Spin Dynamics out of Equilibrium

Master's Thesis in Condensed Matter Physics

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Abstract

In this thesis a Heisenberg spin system, where two spins are coupled antiferromagnetically is investigated. The system is placed in a magnetic field and. A current flows through it. The current can push the system out of equilibirum, due to the emergence of current induced non conservative forces. The equation of motion for the system is theoretically approximated and analyzed. Runaway modes are found to exist for the non equilibrium system, which lead to the system seeking away from the original ground state.
After establishing that runaway modes exist, parameters for the system are chosen, in order to obtain a runaway mode. The precise dynamics of the system, in the pressence of a runaway mode, is found by numerically analyzing the exact equation of motion. The analysis shows that the pressence of a runaway mode causes the antiferromagnetically coupled spins to align parallel in the direction of the magnetic field.

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Chapter 1

Introduction

In all applications it is of vital importance to be able to control physical systems electrically. It is notoriously difficult for spin and nanomagnets to be controlled by electrical currents. The ability to use an electrical current is therefore of great interest.

An electrical current flowing through a molecule extended as a bridge between electrodes and placed in a magnetic field, has shown to be able to break the molecule, due to vibrational modes [1]. The molecule brakes, due to the flowing current, and thereby it acts like a fuse. The combination of electrical current and magnetic field induces non-conservative forces, referred to as ”curl forces” [1]. The ”curl forces” have either a slowing down or speeding up effect on the vibrational modes, and can be regarded as being in either ”headwind” or ”tailwind”. Therefore it is also referred to as ”electron wind” [1].

Spins behave differently compared to vibrations. Vibrations, which can be regarded as springs, have the ability to become more and more violent, as they are driven and eventually overstretch. Spins on the other hand are confined to a sphere and do not possess the feature of overstretching. Instead the spins can move around on the sphere. In the case of a single spin in a magnetic field the motion of the spin on the sphere is precession.

The purpose of thesis will be to investigate and analyse the dynamics of two coupled spins in the presence of curl forces. This is done by first, an expansion close to the ground state, which wil indicate the behaviour around the ground state. Secondly, by utilizing the results obtained from the expansion, the dynamics of the two spins will be found numerically.

Chapter 2

Concepts and Theory

2.1 Classical particle

A particle with mass, m, and moving with a velocity, $\vec{v}$, will have a momentum. The momentum, $\vec{p}$, for the particle is given by the mass times the velocity.

$$\vec{p} = m \cdot \vec{v} \tag{2.1}$$

Newton showed that the force on a particle is equal to the mass, m, times the acceleration, $\vec{a}$.

$$\vec{F} = m \cdot \vec{a} \tag{2.2}$$

From the time derivative of the momentum, it can be seen that, it is equal to the force. This means that the time derivative of the momentum is equal to the force.

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = m \cdot \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = m \cdot \vec{a} = \vec{F} \tag{2.3}$$

The angular momentum for a particle is the cross product between the momentum of the particle and the directional vector, $\vec{r}$,[2]. In figure **2.1** is shown a particle moving in a circular orbit with momentum, $\vec{p}$. This gives rise to an angular momentum for the particle. The angular momentum of the particle is a vector orthogonal to the $\vec{r}$ and $\vec{p}$ vectors, which in this case corresponds to an out of plane vector.

$$\vec{L} = \vec{r} \times \vec{p} \tag{2.4}$$

The torque on the particle is defined as the cross product between the positional vector, $\vec{r}$ and the force, $\vec{F}$.

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{2.5}$$

The torque is a vector, which describes how the rotational motion of an object is changing in time, when influenced by a force field.

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t} \tag{2.6}$$

But if the position is also a function of time the torque is:

$$\vec{\tau} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \vec{p} + \vec{r} \times \frac{\partial \vec{p}}{\partial t} \tag{2.7}$$

Hamilton's equations are [3]

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = -\frac{\partial H}{\partial \vec{r}} \quad \text{and} \quad \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \frac{\partial H}{\partial \vec{p}} \tag{2.8}$$

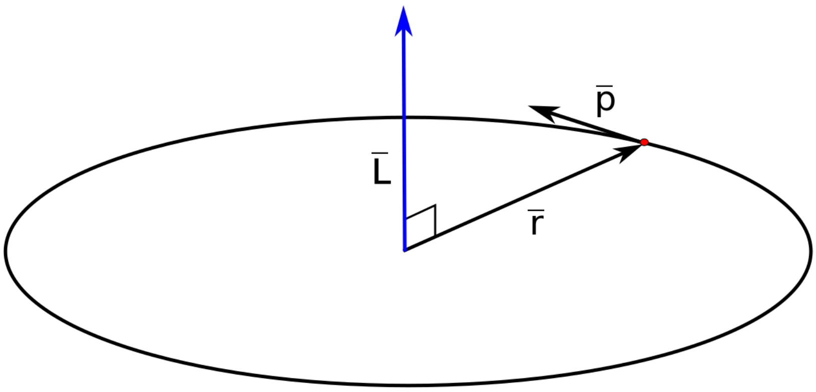

Figure 2.1: Particle moving in circular orbit with radius $|\vec{r}|$ and momentum $\vec{p}$. The resulting angular momentum, $\vec{L}$, is shown as a blue vector perpendicular to the motion of the particle.

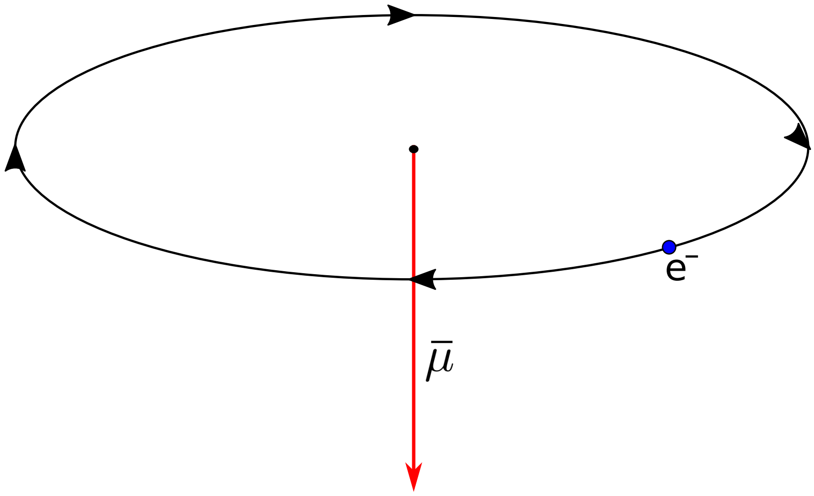

Figure 2.2: An electron moving in a circular orbit generating a current loop, and thereby inducing a magnetic moment, $\vec{\mu}$, perpendicular to the orbit of the electron, shown as a red vector.

Therefore the torque or the change of angular momentum in time can be described by Hamilton's equations.

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = -\vec{r} \times \frac{\partial H}{\partial \vec{r}} + \frac{\partial H}{\partial \vec{p}} \times \vec{p} \tag{2.9}$$

2.1.1 Spin of a single particle

When an electron is moving in a circular orbit, it generates a magnetic moment perpendicular to the orbit [2]. The physics of the magnetic moment is similar to the physics of angular momentum. In both cases the resulting vector from a specific motion is perpendicular to the motion. By comparison of figures **2.1** and **2.2** it is clear that, the physics in the two cases is very much alike.
If the orbiting electron is placed in a magnetic field, the direction of the magnetic moment will change in time corresponding to the classical torque. Therefore the magnetic moment can be described semiclassically as the equivalent to the angular momentum.

$$\vec{\mu} = \gamma\vec{L} = \vec{\mathbf{s}} \tag{2.10}$$

γ is the gyromagnetic ratio [4]. The magnetic moment of the electron is called spin and is represented as a vector $\vec{\mathbf{s}}$.

2.2 Spin 1/2

Spin can be desribed in different ways. In the paragraph above the spin was described semiclassically as a current loop with an associated magnetic moment. In this section spin-1/2 will be described in a quantum mechanical way.

The algebra of spin-1/2 is based on the 2×2 Pauli spin matrices[4].

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.11}$$

These matrices all have the same property that they are hermitian and unitary. Since the matrices are hermitian, they represent observables. Therefore the matrices can be written as a vector

$$\boldsymbol{\sigma} = (\hat{\sigma}_x \, , \, \hat{\sigma}_y \, , \, \hat{\sigma}_z.) \tag{2.12}$$

This vector is confined to a sphere, where the vector goes from the center to the surface of the sphere. All three directions can not be known at the same time. The values of the x- and y-directions can not be known, if the value is known for the z-direction [5]. This gives rise to the cone feature in figure **2.3**.

Another way to see that multiple directions can not be known at the same time, is to consider the superposition of the spin with the eigenstates of the Pauli matrices [5]

$$\begin{aligned} \psi_z^+ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \psi_z^- &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ \psi_x^+ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \psi_x^- &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\ \psi_y^+ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, & \psi_y^- &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \end{aligned} \tag{2.13}$$

Here it can be seen that the ψ_z^+ state can be generated by the two states ψ_x^+ and ψ_x^-

$$\frac{1}{\sqrt{2}} \left(\psi_x^+ + \psi_x^- \right) = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \frac{1}{2} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \psi_z^+ \tag{2.14}$$

In the same fashion the other spin states can also be expressed as a superposition of perpendicular spin states.

2.2.1 Zeeman Effect

An electron is a spin-1/2 particle, which means that it has two possible spin states, as seen in the previous section. Each of the spin states can be distinguished from the others in a magnetic field. The energy of a spin in a magnetic field is given by the so-called the Zeeman energy [4].

$$E = g \, \mu_B \, \vec{m}_s \cdot \vec{B} \tag{2.15}$$

g is the gyromagnetic ratio, which for an electron is ~ 2. μ_B is the Bohr magneton and is defined as $\mu_B = q_e \, \hbar / 2 \, m_e$, with the electron charge and mass q_e and m_e. $\vec{m}_s$ is the direction of the spin, which for an electron can be either up or down corresponding to

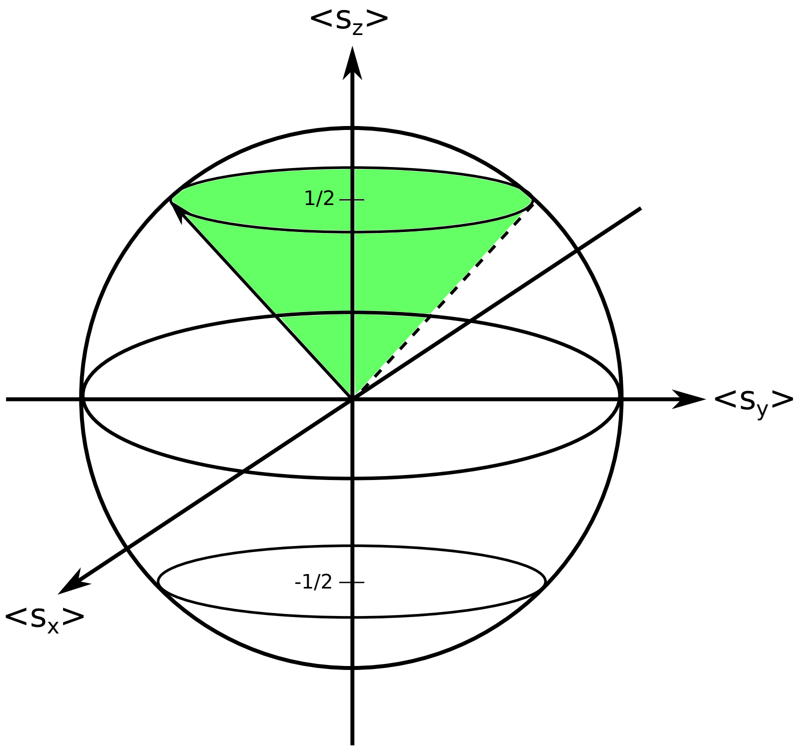

Figure 2.3: A sphere representing the expectation values for the spin direction. Knowing that $\langle s_z \rangle = 1/2$, then $\langle s_x \rangle$ and $\langle s_y \rangle$ can not be known precisely, illustrated by the green cone.

$\pm 1/2$. $\vec{B}$ is the magnetic field. The spin direction and the constants in front can be simplified $g\,\mu_B\,\vec{m}_s = -\vec{\mathbf{s}}$, with the minus originating from the fact that the product of $\mu_B\,\vec{m}_s$ must be negative [4].

$$\hat{\mathcal{H}}_{Zeeman} = -\vec{\mathbf{s}} \cdot \vec{B} \tag{2.16}$$

In figure **2.4** is shown how a magnetic field splits the two energy levels of the electron spin states. It is seen that the spin state parallel to the magnetic field is lowest in energy. The magnetic field is assumed to be along the z-axis and the spin states are pure in the z direction.

2.2.2 Motion of a spin in magnetic field

Instead of having a pure spin state in the z direction, the spin state can be a superposition of all three direction.
A spin in a magnetic field will have a specific motion, because the spin is associated with angular momentum [4]. In order to be able to understand the motion of a more complicated system, this motion needs to be explored further.

The energy of the spin in the magnetic field is given by the Zeeman energy discussed above. The time evolution of the spin can be found with equation **2.8**, if the spin is expressed semiclassically.

$$\begin{aligned} \dot{\vec{\mathbf{s}}} &= -\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{r}\times\vec{p}\right)\cdot\vec{B} = -\left(B\,\vec{\times}\,\vec{r}\right)\times\vec{p} + \vec{r}\times\left(\vec{p}\times\vec{B}\right) \\ \Rightarrow \quad \dot{\vec{\mathbf{s}}} &= -\vec{B}\times\left(\vec{r}\times\vec{p}\right) = -\vec{B}\times\vec{\mathbf{s}} \end{aligned} \tag{2.17}$$

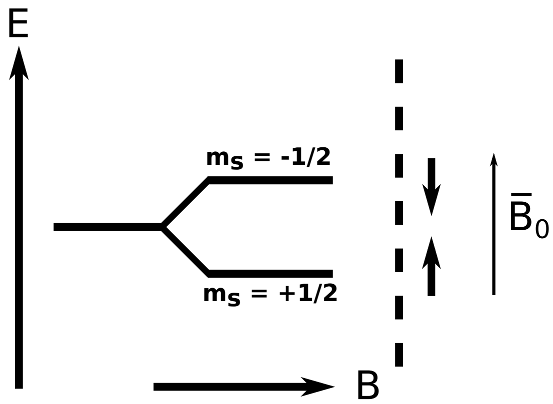

Figure 2.4: The energy splitting of the two electron spin states. The splitting occurs because of the magnetic field. For the case of equation **2.16** it is more preferable to align with the magnetic field, and therefore the spin parallel to the magnetic field will have the lowest energy.

For convenience the magnetic direction is chosen along the z-axis. With the spin having a polar angle θ to the direction of the magnetic field the following is obtained: [4]

$$\begin{aligned} \dot{s}_x &= g\,B\,s_y \\ \dot{s}_y &= -g\,B\,s_x \\ \dot{s}_z &= 0 \end{aligned} \tag{2.18}$$

With g as the gyromagnetic constant.
The solution to these differential equations in spherical coordinates is

$$\begin{aligned} s_x(t) &= |s|\sin(\theta)\cos(g\,B\,t) \\ s_y(t) &= |s|\sin(\theta)\sin(g\,B\,t) \\ s_z(t) &= |s|\cos(\theta) \end{aligned} \tag{2.19}$$

The Larmor precession frequency is defined as

$$\omega_L = g\,B \tag{2.20}$$

The polar angle is not changing in time, but the azimuthal angle is changing. The motion is a precession around the direction of the magnetic field, i.e. the ϕ is oscillating in time, which is shown in figure **2.5**

2.3 Heisenberg spins

The focus in this section will be on a situation, where two electrons interact. They interact not only by the Coulomb force between them, but also through their associated spin.
The two electrons, a and b, are described by their spatial coordinates, $\vec{r}_1$ and $\vec{r}_2$,and the spinstates, χ_t and χ_s. The total wavefunction for the electrons must be antisymmetric, therefore the wavefunctions are composed of both a spatial part and a spin part [**4**]. Given that the spatial part is symmetric, then the spin part must be antisymmetric in order to obtain an overall antisymmetric wavefunction. If instead the spatial part is antisymmetric, then the spin part must be symmetric in order to obtain an overall antisymmetric wavefunction.

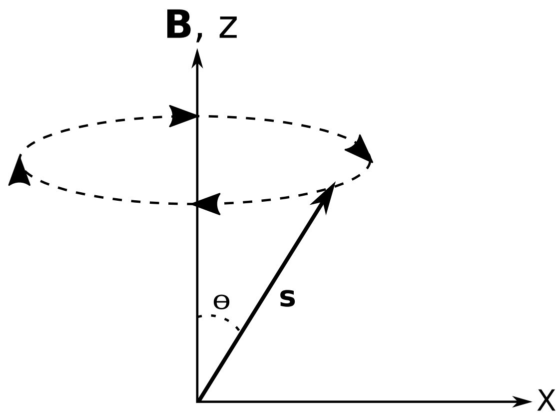

Figure 2.5: A single spin precessing around a magnetic field. The motion of the spin is determined by the direction and magnitude of the magnetic field.

The two wavefunctions for the electrons are [4]

$$\begin{aligned}\Psi_s &= \frac{1}{\sqrt{2}}\left[\psi_a(r_1)\psi_b(r2) + \psi_a(r_2)\psi_b(r1)\right]\chi_s \\ \Psi_t &= \frac{1}{\sqrt{2}}\left[\psi_a(r_1)\psi_b(r2) - \psi_a(r_2)\psi_b(r1)\right]\chi_t\end{aligned} \tag{2.21}$$

The s and t subscripts represents either singlet or triplet spin states, which are

$$\begin{aligned}\chi_t &= \begin{cases} \left|\uparrow\uparrow\right\rangle, & \left|S=1\ m_s=1\right\rangle \\ \frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right), & \left|S=1\ m_s=0\right\rangle \\ \left|\downarrow\downarrow\right\rangle, & \left|S=1\ m_s=-1\right\rangle \end{cases} \\ \chi_s &= \begin{cases} \frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right), & \left|S=0\ m_s=0\right\rangle \end{cases}\end{aligned} \tag{2.22}$$

Where S is the total spin and m_s is the spin projection. The naming of the states is obvious. The triplet state consists of the three situations, where the total spin $S = 1$. The singlet state is the single state, which has total spin $S = 0$.

The energy difference between the two states can be defined as

$$J = \frac{E_s - E_t}{2} = \int \mathrm{d}r_1 \mathrm{d}r_2\, \psi_a^*(r_1)\psi_b^*(r_2)\hat{\mathcal{H}}\psi_a(r_2)\psi_b(r_1) \tag{2.23}$$

This is the exchange integral and J is the exchange constant. The sign of J determines whether the spins want to align or antialign, with respect to each other. If $J > 0$ the singlet state has the highest energy and the triplet state $S = 1$ is preferred. For $J < 0$ the triplet state has the highest energy and the singlet state $S = 0$ is preferred, see figure **2.6**.
The energy of the spin interaction between electrons can be described by the exchange constant between the spins and the alignment of the spins [4].

$$\hat{\mathcal{H}} = -\sum_{i\,j} J_{ij}\,\mathbf{S}_i \cdot \mathbf{S}_j \tag{2.24}$$

In general all electrons interact, but since the exchange constant is position dependent, only electrons in close proximity will have $J \neq 0$, and therefore the energy is often approximated by the nearest neighbour energy.

$$\hat{\mathcal{H}} = -\sum_{\langle i\,j\rangle} J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j \tag{2.25}$$

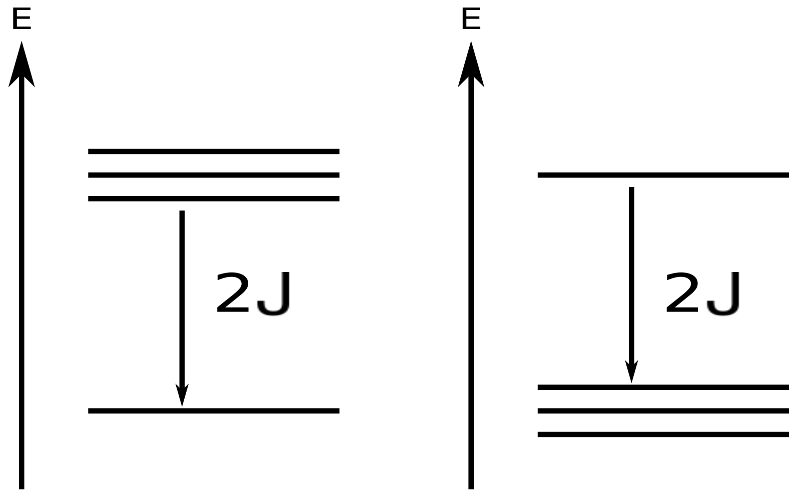

Figure 2.6: The difference in energy depending on the sign of J. The three lines close to each other are the triplet state and the single line is the singlet state. For $J < 0$, the situation on the left is the case. For $J > 0$, the situation on the right is the case.

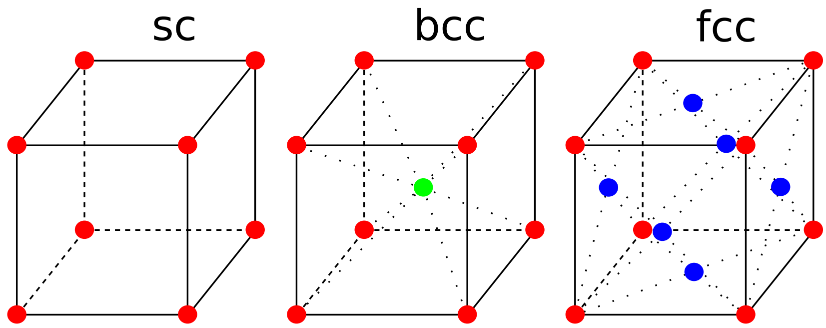

Figure 2.7: Three different crystal structures. **sc** - simple cubic. **bcc** - body centered cubic. The same structure as **sc**, but with an extra atom in the center of the cube, shown as a green dot. **fcc** - face centered cubic. The same structure as **sc**, but with extra atoms on each of the faces of the cube, shown as blue dots.

It is evident that $J > 0$ minimizes the energy for parallel spins (ferromagnetic) and $J < 0$ minimizes the energy for anti parallel spins (antiferromagnetic).

For the case of only two spins interacting the energy is simply

$$\hat{\mathcal{H}} = -J_{1,2}\,\mathbf{S}_1 \cdot \mathbf{S}_2. \tag{2.26}$$

2.4 Metals and impurities

Very often solids are found in crystalline structures, which means that each atom in the bulk material is located in a periodic lattice. A few standard lattices are, as an example, the simple cubic lattice, the body centered cubic lattice and the face centered cubic lattice [6]. In figure **2.7** is shown the three mentioned lattices.
The composition of a pure crystal is such that the lattice only consists of the constituent atoms. Take as an example, NaCl, it only consists of sodium and chloride. The lattice structure of NaCl is simple cubic. The atoms are placed at the corners of the cube, such that each atom has three neighbouring atoms of the other kind, see figure **2.8** (a).

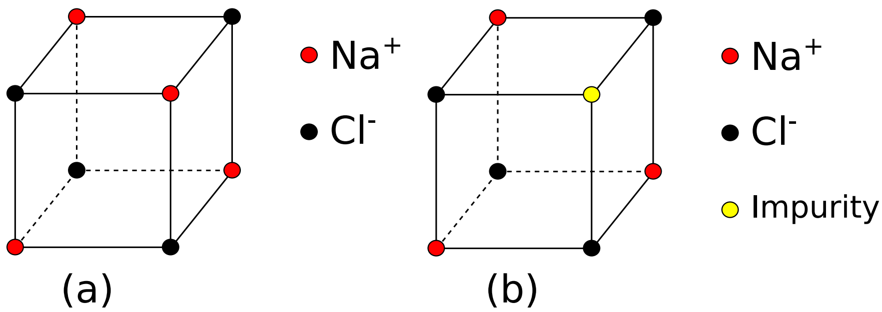

Figure 2.8: (a) The crystal structure of NaCl. The atoms are placed such that not two identical atoms are neighbours. The atom position in the crystal is determined by the charge of the atoms, since the lowest energy configuration is provided by the configuration, where charges of the same kind interact the least. (b) The same crystal structure as in (a), but now one of the atoms is exchanged with an impurity (yellow).

An impure crystal is a crystal, which has one or more atoms changed in the lattice. In figure **2.8** (b) it is shown how an impurity might take the place of one of the sodium atoms and thereby making the crystal impure.

An impurity in a crystal can be an atom with an associated native spin. Not all atoms have a native spin, it depends on how the atomic orbitals are filled with electrons. Electrons, which are spin-1/2 particles, are put into the atomic orbital according to Hunds rules and in some cases, it leaves the atom with a net magnetization [6].

A crystal, which can conduct a current easily is called a metal. A simple way to understand this conduction is to think of the electron furthest from the core of the atom, as being so losely bound that it can be regarded as delocalized from the atom. When every atom contributes with an electron, the sum of these electrons behave as a sea of electrons. The current is passed through the itinerant electrons in the sea [6].

2.5 Coupling of magnetic impurities in metals

In the mid 1900s Ruderman, Kittel, Kasoya and Yosida (RKKY) proposed an exchange mechanism where local magnetic moments in metals can interact through the itinerant electrons.
The interaction between the local magnetic moment and the conduction electrons can be expressed as a delta-function perturbation, with the impurity spin **S** and the spin density of the conduction electrons **s** [7].

$$\hat{\mathcal{H}}_{int} = -J\,\mathbf{S}\cdot\mathbf{s}\,\delta(r) \tag{2.27}$$

The spin up density of the electrons can be shown to be

$$n_{\uparrow}(r) = \frac{k_F^3}{6\,\pi^2}\left(1 - \frac{3\,J\,\mathbf{S}_z\,k_F^3}{2\,\pi\,E_f}\,F(2\,k_F\,r)\right) \tag{2.28}$$

With

$$F(2\,k_F\,r) = \frac{2\,k_F\,r\,\cos(2\,k_F\,r) - \sin(2\,k_F\,r)}{16\,k_F^4\,r^4} \tag{2.29}$$

The spatial dependence of the spin polarization modulation around the impurity is

$$p(r) = \left|\frac{n_{\uparrow}(r) - n_{\downarrow}(r)}{n_{\uparrow}(r) + n_{\downarrow}(r)}\right| = \left|\frac{3\,J\,k_F^3}{E_f}\,F(2\,k_F\,r)\,\mathbf{S}_z\right| \tag{2.30}$$

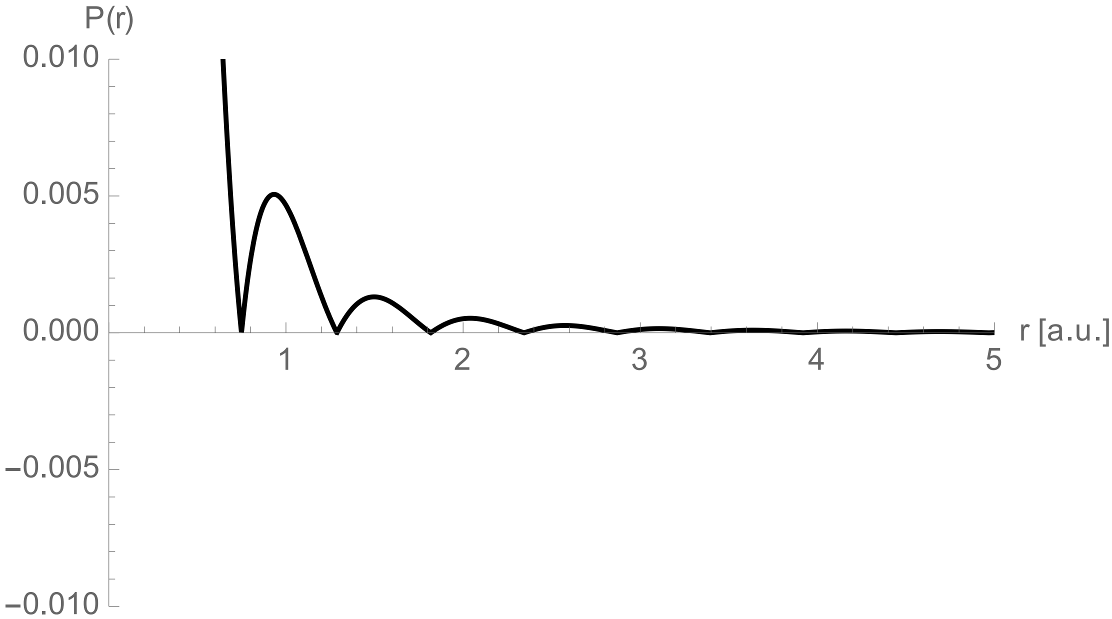

Figure 2.9: The spin polarization moduation. The electron spins close to the impurity will align with the spin of the impurity. As the distance, r, between the impurity and an electron spin goes towards infinity the modulation converges towards zero. This means that an electron far away from the impurity will not feel the impurity.

Figure **2.9** shows the spin modulation. It is clear that the surrounding electrons will polarize close to the impurity, and far from it they will not feel the impurity.
If, instead of a single impurity, there are two impurities separated by a distance R, the interaction is [7]

$$\hat{\mathcal{H}}_{int} = -J\,\mathbf{S}_1 \cdot \mathbf{s}\,\delta(r) - J\,\mathbf{S}_2 \cdot \mathbf{s}\,\delta(r+R) \tag{2.31}$$

The electrons will behave in the same maner as above and polarize close to each impurity, but a coupling between the two impurities emerges from perturbation theory

$$\Delta E_{12} = \frac{J^2\,k_F^6}{4\,\pi^3\,E_F}\,F(2\,k_F\,R)\,\mathbf{S}_1 \cdot \mathbf{S}_2 \tag{2.32}$$

This coupling between the impurities is the RKKY exchange coupling. It is an energy correction, which is proportional to the alignment of the two spin and the distance R between them. The magnetic impurities are coupled through the itinerant electrons in the metal.
In figure **2.10** is shown how the energy correction changes with distance. This results in the impurities aligning either ferromagnetically or antiferromagnetically depending on the distance between them. It is apparent that, if the distance between the two impurities goes towards infinity, the impurities will not have any effect on each other.

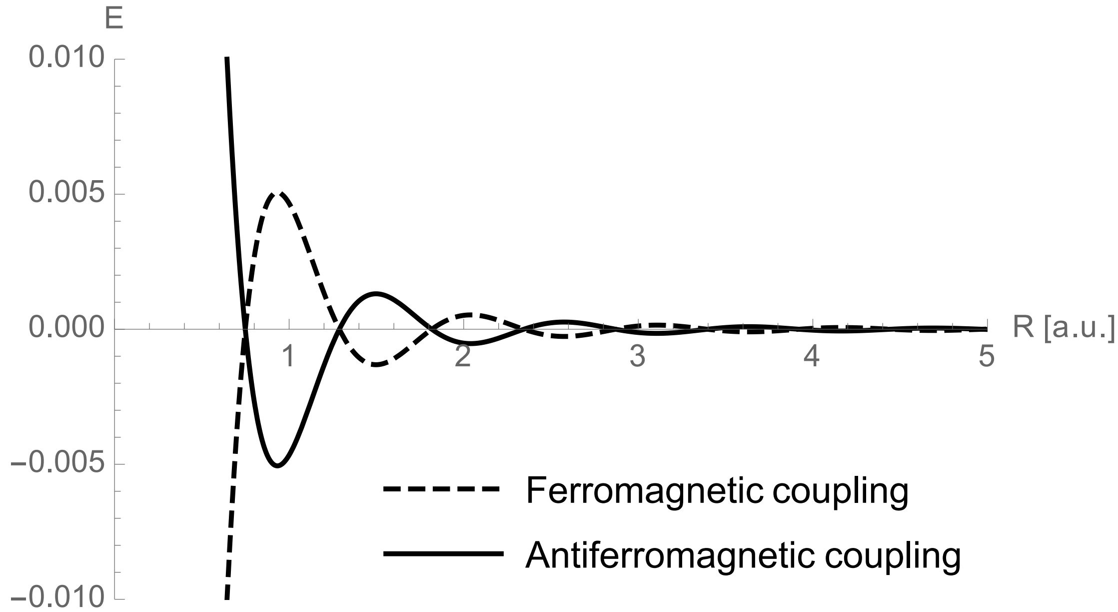

Figure 2.10: The exchange coupling between two magnetic impurities. The distance, R, between the impurities determines whether the spin of the impurities align ferromagnetically or antiferromagnetically. The spins align ferromagnetically, when the dotted line is lowest in energy and antiferromagnetically, when the solid line is lowest in energy.

2.6 Curl forces

It has been shown that an electrical current through a system placed in a magnetic field gives rise to curl forces [1]. In this section a short introduction to the physics of curl forces is introduced.

In a conservative force field the integral of a closed loop around the field will be zero. This implies that no matter which path you choose from point A to point B, the work done by the field will be the same [2]. Imagine an electrostatic field; the electrostatic field can be produced by a single charge, where all field lines are pointing either inwards or outwards with equal magnitudes. In figure **2.11** (a) is shown a positive charge and the field lines pointing outwards. The work done by the field to the two paths shown is the same, because only the start and end point is important.
The curl of the conservative field is zero

$$\nabla \times F(r) = 0 \tag{2.33}$$

In a non-conservative force field the integral of a path determines, how much work has been done by the field. The path chosen from point A to point B is important, imagine the force field produced by wind [2]. In figure **2.11** (b) is shown a force field, which could be a situation generated by wind. The path taken is now important, in order to use as little energy as possible. The green path is moving with the field lines and have tailwind all the way. The blue path moves against the field lines and have headwind all the way. The Blue path is clearly the least favourable, energy wise, imagine riding a bike in headwind versus in tailwind.
The curl of a non-conservative field is non zero

$$\nabla \times F(r) \neq 0 \tag{2.34}$$

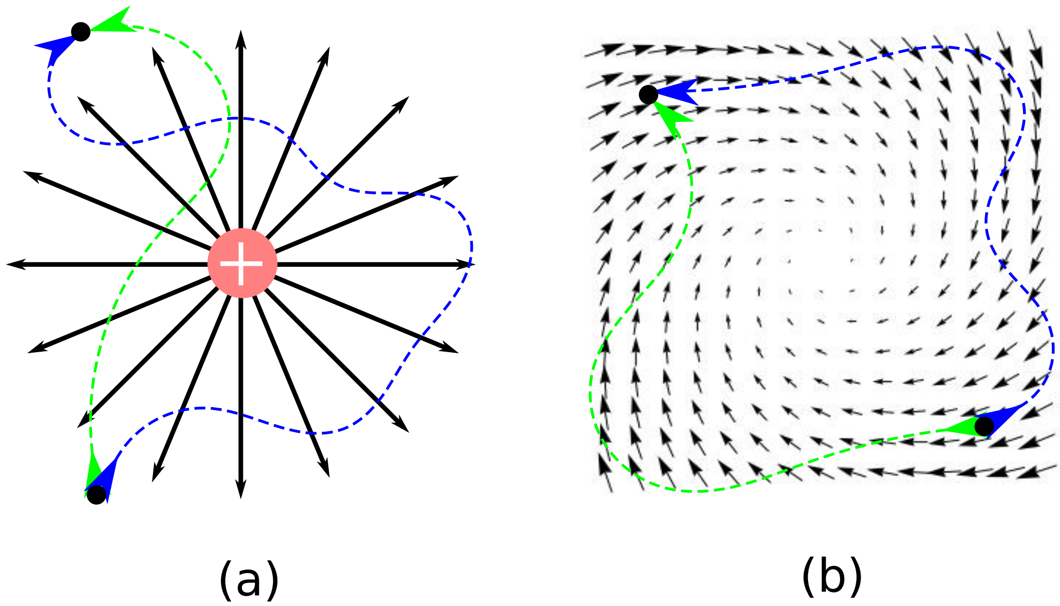

Figure 2.11: (a) A positive charge generating electrical field lines pointing outwards, where the field lines all have the same magnitude. The work done by the field, when moving from one point to another does not depend on the path taken, it only depends on the start and end point. (b) A non-conservative field, which can be thought of as wind. The field is spiraling and the magnitude of the field line differs for two different points. The work done by the field, when moving from one point to another does depend on the path, since you can move in either headwind (green) or tailwind (blue) as illustrated by the two paths.

One could imagine a particle moving under a force $F(r)$, which depends only on position. No dissipative or magnetic forces are taken into account. Newton's equation

$$\ddot{r} = F(r) \tag{2.35}$$

has a solution, $r(t)$, which is the trajectory of the particle.
If the force $F(r)$ fulfills both equation **2.34** and **2.35**, the force is referred to as 'curl force [8].
The dynamics can be illustrated in terms of an electrical curl force, where a time dependent vector potential is used.

$$A(r,t) = t\,F(r) \tag{2.36}$$

The dynamics might be described by the Hamiltonian

$$\hat{\mathcal{H}}(r,p) = \frac{1}{2}\left(p - A(r,t)\right)^2 \tag{2.37}$$

It is seen that the Hamiltonian generates a magnetic force, so that the total force acting on the particle is [8]

$$F_{tot} = F(r) + \nu \times (t\nabla \times F(r)) = F(r) + \nu \times B(r,t) \tag{2.38}$$

This magnetic force is non zero, when the force $F(r)$ satisfies equation **2.34**. This result shows how a time dependent vector field can in fact act as a curl force. If one take the time dependent vector field to be a time dependent magnetic field, the field generates electrical curl forces, which are used to accelerate electrons as for instance happens in a betatron. The time dependent magnetic field also stabilizes the circular orbit of the electrons [8].

2.7 Schrödinger's equation

A highly important equation in physics is the beautiful equation, first desribed by Erwin Schrödinger, known as the Schrödinger equation[5]

$$\frac{i}{\hbar}\frac{\partial\psi}{\partial t} = \hat{\mathcal{H}}\,\psi \tag{2.39}$$

The Schrödinger equation is an imaginary differential equation, which relates the time derivative of a state ψ and the Hamiltonian, $\hat{\mathcal{H}}$. The Hamiltonian and associated states can be expressed in a matrix formalism, first sugested by Paul Dirac, which is called Dirac notation [5]. The $|\psi\rangle$ represents a vector and $\underline{\mathcal{H}}$ represents a matrix.

$$\frac{i}{\hbar}\frac{\partial}{\partial t}\,|\psi\rangle = \underline{\mathcal{H}}\;|\psi\rangle \tag{2.40}$$

The solution to the Schrödinger equation is a set of eigenstates with corresponding eigenvalues. These solutions can be expressed as a single total solution, which is a linear combination of all the eigenstates. The state can be separated in variables, such that the time dependence comes out explicitly as an exponential factor [5]

$$\Psi(\mathrm{t}) = \sum_n \mathrm{e}^{-i\,E_n\mathrm{t}}\,\psi_n \tag{2.41}$$

If instead of an imaginary first order differential equation, it is a real first order differential equation.

$$\frac{\partial}{\partial t}\,|\psi\rangle = \underline{\mathcal{H}}\;|\psi\rangle \tag{2.42}$$

The solution to this differential equation is similar to the solution of the Schrödinger equation. There is no $\hbar$ and no i, which gives the solution

$$\Psi(\mathrm{t}) = \sum_n \mathrm{e}^{E_n\mathrm{t}}\,\psi_n \tag{2.43}$$

The solution is important for this work, because the eigenvalues of specific matrices will be used to understand the dynamics of the system under investigation.

Chapter 3

The System

The system under investigation is a metal with two magnetic impurities. The two impurities are located such that they couple antiferromagnetically. From the Heisenberg Hamiltonian, equation **2.27**, it is known that if the exchange coupling, J, is negative, then the spins will align antiparallel, see figure **3.1**. For now the spins will be treated as a system on its own, but note that the spins are only coupled because of the itinerant electrons in the surroundings.

The two spins are placed in a magnetic field, from which the energy is given by the Heisenberg Hamiltonian.

$$\hat{\mathcal{H}} = J\vec{\mathbf{s}}_1 \cdot \vec{\mathbf{s}}_2 - \vec{B} \cdot (\vec{\mathbf{s}}_1 + \vec{\mathbf{s}}_2) \tag{3.1}$$

Each spin wants to align with the magnetic field, but it also wants to align antiparallel with the other spin, because of the antiferromagnetic coupling between the two. To find the minimal energy for the system, the polar angle needs to be determined. The polar angle describes the alignment of the spin with the magnetic field. $\theta = 0$ is aligned, $\theta = \pi/2$ is perpendicular and $\theta = \pi$ is anti aligned.

The Heisenberg Hamiltonian can be rewritten in terms of trigonometric functions and the polar angle θ. In this case the θ angle is the same for both spins, but in the opposite direction. The following is obtained.

$$E = J\,\mathrm{S}_1\mathrm{S}_2 \cos(2\,\theta) - 2\,B\cos(\theta) \tag{3.2}$$

The lengths of the spins S_1 and S_2 are chosen equal to 1.
The derivative with respect to θ equal to 0 results in an expression for the angle, which gives the lowest energy

$$\begin{aligned} \partial_\theta \mathrm{E} &= 2\,B\sin(\theta) - 2\,J\sin(2\theta) = 0 \\ &\Rightarrow \sin(\theta)\,(2\,B - 4\,J\,\cos(\theta)) = 0 \\ &\Rightarrow 2\,B - 4\,J\,\cos(\theta) = 0 \\ &\Rightarrow \theta_{gs} = \cos^{-1}\left(\frac{B}{2J}\right) \end{aligned} \tag{3.3}$$

It is assumed that the spins will align antiparallel in the azimuthal plane, which results in an angle $\cos^{-1}(\frac{B}{2J})$ in the polar plane. The ground state of the system is thus symmetric around the axis of the magnetic field. The spins orientation are shown in figure **3.2**, with magnetic field along the z-axis.

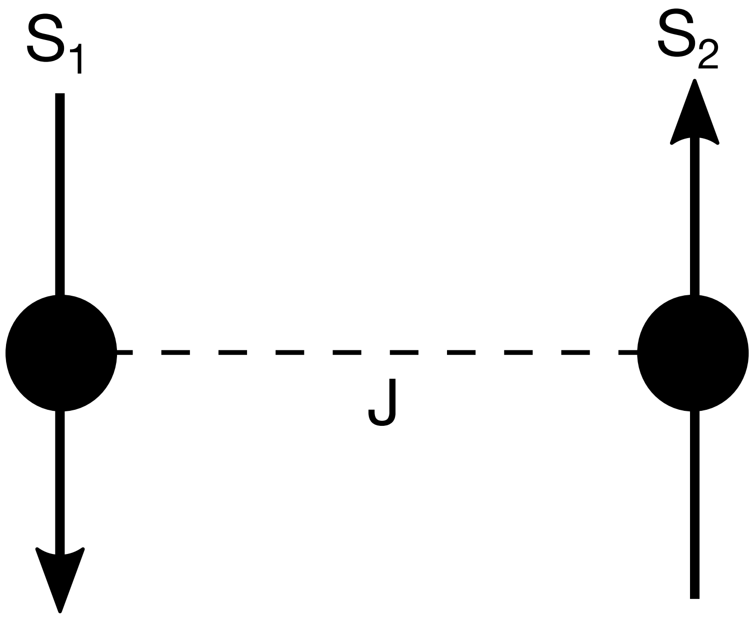

Figure 3.1: Two spins aligned antiferromagnetically, because of the exchange coupling between the them.

3.1 Equation of motion for Heisenberg spins

As was shown in an earlier chapter, the spin and the angular momentum of a particle have similar behaviour. The Hamiltonian, equation **3.1**, can be expressed semi-classically by momentum, $\vec{p}$, and position, $\vec{r}$, only looking at the terms including $\vec{s}_1$.

$$\begin{aligned} H &= J\vec{\mathbf{s}}_1 \cdot \vec{\mathbf{s}}_2 - \vec{B} \cdot (\vec{\mathbf{s}}_1 + \vec{\mathbf{s}}_2) \\ H &= J\vec{\mathbf{s}}_2 \cdot (\vec{r} \times \vec{p}) - \vec{B} \cdot (\vec{r} \times \vec{p}) \\ H &= \vec{h} \cdot (\vec{r} \times \vec{p}) - \vec{B} \cdot (\vec{r} \times \vec{p}) \end{aligned} \tag{3.4}$$

Where $\vec{h}$ can be regarded as a magnetic field, which is felt from the neighbouring spin, i.e. spin $\vec{\mathbf{s}}_1$ feel a magnetic field from spin $\vec{\mathbf{s}}_2$.
From the first section, it was seen that Hamilton's equations can be used to find the time derivative of the angular momentum and thereby the spin. Using the scalar triple product and the Jacobi identity, the calculations are

$$\begin{aligned} \frac{\mathrm{d}\vec{\mathbf{s}}_1}{\mathrm{d}t} &= \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \vec{p} + \vec{r} \times \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \\ &= \frac{\partial H}{\partial p} \times \vec{p} - \vec{r} \times \frac{\partial H}{\partial r} \\ &= \left(\vec{h} \times \vec{r}\right) \times \vec{p} - \left(\vec{B} \times \vec{r}\right) \times \vec{p} - \vec{r} \times \left(\vec{p} \times \vec{h}\right) + \vec{r} \times \left(\vec{p} \times \vec{B}\right) \\ &\Rightarrow \quad \dot{\vec{\mathbf{s}}}_1 = \vec{h} \times (\vec{r} \times \vec{p}) - \vec{B} \times (\vec{r} \times \vec{p}) \end{aligned} \tag{3.5}$$

The time derivative for the spin, given this Hamiltonian, is

$$\frac{\mathrm{d}\vec{\mathbf{s}}_1}{\mathrm{d}t} = J\vec{\mathbf{s}}_2 \times \vec{\mathbf{s}}_1 - \vec{B} \times \vec{\mathbf{s}}_1 \tag{3.6}$$

And similarly for the other spin

$$\frac{\mathrm{d}\vec{\mathbf{s}}_2}{\mathrm{d}t} = J\vec{\mathbf{s}}_1 \times \vec{\mathbf{s}}_2 - \vec{B} \times \vec{\mathbf{s}}_2 \tag{3.7}$$

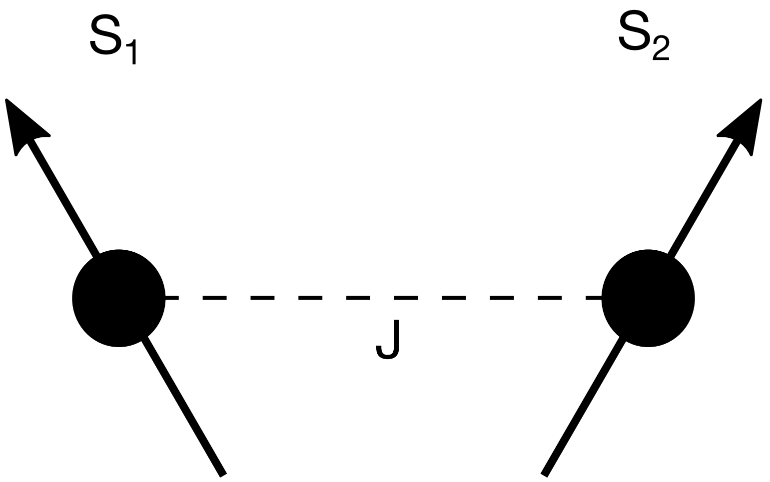

Figure 3.2: The ground state for the system in an equilibrium between exchange and Zeeman energy.

Next is to check how the system behaves, when it is put in the ground state. The spins will have the grundstate angle θ_{gs}. The polar angles for the two spins are the same and, therefore the system can be expressed with a single polar angle θ and trigonometric functions

$$\begin{aligned}\frac{\mathrm{d}\vec{\mathbf{s}}_1}{\mathrm{d}t} &= \vec{\mathbf{s}}_1 \times (\vec{B} - J\vec{\mathbf{s}}_2) \\ &= B\sin(\theta_{gs}) - J\sin(2\theta_{gs}) \\ &= B\sqrt{1-\cos^2(\theta_{gs})} - 2J\sqrt{1-\cos^2(\theta_{gs})}\cos(\theta_{gs}) \\ &= \sqrt{1-\cos^2(\theta_{gs})}(B - 2J\frac{B}{2J}) = 0\end{aligned} \tag{3.8}$$

It is evident that the spins do not change in time, when they are in the ground state. This is an important result, because a single spin in a magnetic field does move, when it has a polar angle, $\theta \neq 0$. A single spin will precess around the magnetic field axis, But when there are two coupled spins in a magnetic field, there exist an equilibrium between the exchange and Zeeman energy, which makes the ground state static.

3.2 Equations of motion for the total system

The equation of motion for the total system will now be investigated. Previously the equation of motion for two coupled spins in a magnetic field was discussed, equation **3.6** and **3.7**. These dynamics are the simplest case for the system. The next term included is the friction, γ. The term is included as a cross product between the velocity of the spin vector with it self. If the system is stable and therefore not moving, there will be no dissipation. The friction is the combined dissipation of the system due to the coupling to the enviroment. If the system is not coupled to the enviroment, there will be no dissipation and it will reduce to the simplest case.

$$\dot{\vec{\mathbf{s}}}_1 = J\,\vec{\mathbf{s}}_2 \times \vec{\mathbf{s}}_1 - \vec{B} \times \vec{\mathbf{s}}_1 + \gamma\,\dot{\vec{\mathbf{s}}}_1 \times \vec{\mathbf{s}}_1 \tag{3.9}$$

$$\dot{\vec{\mathbf{s}}}_2 = J\,\vec{\mathbf{s}}_1 \times \vec{\mathbf{s}}_2 - \vec{B} \times \vec{\mathbf{s}}_2 + \gamma\,\dot{\vec{\mathbf{s}}}_2 \times \vec{\mathbf{s}}_2 \tag{3.10}$$

All three terms are symmetric in the sense that the two equations have the same sign on each of the terms.

The next two terms originates from current. Earlier it was covered that a current runs through the system. This current pushes the system out of equilibrium and curl forces emerge. It has been shown that the current will give rise to the β and σ terms, which are generating the curl forces [1].

$$\dot{\vec{\mathbf{s}}}_1 = (J+\sigma)\,\vec{\mathbf{s}}_2 \times \vec{\mathbf{s}}_1 - \vec{B} \times \vec{\mathbf{s}}_1 + \gamma\,\dot{\vec{\mathbf{s}}}_1 \times \vec{\mathbf{s}}_1 + \beta\,\dot{\vec{\mathbf{s}}}_2 \times \vec{\mathbf{s}}_1 \tag{3.11}$$

$$\dot{\vec{\mathbf{s}}}_2 = (J-\sigma)\,\vec{\mathbf{s}}_1 \times \vec{\mathbf{s}}_2 - \vec{B} \times \vec{\mathbf{s}}_2 + \gamma\,\dot{\vec{\mathbf{s}}}_2 \times \vec{\mathbf{s}}_2 - \beta\,\dot{\vec{\mathbf{s}}}_1 \times \vec{\mathbf{s}}_2 \tag{3.12}$$

The β and σ terms are both antisymmetric in the same sense as above that the terms have opposite signs in the two equations.

The σ term is the same type of cross product as the exchange term, J. Therefore it can be seen that σ behaves as a shift in the exchange energy. The β term is a cross product between the velocity of the other vector and it self. The term acts as a Lorentz force and can be regarded as a kind of magnetic field.

The next step is to iteratively plug in the expressions for $\dot{\vec{s}}_1$ and $\dot{\vec{s}}_2$, such that only first order terms in γ and β will remain.

$$\begin{aligned}\dot{\vec{\mathbf{s}}}_1 = (J+\sigma)\,\vec{\mathbf{s}}_2 \times \vec{\mathbf{s}}_1 - \vec{B} \times \vec{\mathbf{s}}_1 + \gamma\left((J+\sigma)\,\vec{\mathbf{s}}_2 \times \vec{\mathbf{s}}_1 - \vec{B} \times \vec{\mathbf{s}}_1\right) \times \vec{\mathbf{s}}_1 \\ + \beta\left((J-\sigma)\,\vec{\mathbf{s}}_1 \times \vec{\mathbf{s}}_2 - \vec{B} \times \vec{\mathbf{s}}_2\right) \times \vec{\mathbf{s}}_1\end{aligned} \tag{3.13}$$

$$\begin{aligned}\dot{\vec{\mathbf{s}}}_2 = (J-\sigma)\,\vec{\mathbf{s}}_1 \times \vec{\mathbf{s}}_2 - \vec{B} \times \vec{\mathbf{s}}_2 + \gamma\left((J-\sigma)\,\vec{\mathbf{s}}_1 \times \vec{\mathbf{s}}_2 - \vec{B} \times \vec{\mathbf{s}}_2\right) \times \vec{\mathbf{s}}_2 \\ - \beta\left((J+\sigma)\,\vec{\mathbf{s}}_2 \times \vec{\mathbf{s}}_1 - \vec{B} \times \vec{\mathbf{s}}_1\right) \times \vec{\mathbf{s}}_2\end{aligned} \tag{3.14}$$

These two equations are the starting point for the further investigation of the dynamics. First the ground state for the total system will be inspected, and afterwards the dynamics of the system will be explored.

3.2.1 ground state

As seen earlier the angle, θ_{gs}, is the same for both spins, and the system does not move, when it is in the groundsate. Once again it will be assumed that the ground state will be a static and stable point for the system. The γ and β terms vanish, since it is assumed that $\dot{\vec{s}}_1$ and $\dot{\vec{s}}_2$ are both zero, and the following equations are obtained

$$\dot{\vec{\mathbf{s}}}_1 = (J+\sigma)\,\vec{\mathbf{s}}_2 \times \vec{\mathbf{s}}_1 - \vec{B} \times \vec{\mathbf{s}}_1 = 0 \tag{3.15}$$

$$\dot{\vec{\mathbf{s}}}_2 = (J-\sigma)\,\vec{\mathbf{s}}_1 \times \vec{\mathbf{s}}_2 - \vec{B} \times \vec{\mathbf{s}}_2 = 0 \tag{3.16}$$

The first equation can be expressed through trigonometric functions ans the polar angles.

$$(J+\sigma)\,\sin(\theta_2+\theta_1) - B\sin(\theta_1) = 0 \qquad \Rightarrow \qquad \sin(\theta_2+\theta_1) = \frac{B\sin(\theta_1)}{(J+\sigma)} \tag{3.17}$$

This equality can be used in the next equation to obtain

$$(J-\sigma)\,\sin(\theta_1+\theta_2) - B\sin(\theta_2) = 0 \qquad \Rightarrow \qquad B\sin(\theta_2) = (J-\sigma)\sin(\theta_1+\theta_2) \tag{3.18}$$

The ground state has no ϕ dependence and therefore, it is assumed that the spins have an azimuthal angle of π between them. The relation between the polar angles is.

$$\sin(\theta_2) = \frac{J-\sigma}{J+\sigma}\sin(\theta_1) \tag{3.19}$$

From this equation it is clear that the σ introduces a difference between the two angles. When σ is zero the two angles are the same, which agrees with the previous investigation of the simplest case.

Next procedure it to find the two ground state angles θ_1^0 and θ_2^0 startin from Equations **3.15** and **3.16**.

$$0 = (J+\sigma)\sin(\theta_1^0+\theta_2^0) - B\sin(\theta_1^0) \tag{3.20}$$

$$0 = (J-\sigma)\sin(\theta_1^0+\theta_2^0) - B\sin(\theta_2^0) \tag{3.21}$$

Equation **3.20** becomes

$$0 = (J+\sigma)(\cos(\theta_1^0)\sin(\theta_2^0) + \cos(\theta_2^0)\sin(\theta_2^0)) - B\sin(\theta_1^0) \tag{3.22}$$

This can be rewritten with the expression from equation **3.19**

$$0 = (J+\sigma)\left(\frac{J-\sigma}{J+\sigma}\sin(\theta_1^0)\cos(\theta_1^0) + \sin(\theta_1^0)\sqrt{1-\left(\frac{J-\sigma}{J+\sigma}\right)^2(1-\cos^2(\theta_1^0))}\right) - B\sin(\theta_1^0)$$

$$\Rightarrow \left(-(J+\sigma)\sqrt{1-\left(\frac{J-\sigma}{J+\sigma}\right)^2(1-\cos^2(\theta_1^0))}\right)^2 = ((J-\sigma)\cos(\theta_1^0) - B)^2$$

$$\Rightarrow (J+\sigma)^2\left(1-\left(\frac{J-\sigma}{J+\sigma}\right)(1-\cos^2(\theta_1^0))\right) = B^2 + ((J-\sigma)\cos(\theta_1^0))^2 - 2B\,(J-\sigma)\cos(\theta_1^0)$$

$$\Rightarrow B^2 = 2B\,(J-\sigma)\cos(\theta_1^0) + 4\,J\,\sigma \tag{3.23}$$

The same calculations can be done for the other ground state angle, which yields the two grounstate angles

$$\theta_1^0 = \cos^{-1}\left(\frac{B^2-4\,J\,\sigma}{2B\,(J-\sigma)}\right) \tag{3.24}$$

$$\theta_2^0 = \cos^{-1}\left(\frac{B^2+4\,J\,\sigma}{2B\,(J+\sigma)}\right) \tag{3.25}$$

In figure **3.3** is illustrated the system with the two different ground state angles. The two ground state angles are different in the sign of σ, which result in on spin being pushed away from the magnetic field and the other spin pushed towards the magnetic field. If σ is zero, the original simple ground state angle is recovered.

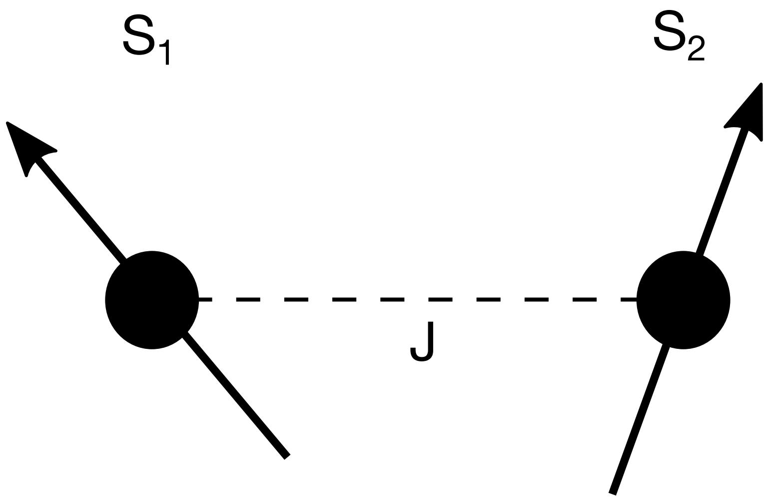

Figure 3.3: The ground state for the system, when the polar angles are shifted, because of the σ term.

3.3 Local coordinates

In order to investigate the spin dynamics, the spins are expressed in a local frame. This frame is spanned by three ortonormal vectors in spherical coordinates

$$\vec{e}_{r\,i} = \begin{pmatrix} \cos(\phi_i)\sin(\theta_i) \\ \sin(\phi_i)\sin(\theta_i) \\ \cos(\theta_i) \end{pmatrix} \quad \text{and} \quad \vec{e}_{\theta\,i} = \begin{pmatrix} \cos(\phi_i)\cos(\theta_i) \\ \sin(\phi_i)\cos(\theta_i) \\ -\sin(\theta_i) \end{pmatrix} \quad \text{and} \quad \vec{e}_{\phi\,i} = \begin{pmatrix} -\sin(\phi_i) \\ \cos(\phi_i) \\ 0 \end{pmatrix} \tag{3.26}$$

This shows that the three vectors have a close resemblence to the spin vector, $\vec{S}$, in spherical coordinates

$$\vec{S}_i = \begin{pmatrix} \cos(\phi_i)\sin(\theta_i) \\ \sin(\phi_i)\sin(\theta_i) \\ \cos(\theta_i) \end{pmatrix} \tag{3.27}$$

The local frame vectors can be generated by the vector, $\vec{S}$

$$\vec{e}_{r\,i} = \frac{\partial \vec{S}_i}{\partial r_i} \quad \text{and} \quad \vec{e}_{\theta\,i} = \frac{\partial \vec{S}_i}{\partial \theta_i} \quad \text{and} \quad \vec{e}_{\phi\,i}\sin(\theta_i) = \frac{\partial \vec{S}_i}{\partial \phi\,i} \tag{3.28}$$

These equations show, how the three orthonormal vectors chosen, is in fact vectors describing how the vector, $\vec{S}$ is changing in each of the spherical coordinates. In figure **3.4** is shown how the coordinate system looks like for a single spin with the local frame placed at the tip of the spin. Furthermore the time derivative of the spin can be expressed by incorporating the local frame vectors

$$\begin{aligned} \frac{\mathrm{d}\vec{S}_i}{\mathrm{d}t} &= \frac{\partial \vec{S}_i}{\partial r_i}\frac{\mathrm{d}r_i}{\mathrm{d}t} + \frac{\partial \vec{S}_i}{\partial \theta_i}\frac{\mathrm{d}\theta_i}{\mathrm{d}t} + \frac{\partial \vec{S}_i}{\partial \phi_i}\frac{\mathrm{d}\phi_i}{\mathrm{d}t} \\ \Rightarrow \dot{\vec{S}}_i &= \vec{e}_{r\,i}\,\dot{r}_i + \vec{e}_{\theta\,i}\,\dot{\theta}_i + \sin(\theta_i)\,\vec{e}_{\phi\,i}\,\dot{\phi}_i \end{aligned} \tag{3.29}$$

Now the equation of motion is described in terms of the local coordinate system vectors.

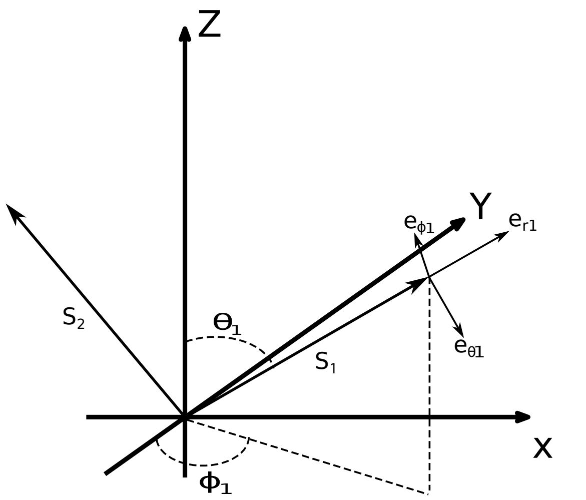

Figure 3.4: Coordinate system showing the cartesian coordinates, thespherical coordinates as well as the local coordinate system composed of the e-vectors describing the change of the spin. The two spins are illustrated as S_1 and S_2.

The direction of $\vec{S}$ is the same as e_r. The magnetic field is along the z-axis and therefore the same direction as e_z.

$$\begin{aligned}
\dot{\vec{s_1}} =& (J+\sigma)\vec{s_2}\times\vec{s_1} - \vec{B}\times\vec{s_1} + \gamma\,\dot{\vec{s_1}}\times\vec{s_1} + \beta\,\dot{\vec{s_2}}\times\vec{s_1} \\
\Rightarrow \dot{\vec{s_1}} =& (J+\sigma)\vec{\mathrm{e}}_{r2}\times\vec{\mathrm{e}}_{r1} - B\,\vec{\mathrm{e}}_z\times\vec{\mathrm{e}}_{r1} + \gamma\,((J+\sigma)\vec{\mathrm{e}}_{r2}\times\vec{\mathrm{e}}_{r1} - B\,\vec{\mathrm{e}}_z\times\vec{\mathrm{e}}_{r1})\times\vec{\mathrm{e}}_{r1} \\
&+\beta\,((J-\sigma)\vec{\mathrm{e}}_{r1}\times\vec{\mathrm{e}}_{r2} - B\,\vec{\mathrm{e}}_z\times\vec{\mathrm{e}}_{r2})\times\vec{\mathrm{e}}_{r1}
\end{aligned} \tag{3.30}$$

The following expressions will be usefull, when it is time to calculate the equations of motion in a little while

$$\begin{aligned}
\vec{\mathrm{e}}_{r2}\cdot\vec{\mathrm{e}}_{\theta1} &= \cos(\theta_1)\sin(\theta_2)\cos(\phi_1-\phi_2) - \cos(\theta_2)\sin(\theta_1) \\
\vec{\mathrm{e}}_{r2}\cdot\vec{\mathrm{e}}_{\phi1} &= -\sin(\theta_2)\sin(\phi_1-\phi_2) \\
\vec{\mathrm{e}}_{\phi2}\cdot\vec{\mathrm{e}}_{\phi1} &= \cos(\phi_1-\phi_2) \\
\vec{\mathrm{e}}_{\phi2}\cdot\vec{\mathrm{e}}_{\theta1} &= \cos(\theta_1)\sin(\phi_1-\phi_2)
\end{aligned} \tag{3.31}$$

In order to simplify the equation of motion, it is necessary to express e_{r2} and e_z in the s_1 local frame. This can be done by performing a basis change of the vectors to the local frame coordinates, starting with $\vec{\mathrm{e}}_z$.

$$\vec{\mathrm{e}}_z = (\vec{\mathrm{e}}_z\cdot\vec{\mathrm{e}}_{r1})\vec{\mathrm{e}}_{r1} + (\vec{\mathrm{e}}_z\cdot\vec{\mathrm{e}}_{\theta1})\vec{\mathrm{e}}_{\theta1} + (\vec{\mathrm{e}}_z\cdot\vec{\mathrm{e}}_{\phi1})\vec{\mathrm{e}}_{\phi1} \tag{3.32}$$

It is used that $\vec{\mathrm{e}}_z$ is $\begin{pmatrix}0\\0\\1\end{pmatrix}$ and that the local coordinate vectors, $\vec{\mathrm{e}}_{r1}$, $\vec{\mathrm{e}}_{\theta1}$ and $\vec{\mathrm{e}}_{\phi1}$ are defined as above. The $\vec{\mathrm{e}}_z$ vector expressed in the local frame is thereby obtained.

$$\vec{\mathrm{e}}_z = \cos(\theta_1)\vec{\mathrm{e}}_{r1} - \sin(\theta_1)\vec{\mathrm{e}}_{\theta1} \tag{3.33}$$

In the same fashion a basis change of $\vec{\mathrm{e}}_{r2}$ to the local coordinate system is performed

$$\vec{\mathrm{e}}_{r2} = (\vec{\mathrm{e}}_{r2}\cdot\vec{\mathrm{e}}_{r1})\vec{\mathrm{e}}_{r1} + (\vec{\mathrm{e}}_{r2}\cdot\vec{\mathrm{e}}_{\theta1})\vec{\mathrm{e}}_{\theta1} + (\vec{\mathrm{e}}_{r2}\cdot\vec{\mathrm{e}}_{\phi1})\vec{\mathrm{e}}_{\phi1} \tag{3.34}$$

When these basis changes are done, the total equation of motion can be expressed in terms of $\vec{e}_{r1}$

$$\begin{aligned} \dot{\vec{S_1}} = \quad & (J+\sigma)\left[(\vec{e}_{r2}\cdot\vec{e}_{r1})\vec{e}_{r1} + (\vec{e}_{r2}\cdot\vec{e}_{\theta1})\vec{e}_{\theta1} + (\vec{e}_{r2}\cdot\vec{e}_{\phi1})\vec{e}_{\phi1}\right]\times\vec{e}_{r1} \\ & -B(\cos(\theta_1)\vec{e}_{r1} - \sin(\theta_1)\vec{e}_{\theta1})\times\vec{e}_{r1} \\ & +\gamma\{\ (J+\sigma)\left[(\vec{e}_{r2}\cdot\vec{e}_{r1})\vec{e}_{r1} + (\vec{e}_{r2}\cdot\vec{e}_{\theta1})\vec{e}_{\theta1} + (\vec{e}_{r2}\cdot\vec{e}_{\phi1})\vec{e}_{\phi1}\right]\times\vec{e}_{r1} \\ & \quad -B\left[\cos(\theta_1)\vec{e}_{r1} - \sin(\theta_1)\vec{e}_{\theta1}\right]\times\vec{e}_{r1}\ \}\times\vec{e}_{r1} \\ & +\beta\{\ (J-\sigma)\,\vec{e}_{r1}\times\left[(\vec{e}_{r2}\cdot\vec{e}_{r1})\vec{e}_{r1} + (\vec{e}_{r2}\cdot\vec{e}_{\theta1})\vec{e}_{\theta1} + (\vec{e}_{r2}\cdot\vec{e}_{\phi1})\vec{e}_{\phi1}\right] \\ & \quad -B\left[\cos(\theta_2)\vec{e}_{r2} - \sin(\theta_2)\vec{e}_{\theta2}\right]\times\vec{e}_{r2}\ \}\times\vec{e}_{r1} \end{aligned} \tag{3.35}$$

This is a long expression and to simplify it even more, the cross products are calculated, which yield the directions of each of the terms.

$$\begin{aligned} \dot{\vec{S_1}} = & (J+\sigma)\left[-(\vec{e}_{r2}\cdot\vec{e}_{\theta1})\vec{e}_{\phi1} + (\vec{e}_{r2}\cdot\vec{e}_{\phi1})\vec{e}_{\theta1}\right] - B\sin(\theta1)\vec{e}_{\phi1} \\ & -\gamma\ \left[(J+\sigma)\left[(\vec{e}_{r2}\cdot\vec{e}_{\theta1})\vec{e}_{\theta1} + (\vec{e}_{r2}\cdot\vec{e}_{\phi1})\vec{e}_{\phi1}\right] + B\sin(\theta1)\vec{e}_{\theta1}\right] \\ & +\beta\ \left[(J-\sigma)\left[(\vec{e}_{r2}\cdot\vec{e}_{\theta1})\vec{e}_{\theta1} + (\vec{e}_{r2}\cdot\vec{e}_{\phi1})\vec{e}_{\phi1}\right] - B\sin(\theta2)\left[(\vec{e}_{\phi2}\cdot\vec{e}_{\phi1})\vec{e}_{\theta1} - (\vec{e}_{\phi2}\cdot\vec{e}_{\theta1})\vec{e}_{\phi1}\right]\right] \end{aligned} \tag{3.36}$$

From equation **3.29** it is known that the local frame vectors act as the unit vector constituents of the total spin vector. With these constituents the equation of motion can be seperated into three different directions, $\vec{e}_{r1}$, $\vec{e}_{\theta1}$ and $\vec{e}_{\phi1}$.

In equation **3.36** it is seen that there is no term with the direction $\vec{e}_{r1}$, which means that the length of the spin is not changing in time. By equating equation **3.36** to **3.29** the following equality is obtained.

$$\begin{aligned} \vec{e}_{\theta1}\,\dot{\theta_1} + \sin(\theta_1)\,\vec{e}_{\phi1}\,\dot{\phi_1} = \quad & (J+\sigma)(-(\vec{e}_{r2}\cdot\vec{e}_{\theta1})\vec{e}_{\phi1} + (\vec{e}_{r2}\cdot\vec{e}_{\phi1})\vec{e}_{\theta1}) - B\sin(\theta_1)\,\vec{e}_{\phi1} \\ & -\gamma\ \left[(J+\sigma)((\vec{e}_{r2}\cdot\vec{e}_{\theta1})\vec{e}_{\theta1} + (\vec{e}_{r2}\cdot\vec{e}_{\phi1})\vec{e}_{\phi1}) + B\sin(\theta_1)\,\vec{e}_{\theta1}\right] \\ & +\beta\ \left[(J-\sigma)((\vec{e}_{r2}\cdot\vec{e}_{\theta1})\vec{e}_{\theta1} + (\vec{e}_{r2}\cdot\vec{e}_{\phi1})\vec{e}_{\phi1})\right. \\ & \left. -B\sin(\theta_2)((\vec{e}_{\phi2}\cdot\vec{e}_{\phi1})\vec{e}_{\theta1} - (\vec{e}_{\phi2}\cdot\vec{e}_{\theta1})\vec{e}_{\phi1})\ \right] \end{aligned} \tag{3.37}$$

Taking each constituent separately, gives an equation with $\vec{e}_{\theta1}$ direction and another with $\vec{e}_{\phi1}$ direction and so forth. The following equations are obtained.

$$\begin{aligned} \vec{e}_{\theta1}\,\dot{\theta_1} = \quad & (J+\sigma)(\vec{e}_{r2}\cdot\vec{e}_{\phi1})\,\vec{e}_{\theta1} - \gamma\left[(J+\sigma)(\vec{e}_{r2}\cdot\vec{e}_{\theta1}) + B\sin(\theta_1)\right]\,\vec{e}_{\theta1} \\ & +\beta\ \left[(J-\sigma)(\vec{e}_{r2}\cdot\vec{e}_{\theta1}) - B\sin(\theta_2)(\vec{e}_{\phi2}\cdot\vec{e}_{\phi1})\right]\,\vec{e}_{\theta1} \end{aligned} \tag{3.38}$$

$$\begin{aligned}\vec{e}_{\theta 2}\,\dot{\theta}_2 = & \;(J-\sigma)(\vec{e}_{r1}\cdot\vec{e}_{\phi 2})\,\vec{e}_{\theta 2} - \gamma\,(J-\sigma)(\vec{e}_{r1}\cdot\vec{e}_{\theta 2}) + B\mathrm{sin}(\theta_2)\,\vec{e}_{\theta 2} \\ & - \beta\,[(J+\sigma)(\vec{e}_{r1}\cdot\vec{e}_{\theta 2}) - B\sin(\theta_1)(\vec{e}_{\phi 1}\cdot\vec{e}_{\phi 2})]\;\vec{e}_{\theta 1}\end{aligned} \tag{3.39}$$

$$\begin{aligned}\vec{e}_{\phi 1}\,\dot{\phi}_1 = & -\frac{(J+\sigma)(\vec{e}_{r2}\cdot\vec{e}_{\theta 1}) - B\mathrm{sin}(\theta_1) - \gamma\,(J+\sigma)(\vec{e}_{r2}\cdot\vec{e}_{\phi 1})}{\mathrm{sin}(\theta_1)}\,\vec{e}_{\phi 1} \\ & + \frac{\beta\,[(J-\sigma)(\vec{e}_{r2}\cdot\vec{e}_{\phi 1}) + B\sin(\theta_2)(\vec{e}_{\phi 2}\cdot\vec{e}_{\theta 1})]}{\sin(\theta_1)}\,\vec{e}_{\phi 1}\end{aligned} \tag{3.40}$$

$$\begin{aligned}\vec{e}_{\phi 2}\,\dot{\phi}_2 = & -\frac{(J-\sigma)(\vec{e}_{r1}\cdot\vec{e}_{\theta 2}) - B\mathrm{sin}(\theta_2) - \gamma\,[(J-\sigma)(\vec{e}_{r1}\cdot\vec{e}_{\phi 2})]}{\mathrm{sin}(\theta_2)}\,\vec{e}_{\phi 2} \\ & - \frac{\beta\,[(J+\sigma)(\vec{e}_{r1}\cdot\vec{e}_{\phi 2}) + B\sin(\theta_1)(\vec{e}_{\phi 1}\cdot\vec{e}_{\theta 2})]}{\sin(\theta_2)}\,\vec{e}_{\phi 2}\end{aligned} \tag{3.41}$$

Which can be simplified by dividing the directions $\vec{e}_{\theta 1}$, $\vec{e}_{\theta 2}$, $\vec{e}_{\phi 1}$ and $\vec{e}_{\phi 2}$ on each side of the equal sign and using the scalar products in equation **3.31**

$$\begin{aligned}\dot{\theta}_1 = & \;-(J+\sigma)\mathrm{sin}(\theta_2)\mathrm{sin}(\phi_1-\phi_2) \\ & - \gamma\,[(J+\sigma)\,[\cos(\theta_1)\mathrm{sin}(\theta_2)\mathrm{cos}(\phi_1-\phi_2) - \cos(\theta_2)\mathrm{sin}(\theta_1)] + B\sin(\theta 1)] \\ & + \beta\,[(J-\sigma)\,[\cos(\theta_1)\mathrm{sin}(\theta_2)\mathrm{cos}(\phi_1-\phi_2) - \cos(\theta_2)\mathrm{sin}(\theta_1)] - B\sin(\theta_2)\mathrm{cos}(\phi_1-\phi_2)]\end{aligned} \tag{3.42}$$

$$\begin{aligned}\dot{\theta}_2 = & \;(J-\sigma)\mathrm{sin}(\theta_1)\mathrm{sin}(\phi_1-\phi_2) \\ & - \gamma\,[(J-\sigma)\,[\cos(\theta_2)\mathrm{sin}(\theta_1)\mathrm{cos}(\phi_1-\phi_2) - \cos(\theta_1)\mathrm{sin}(\theta_2)] + B\sin(\theta 2)] \\ & - \beta\,[(J+\sigma)\,[\cos(\theta_2)\mathrm{sin}(\theta_1)\mathrm{cos}(\phi_1-\phi_2) - \cos(\theta_1)\mathrm{sin}(\theta_2)] - B\sin(\theta_1)\mathrm{cos}(\phi_1-\phi_2)]\end{aligned} \tag{3.43}$$

$$\begin{aligned}\dot{\phi}_1 = & \;\frac{-(J+\sigma)\,[\cos(\theta_1)\mathrm{sin}(\theta_2)\mathrm{cos}(\phi_1-\phi_2) - \cos(\theta_2)\mathrm{sin}(\theta_1)] - B\sin(\theta_1)}{\mathrm{sin}(\theta_1)} \\ & - \frac{\gamma\,(J+\sigma)(-\mathrm{sin}(\theta_2)\mathrm{sin}(\phi_1-\phi_2))}{\mathrm{sin}(\theta_1)} \\ & + \frac{\beta\,[(J-\sigma)(-\mathrm{sin}(\theta_2)\mathrm{sin}(\phi_1-\phi_2)) + B\sin(\theta_2)\mathrm{cos}(\theta_1)\mathrm{sin}(\phi_1-\phi_2)]}{\mathrm{sin}(\theta_1)}\end{aligned} \tag{3.44}$$

$$\begin{aligned}\dot{\phi}_2 = & \;\frac{-(J-\sigma)\,[\cos(\theta_2)\mathrm{sin}(\theta_1)\mathrm{cos}(\phi_1-\phi_2) - \cos(\theta_1)\mathrm{sin}(\theta_2)] - B\sin(\theta_2)}{\mathrm{sin}(\theta_2)} \\ & - \frac{\gamma\,(J-\sigma)(-\mathrm{sin}(\theta_1)\mathrm{sin}(\phi_1-\phi_2))}{\mathrm{sin}(\theta_2)} \\ & + \frac{\beta\,[(J+\sigma)(-\mathrm{sin}(\theta_1)\mathrm{sin}(\phi_1-\phi_2)) + B\sin(\theta_1)\mathrm{cos}(\theta_2)\mathrm{sin}(\phi_1-\phi_2)]}{\mathrm{sin}(\theta_2)}\end{aligned} \tag{3.45}$$

These four equations of motion will be the basis of the further investigation of the system. It is clearly some long expressions which will need some special treatment.

Chapter 4

Analysis of the ground state

The equations of motion that have been found for the polar and azimuthal angles are not easily solvable, since there are 4 equations with 4 unknowns. Instead of trying to solve the equations exactly, a linearization close to the ground state of the system will be performed, by giving the system a little push away from the stable point. The dynamics close to the ground state will be found by this procedure.

In order to easily get an overview of the different term's influence on the behaviour of the dynamics, the expansion is done in steps. Starting with looking at the system without friction and without the β term.
After expanding all terms in the total equations of motion, a matrix will be generated, which describes the dynamics of the system.

4.1 Expansion

Close to the ground state of the system the angles can be expressed as the ground state angles with a small deviation. The situation without a deviation, is the ground state and the system will not move. In addition, the time derivatives of the angles are equal to the time derivative of the deviation, since the ground state angles do not depend on time.

$$\begin{aligned}
\theta_1 &= \theta_1^0 + \delta\theta_1 & &\Rightarrow \dot{\theta}_1 = \delta\dot{\theta}_1 \\
\theta_2 &= \theta_2^0 + \delta\theta_2 & &\Rightarrow \dot{\theta}_2 = \delta\dot{\theta}_2 \\
\phi_1 &= \phi_1^0 + \delta\phi_1 & &\Rightarrow \dot{\phi}_1 = \delta\dot{\phi}_1 \\
\phi_2 &= \phi_2^0 + \delta\phi_2 & &\Rightarrow \dot{\phi}_2 = \delta\dot{\phi}_2 \\
\phi_1 - \phi_2 &= \pi + \delta\phi_1 - \delta\phi_2 & &
\end{aligned}$$

4.1.1 Oscillator expansion

From equations **3.42**, **3.43**, **3.44** and **3.45** the 4 equations of motion are obtained. First the case, where both γ and β are equal to zero is expanded. The expressions for the angles close to the ground state, as defined above, are plugged in.

$$\delta\dot{\theta}_1 = -(J+\sigma)\sin\left(\theta_2^0 + \delta\theta_2\right)\sin\left(\pi + \delta\phi_1 - \delta\phi_2\right) \tag{4.1}$$

$$\delta\dot{\theta}_2 = \quad (J-\sigma)\sin\left(\theta_1^0 + \delta\theta_1\right)\sin\left(\pi + \delta\phi_1 - \delta\phi_2\right) \tag{4.2}$$

$$\delta\dot{\phi}_1 = -B + (J+\sigma)\left[\cos\left(\theta_2^0 + \delta\theta_2\right) - \cos\left(\pi + \delta\phi_1 - \delta\phi_2\right)\cot\left(\theta_1^0 + \delta\theta_1\right)\sin\left(\theta_2^0 + \delta\theta_2\right)\right] \tag{4.3}$$

$$\delta\dot{\phi}_2 = -B + (J-\sigma)\left[\cos\left(\theta_1^0+\delta\theta_1\right) - \cos\left(\pi+\delta\phi_1-\delta\phi_2\right)\cot\left(\theta_2^0+\delta\theta_2\right)\sin\left(\theta_1^0+\delta\theta_1\right)\right] \tag{4.4}$$

Now the equations are expanded to first order in δ around 0.

$$\begin{aligned}\delta\dot{\theta}_1 = &-(J+\sigma)\sin\left(\theta_2^0\right)\sin(\pi) - (J+\sigma)\cos\left(\theta_2^0\right)\ \delta\theta_2\sin(\pi)\\ &-(J+\sigma)\sin\left(\theta_2^0\right)\cos(\pi)\ (\delta\phi_1-\delta\phi_2)\end{aligned} \tag{4.5}$$

$$\begin{aligned}\delta\dot{\theta}_2 = &\quad (J-\sigma)\sin\left(\theta_1^0\right)\sin(\pi) + (J-\sigma)\cos\left(\theta_1^0\right)\ \delta\theta_1\sin(\pi)\\ &+(J-\sigma)\sin\left(\theta_1^0\right)\cos(\pi)\ (\delta\phi_1-\delta\phi_2)\end{aligned} \tag{4.6}$$

$$\begin{aligned}\delta\dot{\phi}_1 = -B+&(J+\sigma)\left[\cos\left(\theta_2^0\right) - \cos(\pi)\cot\left(\theta_1^0\right)\sin\left(\theta_2^0\right) - \sin\left(\theta_2^0\right)\delta\theta_2\right.\\ &\left.+\sin(\pi)\cot\left(\theta_1^0\right)\sin\left(\theta_2^0\right)(\delta\phi_1-\delta\phi_2)\ \right]\\ &+(J+\sigma)\left[\cos(\pi)\left(1+\cot^2\left(\theta_1^0\right)\right)\sin\left(\theta_2^0\right)\delta\theta_1 - \cos(\pi)\cot\left(\theta_1^0\right)\cos\left(\theta_2^0\right)\delta\theta_2\right]\end{aligned} \tag{4.7}$$

$$\begin{aligned}\delta\dot{\phi}_2 = -B+&(J-\sigma)\left[\cos\left(\theta_1^0\right) - \cos(\pi)\cot\left(\theta_2^0\right)\sin\left(\theta_1^0\right) - \sin\left(\theta_1^0\right)\delta\theta_1\right.\\ &\left.+\sin(\pi)\cot\left(\theta_2^0\right)\sin\left(\theta_1^0\right)(\delta\phi_1-\delta\phi_2)\ \right]\\ &+(J-\sigma)\left[\cos(\pi)\left(1+\cot^2\left(\theta_2^0\right)\right)\sin\left(\theta_1^0\right)\delta\theta_2 - \cos(\pi)\cot\left(\theta_2^0\right)\cos\left(\theta_1^0\right)\delta\theta_1\right]\end{aligned} \tag{4.8}$$

From the beginning it was found that, the ground state is not changing in time. Because of this behaviour of the ground state, all zero order terms are zero, and only the linear parts in δ remain. The equations of motion can be simplified to the following expressions.

$$\delta\dot{\theta}_1 = \quad (J+\sigma)\sin\left(\theta_2^0\right)(\delta\phi_1-\delta\phi_2) \tag{4.9}$$

$$\delta\dot{\theta}_2 = -(J-\sigma)\sin\left(\theta_1^0\right)(\delta\phi_1-\delta\phi_2) \tag{4.10}$$

$$\delta\dot{\phi}_1 = (J+\sigma)\left[-\sin\left(\theta_2^0\right)(\delta\theta_1+\delta\theta_2) - \cot^2\left(\theta_1^0\right)\sin\left(\theta_2^0\right)\delta\theta_1 + \cot\left(\theta_1^0\right)\cos\left(\theta_2^0\right)\delta\theta_2\right] \tag{4.11}$$

$$\delta\dot{\phi}_2 = (J-\sigma)\left[-\sin\left(\theta_1^0\right)(\delta\theta_1+\delta\theta_2) - \cot^2\left(\theta_2^0\right)\sin\left(\theta_1^0\right)\delta\theta_2 + \cot\left(\theta_2^0\right)\cos\left(\theta_1^0\right)\delta\theta_1\right] \tag{4.12}$$

Pluggin in the ground state angles θ_1^0 and θ_2^0 from equations **3.24** and **3.25** gives

$$\delta\dot{\theta}_1 = \quad (J+\sigma)\sqrt{1-\frac{\left(B^2+4J\sigma\right)^2}{4B^2\left(J+\sigma\right)^2}}\,(\delta\phi_1-\delta\phi_2) \tag{4.13}$$

$$\delta\dot{\theta}_2 = -\ (J-\sigma)\sqrt{1-\frac{\left(B^2-4J\sigma\right)^2}{4B^2\left(J-\sigma\right)^2}}\,(\delta\phi_1-\delta\phi_2) \tag{4.14}$$

$$\delta\dot{\phi}_1 = \frac{B\left[B^2\delta\theta_1 - 2J^2\left(\delta\theta_1+\delta\theta_2\right) + 4J\sigma\delta\theta_1 - 2\sigma^2\left(\delta\theta_1+\delta\theta_2\right)\right]}{\sqrt{-\left(B^2-4J^2\right)\left(B^2-4\sigma^2\right)}} \tag{4.15}$$

$$\delta\dot{\phi}_2 = \frac{B\left[B^2\delta\theta_2 - 2J^2\left(\delta\theta_1+\delta\theta_2\right) - 4J\sigma\delta\theta_2 - 2\sigma^2\left(\delta\theta_1+\delta\theta_2\right)\right]}{\sqrt{-\left(B^2-4J^2\right)\left(B^2-4\sigma^2\right)}} \tag{4.16}$$

From these four equations it is now possible to generate a matrix spanned by $\delta\theta_1$, $\delta\theta_2$, $\delta\phi_1$ and $\delta\phi_2$, such that

$$\begin{pmatrix}\delta\dot{\theta}_1\\ \delta\dot{\theta}_2\\ \delta\dot{\phi}_1\\ \delta\dot{\phi}_2\end{pmatrix} = \underline{M_{4\times4}}\begin{pmatrix}\delta\theta_1\\ \delta\theta_2\\ \delta\phi_1\\ \delta\phi_2\end{pmatrix} \tag{4.17}$$

This equation has the form as mentioned earlier, it is a real first order differential equation, see equation **2.42**.

$$\underline{M_1} = \begin{pmatrix} 0 & 0 & (J+\sigma)\sqrt{1-\frac{(B^2+4J\sigma)^2}{4B^2(J+\sigma)^2}} & -(J+\sigma)\sqrt{1-\frac{(B^2+4J\sigma)^2}{4B^2(J+\sigma)^2}} \\ 0 & 0 & -(J-\sigma)\sqrt{1-\frac{(B^2-4J\sigma)^2}{4B^2(J-\sigma)^2}} & (J-\sigma)\sqrt{1-\frac{(B^2-4J\sigma)^2}{4B^2(J-\sigma)^2}} \\ \frac{B\left(-2J^2+4\sigma J-2\sigma^2\right)}{\sqrt{-(B^2-4J^2)(B^2-4\sigma^2)}} & \frac{B\left(B^2-2J^2-2\sigma^2\right)}{\sqrt{-(B^2-4J^2)(B^2-4\sigma^2)}} & 0 & 0 \\ \frac{B\left(B^2-2J^2-2\sigma^2\right)}{\sqrt{-(B^2-4J^2)(B^2-4\sigma^2)}} & \frac{B\left(-2J^2-4\sigma J-2\sigma^2\right)}{\sqrt{-(B^2-4J^2)(B^2-4\sigma^2)}} & 0 & 0 \end{pmatrix} \tag{4.18}$$

For values $J = 1.0$, $\sigma = 0.01$ and $B = 1.0$ the eigenfrequencies for the matrix are

$$E_1 = -0.1\,i \quad , \quad E_2 = 0.1\,i \quad , \quad E_3 = 0.0 \quad , \quad E_4 = 0.0 \tag{4.19}$$

The meaning of the value of these eigenfrequencies will be elaborated later in section **4.2**.

4.1.2 Friction expansion

Now the γ terms will be expanded. As mentioned earlier the γ represents the friction in the system. The 4 equations of motion are obtained from equations **3.42**, **3.43**, **3.44** and **3.45** and the ground state angles with a small deviation are used.

$$\begin{aligned}\delta\dot{\theta}_1 = {} & \gamma\left[-B\sin(\theta_1^0+\delta\theta_1) + (J+\sigma)\left[\,\cos(\theta_2^0+\delta\theta_2)\sin(\theta_1^0+\delta\theta_1)\right.\right.\\ & \left.\left. - \cos(\pi+\delta\phi_1-\delta\phi_2)\cos(\theta_1^0+\delta\theta_1)\sin(\theta_2^0+\delta\theta_2)\,\right]\right]\end{aligned} \tag{4.20}$$

$$\begin{aligned}\delta\dot{\theta}_2 = {} & \gamma\left[-B\sin(\theta_2^0+\delta\theta_2) + (J-\sigma)\left[\cos(\theta_1^0+\delta\theta_1)\sin(\theta_2^0+\delta\theta_2)\right.\right.\\ & \left.\left. - \cos(\pi+\delta\phi_1-\delta\phi_2)\cos(\theta_2^0+\delta\theta_2)\sin(\theta_1^0+\delta\theta_1)\,\right]\right]\end{aligned} \tag{4.21}$$

$$\delta\dot{\phi}_1 = \gamma(J+\sigma)\csc(\theta_1^0+\delta\theta_1)\sin(\theta_2^0+\delta\theta_2)\sin(\pi+\delta\phi_1-\delta\phi_2) \tag{4.22}$$

$$\delta\dot{\phi}_2 = -\gamma(J-\sigma)\csc(\theta_2^0+\delta\theta_2)\sin(\theta_1^0+\delta\theta_1)\sin(\pi+\delta\phi_1-\delta\phi_2) \tag{4.23}$$

An expansion to first order in δ around 0 is performed. The zeroth order terms also cancel out now, because the ground state is not moving.

$$\delta\dot{\theta}_1 = \gamma\left[(J+\sigma)\cos\left(\theta_1^0+\theta_2^0\right)(\delta\theta_1+\delta\theta_2) - B\cos\left(\theta_1^0\right)\delta\theta_1\right] \tag{4.24}$$

$$\delta\dot{\theta}_2 = \gamma\left[(J-\sigma)\cos\left(\theta_1^0+\theta_2^0\right)(\delta\theta_1+\delta\theta_2) - B\cos\left(\theta_2^0\right)\delta\theta_2\right] \tag{4.25}$$

$$\delta\dot{\phi}_1 = -\gamma\,(J+\sigma)\csc\left(\theta_1^0\right)\sin\left(\theta_2^0\right)(\delta\phi_1-\delta\phi_2) \tag{4.26}$$

$$\delta\dot{\phi}_1 = \gamma\,(J-\sigma)\csc\left(\theta_2^0\right)\sin\left(\theta_1^0\right)(\delta\phi_1-\delta\phi_2) \tag{4.27}$$

Which, when the ground state angles θ_1^0 and θ_2^0 from equations **3.24** and **3.25** are plugged in, becomes.

$$\delta\dot{\theta}_1 = -\frac{\gamma(-B^2\,\delta\theta_2 + 2(J^2+\sigma^2)(\delta\theta_1+\delta\theta_2) - 4\,J\,\sigma\,\delta\theta_1)}{2(J-\sigma)} \tag{4.28}$$

$$\delta\dot{\theta}_2 = -\frac{\gamma(-B^2\,\delta\theta_1 + 2(J^2+\sigma^2)(\delta\theta_1+\delta\theta_2) + 4\,J\,\sigma\,\delta\theta_2)}{2(J+\sigma)} \tag{4.29}$$

$$\delta\dot{\phi}_1 = -\gamma(J-\sigma)(\delta\phi_1-\delta\phi_2) \tag{4.30}$$

$$\delta\dot{\phi}_2 = \quad \gamma(J+\sigma)(\delta\phi_1-\delta\phi_2) \tag{4.31}$$

To fulfill equation **4.17**, a matrix, spanned by $\delta\theta_1$, $\delta\theta_2$, $\delta\phi_1$ and $\delta\phi_2$, is generated in the same maner as before.

$$\underline{M_2} = \begin{pmatrix} -\gamma(J-\sigma) & \frac{\gamma\left(B^2-2J^2-2\sigma^2\right)}{2(J-\sigma)} & 0 & 0 \\ \frac{\gamma\left(B^2-2J^2-2\sigma^2\right)}{2(J+\sigma)} & -\gamma(J+\sigma) & 0 & 0 \\ 0 & 0 & -\gamma(J-\sigma) & \gamma(J-\sigma) \\ 0 & 0 & \gamma(J+\sigma) & -\gamma(J+\sigma) \end{pmatrix} \tag{4.32}$$

For values $J = 1.0$, $\sigma = 0.01$, $B = 1.0$ and $\gamma = 0.01$ the eigenfrequencies for the matrix are

$$E_1 = -0.02 \quad , \quad E_2 = -0.015 \quad , \quad E_3 = -0.005 \quad , \quad E_4 = 0.0 \tag{4.33}$$

The meaning of the value of these eigenfrequencies will be elaborated later in section **4.2**.

4.1.3 Beta expansion

The last part of the expansion is the terms including β. Again the 4 equations of motion are obtained from equations **3.42**, **3.43**, **3.44** and **3.45**. Just like above the expressions with the ground state angles and a small deviation are used.

$$\begin{aligned}\delta\dot{\theta}_1 = \beta \big[&-(J-\sigma)\cos(\theta_2^0+\delta\theta_2)\sin(\theta_1^0+\delta\theta_1) \\ &- \big[B-(J-\sigma)\cos(\theta_1^0+\delta\theta_1)\big]\cos(\pi+\delta\phi_1-\delta\phi_2)\sin(\theta_2^0+\delta\theta_2)\ \big]\end{aligned} \tag{4.34}$$

$$\begin{aligned}\delta\dot{\theta}_2 = -\beta \big[&-(J+\sigma)\cos(\theta_1^0+\delta\theta_1)\sin(\theta_2^0+\delta\theta_2) \\ &- \ \big[B-(J+\sigma)\cos(\theta_2^0+\delta\theta_2))\cos(\pi+\delta\phi_1-\delta\phi_2)\sin(\theta_1^0+\delta\theta_1)\big]\end{aligned} \tag{4.35}$$

$$\delta\dot{\phi}_1 = \beta\left[B\cos(\theta_1^0+\delta\theta_1)-(J-\sigma)\right]\csc(\theta_1^0+\delta\theta_1)\sin(\theta_2^0+\delta\theta_2)\sin(\pi+\delta\phi_1-\delta\phi_2) \tag{4.36}$$

$$\delta\dot{\phi}_2 = -\beta\left[(J-\sigma)-B\cos(\theta_2^0+\delta\theta_2)\right]\csc(\theta_2^0+\delta\theta_2)\sin(\theta_1^0+\delta\theta_1)\sin(\pi+\delta\phi_1-\delta\phi_2) \tag{4.37}$$

As before the equations are expanded to first order in δ around 0. Once again the zero order terms all cancel out, because of the definition of the ground state.

$$\delta\dot{\theta}_1 = \beta\left[B\cos\left(\theta_2^0\right)\delta\theta_2-(J-\sigma)\cos\left(\theta_1^0+\theta_2^0\right)\right] \tag{4.38}$$

$$\delta\dot{\theta}_2 = \beta\left[-B\cos\left(\theta_1^0\right)\delta\theta_1+(J+\sigma)\cos\left(\theta_1^0+\theta_2^0\right)\right] \tag{4.39}$$

$$\delta\dot{\phi}_1 = -\beta\left[-J+\sigma+B\cos\left(\theta_1^0\right)\right]\csc\left(\theta_1^0\right)\sin\left(\theta_2^0\right)\left(\delta\phi_1-\delta\phi_2\right) \tag{4.40}$$

$$\delta\dot{\phi}_2 = \beta\left[J+\sigma-B\cos\left(\theta_2^0\right)\right]\csc\left(\theta_2^0\right)\sin\left(\theta_1^0\right)\left(\delta\phi_1-\delta\phi_2\right) \tag{4.41}$$

Plugging in the ground state angles θ_1^0 and θ_2^0 from equations **3.24** and **3.25** gives.

$$\delta\dot{\theta}_1 = \frac{\beta(-B^2+2(J^2+\sigma^2)(\delta\theta_1+\delta\theta_2)+4J\sigma\,\delta\theta_2)}{2(J+\sigma)} \tag{4.42}$$

$$\delta\dot{\theta}_2 = \frac{-\beta(-B^2+2(J^2+\sigma^2)(\delta\theta_1+\delta\theta_2)-4J\sigma\,\delta\theta_2)}{2(J-\sigma)} \tag{4.43}$$

$$\delta\dot{\phi}_1 = \frac{-\beta(B^2-2(J^2+\sigma^2))(\delta\phi_1+\delta\phi_2)}{2(J+\sigma)} \tag{4.44}$$

$$\delta\dot{\phi}_2 = \frac{-\beta(B^2-2(J^2+\sigma^2))(\delta\phi_1+\delta\phi_2)}{2(J-\sigma)} \tag{4.45}$$

Again a matrix spanned by $\delta\theta_1$, $\delta\theta_2$, $\delta\phi_1$ and $\delta\phi_2$ is generated, such that it fulfills equation **4.17**

$$\underline{M_3} = \beta \begin{pmatrix} \frac{2(J^2+\sigma^2)-B^2}{2(J+\sigma)} & \frac{2J^2+4J\sigma+2\sigma^2}{2(J+\sigma)} & 0 & 0 \\ \frac{-2J^2+4J\sigma-2\sigma^2}{2(J-\sigma)} & \frac{B^2-2(J^2+\sigma^2)}{2(J-\sigma)} & 0 & 0 \\ 0 & 0 & -\frac{B^2-2(J^2+\sigma^2)}{2(J+\sigma)} & \frac{B^2-2(J^2+\sigma^2)}{2(J+\sigma)} \\ 0 & 0 & -\frac{B^2-2(J^2+\sigma^2)}{2(J-\sigma)} & \frac{B^2-2(J^2+\sigma^2)}{2(J-\sigma)} \end{pmatrix} \tag{4.46}$$

For values $J = 1.0$, $\sigma = 0.01$, $B = 1.0$, $\gamma = 0.01$ and $\beta = 0.1$ the eigenfrequencies for the matrix are

$$E_1 = -0.0005 + 0.087i \quad , \quad E_2 = -0.0005 - 0.087i \quad , \quad E_3 = -0.001 \quad , \quad E_4 = 0.0 \tag{4.47}$$

The meaning of the value of these eigenfrequencies will be elaborated later in section **4.2**.

4.2 Total system

Adding the three matrices M_1, M_2 and M_3 together, will provide the total matrix describing the dynamics of the system close to the ground state.

$$\underline{M} = \begin{pmatrix} \frac{\beta(2(J^2+\sigma^2)-B^2)}{2(J+\sigma)} - \gamma(J-\sigma) & \frac{\beta(2J^2+4\sigma J+2\sigma^2)}{2(J+\sigma)} + \frac{\gamma(B^2-2J^2-2\sigma^2)}{2(J-\sigma)} & (J+\sigma)\sqrt{1-\frac{(B^2+4J\sigma)^2}{4B^2(J+\sigma)^2}} & (-J-\sigma)\sqrt{1-\frac{(B^2+4J\sigma)^2}{4B^2(J+\sigma)^2}} \\ \frac{\gamma(B^2-2J^2-2\sigma^2)}{2(J+\sigma)} + \frac{\beta(-2J^2+4\sigma J-2\sigma^2)}{2(J-\sigma)} & \frac{\beta(B^2-2(J^2+\sigma^2))}{2(J-\sigma)} - \gamma(J+\sigma) & (\sigma-J)\sqrt{1-\frac{(B^2-4J\sigma)^2}{4B^2(J-\sigma)^2}} & (J-\sigma)\sqrt{1-\frac{(B^2-4J\sigma)^2}{4B^2(J-\sigma)^2}} \\ \frac{B(-2J^2+4\sigma J-2\sigma^2)}{\sqrt{(4J^2-B^2)(B^2-4\sigma^2)}} & \frac{B(B^2-2J^2-2\sigma^2)}{\sqrt{(4J^2-B^2)(B^2-4\sigma^2)}} & \frac{\beta(2(J^2+\sigma^2)-B^2)}{2(J+\sigma)} - \gamma(J-\sigma) & \gamma(J-\sigma) + \frac{\beta(B^2-2(J^2+\sigma^2))}{2(J+\sigma)} \\ \frac{B(B^2-2J^2-2\sigma^2)}{\sqrt{(4J^2-B^2)(B^2-4\sigma^2)}} & \frac{B(-2J^2-4\sigma J-2\sigma^2)}{\sqrt{(4J^2-B^2)(B^2-4\sigma^2)}} & \gamma(J+\sigma) + \frac{\beta(2(J^2+\sigma^2)-B^2)}{2(J-\sigma)} & \frac{\beta(B^2-2(J^2+\sigma^2))}{2(J-\sigma)} - \gamma(J+\sigma) \end{pmatrix} \tag{4.48}$$

For the values $J = 1.0$, $B = 1.0$, $\sigma = 0.01$, $\gamma = 0.01$ and $\beta = 0.1$ the eigenfrequencies are

$$E_1 = -0.013 + 1.0i \quad , \quad E_2 = -0.013 - 1.0i \quad , \quad E_3 = -0.017 \quad , \quad E_4 = 0.0 \tag{4.49}$$

As mentioned earlier the behaviour of the mode can be described with equation **2.43**. This is a case, where the eigenfrequencies consists of both a real and an imaginary part. The imaginary part makes the mode oscillate in time, while the real part is responsible for the decay or drive of the mode.
The sign of the imaginary part has no effect on the behaviour of the system, as far as this study is concerned. The sign of the real part, on the other hand, makes a big difference. If the value is negative, then the mode is decaying towards the stable ground state. If the value is positive, it corresponds to a mode running away from the ground state.

In the simple case, where the friction and the β term are excluded, it is seen from equation **4.19** that the eigenfrequencies are either zero or imaginary. As expected the simple case is not decaying, but only oscillating in time.
When including friction to the system, it is seen from equation **4.33** that the eigenfrequencies are negative and real. Just as expected, the system decays towards the ground

state, when friction is present.
The eigenfrequencies for the β term is shown in equation **4.47**. The eigenfrequencies have both a real part and an imaginary part. The real part of each of the eigenfrequencies is negative, which means that the system wants to decay towards the ground state.

Lastly the total matrix for the system close to the ground state is inspected. From equation **4.49**, it is seen that the eigenfrequencies consist of a real and imaginary part. This is clearly because each of the constituent matrices have either real or imaginary eigenfrequencies, or in the last case both. This means that the total system is oscillating in time, but also decaying towards the stable ground state.

The next endeavour will be to investigate, whether or not a possibility exists that, the system does not decay back towards the ground state.

4.3 β and σ relevance

This next section will look at, how the eigenfrequencies of the total matrix for the system change, when the parameters are changed. This is done, in order to get a better understanding of what is going on close to the ground state of the system. The focus will be on the two parameters β and σ, because these two parameters arise from the current that flows through the system. As mentioned earlier, the parameters β and σ can generate curl forces.
From the matrix in equation **4.48** the eigenfrequencies were found numerically. It is the decay of the eigenmodes and not their oscillation in time, which is interesting for this study. The real part of the eigenfrequency is responsible for the decay of the eigenmodes, therefore the real part of the eigenfrequencies will be in focus. For now the imaginary parts of the eigenfrequencies are excluded.

The next step is to take a sweep in each of the parameters β and σ and make a contour plot for the highest value of the eigenfrequencies. The eigenfrequency equal to zero will not be included, because it does not change and therefore is not interesting in this investigation. The highest valued eigenfrequency is an indicator of whether or not, there exists a positive eigenfrequency.

Depending on the size of B and J, the size of σ has to be so that the argument of the inverse cosine does not exceed the value $||1||$.

$$
\begin{aligned}
\theta_1^0 &= \cos^{-1}\left(\frac{B^2 - 4J\sigma}{2B(J-\sigma)}\right) \\
\theta_2^0 &= \cos^{-1}\left(\frac{B^2 + 4J\sigma}{2B(J+\sigma)}\right)
\end{aligned}
\tag{4.50}
$$

With that in mind B, J and γ are chosen and kept fixed. In this way it will be clear how β and σ affect the eigenfrequencies.

$$
J = 1.0 \quad , \quad B = 1.0 \quad , \quad \gamma = 0.01 \tag{4.51}
$$

In figure **4.1** is shown how the highest valued eigenfrequency change, when the two parameters are changed. It is obvious that there are two distinct regions. One region is blue and corresponds to a negative eigenfrequency. The other region is red and corresponds to a positive eigenfrequency. The eigenmodes with parameters in the blue region are damped, which means that the spins will decay back to the ground state. In the red

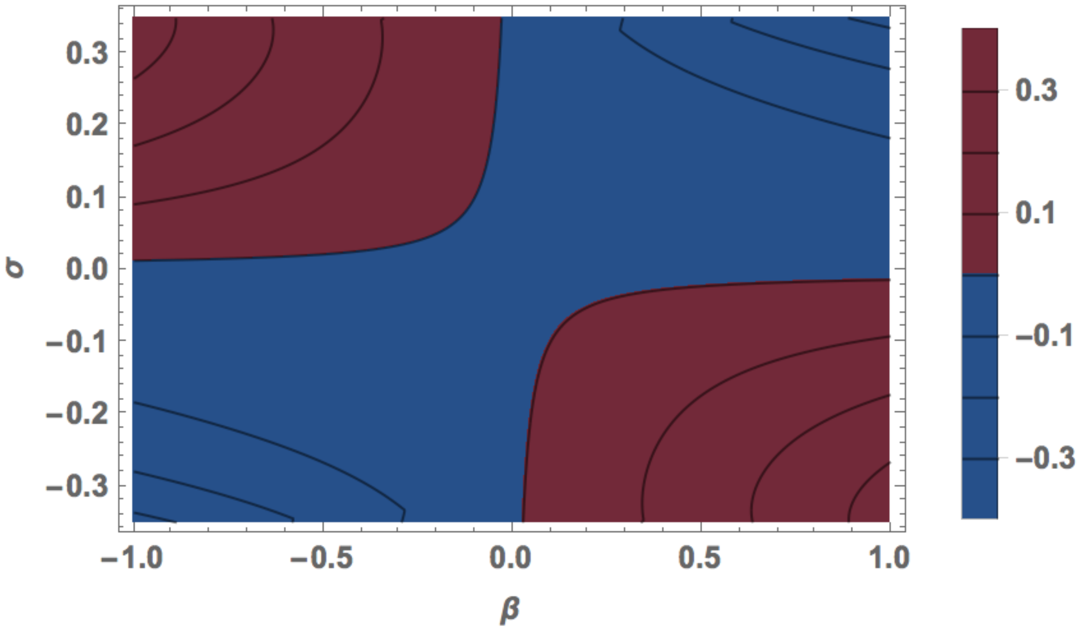

Figure 4.1: A contour plot showing for which values of β and σ the real part of the eigenfrequency is either positive or negative for $\gamma = 0.01$. The red region indicates that the eigenfrequency is positive, which corresponds to a runaway mode. The blue region corresponds to the situation, where the eigenmode decays back towards the ground state. Only the sign of the eigenfreguency is clear from the contour plot.

region, where the eigenmodes have positive eigenfrequencies, the spins will seek away from the ground state, such a mode will be referred to as a runaway mode.
So there exist values for β and σ, where the ground state is no longer the desired destination for the spins. This is the behaviour, this study is looking for.

Now the values for β and σ are chosen, so that an eigenmode, which will seek away from the ground state is obtained. After inspecting the contour plot, the values chosen are.

$$J = 1.0 \quad , \quad B = 1.0 \quad , \quad \sigma = 0.1 \quad , \quad \gamma = 0.01 \quad , \quad \beta = -0.2 \tag{4.52}$$

From the total matrix, equation **4.48**, the following eigenfrequencies are calculated

$$E_1 = -0.0065 \quad , \quad E_2 = -0.0065 \quad , \quad E_3 = 0.014 \quad , \quad E_4 = 0.0 \tag{4.53}$$

These eigenfrequencies show that at the same time, it is possible to have a mode, which is damped and seeks back to the ground state, but also a mode, which seeks away from the ground state.

4.3.1 γ relevance

Now that the influence on the eigenfrequencies by β and σ has been observed, the influence by changing γ will be explored. First is observed how a change from 0.01 to 0.1 will affect the eigenfrequencies.

When comparing figure **4.1** with figure **4.2**, it can be seen that the blue region is bigger in the case of bigger γ. This can be understood physically as, when the friction in the system is increased, there will be an increase in how much the system would like to go

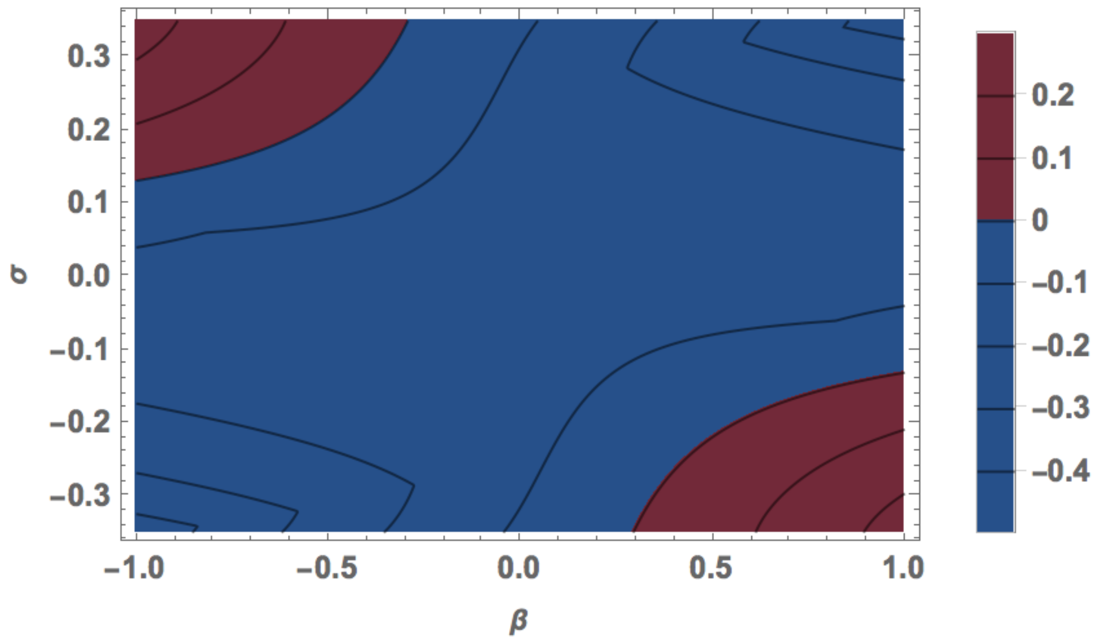

Figure 4.2: A contour plot showing for which β and σ values the real part of the eigenfrequency is either positive or negative for $\gamma = 0.1$. The red region indicates that the eigenfrequency is positive, which corresponds to a runaway mode. The blue region corresponds to the situation, where the eigenmode decays back towards the ground state. Only the sign of the eigenfreguency is clear from the contour plot.

back to the ground state. It will be more difficult to get a runaway mode, and that is exactly what is seen. In order to make the system run away from the ground state, higher values for β and σ are needed, which correspond to more current running through the system.

If γ is decreased to 0.001, figure **4.3** shows that the blue region gets smaller compared to that in figure **4.1**. It will be easier to get a runaway mode, because the values of β and σ need not to be as high, which corresponds to less current running through the system.

4.3.2 B and J relevance

It has now been shown how γ affects the system. Next it is time to investigate how the size of B and J affect the system. Recall that the polar ground state angles for the system is

$$\begin{aligned}\theta_1^0 &= \cos^{-1}\left(\frac{B^2 - 4J\sigma}{2B(J-\sigma)}\right) \\ \theta_2^0 &= \cos^{-1}\left(\frac{B^2 + 4J\sigma}{2B(J+\sigma)}\right)\end{aligned} \tag{4.54}$$

The arccos function is only defined for an argument between -1 and 1, which for $\sigma = 0$ corresponds to

$$-2J < B < 2J \tag{4.55}$$

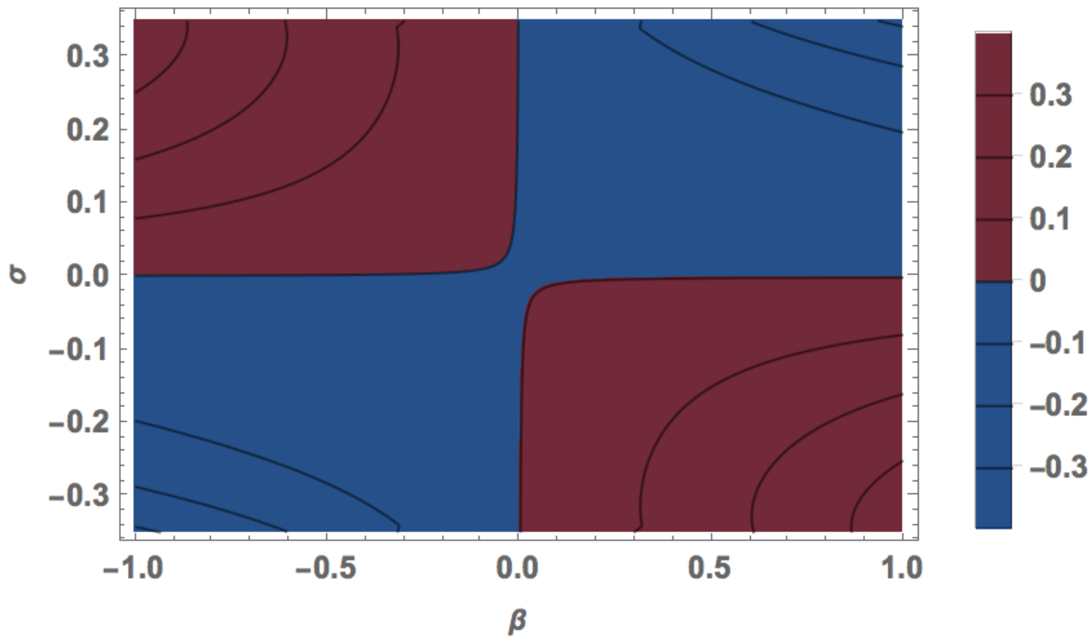

Figure 4.3: A contour plot showing for which β and σ values the real part of the eigenfrequency is either positive or negative for $\gamma = 0.001$. The red region indicates that the eigenfrequency is positive, which corresponds to a runaway mode. The blue region corresponds to the situation, where the eigenmode decays back towards the ground state. Only the sign of the eigenfreguency is clear from the contour plot.

When the values for B and J are altered and with $\sigma = 0$, the ground state must change.

$$\theta^0 = \cos^{-1}\left(\frac{B}{2\,J}\right) \tag{4.56}$$

Which for the two cases under investigation is

$$\theta^0 = \begin{cases} 24.6^\circ & \text{for} \quad J = 0.55\,, \quad B = 1.0\,, \quad \gamma = 0.01 \\ 82.8^\circ & \text{for} \quad J = 1.0\,, \quad B = 0.25\,, \quad \gamma = 0.01 \end{cases} \tag{4.57}$$

Here it is evident that the ground state angle is in fact closer to the magnetic field, when the Zeeman energy is greater than the exchange energy.
It is also clear that, when the exchange energy is greater than the Zeeman energy, the ground state angle is further from the magnetic field.

In figure **4.4** is shown a situation, where $B = 1.0$, $J = 0.55$ and $\gamma = 0.01$. This corresponds to a situation, where the Zeeman energy is greater than the exchange energy between the spins. It is therfore more favourable to align closer towards the magnetic field, and thereby align the spins almost parallel.
The red region is dominating the contour plot, which means that for almost any value σ and β, there exists at least one positive eigenfrequency. With the positive eigenfrequency, the eigenmode wants to seek away from the ground state. Only in the blue region is the ground state a stable point, where the eigenfrequencies are all negative.

In figure **4.5** is shown a situation, where $B = 0.25$, $J = 1.0$ and $\gamma = 0.01$. This corresponds to a situation, where the exchange energy between the spins is greater than the Zeeman energy. It is therefore more favourable to not align closely to the magnetic

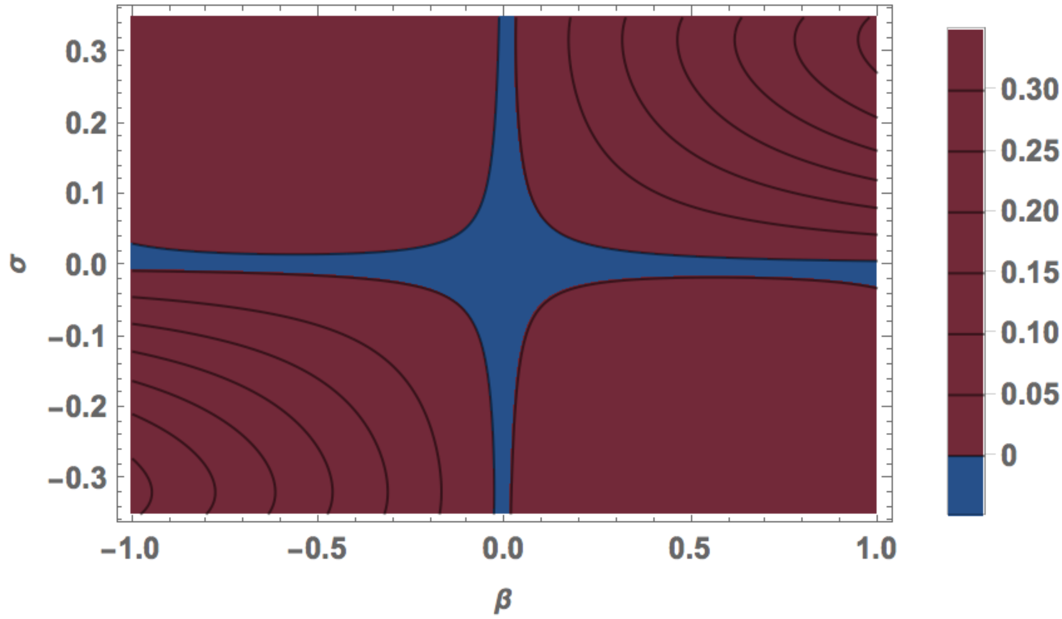

Figure 4.4: A contour plot showing the highest valued eigenfrequency. The red region indicates that the eigenfrequency is positive, which corresponds to a runaway mode. The blue region corresponds to the situation, where the eigenmode decays back towards the ground state. The values used are $B = 1.0$, $J = 0.55$ and $\gamma = 0.01$. Only the sign of the eigenfreguency is clear from the contour plot.

field, but instead align the spins almost antiparallel.
It is seen that this contourplot is almsot identical to that in figure **4.1**. The σ axis is in figure **4.5** scaled differently compared to figure **4.1**. This sugests that changing the value of B, is only scaling the energy for the system.
The values needed to examine the situation was chosen in equation **4.52**. Next up will be to try and understand the actual dynamics of the system.

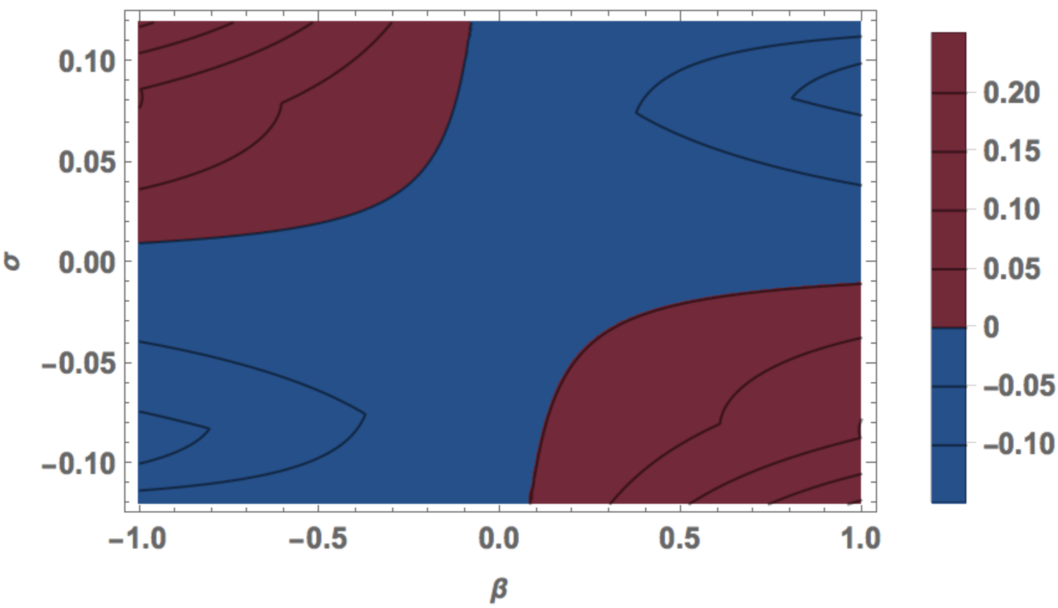

Figure 4.5: A contour plot showing the highest valued eigenfrequency. The red region indicates that the eigenfrequency is positive, which corresponds to a runaway mode. The blue region corresponds to the situation, where the eigenmode decays back towards the ground state. The values used are $B = 0.25$, $J = 1.0$ and $\gamma = 0.01$. Only the sign of the eigenfreguency is clear from the contour plot.

Chapter 5

Numerical analysis

5.1 Exact solution

In this section the system will be investigated by means of numerics. The differential equations will be solved numerically, observing the behaviour around the ground state, and see how a runaway mode will affect the dynamics of the system.

5.1.1 Simple cases

The simplest case, where two spins are coupled in a magnetic field wihtout friction included and the β term equal zero, is the first to be studied.
The four coupled differential equations from equations **3.42**, **3.43**, **3.44** and **3.45** needs to be solved. Choosing initial conditions, so the system starts in the ground state.

$$\theta_1(t=0) = \cos^{-1}(\frac{B^2 - 4J\sigma}{2B(J-\sigma)}) \quad , \quad \theta_2(t=0) = \cos^{-1}(\frac{B^2 + 4J\sigma}{2B(J+\sigma)}) \quad , \quad \phi_1(t=0) - \phi_2(t=0) = \pi \tag{5.1}$$

In the numerical analysis for this section the values used are

$$J = 1.0 \qquad B = 1.0 \qquad \gamma = 0.1 \qquad \beta = 0.0 \qquad \sigma = 0.1 \tag{5.2}$$

And the push away from the ground state in each of the situations is

$$\delta\theta_1 = 0.1 \qquad \delta\theta_2 = 0.1 \qquad \delta\phi_1 = 0.1 \qquad \delta\phi_2 = 0.1 \tag{5.3}$$

In figure **5.1**, is shown the time evolution of the system. The y-axis is cosine to the angles, in this way the spherical behaviour of the spins can easily be seen. The main point from this figure, is that the angles do not change in time. This is exactly what was found earlier that the ground state is a stable point.
In figure **5.2** the system is pushed away from the ground state, and the system oscillates in time. Each of the azimuthal angles moves as the precession for a single spin in a magnetic field, see figure **2.5**. The two azimuthal angles are moving with a phase difference of π relative to each other.
The polar angles also oscillate, but they do so around the ground state. The total motion is the spins precessing around the magnetic field direction with phase difference π, while they each wobble around the ground state angle, $\theta_g s$.

When friction is included in the system, it can be seen that the system relaxes back towards the ground state, see figure **5.3**. The oscillatory motions of both the azimuthal angles and the polar angles are slowing down. When the polar angles are nearing the ground state and stops moving, the azimuthal angles also stop their precession. In the end, the friction will cause the system to stop moving and end back in the situation like that in figure **5.1**.

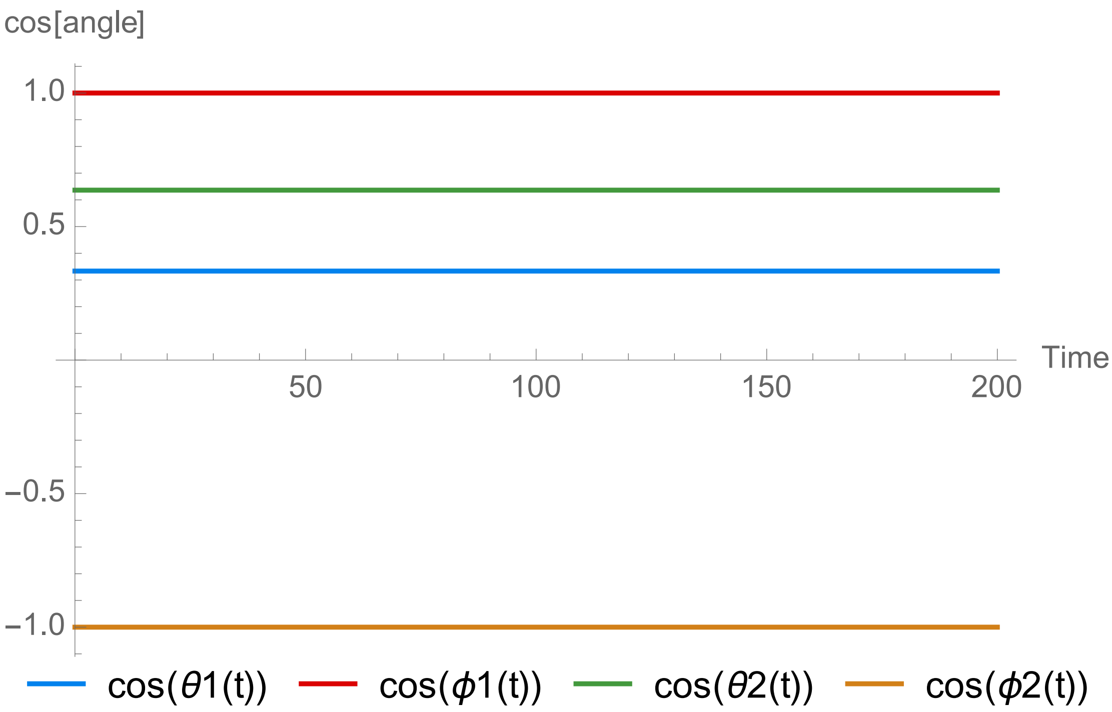

Figure 5.1: The time evolution of the system, when initially put in the ground state. The change in time is zero for all angles, which means that the system is not moving. There is a phase difference of π between the two spin in the azimuthal plane. The angles in the polar plane are split symmetrically around the angle, $\cos\left[(B^2 \pm 4\,J\,\sigma)/(2\,B\,(J \pm \sigma)\right]$.
$J = 1.0 \qquad B = 1.0 \qquad \gamma = 0.0 \qquad \beta = 0.0 \qquad \sigma = 0.1$

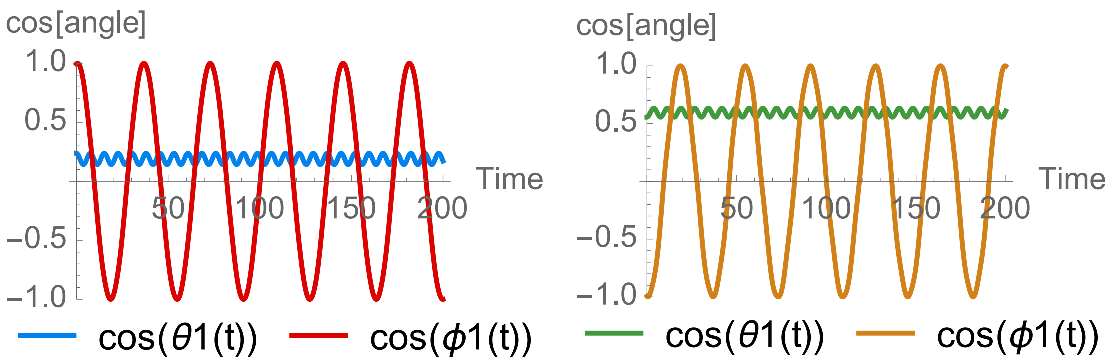

Figure 5.2: The oscillating motion of the system when pushed away from the ground state. The motions for both spins are the same, again the spins are π out of phase in the azimuthal plane. The angles in the polar plane are split symmetrically, but oscillates around the angle, $\cos\left[(B^2 \pm 4\,J\,\sigma)/(2\,B\,(J \pm \sigma)\right]$.
$J = 1.0 \qquad B = 1.0 \qquad \gamma = 0.0 \qquad \beta = 0.0 \qquad \sigma = 0.1$

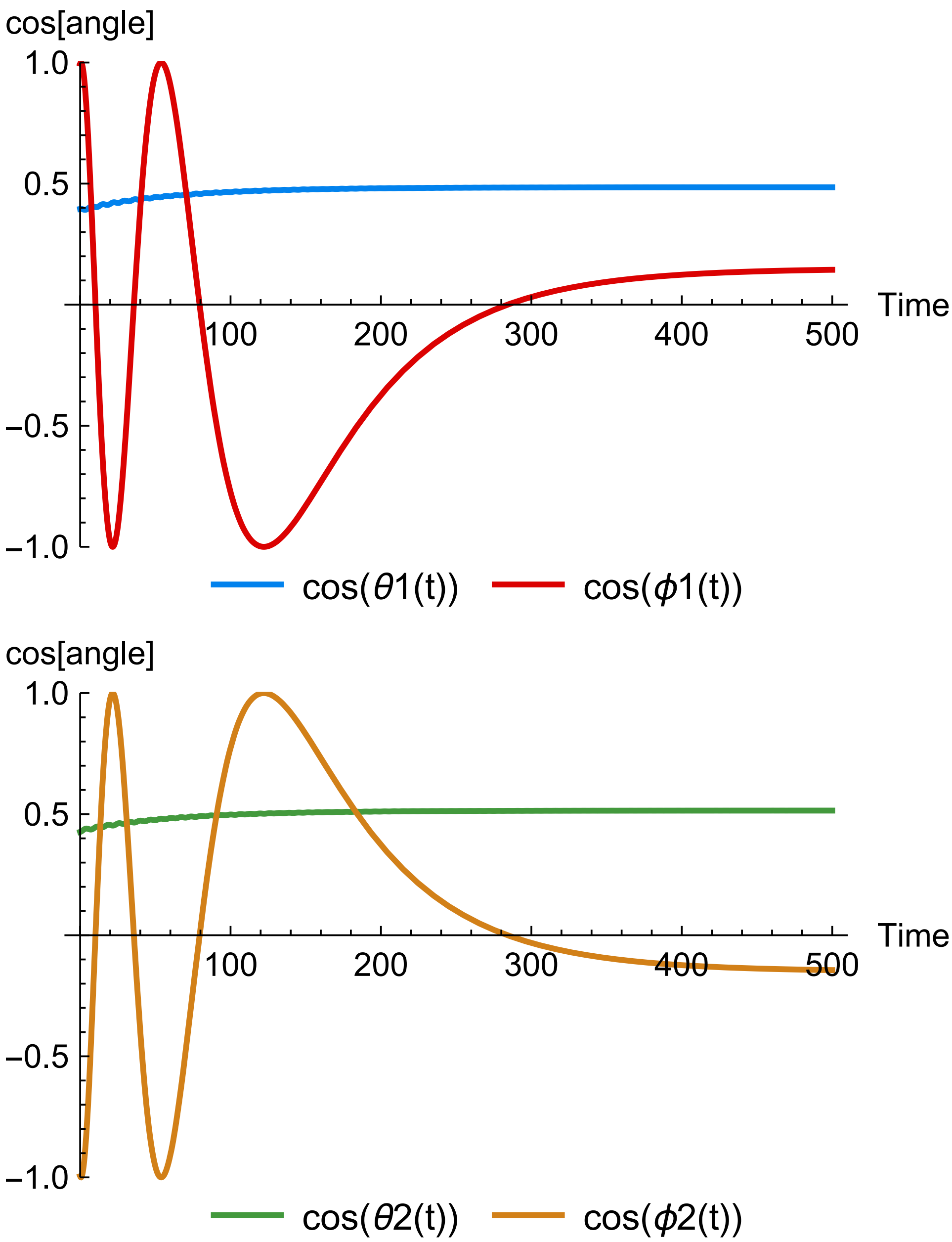

Figure 5.3: The oscillating motion relaxing towards the ground state. The friction relaxes the system, and it will stop moving in the end. The oscillation in the azimuthal plane gets slower as the polar oscillation loses amplitude.
$J = 1.0 \qquad B = 1.0 \qquad \gamma = 0.1 \qquad \beta = 0.0 \qquad \sigma = 0.1$

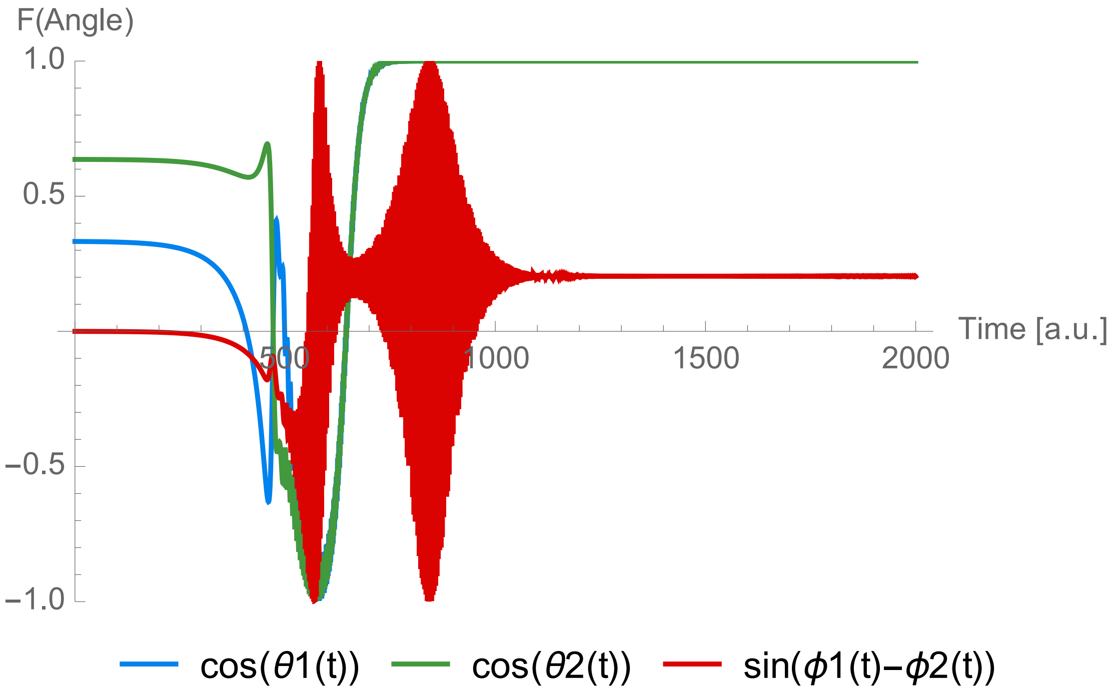

Figure 5.4: Time evolution of the runaway mode, when a runaway mode is present. F is a function, which is either cosine or sine. The polar angles start close to the ground state and as time goes, they becomes chaotic and end up with an angle $\theta = 0$. The phase difference between ϕ_1 and ϕ_2 is π, but as the polar angles become chaotic, the azimuthal difference also becomes chaotic before converging towards an angle $\sim \pi \pm 0.2$ radians.

5.1.2 Complicated cases

In order to investigate the more complicated case, the non-equilibrium term, β, is included to the system. From the expansion found in the previous section, it was clear that runaway modes can be present - this will be explored numerically now.

How two different modes evolve in time can be observed by using the corresponding eigenmodes as the perturbation. A single mode is investigated at a time. This is done by multiplying the eigenmode by a small number, η, and using this as the perturbation away from the ground state.

$$\begin{pmatrix} \theta_1(0) \\ \theta_2(0) \\ \phi_1(0) \\ \phi_2(0) \end{pmatrix} = \begin{pmatrix} \theta_1^0 \\ \theta_2^0 \\ \phi_1^0 \\ \phi_2^0 \end{pmatrix} + \eta \begin{pmatrix} \delta\theta_1 \\ \delta\theta_2 \\ \delta\phi_1 \\ \delta\phi_2 \end{pmatrix} \tag{5.4}$$

The following values are used for the analysis

$$J = 1.0 \qquad B = 1.0 \qquad \gamma = 0.01 \qquad \beta = -0.2 \qquad \sigma = 0.1 \tag{5.5}$$

The time evolution of the runaway mode is shown in figure **5.4**. The cosine to the polar angles are plotted as well as sine to the difference in the azimuthal angles. Both polar angles start almost not moving, but as time goes they both go through a chaotic period, before they converge towards an angle $\theta = 0$; the direction along the magnetic field. The difference between the azimuthal angles begins at π, but undergoes a chaotic period alongside the polar angles. The difference ends up at an angle difference $\sim \pi \pm 0.2$ radians.

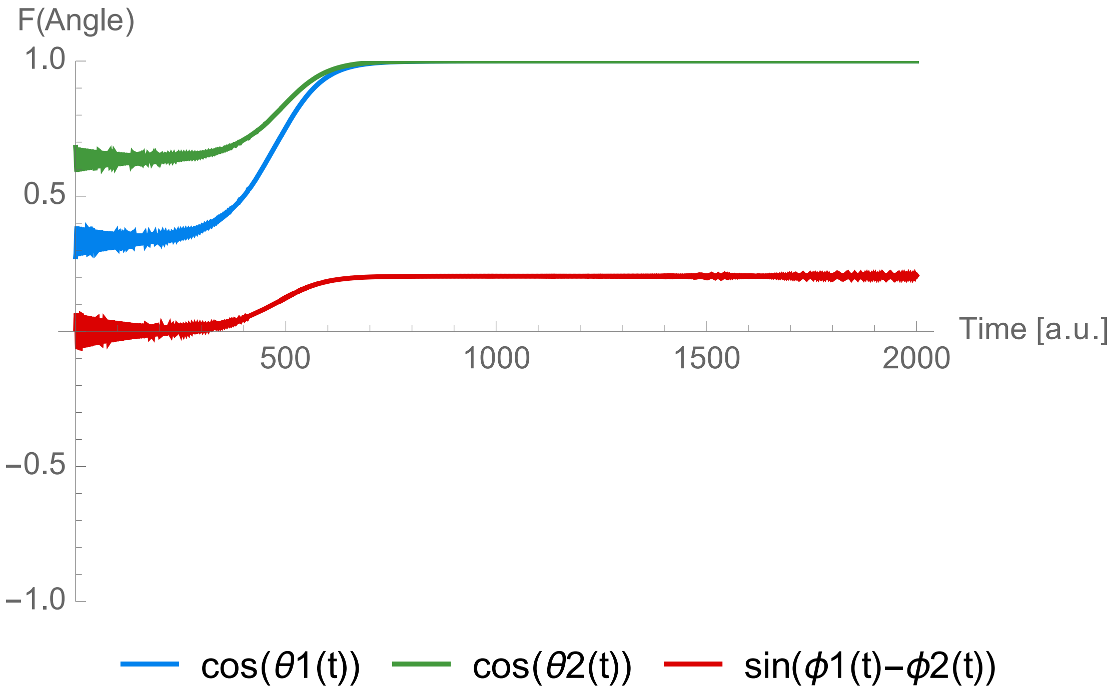

Figure 5.5: Time evolution of a damped mode, when a runaway mode is present. F is a function, which is either cosine or sine. The polar angles oscillate around the ground state, they seek towards the North Pole. The difference between phi_1 and ϕ_2 oscillates around zero and then converges towards an angle $\sim \pi \pm 0.2$ radians, when the polar angles is close to the North Pole.

The damped mode, is shown in figure **5.5**. The polar angles behave as they did in the situation without the β term, but after some time the behaviour changes. The angles converge towards the North Pole, just as they did for the runaway mode. likewise the azimuthal difference converges towards an angle difference $\sim \pi \pm 0.2$ radians, just as for the runaway mode.

In figure **5.6** is shown the system, when β is changed from a negative value to a positive value. Both the polar angles and the difference in the azimuthal plane oscillates at first, but decays back towards the ground state. The system decays towards the ground state, because, as was seen in figure **4.1**, there is no runaway mode for these values.

In figures **5.7** and **5.8** are shown a real space representation of the spins. The azimuthal plane is seen to the left and the polar plane to the right.
Figure **5.7** shows the spins as they are in the situation in figure **5.6**. The spins are shown for the time equal to 1500. It is seen that the spins have an angle of π between them in the azimuthal plane. In the polar plan the spins are close to the same angle, but are shifted by σ. This means that figure **5.7** shows how the ground state looks like.
Figure **5.8** shows the spins as they are in the situation in figure **5.4**. The spins are shown for the time equal to 1500. Both spins are pointing towards the North Pole. In the azimuthal plane only the tips of the spins are seen, which means that they are pointing upwards. In the polar plane is seen that both spins have an angle equal to zero, which also means that they are pointing towards the North Pole.

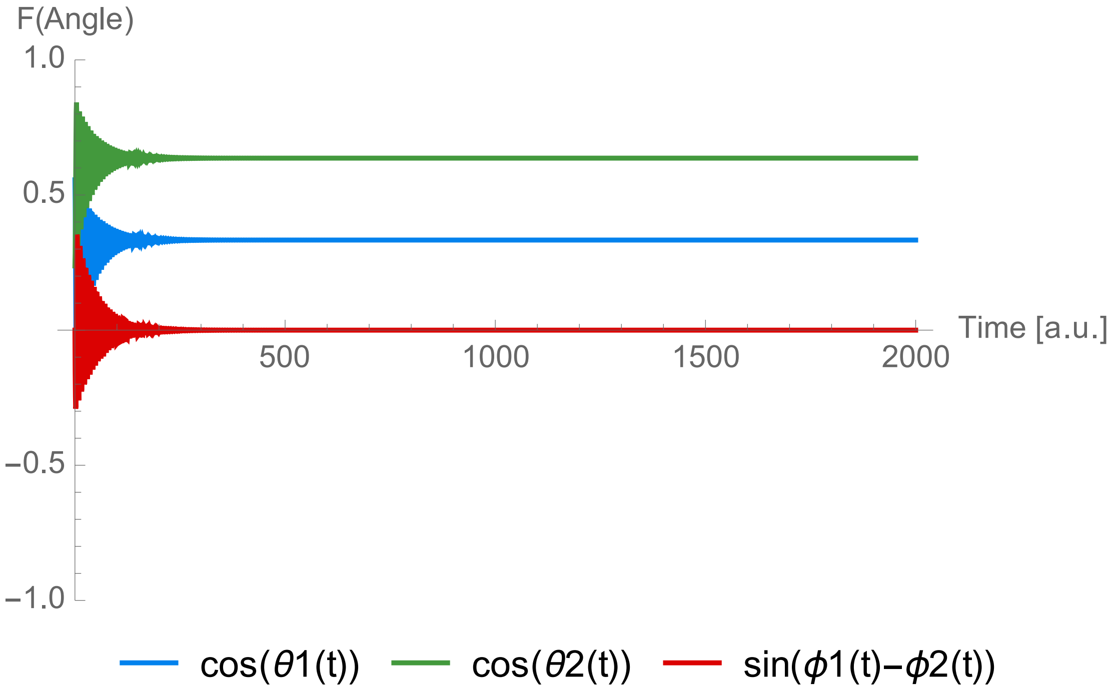

Figure 5.6: Time evolution of a damped mode, when a runaway mode is not present. F is a function, which is either cosine or sine. The polar angles oscillate around the ground state, they seek towards the ground state. The ϕ difference oscillates around zero and then converges towards an angle different than π, when the polar angles is close to the ground state. Values used are $J = 1.0 \quad B = 1.0 \quad \gamma = 0.01 \quad \beta = 0.2 \quad \sigma = 0.1$

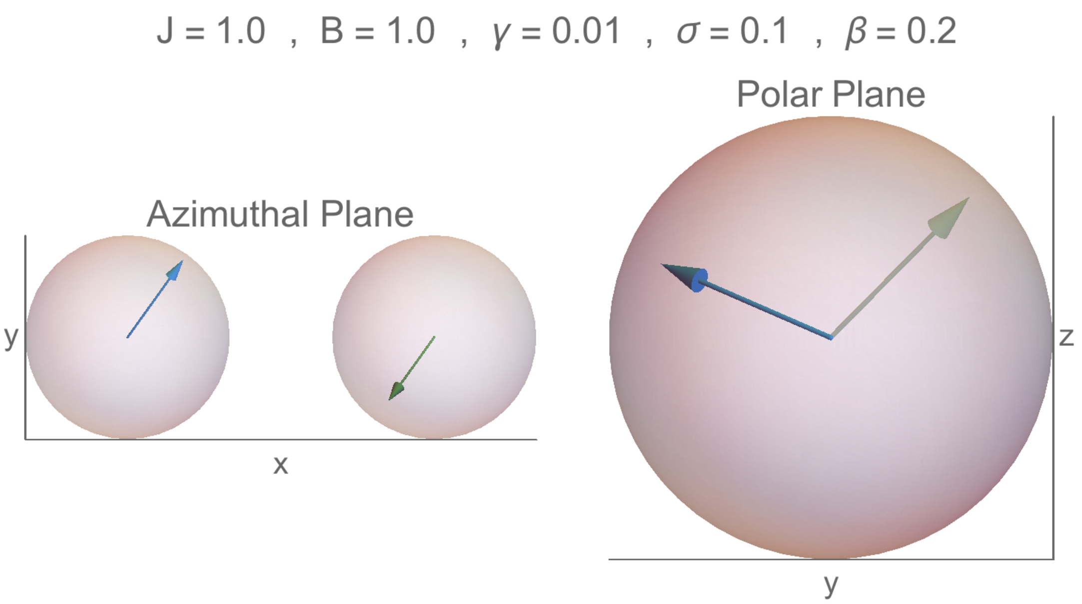

Figure 5.7: A real space representation of the spins, when the system is in the ground state. To the left is shown the azimuthal plane and the polar plane is shown to the right. The coordinates of the spins are taken from figure **5.6** at time equal to 1500. The spins have a difference in the azimuthal angle of π and have an almost similar angle in the polar plane, but shifted, because of σ.

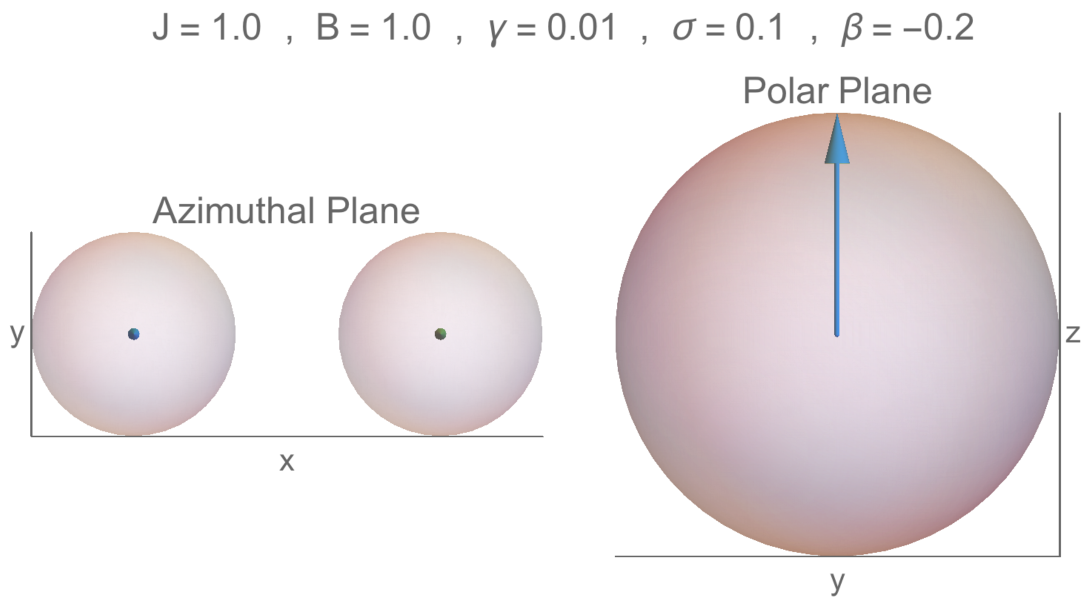

Figure 5.8: A real space representation of the spins, when a runaway mode is present. To the left is shown the azimuthal plane and the polar plane is shown to the right. The coordinates of the spins are taken from figure **5.4** at time equal to 1500. The spins are pointing towards the North Pole, along the magnetic field direction.

The spins want to align with each other at the North Pole, when a runaway mode is present. This is very interesting, since the spins couple antiferromagnetically. The antiferromagnetic nature should also ensure that the spins would like an azimuthal phase difference of π, but it is seen that they prefer a different angle than that.
It is clear that further investigation of the North Pole is needed, in order to understand why the spins want to align with each other and the magnetic field. Also the feature of the azimuthal difference will need some attention.

5.1.3 B and J effect

Before investigating the North Pole, the system behaviour, when changing B and J, needs to be examined.
Earlier it was shown that changing B and J, meant that the ground state angle was different, and a runaway mode could be obtained by almost any value σ and β.
A numerical simulation will show how the system behaves for these other values of B and J. In the following simulations the perturbations away from the ground state has been done with the following values.

$$\delta\theta_1 = 0.1 \qquad \delta\theta_2 = 0.1 \qquad \delta\phi_1 = 0.1 \qquad \delta\phi_2 = 0.1 \tag{5.6}$$

In figure **5.9** is shown the time evolution for the case, where the values used are.

$$J = 0.55 \qquad B = 1.0 \qquad \gamma = 0.01 \qquad \beta = 0.0 \qquad \sigma = 0.1 \tag{5.7}$$

This shows that the polar ground state angle is close to alignment with the magnetic field. They align close to the North Pole, because the Zeeman energy is greater than the exchange energy. After the angles have been pushed, they decay back towards the ground state. The difference in the azimuthal plane oscillates at first, but as the polar angles converge towards the ground state, the difference itself converges towards π.

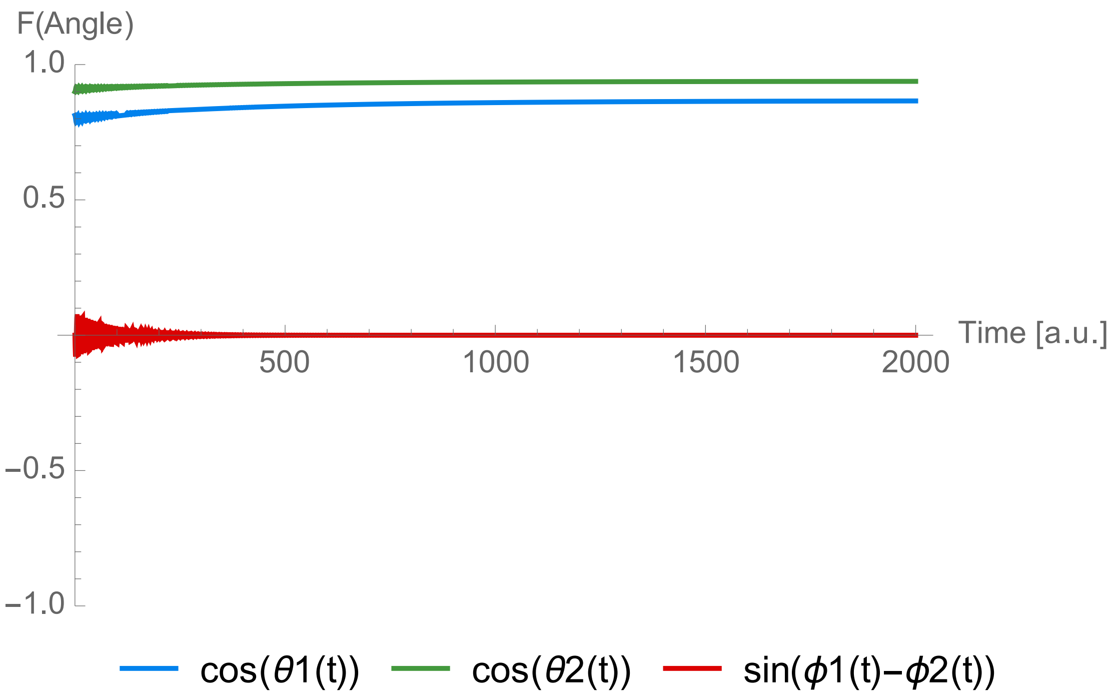

Figure 5.9: A plot of the time evolution of a damped mode for the case, where $B > J$. F is a function, which is either cosine or sine. The polar angles are pushed away from their ground state, but seek back towards it. The azimuthal difference between the two spins converge towards π, as the polar angles get close to their goundstate.

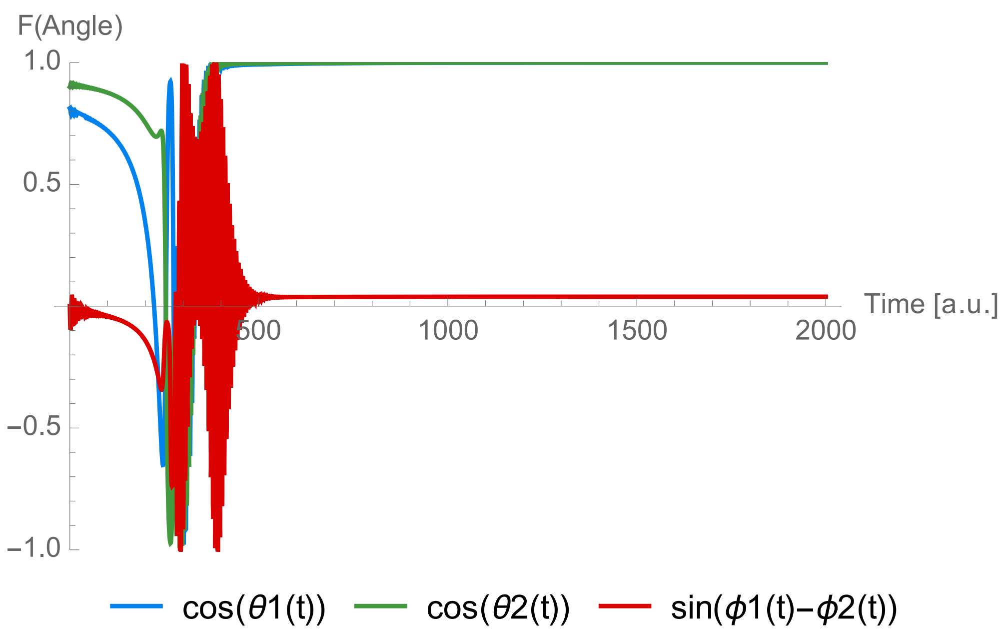

Figure 5.10: A plot of the time evolution of a runaway mode for the case, where $B > J$. F is a function, which is either cosine or sine. The polar angles are pushed away from their ground state, and seek towards the North Pole. The azimuthal difference between the two spins converge towards an angle different than π, as the polar angles get close to the North Pole.

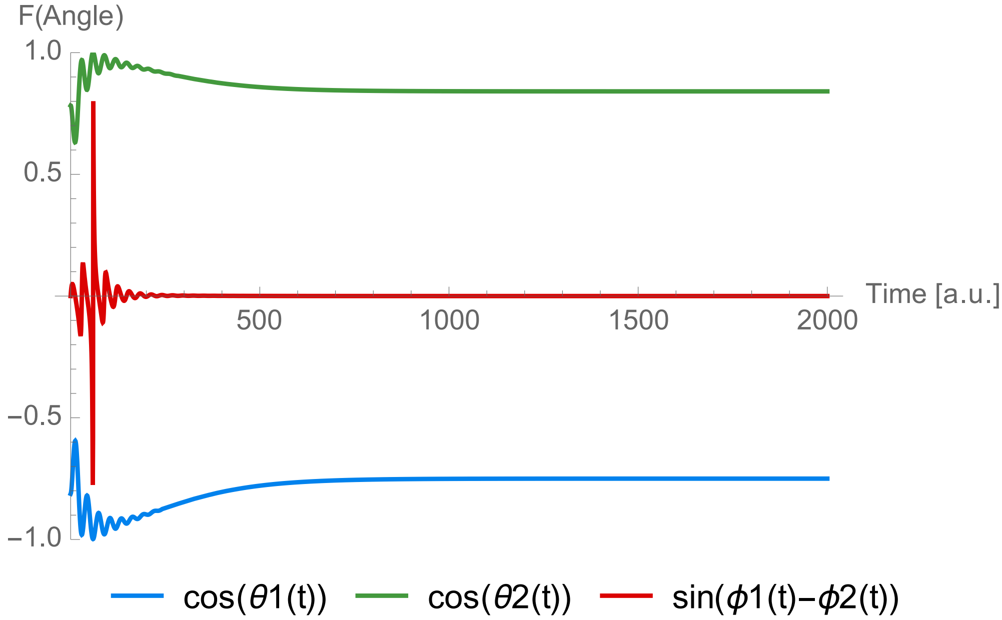

Figure 5.11: A plot of the time evolution of a damped mode for the case, where $B < J$. F is a function, which is either cosine or sine. The polar angles are pushed away from their ground state, but seek back towards it. The azimuthal difference between the two spins converge towards π, as the polar angles get close to their goundstate.

In figure **5.10** is shown the time evolution for the case, where the values used are.

$$J = 0.55 \qquad B = 1.0 \qquad \gamma = 0.01 \qquad \beta = -0.2 \qquad \sigma = 0.1 \tag{5.8}$$

The polar ground state is again close to the magnetic field direction. When time goes, the polar angles get chaotic, before they both converge towards the polar angle $\theta = 0$, pointing towards the North Pole. The difference between the azimuthal angles goes through a chaotic period, before it converges. The difference converges towards an angle difference $\phi_1 - \phi_2 \sim \pi \pm 0.04$ radians.
In figure **5.11** is shown the time evolution for the case, where the values used are

$$J = 1.0 \qquad B = 0.25 \qquad \gamma = 0.01 \qquad \beta = 0.0 \qquad \sigma = 0.1 \tag{5.9}$$

The polar ground state angles are aligned almost antiferromagnetically. The polar angles align like this, because the exchange energy is greater than the Zeeman energy. The polar angles decay towards the ground state, while the difference in the azimuthal plane converges towards π.

In figure **5.12** is shown the time evolution for the case, where the values used are.

$$J = 1.0 \qquad B = 0.25 \qquad \gamma = 0.01 \qquad \beta = -0.2 \qquad \sigma = 0.1 \tag{5.10}$$

It is difficult to see the exact behaviour of the system from time equal to zero and until time ~300. The important part is that the polar angles both converges towards the North Pole after som chaotic period. The azimuthal difference is also chaotic at first, but as the polar angles align with the magnetic field, the difference in the azimuthal plane converges towards and angle difference $\phi_1 - \phi_2 \sim \pi \pm 0.05$ radians.

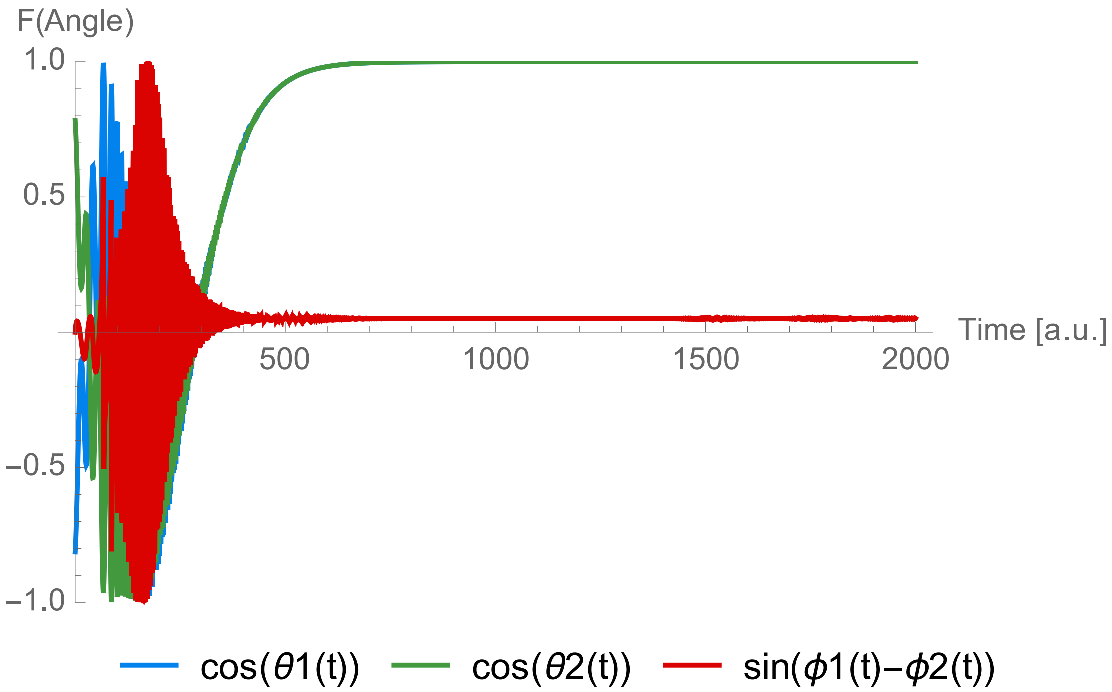

Figure 5.12: A plot of the time evolution of a runaway mode for the case, where $B < J$. F is a function, which is either cosine or sine. The polar angles are pushed away from their ground state. The behaviour is difficult to see, but both spins seek towards the North Pole. The azimuthal difference between the two spins start out chaotic, but converge towards an angle different than π, as the polar angles get close to the North Pole.

To conclude, the alteration of either B or J, only changes the physics of the system in terms of the ground state. The physics of the damped and runaway situation is not affected by it.
Now it is time to investigate the North Pole.

5.2 Behaviour around the pole

In the previous section it was shown that, the system out of equilibrium will end up pointing towards the North Pole, when a runaway mode is present. The next thing to determine is therefore, whether or not the North Pole in that case is a stable point.
This is done by expanding around the North Pole to see how the equations of motion evolve. If the North Pole is a stable point, the spins will seek back towards the pole, after having been perturbed away from it.
From the numerical solution it was found that, the spins tend to align with magnetic field, and therefore it is expected that the North Pole is in fact a stable point in the expansion around the pole.

5.2.1 Expansion around the North Pole

Earlier the expansion around the ground state was made, in order to find the dynamics close to it. To find the dynamics around the North Pole, the same procedure of expanding close to the pole is used.
In order to show what each term looked like in the ground state expansion, the constituent terns were not mixed in the earlier calculations. To make the expansion clearer, equations **3.42**, **3.43**, **3.44** and **3.45** are rewriten.

$$\begin{aligned}\dot{\theta}_1 = &-\sin(\theta_2)\left[\cos(\phi_1-\phi_2)\left[\beta B+\cos(\theta_1)\left(\sigma(\beta+\gamma)+J(\gamma-\beta)\right)\right]+(J+\sigma)\sin(\phi_1-\phi_2)\right]\\ &+\sin(\theta_1)\left[\cos(\theta_2)\left(\sigma(\beta+\gamma)+J(\gamma-\beta)\right)-B\gamma\right]\end{aligned} \tag{5.11}$$

$$\begin{aligned}\dot{\theta}_2 = &\quad\sin(\theta_1)\left[\cos(\phi_1-\phi_2)\left[\beta B-\cos(\theta_2)\left(\sigma(\beta-\gamma)+J(\beta+\gamma)\right)\right]+(J-\sigma)\sin(\phi_1-\phi_2)\right]\\ &+\sin(\theta_2)\left[\cos(\theta_1)\left(\sigma(\beta-\gamma)+J(\beta+\gamma)\right)-B\gamma\right]\end{aligned} \tag{5.12}$$

$$\begin{aligned}\dot{\phi}_1 = &-\sin(\theta_2)\left[\sin(\phi_1-\phi_2)\left[-\sigma(\beta+\gamma)-\beta B\cos(\theta_1)+J(\beta-\gamma)\right]+\cos(\theta_1)(J+\sigma)\cos(\phi_1-\phi_2)\right]\\ &+\sin(\theta_1)\left[-B+\cos(\theta_2)(J+\sigma)\right]\end{aligned} \tag{5.13}$$

$$\begin{aligned}\dot{\phi}_2 = &-\sin(\theta_1)\left[\sin(\phi_1-\phi_2)\left[\sigma(\beta-\gamma)+\beta B\cos(\theta_2)+J(\beta+\gamma)\right]+\cos(\theta_2)(J-\sigma)\cos(\phi_1-\phi_2)\right]\\ &+\sin(\theta_2)\left[-B-\cos(\theta_1)(\sigma-J)\right]\end{aligned} \tag{5.14}$$

The difference $\phi_1-\phi_2$ is not well defined at the North Pole, therefore the difference in the azimuthal plane is not expanded, but kept as a variable, $\delta\phi$.
The polar angles around the North Pole are, however, expanded, i.e. $\theta_1 = 0+\delta\theta_1$ and $\theta_2 = 0+\delta\theta_2$

$$\begin{aligned}\delta\dot{\theta}_1 = & -\sin(0+\delta\theta_2)\left[\cos(\delta\phi)\left[\beta B + \cos(0+\delta\theta_1)(\sigma(\beta+\gamma) + J(\gamma-\beta))\right] + (J+\sigma)\sin(\delta\phi)\right] \\ & + \sin(0+\delta\theta_1)\left[\cos(0+\delta\theta_2)(\sigma(\beta+\gamma)+J(\gamma-\beta)) - B\gamma\right]\end{aligned} \tag{5.15}$$

$$\begin{aligned}\delta\dot{\theta}_2 = & \quad \sin(0+\delta\theta_1)\left[\cos(\delta\phi)\left[\beta B - \cos(0+\delta\theta_2)(\sigma(\beta-\gamma) + J(\beta+\gamma))\right] + (J-\sigma)\sin(\delta\phi)\right] \\ & + \sin(0+\delta\theta_2)\left[\cos(0+\delta\theta_1)(\sigma(\beta-\gamma)+J(\beta+\gamma)) - B\gamma\right]\end{aligned} \tag{5.16}$$

$$\begin{aligned}\delta\dot{\phi}_1 = & -\sin(0+\delta\theta_2)\left[\sin(\delta\phi)\left[-\sigma(\beta+\gamma) - \beta B\cos(0+\delta\theta_1) + J(\beta-\gamma)\right]\right. \\ & \left. + \cos(0+\delta\theta_1)(J+\sigma)\cos(\delta\phi)\right] \\ & + \sin(0+\delta\theta_1)\left[-B + \cos(0+\delta\theta_2)(J+\sigma)\right]\end{aligned} \tag{5.17}$$

$$\begin{aligned}\delta\dot{\phi}_2 = & -\sin(0+\delta\theta_1)\left[\sin(\delta\phi)\left[\sigma(\beta-\gamma) + \beta B\cos(0+\delta\theta_2) + J(\beta+\gamma)\right]\right. \\ & \left. + \cos(0+\delta\theta_2)(J-\sigma)\cos(\delta\phi)\ \right] \\ & + \sin(0+\delta\theta_2)\left[-B - \cos(0+\delta\theta_1)(\sigma - J)\right]\end{aligned} \tag{5.18}$$

The following equations are obtained after the expansion, noting that since the North Pole is a static point, the zero order terms cancels out.

$$\begin{aligned}\delta\dot{\theta}_1 = & -\delta\theta_2\left[\cos(\delta\phi)(\sigma(\beta+\gamma) + \beta B + J(\gamma-\beta)) + \sin(\delta\phi)(J+\sigma)\right] \\ & -\delta\theta_1\left[-\sigma(\beta+\gamma) + B\gamma + J(\beta-\gamma)\right]\end{aligned} \tag{5.19}$$

$$\begin{aligned}\delta\dot{\theta}_2 = & \quad \delta\theta_1\left[\cos(\delta\phi)(\sigma(\gamma-\beta) + \beta B - J(\beta+\gamma)) + \sin(\delta\phi)(J-\sigma)\right] \\ & +\delta\theta_2\left[\sigma(\beta-\gamma) - B\gamma + J(\beta+\gamma)\right]\end{aligned} \tag{5.20}$$

$$\begin{aligned}\delta\dot{\phi}_1 = & \quad \delta\theta_2\left[\sin(\delta\phi)\left[\sigma(\beta+\gamma) + \beta B + J(\gamma-\beta)\right] + \cos(\delta\phi)(-J-\sigma)\right] \\ & +\delta\theta_1(-B + J + \sigma)\end{aligned} \tag{5.21}$$

$$\begin{aligned}\delta\dot{\phi}_2 = & -\delta\theta_1\left[\sin(\delta\phi)\left[-\sigma(\gamma-\beta) + \beta(-B) + J(\beta+\gamma)\right] + \cos(\delta\phi)(J-\sigma)\right] \\ & -\delta\theta_2(B - J + \sigma)\end{aligned} \tag{5.22}$$

From these equations of motion it is possible to generate a matrix desribing the motion of the spins around the north pole.

$$\begin{pmatrix} -(J(\beta-\gamma) + B\gamma - (\beta+\gamma)\sigma) & -((B\beta + J(\gamma-\beta) + (\beta+\gamma)\sigma)\cos(\delta\phi) + (J+\sigma)\sin(\delta\phi)) & 0 & 0 \\ (B\beta - J(\beta+\gamma) + (\gamma-\beta)\sigma)\cos(\delta\phi) + (J-\sigma)\sin(\delta\phi) & -B\gamma + J(\beta+\gamma) + (\beta-\gamma)\sigma & 0 & 0 \\ -B + J + \sigma & (-J-\sigma)\cos(\delta\phi) + (B\beta + J(\gamma-\beta) + (\beta+\gamma)\sigma)\sin(\delta\phi) & 0 & 0 \\ -((J-\sigma)\cos(\delta\phi) + (-B\beta + J(\beta+\gamma) - (\gamma-\beta)\sigma)\sin(\delta\phi)) & -(B - J + \sigma) & 0 & 0 \end{pmatrix} \tag{5.23}$$

The eignevalues for this matrix are.

$$\begin{aligned}E_1 = & \beta\sigma - B\gamma + \gamma J - \frac{1}{\sqrt{2}}\{\beta^2\sigma^2 - \beta^2B^2 - 2\beta B\gamma\sigma + 2\beta^2BJ + \gamma^2\sigma^2 + \beta^2J^2 + \gamma^2J^2 - J^2 + \sigma^2 \\ & + \cos(2\delta\phi)\left[\sigma^2\left(\beta^2-\gamma^2-1\right) - \beta^2B^2 - 2\beta B\gamma\sigma + 2\beta J(\beta B + 2\gamma\sigma) + J^2\left(-\beta^2+\gamma^2+1\right)\right] \\ & - 2\beta\sin(2\delta\phi)\left(BJ - J^2 - \sigma^2\right)\}^{1/2}\end{aligned} \tag{5.24}$$

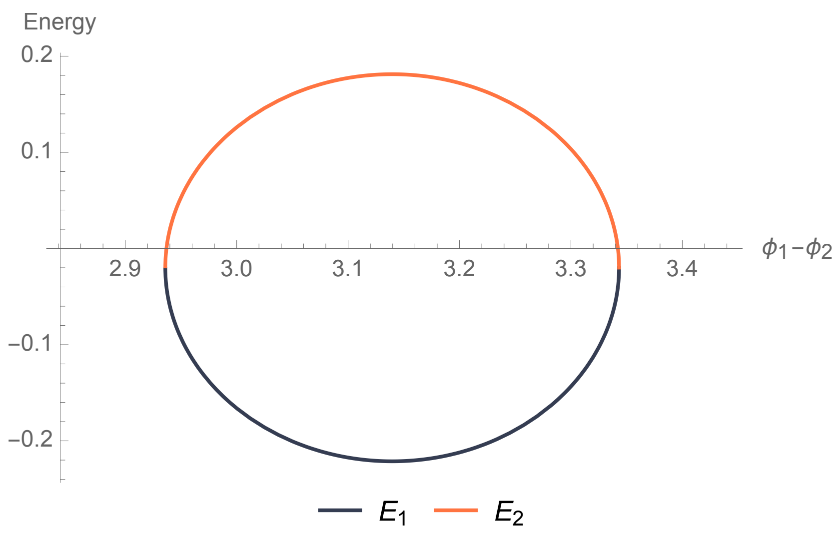

Figure 5.13: The eigenfrequencies as a function of difference in the azimuthal plane between the two spins with $\beta = -0.2$. The eigenfrequencies are obtained from the matrix describing the dynamics close to the North Pole. The eigenfrequencies are simultaneous negative for an angle difference of $\phi_1 - \phi_2 \sim \pi \pm 0.2$ radians

$$E_2 = \beta\sigma - B\gamma + \gamma J + \frac{1}{\sqrt{2}}\{\beta^2\sigma^2 - \beta^2B^2 - 2\beta B\gamma\sigma + 2\beta^2 BJ + \gamma^2\sigma^2 + \beta^2J^2 + \gamma^2J^2 - J^2 + \sigma^2 + \cos(2\delta\phi)\left[\sigma^2\left(\beta^2 - \gamma^2 - 1\right) - \beta^2B^2 - 2\beta B\gamma\sigma + 2\beta J(\beta B + 2\gamma\sigma) + J^2\left(-\beta^2 + \gamma^2 + 1\right)\right] - 2\beta\sin(2\delta\phi)\left(BJ - J^2 - \sigma^2\right)\}^{1/2} \tag{5.25}$$

$$E_3 = 0 \quad , \quad E_4 = 0 \tag{5.26}$$

In order for the North Pole to be a stable point, the system must go back to the starting point, the North Pole. For this to happen, both eigenfrequencies must be negative at the same time.
In figure **5.13** is shown the eigenfrequencies as a function of the azimuthal difference for the values $J = 1.0 \quad B = 1.0 \quad \gamma = 0.1 \quad \beta = -0.2 \quad \sigma = 0.1$. It is seen that there exist at least one angle, which allows for both eigenfrequencies to be negative simultaneously.

The two points in the azimuthal difference, where the eigenfrequencies become negative, corresponds to the same angle between the spins, as illustrated in figure **5.14**. As is seen in the figure, it depends on which way the angle is measured.

5.2.2 Numerical solution around North pole

To see how the system behaves around the North Pole, the four coupled equations of motion, equations **3.42**, **3.43**, **3.44** and **3.45**, needs to be solved numerically with a small pertubation away from the North pole.

In figure **5.15** the eigenfrequencies are shown as a function of $\delta\phi$ for the values

$$J = 1.0 \qquad B = 1.0 \qquad \gamma = 0.1 \qquad \beta = 0.2 \qquad \sigma = 0.1 \tag{5.27}$$

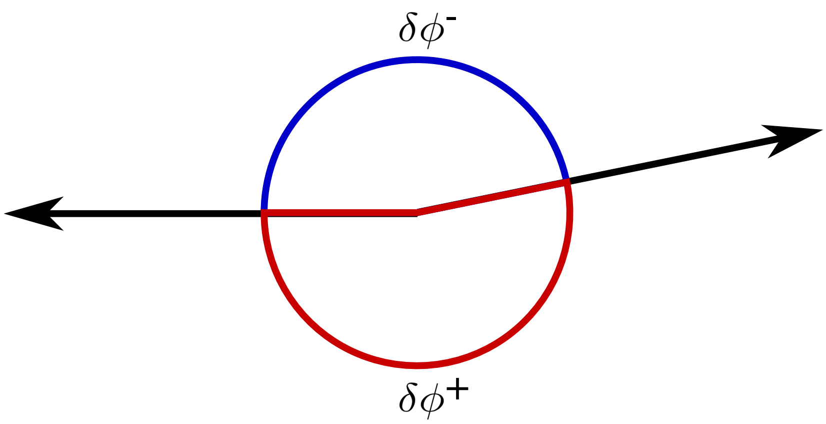

Figure 5.14: A look at the azimuthal plane for the two spins. The two angles correspond to the angles where the eigenfrequencies are simulaneous negative. The ϕ^+ (red) corresponds to the $\pi + 0.2$ angle, whereas the ϕ^- (blue) correspond to the $\pi - 0.2$ angle.

There are no simultaneous negative eigenfrequencies and therefore, it indicates that the North Pole is not a stable point. In figure **5.16** is shown the time evolution for two spins, when they start at the North Pole with a small push and the values defined above.
The plot shows how the cosine to the polar angles evolve, and how the sine to the difference in the azimuthal plane between the two spins evolve. At time equal to zero and the polar angles close to zero, the sine to the difference is not well defined and it is unclear, how it behaves. As the polar angles converges towards the ground state, the azimuthal difference converges towards an angle π. For these values, the North Pole is not a stable point.

In figure **5.13** is shown the eigenfrequencies as a function of $\delta\phi$ for the values

$$J = 1.0 \qquad B = 1.0 \qquad \gamma = 0.1 \qquad \beta = -0.2 \qquad \sigma = 0.1 \tag{5.28}$$

There exists simultaneous negative eigenfrequencies, which indicates that the North Pole is a stable point. In figure **5.17** is shown the time evolution of the spins for the values defined above.
At the time equal to zero it is seen that the sine to the difference in azimuthal plane is chaotic, but as time goes the difference converges towards a specific angle, even though the azimuthal difference is ill defined at the pole. The angle, which the difference converges towards, is $\sim \pi \pm 0.2$ radians.

The eigenfrequencies as a function of $\delta\phi$ is shown in figure **5.18** for the values

$$J = 1.0 \qquad B = 1.0 \qquad \gamma = 0.1 \qquad \beta = -0.4 \qquad \sigma = 0.1 \tag{5.29}$$

There exists simultaneous negative eigenfrequencies, which indicates that the North Pole is a stable point. In figure **5.19** the time evolution of the spins for the values defined above is shown .
At time equal to zero, when the polar angles are pushed slightly away from the North Pole, the azimuthal angle difference is chaotic. As the polar angles converge towards the North Pole, the azimuthal difference converges towards and angle $\sim \pi \pm 0.4$ radians.
The angle difference in figure **5.18** is greater than the angle difference in figure **5.13**. The azimuthal angle difference is also greater in figure **5.19** compared to that in figure **5.17**.

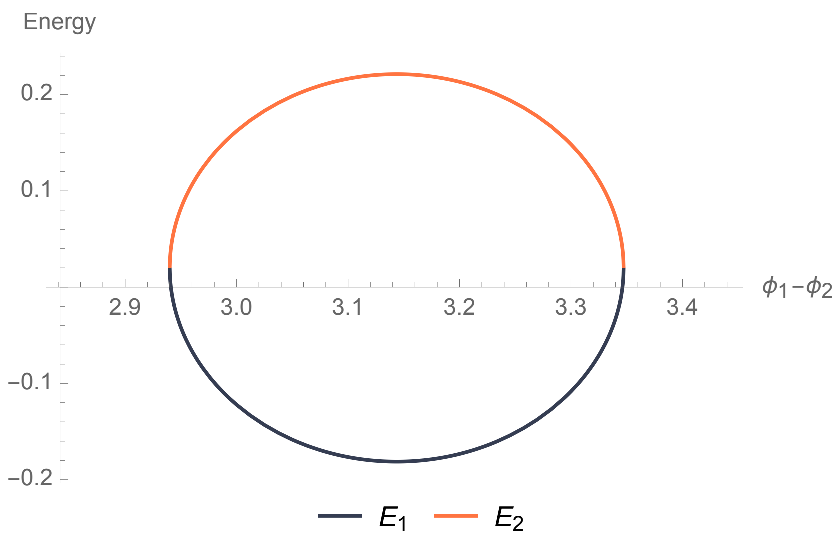

Figure 5.15: The eigenfrequencies as a function of difference in the azimuthal plane between the two spins with $\beta = 0.2$. The eigenfrequencies are obtained from the matrix describing the dynamics close to the North Pole. The eigenfrequencies are not simultaneous negative for any angle difference of $\phi_1 - \phi_2$.

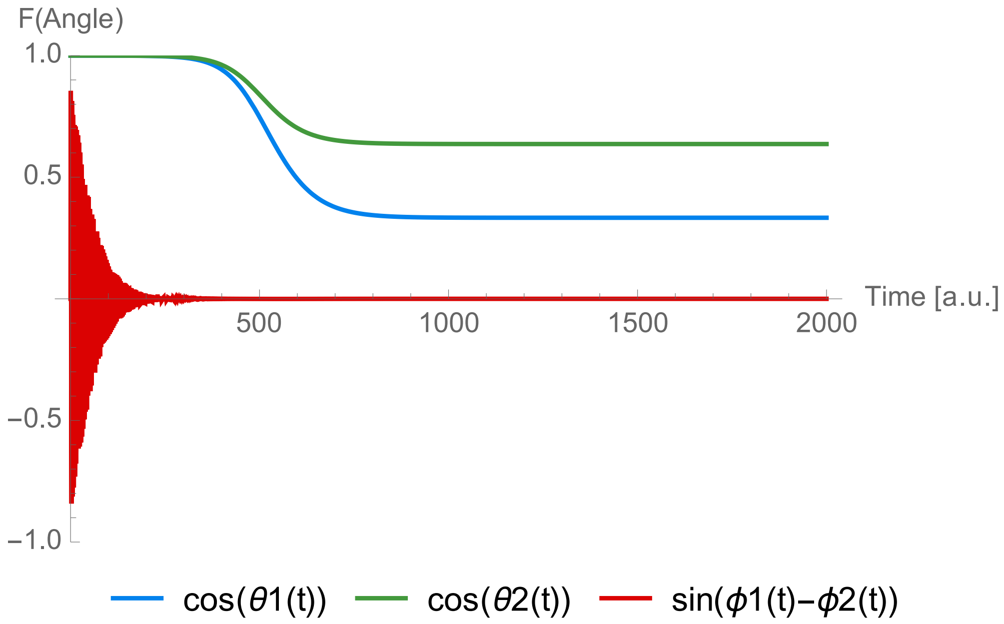

Figure 5.16: The time evolution near the North Pole with values $\beta = 0.2$ and $\sigma = 0.1$. F is a function, which is either cosine or sine. The spins start out at the North Pole and the azimuthal difference between the spins is chaotic. The polar angles for both spins relax toward the original ground state. As the polar angles approach the ground state, the difference between the ϕ angles converge towards π.

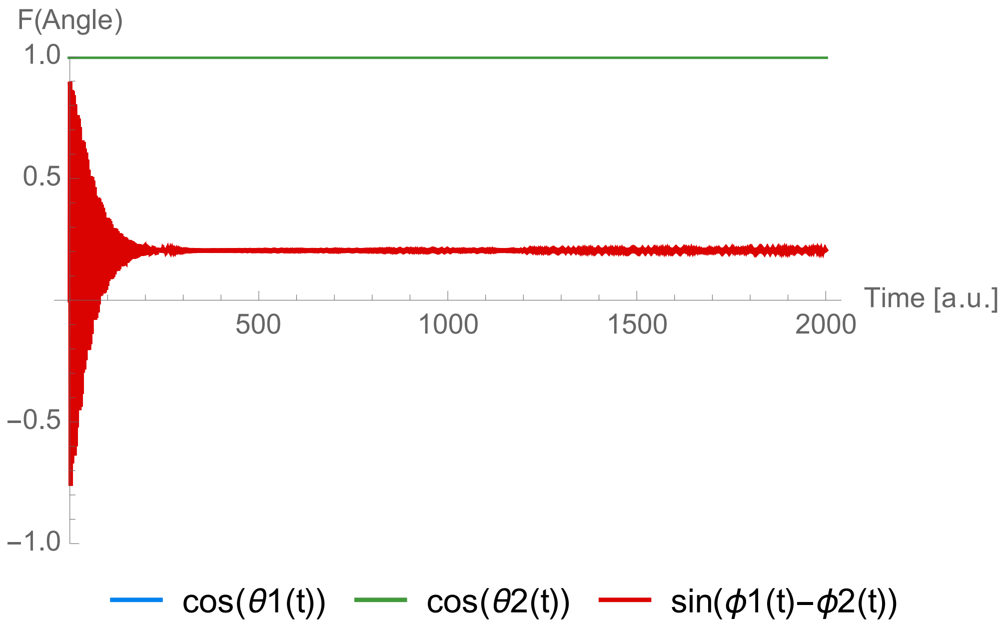

Figure 5.17: The time evolution near the North Pole with values $\beta = -0.2$ and $\sigma = 0.1$. F is a function, which is either cosine or sine. The spins start out at the North Pole and the azimuthal difference between the spins is chaotic. The polar angles for both spins seek back towards the pole. As the polar angles approach a stable situation at the pole, the ϕ difference between the spins converge towards an angle of $\phi_1 - \phi_2 \sim \pi \pm 0.2$ radians.

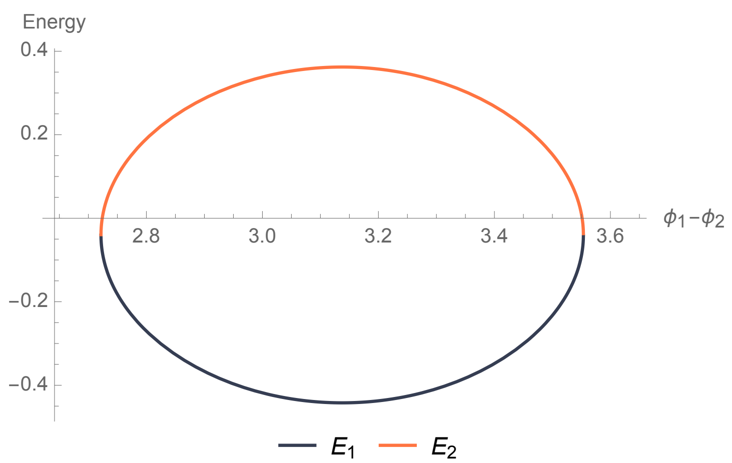

Figure 5.18: The eigenfrequencies as a function of difference in the azimuthal plane between the two spins with $\beta = -0.4$. The eigenfrequencies are obtained from the matrix describing the dynamics close to the North Pole. The eigenfrequencies are simultaneous negative for an angle difference of $\phi_1 - \phi_2 \sim \pi \pm 0.4$ radians.

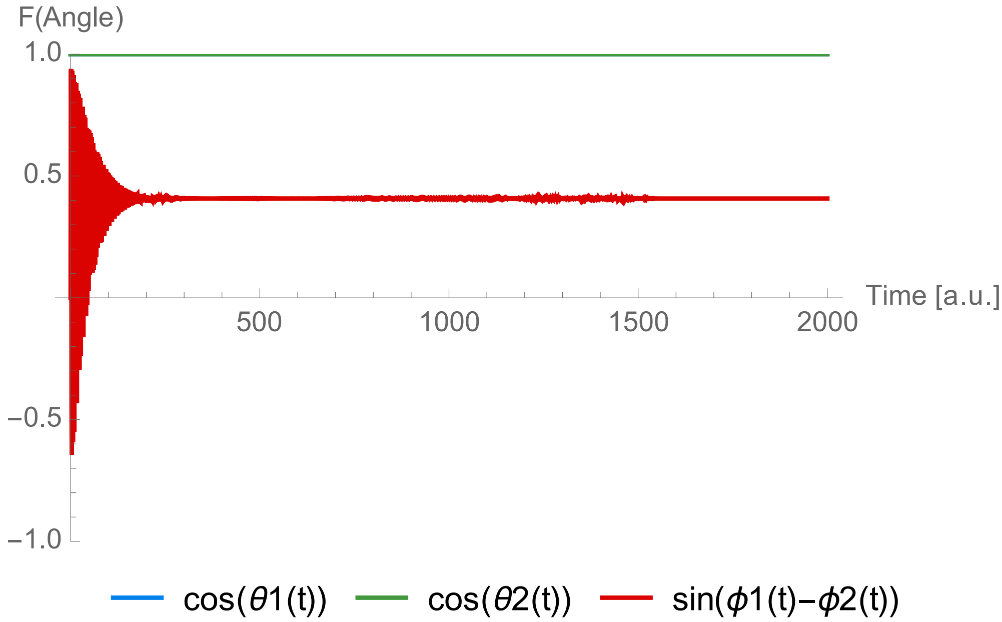

Figure 5.19: The time evolution near the North Pole with values $\beta = -0.4$ and $\sigma = 0.1$. F is a function, which is either cosine or sine. The spins start out at the North Pole and the azimuthal difference between the spins is chaotic. The polar angles for both spins seek back towards the pole. As the polar angles approach a stable situation at the pole, the ϕ difference between the spins converge towards an angle of $\phi_1 - \phi_2 \sim \pi \pm 0.4$ radians.

5.2.3 Azimuthal contourplot

It is now clear that the value of β does in fact determine the azimuthal difference between the two spins, and that the North Pole is stable, only when there is an azimuthal angle different than π between the spins.
Figure **5.20** shows a contour plot of the energy found from the matrix in equation **5.23** as a function of β and the azimuthal difference in $\delta\phi$, for the values

$$J = 1.0 \qquad B = 1.0 \qquad \gamma = 0.01 \qquad \sigma = 0.1 \tag{5.30}$$

It shows the maximum of the eigenfrequencies, which means that when it is negative both eigenfrequencies are negative. For $\beta > 0$, there exist no angle, which makes the eigenfrequencies negative, thereby making the North Pole a stable point.
For $\beta < 0$ there are two dark blue regions, where the eigenfrequencies are negative and as such the North Pole is stable here. As the magnitude of β gets bigger, it is observed that there must be a smaller difference between the spins in the azimuthal plane.
That is exactly, what is observed from the eigenfrequencies and the numerics for the dynamics around the North Pole.

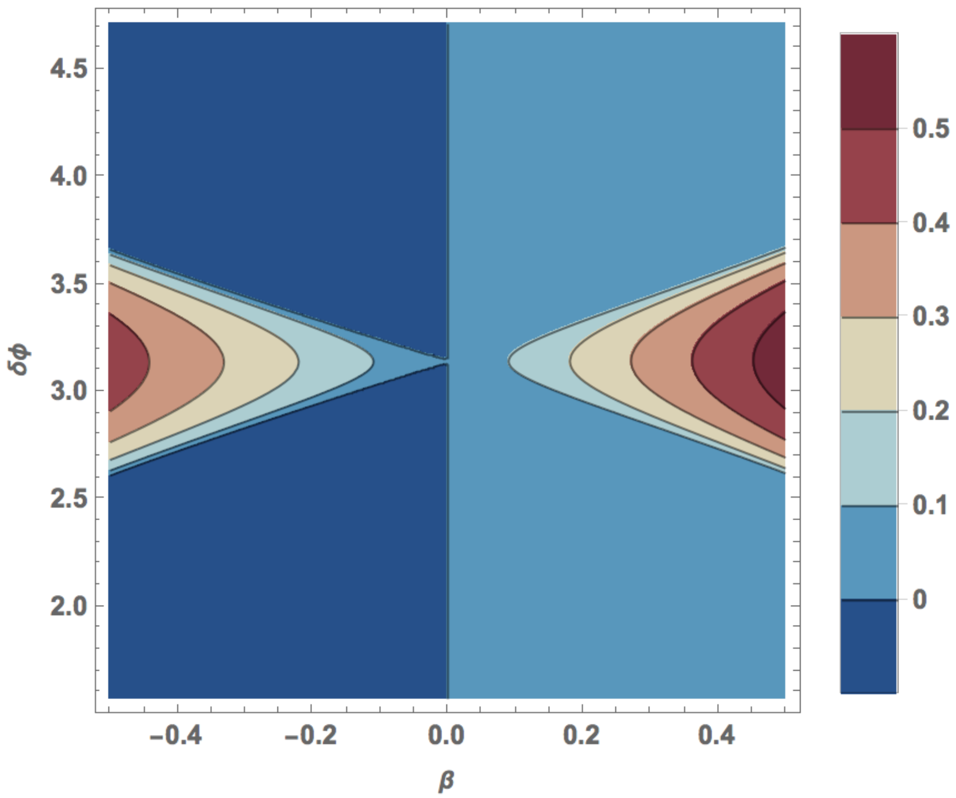

Figure 5.20: A contourplot for β and difference in th azimuthal plane $\delta\phi$ with σ=0.1. The contourplot shows, for which values of β and the azimuthal angle difference $\delta\phi$ there exists two simultaneous negative eigenfrequencies. It is seen that, when β gets bigger, the angle between the two spins in the azimuthal plane must be smaller, in order to obtain the negative eigenfrequencies.

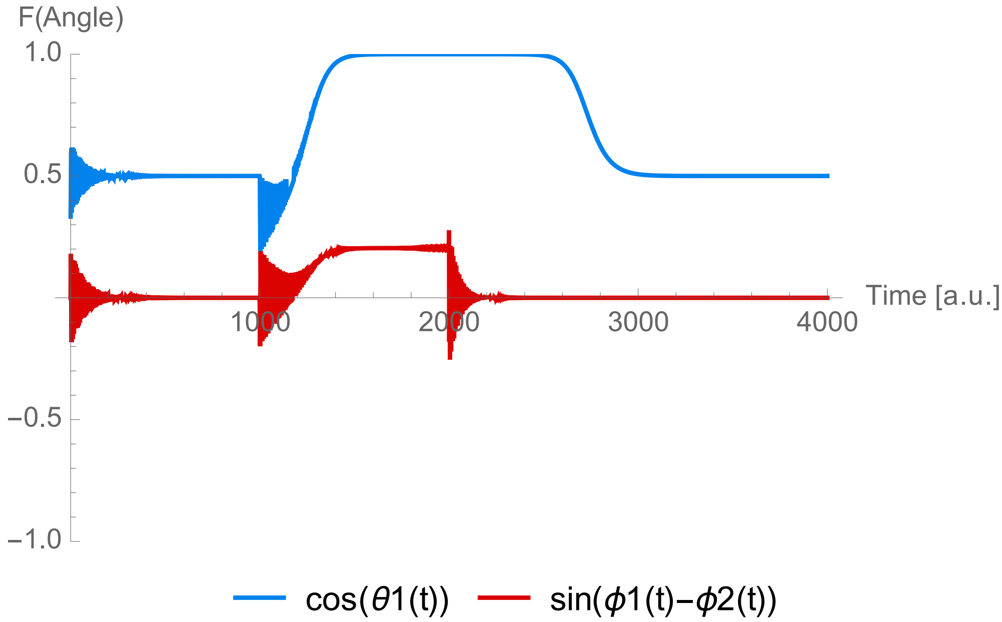

Figure 5.21: The total time evolution. F is a function, which is either cosine or sine. The system starts without current and stabilizes around the ground state. The current is turned on at time equal to 1000, resulting in the spin aligning along the magnetic field at the North Pole, while the azimuthal difference converge towards an angle different than π. The current is turned off again at time equal to 2000, and the spin decays back to the original ground state.

5.3 Current

It has been shown how the β term can affect the dynamics of the spin system. By tuning the β term it was possible to make the North Pole a stable point for both spins.
Now a simulation of the system starting out near the ground state without current is considered, i.e. $\beta(t = 0) = 0$ and $\sigma(t = 0) = 0$, and turning the current on at a later point. The evolution is shown in figure **5.21**.

When there is no current flowing through the system, the system wants to decay towards the ground state, with the spins having equal polar angles and an azimuthal difference of π. This is what is seen in the interval $0 > t > 1000$.
Now the current is turned on and the system becomes non-equilibrium in the interval $1000 > t > 2000$. The spins seek towards the North Pole and align with the magnetic field. As the polar angles become close to 0, the difference in the azimuthal plane approaches an angle shifted away from π.
Lastly the current is turned off again in the interval $2000 > t > 4000$. The spins relax back to the ground state and the original azimuthal difference is restored for the system.

This final simulation is interesting, because it shows, how it is possible to manipulate two anitferromagnetically coupled spins to align along the magnetic field, when a suitable current is passed through the system.

Chapter 6

Conclusion

In this thesis the dynamics of a system consisting of two antiferromagnetically coupled spins, in the presence of curl forces, have been investigated.

By an expansion around the ground state of the system, it was possible to generate a matrix, which describes the dynamics of the system close to the ground state. Using the eigenfrequencies for the matrix, a contourplot was made, in order to determine the possibility that a runaway mode exists.
It was found that, in the regime, where current induced curl forces is present, there exists at least one runaway mode.

A numerical analysis was made, in order to find the actual dynamics of the spins. The dynamics was found both with and without a runaway mode present. Without the runaway mode, the system always decays back to the stable and static ground state, with an azimuthal angle between the spins of π. When a runaway mode is present, the system never decays back to the ground state. The runaway mode is responsible for the system seeking towards a polar angle equal to zero along the magnetic field. Alongside the alignment of the spins with the magnetic field, the azimuthal angle between the spins also gets smaller. Even though the difference in the azimuthal angle is not well defined close to the polar angle equal to zero, an expansion around that point showed that, the squeezing in the azimuthal plane stabilized the North Pole.

It has been shown that by applying an electrical current, a system consisting of spins can be aligned along a magnetic field. The emergent curl forces are critical, in order to control the physical system electrically.

Further studies should try and adress the behavoiur of the azimuthal angles, when the spins align with magnetic field, because even though the azimuthal difference $\delta\phi$ is ill defined at polar angles equal to zero, it is critical to have a squeezing in the angle between the spins, in order for the North Pole to be a stable point.

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