integral that appears in the action and identify the potential

$$V(x) = \frac{\sqrt{\lambda}}{4\pi x_3} \frac{r_1}{\sqrt{\Lambda^2+1}} \left[\Lambda r_1 - \frac{r_1}{\sqrt{\Lambda^2+1}} \int_\infty^1 \frac{dv}{v^2\sqrt{v^4 - \frac{1}{\Lambda^2+1}}} \right]^2 . \tag{5.3}$$

This expression is general to all three cases, however for the first D3-D7 setup the angle Λ cannot be inverted to be written in terms of the fluxes k_1, k_2 in closed form. Since it is possible to find the potential exactly for the D3-D5 and the second D3-D7 setup it would be interesting to try to match it on the gauge theory side. With the perturbative techniques employed here on the gauge theory side this is of course not possible, however new work on localization [15] could make this possible for the D3-D5 setup if applied to Wilson lines.

References

[1] G. 't Hooft, A Planar Diagram Theory for Strong Interactions, Nucl. Phys. B 72 (1974) 461. doi:10.1016/0550-3213(74)90154-0.

[2] J. Maldacena, The large-n limit of superconformal field theories and supergravity, International journal of theoretical physics 38 (4) (1999) 1113–1133. arXiv:9711200.

[3] M. de Leeuw, A. C. Ipsen, C. Kristjansen, M. Wilhelm, One-loop Wilson loops and the particle-interface potential in AdS/dCFT, Vol. B768, 2017. arXiv:1608.04754, doi:10.1016/j.physletb.2017.02.047.

[4] A. Gimenez Grau, C. Kristjansen, M. Volk, M. Wilhelm, A quantum check of non-supersymmetric AdS/dCFT, JHEP 01 (2019) 007. arXiv:1810.11463, doi:10.1007/JHEP01(2019)007.

[5] A. Gimenez-Grau, C. Kristjansen, M. Volk, M. Wilhelm, A quantum framework for AdS/dCFT through fuzzy spherical harmonics on S^4, JHEP 04 (2020) 132. arXiv:1912.02468, doi:10.1007/JHEP04(2020)132.

[6] I. Buhl-Mortensen, M. de Leeuw, A. C. Ipsen, C. Kristjansen, M. Wilhelm, A quantum check of AdS/dCFT, JHEP 01 (2017) 098. arXiv:1611.04603, doi:10.1007/JHEP01(2017)098.

[7] N. R. Constable, R. C. Myers, O. Tafjord, The noncommutative bion core, Phys. Rev. D61 (2000) 106009. arXiv:hep-th/9911136, doi:10.1103/PhysRevD.61.106009.

[8] C. Kristjansen, G. W. Semenoff, D. Young, Chiral primary one-point functions in the D3-D7 defect conformal field theory, JHEP 01 (2013) 117. arXiv:1210.7015, doi:10.1007/JHEP01(2013)117.

[9] N. R. Constable, R. C. Myers, O. Tafjord, Non-abelian brane intersections, JHEP 06 (2001) 023. arXiv:hep-th/0102080, doi:10.1088/1126-6708/2001/06/023.

[10] K. Nagasaki, H. Tanida, S. Yamaguchi, Holographic interface-particle potential, JHEP 01 (2012) 139. arXiv:1109.1927, doi:10.1007/JHEP01(2012)139.

[11] M. de Leeuw, C. Kristjansen, G. Linardopoulos, One-point functions of non-protected operators in the SO(5) symmetric D3–D7 dCFT, J. Phys. A50 (25) (2017) 254001. arXiv:1612.06236, doi:10.1088/1751-8121/aa714b.

[12] F. W. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, NIST handbook of mathematical functions hardback and CD-ROM, Cambridge university press, 2010.

[13] R. C. Myers, M. C. Wapler, Transport properties of holographic defects, JHEP 12 (2008) 115. arXiv:0811.0480, doi:10.1088/1126-6708/2008/12/115.

[14] N. Drukker, D. J. Gross, H. Ooguri, Wilson loops and minimal surfaces, Phys. Rev. D60 (1999) 125006. arXiv:hep-th/9904191, doi:10.1103/PhysRevD.60.125006.

[15] Y. Wang, Taming Defects in $\mathcal{N} = 4$ Super-Yang-Mills (3 2020). arXiv:2003.11016.