Quantum teleportation between oscillators interacting with light in cascades

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This thesis has been submitted for
the degree of Cand. Scient. in Physics
May 22nd 2023

Abstract

This thesis examines quantum teleportation between two quantum harmonic oscillators, call them system 1 and 2, coupled to a traveling light field in a cascaded setup from system 1 to 2 and measured using a homodyne measuring scheme. The interaction between the light and systems is described by a combination of beam-splitter and two-mode-squeezing interactions. The teleportation will transfer the state from system 2 to 1, using the measured homodyne signal to do a feedback on system 1. For purposes of characterizing and optimizing the protocol The initial state of system 2 is taken to be a coherent state drawn from a Gaussian distribution with zero mean and finite width, and the initial state of system 1 is assumed to be a thermal state. The purpose of this thesis is to investigate how well this can be implemented in a realistic setup in which noise and errors are present. This includes thermal noise from the oscillators interacting with the environment and vacuum noise arising from optical losses. This will be analyzed using the Heisenberg-Langevin formalism to account for the quantum noise processes. To characterize how well the teleportation protocol works, the concept of fidelity is used. An analytical expression for the fidelity is found in terms of the feedback gain envelope and the drive pulses driving the systems. We then want to maximize this expression in terms of the quantities mentioned, however this is not possible analytically, so numerical methods are used instead. This will be done by making an ansatz for the shape for the drive pulses and then the optimal feedback gain will solved for numerically for a given set of experimental parameters from the QUANTOP lab at NBI. We predict a fidelity of 0.79, which is above the corresponding classical fidelity of 0.56.

Acknowledgments

I wish to thank Emil, who is an great advisor and has helped me through, not just this project but also my bachelor's and first-year project. He is a talented physicist, who is always ready to talk and help his students. I want to give a thanks to my friends: Nikolaj, with whom I have spent the last 5 years discussing about physics and Star Wars. Unfortunately I also had to suffer through his bad puns. Jesper (The LaTeX-Wizard) who is always ready to help with any latex related questions, and Mikkel, who always some exciting board game ready. Finally, I would like to thank my little brother, Mads, and my parents, Preben and Anita, for always believing in me and supporting me however they can.

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Chapter 1

Introduction

> “ Do you just put the word "quantum" in front of everything?
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> Scott lang
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> - *Ant-Man and the Wasp,* Kevin Feige & Stephen Broussard

In recent years quantum technologies e.g. quantum- computers, communication and networks, have become a focus for many research groups and technology companies. A key ingredient in many of these new technologies is the ability to transfer quantum states without letting them decohere and thereby destroying their quantum properties. This can be done by sending the state through causal quantum channel, for example, sending a qubit encoded in a photon by using a fiber optics cable, however it can still decohere or even be absorbed. This is why, *quantum teleportation,* which utilizes the non-local properties of quantum entanglement is preferred. The first teleportation protocol was proposed by Bennett *et al.* in 1993 [2], however this paper deals only with a qubit-like state; in the article they use a spin-$\frac{1}{2}$ particle. In 1994 Vaidman [12] proposed a teleportation protocol for continuous variables, such as the amplitude and phase quadrature of light. The first realization of Bennett's protocol was done in 1997 by Zeilinger's group [3]. And the first unconditional quantum teleportation of continuous variables was done in 1998 by Polzik's group [6].

Quantum teleportation is a central elementary operation in quantum networks, not just for sending states between, e.g, quantum computers. For example, if one has made a complicated state to be stored for later use then it could be teleported to a quantum memory system. It can also be used for enabling the entanglement between two distant nodes by using quantum repeaters. These are a series of intermediate subsystems between the nodes where it is easier to generate entanglement between each neighboring repeater. Quantum teleportation can be used to teleport part of an entangled state from one quantum repeater to the next one and thereby entangling the prior one with the next one; this is called entanglement swapping.

Since quantum teleportation is a crucial component in many quantum technologies, there is an interest in developing teleportation protocols which functions even when there is errors and noise presence, as in a realistic setup. This thesis will investigate quantum teleportation between quantum harmonic oscillators coupled to a traveling light field. The light first gets entangled with the receiving system after which it interacts with the input system, the system whose state we want to transfer, and then gets measured with a homodyne detection. This is analyzed in the presence of thermal noise and optical losses. The figure of merit used to quantify how well the teleportation protocol works, is a quantity called the fidelity, defined as the overlap between to the state being teleported and the resulting state of the receiving system. This is a useful measure since it tells you how on average how similarly the resultant state at the receiving system is to the input state. More specifically we are interested in the average fidelity after doing the teleportation protocol many time, where each time the input state is a new random coherent state selected from a Gaussian distribution with zero mean and with width $\bar{n}$. For an arbitrary input state, $\bar{n} \to \infty$, if the fidelity is below $\frac{1}{2}$ then the transfer can be done be equally as well with just local operations and classical communication, no need for entanglement. At above a half the protocol is above the classical threshold and the closer the fidelity is to 1 the better. So the goal is to maximize the fidelity of our teleportation protocol; this will be done though both analytical and numerical means.

The thesis is structured as follows:

- In Chapter 2 we present the basic idea behind quantum teleportation and show how this can be done for both discrete and continuous variables and define the fidelity used as a figure of merit for the protocol.
- Chapter 3 introduces the necessary theoretical framework for the rest of the thesis. This includes: Heisenberg-Langevin formalism, input-output formalism and the equations of motion for the systems.
- In Chapter 4 we derive an expression for the total added noise in terms of the pulses driving the systems and the feedback gain for the measured signal, which will be further treated numerically in the next chapter. This is done in the case where the state to be teleported comes from a restricted family of states and the receiving system is in a known thermal state, also in the presence of optical losses and thermal noise. Finally we consider the ideal case of no losses and noise, and where an arbitrary state is being teleported to an unknown state, which will serve as inspiration in the next chapter.
- In Chapter 5 the expression for the total noise is discretized and the optimal filter is solved for numerically by using ansatz function for the drive pulses, after which the fidelity is calculated.
- In Chapter 6 we conclude on the results obtained in this thesis and future paths are indicated.

Chapter 2

Theory of Quantum Teleportation

In this chapter, we start by giving a qualitative description of what quantum teleportation is and how it works in principle. Next, we will give a more detailed description of how the teleportation protocol can be performed, with both discrete and continuous variables. Then we introduce the concept of fidelity.

2.1 Quantum Teleportation: Discrete vs. Continuous

Quantum teleportation is the process of transferring an unknown quantum state of one system to another, by using entanglement to a third, auxiliary system and Bell measurement on the source and auxiliary systems. It is important that it is a unknown state being transferred or else the state could just be conveyed over a classical channel (e.g. via telephone).The usual process for the teleportation protocol, due to Bennett *et al.* [2], goes as follows (see Fig. 2.1): We have three parties called, "Alice", "Bob" and "Charlie", each possessing a quantum system. Charlie wants to transfer his (unknown) state to Bob with the help of Alice. It starts with Bob and Alice entangling their systems then Alice sends her system to Charlie, so that a joint, Bell measurement can be made on Charlie's and Alice's systems. Then via a classical communication channel Charlie sends the result of this measurement to Bob, which then, based on this information, performs a local operation on his system to finish the transfer of Charlie's initial state to his system. Charlie's and Alice's systems are now in a mixed state. A necessary criterion for a perfect teleportation process (i.e. Bob's state is identical to Charlie's initial state) is, that Alice and Bob need to share a maximally entangled state, which is basically impossible in the real world. Also no information must leaks out, e.g., via losses or no dissipation.

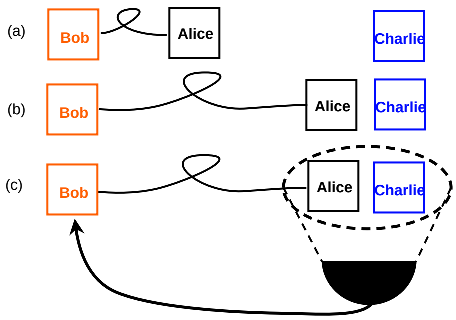

Figure 2.1: Quantum Teleportation protocol. (a) Bob and Alice gets entangled (represented by the curly line). (b) Alice travels to Charlie. (c) a Bell measurement is made on Alice and Charlie, the result is then communicated classically to Bob.

2.1.1 Discrete-variable Teleportation

We will give a mathematical derivation for the discrete case of teleportation (this derivation is again due to Bennett *et al.*[2]) and we will do this in the context of a two-level system with the basis $\{|0\rangle, |1\rangle\}$. We take Charlie's state to be in an arbitrary superposition $|\psi\rangle_C = \alpha |0\rangle_C + \beta |1\rangle_C$, where α and β are unknown coefficients. Alice and Bob then needs to share an entangled state, we will assume that to be a maximally entangled state. For specificity we will assume that they share the Bell state $|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B)$. We then have that the total state is

$$|\psi\rangle_C \otimes |\Phi\rangle_{AB} = (\alpha |0\rangle_C + b |1\rangle_C) \otimes \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B) \tag{2.1}$$

By using the fact that we can rewrite a 2-qubit state as a superposition of Bell states using the relations:

$$|0\rangle |0\rangle = \frac{1}{\sqrt{2}}\left(\left|\Phi^+\right\rangle + |\Phi^-\rangle\right) \tag{2.2a}$$

$$|0\rangle |1\rangle = \frac{1}{\sqrt{2}}\left(\left|\Psi^+\right\rangle + |\Psi^-\rangle\right) \tag{2.2b}$$

$$|1\rangle |0\rangle = \frac{1}{\sqrt{2}}\left(\left|\Psi^+\right\rangle - |\Psi^-\rangle\right) \tag{2.2c}$$

$$|1\rangle |1\rangle = \frac{1}{\sqrt{2}}\left(\left|\Phi^+\right\rangle - |\Phi^-\rangle\right) \tag{2.2d}$$

We can then write the total state (Eq. (2.1)) as [(1)]:

$$\begin{aligned}|\psi\rangle_C \otimes |\Phi\rangle_{AB} = \frac{1}{2}\Big[&\left|\Phi^+\right\rangle_{CA} \otimes (\alpha |0\rangle_B + \beta |1\rangle_B) + |\Phi^-\rangle_{CA} \otimes (\alpha |0\rangle_B - \beta |1\rangle_B) \\ &+ \left|\Psi^+\right\rangle_{CA} \otimes (\alpha |1\rangle_B + \beta |0\rangle_B) + |\Psi^-\rangle_{CA} \otimes (\alpha |1\rangle_B - \beta |0\rangle_B)\Big]\end{aligned} \tag{2.3}$$

(1) Here is one of the terms $|0\rangle_C \otimes (|0\rangle_A |0\rangle_B) = (|0\rangle_C |0\rangle_A) \otimes |0\rangle_B = \frac{1}{\sqrt{2}}(|\Phi^+\rangle_{CA} + |\Phi^-\rangle_{CA}) \otimes |0\rangle_B$, the rest of the terms follows similarly.

So now we can easily see what happens when a, joint measurement is done on Charlie and Alice in the Bell basis, they will become entangled instead, as one of the four Bell states with equal probability. Bob's state collapses into either the state we wanted to teleport, namely $\alpha |0\rangle_B + \beta |1\rangle_B$ or a unitarily transformed version of that state. This means that after the measurement, Charlie can send the result to Bob allowing him to perform the necessary inverse transformation, corresponding to Pauli-operations. Thus completing the teleportation protocol.

The Bell measurement is an important step of the teleportation protocol since it is here the teleportation happens. When making the Bell measurement it is important to *not* be available to distinguish Alice's and Charlie's states, or else that would violate the no cloning theorem. Since then we could keep Charlie's state and transfer it to Bob, thereby having made a copy of an unknown quantum state(2) (for a proof of the no cloning theorem see Appendix A). Actually we are not even interested in Alice's state itself but rather the information she has on Bob.

Note that we didn't put any restrictions on α and β, so suppose Charlie shares an entangled state with Diana, $|\psi\rangle_C = \frac{1}{\sqrt{2}}\left(|0\rangle_D |0\rangle_C + |1\rangle_D |1\rangle_C\right) = \alpha |0\rangle_C + \beta |1\rangle_C$ where $\alpha = \frac{1}{\sqrt{2}} |0\rangle_D$ and $\beta = \frac{1}{\sqrt{2}} |1\rangle_D$. Then after the teleportation protocol Bob's state is $|\psi\rangle_B = \alpha |0\rangle_B + \beta |1\rangle_B = \frac{1}{\sqrt{2}}\left(|0\rangle_D |0\rangle_B + |1\rangle_D |1\rangle_B\right)$ and now Bob shares an entangled state with Diana, this is the entanglement swapping mentioned in the introduction.

2.1.2 Continuous-variable Teleportation

The first person to consider teleportation of continuous-variable states was Lev Vaidman [12]. However, we will follow the approach from Ref. [10]. For the Continuous case we will take; Alice, Bob and Charlies systems to have a pair of conjugate continuous degrees of freedom (e.g. position and momentum or amplitude and phase quadratures of light), $\hat{q}_i, \hat{p}_i, i \in \{A, B, C\}$. Again Bob and Alice, start by creating an entangled state so that the variables of their systems are (anti-)correlated like this,

$$\hat{q}_A - \hat{q}_B = 0, \quad \hat{p}_A + \hat{p}_B = 0 \tag{2.4}$$

Then a Bell measurement is performed on Charlie's and Alice's states, by first mixing their quadratures with a balanced beam-splitter transformation:

$$\hat{q}_\pm = \frac{1}{\sqrt{2}}\left(\hat{q}_A \pm \hat{q}_C\right), \quad \hat{p}_\pm = \frac{1}{\sqrt{2}}\left(\hat{p}_A \pm \hat{p}_C\right) \tag{2.5}$$

Then using a homodyne detection to measure the conjugate pair $\hat{q}_-$ and $\hat{p}_+$, denoting the outcomes with (q_-, p_+). Which causes the collapse of the quadratures in Eq. (2.5) to

$$\hat{q}_A = \hat{q}_C + \sqrt{2}\, q_-, \quad \hat{p}_A = -\hat{p}_C + \sqrt{2}\, p_+ \tag{2.6}$$

(2) Although perfect cloning is impossible, imperfect cloning is possible with a fidelity (see Section 2.1.3 for what fidelity is) up to 5/6 [4, p. 7 (A.(2))]

We can combine Eq. (2.4) and Eq. (2.6) to get that Bob's state is now:

$$\hat{q}_B = \hat{q}_C + \sqrt{2}\, q_-, \quad \hat{p}_B = \hat{p}_C - \sqrt{2}\, p_+ \tag{2.7}$$

Charlie can then communicate the outcomes of (q_-, p_+) to Bob via a classical channel and thereby Bob can do a feedback on his state to shift the quadratures appropriately to match that of Charlie

$$\hat{q}'_B = \hat{q}_B - \sqrt{2}\, q_- = \hat{q}_C, \quad \hat{p}'_B = \hat{p}_B + \sqrt{2}\, p_+ = \hat{p}_C. \tag{2.8}$$

And thereby completing the teleportation protocol.

2.1.3 Fidelity

Fidelity is a concept used to quantify how different two quantum states are or one can think of it as the probability of producing a desired state. This is a useful concept when doing the teleportation protocol since there is going to be noise or errors in the system, so after doing the feedback, Bob's final state might not be equal to Charlie's initial state. The fidelity can then be used to tell us how well the teleportation went. The fidelity F' for two states is defined so that $0 \leq F' \leq 1$, where a fidelity of 1 means they are the same state and fidelity of 0 means they are completely different i.e. they are orthogonal. Say, the state we want to have is $|\Psi_{\text{ideal}}\rangle$ and the state we actually produced is $|\Psi\rangle$ then fidelity can be defined as:

$$F' = |\langle \Psi_{\text{ideal}} \,|\, \Psi \rangle|^2 \tag{2.9}$$

or, more generally, if the state we made is described by a density matrix ρ:

$$F' = \langle \Psi_{\text{ideal}} | \, \rho \, | \Psi_{\text{ideal}} \rangle \tag{2.10}$$

In this project on continuous-variable teleportation the fidelity we are interested in is actually the average fidelity over many runs of the protocol, where in each run Charlie "picks" a random state from a set of states described by a Gaussian distribution. The fidelity here is the same as in Ref. [8, p.42, Eq.88] with $\Delta X^2_{A,\text{out}} = \Delta P^2_{A,\text{out}}$

$$F = \int \mathrm{d}^2\alpha P_{\bar{n}}(\alpha) \langle \alpha | \, \rho_B \, | \alpha \rangle = \frac{1}{1 + N_{\text{add}}^{\text{total}}}. \tag{2.11}$$

Where $|\alpha\rangle$ is Charlie's coherent state of a single run of the protocol, $P_{\bar{n}}(\alpha) = \frac{1}{\bar{n}\sqrt{2\pi}} e^{-|\alpha|^2/(2\bar{n}^2)}$ is the probability distribution from which Charlie picks his states, with a width $\bar{n}$ and ρ_B is the density matrix for Bob's state. $N_{\text{add}}^{\text{total}}$ is the total added noise to the system, defined as the variance of the Bob's state after feedback state minus Charlie's initial state and then averaged over the (classical) Gaussian distribution.

In the classical limit, meaning we don't utilize entanglement, then the fidelity becomes [8, p. 30]:

$$F_{\text{cl}} = \frac{1 + \bar{n}}{1 + 2\bar{n}} \tag{2.12}$$

where $\bar{n}$ is the width of the distribution of states Charlies can pick from. At first sight one might think that perfect teleportation is possible since $F_{\text{cl}} = 1$ for $\bar{n} = 0$. However, when $\bar{n} = 0$ means that the distribution has zero width which means it is just a single state. Then it is correct, since if you know the state, you can of course transfer the state by just sending what the state is with a classical channel. We are more interested in the case for a completely arbitrary state, $\bar{n} \longrightarrow \infty$, then we have $F_{\text{cl}} = 1/2$, which is the best one can do in the classical limit, meaning that half the time the teleportation protocol is going to fail. So, when doing the analysis in the none classical limit we aim to get at least a fidelity greater than a half.

CHAPTER 3

Hamiltonian, Equations of Motion and Input-Output Relations

We are going to introduce a mathematical description for the systems which are going to acts Alice, Bob and Charlie. We are considering a system for two separate oscillators 1 and 2 with the same resonance frequency Ω, in a cascaded setup in which they will interact with a common traveling light field with carrier frequency ω_L. Each oscillator also interacts with its own thermal bath, which is uncorrelated from the input light field, meaning that information can leak out and in turn noise leaks in. We will set $\hbar = 1$ for all calculations.

First we will derive the quantum Langevin equation (QLE) for Markovian reservoirs in a rotating wave approximating, which introduces the input noise operator, from the QLE the Heisenberg-Langevin Equation of motion (EOM) can easily be derived. We motivate the time-reversed quantum Langevin equation by defining the output noise operator, which when combined with the input noise operator the input-output relation can be derived. Then looking at the case where the light interacts with a oscillator in a cavity and using the the Heisenberg-Langevin EOM, together with the input-output relation to derive the full input-output relations of the entire setup of the two systems and a detector and the EOM of the oscillator.

3.1 Heisenberg-Langevin Equation

Here we are going to introduce the Heisenberg-Langevin equation, which is needed when modeling the dynamics of an open quantum system, i.e, a quantum system interacting with an environment, here the environments are thermal baths and the light field. The need for a new formalism comes from the fact, that if one tried to just add a decay term to the equations of motion for the system (as a result of interacting with the environment) then one find that the commutation relations are not preserved for all time. The derivation of the Heisenberg-Langevin equation is usually done by assuming that our system is interacting with a reservoir which is modeled as a large number (approximately infinite) of bosonic harmonic oscillators. The interaction will be assumed to be weak, meaning we can write it in a bilinear form, $\hat{H}_{\text{int}} \sim \hat{R}\hat{S}^\dagger + \hat{S}\hat{R}^\dagger$, where $\hat{S}$ and $\hat{R}$ is the system and reservoir annihilation operators respectively. We consider a Hamiltonian of the form [7, p.2-3]

$$\hat{H} = \hat{H}_{\text{sys}} + \hat{H}_{\text{res}} + \hat{H}_{\text{sys-res}} \tag{3.1a}$$

$$\hat{H}_{\text{res}} = \int_{-\infty}^{\infty} \mathrm{d}\omega \omega \hat{R}^\dagger(\omega)\hat{R}(\omega) \tag{3.1b}$$

$$\hat{H}_{\text{int}} = i\int_{-\infty}^{\infty} \mathrm{d}\omega \kappa(\omega)\left(\hat{R}^\dagger(\omega)\hat{S}(t) - \hat{S}^\dagger(t)\hat{R}(\omega)\right) \tag{3.1c}$$

Where $\hat{H}_{\text{sys}}$ is a general system Hamiltonian, $\hat{H}_{\text{res}}$ is the free evolution for the reservoir and $\hat{H}_{\text{int}}$ is the interaction Hamiltonian and $\kappa(\omega)$ is the coupling strength between the system and reservoir. The variable ω indicates the infinitely many reservoir operators spans all possible frequencies, and it is assumed they are independent degrees of freedom. Therefore the reservoir operator satisfies

$$\left[\hat{R}(\omega), \hat{R}^\dagger(\omega')\right] = \delta(\omega - \omega') \tag{3.2}$$

This Hamiltonian is in a rotating frame with respect to one of the high frequency $\tilde{\Omega}$ reservoir modes compared to the system frequency Ω, $\tilde{\Omega} \gg \Omega$. The lower bound of integration in Eq. (3.1b) and Eq. (3.1c) is $-\tilde{\Omega}$ but is replace with $-\infty$. We find the Heisenberg equations of motion (EOM) for the reservoir $\hat{R}$ and an arbitrary system operator $\hat{O}$ from Eq. (3.1)

$$\dot{\hat{R}}(\omega) = -i\omega\hat{R}(\omega) + \kappa(\omega)\hat{S}(t) \tag{3.3}$$

$$\dot{\hat{O}}(t) = -i\left[\hat{O}(t), \hat{H}_{\text{sys}}\right] + \int_{-\infty}^{\infty} \mathrm{d}\omega\kappa(\omega)\left(\hat{R}^\dagger(\omega)\left[\hat{O}(t), \hat{S}(t)\right] - \left[\hat{O}(t), \hat{S}^\dagger(t)\right]\hat{R}(\omega)\right) \tag{3.4}$$

we can then solve Eq. (3.3)

$$\hat{R}(\omega) = e^{-i\omega(t-t_0)}R_0(\omega) + \kappa(\omega)\int_{t_0}^{t} \mathrm{d}t' e^{-i\omega(t-t')}\hat{S}(t'), t \geq t_0. \tag{3.5}$$

Where $\hat{R}_0(\omega)$ is $\hat{R}(\omega)$ at $t = t_0$. Inserting it back into Eq. (3.4) and making the first Markov approximation $\kappa(\omega) = \sqrt{\gamma/\pi}$, where γ is the HWHM decay rate, physically this means that the noise processes do not depend on what happened at previous times and this is often referred to as the reservoir having no memory. Then using the properties of the Dirac-delta function

$$\int_{-\infty}^{\infty} \mathrm{d}\omega e^{-i\omega(t-t')} = 2\pi\delta(t-t') \tag{3.6}$$

and

$$\int_{t_0}^{t} \mathrm{d}t'\delta(t-t')\hat{S}(t') = \frac{1}{2}\hat{S}(t) \tag{3.7}$$

we get the RWA Markovian quantum Langevin equation:

$$\dot{\hat{O}}(t) = -i\left[\hat{O}(t), \hat{H}_{\text{sys}}\right] - \left[\hat{O}(t), \hat{S}^\dagger(t)\right]\left(\gamma\hat{S}(t) + \sqrt{2\gamma}\hat{R}_{\text{in}}(t)\right) + \left(\gamma\hat{S}^\dagger(t) + \sqrt{2\gamma}\hat{R}^\dagger_{\text{in}}(t)\right)\left[\hat{O}(t), \hat{S}(t)\right] \tag{3.8}$$

where we have introduced the input field operator $\hat{R}_{\text{in}}$

$$\hat{R}_{\text{in}}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega e^{-i\omega(t-t_0)}\hat{R}_0(\omega) \tag{3.9}$$

that satisfies

$$\left[\hat{R}_{\text{in}}(t), \hat{R}_{\text{in}}(t')\right] = \delta(t-t').$$

For our purposes it is enough to only look at the case where $\hat{O} = \hat{S} \in \{\hat{b}_1, \hat{b}_2, \hat{a}_1, \hat{a}_2\}$, using this in Eq. (3.8) we then get the RWA Markovian Heisenberg-Langevin equation:

$$\dot{\hat{S}}(t) = -i\left[\hat{S}(t), \hat{H}_{\text{sys}}\right] - \gamma\hat{S}(t) - \sqrt{2\gamma}\hat{R}_{\text{in}}(t). \tag{3.10}$$

From this we see there is a decay term $-\gamma\hat{S}(t)$ as one would expect, but there is also a fluctuation term $-\sqrt{2\gamma}\hat{R}_{\text{in}}(t)$. $\hat{R}_{\text{in}}(t)$ can be interpreted as the influx of noise associated with information leaking out, this is consistent with the quantum fluctuation-dissipation theorem [9]. In the case, where it is an incoherent state e.g. thermal state it would correspond to thermal noise and if $\hat{R}_{\text{in}}(t)$ is a coherent state instead, this would correspond to vacuum fluctuations.

In Section 3.1.2 we derive the relation between the in going noise $\hat{R}_{\text{in}}$ and the system $\hat{S}$ with the outgoing noise $\hat{R}_{\text{out}}$, known as the input-output relation. However, physically we will not have access to $\hat{R}_{\text{out}}$ for all types of reservoir, e.g. mechanical damping. But if the system couples to an optical reservoir we do have access to $\hat{R}_{\text{out}}$, since we can measure the light going out, and this is exactly how we will infer the motion of the oscillators. By doing a similar derivation to Eq. (3.8) we can derive the time-reversed quantum Langevin equation. We start by consider the solution to reservoir operator with the boundary condition at the final time

$$\hat{R}(\omega) = e^{-i\omega(t-t_1)}\hat{R}_1(\omega) - \kappa(\omega)\int_t^{t_1} \mathrm{d}t' e^{-i\omega(t-t')}\hat{S}(t'),\ t \le t_1 \tag{3.11}$$

and similarly to before $\hat{R}_1(\omega) = \hat{R}(\omega)\big|_{t=t_1}$. We also define the output field operator

$$\hat{R}_{\text{out}}(t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega e^{-i\omega(t-t_1)}\hat{R}_1(\omega). \tag{3.12}$$

Then following the same procedure as before, we get Eq. (3.8) but with $\gamma\hat{S}(t) \to -\gamma\hat{S}(t)$ and $\hat{R}_{\text{in}}(t) \to \hat{R}_{\text{out}}(t)$.

The input-output relation, which relates the input, output and system operators, can be derived by integrating Eq. (3.5) and Eq. (3.11) separately over all reservoir operators. Starting with forward evolving solution

$$\int_{-\infty}^{\infty} \mathrm{d}\omega\hat{R}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}\omega e^{-i\omega(t-t_0)}R_0(\omega) + \sqrt{\frac{\gamma}{\pi}}\int_{t_0}^{t} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}\omega e^{-i\omega(t-t')}\hat{S}(t') \tag{3.13}$$

$$= \sqrt{2\pi}\hat{R}_{\text{in}}(t) + \sqrt{\gamma\pi}\hat{S}(t) \tag{3.14}$$

doing the same calculation, but for the backwards evolving solution

$$\int_{-\infty}^{\infty} \mathrm{d}\omega \hat{R}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}\omega e^{-i\omega(t-t_1)} R_1(\omega) - \sqrt{\frac{\gamma}{\pi}} \int_{t}^{t_1} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}\omega e^{-i\omega(t-t')} \hat{S}(t') \tag{3.15}$$

$$= \sqrt{2\pi} \hat{R}_{\text{out}}(t) - \sqrt{\gamma\pi} \hat{S}(t) \tag{3.16}$$

from which we can easily get the input-output relation

$$\hat{R}_{\text{out}}(t) = \hat{R}_{\text{in}}(t) + \sqrt{2\gamma} \hat{S}(t). \tag{3.17}$$

These results, Eq. (3.10) and Eq. (3.17), will be used the later sections to derive the EOM of the oscillators and the input-output relation of the whole setup. Which is going to give us the signal we need to measure in order to complete the teleportation protocol by doing a feedback on Bob's system.

3.1.1 Interaction Hamiltonian

Here we are going to give a qualitative description of the light-oscillator interaction and later a more detailed description will be given. The local interaction between each of the oscillators and the local light field is described by the effective bilinear Hamiltonian of the form

$$\hat{H}_{\text{int},i} \sim \mu_i \left(\hat{a}_i^\dagger \hat{b}_i + \hat{a}_i \hat{b}_i^\dagger \right) + \nu_i \left(\hat{a}_i^\dagger \hat{b}_i^\dagger + \hat{a}_i \hat{b}_i \right), \; i \in \{1, 2\}. \tag{3.18}$$

Where μ_i, ν_i are non-negative real numbers characterizing the strength for each interaction. $\hat{b}_i^\dagger, \hat{b}_i$ and $\hat{a}_i^\dagger, \hat{a}_i$ are the creation and annihilation operators for the oscillators and light field respectively. which all satisfies the canonical commutation relations:

$$\left[\hat{b}(t), \hat{b}^\dagger(t)\right] = 1, \; \left[\hat{a}_i(t), \hat{a}_j^\dagger(t')\right] = \delta_{ij}\delta(t - t') \tag{3.19a}$$

$$\left[\hat{b}_i, \hat{b}_j\right] = \left[\hat{b}_i^\dagger, \hat{b}_j^\dagger\right] = \left[\hat{a}_i, \hat{a}_j\right] = \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = 0. \tag{3.19b}$$

The first term in the Hamiltonian is a beam-splitter (BS) interaction and express the energy transfer between the field and oscillator. The second term two-mode squeezing (TMS) interaction and describes the joint (de-)excitations of the oscillator and field modes, which can be used to generate entanglement between the light and oscillator. This type of Hamiltonian can be achieve by having the light couple to the oscillator with $\hat{a}^\dagger\hat{a}(\hat{b} + \hat{b}^\dagger)$ in a rotating frame with respect to the laser light frequency ω_L. We can then linearize the interaction by splitting the light into a classical coherent part α and quantum fluctuations $\delta\hat{a}$ part, $\hat{a} = \alpha + \delta\hat{a}$. we get the interaction to be the form $\delta\hat{a}^\dagger\hat{b} + \delta\hat{a}^\dagger\hat{b}^\dagger$ + H.C. Then we can get the BS or TMS interaction by detuning the laser and doing a rotating wave approximation. Furthermore the interaction Eq. (3.18) is characterized by the readout rate/drive pulse Γ_i and optical broadening parameter $\zeta_i \in [-1, 1]$ which also indicates if the interaction is red detuned ($\zeta_i > 0$) or blue detuned ($\zeta_i < 0$). The readout rate and broadening parameter can be expressed in terms of the interaction weights:

$$\Gamma_i = \frac{(\mu_i + \nu_i)^2}{2}, \tag{3.20a}$$

$$\zeta_i = \frac{\mu_i - \nu_i}{\mu_i + \nu_i}, \tag{3.20b}$$

$$\gamma_{\text{opt},i} = \zeta_i \Gamma_i = \frac{\mu_i^2 - \nu_i^2}{2} \tag{3.20c}$$

$$\mu_j = \sqrt{2\Gamma_j}\,\frac{1+\zeta_j}{2}, \quad \nu_j = \sqrt{2\Gamma_j}\,\frac{1-\zeta_j}{2} \tag{3.20d}$$

Since each oscillator is coupled to a thermal bath they will experience damping denoted by the (HWHM) line width γ_i, the total decay rate of the oscillators including the contribution from the optical broadening term, $\check{\gamma}_i \equiv \gamma_i + \gamma_{\text{opt},i} = \gamma_i + \zeta_i \Gamma_i$. All of the quantities defined above are time dependent coming from $\alpha(t)$ being time dependent.

3.1.2 Input-Output Relations

Here we will go through the main steps to get to the input-output relation for the quantum light that carries information about the oscillators, but not for the carrier light. Instead we assume the interactions between the light and oscillators will induce pairs of narrow sidebands that are well-separated and centered at $\omega_L \pm \Omega$, defined for the oscillators resonance frequency Ω. From this we can define independent upper- and lower-sideband

$$\hat{a}_{\text{in/out},\pm}(t) = e^{\pm i\Omega t}\hat{a}_{\text{in/out}}(t) \tag{3.21}$$

which satisfies

$$\left[a_{\text{in/out},\pm}(t), a^\dagger_{\text{in/out},\pm}(t')\right] = \delta(t-t'). \tag{3.22}$$

In order to properly justify the input-output relations for the sidebands we have to go back and linearize the optomechanical Hamiltonian as mentioned in the previous section.

$$\hat{\tilde{H}} = \hat{\tilde{H}}_0 + \hat{\tilde{H}}_{\text{int}} \tag{3.23}$$

$$\hat{\tilde{H}}_0 = \Omega \hat{b}^\dagger \hat{b} + \omega_{\text{cav}} \hat{a}^\dagger \hat{a} \tag{3.24}$$

$$\hat{\tilde{H}}_{\text{int}} = g_0 \hat{\hat{a}}^\dagger \hat{\hat{a}} \left(\hat{b} + \hat{b}^\dagger\right) \tag{3.25}$$

$$\rightarrow g_0 \alpha \left(\delta\hat{a} + \delta\hat{a}^\dagger\right)\left(\hat{b} + \hat{b}^\dagger\right) \tag{3.26}$$

where we have linearized the interaction Hamiltonian and assumed α is real and positive. Using the Heisenberg-Langevin EOM Eq. (3.10) for the case of light interacting with an oscillator in a cavity, $\hat{S} \rightarrow \hat{a}, \hat{H}_{\text{sys}} \rightarrow \hat{\tilde{H}}, \hat{R}_{\text{in}} \rightarrow -\hat{a}_{\text{in}}, \gamma \rightarrow \kappa$, we have the EOM of the intracavity field in a rotating frame with respect to the cavity frequency ω_L, $\Delta = \omega_{\text{cav}} - \omega_L$ is

$$\dot{\hat{a}} = (-i\Delta - \kappa)\hat{a} - ig_0\alpha\left(\hat{b}_I e^{-i\Omega t} + \hat{b}_I^\dagger e^{i\Omega t}\right) + \sqrt{2\kappa}\,\hat{a}_{\text{in}} \tag{3.27}$$

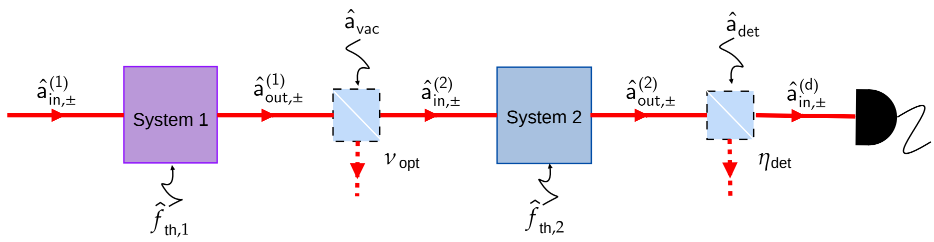

Figure 3.1: Beam-splitter model for optical losses and detection inefficiency. As the light field (red line) travels it interactions with the systems and along the way it encounters virtual beam-splitter (with power transmission coefficient ν_{opt} and η_{det}), where some is lost (dash red line) and vacuum noise (black curves) enters. Also depicted is the thermal noise $\hat{f}_{\text{th},1/2}$ entering the systems. Adapted from [1].

where we have introduced the slowly varying variable $\hat{b}_I = \hat{b}e^{i\Omega t}$ which is slow compared to $1/\kappa$. Formally integrating this we find the solution to be

$$\hat{a}(t) = e^{(-i\Delta-\kappa)(t-t_0)}\hat{a}(t_0) + \int_{t_0}^{t} dt' e^{(-i\Delta-\kappa)(t-t')}\left[-ig_0\alpha\left(\hat{b}_I(t')e^{-i\Omega t'} + \hat{b}_I^\dagger(t')e^{i\Omega t'}\right) + \sqrt{2\kappa}\,\hat{a}_{\text{in}}(t')\right] \tag{3.28}$$

$$\approx -ig_0\alpha\left(\frac{\hat{b}_I(t)e^{-i\Omega t}}{\kappa - i[\Omega-\Delta]} + \frac{\hat{b}_I^\dagger(t)e^{i\Omega t}}{\kappa - i[-\Omega-\Delta]}\right) + \sqrt{2\kappa}\int_{-\infty}^{t} dt' e^{(-i\Delta-\kappa)(t-t')}\hat{a}_{\text{in}}(t') \tag{3.29}$$

Where we have taking the limit $t_0 \to \infty$ and used the fact that $\hat{b}_I$ varies slowly to approximate it as $\hat{b}_I(t') \approx \hat{b}_I(t)$, then $\hat{a}(t)$ can be inserted into input-output relation Eq. (3.37) with the appropriate quantities

$$\hat{a}_{\text{out}}(t) = -\hat{a}_{\text{in}}(t) + \sqrt{2\kappa}\,\hat{a}(t). \tag{3.30}$$

After properly redefining the operators (see Appendix B) we get

$$\hat{a}_{\text{out}}(t) = -\hat{a}_{\text{in}}(t) - i[\mu(t)\hat{b}_I(t)e^{-i\Omega t} + \nu(t)\hat{b}_I^\dagger(t)e^{i\Omega t}] \tag{3.31}$$

multiplying it through with $e^{\pm i\Omega t}$ in order to use the sidebands Eq. (3.21), then using a rotating wave approximating (RWA) to discard terms with $e^{\pm 2i\Omega t}$ under the assumption that $\mu^2, \nu^2 \ll \Omega$. We then arrive at the input-output relations for the sideband annihilation operators are

$$\hat{a}_{\text{out},+}(t) = -\hat{a}_{\text{in},+}(t) - i\mu(t)\hat{b}_I(t) \tag{3.32a}$$
$$\hat{a}_{\text{out},-}(t) = -\hat{a}_{\text{in},-}(t) - i\nu(t)\hat{b}_I^\dagger(t). \tag{3.32b}$$

Here we that the upper-sideband couples to $\hat{b}_I(t)$ and the lower-sideband couples to $\hat{b}_I^\dagger(t)$. However these input-output relations are only enough when the light interacts with a single system. For the full setup we want to consider, the light interacts with two systems and a detector with losses in between. To model this, we use a virtual beam-splitter model, where virtual beam-splitters are placed between the systems and between the last system and the detector see Fig. 3.1. When the input light field $\hat{a}^{(1)}_{\text{in},\pm}(t)$ reaches the first system we have the upper-sideband $\hat{a}^{(1)}_{\text{in},+}(t)$ couples to $\hat{b}^{(1)}_I(t)$ and the lower-sideband $\hat{a}^{(1)}_{\text{in},-}(t)$ couples to $\hat{b}^{(1)\dagger}_I(t)$ then use the input-output relations Eq. (3.32) we the out going light from system 1 is

$$\hat{a}^{(1)}_{\text{out},+}(t) = -\hat{a}^{(1)}_{\text{in},+}(t) - i\mu_1(t)\hat{b}^{(1)}_I(t) \tag{3.33a}$$

$$\hat{a}^{(1)}_{\text{out},-}(t) = -\hat{a}^{(1)}_{\text{in},-}(t) - i\nu_1(t)\hat{b}^{(1)\dagger}_I(t). \tag{3.33b}$$

Then before the light gets to the second system, some of the light is lost, this is modeled by having the light passing through a beam-splitter with a power transmission coefficient ν_{opt} together with vacuum fluctuations $\hat{a}_{\text{vac},\pm}(t)$, the light that gets to the second system is then

$$\hat{a}^{(2)}_{\text{in},\pm}(t) = -\sqrt{\nu_{\text{opt}}}\,\hat{a}^{(1)}_{\text{out},\pm}(t) - \sqrt{1-\nu_{\text{opt}}}\,\hat{a}_{\text{vac},\pm}(t). \tag{3.34}$$

Where the minus sign on the RHS is a convenient. This is then the input light for the second and the upper- and lower-sidebands $\hat{a}^{(2)}_{\text{in},\pm}(t)$ interacts with $\hat{b}^{(2)}_I(t)$ and $\hat{b}^{(2)\dagger}_I(t)$ respectively , just like for the first system Eq. (3.33), and the outgoing light becomes

$$\hat{a}^{(2)}_{\text{out},+}(t) = -\hat{a}^{(2)}_{\text{in},+}(t) - i\mu_2(t)\hat{b}^{(2)}_I(t) \tag{3.35a}$$

$$\hat{a}^{(2)}_{\text{out},-}(t) = -\hat{a}^{(2)}_{\text{in},-}(t) - i\nu_2(t)\hat{b}^{(2)\dagger}_I(t). \tag{3.35b}$$

Then again before the light reaches the detector it encounters a virtual beam-splitter with power transmission coefficient η_{det} and vacuum fluctuations $\hat{a}_{\text{det},\pm}(t)$

$$\hat{a}^{(\text{d})}_{\text{in},\pm}(t) = \sqrt{\eta_{\text{det}}}\,\hat{a}^{(2)}_{\text{out},\pm}(t) + \sqrt{1-\eta_{\text{det}}}\,\hat{a}_{\text{det},\pm}(t) \tag{3.36}$$

Here we list most of the input-output relations which will be needed later on.

$$\hat{a}^{(2)}_{\text{in},\pm}(t) = -\sqrt{\nu_{\text{opt}}}\,\hat{a}^{(1)}_{\text{out},\pm}(t) - \sqrt{1-\nu_{\text{opt}}}\,\hat{a}_{\text{vac},\pm}(t) \tag{3.37a}$$

$$\hat{a}^{(j)}_{\text{out},+}(t) = -\hat{a}^{(j)}_{\text{in},+}(t) - i\mu_j(t)\hat{b}^{(j)}_I(t) \tag{3.37b}$$

$$\hat{a}^{(j)}_{\text{out},-}(t) = -\hat{a}^{(j)}_{\text{in},-}(t) - i\nu_j(t)\hat{b}^{(j)\dagger}_I(t) \tag{3.37c}$$

3.1.3 Equations of Motion

To arrive at the correct equations of motion for the oscillators we again start with the linearize optomechanical Hamiltonian Eq. (3.23) and using the Heisenberg-Langevin EOM Eq. (3.10) with $\hat{S} \to \hat{b}$ and $\hat{R}_{\text{in}} \to -\hat{f}_{\text{th}}$

$$\begin{aligned}\dot{\hat{b}}(t) &= [-i\Omega - \gamma]\hat{b}(t) - ig_0[\alpha^*\hat{a}(t) + \alpha\hat{a}^\dagger(t)] + \sqrt{2\gamma}\,\hat{f}_{\text{th}}(t) \Rightarrow \\ \dot{\hat{b}}_I(t) &= -\gamma\hat{b}_I(t) - ig_0e^{i\Omega t}[\alpha^*\hat{a}(t) + \alpha\hat{a}^\dagger(t)] + \sqrt{2\gamma}\,\hat{f}_{\text{th}}(t)\end{aligned}$$

and again we have switched to the slowly varying variable $\hat{b}_I = \hat{b}e^{i\Omega t}$. Then inserting the solution for the light field $\hat{a}(t)$ with the necessary assumption Eq. (3.29) and doing a RWA to discard terms with $e^{\pm 2i\Omega t}$ we get

$$\dot{\hat{b}}_I(t) = -\gamma\hat{b}_I(t) - \frac{(g_0\alpha)^2}{\kappa}\left(\frac{\kappa}{\kappa - i[\Omega - \Delta]} - \frac{\kappa}{\kappa + i[-\Omega - \Delta]}\right)\hat{b}_I(t) + (\text{opt. noise}) + \sqrt{2\gamma}\hat{f}_{\text{th}}(t).$$

Where the real and imaginary part of

$$\left(\frac{\kappa}{\kappa - i[\Omega - \Delta]} - \frac{\kappa}{\kappa + i[-\Omega - \Delta]}\right),$$

are the optical boarding and optical spring shift respectively. We are only going to focus on the optical boarding and assume the shift from the optical spring effect is compensated by other means and/or absorbed in a redefined Ω.

$$\dot{\hat{b}}_I(t) = -\left[\gamma + \frac{\mu^2 - \nu^2}{2}\right]\hat{b}_I(t) + (\text{opt. noise}) + \sqrt{2\gamma}\hat{f}_{\text{th}}(\text{t})$$

Then including the optical noise, making a RWA to approximate $\hat{a}_{\text{in},\pm}(t') \approx \hat{a}_{\text{in},\pm}(t)$ and making the proper redefinitions as before, we get the EOM for oscillators are

$$\dot{\hat{b}}_I^{(j)}(t) = -\check{\gamma}_j(t)\hat{b}_I^{(j)}(t) + i[\mu_j(t)\hat{a}_{\text{in},+}^{(j)}(t) + \nu_j(t)\hat{a}_{\text{in},-}^{(j)\dagger}(t)] + \sqrt{2\gamma_j}\hat{f}_{\text{th}}^{(j)}(t). \quad (3.38)$$

The solution can be found by formal integration

$$\hat{b}_I^{(j)}(t) = e^{-\int_{t_0}^{t}\text{d}t'\check{\gamma}_j(t')}\hat{b}_I^{(j)}(t_0) + \int_{t_0}^{t}\text{d}t' e^{-\int_{t'}^{t}\text{d}t''\check{\gamma}_j(t'')}\hat{f}_j(t') \quad (3.39)$$

$$\hat{f}_j(t) = i[\mu_j(t)\hat{a}_{\text{in},+}^{(j)}(t) + \nu_j(t)\hat{a}_{\text{in},-}^{(j)\dagger}(t)] + \sqrt{2\gamma_j}\hat{f}_{\text{th}}^{(j)}(t) \quad (3.40)$$

We choose two different boundary conditions of the two oscillators, for system 1: $t_0 = T$ and for system 2: $t_0 = 0$

$$\hat{b}_I^{(1)}(t) = e^{\int_t^T \text{d}t'\check{\gamma}_1(t')}\hat{b}_I^{(1)}(T) - \int_t^T \text{d}t' e^{-\int_{t'}^{t}\text{d}t''\check{\gamma}_1(t'')}\hat{f}_1(t') \quad (3.41)$$

$$\hat{b}_I^{(2)}(t) = e^{-\int_0^t \text{d}t'\check{\gamma}_2(t')}\hat{b}_I^{(2)}(0) + \int_0^t \text{d}t' e^{-\int_{t'}^{t}\text{d}t''\check{\gamma}_2(t'')}\hat{f}_2(t') \quad (3.42)$$

We do this since the initial state of system 2 is the state we want to teleport and it will be teleported to system 1 at the end of the protocol.

Chapter 4

Continuous-variable teleportation between two cascaded systems

> Mathematics is not just a language. Mathematics is a language plus reasoning.
>
> Richard Phillips Feynman

In this chapter we will see how the Bell measurement arises from the measurement signal Eq. (3.36) and how it can be used for the feedback on Bob's system. Then in the general case for optical losses and thermal noise, and where Charlie can only teleport states from a restricted set of sets and where Bob's state is in a known thermal initial state. An expression for the total noise and an equation for the optimal filter is found. Since these equations cannot be solved analytically, therefore they will need to be treated further with numerical methods. To help facilitate this a treatment of a more ideal case will be considered, where Charlie can any arbitrary state and Bob's state is initial in an unknown state.

4.1 Deriving the measurement signal with feedback

In this section, it will be shown how the measurement signal Eq. (3.36) that has been obtained from the light after it as interacted with both oscillators, by doing a homodyne measurement, corresponds to doing the Bell measurement of the teleportation protocol. Where Alice's state is the light emanating from Bob's system and Charlie's state is the second oscillator. This signal is used to do the necessary feedback on Bob's state, the first oscillator, in order to complete the protocol.

The complex measurement current operator from the homodyne detection is defined from the detection field Eq. (3.36) as:

$$\hat{\mathcal{M}}(t) = \frac{1}{\sqrt{\eta_{\text{det}}}} \frac{\hat{a}^{(\text{signal})}_{\text{in},+}(t) - \hat{a}^{\dagger(\text{signal})}_{\text{in},-}(t)}{\sqrt{2}i} = \frac{\hat{a}^{(2)}_{\text{out},+}(t) - \hat{a}^{(2)\dagger}_{\text{out},-}(t)}{\sqrt{2}i} + \sqrt{\frac{1}{\eta_{\text{det}}} - 1}\, \frac{\hat{a}_{\text{det},+}(t) - \hat{a}^{\dagger}_{\text{det},-}(t)}{\sqrt{2}i} \tag{4.1}$$

Which is the output phase quadrature from the second system with some loss before the detector characterized by the detection efficiency η_{det} see Fig. 3.1. Note that it is non-hermitian, however $\hat{\mathcal{M}}$ and $\hat{\mathcal{M}}^\dagger$ are related to the sine and cosine component of the measurement current demodulated at, the oscillators resonance frequency, ω_0, $\hat{I}^c \sim \hat{\mathcal{M}} + \hat{\mathcal{M}}^\dagger, \hat{I}^s \sim \left(\hat{\mathcal{M}} - \hat{\mathcal{M}}^\dagger\right)/i$. We have assumed the sidebands are well separated and therefore can treat them as independent variables, so they are uncorrelated and we assume they are both in the ground state, so both have mean zero and the same variance. Therefore, we only need to consider one of them. Using the input-output relation Eq. (3.37) and the relation between Γ, μ, ν Eq. (3.20a) the measurement current can be written in terms of the optical input fields and the oscillator readout as

$$\begin{aligned} \hat{\mathcal{M}}(t) = &-\sqrt{\nu_{\text{opt}}\Gamma_1(t)}\,\hat{b}^{(1)}_I(t) - \sqrt{\Gamma_2(t)}\,\hat{b}^{(2)}_I(t) - \sqrt{\nu_{\text{opt}}}\,\frac{\hat{a}^{(1)}_{\text{in},+}(t) - \hat{a}^{(1)\dagger}_{\text{in},-}(t)}{\sqrt{2}i} \\ &+ \sqrt{1-\nu_{\text{opt}}}\,\frac{\hat{a}_{\text{vac},+}(t) - \hat{a}^{\dagger}_{\text{vac},-}(t)}{\sqrt{2}i} + \sqrt{\frac{1}{\eta_{\text{det}}} - 1}\,\frac{\hat{a}_{\text{det},+}(t) - \hat{a}^{\dagger}_{\text{det},-}(t)}{\sqrt{2}i}. \end{aligned} \tag{4.2}$$

The first two terms are the contribution from the dynamics of the oscillators. The third term is the shot noise coming from the fluctuations of the traveling light field. And the last two terms are the vacuum noises from optical losses. The motion of the second oscillator Eq. (3.38) ($j = 2$) depends the on the motion of the first oscillator by being driven by the readout signal from the first oscillator. It turns out that the calculations can be made easier if one uses a continuous, measurement-based feedback on the second oscillator to eliminate the dependence of the first oscillator in Eq. (3.38) ($j = 2$). The measurement current becomes:

$$\begin{aligned} \hat{\mathcal{M}}_{\text{v}}(t) = &-\sqrt{\nu_{\text{opt}}\Gamma_1(t)}\,\hat{b}^{(1)}_I(t) - \sqrt{\Gamma_2(t)}\,\hat{b}^{(2\text{v})}_I(t) - \sqrt{\nu_{\text{opt}}}\,\frac{\hat{a}^{(1)}_{\text{in},+}(t) - \hat{a}^{(1)\dagger}_{\text{in},-}(t)}{\sqrt{2}i} \\ &+ \sqrt{1-\nu_{\text{opt}}}\,\frac{\hat{a}_{\text{vac},+}(t) - \hat{a}^{\dagger}_{\text{vac},-}(t)}{\sqrt{2}i} + \sqrt{1/\eta_{\text{det}} - 1}\,\frac{\hat{a}_{\text{det},+}(t) - \hat{a}^{\dagger}_{\text{det},-}(t)}{\sqrt{2}i} \end{aligned} \tag{4.3}$$

where the superscript v indicates that the variable is altered by the presence of feedback. The feedback on the second oscillator is conditioned on the new measurement current $\hat{\mathcal{M}}_{\text{v}}(t)$. Physically this would correspond to applying a force to the oscillator thereby changing the motion and hence this would change the measurement current. However it is more practical to just implement this in the post processing. The EOM Eq. (3.38) with feedback then becomes

$$\dot{\hat{b}}^{(2\text{v})}_I(t) = -\check{\gamma}_2(t)\hat{b}^{(2\text{v})}_I(t) + i[\mu_2(t)\hat{a}^{(2)}_{\text{in},+}(t) + \nu_2(t)\hat{a}^{(2)\dagger}_{\text{in},-}(t)] + \sqrt{2\gamma_2}\,\hat{f}^{(2)}_{\text{th}}(t) + F_{\text{v}}(t)\hat{\mathcal{M}}_{\text{v}}(t) \tag{4.4}$$

Where F_{v} is the feedback gain. using the input-output relation Eq. (3.37) we have

$$\begin{aligned}\dot{\hat{b}}_I^{(2\text{v})}(t) &= -\check{\gamma}_2(t)\hat{b}_I^{(2\text{v})}(t) - \sqrt{\nu_{\text{opt}}}\left(\mu_1(t)\mu_2(t) - \nu_1(t)\nu_2(t)\right)\hat{b}_I^{(1)}(t) \\ &\quad + i\left[\sqrt{\nu_{\text{opt}}}\left(\mu_2(t')\hat{a}_{\text{in},+}^{(1)}(t') + \nu_2(t)\hat{a}_{\text{in},-}^{(1)\dagger}(t')\right)\right. \\ &\quad \left.-\sqrt{1-\nu_{\text{opt}}}\left(\mu_2(t')\hat{a}_{\text{vac},+}^{(1)}(t') + \nu_2(t')\hat{a}_{\text{vac},-}^{(1)\dagger}(t')\right)\right] + F_{\text{v}}(t)\hat{\mathcal{M}}_{\text{v}}(t) \qquad (4.5)\end{aligned}$$

Where on the second term we now can see how the motion of the second oscillator depends on the first. To figure out the feedback needed to cancel this dependence. We insert the feedback measurement current Eq. (4.3), into the equation for the second oscillator with feedback Eq. (4.4) and using μ_j, ν_j in terms of Γ_j and ζ_j Eq. (3.20d)

$$\begin{aligned}\dot{\hat{b}}_I^{(2\text{v})}(t) &= -\left(F_{\text{v}}(t)\sqrt{\nu_{\text{opt}}\Gamma_1(t)} + \sqrt{\nu_{\text{opt}}}\sqrt{\Gamma_1(t)\Gamma_2(t)}(\zeta_1+\zeta_2)\right)\hat{b}_I^{(1)}(t) \\ &\quad - \left(F_{\text{v}}(t)\sqrt{\Gamma_2(t)} + \check{\gamma}_2(t)\right)\hat{b}_I^{(2\text{v})}(t) + \sqrt{2\gamma_2}\hat{f}_{\text{th}}^{(2)}t) \\ &\quad + i\sqrt{\nu_{\text{opt}}\Gamma_2(t)}\,\frac{\hat{a}_{\text{in},+}^{(1)}(t) + \hat{a}_{\text{in},-}^{(1)\dagger}(t)}{\sqrt{2}} - \left(\zeta_2\sqrt{\nu_{\text{opt}}\Gamma_2(t)} + F_{\text{v}}(t)\sqrt{\nu_{\text{opt}}}\right)\frac{\hat{a}_{\text{in},+}^{(1)}(t) - \hat{a}_{\text{in},-}^{(1)\dagger}(t)}{\sqrt{2}i} \\ &\quad + F_{\text{v}}(t)\sqrt{\frac{1}{\eta_{\text{det}}} - 1}\,\frac{\hat{a}_{\text{det},+}(t) - \hat{a}_{\text{det},-}^{\dagger}(t)}{\sqrt{2}i} - i\sqrt{(1-\nu_{\text{opt}})\Gamma_2(t)}\,\frac{\hat{a}_{\text{vac},+}(t) + \hat{a}_{\text{vac},-}^{\dagger}(t)}{\sqrt{2}} \\ &\quad + \left(\zeta_2\sqrt{(1-\nu_{\text{opt}})\Gamma_2(t)} + F_{\text{v}}(t)\sqrt{1-\nu_{\text{opt}}}\right)\frac{\hat{a}_{\text{vac},+}(t) - \hat{a}_{\text{vac},-}^{\dagger}(t)}{\sqrt{2}i}. \qquad (4.6)\end{aligned}$$

We see that is we choose $F_{\text{v}}(t)$ to be

$$F_{\text{v}}(t) = -\sqrt{\Gamma_2(t)}(\zeta_1+\zeta_2) \qquad (4.7)$$

the prefactor for the first oscillator $\hat{b}_I^{(1)}(t)$ becomes zero. Then inserting this choice for the gain function $F_{\text{v}}(t)$ back into the equation of the second oscillator Eq. (4.6) we get the equation:

$$\dot{\hat{b}}_I^{(2\text{v})}(t) = -\check{\gamma}_{2\text{v}}(t)\hat{b}_I^{(2\text{v})}(t) + \hat{f}_{2\text{v}}(t) \Rightarrow \qquad (4.8)$$

$$\hat{b}_I^{(2\text{v})}(t) = e^{-\int_0^t \text{d}t'\check{\gamma}_{2\text{v}}(t')}\hat{b}_I^{(2\text{v})}(0) + \int_0^t \text{d}t' e^{-\int_{t'}^t \text{d}t''\check{\gamma}_{2\text{v}}(t'')}\hat{f}_{2\text{v}}(t') \qquad (4.9)$$

Where

$$\check{\gamma}_{2\text{v}}(t) \equiv F_{\text{v}}(t)\sqrt{\Gamma_2(t)} + \check{\gamma}_2(t) = \check{\gamma}_2(t) - \Gamma_2(t)(\zeta_1+\zeta_2) = \gamma_2 - \zeta_1\Gamma_2(t) \qquad (4.10)$$

Is the new total decay rate, which interestingly does not depend on the broadening parameter for the second system but rather the first and the new Langevin force

$$\begin{aligned}\hat{f}_{2\text{v}}(t) &\equiv i\sqrt{\nu_{\text{opt}}\Gamma_2(t)}\,\frac{\hat{a}_{\text{in},+}^{(1)}(t) + \hat{a}_{\text{in},-}^{(1)\dagger}(t)}{\sqrt{2}} + \zeta_1\sqrt{\nu_{\text{opt}}\Gamma_2(t)}\,\frac{\hat{a}_{\text{in},+}^{(1)}(t) - \hat{a}_{\text{in},-}^{(1)\dagger}(t)}{\sqrt{2}i} \\ &\quad - i\sqrt{(1-\nu_{\text{opt}})\Gamma_2(t)}\,\frac{\hat{a}_{\text{vac},+}(t) + \hat{a}_{\text{vac},-}^{\dagger}(t)}{\sqrt{2}} - \zeta_1\sqrt{(1-\nu_{\text{opt}})\Gamma_2(t)}\,\frac{\hat{a}_{\text{vac},+}(t) - \hat{a}_{\text{vac},-}^{\dagger}(t)}{\sqrt{2}i} \\ &\quad + \sqrt{2\gamma_2}\hat{f}_{\text{th}}^{(2)}(t) - \sqrt{\Gamma_2(t)}(\zeta_1+\zeta_2)\sqrt{\frac{1}{\eta_{\text{det}}} - 1}\,\frac{\hat{a}_{\text{det},+}(t) - \hat{a}_{\text{det},-}^{\dagger}(t)}{\sqrt{2}i} \qquad (4.11)\end{aligned}$$

which almost also does not depend on ζ_2 expect for the last term. The complex measurement current $\hat{\mathcal{M}}_v$ Eq. (4.3) contains information about the oscillators which we want to extract, but it also contains the optical noise from the vacuum and thermal noise which we do not want. So to get as much information as possible from the measurement, we look at the measurement current $\mathcal{M}_v(t)$ Eq. (4.3) filtered by a mode function $f_v(t)$. This is then going to constitute the Bell measurement used to transfer the state from system 2 to system 1. Using the solutions Eq. (3.41) with the boundary condition at the end of the protocol and Eq. (4.9) with boundary condition at the start of the protocol

$$\hat{O} = \int_0^T \mathrm{d}t f_v(t)\hat{\mathcal{M}}_v(t) = M_2 \hat{b}_I^{(2v)}(0) - M_1 \hat{b}_I^{(1)}(T) + \hat{\mathcal{N}} \tag{4.12}$$

where the transfer coefficients $M_{1,2}$ are

$$M_2 \equiv -A_{2v}(0) \tag{4.13a}$$
$$M_1 \equiv \sqrt{\nu_{\mathrm{opt}}}\tilde{A}_1(T) \tag{4.13b}$$

$$A_{2v}(t) \equiv \int_t^T \mathrm{d}t' f_v(t')\sqrt{\Gamma_2(t')}\, e^{-\int_t^{t'} \mathrm{d}\tau \check{\gamma}_{2v}(\tau)} \tag{4.14a}$$
$$\tilde{A}_1(t) \equiv \int_0^t \mathrm{d}t' f_v(t')\sqrt{\Gamma_1(t')}\, e^{\int_{t'}^t \mathrm{d}\tau \check{\gamma}_1(\tau)} \tag{4.14b}$$

and we The optimal value of M_2 depends on the distribution of states that can be teleported and the optimal value of M_1 depends on the state being teleported to. $\hat{\mathcal{N}}$ is the total noise operator with all the input fields and it has the form $\hat{\mathcal{N}} = \sum_j H_j(t)\hat{O}_{\mathrm{in},j}(t)$. For the full expression of $\hat{\mathcal{N}}$ see Appendix C.

The filtered measurement signal Eq. (4.12) is what is used to transfer the initial state of system 2 to the final state of system 1 by making a feedback corresponding to $\hat{O}$[(1)], so after the teleportation protocol is done then system 1 is

$$\hat{b}_I^{(1),\mathrm{Tele}} = \hat{b}_I^{(1)}(T) + \hat{O} \tag{4.15}$$
$$= (1 - M_1)\, \hat{b}_I^{(1)}(T) + M_2 \hat{b}_I^{(2)}(0) + \hat{\mathcal{N}}. \tag{4.16}$$

Since we assumed the initial state of system 2 is picked from a Gaussian distribution of coherent states, we can write it as a classical part and a quantum fluctuation part:

$$\hat{b}_I^{(2)}(0) = \bar{b}_{\bar{n}}^{(2)} + \delta\hat{b}_I^{(2)}(0). \tag{4.17}$$

From Section 2.1.3 we defined the figure of merit for the teleportation protocol as

$$F = \frac{1}{1 + N_{\mathrm{add}}^{\mathrm{total}}} \tag{4.18}$$

(1) Since the fidelity is defined as the average fidelity over running the teleportation protocol many time, $\hat{O}$ is a stochastic operator when looking at all the runs. However for a single run of the protocol, one can think of $\hat{O}$ has just being a complex number so that the feedback corresponds to a displacement in phase space.

where the total added noise $N_{\text{add}}^{\text{total}}$ is the variance of the difference between the teleported state and original (classical) state. So we are averaging over both quantum operators and a classical stochastic variable $\bar{b}_{\bar{n}}^{(2)}$, since there are two stochastic processes, the randomness from quantum mechanics i.e. zero-point motion of the oscillators and the random selection of coherent states. Then $N_{\text{add}}^{\text{total}}$ basically tells you how much noise has been added during the protocol.

$$N_{\text{add}}^{\text{total}} = \text{Var}\left[\frac{\hat{b}_I^{(1),\text{Tele}} - \bar{b}_{\bar{n}}^{(2)}}{\sqrt{2}} + \text{H.c.}\right]_{\bar{n}} - \frac{1}{2} \tag{4.19}$$

where the variance is done with both a averaging of quantum states and of a (classical) Gaussian distribution with width $\bar{n}$.

$$\text{Var}\left[\hat{O}\right]_{\bar{n}} = \left\langle \hat{O}^2 \right\rangle_{\bar{n}} - \left\langle \hat{O} \right\rangle_{\bar{n}}^2 \tag{4.20}$$

However, we are working with operators of zero mean, so we just have

$$\text{Var}\left[\hat{O}\right]_{\bar{n}} = \left\langle \hat{O}^2 \right\rangle_{\bar{n}}$$

4.2 Teleportation of a state from a restricted family to an oscillator in a thermal initial state

The teleportation protocol here assumes that Charlie's state, which is the initial state of the second system, comes from a classical Gaussian distribution in the x-p phase space $\exp[-(\langle x\rangle^2 + \langle p\rangle^2)/(2\bar{n})]$ with a width $\bar{n}$. However in the actual implementation of the teleportation protocol it is enough that the state being teleported is from a finite family of states. Also if the state is excited enough it might not behave as a harmonic oscillator. We also assume that Bob's state, the state of the first system is in a known thermal initial state. Here we will be looking at the case, where the states being teleported comes from a finite distribution i.e. finite width $\bar{n}$, and the initial state of the target oscillator also has a finite width given by the thermal occupation number $\bar{n}_1$. We also allow $M_1 \neq 1$ and $M_2 \neq 1$, since in this case it is generally not optimal to have them equal to one[(2)]. It will also be in a realistic setup (see Fig. 3.1) with optical losses, as characterized by the transmission and detection efficiencies $\nu_{\text{opt}} < 1, \eta_{\text{det}} < 1$ and with thermal noise, decay rates $\gamma_1 > 0, \gamma_2 > 0$ and associated thermal bath occupancies n_1 and n_2.

For a given set of system parameters and drive pulses $\Gamma_{1,2}(t)$ there is a optimal filter function $f_{\text{v}}(t)$. For Gaussian quantum system the filter function can be obtained in a similar manner as, e.g., the Kalman filter for the prediction/filtering of a classical signal based on a noisy, continuous measurement over a finite time interval. By discretizing time this amounts then to solve a linear system for the optimal filter f_{v}, as we will see below. For drive pulses, since in the general case it is not possible to analytically the determine $\Gamma_{1,2}(t)$, the method for finding the drive pulses will be to use an ansatz function with

(2) Expect for in the limit when $\int_0^T dt\Gamma_1(t) \to \infty$, then the optimal value for M_1 will approach 1. Independent on $\bar{n}_1$, since the variance of $b^{(1)}(T) + \text{H.C.}$ goes to infinity in this limit.

some adjustable parameters that will be determined numerically. The type of ansatz function will be motivated by theoretical results from Section 4.2.3.

4.2.1 Deriving the Noise and Finding the Optimal Filter

The overall goal of this project is to maximize the fidelity Eq. (4.18), this is equivalent to minimizing the total added noise $N_{\text{add}}^{\text{total}}$. To this end we will seek a analytic expression for $N_{\text{add}}^{\text{total}}$ in terms of the drive pulses $\Gamma_{1,2}(t)$ and the filter function $f_{\text{v}}(t)$, in a form that lends itself to be optimized. Which we then optimize analytically with respect to $f_{\text{v}}(t)$ and get a expression for $N_{\text{add}}^{\text{total}}$ in terms of the optimal filter function $f_{\text{v}}^{\text{opt}}(t)$. However, solving the equation for $f_{\text{v}}^{\text{opt}}(t)$ and minimizing $N_{\text{add}}^{\text{total}}$ with respect to $\Gamma_{1,2}(t)$ cannot be analytically in this case and will have to be done numerically.

When finding an expression of the total noise Eq. (4.19) it is more convenient to write the annihilation operator of system 1 appearing in Eq. (4.16) in terms of it's initial condition, so we can incorporate our knowledge of the initial state of system 1

$$\hat{b}_I^{(1)}(T) = e^{-\int_0^T \mathrm{d}t\check{\gamma}_1(t)}\hat{b}_I^{(1)}(0) + \int_0^T \mathrm{d}t e^{-\int_t^T \mathrm{d}t'\check{\gamma}_1(t')}\hat{f}_1(t) \tag{4.21}$$

and the second term of the RHS is combined with the noise operator $\hat{\mathcal{N}}$ in Eq. (4.16). This lead to the teleported state becoming

$$\hat{b}_I^{(1),\text{Tele}} = (1 - M_1)\, e^{-\int_0^T \mathrm{d}t\check{\gamma}_1(t)}\hat{b}_I^{(1)}(0) + M_2\bar{b}_{\bar{n}}^{(2)} + M_2\delta\hat{b}_I^{(2)}(0) + \hat{\mathcal{N}}.$$

Which is used to determine the total added noise Eq. (4.19):

$$\begin{aligned}
N_{\text{add}}^{\text{total}} &= \text{Var}\left[\frac{\hat{b}_I^{(1),\text{Tele}} - \bar{b}_{\bar{n}}^{(2)}}{\sqrt{2}} + \text{H.c.}\right]_{\bar{n}} - \frac{1}{2} \\
&= (1 - M_1)^2\, e^{-2\int_0^T \mathrm{d}t\check{\gamma}_1(t)}\hat{b}_I^{(1)}(0)\text{Var}\left[\frac{\hat{b}_I^{(1)}(0) + \hat{b}_I^{(1)\dagger}(0)}{\sqrt{2}}\right]_{\bar{n}_1} + (1 - M_2)^2\,\text{Var}\left[\frac{\bar{b}_{\bar{n}}^{(2)} + \bar{b}_{\bar{n}}^{(2)\dagger}}{\sqrt{2}}\right]_{\bar{n}} \qquad (4.22) \\
&+ M_2^2\text{Var}\left[\frac{\hat{b}_I^{(1)}(0) + \hat{b}_I^{(1)\dagger}(0)}{\sqrt{2}}\right]_{\bar{n}} + \text{Var}\left[\frac{\hat{\mathcal{N}} + \hat{\mathcal{N}}^\dagger}{\sqrt{2}}\right] - \frac{1}{2} \qquad (4.23) \\
&= \bar{n}\,(1 - M_2)^2 - \frac{1}{2}\left(1 - M_2^2\right) + \left(\bar{n}_1 + \frac{1}{2}\right)(1 - M_1)^2\, e^{-2\int_0^T \mathrm{d}t\check{\gamma}_1(t)} + N_{\text{add}} \qquad (4.24)
\end{aligned}$$

where we have used for the thermal noise operators

$$\left\langle \hat{f}_{\text{th},x}^{(j)}(t)\hat{f}_{\text{th},x}^{(j)}(t')\right\rangle_{\bar{n}} = \left(n_j + \frac{1}{2}\right)\delta(t - t'),\ \hat{f}_{\text{th},x}^{(j)}(t) = \frac{\hat{f}_{\text{th},j}(t) + \hat{f}_{\text{th},j}^\dagger(t)}{\sqrt{2}}, j \in \{1, 2\} \tag{4.25}$$

where n_j is the occupancy for the thermal bath of the j'th oscillator, and for the optical noise operators we have

$$\left\langle \hat{x}_k^{(j)}(t)\hat{x}_k^{(j)}(t')\right\rangle_{\bar{n}} = \frac{1}{2}\delta(t-t'),\ \hat{x}_k^{(j)}(t) = \frac{\hat{a}_{k,+}^{(j)}(t) + \hat{a}_{k,-}^{(j)\dagger}(t)}{\sqrt{2}\,i} + \text{H.c.}$$

$$\left\langle \hat{p}_k^{(j)}(t)\hat{p}_k^{(j)}(t')\right\rangle_{\bar{n}} = \frac{1}{2}\delta(t-t'),\ \hat{p}_k^{(j)}(t) = \frac{\hat{a}_{k,+}^{(j)}(t) - \hat{a}_{k,-}^{(j)\dagger}(t)}{\sqrt{2}\,i} + \text{H.c.}$$

Where $k \in \{\text{in}\}$ and $j \in \{1,2\}$ or $k \in \{\text{vac}, \text{det}\}$ and $j \in \emptyset$. And

$$\bar{n} = \text{Var}\left[\frac{\bar{b}_{\bar{n}}^{(2)} + \bar{b}_{\bar{n}}^{(2)\dagger}}{\sqrt{2}}\right]_{\bar{n}} \tag{4.26a}$$

$$\bar{n}_1 + \frac{1}{2} = \text{Var}\left[\frac{\hat{b}_I^{(1)}(0) + \hat{b}_I^{(1)\dagger}(0)}{\sqrt{2}}\right]_{\bar{n}_1} \tag{4.26b}$$

are the the width of the classical distribution of coherent input states for the second and the full variance of first system (including ground-state fluctuations) respectively. The last term in Eq. (4.19) is giving by

$$\begin{aligned} N_{\text{add}} &= \text{Var}\left[\frac{\hat{\mathcal{N}} + \hat{\mathcal{N}}^\dagger}{\sqrt{2}}\right] \\ &= \left(\frac{1}{\eta_{\text{det}}} - 1\right)\frac{1}{2}\int_0^T \mathrm{d}t B_{12}^2(t) + \sum_{j=1}^{2}\int_0^T \mathrm{d}t C_j^2(t) \qquad (4.27) \\ &+ 2\gamma_{\text{d},1}\int_0^T \mathrm{d}t D_1^2(t) + 2\gamma_{\text{d},2}\int_0^T \mathrm{d}t A_{2\text{v}}^2(t) \\ &+ \left(1-\nu_{\text{opt}}\right)\frac{1}{2}\int_0^T \mathrm{d}t B_1^2(t) + \left(1-\nu_{\text{opt}}\right)\frac{1}{2}\int_0^T \mathrm{d}t A_{2\text{v}}^2(t)\Gamma_2(t) \qquad (4.28) \end{aligned}$$

and contains all of the noise from optical losses and thermal noise driving the system during the protocol, $\gamma_{\text{d},j} = \gamma_j\left(n_j + 1/2\right)$ is the corresponding thermal decoherence rate of oscillator j. For the definitions of the temporal mode functions B_1, B_{12}, C_j and D_1 in Eq. (4.28) see Appendix C.

The goal is to minimize the noise Eq. (4.24) with respect to filter f_v and the drive pulses $\Gamma_{1,2}$. This is in general a hard problem, since the dependence of the drive pulse is non-linear. However, one may notice that it is quadratic in f_v, which means it can be relatively easily minimized with respect to f_v, since the derivative will be linear in f_v. If we reorganize this equation according to its f_v dependence, then the noise can be written as:

$$N_{\text{add}}^{\text{total}} = \int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' f_\text{v}(t)\mathcal{A}_{\text{sym}}(t,t')f_\text{v}(t') + \int_0^T \mathrm{d}t C(t) f_\text{v}(t) + \tilde{N} \tag{4.29}$$

Where $\mathcal{A}_{\text{sym}}(t,t') = \mathcal{A}_{\text{sym}}(t',t)$ is a symmetric function[3] under the exchange $t \leftrightarrow t'$. Upon discretization of the time coordinates, the first term can be viewed as a "sandwich" product with a vector $\vec{f}$ and matrix $\mathbf{M}$, $\vec{f}^T\mathbf{M}\vec{f}$. The second term is like a dot product between two vectors, $\vec{b} \cdot \vec{f}$ and the last term is a constant with respect to $\vec{f}$. We are going to figure out what $\mathcal{A}_{\text{sym}}$, C and $\tilde{N}$ are in the next section, Section 4.2.2. Since the filter f_{v} is "hidden" inside integrals with other functions, it is not so obvious how one would arrive at this form, although it can be achieved using appropriate changes of integration order. For now we will find the noise in terms of the optimal filter $f_{\text{v}}^{\text{opt}}$ for given drive pulse envelopes $\Gamma_j(t)$. The optimal filter is defined as the one minimizing the total added noise $N_{\text{add}}^{\text{total}}$, so we can take the functional derivative of the noise Eq. (4.29) with respect to f_{v}

$$\left.\frac{\delta N_{\text{add}}^{\text{total}}}{\delta f_{\text{v}}(\tau)}\right|_{f_{\text{v}}^{\text{opt}}} = 2\int_0^T \mathrm{d}t' \mathcal{A}_{\text{sym}}(t,t') f_{\text{v}}^{\text{opt}}(t') + C(t) = 0 \tag{4.30}$$

Since Eq. (4.29) is quadratic in f_{v}, its derivative Eq. (4.30) is linear in $f_{\text{v}}$and results in the discretized form $\mathbf{M}\vec{f} = \vec{b}$, so this is our equation for $f_{\text{v}}^{\text{opt}}$, which can be solved using numerical methods. To find a convenient expression for the total noise evaluated using the optimal filter $f_{\text{v}}^{\text{opt}}$ we multiply Eq. (4.30) by $f_{\text{v}}^{\text{opt}}(t)$ and integrate with respect to t from 0 to T, leading to

$$\int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' f_{\text{v}}^{\text{opt}}(t) \mathcal{A}_{\text{sym}}(t,t') f_{\text{v}}^{\text{opt}}(t') = -\frac{1}{2}\int_0^T \mathrm{d}t C(t) f_{\text{v}}^{\text{opt}}(t). \tag{4.31}$$

Substituting $f_{\text{v}} = f_{\text{v}}^{\text{opt}}$in Eq. (4.29) and using Eq. (4.31) we get the noise

$$N_{\text{add}}^{\text{total}} = \int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' f_{\text{v}}^{\text{opt}}(t) \mathcal{A}_{\text{sym}}(t,t') f_{\text{v}}^{\text{opt}}(t') + \int_0^T \mathrm{d}t C(t) f_{\text{v}}^{\text{opt}}(t) + \tilde{N} \tag{4.32}$$

$$= -\frac{1}{2}\int_0^T \mathrm{d}t C(t) f_{\text{v}}^{\text{opt}}(t) + \int_0^T \mathrm{d}t C(t) f_{\text{v}}^{\text{opt}}(t) + \tilde{N} \tag{4.33}$$

$$= \frac{1}{2}\int_0^T \mathrm{d}t C(t) f_{\text{v}}^{\text{opt}}(t) + \tilde{N} \tag{4.34}$$

We will use this expression in Chapter 5 to calculate the noise numerically after having solved for $f_{\text{v}}^{\text{opt}}(t)$.

(3) We only need to consider a symmetric function since,

$$\int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' f_{\text{v}}(t)\mathcal{A}(t,t') f_{\text{v}}(t') = \frac{1}{2}\left(\int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' f_{\text{v}}(t)\mathcal{A}(t,t') f_{\text{v}}(t') + \int_0^T \mathrm{d}t' \int_0^T \mathrm{d}t f_{\text{v}}(t')\mathcal{A}(t',t) f_{\text{v}}(t)\right)$$
$$= \int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' f_{\text{v}}(t) \frac{\mathcal{A}(t,t') + \mathcal{A}(t',t)}{2} f_{\text{v}}(t')$$

.

4.2.2 Deriving $\mathcal{A}_{\text{sym}}$, C and $\tilde{N}$eriving A, C and N

In order to find $\mathcal{A}_{\text{sym}}$, C and $\tilde{N}$ we notice that Eq. (4.29) is a kind of (exact) Taylor expansion of the noise:

$$\tilde{N} = N_{\text{add}}^{\text{total}}\Big|_{f_{\text{v}}=0} \tag{4.35}$$

$$C(\tau) = \frac{\delta N_{\text{add}}^{\text{total}}}{\delta f_{\text{v}}(\tau)}\Bigg|_{f_{\text{v}}=0} \tag{4.36}$$

$$\mathcal{A}_{\text{sym}}(\tau,\tau') = \frac{1}{2}\,\frac{\delta^2 N_{\text{add}}^{\text{total}}}{\delta f_{\text{v}}(\tau')\delta f_{\text{v}}(\tau)}\Bigg|_{f_{\text{v}}=0} \tag{4.37}$$

We insert into these formulas the expression for the noise in Eq. (4.24) . For the constant term we get:

$$\tilde{N} = \bar{n} - \frac{1}{2} + \left(\bar{n}_1 + \frac{1}{2}\right) e^{-2\int_0^T \mathrm{d}t \check{\gamma}_1(t)} + \int_0^T dt \left(2\gamma_{\text{d},1} + \frac{1}{2}\left(1+\zeta_1^2\right)\Gamma_1(t)\right) e^{-2\int_t^T \mathrm{d}t' \check{\gamma}_1(t')} \tag{4.38}$$

for the "vector" part we get:

$$\begin{aligned} C(s) = &-2\bar{n}\frac{\delta A_{2\text{v}}(0)}{\delta f_{\text{v}}(s)} - 2\sqrt{\nu_{\text{opt}}}\left(\bar{n}_1 + \frac{1}{2}\right)\frac{\delta \tilde{A}_1(T)}{\delta f_{\text{v}}(s)} e^{-2\int_0^T \mathrm{d}t\check{\gamma}_1(t)} \\ &+ \int_0^T dt \left[\sqrt{\Gamma_1(t)}\left\{\frac{\delta C_1(t)}{\delta f_{\text{v}}(s)} + \sqrt{\nu_{\text{opt}}}\zeta_1 A_{12}(t,s)\right\} + 4\gamma_{\text{d},1}\frac{\delta D_1(t)}{\delta f_{\text{v}}(s)}\right] e^{-\int_t^T \mathrm{d}t'\check{\gamma}_1(t')} \\ &+ \zeta_1\sqrt{\nu_{\text{opt}}}\sqrt{\Gamma_1(s)}\,e^{-\int_s^T \mathrm{d}t'\check{\gamma}_1(t')} \end{aligned} \tag{4.39}$$

where

$$A_{12}(t,s) = \zeta_1\left(\sqrt{\Gamma_1(t)}\left\{\frac{\delta \tilde{A}_1(t)}{\delta f_{\text{v}}(s)} - \frac{\delta \tilde{A}_1(T)}{\delta f_{\text{v}}(s)} e^{-\int_t^T \mathrm{d}t'\check{\gamma}_1(t')}\right\} + \sqrt{\Gamma_2(t)}\frac{\delta A_{2\text{v}}(t)}{\delta f_{\text{v}}(s)}\right) \tag{4.40}$$

and for the "matrix" part we get:

$$\begin{aligned} \mathcal{A}_{\text{sym}}(s,s') = &\,\nu_{\text{opt}}\left(\bar{n}_1 + \frac{1}{2}\right) e^{-2\int_0^T \mathrm{d}t\check{\gamma}_1(t)}\frac{\delta \tilde{A}_1(T)}{\delta f_{\text{v}}(s)}\frac{\delta \tilde{A}_1(T)}{\delta f_{\text{v}}(s')} + \frac{1}{2}(2\bar{n}+1)\frac{\delta A_{2\text{v}}(0)}{\delta f_{\text{v}}(s)}\frac{\delta A_{2\text{v}}(0)}{\delta f_{\text{v}}(s')} \\ &+ \frac{1}{2}\int_0^T \mathrm{d}t\left[\left(\frac{1}{\eta_{\text{det}}} - 1\right)\frac{\delta B_{12}(t)}{\delta f_{\text{v}}(s)}\frac{\delta B_{12}(t)}{\delta f_{\text{v}}(s')} + 4\gamma_{\text{d},2}\frac{\delta A_{2\text{v}}(t)}{\delta f_{\text{v}}(s)}\frac{\delta A_{2\text{v}}(t)}{\delta f_{\text{v}}(s')}\right. \\ &+ \left(1-\nu_{\text{opt}}\right)\frac{\delta B_1(t)}{\delta f_{\text{v}}(s)}\frac{\delta B_1(t)}{\delta f_{\text{v}}(s')} + \left(1-\nu_{\text{opt}}\right)\Gamma_2(t)\frac{\delta A_{2\text{v}}(t)}{\delta f_{\text{v}}(s)}\frac{\delta A_{2\text{v}}(t)}{\delta f_{\text{v}}(s')} \\ &\left.+\frac{\delta C_1(t)}{\delta f_{\text{v}}(s)}\frac{\delta C_1(t)}{\delta f_{\text{v}}(s')} + \frac{\delta C_2(t)}{\delta f_{\text{v}}(s)}\frac{\delta C_2(t)}{\delta f_{\text{v}}(s')} + 4\gamma_{\text{d},1}\frac{\delta D_1(t)}{\delta f_{\text{v}}(s)}\frac{\delta D_1(t)}{\delta f_{\text{v}}(s')}\right]. \end{aligned} \tag{4.41}$$

Some of the terms in the function $\mathcal{A}_{\text{sym}}$ contains Dirac delta functions, which will be a problem for the numerical calculations, the terms containing the delta functions are:

$$\begin{aligned} \int_0^T \mathrm{d}t\frac{\delta B_{12}(t)}{\delta f_{\text{v}}(s)}\frac{\delta B_{12}(t)}{\delta f_{\text{v}}(s')} = &\,\delta(s-s') - (\zeta_1+\zeta_2)\left(\sqrt{\Gamma_2(s)}\frac{\delta A_{2\text{v}}(s)}{\delta f_{\text{v}}(s')} + \sqrt{\Gamma_2(s')}\frac{\delta A_{2\text{v}}(s')}{\delta f_{\text{v}}(s)}\right) \\ &+ (\zeta_1+\zeta_2)^2\int_0^T \mathrm{d}t\Gamma_2(t)\frac{\delta A_{2\text{v}}(t)}{\delta f_{\text{v}}(s)}\frac{\delta A_{2\text{v}}(t)}{\delta f_{\text{v}}(s')} \end{aligned} \tag{4.42}$$

$$\int_0^T \mathrm{d}t \frac{\delta B_1(t)}{\delta f_v(s)} \frac{\delta B_1(t)}{\delta f_v(s')} = \delta(s-s') + \zeta_1 \left(\sqrt{\Gamma_2(s)} \frac{\delta A_{2v}(s)}{\delta f_v(s')} + \sqrt{\Gamma_2(s')} \frac{\delta A_{2v}(s')}{\delta f_v(s)} \right) + \zeta_1^2 \int_0^T \mathrm{d}t \Gamma_2(t) \frac{\delta A_{2v}(t)}{\delta f_v(s)} \frac{\delta A_{2v}(t)}{\delta f_v(s')} \tag{4.43}$$

$$\int_0^T \mathrm{d}t \frac{\delta C_2(t)}{\delta f_v(s)} \frac{\delta C_2(t)}{\delta f_v(s')} = \nu_{\text{opt}} \left(\delta(s-s') + A_{12}(s,s') + A_{12}(s',s) + \int_0^T \mathrm{d}t A_{12}(t,s) A_{12}(t,s') \right) \tag{4.44}$$

We can then separate out the delta functions and split the function up into two parts

$$\mathcal{A}_{\text{sym}}(s,s') = \bar{\mathcal{A}}_{\text{sym}}(s,s') + \delta(s-s')\mathcal{A}^{\delta}_{\text{sym}}(s) \tag{4.45}$$

where

$$\begin{aligned}\bar{\mathcal{A}}_{\text{sym}}(s,s') = {}& \nu_{\text{opt}} \left(\bar{n}_1 + \frac{1}{2} \right) e^{-2\int_0^T \mathrm{d}t \check{\gamma}_1(t)} \frac{\delta \tilde{A}_1(T)}{\delta f_v(s)} \frac{\delta \tilde{A}_1(T)}{\delta f_v(s')} + \frac{1}{2} (2\bar{n}+1) \frac{\delta A_{2v}(0)}{\delta f_v(s)} \frac{\delta A_{2v}(0)}{\delta f_v(s')} \\ & \times \frac{1}{2} \int_0^T \mathrm{d}t \left[\left\{ \left(\frac{1}{\eta_{\text{det}}} - 1 \right) (\zeta_1 + \zeta_2)^2 \Gamma_2(t) + \left(1+\zeta_1^2\right) \left(1-\nu_{\text{opt}}\right) \Gamma_2(t) + 4\gamma_{\text{d},2} \right\} \frac{\delta A_{2v}(t)}{\delta f_v(s)} \frac{\delta A_{2v}(t)}{\delta f_v(s')} \right. \\ & \left. + \frac{\delta C_1(t)}{\delta f_v(s)} \frac{\delta C_1(t)}{\delta f_v(s')} + 4\gamma_{\text{d},1} \frac{\delta D_1(t)}{\delta f_v(s)} \frac{\delta D_1(t)}{\delta f_v(s')} + \nu_{\text{opt}} A_{12}(t,s) A_{12}(t,s') \right] \\ & + \frac{1}{2} \left(\left[\left(1-\nu_{\text{opt}}\right) \zeta_1 - \left(\frac{1}{\eta_{\text{det}}} - 1 \right) (\zeta_1+\zeta_2) \right] \left[\sqrt{\Gamma_2(s)} \frac{\delta A_{2v}(s)}{\delta f_v(s')} + \sqrt{\Gamma_2(s')} \frac{\delta A_{2v}(s')}{\delta f_v(s)} \right] \right) \\ & + \frac{1}{2} \nu_{\text{opt}} \left(A_{12}(s,s') + A_{12}(s',s) \right) \end{aligned} \tag{4.46}$$

and

$$\mathcal{A}^{\delta}_{\text{sym}}(s) = \frac{1}{2} \left[\left(\frac{1}{\eta_{\text{det}}} - 1 \right) + \left(1-\nu_{\text{opt}}\right) + \nu_{\text{opt}} \right] = \frac{1}{2\eta_{\text{det}}} \tag{4.47}$$

4.2.3 Teleportation between oscillators in completely unknown initial states

In this section we will take a quick look at a special case where Charlie can teleport a completely arbitrary state and Bob's initial is unknown. This is an interesting case to look at since then the whole problem can be solved analytically and the solutions could serve as inspiration when choosing the ansatz for $\Gamma_{1,2}(t)$. If Charlie wants to teleport any arbitrary state that means the distribution becomes infinitely wide, $\bar{n} \to \infty$, and forces $M_2 = 1$ so that the noise Eq. (4.24) does not diverge. If the initial state of the oscillator being teleported to is unknown means the corresponding distribution also has an infinite width, $\bar{n}_1 \to \infty$, (for a thermal initial state, it is also described by a Gaussian) and forces $M_1 = 1$ so the noise again does not diverge. The analysis of this limit was done in Ref. [5] in the ideal case of no optical losses $\left(\nu_{\text{opt}} = \eta_{\text{det}} = 1\right)$ and no thermal noise ($\gamma_1 = \gamma_2 = 0$), and for a constant drive pulse $\Gamma_1(t) = \Gamma_1$ on the first system without loss of generality. This is possible since if there are no noise processes, then the only processes to set the time scale are the drive pulses and only the relative time between them matters, so we can set one of them to be the reference, meaning it is constant in is own frame. With these assumption we get that the total noise Eq. (4.24) becomes

$$N_{\text{add}}^{\text{total}} = \int_0^T dt' \frac{B_X^2(t') + B_P^2(t')}{2} \tag{4.48}$$

where

$$B_P(t) = f_v(t) + \zeta_1 \left\{ A_{2v}(t)\sqrt{\Gamma_2(t)} + \tilde{A}_1(t)\sqrt{\Gamma_1} \right\} \tag{4.49}$$

$$B_X(t) = A_{2v}(t)\sqrt{\Gamma_2(t)} - \tilde{A}_1(t)\sqrt{\Gamma_1} \tag{4.50}$$

$$A_{2v}(t) = \int_t^T \mathrm{d}t' f_v(t')\sqrt{\Gamma_2(t')}\, e^{\zeta_1 \int_t^{t'} \mathrm{d}t'' \Gamma_2(t'')} \tag{4.51}$$

$$\tilde{A}_1(t) = \int_0^t \mathrm{d}t' f_v(t')\sqrt{\Gamma_1}\, e^{\zeta_1 \Gamma_1 (t'-t)}. \tag{4.52}$$

Then in order to enforce the constraint $M_1 = M_2 \equiv M = 1$ we use a Lagrange multiplier λ

$$\mathcal{I} = N_{\text{add}}^{\text{total}} + \lambda(M - 1) \tag{4.53}$$

this can be minimized analytically with respect to f_v and Γ_2 using variational calculus. The result they obtained was:

$$f_v(t) = \zeta_1 \sqrt{\Gamma_1} \frac{e^{-\zeta_1 \Gamma_1 t}}{\sinh(\zeta_1 \Gamma_1 T)} \tag{4.54}$$

$$\Gamma_2(t) = \Gamma_1 \frac{\sinh^2[\zeta_1 \Gamma_1 (t - T)]}{\sinh^2(\zeta_1 \Gamma_1 T) - \sinh^2[\zeta_1 \Gamma_1 (t - T)]} \tag{4.55}$$

$$N_{\text{add}}^{\text{total}} = \frac{\zeta_1}{1 - e^{-2\zeta_1 \Gamma_1 T}} \tag{4.56}$$

We see that for the ideal case it is possible to solve the problem analytically, and get that the filter is a decaying exponential and the drive pulse is also a decaying function however it diverges as $t \to 0$. For the noise we see that if the light-interaction type for the first system is an equal combination of beamsplitter and two-mode-squeezing interactions, i.e. $\zeta_1 = 0$, we see that the noise $N_{\text{add}}^{\text{total}}$, after having used l'Hôpitals rule, is $N_{\text{add}}^{\text{total}} = \frac{1}{2\Gamma_1 T}$ which goes to zero as $\Gamma_1 T \to \infty$. If it is blue detuned i.e. $\zeta_1 < 0$, then again $N_{\text{add}}^{\text{total}} \to 0$ as $\Gamma_1 T \to \infty$. So for the ideal case it is possible to make a perfect teleportation protocol. See that in the case where one of the drive pulses is constant (Γ_1 in this case), the other ($\Gamma_2(t)$) is an decaying function. Although in the ideal case here $\Gamma_2(t)$ diverges at $t = 0$ and therefore cannot be implemented neither physically nor numerical. It can still serves as inspiration when analyzing the problem numerically, like having one of the drive pulses be decaying, and if the function is modified a bit can also be used as an ansatz.

CHAPTER 5

Numerical optimization of teleportation

> Beep-bee-bee-boop-bee-doo-weep.
>
> R2-D2
>
> - *Star Wars,* George Lucas

In order to analyze the problem further we need to use numerical methods, since the equation for the optimal filter, Eq. (4.30) and optimizing of the drive pulses $\Gamma_{1,2}(t)$, cannot be solved analytically. First, we are going to take our equations for the total added noise $N_{\text{add}}^{\text{total}}$ and optimal filter function $f_{\text{v}}^{\text{opt}}$ derived in a previous section, Section 4.2, and discretize them. Then we will be looking at numerically minimizing $N_{\text{add}}^{\text{total}}$ by using four different ansatz functions for $\Gamma_{1,2}(t)$ and see how the fidelity changes. Since we will not be searching the entire space of continuous functions, we are not going to be finding the global optimum. The numerical calculations are done in the programming language Python and the minimization is done using the library SciPy. The run time of pair of ansatz is about 5-8 hours starting with discretizing the time into 60 points. The code can be found on https://github.com/yodaqwq/Quantum-teleportation.git

5.1 Discretizing the problem

In order to treat the problem with numerical methods, we will first make it dimensionless and then make the dimensionless time discrete. Taking the equation for the total noise Eq. (4.29) and making the substitution $u^{(\prime)} = \frac{t^{(\prime)}}{T}$, $\mathrm{d}t^{(\prime)} = T\mathrm{d}u^{(\prime)}$ we then get

$$N_{\text{add}}^{\text{total}} = \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}u' \tilde{f}_{\text{v}}(u)\tilde{\mathcal{A}}_{\text{sym}}(u,u')\tilde{f}_{\text{v}}(u') + \int_0^1 \mathrm{d}u \tilde{C}(u)\tilde{f}_{\text{v}}(u) + \tilde{N} \tag{5.1}$$

Where we have defined the dimensionless functions

$$\tilde{f}_{\mathrm{v}}(u) \equiv \sqrt{T} f_{\mathrm{v}}(Tu) \tag{5.2}$$

$$\tilde{\mathcal{A}}_{\mathrm{sym}}(u,u') \equiv T\mathcal{A}_{\mathrm{sym}}(Tu,Tu') \tag{5.3}$$

$$\tilde{C}(u) \equiv \sqrt{T}\, C(Tu) \tag{5.4}$$

Then to find the optimal filter we use Eq. (4.30) in its dimensionless form

$$2\int_0^1 \mathrm{d}u' \tilde{\mathcal{A}}_{\mathrm{sym}}(u,u')\tilde{f}_{\mathrm{v}}^{\mathrm{opt}}(u') + \tilde{C}(u) = 0 \tag{5.5}$$

we then insert $\tilde{\mathcal{A}}_{\mathrm{sym}}$ Eq. (4.45) in it's dimensionless form

$$\tilde{\mathcal{A}}^{\delta}_{\mathrm{sym}}(u)\tilde{f}_{\mathrm{v}}^{\mathrm{opt}}(u) + \int_0^1 \mathrm{d}u' \tilde{\bar{\mathcal{A}}}_{\mathrm{sym}}(u,u')\tilde{f}_{\mathrm{v}}^{\mathrm{opt}}(u') = -\frac{1}{2}\tilde{C}(u) \tag{5.6}$$

and discretize it with respect to the dimensionless time coordinate

$$\tilde{\mathcal{A}}^{\delta}_{\mathrm{sym}}(u_i)\tilde{f}_{\mathrm{v}}^{\mathrm{opt}}(u_i) + \sum_{j=0}^{N} h_j \tilde{\bar{\mathcal{A}}}_{\mathrm{sym}}(u_i,u_j)\tilde{f}_{\mathrm{v}}^{\mathrm{opt}}(u_j) = -\frac{1}{2}\tilde{C}(u_i) \tag{5.7}$$

where we have switched from continuous variables u to discrete variables u_i with $i \in \{0,\ldots,N\}$ and converted the integral to a Riemann sum with a (non-)uniform step-size h_j. We can then rewrite is as:

$$the\tilde{\mathcal{A}}^{\delta}_{\mathrm{sym}}(u_i)\,(h_i)^{-1}\left(h_i\tilde{f}_{\mathrm{v}}^{\mathrm{opt}}(u_i)\right) + \sum_{j=0}^{N}\tilde{\bar{\mathcal{A}}}_{\mathrm{sym}}(u_i,u_j)\left(h_j\tilde{f}_{\mathrm{v}}^{\mathrm{opt}}(u_j)\right) = -\frac{1}{2}\tilde{C}(u_i) \Leftrightarrow \tag{5.8}$$

$$\sum_{j=0}^{N}\tilde{\boldsymbol{\mathcal{A}}}^{\mathrm{sym},\delta}_{ij}\sum_{l=0}^{N}[\mathbf{h}^{-1}]_{jl}\,\vec{\tilde{f}}^{\mathrm{opt}}_{\mathrm{v},l} + \sum_{j=0}^{N}\tilde{\bar{\boldsymbol{\mathcal{A}}}}^{\mathrm{sym}}_{ij}\,\vec{\tilde{f}}^{\mathrm{opt}}_{\mathrm{v},j} = \vec{\tilde{C}}_i \Leftrightarrow \tag{5.9}$$

$$\underbrace{\left(\tilde{\boldsymbol{\mathcal{A}}}^{\mathrm{sym},\delta}\mathbf{h}^{-1} + \tilde{\bar{\boldsymbol{\mathcal{A}}}}^{\mathrm{sym}}\right)}_{\mathbf{M}}\vec{\tilde{f}}^{\mathrm{opt}}_{\mathrm{v}} = \vec{\tilde{C}} \Leftrightarrow \tag{5.10}$$

$$\vec{\tilde{f}}^{\mathrm{opt}}_{\mathrm{v}} = \mathbf{M}^{-1}\vec{\tilde{C}} \tag{5.11}$$

where the entries of the (discrete) matrices and vectors are

$$\tilde{\bar{\boldsymbol{\mathcal{A}}}}^{\mathrm{sym}}_{ij} \equiv \tilde{\bar{\boldsymbol{\mathcal{A}}}}_{\mathrm{sym}}(u_i,u_j) \tag{5.12}$$

$$\tilde{\boldsymbol{\mathcal{A}}}^{\mathrm{sym},\delta}_{ij} \equiv \delta_{ij}\tilde{\mathcal{A}}^{\delta}_{\mathrm{sym}}(u_i), \tag{5.13}$$

$$\mathbf{h}_{ij} \equiv \delta_{ij}h_i \tag{5.14}$$

$$\vec{\tilde{C}}_i \equiv -\frac{1}{2}\tilde{C}(u_i), \tag{5.15}$$

$$\vec{\tilde{f}}^{\mathrm{opt}}_{\mathrm{v},i} \equiv \sum_{k=0}^{N}\mathbf{h}_{ik}\tilde{f}_{\mathrm{v}}^{\mathrm{opt}}(u_k) = h_i\tilde{f}_{\mathrm{v}}^{\mathrm{opt}}(u_j) \tag{5.16}$$

We can make Eq. (4.34) for the added noise resulting from applying the optimal filter dimensionless and discretize it:

$$N_{\text{add}}^{\text{total}} = \frac{1}{2}\int_0^1 \mathrm{d}u\tilde{C}(u)\tilde{f}_{\mathrm{v}}^{\text{opt}}(u) + \tilde{N} \tag{5.17}$$

$$= \frac{1}{2}\vec{\tilde{C}}\vec{\tilde{f}}_{\mathrm{v}}^{\text{opt}} + \tilde{N} \tag{5.18}$$

We then have a method for a given pair of drive pulses $\Gamma_j(t)$ and experimental parameters to find the optimal filter Eq. (5.11) and calculate the total noise Eq. (5.18) and in turn the fidelity Eq. (4.18).

5.2 Numerical Results

We are now finally in a position to actually calculate the fidelity using the results from the previous section. For a specific set of drive pulses $\Gamma_{1,2}(t)$ and experimental parameters, see Table 5.1 for all the different parameters. It is assume the second system is in the thermal ground state and slightly red detuned, the first system is in a very excited thermal state and is blue detuned and the protocol takes a relative long amount of time compared to the drive pulses. The optimal filter $\vec{\tilde{f}}_{\mathrm{v}}^{\text{opt}}$ can be obtain using Eq. (5.11) and the total added noise is then calculated using Eq. (5.18), from which one can easily get the fidelity $F = \frac{1}{1+N_{\text{add}}^{\text{total}}}$. Here we will look at four pairs of ansatz for the drive pulses $\Gamma_{1,2}$ and using the same parameters in all four cases. The drive pulses are subject to some physically constrains, e.g. the pulses has to be less that the maximum value possible in the lab. First however, we will quickly look at the physical system these calculations was inspired from.

5.2.1 Physical setup

This project is inspired by the experiment done in [11]. Here Bob corresponds to an oscillating membrane in a optical cavity, with vibration modes as degrees of freedom. Charlie corresponds to an ensemble of cesium atoms in a magnetic field with a collective spin as degrees of freedom and Alice is a traveling light field with amplitude and phase quadratures as degrees of freedom. See Fig. 5.1 for an idealized version of the setup.

5.2.2 Numerical calculations

The gist of the algorithm is to make ansatzes for the drive pulses of the two systems $\Gamma_{1,2}(t)$, with some number of adjustable parameters. Then optimize those parameter by minimizing the noise. For the basic structure of the program see the flowchart in Fig. 5.2. The program starts with defining the parameters characterizes the problem and the ansatzes for $\Gamma_{1,2}(t)$(see Table 5.1). An initial guess for the parameters of the ansatzes is made and the dimensionless time variable is discretized either uniformly or not, and a matrix $\mathbf{h}$ with the step-sizes on the diagonal is defined. Next the quantities $\mathbf{M} = \tilde{\tilde{\mathcal{A}}}^{\text{sym}} + \tilde{\mathcal{A}}^{\text{sym},\delta}\mathbf{h}^{-1}$, $\vec{\tilde{C}}$ and $\tilde{N}$ are calculated, see Section 5.1 and Section 4.2.2 for how they are defined. This is used to calculate the optimal filter

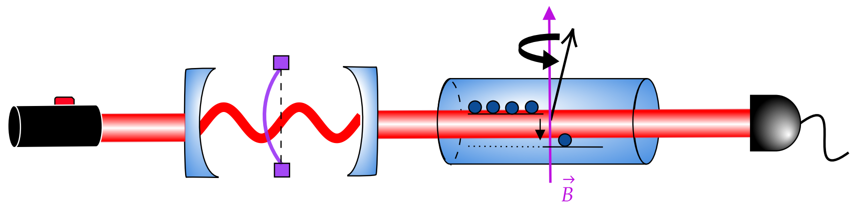

Figure 5.1: Setup of the physical system. From left to right: A laser light field, an optomechanical cavity with a membrane in the middle, a glass cell with cesium atoms, acting like a large collective spin, with a constant magnetic field over it and finally a detector. Adapted from [1].

$\vec{\tilde{f}}_v^{\text{opt}} = \mathbf{M}^{-1}\vec{\tilde{C}}$. Then the program checks if the array containing the values of $\vec{\tilde{f}}_v^{\text{opt}}$ is "smooth" enough, by looking at adjacent values and calculating the relative difference. If this is larger than a predefined tolerance more points will be added to the discretized time array and the optimal filter is calculated again. Then the total added noise, $N_{\text{add}}^{\text{total}} = \frac{1}{2}\vec{\tilde{C}}\vec{\tilde{f}}_v^{\text{opt}} + \tilde{N}$ is calculated and checked if this is a minimum, if yes then the program is done and if not then new a guess for the ansatz parameters is made and the optimal filter is calculated again. The numerical calculations are done using the ansatzes and values for the system parameters in Table 5.1.

For the first run, we took inspiration from the ideal case in Section 4.2.3 by having one of the drive pulses be constant in this case Γ_2 and the other be a decaying function. To start with the simplest decaying function, an exponential, was used as an ansatz (written in terms of suitable dimensionless parameter combinations)

$$\Gamma_1(t)T = (aT)\, e^{(r_1 T)\left(\frac{t}{T}\right)}. \tag{5.19}$$

Here the amplitude was fixed to the maximum experimental value $aT = \Gamma_{1,\text{max}}T$ and the rate $r_1 < 0$ was allowed to vary. Here a minimum was found at $r_1 T = -13.16$ with a fidelity of 0.58 which is just above the classical limit of $\frac{5}{9} \approx 0.56$ for $\bar{n} = 4$ see Eq. (2.12). See Fig. 5.3 for plots of the drive pulses and filter.

For the second run, as an extension of the previous ansatz, we consider for $\Gamma_1(t)$ the combination of a decaying and a rising exponential.

$$\Gamma_1(t)T = (aT)\, e^{(r_1 T)\left(\frac{t}{T}\right)} + (bT)\, e^{(r_2 T)\left(\frac{t}{T}\right)} \tag{5.20}$$

and this time all four parameters were allowed to vary. However, with some constrains:

$$0 < \Gamma_1(t) \le \Gamma_{1,\text{max}}, \quad \forall t \in [0, T] \tag{5.21a}$$

$$r_1 < 0 \tag{5.21b}$$

$$r_2 > 0 \tag{5.21c}$$

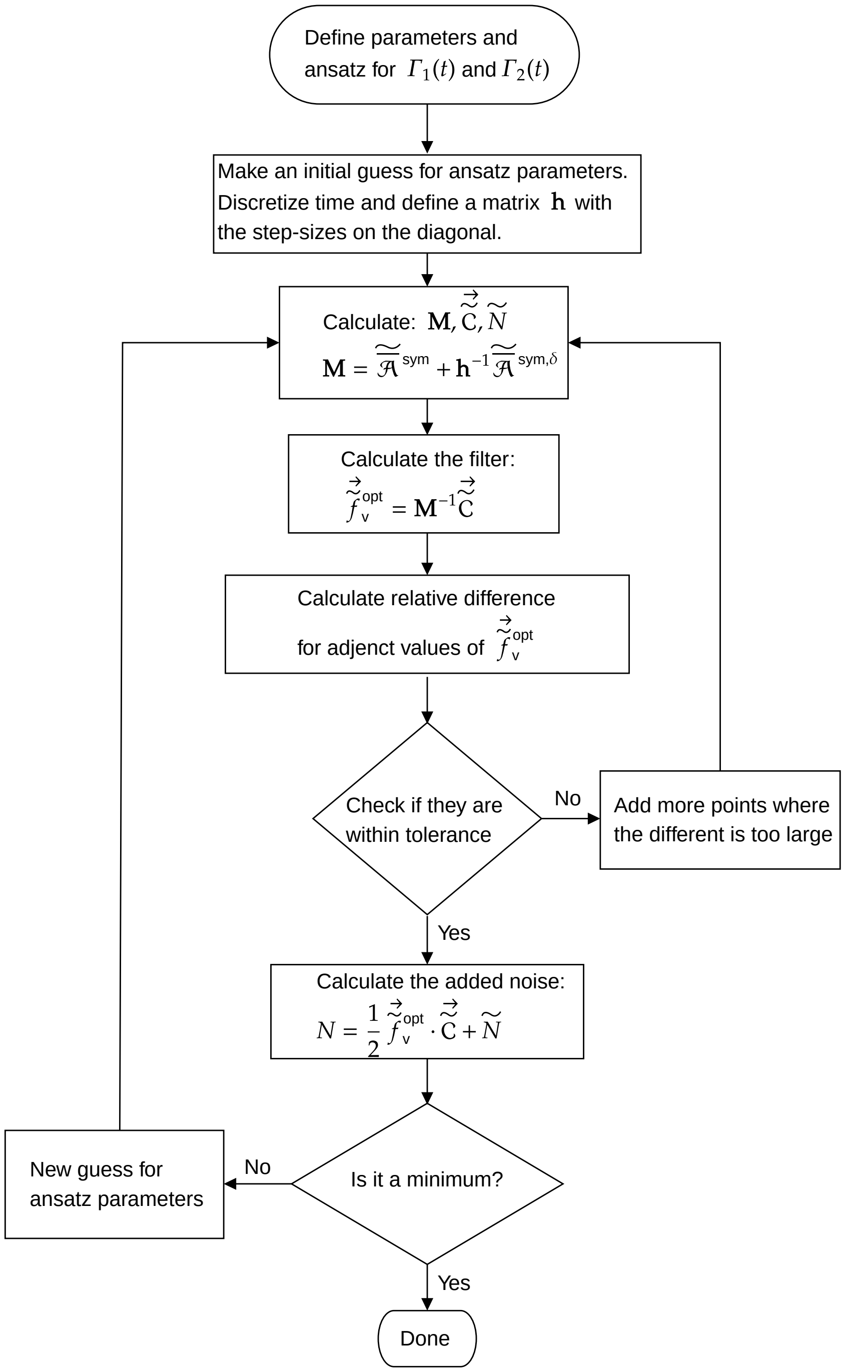

Figure 5.2: Flowchart of the optimization algorithm of the program. For each new guess of parameters for the ansatz the program finds the optimal $\vec{\tilde{f}}_{\text{v}}^{\text{opt}}$ and it also iterative refines the discretized time.

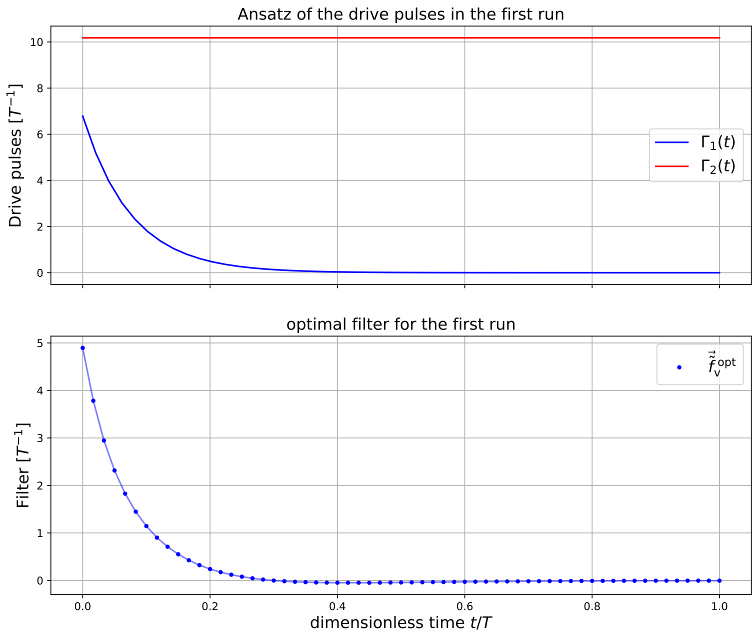

Figure 5.3: Drive pulses and filter for the first run.

The solution found for this ansatz is basically the same as the previous one with $b \approx 0$, $aT \approx \Gamma_{1,\max}T$ and $r_1T = -13.38$.

For the third run, we then tried an ansatz directly inspired by the ideal case Eq. (4.55). However, it has been modified so it does not diverge at $t = 0$

$$\Gamma_1(t)T = \frac{\sinh^2\left(r_1T\left(1-\frac{t}{T}\right)\right)}{\sinh^2\left(r_1T\right)\left(1+\frac{1}{aT}\right)-\sinh^2\left(r_1T\left(1-\frac{t}{T}\right)\right)}. \tag{5.22}$$

With its maximum at $t = 0$, $\Gamma_1(0)T = aT$. Again the amplitude was fixed to the maximum $aT = \Gamma_{1,\max}T$ and the rate r_1 was adjustable. Here the sign of r_1 does not matter since $\sinh^2(x)$ is even. This ansatz turns out to be not quite as good as the exponential ansatz, with a fidelity of 0.56 with $r_1T = 2.54$. See Fig. 5.4 for plots of the drive pulses and filter.

In the ideal case it was sufficient to only have one of the drive pulses to be time dependent, since the only processes to set the time scale were the drive pulses. However this is not the case when we include thermal noise since it introduces a separate time scale and it is therefore not sufficient just to optimize the relative time scale between the drive pulses.

So for the fourth run, both drive pulses was taken to be time dependent and a single exponential function was used for both $\Gamma_1(t)$ and $\Gamma_2(t)$

$$\Gamma_1(t)T = (aT)\, e^{(r_1T)\left(\frac{t}{T}\right)} \tag{5.23a}$$

$$\Gamma_2(t)T = (bT)\, e^{(r_2T)\left(\frac{t}{T}-1\right)} \tag{5.23b}$$

$$0 < \Gamma_j(t) \leq \Gamma_{j,\max}, \quad \forall t \in [0,T],\ j \in \{1,2\}. \tag{5.23c}$$

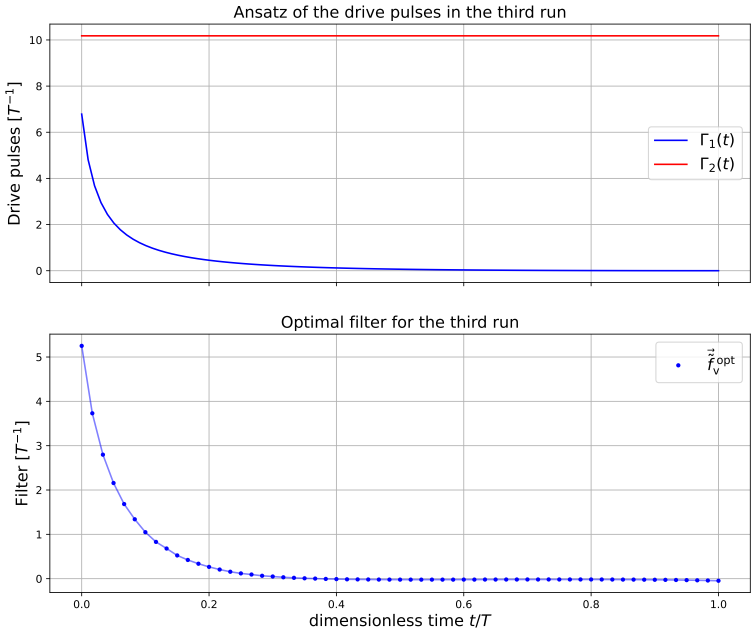

Figure 5.4: Drive pulses and filter for the second run.

With $\Gamma_1(t)$ taken to be decaying, $r_1 < 0$ and $\Gamma_2(t)$ to be growing, $r_2 > 0$. The amplitudes a and b where fixed to their maximum value consistent with $\Gamma_j(t) \leq \Gamma_{j,\max}$ Eq. (5.23c), $aT = \Gamma_{1,\max}T$, $bT = \Gamma_{2,\max}T$. The reason for the shift in $\Gamma_2(t)T$ is so it reaches the value of bT at the end of the protocol. This ansatz worked much better than the previous ansatzes, with a fidelity of 0.79 for $r_1T = -1.49$ and $r_2T = 5.59$, which is significantly above from the classical limit of 0.56 for $\bar{n} = 4$. See Fig. 5.5 for plots of the drive pulses and filter.

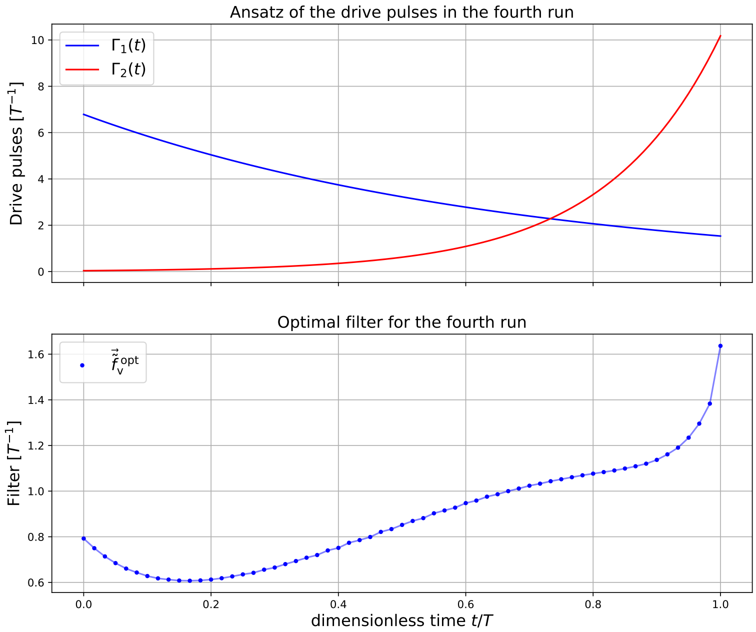

Figure 5.5: Drive pulses and filter for the fourth run.

	I	II	III	IV
$\Gamma_1(t)$	$ae^{r_1 t}$	$ae^{r_1 t} + be^{r_2 t}$	$\frac{1}{T}\frac{\sinh^2(r_1(T-t))}{\sinh^2(r_1 T)(1+1/aT)-\sinh^2(r_1(T-t))}$	$ae^{r_1 t}$
$\Gamma_2(t)$	$\Gamma_{2,\max}$	$\Gamma_{2,\max}$	$\Gamma_{2,\max}$	$be^{r_2(t-T)}$
r_1	$-13.16/T$	$-13.38/T$	$2.54/T$	$-1.49/T$
a	$\Gamma_{1,\max} = 6.79/T$	$6.79/T$	$\Gamma_{1,\max}$	$6.79/T$
r_2	N/A	$6.2/T$	N/A	$5.59/T$
b	N/A	$-2.4 \times 10^{-14}/T$	N/A	$\Gamma_{2,\max} = 10.18/T$
F	0.58	0.58	0.56	0.79
F_{cl}	0.56	0.56	0.56	0.56
$\Gamma_{1,\max}$	$2\pi \times 40$kHz	$2\pi \times 40$kHz	$2\pi \times 40$kHz	$2\pi \times 40$kHz
$\Gamma_{2,\max}$	$2\pi \times 60$kHz	$2\pi \times 60$kHz	$2\pi \times 60$kHz	$2\pi \times 60$kHz
γ_2	$2\pi \times 85$Hz	$2\pi \times 85$Hz	$2\pi \times 85$Hz	$2\pi \times 85$Hz
n_2	0	0	0	0
ζ_2	0.03	0.03	0.03	0.03
γ_1	$2\pi \times 2.1$mHz	$2\pi \times 2.1$mHz	$2\pi \times 2.1$mHz	$2\pi \times 2.1$mHz
n_1	173×10^3	173×10^3	173×10^3	173×10^3
$\bar{n}_1$	1/2	1/2	1/2	1/2
$\bar{n}$	4	4	4	4
ζ_1	−0.3	−0.3	−0.3	−0.3
ν_{opt}	75%	75%	75%	75%
η_{det}	95%	95%	95%	95%
T	$27\mu s$	$27\mu s$	$27\mu s$	$27\mu s$

Table 5.1: Table of parameters. The parameters are inspired by the experimental parameters at QUANTOP NBI. All values are rounded to two decimal places.

Chapter 6

Conclusion & Outlook

> “The difference between screwing around and science is writing it down.
>
> Adam Savage

6.1 Conclusion

In this project we have examined quantum teleportation between two quantum oscillators coupled to a unidirectional light field in the presence of thermal noise and optical losses. This was done with a combination of analytical and numerical methods. An analytical expression for the total added noise was found and subsequently attempted minimized using numerical methods. This involved using ansatzes for the drive pulses , whereas the corresponding optimal filter function for determining the final feedback could be determined exactly (within numerical precision). Four runs were done with different pairs of ansatzes for the two systems. It was found that the last ansatz of two exponentials gave the highest fidelity of 0.79, where the corresponding classical fidelity bound is 0.56. Which bodes well for a future teleportation experiment of QUANTOP's macroscopic hybrid Fig. 5.1. In the future it is worth trying more ansatzes and also different experimental parameters to see how the fidelity will change as function of e.g. temperature or the duration T of the experiment. The program made here can serve as a useful tool for optimizing drive-pulses of future teleportation experiments with the purpose of maximizing the fidelity under the presence of the aforementioned sources of noise.

Appendix

A

Proof of the No Cloning Theorem

Suppose we have a arbitrary unknown state $|\psi\rangle$which we wish to make a copy of. For simplicity we will do it for a two level system, so an arbitrary state is $|\psi\rangle = a|0\rangle + b|1\rangle$ .To do so we suppose there exists a unitary operator U such that $U\left(|\psi\rangle_A |e\rangle_B\right) = |\psi\rangle_A |\psi\rangle_B$. We can proof there exists no such operator, start by using U first:

$$U\left(|\psi\rangle_A |e\rangle_B\right) = |\psi\rangle_A |\psi\rangle_B = (a|0\rangle_A + b|1\rangle_A)(a|0\rangle_B + b|1\rangle_B) \tag{A.1}$$

$$= a^2|0\rangle_A|0\rangle_B + ab|0\rangle_A|1\rangle_B + ba|1\rangle_A|0\rangle_B + b^2|1\rangle_A|1\rangle_B \tag{A.2}$$

Then write out $|\psi\rangle_A$before using U:

$$U\left(|\psi\rangle_A |e\rangle_B\right) = U\left((a|0\rangle_A + b|1\rangle_A)|e\rangle_B\right) = U\left(a|0\rangle_A|e\rangle_B + b|1\rangle_A|e\rangle_B\right) \tag{A.3}$$

$$= (a|0\rangle_A|0\rangle_B + b|1\rangle_A|1\rangle_B) \tag{A.4}$$

$$\neq a^2|0\rangle_A|0\rangle_B + ab|0\rangle_A|1\rangle_B + ba|1\rangle_A|0\rangle_B + b^2|1\rangle_A|1\rangle_B \tag{A.5}$$

$$= U\left(|\psi\rangle_A|e\rangle_B\right) \Rightarrow \tag{A.6}$$

$$U\left(|\psi\rangle_A|e\rangle_B\right) \neq U\left(|\psi\rangle_A|e\rangle_B\right) \tag{A.7}$$

Which is an obvious contradiction.

Appendix

B

Adiabatic elimination of light in a cavity

Using Eq. (3.10) for the case of light interacting with an oscillator in a cavity, $\hat{S} = \hat{a}, \hat{R}_{\text{in}} = \hat{a}_{\text{in}}, \gamma \longrightarrow \kappa$ and

$$\hat{H}_i = \hat{H}_{0,i} + \tilde{\hat{H}}_{\text{int},i} \tag{B.1}$$

$$\hat{H}_{0,i} = \Omega \hat{b}_i^\dagger \hat{b}_i + \omega_{\text{cav},i} \hat{a}_i^\dagger \hat{a}_i \tag{B.2}$$

$$\begin{aligned}\tilde{\hat{H}}_{\text{int}} &= g_0 \tilde{\hat{a}}^\dagger \tilde{\hat{a}} \left(\hat{b} + \hat{b}^\dagger\right) \\ &\longrightarrow g_0 \left(\alpha^* \hat{a} + \alpha \hat{a}^\dagger\right)\left(\hat{b} + \hat{b}^\dagger\right)\end{aligned}$$

where we have linearized the interaction Hamiltonian. We have the EOM of the light in rotating frame with respect to the laser frequency ω_L , $\Delta = \omega_{\text{cav}} - \omega_L$ is

$$\begin{aligned}\dot{\hat{a}} &= (-i\Delta - \kappa)\hat{a} - i g_0 \alpha \left(\hat{b} + \hat{b}^\dagger\right) + \sqrt{2\kappa}\,\hat{a}_{\text{in}} \\ &= (-i\Delta - \kappa)\hat{a} - i g_0 \alpha \left(\hat{b}_I e^{-i\Omega t} + \hat{b}_I^\dagger e^{i\Omega t}\right) + \sqrt{2\kappa}\,\hat{a}_{\text{in}}\end{aligned}$$

where $\hat{b}_I(t)$ varies slowly compared to $1/\kappa$ the solution is then

$$\begin{aligned}\hat{a}(t) = {}& e^{(-i\Delta-\kappa)(t-t_0)}\hat{a}(t_0) - i g_0 \alpha e^{(-i\Delta-\kappa)t} \int_{t_0}^{t} dt' \Big(\hat{b}_I(t') e^{(i[\Delta-\Omega]+\kappa)t'} + \hat{b}_I^\dagger(t') e^{(i[\Delta+\Omega]+\kappa)t'} \\ & + \sqrt{2\kappa} \int_{t_0}^{t} dt' e^{(-i\Delta-\kappa)(t-t')} \hat{a}_{\text{in}}(t') \\ \xrightarrow{t_0 \to -\infty \text{ and } \hat{b}_I(t') \approx \hat{b}_I(t)} & - i g_0 \alpha \left(\frac{\hat{b}_I(t) e^{-i\Omega t}}{\kappa - i[\Omega - \Delta]} + \frac{\hat{b}_I^\dagger(t) e^{i\Omega t}}{\kappa - i[-\Omega - \Delta]}\right) + \sqrt{2\kappa} \int_{-\infty}^{t} dt' e^{(-i\Delta-\kappa)(t-t')} \hat{a}_{\text{in}}(t')\end{aligned}$$

Then using the input-output relation Eq. (3.37)

$$\begin{aligned}\hat{a}_{\text{out}}(t) &= -\hat{a}_{\text{in}}(t) + \sqrt{2\kappa}\,\hat{a}(t) \\ &= -\hat{a}_{\text{in}}(t) + 2\kappa \int_{-\infty}^{t} dt' e^{(-i\Delta-\kappa)(t-t')} \hat{a}_{\text{in}}(t') - i\sqrt{2\kappa}\, g_0 \alpha \left(\frac{\hat{b}_I(t) e^{-i\Omega t}}{\kappa - i[\Omega - \Delta]} + \frac{\hat{b}_I^\dagger(t) e^{i\Omega t}}{\kappa - i[-\Omega - \Delta]}\right).\end{aligned}$$

Then we can the input-output relation for the sidebands by using

$$a_{\text{in/out},\pm}(t) = e^{\pm i\Omega t} a_{\text{in/out}}(t)$$

and doing a RWA (discarding terms with $e^{\pm 2i\Omega t}$) and assuming $\mu^2, \nu^2 \ll \Omega$ and α is real and positive

$$\begin{aligned}
\hat{a}_{\text{out},+}(t) \equiv \hat{a}_{\text{out}}(t)e^{i\Omega t} &= -\hat{a}_{\text{in},+}(t) + 2\kappa e^{i\Omega t}\int_{-\infty}^{t} dt' e^{(-i\Delta-\kappa)(t-t')}\hat{a}_{\text{in}}(t') \\
&- i\sqrt{2\kappa}\, g_0\alpha\left(\frac{\hat{b}_I(t)}{\kappa - i[\Omega-\Delta]} + \frac{\hat{b}_I^\dagger(t)e^{2i\Omega t}}{\kappa - i[-\Omega-\Delta]}\right) \\
&\approx e^{2i\theta_{\text{cav}}(\Omega)}\hat{a}_{\text{in},+}(t) - i\overbrace{\left|\frac{\sqrt{2\kappa}\, g_0\alpha}{\kappa - i[\Omega-\Delta]}\right|}^{\mu\equiv} e^{i\theta_{\text{cav}}(\Omega)}\hat{b}_I(t)
\end{aligned}$$

and similary

$$\begin{aligned}
\hat{a}_{\text{out},-}(t) \equiv \hat{a}_{\text{out}}(t)e^{-i\Omega t} &= -\hat{a}_{\text{in},-}(t) + 2\kappa e^{-i\Omega t}\int_{-\infty}^{t} dt' e^{(-i\Delta-\kappa)(t-t')}\hat{a}_{\text{in}}(t') \\
&- i\sqrt{2\kappa}\, g_0\alpha\left(\frac{\hat{b}_I(t)e^{-2i\Omega t}}{\kappa - i[\Omega-\Delta]} + \frac{\hat{b}_I^\dagger(t)}{\kappa - i[-\Delta-\Omega]}\right) \\
&\approx e^{2i\theta_{\text{cav}}(-\Omega)}\hat{a}_{\text{in},-}(t) - i\overbrace{\left|\frac{\sqrt{2\kappa}\, g_0\alpha}{\kappa - i[-\Omega-\Delta]}\right|}^{\nu\equiv} e^{i\theta_{\text{cav}}(-\Omega)}\hat{b}_I^\dagger(t)
\end{aligned}$$

where

$$\theta_{\text{cav}}(\omega) \equiv \text{Arg}\left[\frac{\kappa}{\kappa - i(\omega-\Delta)}\right]$$

for now we have

$$\begin{aligned}
\hat{a}_{\text{out},+}(t) &= e^{2i\theta_{\text{cav}}(\Omega)}\hat{a}_{\text{in},+}(t) - i\mu e^{i\theta_{\text{cav}}(\Omega)}\hat{b}_I(t) \\
\hat{a}_{\text{out},-}(t) &= e^{2i\theta_{\text{cav}}(-\Omega)}\hat{a}_{\text{in},-}(t) - i\nu e^{i\theta_{\text{cav}}(-\Omega)}\hat{b}_I^\dagger(t)
\end{aligned}$$

Then the EOM of the oscillators is

$$\begin{aligned}
\dot{\hat{b}}(t) &= [-i\Omega - \gamma]\hat{b}(t) - ig_0[\alpha^*\hat{a}(t) + \alpha\hat{a}^\dagger(t)] \Rightarrow \\
\dot{\hat{b}}_I(t) &= -\gamma\hat{b}_I(t) - ig_0e^{i\Omega t}[\alpha^*\hat{a}(t) + \alpha\hat{a}^\dagger(t)] \\
&\overset{\text{RWA}}{\approx} -\gamma\hat{b}_I(t) - \frac{g_0\alpha}{\sqrt{2\kappa}}e^{i\Omega t}(\mu e^{i\theta_{\text{cav}}(\Omega)} - \nu e^{-i\theta_{\text{cav}}(-\Omega)})\hat{b}_I(t) + (\text{opt. noise}) \\
&= -\gamma\hat{b}_I(t) - \frac{(g_0\alpha)^2}{\kappa}\left(\frac{\kappa}{\kappa - i[\Omega-\Delta]} - \frac{\kappa}{\kappa + i[-\Omega-\Delta]}\right)\hat{b}_I(t) + (\text{opt. noise})
\end{aligned}$$

The optical broadening arises from the real part of

$$\left(\frac{\kappa}{\kappa - i[\Omega-\Delta]} - \frac{\kappa}{\kappa + i[-\Omega-\Delta]}\right),$$

whereas its imaginary part gives the optical spring shift. We here focus on the optical broadening and assume that the optical spring shift is compensated by other means and/or absorbed in a redefined Ω.

$$\begin{aligned}\dot{\hat{b}}_I(t) &= -\left[\gamma + \frac{\mu^2 - \nu^2}{2}\right]\hat{b}_I(t) - ig_0\alpha\sqrt{2\kappa}\left[\int_{-\infty}^{t} dt' e^{(i(\Omega-\Delta)-\kappa)(t-t')}\hat{a}_{\text{in},+}(t')\right.\\ &\quad \left. + \int_{-\infty}^{t} dt' e^{(i(\Omega+\Delta)-\kappa)(t-t')}\hat{a}^\dagger_{\text{in},-}(t')\right]\\ \text{[RWA]} \xrightarrow{\hat{a}_{\text{in},\pm}(t')\approx\hat{a}_{\text{in},\pm}(t)} &\approx -\left[\gamma + \frac{\mu^2 - \nu^2}{2}\right]\hat{b}_I(t) - i\left[\hat{a}_{\text{in},+}(t)\mu e^{i\theta_{\text{cav}}(\Omega)} + \hat{a}^\dagger_{\text{in},-}(t)\nu e^{-i\theta_{\text{cav}}(-\Omega)}\right]\end{aligned}$$

The phase factors $e^{i\theta_{\text{cav}}(\pm\Omega)}$ can be made to disappear by proper redefinition of the light and oscillator operators.

$$\begin{aligned}\hat{a}'_{\text{in},\pm}(t) &\equiv e^{i[\theta_{\text{cav}}(\Omega)+\theta_{\text{cav}}(-\Omega)]/2}\hat{a}_{\text{in},\pm}(t)\\ \hat{b}'_I(t) &\equiv e^{-i[\theta_{\text{cav}}(\Omega)-\theta_{\text{cav}}(-\Omega)]/2}\hat{b}_I(t)\\ \hat{a}'_{\text{out},+}(t) &\equiv e^{-i[\theta_{\text{cav}}(\Omega)+\theta_{\text{cav}}(-\Omega)]/2}\hat{a}_{\text{out},+}(t)\end{aligned}$$

Then we get

$$\begin{aligned}\dot{\hat{b}}'_I(t) &= -\left[\gamma + \frac{\mu^2 - \nu^2}{2}\right]\hat{b}'_I(t) - i\left[\mu\hat{a}'_{\text{in},+}(t) + \nu\hat{a}'^\dagger_{\text{in},-}(t)\right]\\ &= -\check{\gamma}(t)\hat{b}'_I(t) - i\left[\mu(t)\hat{a}'_{\text{in},+}(t) + \nu(t)\hat{a}'^\dagger_{\text{in},-}(t)\right]\end{aligned}$$

$$\begin{aligned}\hat{a}'_{\text{out},+}(t) &= e^{i[\theta_{\text{cav}}(\Omega)-\theta_{\text{cav}}(-\Omega)]}\left[\hat{a}'_{\text{in},+}(t) - i\mu(t)\hat{b}'_I(t)\right]\\ \hat{a}'_{\text{out},-}(t) &= e^{-i[\theta_{\text{cav}}(\Omega)-\theta_{\text{cav}}(-\Omega)]}\left[\hat{a}'_{\text{in},-}(t) - i\nu(t)\hat{b}'^\dagger_I(t)\right]\end{aligned}$$

The final phase factor can be absorbed into the (complex) measurement current and in the main text the operators are renamed without the primes.

Appendix C

Definitions of functions

$$M_1 = \sqrt{\nu_{\text{opt}}}\tilde{A}_1(T) \tag{C.1}$$

$$M_2 = -A_{2\text{v}}(0) \tag{C.2}$$

$$\tilde{A}_1(t) = \int_0^t \mathrm{d}t' f_{\text{v}}(t')\sqrt{\Gamma_1(t')}\, e^{\int_{t'}^{t} \mathrm{d}\tau \check{\gamma}_1(\tau)} \tag{C.3}$$

$$A_{2\text{v}}(t) = \int_t^T \mathrm{d}t' f_{\text{v}}(t')\sqrt{\Gamma_2(t')}\, e^{-\int_t^{t'} \mathrm{d}\tau \check{\gamma}_{2\text{v}}(\tau)} \tag{C.4}$$

$$B_{12}(t) = f_{\text{v}}(t) - (\zeta_1 + \zeta_2)\sqrt{\Gamma_2(t)}\, A_{2\text{v}}(t) \tag{C.5}$$

$$B_1(t) = f_{\text{v}}(t) + \zeta_1\sqrt{\Gamma_2(t)}\, A_{2\text{v}}(t) \tag{C.6}$$

$$B_P(t) = f_{\text{v}}(t) + \zeta_1 \left\{A_{2\text{v}}(t)\sqrt{\Gamma_2(t)} + \tilde{A}_1(t)\sqrt{\Gamma_1(t)}\right\} \tag{C.7}$$

$$B_X(t) = A_{2\text{v}}(t)\sqrt{\Gamma_2(t)} - \tilde{A}_1(t)\sqrt{\Gamma_1(t)} \tag{C.8}$$

$$C_1(t) = (1 - M_1)\, e^{-\int_t^T \mathrm{d}t' \check{\gamma}_1(t')}\sqrt{\Gamma_1(t)} + \sqrt{\nu_{\text{opt}}}B_X(t) \tag{C.9}$$

$$C_2(t) = \zeta_1\, (1 - M_1)\, e^{-\int_t^T \mathrm{d}t' \check{\gamma}_1(t')}\sqrt{\Gamma_1(t)} + \sqrt{\nu_{\text{opt}}}B_P(t) \tag{C.10}$$

$$D_1(t) = (1 - M_1)\, e^{-\int_t^T \mathrm{d}t' \check{\gamma}_1(t')} + \sqrt{\nu_{\text{opt}}}\tilde{A}_1(t) \tag{C.11}$$

$$\begin{aligned} N_{\text{add}} = &\left(\frac{1}{\eta_{\text{det}}} - 1\right)\frac{1}{2}\int_0^T \mathrm{d}t B_{12}^2(t) + \sum_{j=1}^{2}\int_0^T \mathrm{d}t C_j^2(t) \\ &+ 2\gamma_{\text{d},1}\int_0^T \mathrm{d}t D_1^2(t) + 2\gamma_{\text{d},2}\int_0^T \mathrm{d}t A_{2\text{v}}^2(t) \\ &+ \left(1 - \nu_{\text{opt}}\right)\frac{1}{2}\int_0^T \mathrm{d}t B_1^2(t) + \left(1 - \nu_{\text{opt}}\right)\frac{1}{2}\int_0^T \mathrm{d}t A_{2\text{v}}^2(t)\Gamma_2(t) \end{aligned} \tag{C.12}$$

$$
\begin{aligned}
\hat{\mathcal{N}} = &\int_0^T \mathrm{d}t \sqrt{\frac{1}{\eta_{\mathrm{det}}} - 1}\, B_{12}(t) \frac{\hat{a}^{(2)}_{\mathrm{det},+}(t) - \hat{a}^{(2)\dagger}_{\mathrm{det},-}(t)}{\sqrt{2}\, i} \\
&- \sqrt{2\gamma_2} \int_0^T \mathrm{d}t A_{2\mathrm{v}}(t) \hat{f}^{(2)}_{\mathrm{th}}(t) + \sqrt{2\gamma_1} \int_0^T \mathrm{d}t D_1(t) \hat{f}^{(1)}_{\mathrm{th}}(t) \\
&+ \int_0^T \mathrm{d}t C_1(t) \frac{\hat{a}^{(1)}_{\mathrm{in},+}(t) + \hat{a}^{(1)\dagger}_{\mathrm{in},-}(t)}{\sqrt{2}\, i} - \int_0^T \mathrm{d}t C_2(t) \frac{\hat{a}^{(1)}_{\mathrm{in},+}(t) - \hat{a}^{(1)\dagger}_{\mathrm{in},-}(t)}{\sqrt{2}\, i} \\
&+ \int_0^T \mathrm{d}t \sqrt{1 - \nu_{\mathrm{opt}}}\, B_1(t) \frac{\hat{a}^{(1)}_{\mathrm{vac},+}(t) - \hat{a}^{(1)\dagger}_{\mathrm{vac},-}(t)}{\sqrt{2}\, i} - \int_0^T \mathrm{d}t \sqrt{1 - \nu_{\mathrm{opt}}}\, A_{2\mathrm{v}}(t) \sqrt{\Gamma_2(t)} \frac{\hat{a}^{(1)}_{\mathrm{vac},+}(t) + \hat{a}^{(1)\dagger}_{\mathrm{vac},-}(t)}{\sqrt{2}\, i}
\end{aligned}
\tag{C.13}
$$

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