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Magnetically-dominated magnetosphere of a near-NHEK Kerr Black Hole

A perturbative approach

Master's thesis

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Abstract

With the continuously growing interest in active galactic nuclei (AGN) and the propagation of their associated jets, the subject of energy extraction from rotating Kerr black holes is still highly relevant today. By now, several general relativistic magnetohydrodynamic simulations validate the Blandford-Znajek process which is a black hole energy extraction mechanism first proposed by Blandford and Znajek in 1977 [1] that was greatly inspired by the similar case of the pulsar magnetosphere. Unlike the example of a slowly rotating black hole, which Blandford and Znajek considered, most observed AGN are found to spin at rates close to extremality. This means that it is necessary to construct a new perturbative approach for near-extreme and extreme black holes much like Menon and Dermer did in the Kerr background [2]. This thesis reviews the near-horizon extreme Kerr (NHEK) limit of rotating black holes. Consequently, an expansion of the field in orders of the scaling parameter λ is performed in order to construct a magnetically-dominated field strength from the NHEK attractor solution found by Camilloni et al. [3]. Although this solution is null everywhere, calculating the first two post-NHEK order corrections to the field allows one to show that it is indeed possible to construct a magnetically-dominated magnetosphere, at least in some regions. The near-NHEK limit of the Kerr solution, which is more astrophysically correct than the NHEK limit, is also reviewed. Again, the field is expanded but now in orders of λ and σ which measures the deviation from extremality ($\sigma = 0$). With a novel expansion of the field variables, I find corrections to the near-NHEK attractor to the second order in λ (and σ), and present the field variables that are determined along the way. I show that the field strength is null in the first order in λ and present the general field strength in the second order in λ with field variables that are left unfixed. Lastly, the novel general expansion is compared with a known solution [4].

Contents

Introduction

Several near-extreme Kerr black holes thought to be the centre of active galactic nuclei (AGN) have been observed throughout the Universe. These possess rotations high enough to drive the astrophysical jets that are likely candidates for sites of creation of highly energetic particles such as high-energy neutrinos. The neutrinos have been observed by detectors such as IceCube [5] located on the South Pole, and are ideal messengers carrying information about the Universe which is otherwise not available to us. Generally, information is lost along the way as it travels through various astrophysical objects before being observed by detectors on Earth, thus making these neutrinos highly valuable to us. With the publication of the first ever image taken of a black hole as late as in 2019 [6] it is evident that there is still much to be studied. The image which in fact is not of the supermassive black hole M87* itself but rather the surrounding accretion disc or emitted jet only fuels the interest in investigating AGN and their central black holes even further. It is thus of great interest to further understand the mechanics of near-extreme and extreme Kerr black holes with mass M and angular momentum J (with $J = M^2$ in the extreme limit) such that we may better understand processes like the creation of the relativistic jet or the emitted high-energy neutrinos, and the process of energy extraction from the highly spinning black hole possibly through said jets [7]. Several models have been proposed in order to describe the magnetosphere of the black hole. In order to construct a magnetosphere which is astrophysically acceptable it must be magnetically-dominated. This means that the field strength of the electromagnetic field $F^{\mu\nu}$ surrounding the Kerr black hole must be positive $F^2 > 0$, or equivalently $B^2 > E^2$ where B_i and E_i are the magnetic field and the electric field, respectively. The approach which has proved to be best equipped to describe this near-extreme and extreme Kerr background is the force-free (FF) limit of ideal magnetohydrodymanics (MHD) governed by force-free electrodynamics (FFE) which is a regime of Maxwell's equations. It is a closed set of equations describing the electromagnetic field and the force-free plasma within the black hole magnetosphere [8]. The plasma is said to be FF when its associated energy-momentum tensor is dominated by the electromagnetic energy versus the matter energy such that the Lorentz force density in effect vanishes which is where the term "force-free" originates from. This was initially inspired by the discovery of a FF plasma-filled pulsar magnetosphere in 1969 by Goldreich and Julian [9] allowing for the energy extraction through the relativistic jets originating from the rotating neutron star. In 1977 Blandford and Znajek applied this idea to the slowly rotating limit of a black hole as they constructed the Blandford-Znajek (BZ) process. This is an electromagnetic Penrose-like process describing the energy extraction from a rotating black hole immersed in a magnetic field supported by the surrounding accretion disc. The presence of the accretion disc gives rise to a force-free plasma in some parts of the magnetosphere, and consequently the electromagnetic conversion of the black hole angular momentum into energy that as mentioned may power the relativistic jet.

Due to the complexity of the non-linear equations produced by FFE in the Kerr background it is difficult to obtain a stationary and axisymmetric analytical solution for a magnetically-dominated magnetosphere and in fact only one set of such solutions has been found by Menon and Dermer [2, 10] and by Brennan et al. [11]. Both share the property of being null everywhere ($F^2 = 0$). This motivates the reason to perturbatively move slightly away from the horizon of the extreme Kerr black hole, a region governed by the near horizon extreme Kerr (NHEK) geometry which possesses some mathematical qualities allowing for the simplification of the problem such that an exact solution can be found.

In this thesis I will initially be introducing the general relativistic (GR) Einstein field equations as well as the Schwarzschild and Kerr solutions before reviewing the FFE equations. I will then review existing literature in order to derive the NHEK attractor solution recently found by Camilloni et al. [3] which is null. From this, a stationary and axisymmetric solution in the Kerr background that is magnetically-dominated can be found using perturbation theory by computing the n-post order corrections. I will only be presenting the method with which the first and second post-NHEK order corrections are obtained as it is not necessary to continue to higher orders in order to recover positive regions of F^2. Subsequently, I will move on to the near-NHEK geometry as presented in [12] which is used to study the region close to the horizon of a near-extreme Kerr black hole, i.e. a rotating black hole that deviates slightly from extremality. I will derive the near-NHEK attractor solution which is used as a base for perturbing away from the near-NHEK region, again with the purpose of constructing a FF magnetosphere in the near-NHEK region. As in the

NHEK limit of the Kerr metric I will only consider the first two post-near-NHEK corrections and I will be presenting the various attempts that were made to procure the appropriate solution. In this case I will also show that the perturbation which we found reduces to the Menon-Dermer class of solutions and I will introduce the concept of light surfaces and analyse whether these critical surfaces help find the desired outcome. Lastly I will compare our result with another found by Pompili [4] in order to study the similarities.

Chapter 1

Black holes

In this Chapter I will be introducing two exact known solutions to the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.0.1}$$

which relate the curvature, or geometry, of spacetime through the Ricci tensor $R_{\mu\nu}$, and subsequently the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$, to matter via the energy-momentum tensor $T_{\mu\nu}$. I will be fixing the geometric units $G = c = 1$ and follow the signature $(-, +, +, +)$ throughout the thesis for the metric $g_{\mu\nu}$.

The first exact and most simple solution to Einstein's equations is *the Schwarzschild solution* which describes the spacetime surrounding a static and symmetrical black hole with the following metric in spherical coordinates (t, r, θ, ϕ)

$$ds^2 = -\left(1 - \frac{r_0}{r}\right)dt^2 + \frac{1}{1 - r_0/r}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{1.0.2}$$

where M is the mass of the black hole and $r_0 = 2M$ is known as the Schwarzschild radius. The metric admits a timelike Killing vector field $K = \partial_t$ which is evident since the components of the metric are time-independent. Killing vector fields (or just Killing vectors) are four-vectors ξ which satisfy the condition known as the Killing equation

$$0 = D_\mu \xi_\nu + D_\nu \xi_\mu, \tag{1.0.3}$$

where D_μ is the covariant derivative, which means that ξ is a Killing vector if the Lie derivative with respect to ξ of the metric vanishes[1]

$$(\mathcal{L}_\xi g)_{\mu\nu} = 0. \tag{1.0.4}$$

[1]Recall that $(\mathcal{L}_X g)_{\mu\nu} = X^\rho \partial_\rho(g_{\mu\nu}) + (\partial_\mu X^\rho)g_{\rho\nu} + (\partial_\nu X^\rho)g_{\mu\rho}$ for an arbitrary vector field $X = X^\mu \partial_\mu$.

As mentioned, this is true for the vector field ∂_t in the Schwarzschild solution. Killing vectors are interesting since they are infinitesimal generators of the *isometry* of the metric, meaning that they preserve the metric. This allows us to conclude that the metric is stationary and invariant under time-translations, and due to the metric's spherical symmetry there also exists three non-commuting spacelike Killing fields

$$R = \partial_\phi, \quad S = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi, \quad T = \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi. \tag{1.0.5}$$

The metric is asymptotically flat at infinity which means that for $r \to \infty$ or $r_0 \to 0$ the Schwarzschild metric reduces to the metric describing flat space in spherical coordinates. Since the existence of a Killing vector also implies the existence of a conserved quantity (the metric is preserved), let us consider the physical meaning of the symmetries they also then represent in flat spacetime, due to the metric being asymptotically flat. In flat spacetime, invariance under time-translation results in the conservation of energy, while invariance under spatial rotations results in the conservation of angular momentum. This means that the Killing vector K is associated with the conservation of energy, and the Killing vector R is associated with conservation of the magnitude of angular momentum [13]. This is true even for the Schwarzschild metric and not only in flat spacetime.

Another important feature of the metric is the two singularities found at $r = r_0$ and $r = 0$, where the former becomes evident since the component $g_{tt} \to 0$ and $g_{rr} \to \infty$ for $r = r_0$, however, this turns out to be a coordinate singularity which can be removed by the appropriate parametrisation. For example using Eddington-Finkelstein coordinates[2] (v, r, θ, ϕ) in which the metric reduces to [14]

$$ds^2 = -\left(1 - \frac{r_0}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2, \tag{1.0.6}$$

with the metric of a unit two-sphere $d\Omega = d\theta^2 + \sin^2\theta d\phi^2$. From Eq. (1.0.6) one can infer that the singularity at $r = 0$ is a true singularity where the curvature of the spacetime approaches infinity. However, the coordinate singularity at $r = r_0$ is of no less physical importance as it defines the surface known as the event horizon, marking the boundary (for fixed t) at which not even light can escape the gravitational pull of the black hole. In fact, it is the spacetime in the region $r \leq r_0$ that constitutes what we call the Schwarzschild black hole itself whereas the metric in the region outside of the event horizon $r > r_0$ is a solution to the Einstein equations in vacuum $R_{\mu\nu} = 0$, since $T_{\mu\nu} = 0$ when no matter is present. As a consequence hereof, the Schwarzschild solution is *unique* since it satisfies the following properties [14]

[2]For ingoing coordinates $v = t + r^*$ where $r^* = r + r_0 \log\left(\frac{r}{r_0} - 1\right)$.

- The metric is asymptotically flat.
- The metric is stationary.
- There exists an event horizon.
- Outside of the event horizon, the metric is a solution to the vacuum Einstein equations.

The Schwarzschild solution has historically been used to model slowly spinning objects such as the Sun and the Earth [13] and also in 1977 when Blandford and Znajek proposed a perturbative expansion method for finding approximate solutions for slowly rotating black holes [1] which serves as the inspiration for the basis of this thesis. However, more recent observational findings based on the analysis of the continuum spectrum of the accretion disc surrounding AGN suggest that most astrophysical black holes rotate at a rate closer to the extremal limit $J = M^2$ [15]. This motivates the introduction of another solution to the Einstein field equations, namely the Kerr solution which allows for the evaluation of the NHEK and near-NHEK regions in later Chapters.

1.1 Kerr black holes

The Kerr metric is a solution to Einstein's field equations which describes the spacetime of a rotating (Kerr) black hole thought to be more astrophysically realistic than the non-rotating Schwarzschild counterpart. It is generally believed that Kerr black holes are birthed through the gravitational collapse of spinning massive stars, other compact objects or a collision of binary systems, all of which posses non-zero angular momenta, resulting in a black hole with mass M and angular momentum J. The Kerr metric in Boyer-Lindquist (BL) coordinates (t, r, θ, ϕ) is

$$\begin{aligned} ds^2 = &-\left(1 - \frac{r_0 r}{\Sigma}\right)dt^2 - \frac{2r_0 r}{\Sigma} a \sin^2\theta dt d\phi \\ &+ \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \end{aligned} \tag{1.1.1}$$

with the spin parameter $a = J/M$, Schwarzschild radius $r_0 = 2M$, and parameters

$$\begin{aligned} \Sigma(r,\theta) &= r^2 + a^2\cos^2\theta, \\ \Delta(r) &= (r - r_+)(r - r_-). \end{aligned} \tag{1.1.2}$$

The metric contains a ring singularity at $r = 0$, $\theta = \pi/2$ stemming from the divergence of the metric at $\Sigma = 0$, and two singular surfaces which are evident since the component g_{rr}

diverges at $\Delta = 0$, with Δ vanishing at the radii $r_\pm$ [13]

$$r_\pm = \frac{r_0}{2}\left(1 \pm \sqrt{1 - \frac{4a^2}{r_0^2}}\right). \tag{1.1.3}$$

The two surfaces $r_\pm$ are, however, only coordinate singularities and can thus be removed following the appropriate parametrisation as we also saw with the Schwarzschild solution. The two surfaces are null hypersurfaces that can be thought of as a collection of null geodesics [13], where the innermost null surface is the inner event horizon located at $r = r_-$, surrounded by the outer event horizon at $r = r_+$ denoting the point at which it is impossible for massive and non-massive objects to escape the gravitational pull of the black hole. It is common practise to refer to the surface at $r = r_+$ as *the* event horizon, and I will be following this convention in the remainder of the thesis as my main focus does not lie in the spacetime within the event horizon but rather the region immediately outside of it.

From Eq. (1.1.1) it is evident that the Kerr metric is equipped with two Killing vector fields since the metric coefficients are independent of the coordinates (t, ϕ). The Killing vectors $K = \partial_t$ and $R = \partial_\phi$ are generators of time-translation symmetry and axisymmetry [16], respectively, revealing the Kerr metric to be stationary as well as axisymmetric and since it is asymptotically flat for $r \to \infty$, the Kerr solution is also *unique*. The two commuting vectors generate the Kerr isometry group $\mathbb{R} \times U(1)$ which I will return to when discussing the NHEK region, and they combine to form the co-rotating vector field

$$\chi = \partial_t + \Omega_H \partial_\phi, \tag{1.1.4}$$

with Ω_H being the angular velocity of the black hole, or more precisely the horizon

$$\Omega_H = \frac{a}{r_+^2 + a^2}. \tag{1.1.5}$$

Furthermore, the Killing vectors will span a surface which is denoted as the toroidal surface, while the surface orthogonal to this is the poloidal surface spanned by (r, θ). This allows for the decomposition of the metric into a toroidal component g_T and poloidal component g_P with the determinant of the metric being a product of the two [3]

$$\begin{aligned} g_T = g_{tt}g_{\phi\phi} - g_{t\phi}^2 = -\Delta \sin^2\theta, \quad g_P = g_{rr}g_{\theta\theta} = \frac{\Sigma^2}{\Delta}, \\ \implies g = g_T \cdot g_P = -\Sigma^2 \sin^2\theta. \end{aligned} \tag{1.1.6}$$

Another important feature of the Kerr black hole is the surface surrounding the event horizon known as the *ergosphere* which intersects with $r = r_+$ at the poles of the rotational axis

($\sin\theta = 0$) as seen in Figure 1.1. It is found at the surface where the norm of the timelike Killing vector is null $K^\mu K_\mu = g_{tt} = 0$, i.e. $\Sigma = r_0 r$, leading to the equation

$$r^2 - r_0 r + a^2 \cos^2\theta = 0 \tag{1.1.7}$$

with the following solutions

$$r = \frac{r_0}{2} \pm \frac{r_0}{2}\sqrt{1 - \frac{4a^2}{r_0^2}\cos^2\theta} \tag{1.1.8}$$

allowing us to define the boundary of the ergosphere $r = r_{\text{ergo}}$ as

$$r_{\text{ergo}} = \frac{r_0}{2} + \frac{r_0}{2}\sqrt{1 - \frac{4a^2}{r_0^2}\cos^2\theta}. \tag{1.1.9}$$

Inside this region, the *ergoregion* $r_+ < r < r_{\text{ergo}}$, the Killing vector changes from being time-like $K^\mu K_\mu < 0$ to space-like $K^\mu K_\mu > 0$ meaning that while it is still possible to escape the gravitational pull of the black hole, you cannot stand still and must instead follow the rotation of the black hole in the increasing ϕ-direction with the same angular velocity as the horizon. This effect is known as *frame-dragging* as it results in the dragging of the local inertial frame (the ergosphere) and it motivates the introduction of co-rotating coordinates which I will present alongside the NHEK limit in a following CHapter. In order to see the frame-dragging effect explicitly, let us consider a massless particle, i.e. a photon, that has entered the ergosphere in the ϕ-direction at an arbitrary radius r in the equatorial plane ($\theta = \pi/2$) of the black hole. Since the photon is moving in the ϕ-direction its momentum will not contain any components in the r- and θ-direction thus reducing the null trajectory of the photon to

$$\begin{aligned} 0 = ds^2 &= g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2, \\ &= \frac{d\phi^2}{dt^2} + 2\frac{g_{t\phi}}{g_{\phi\phi}}\frac{d\phi}{dt} + \frac{g_{tt}}{g_{\phi\phi}}, \end{aligned} \tag{1.1.10}$$

which is solved for $d\phi/dt$, yielding [14]

$$\frac{d\phi}{dt} := \Omega_\pm(r,\theta) = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}} = \frac{-g_{t\phi} \pm \sin\theta\sqrt{\Delta}}{g_{\phi\phi}}. \tag{1.1.11}$$

Inside the ergoregion $g_{tt} > 0$, $g_{t\phi} > 0$, and $g_{\phi\phi} > 0$ such that $\Omega_\pm(r,\theta) > 0$, meaning that the photon is bounded by

$$\Omega_-(r,\theta) \le \frac{d\phi}{dt} \le \Omega_+(r,\theta), \tag{1.1.12}$$

with $d\phi/dt$ being the angular velocity of the photon inside the ergoregion. For massive particles, that necessarily move slower than the photon, the upper and lower limit is instead

$$\Omega_-(r,\theta) < \frac{d\phi}{dt} < \Omega_+(r,\theta), \tag{1.1.13}$$

and if we consider the limit $r \to r_+$ it becomes evident that the upper and lower limit asymptote to the angular velocity of the horizon

$$\Omega_\pm(r_+,\theta) = -\frac{g_{t\phi}}{g_{\phi\phi}}\bigg|_{r=r_+} = \frac{ar_0r_+}{(a^2+r_+^2)^2} = \frac{a}{r_0r_+}, \tag{1.1.14}$$

since $0 = \Delta(r_+) = r_+^2 + a^2 - r_0r_+$. Thus we see that particles inside the ergosphere will move with the horizon as the black hole rotates.

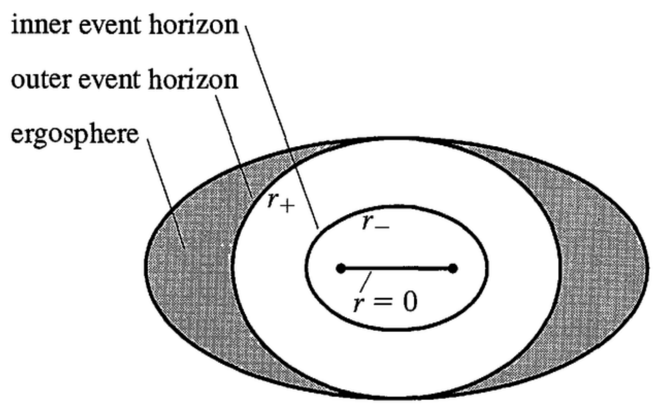

Figure 1.1: Illustration of a Kerr black hole with a ring singularity at $r = 0$, $\theta = \pi/2$ surrounded by the inner horizon at $r = r_-$ and outer horizon at $r = r_+$. The ergoregion (grey) which is bounded by the ergosphere and outer event horizon marks the region in which everything must co-rotate with the black hole. Modified Figure 6.7 from [13].

As I will be considering the extreme and near-extreme limit of a Kerr black hole it is important to establish exactly when the Kerr black hole reaches extremality. The Kerr black hole angular momentum is bounded by the condition $a \leq r_0/2$ with the black hole reaching extremality at $a = r_0/2$. This results in a maximal angular momentum of $|J| = M^2$ and $r_+ = r_- = r_0/2$, as well as the angular velocity of the event horizon reducing to $\Omega_H = r_0^{-1}$ [3]. The angular momentum of the black hole is limited by the cosmic censorship hypothesis [13] which states that a *naked singularity*, i.e. a singularity not surrounded by an event horizon, cannot be formed by gravitational collapse since we cannot allow the curvature singularity to affect the surrounding spacetime. If $a > r_0/2$ then $\Delta > 0$ and no event horizon would be present which is forbidden.

1.2 Black hole energy extraction

Before concretising what the extremal boundary of the Kerr black hole entails let us first consider exactly how energy can be extracted from within the ergosphere of rotating black holes. Perhaps the most well known method of extraction is the *Penrose process* which can be visualised by considering a fairly straightforward example inspired in part by literature by Carroll and Harmark [13, 14]. Consider a massive particle "a" with mass m_a travelling from infinity, moving along a geodesic with positive energy since it is moving forward in time. The energy of the particle will be

$$E_a = -K_\mu p_a^\mu > 0 \tag{1.2.1}$$

since we know that the conserved quantity of the Killing vector K is energy, with $p_a^\mu = m_a\, dx^\mu/d\tau$ being the four-momentum. The minus sign in E_a is needed due to the timelike nature of the Killing vector K at infinity to ensure that the energy is in fact positive. Inside the ergoregion of the Kerr black hole, the Killing vector becomes spacelike meaning that it is possible to imagine a particle for which

$$E = -K_\mu p^\mu < 0, \tag{1.2.2}$$

and thus returning to the example, we imagine that the particle "a" enters the ergosphere and splits into two particles "b" and "c" near the outer event horizon as seen in Figure 1.2. Particle "c" escapes the ergosphere while particle "b" enters the event horizon; both are assumed to follow geodesics such that energy and momentum conservation can be considered

$$E_a = E_b + E_b, \quad p_a = p_b + p_c. \tag{1.2.3}$$

Since "c" is outside the ergoregion it must necessarily have $E_c > 0$ but it is possible for particle "b" to have negative energy $E_b < 0$ if its angular momentum

$$L_b = R_\mu p_b^\mu < 0 \tag{1.2.4}$$

or, more precisely, if "b" has the exact same angular momentum as the black hole but with an opposite sign. This would mean that $E_c > E_a$ which can be interpreted as having extracted rotational energy from the black hole by decreasing its angular momentum. To see this explicitly, consider the momentum of particle "b" crossing the event horizon

$$\begin{aligned} 0 > p_b^\mu \chi_\mu &= p_b^\mu K_\mu + \Omega_H \, p_b^\mu R_\mu \\ &= -E_b + \Omega_H \, L_b \\ &\implies L_b < \frac{E_b}{\Omega_H} \end{aligned} \tag{1.2.5}$$

since $E_b < 0$ and Ω_H is positive, having used the definition of χ from Eq. (1.1.4). Since we know that $\delta M = E_b$ and $\delta J = L_b$ due to energy-momentum conservation, this means

$$\delta J < \frac{\delta M}{\Omega_H}, \tag{1.2.6}$$

demonstrating that the angular momentum of the black hole has decreased.

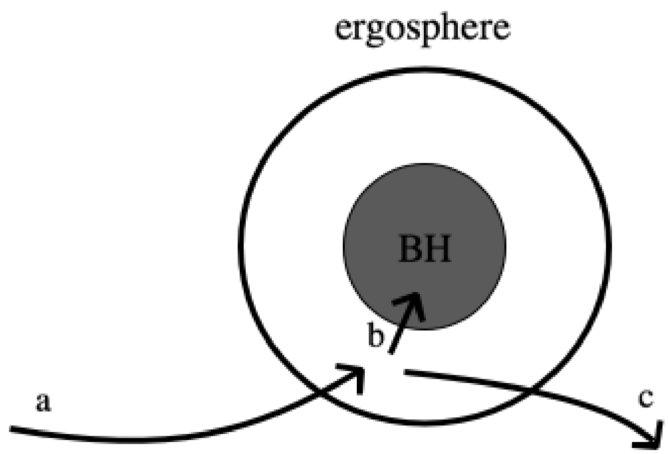

Figure 1.2: Sketch of the Penrose process in which a massive particle "a" enters the ergoregion (top view) and splits into two; particle "c" that escapes the ergosphere with positive energy and particle "b" that enters the event horizon with negative energy or negative angular momentum thus extracting rotational energy from the black hole.

1.2.1 The Blandford-Znajek process

Another newer model is the *Blandford-Znajek process* which is widely regarded as the most promising model for describing the driving force behind the energy extraction of rotating black holes through relativistic jets of astrophysical phenomena such as AGN (pictured in Figure 1.3), gamma-ray bursts and so forth. General relativistic magnetohydrodynamic (GRMHD) simulations have been done which support the force-free approximation and specifically the BZ process [17] thus further confirming the validity of this model. Unlike the Penrose process, which is mechanical, the BZ process is purely electromagnetic requiring $\Omega < \Omega_H$. The outgoing flow of electromagnetic wind (the electromagnetic field) in the magnetosphere is analogous to the outgoing particle with positive energy[3] in the Penrose process, and likewise the ingoing flow of electromagnetic wind is analogous to the ingoing particle with negative energy[4] [18]. From the *"no-hair" theorem* [19] we know that a black hole itself cannot generate a magnetic field, which makes the presence of a magnetised accretion disc consisting of ionised plasma necessary in order to account for the electromagnetic field present in the black hole magnetosphere [1]. Inspired by the apparent advantages in modelling the energy extraction inside pulsar magnetospheres using force-free electrodynamics, Blandford and Znajek applied this concept to slowly rotating black holes. Provided that the magnetic field and angular momentum is large enough, charged particles will be accelerated along the magnetic field lines that are "frozen-in" the plasma and are twisted due to the frame-dragging effect. This will result in the radiation of curvature energy, igniting a cascade-like continuous creation of electron-positron pairs, in turn leading to an approximately force-free plasma near the horizon. Finally, this allows for energy extraction along the magnetic field lines that are threading the black hole, through the relativistic jets at the magnetic poles where the FF approximation breaks down.

The origin of the electromotive force driving the currents in the BZ process is not entirely clear and has been suggested to be the event horizon by Thorne et al. [21] who used the *membrane paradigm* to argue their case. In the membrane paradigm, the black hole horizon is considered to be a spherical rotating conductor with finite resistivity from which the magnetic field lines originate and transfer kinetic energy to an outward pointing Poynting flux and matter flux [22]. However, this is not a very physically credible theory due to the causal disconnection of the horizon from the black hole exterior. Instead it has been found by GRMHD numerical simulations of the black hole magnetosphere by Komissarov and Ruiz et al. [23, 24] that the ergosphere is the origin of the electromotive force, which is also argued by Toma et al. [25] who find that the open magnetic field lines (lines that leave the magnetosphere and extend towards infinity)[5] crossing the ergosphere will have an

[3] At infinity.

[4] Also at infinity.

[5] I will return to the topology of the magnetic field lines in Sections 2.1 and 2.5.

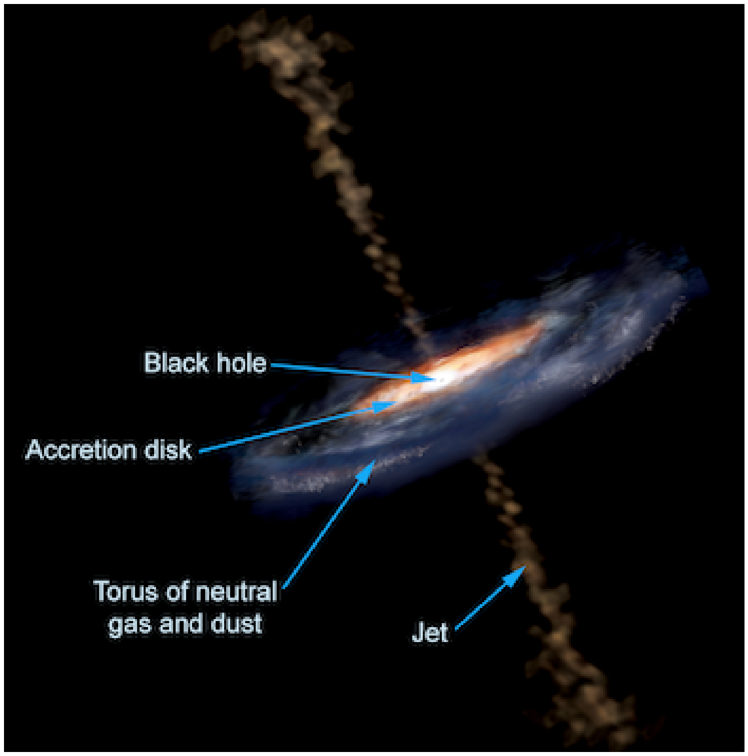

Figure 1.3: Artist's illustration of an AGN enveloped by an accretion disc of ionised plasma (as well as neutral matter) and with relativistic jets originating at the magnetic poles. Image taken from [20].

electric field perpendicular to the magnetic field with $E^2 > B^2$ thus driving the poloidal currents across the magnetic field lines, giving rise to an electromotive force and an outward pointing Poynting flux where $\Omega > 0$. This allows for the extraction of energy from the black hole.

Chapter 2

Force-free electrodynamics

In this Chapter I will be presenting force-free electrodynamics which lay the foundation for the framework wherein it should be possible to construct a magnetically-dominated, FF magnetosphere surrounding both an extreme and near-extreme Kerr black hole. Before turning to the black hole case though, let us first briefly consider the force-free approximation of a pulsar magnetosphere in greater detail, which is what initially opened for the possibility of the force-free approach to the rotating black hole magnetosphere.

2.1 The pulsar magnetosphere

In 1969 Goldreich and Julian (GJ) confirmed the existence of a force-free magnetosphere surrounding a rotating neutron star, or pulsar, allowing for electromagnetic driven winds inside the magnetosphere to extract rotational energy. They achieved this by initially considering the more conceptually simple case of a pulsar surrounded by vacuum [9]. A pulsar rotating with angular velocity $\mathbf{\Omega}$[1] can be considered to be an approximately perfect conductor whose rotation will induce an electric current given by Ohm's law

$$\mathbf{E} + (\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{B} = 0. \tag{2.1.1}$$

This allows one to calculate the Lorentz invariant quantity $\mathbf{E} \cdot \mathbf{B}$ as

$$\begin{aligned} &\mathbf{E} \cdot \mathbf{B} + ((\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{B}) \cdot \mathbf{B} = 0 \\ \implies &\mathbf{E} \cdot \mathbf{B} = 0, \end{aligned} \tag{2.1.2}$$

[1]Here and in the following bold text implies that the quantity is a spatial vector.

however, while $\mathbf{E} = 0$ inside of a perfect conductor, there will be an electric field present $E_{\|}$ that is perpendicular to the surface just outside the pulsar [26] and thus, since it can be shown that the gravitational force on the charges is negligible compared to the Coulomb forces exerted on them[2], the vacuum will effectively be filled with plasma, breaking down the vacuum approximation. The origin of the electromotive force and fluxes is the pulsar itself, unlike in the black hole case where the horizon cannot take on this role due to the issue of causality. In order to ensure that the plasma-filled magnetosphere, which we now know must be present, is force-free with $\mathbf{E}\cdot\mathbf{B} = 0$, then the perpendicular electric field must be "screened" and this condition was found by GJ to be satisfied at the Goldreich-Julian charge density

$$\rho = \nabla \cdot \mathbf{E} \propto -2\boldsymbol{\Omega} \cdot \mathbf{B}, \tag{2.1.3}$$

where the plasma co-rotates with the pulsar, which is necessary in order to ensure continuous screening of the longitudinal electric field. This rotation is bounded by the speed of light, leading to the surface known as the light cylinder with radius

$$R_L = \frac{1}{\Omega}, \tag{2.1.4}$$

beyond which the magnetic field lines in the plasma no longer can co-rotate with the pulsar, meaning that the magnetosphere does not stretch beyond $r = R_L$. As we shall soon see in Section 2.5, the magnetic field lines in the pulsar magnetosphere differ slightly from the lines inside the black hole magnetosphere, as the pulsar lines are allowed to be both open and closed without further restrictions. Closed magnetic field lines will intersect with the surface of the rotating neutron star twice, while open lines (in theory) extend to infinity, exiting the magnetosphere. This is illustrated in Figure 2.1. When the magnetosphere is magnetically-dominated, the magnetic field lines will be "frozen-in" the plasma and have to move with the plasma as per Alfvén's theorem [19]. This is also the case in the black hole magnetosphere, where the effect allows for energy extraction along the open magnetic field lines which follow the plasma out through the jets.

Now that we know that it is indeed possible to construct a force-free magnetosphere surrounding a pulsar, and that the same idea has been used for rotating black holes by e.g. Blandford and Znajek, I will continue to a more in-depth walk-through of force-free electrodynamics and its usages regarding Kerr black holes.

[2]This means that the effects of GR are insignificant in the case of the pulsar magnetosphere.

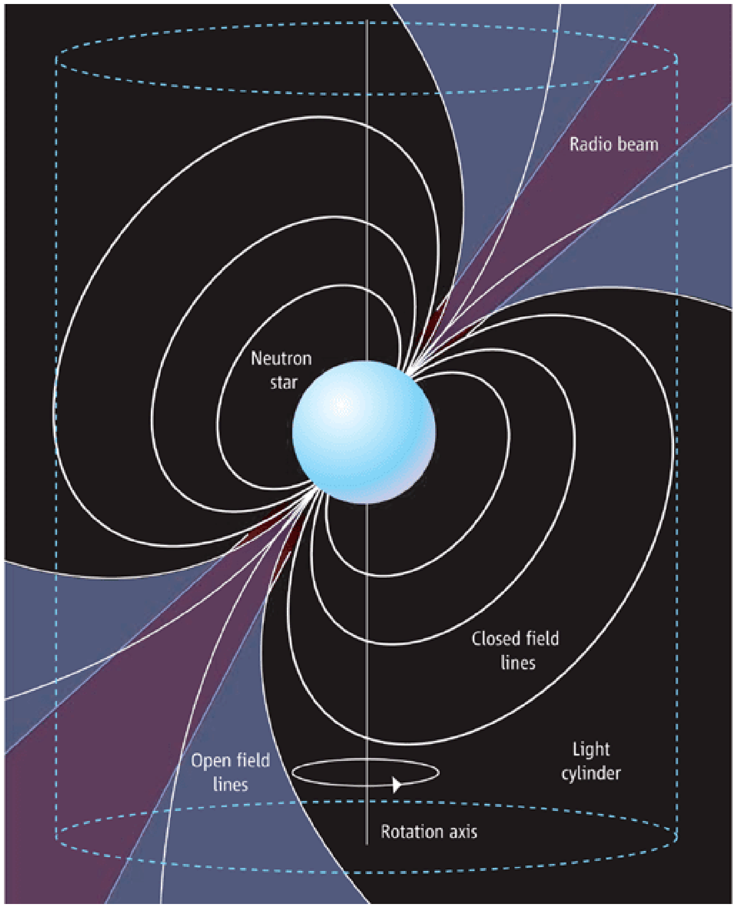

Figure 2.1: Illustration of a pulsar with allowed magnetic field line topology accounting for both open and closed lines. The light cylinder serves as the outer boundary of the magnetosphere's reach. Image taken from [27].

2.2 Equations of FFE

Force-free electrodynamics is an regime of Maxwell's equations which are

$$D_\mu F^{\mu\nu} = j^\nu, \quad D_{[\rho} F_{\mu\nu]} = 0, \tag{2.2.1}$$

where the square brackets denote antisymmetry of the indices. The electromagnetic field strength $F_{\mu\nu}$ is composed of the gauge potential A_μ such that

$$F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} \tag{2.2.2}$$

and j^ν is the four-current. The electromagnetic energy-momentum tensor

$$T^{\mu\nu}_{\text{EM}} = F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}, \tag{2.2.3}$$

together with the contribution from the matter distribution constitutes the total energy-momentum tensor

$$T^{\mu\nu} = T^{\mu\nu}_{\text{EM}} + T^{\mu\nu}_{\text{matter}}, \tag{2.2.4}$$

and conservation of energy and momentum demands that the total energy-momentum tensor is conserved [28]

$$\begin{aligned} 0 &= D_\mu T^{\mu\nu} \\ &= D_\mu T^{\mu\nu}_{\text{EM}} + D_\mu T^{\mu\nu}_{\text{matter}} \\ &= -F_{\mu\nu}j^\nu + D_\mu T^{\mu\nu}_{\text{matter}} \end{aligned} \tag{2.2.5}$$

from which the *force-free condition* can be inferred

$$F_{\mu\nu}D_\rho F^{\rho\nu} = F_{\mu\nu}j^\nu = 0. \tag{2.2.6}$$

This is the equivalent of the invariant $\mathbf{E}\cdot\mathbf{B} = 0$ in the pulsar magnetosphere (flat spacetime), and I will again highlight that this condition means that the magnetic field is so strong that it greatly surpasses the energy contribution of the plasma. Consequently, the dynamics are governed entirely by the magnetic field lines while the plasma just serves to realise the FF condition by screening the electric field in the direction of the magnetic field [29]. Thus the equations of FFE are Maxwell's equations Eq. (2.2.1) combined with the FF condition Eq. (2.2.6), and they can be used to describe plasma whose energy-momentum tensor is dominated by the electromagnetic energy compared to the energy of the matter with

$$F^2 = F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(B^2 - E^2) > 0 \tag{2.2.7}$$

or equivalently $B^2 > E^2$. To really emphasise this important feature, I will present the nature of the Lorentz invariant quantity F^2 as follows

$F^2 > 0$: *magnetically dominated*,

$F^2 < 0$: *electrically dominated*,

$F^2 = 0$: *null*.

In flat spacetime only the magnetically-dominated solution is stable, describing a FF plasma in equilibrium. This is why we are solely interested in electromagnetic field strengths that are magnetically-dominated.

Since the force-free equations are non-linear, as shall soon become more evident, it is difficult to find exact analytical solutions, and still today only a handful are known. An example of a solution in the Schwarzschild spacetime is [30], in Kerr spacetime by [2, 10, 11, 1], and in the NHEK limit of Kerr spacetime by for example [28, 31].

2.3 Determinism

It is now fitting to ask ourselves how we are certain that the FFE equations are able to accurately determine the evolution of the field based only on initial data, and why we should expect it to be a nice approximation to work with. As found by Gralla et al. [29], the equations are deterministic provided that the field is magnetically-dominated with $F^2 = 1/2\,(B^2 - E^2) > 0$ which they demonstrate by making a 3+1 decomposition of the flat spacetime such that the FF condition takes the form

$$\mathbf{E} \cdot \mathbf{j} = 0, \quad \rho\mathbf{E} + \mathbf{j} \times \mathbf{B} = 0, \tag{2.3.1}$$

again implying

$$\mathbf{E} \cdot \mathbf{B} = 0, \tag{2.3.2}$$

as shown in the derivation of Eq. (2.1.2). By taking the cross product of the second equality in Eq. (2.3.1) with $\mathbf{B}$, one finds

$$\begin{aligned} &\rho\mathbf{E} \times \mathbf{B} + j_\perp B^2 = 0 \implies \\ &j_\perp = (\boldsymbol{\nabla} \cdot \mathbf{E}) \frac{\mathbf{E} \times \mathbf{B}}{|B|^2}, \end{aligned} \tag{2.3.3}$$

since $\rho = (\boldsymbol{\nabla} \cdot \mathbf{E})$. The component $j_\parallel$ can be found using Maxwell's time evolution equations

$$\partial_t \mathbf{E} = \boldsymbol{\nabla} \times \mathbf{B} - \mathbf{j}, \tag{2.3.4a}$$

$$\partial_t \mathbf{B} = -\boldsymbol{\nabla} \times \mathbf{E}, \tag{2.3.4b}$$

such that the time derivative of Eq. (2.3.2) yields

$$\begin{aligned} 0 &= \partial_t(\mathbf{E} \cdot \mathbf{B}) = \partial_t(\mathbf{E}) \cdot \mathbf{B} + \mathbf{E} \cdot \partial_t(\mathbf{B}) \\ &= (\boldsymbol{\nabla} \times \mathbf{B}) \cdot \mathbf{B} - \mathbf{j} \cdot \mathbf{B} - \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{E}), \end{aligned} \tag{2.3.5}$$

meaning that

$$j_\perp = \frac{(\boldsymbol{\nabla} \times \mathbf{B}) \cdot \mathbf{B} - \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{E})}{|B|^2} \cdot \mathbf{B}. \tag{2.3.6}$$

This shows that if $\mathbf{B} = 0$ then one cannot solve for the current $\mathbf{j}$ in terms of the fields and their spatial derivatives, so one must have $\mathbf{B} \neq 0$ which is true in the magnetically-dominated case. From this one can infer that $\mathbf{B}$ is non-zero in all reference frames and that $\mathbf{E} \cdot \mathbf{B} = 0$, and consequently Eq. (2.2.6), is preserved as time evolves. I should note that a similar derivation should be possible in curved geometry with a 3+1 decomposition of the spacetime (see the review by Gralla et al. [29] for reference).

Thus we know that the FFE equations are deterministic in flat spacetime given $B^2 - E^2 > 0$, however, since neither the NHEK nor the near-NHEK geometry is asymptotically flat, we cannot say for sure that the null solution $F^2 = 0$ or electrically-dominated solution $F^2 < 0$ is unphysical in this regime. For now let us assume that the only physically relevant solution in the near-NHEK and NHEK regime is magnetically-dominated, and continue to the derivation of the stream equation which is important for understanding the physics of the magnetosphere.

2.4 Field variables and Kerr field strength

Let us now consider the FFE equations for a field in the Kerr background. Since the Kerr black hole is stationary and axisymmetric it is appropriate to assume that the same is true for the electromagnetic field, with axisymmetry around the same axis of rotation as the black hole, thus allowing for the gauge choice $\partial_t A_\mu = \partial_\phi A_\mu = 0$. In order to characterise the field strength and in turn the magnetosphere, *the magnetic flux* $\psi(r,\theta)$ and *poloidal current* $I(r,\theta)$ of the magnetic field lines are defined as follows

$$\psi = A_\phi, \quad I = \sqrt{-g} F^{r\theta}. \tag{2.4.1}$$

Let us consider the toroidal components of Eq. (2.2.6), first $\mu = t$

$$\begin{aligned} 0 = F_{t\nu} j^\nu &= F_{tr} j^r + F_{t\theta} j^\theta + F_{t\phi} j^\phi \\ &= (\partial_t A_r - \partial_r A_t) j^r + (\partial_t A_\theta - \partial_\theta A_t) j^\theta + (\partial_t A_\phi - \partial_\phi A_t) j^\phi \\ &= -\partial_r A_t j^r - \partial_\theta A_t j^\theta \end{aligned} \tag{2.4.2}$$

which means that

$$-\partial_r A_t j^r = \partial_\theta A_t j^\theta, \tag{2.4.3}$$

and secondly for $\mu = \phi$

$$0 = F_{\phi\nu} j^\nu = F_{\phi t} j^t + F_{\phi r} j^r + F_{\phi\theta} j^\theta$$

$$= (\partial_\phi A_t - \partial_t A_\phi)j^t + (\partial_\phi A_r - \partial_r A_\phi)j^r + (\partial_\phi A_\theta - \partial_\theta A_\phi)j^\theta$$
$$= -\partial_r A_\phi j^r - \partial_\theta A_\phi j^\theta \tag{2.4.4}$$

or

$$-\partial_r A_\phi j^r = \partial_\theta A_\phi j^\theta \tag{2.4.5}$$

such that when you divide Eq. (2.4.3) by Eq. (2.4.5) and recall that $A_\phi = \psi$ you find the following integrability condition

$$\partial_r A_t \partial_\theta \psi = \partial_\theta A_t \partial_r \psi \tag{2.4.6}$$

from which we infer that $A_t = A_t(\psi)$. This allows one to define *the angular velocity of the magnetic field lines* $\Omega(r,\theta)$ as

$$\Omega = -\frac{\partial_r A_t}{\partial_r \psi} = -\frac{\partial_\theta A_t}{\partial_\theta \psi} \tag{2.4.7}$$

and in order to find an integrability condition concerning Ω I differentiate Eq. (2.4.7) as follows

$$\partial_\theta(\partial_r A_t) = -\partial_\theta(\Omega)\partial_r\psi - \Omega\partial_\theta(\partial_r\psi), \tag{2.4.8a}$$
$$\partial_r(\partial_\theta A_t) = -(\partial_r\Omega)\partial_\theta\psi - \Omega\partial_r(\partial_\theta\psi) \tag{2.4.8b}$$

which, since $\partial^2/(\partial\theta\partial r) = \partial^2/(\partial r\partial\theta)$, yields the following when divided by each other

$$\partial_r\Omega\partial_\theta\psi = \partial_\theta\Omega\partial_r\psi \tag{2.4.9}$$

meaning that $\Omega = \Omega(\psi)$. From the $\mu = \phi$ component of Eq. (2.2.6) alone, that is $-\partial_r\psi j^r = \partial_\theta\psi j^\theta$, it is possible to find an integrability condition for I using the definition of the poloidal current Eq. (2.4.1) and j^μ

$$-\partial_r\psi j^r = -\partial_r\psi D_\mu F^{\mu r} = -\partial_r\psi(-\partial_\theta F^{r\theta}) = \partial_r\psi\frac{1}{\sqrt{-g}}\partial_\theta I$$
$$= \partial_\theta\psi j^\theta = \partial_\theta\psi\frac{1}{\sqrt{-g}}\partial_r I \tag{2.4.10}$$

which reduces to

$$\partial_r I\partial_\theta\psi = \partial_\theta I\partial_r\psi \tag{2.4.11}$$

thus showing that also $I = I(\psi)$. These field variables ψ, Ω, I can be shown to be connected by *the stream equation* found from the $\mu = r, \theta$ components of Eq. (2.2.6). For $\mu = r$

$$0 = F_{r\nu}j^\nu = F_{rt}j^t + F_{r\theta}j^\theta + F_{r\phi}j^\phi = \partial_r A_t j^t + F_{r\theta}j^\theta + \partial_r\psi j^\phi$$
$$= -\Omega\partial_r\psi\left(\partial_r F^{tr} + \partial_\theta F^{t\theta}\right) + F_{r\theta}\frac{1}{\sqrt{-g}}\partial_r I + \partial_r\left(\partial_r F^{\phi r} + \partial_\theta F^{\phi\theta}\right)$$

$$= \partial_r \left(\sqrt{-g}F^{\phi r}\right) + \partial_\theta \left(\sqrt{-g}F^{\phi\theta}\right) - \Omega\partial_r \left(\sqrt{-g}F^{tr}\right) + \partial_\theta \left(\sqrt{-g}F^{t\theta}\right) + F_{r\theta}\frac{\partial_r I}{\partial_r \psi}$$

$$= \partial_\rho \left(\sqrt{-g}F^{\phi\rho}\right) - \Omega\partial_\rho \left(\sqrt{-g}F^{t\rho}\right) + F_{r\theta}\frac{\partial_r I}{\partial_r \psi} \tag{2.4.12}$$

and likewise for $\mu = \theta$ I find

$$0 = \partial_\rho \left(\sqrt{-g}F^{\phi\rho}\right) - \Omega\partial_\rho \left(\sqrt{-g}F^{t\rho}\right) + F_{r\theta}\frac{\partial_\theta I}{\partial_\theta \psi}. \tag{2.4.13}$$

From Eq. (2.4.11) we know that these two equations, Eqs. (2.4.12)-(2.4.13), are consistent such that the stream equation reads

$$\partial_\rho(\sqrt{-g}F^{\phi\rho}) - \Omega\partial_\rho(\sqrt{-g}F^{t\rho}) + F_{\theta r}\frac{dI}{d\psi} = 0, \tag{2.4.14}$$

with the stationary and axisymmetric F in Kerr spacetime which one can always write in the following form [3, 29]

$$F = \frac{\Sigma I(\psi)}{\Delta \sin\theta} dr \wedge d\theta + d\psi \wedge (d\phi - \Omega(\psi)dt) \tag{2.4.15}$$

using Eq. (1.1.6). Hereinafter F refers to the electromagnetic field strength as a two-form $F = F_{\mu\nu}dx^\mu \wedge dx^\nu$, as it is convenient to present the results using differential forms to obtain a more elegant and compact notation. Eq. (2.4.14) is difficult to solve given the integrability conditions and the demand that the field strength is magnetically-dominated, however, from Eq. (2.4.15) it is evident that we are able to construct the field strength, and thereby the magnetosphere, by explicitly computing the field variables ψ, I, and Ω in the NHEK as well as near-NHEK limit of the Kerr metric, which is thus one of the main goals of this thesis.

Before I move on to the perturbative expansion of the Kerr metric, however, it is important to briefly consider the question of regularity regarding the field strength at the event horizon.

2.5 The question of regularity

In order to ensure that the stationary, axisymmetric electromagnetic field strength F is a physically acceptable solution, we must demand regularity of F at the future event horizon. This is not needed for the past horizon as astrophysical black holes do not possess these [32]. Since the aim of this thesis is to find purely physically acceptable solutions concerning rotating black holes I will not consider the past event horizon. Thus whenever the event horizon is mentioned it is to be understood as the *future* event horizon.

The Znajek condition presented by Blandford and Znajek in 1977 is used to ensure regularity at the horizon and relates the field variables in the following way [3]

$$\left(I\Sigma - \Lambda\Gamma(r^2 + a^2)(\Omega - \Omega_H)\partial_\theta\psi\right)|_{r=r_+} = 0, \tag{2.5.1}$$

which when evaluated at $r = r_+$ reduces the Znajek condition to

$$I_0 = \frac{\Lambda}{r_0}(r_0\Omega_0 - 1)(\partial_\theta\psi)_0 \tag{2.5.2}$$

in the extreme limit in which $r_+ = r_0/2$. Here, as well as in the remainder of the thesis, $(\partial_\theta\psi)_0$, Ω_0, and I_0 indicates that the variables ψ, Ω, and I have been evaluated at the event horizon. In the extremal case Eq. (2.5.2) cannot stand alone and a second condition is necessary in order to ensure regularity of the field at the event horizon

$$(\partial_r I_0) = \frac{\Lambda}{r_0}\left[(r_0\Omega_0 - 1)(\partial_r\partial_\theta\psi)_0 + \left(r_0(\partial_r\Omega)_0 - \Lambda^2\Gamma\Omega_0 + \frac{2}{r_0\Gamma}\right)(\partial_\theta\psi)_0\right]. \tag{2.5.3}$$

I will be referring to Eq. (2.5.2) as the first Znajek condition and Eq. (2.5.3) as the second Znajek condition in this thesis.

In Section 1.1 I briefly mentioned the ”no-hair” theorem which can be paired with the regularity condition to make some comments about the magnetic field lines in the magnetosphere. Inside the magnetosphere we, as with the pulsar, distinguish between open and closed field lines, where by closed lines we refer to the field lines that intersect the horizon $r = r_+$ twice. Unlike in the pulsar magnetosphere where both open and closed field lines are freely allowed, the ”no-hair” theorem ensures that only closed lines which intersect with non-force-free regions $\mathbf{E} \cdot \mathbf{B} \neq 0$ such as an accretion disc, can exist. For closed field lines in a force-free region we deduce from Eq. (2.5.1) that $I = 0$ and $\Omega_H = \Omega$ with allowed lines illustrated in Figure 2.2.

As also mentioned in Section 1.1 it is possible to extract angular momentum and energy from the rotating black hole by the BZ process with the negative inflow (which can thus be considered an outflow) of flux across the event horizon given by [3]

$$\frac{dE}{dt} = 2\pi \int_0^\pi d\theta\ \Omega_+(\psi)\left(\Omega_H - \Omega_+(\psi)\right)(\partial_\theta\psi_+)^2 \sqrt{\frac{g_{\phi\phi}}{g_{\theta\theta}}}\Bigg|_{r=r_+}, \tag{2.5.4a}$$

$$\frac{dL}{dt} = 2\pi \int_0^\pi d\theta\ \left(\Omega_H - \Omega_+(\psi)\right)(\partial_\theta\psi_+)^2 \sqrt{\frac{g_{\phi\phi}}{g_{\theta\theta}}}\Bigg|_{r=r_+} \tag{2.5.4b}$$

where $\Omega_+ = \Omega(r_+, \theta)$, and similarly $\psi_+ = \psi(r_+, \theta)$. The expressions can be found by integrating the energy-momentum tensor and implementing the condition (2.5.1), i.e. integrating $T^{\mu\nu}K_\mu$ in order to find the electromagnetic energy flux and $T^{\mu\nu}R_\mu$ in order to

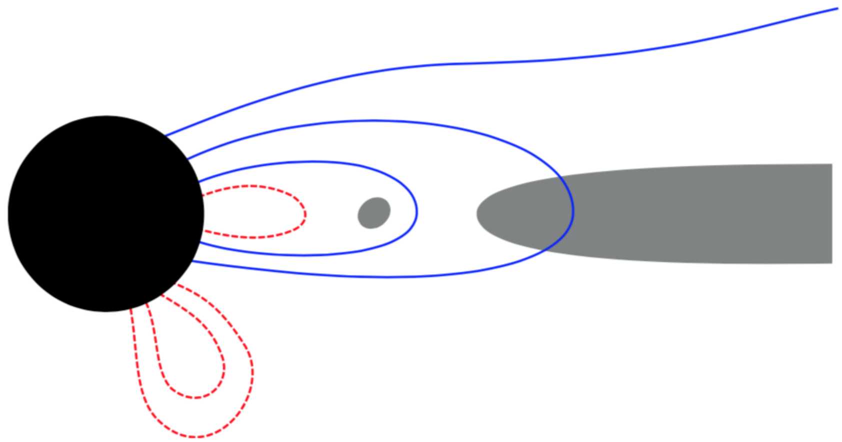

Figure 2.2: Illustration of allowed and forbidden closed magnetic field lines in the force-free Kerr black hole magnetosphere marked by blue solid lines and red dashed lines, respectively. Open field lines (blue solid, only intersecting the black hole once) are also allowed. The grey area is the accretion disc. Taken from [29].

find the angular momentum flux [1]. From this we see that we must have $\Omega_+ < 0 < \Omega_H$ to ensure an outflow of energy, with the maximal energy extraction reached at

$$\Omega_+ = \frac{\Omega_H}{2}, \tag{2.5.5}$$

as found by Blandford and Znajek [1] at the first order in their perturbation theory for small $\alpha = a/M$. In fact, in order to find an exact solution in the Schwarzschild metric which also satisfied the first Znajek condition, Blandford and Znajek perturbatively expanded the field in powers of α for slowly spinning black holes. However, as we know, this is not representative of observed AGN with much lager angular momentum than Blandford and Znajek assumed. Instead the rotation of the black holes approach near-extremality which is exactly why it is relevant to consider new perturbative approaches that are valid around extremality.

Chapter 3

The NHEK limit

Having studied the environment of FFE, we are now prepared to consider the region near the horizon of an extreme Kerr black hole by exploring the NHEK geometry in order to find the so-called NHEK attractor solution. This is the leading order of the expansion of the field in λ (scaling parameter) which will then be supplemented by corrections up to the second post order before we are ready to analyse the sign of the invariant F^2.

3.1 NHEK geometry

In order to study the region close to the horizon, a set of co-rotating coordinates must be introduced as mentioned in Section 1.1, which take the angular velocity of the horizon into account [3]

$$t' = \Omega_H t, \quad r' = \frac{r - r_+}{r_+}, \quad \Phi = \phi - \Omega_H t. \tag{3.1.1}$$

This is not sufficient, however, as we also wish to zoom into the NHEK region at the same time as one co-rotates with the horizon and thus, with the extreme condition for which $\Omega_H|_{a=r_0/2} = r_0^{-1}$, the scaling-coordinates (T, R, θ, Φ) are defined as [3]

$$T = \lambda r' = \frac{\lambda}{r_0} t, \quad R = \frac{r'}{\lambda} = \frac{2r - r_0}{\lambda r_0}, \quad \Phi = \phi - \frac{t}{r_0} = \phi - \frac{T}{\lambda}, \tag{3.1.2}$$

that span $T \in (-\infty, +\infty)$, $R \in [0, \infty)$, $\theta \in [0, \pi]$, and $\Phi \sim \Phi + 2\pi$. Using the following expansion in the scaling parameter λ

$$g = \sum_{n=0}^{\infty} \lambda^n g^{(n)}, \tag{3.1.3}$$

the NHEK geometry is obtained by substituting the coordinates t, r, ϕ with T, R, Φ in the Kerr metric Eq. (1.1.1). The coordinates (T, R, θ, Φ) are then held fixed while taking the limit $\lambda \to 0$ yielding to the leading order in λ

$$ds^2 = \frac{r_0^2}{2}\Gamma(\theta)\left[-R^2 dT^2 + \frac{dR^2}{R^2}d\theta^2 + \Lambda(\theta)(d\Phi + RdT)^2\right] \tag{3.1.4}$$

with the defined functions $\Gamma(\theta)$ and $\Lambda(\theta)$ for simplicity

$$\Gamma(\theta) := \frac{1+\cos^2\theta}{2}, \quad \Lambda(\theta) := \frac{2\sin\theta}{1+\cos^2\theta}. \tag{3.1.5}$$

The event horizon has been relocated to $R = 0$ as is evident from when the component $g_{TT} = 0$. Since the NHEK geometry is a non-singular scaling limit of the extreme Kerr metric it will inherently solve the vacuum Einstein equations thus rendering it a spacetime in its own right [16]. However, the spacetime is no longer asymptotically flat as the Kerr solution is. The so-called NHEK "throat", which originates near the horizon, ends at $R = \infty$ where the near-horizon geometry is exchanged for the asymptotically flat spacetime that is "glued" on to the NHEK geometry [33].

3.1.1 Isometries

While enjoying the same symmetry as the Kerr metric with the Killing vectors $K = \partial_T$ and $R = \partial_\Phi$ generating the time-translation symmetry and axisymmetry, Eq. (3.1.4) is also invariant under the rescaling

$$R \to cR, \quad T \to \frac{T}{c}, \tag{3.1.6}$$

for any constant c. This means that the metric is self-similar in the (near-horizon) extreme Kerr limit. The Killing vector R is the generator of the $U(1)$ rotational symmetry of the metric, while K now is a part of an enhanced $\mathrm{SO}(2,1)$ symmetry group [28]

$$H_0 = T\partial_T - R\partial_R, \tag{3.1.7a}$$

$$H_+ = \sqrt{2}\,\partial_T, \tag{3.1.7b}$$

$$H_- = \sqrt{2}\left[\frac{1}{2}\left(T^2 + \frac{1}{R^2}\right)\partial_T - TR\partial_R - \frac{1}{R}\partial_\Phi\right], \tag{3.1.7c}$$

with the following commutation relations

$$[H_0, H_\pm] = \mp H_\pm, \quad [H_+, H_-] = 2H_0, \tag{3.1.8a}$$

$$[R, H_0] = [R, H_\pm] = 0. \tag{3.1.8b}$$

This means that the isometry group of the NHEK metric is enhanced from the Kerr isometry group $\mathbb{R} \times U(1)$ to $\mathrm{SO}(2,1) \times U(1)$ allowing for the possibility of finding analytical solutions within the regime of FFE, one of which is the NHEK attractor solution.

3.1.2 Particle orbits

Let us briefly consider the motion of particles in the extreme Kerr background. The following is another example of why the NHEK spacetime is nice to work with if you want to consider the BZ process and magnetosphere of a rotating black hole. A simple way of modelling accretion onto the extreme Kerr black hole is by considering the accretion of matter in form of particles following the prograde, circular geodesics at some radius r (in BL coordinates) [16]

$$u_s = u_s^\mu \chi_\mu = u_s^t(\partial_t + \Omega_s \partial_\phi) \tag{3.1.9}$$

with

$$u_s^t = \frac{r^{3/2} + aM^{1/2}}{\sqrt{r^3 - 3Mr^2 + 2aM^{1/2}r^{3/2}}}, \quad \Omega_s = \frac{M^{1/2}}{r^{3/2} + aM^{1/2}}. \tag{3.1.10}$$

The particles will at some point reach the innermost stable circular orbit (ISCO) effectively marking the end of the accretion disc, after which they will plunge into the black hole. To the leading order in the extremal limit, the ISCO lies at

$$\lim_{a \to r_0/2} r_{\mathrm{ISCO}} = \lim_{a \to r_0/2} r_+, \tag{3.1.11}$$

which is inconsistent since the trajectories on the event horizon are null geodesics while they are timelike on the ISCO. This is a result of the Kerr metric failing to resolve the spacetime of the NHEK region accurately, which is one of the motivating factors of working in the NHEK limit instead. In the NHEK spacetime, the particles on the ISCO will co-rotate with the black hole as we already know, with stable orbits extending down into the throat until particles are perturbed from the ISCO geodesic and fall into the black hole.

3.2 The NHEK attractor solution

The Kerr field strength Eq. (2.4.15) which is stationary, axisymmetric and regular on the event horizon, as well as the four-current are expanded in a similar fashion

$$F = \sum_{n=-1}^{\infty} \lambda^n F^{(n)}, \tag{3.2.1a}$$

$$j = j^\mu \partial_\mu = \sum_{n=-1}^{\infty} \lambda^n j^{(n)}. \tag{3.2.1b}$$

The leading orders $F^{(-1)}$ and $j^{(-1)}$ constitute the self-similar NHEK attractor solution in the NHEK geometry Eq. (3.1.4). The field variables ψ, $\Omega(\psi)$, and $I(\psi)$ will be Taylor expanded as follows such that it is ensured that they will be regular near the horizon [3]

$$\psi(r,\theta) = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{r_0}{2}\lambda R\Big)^n (\partial_r^{(n)}\psi)_0 = \psi_0(\theta) + \Big(\frac{r_0}{2}R\psi_1(\theta)\Big)\lambda + \mathcal{O}(\lambda^2), \tag{3.2.2a}$$

$$I(r,\theta) = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{r_0}{2}\lambda R\Big)^n (\partial_r^{(n)}I)_0 = I_0(\theta) + \Big(\frac{r_0}{2}RI_1(\theta)\Big)\lambda + \mathcal{O}(\lambda^2), \tag{3.2.2b}$$

$$\Omega(r,\theta) = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{r_0}{2}\lambda R\Big)^n (\partial_r^{(n)}\Omega)_0 = \Omega_0(\theta) + \Big(\frac{r_0}{2}R\Omega_1(\theta)\Big)\lambda + \mathcal{O}(\lambda^2), \tag{3.2.2c}$$

where $\psi_n = \psi_n(\theta) := (\partial_r^{(n)}\psi)_0$ is the n-th radial derivative of ψ evaluated at the event horizon of extreme Kerr spacetime. The variables I_n, ψ_n, Ω_n with $n > 0$ are to be derived later as functions of I_0, ψ_0, and Ω_0. In the following I will use "prime" to denote derivatives with respect to θ, assuming $\psi_0' \neq 0$ since otherwise the NHEK attractor solution would vanish at the leading order in NHEK and the field would also be electrically-dominated which is not the case that is being investigated in this thesis. The leading order $F^{(-1)}$ is found by expansion in the limit $\lambda \to 0$ as

$$F^{(-1)} = \left[\frac{r_0 I_0}{\Lambda}\frac{dR}{R^2} + (r_0\Omega_0 - 1)\psi_0'\, dT\right] \wedge d\theta \tag{3.2.3a}$$

$$= \frac{r_0 I_0}{\Lambda} d\left(T - \frac{1}{R}\right) \wedge d\theta \tag{3.2.3b}$$

with $j^{(-1)}$ found using

$$j^\nu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right), \tag{3.2.4}$$

yielding

$$j^{(-1)} = \frac{4}{r_0^3}\frac{1}{\Gamma^2\Lambda}\left[\partial_\theta\Big(\frac{\Lambda}{r_0}(r_0\Omega_0 - 1)\psi_0'\Big)\Big(\frac{\partial_T}{R^2} - \frac{\partial_\Phi}{R}\Big) - I_0'\partial_R\right]. \tag{3.2.5}$$

Next, the Bianchi identity, FF condition, and F^2 are expanded in λ as follows

$$dF = \sum_{n=-1}^{\infty} \lambda^n (dF)^{(n)} = 0, \tag{3.2.6a}$$

$$F \cdot j = \sum_{n=-2}^{\infty} \lambda^n (F\cdot j)^{(n)} = 0, \tag{3.2.6b}$$

$$F^2 = \sum_{n=-2}^{\infty} \lambda^n (F^2)^{(n)} = \sum_{n=-1}^{\infty} \lambda^n (F^{(n)})^2. \tag{3.2.6c}$$

Both the Binachi identity $dF = 0$ as well as the FF condition Eq. (2.2.6) are satisfied in the leading order due to the first Znajek condition Eq. (2.5.2), with

$$(F \cdot j)^{(-2)} = F^{(-1)} \cdot j^{(-1)} = \frac{2}{r_0^2 R^2} \frac{1}{\Lambda^2 \Gamma^2} \partial_\theta \left[I_0^2 - \frac{\Lambda^2}{r_0^2} (r_0 \Omega_0 - 1)^2 (\psi_0')^2 \right] d\theta = 0. \tag{3.2.7}$$

The first Znajek condition is also imposed to show that the field is null

$$(F^{(-1)})^2 = (F^2)^{(-2)} = \frac{8}{r_0^2 R^2 \, \Gamma^2 \Lambda^2} \left[I_0^2 - \frac{\Lambda^2}{r_0^2} (r_0 \Omega_0 - 1)^2 (\psi_0')^2 \right] = 0. \tag{3.2.8}$$

Thus, we have found the self-similar, FF and null NHEK attractor solution which is the starting point of our perturbation away from the horizon.

3.3 Post-NHEK order corrections

After having found the NHEK attractor solution it is possible to perturbatively move away from it in order to reconstruct a FF field F in the extreme Kerr background by computing post-NHEK order corrections to the FF field in the NHEK geometry. This is motivated by the fact that despite the leading order being null $(F^2)^{(-2)} = 0$, it should still be possible to construct a magnetically-dominated FF field by going to higher orders in λ as shown explicitly in [3]. As mentioned earlier, I will not be continuing beyond the second order correction in this thesis.

3.3.1 1st order correction

The first post-NHEK order of the field strength with $n = 0$ is found to be

$$\begin{aligned} F^{(0)} =& \frac{r_0^2}{2} \frac{I_0}{\Lambda} \frac{\psi_1}{\psi_0'} \, dT \wedge dR + \frac{r_0^2}{2} \left(\frac{I_0}{\Lambda} \frac{\psi_1'}{\psi_0'} + \psi_0' \Omega_1 \right) R \, dT \wedge d\theta \\ &+ \frac{r_0}{2} \left(\frac{2}{\Gamma} \frac{I_0}{\Lambda} + r_0 \frac{I_1}{\Lambda} \right) \frac{dR}{R} \wedge d\theta + \psi_0' \, d\theta \wedge d\Phi. \end{aligned} \tag{3.3.1}$$

It is scale-invariant (self-similar) under the rescaling $T \to T/c$, $R \to cR$ and contains the unknown field variables ψ_1, Ω_1 and I_1 that depend on ψ_0 and Ω_0, and in turn also the current $I_0 = I_0(\psi_0, \Omega_0)$ as evident from the first Znajek condition Eq. (2.5.2). The unknown variables can be found using the Bianchi identity and FF condition but before I turn to these I will present the correction to the four-current $j^{(0)}$ which is composed as follows

$$j^{(0)} = j^T_{(0)} \frac{\partial_T}{R} + j^R_{(0)} R \partial_R + j^\theta_{(0)} \partial_\theta + j^\Phi_{(0)} \partial_\Phi, \tag{3.3.2}$$

with the components given by

$$j^T_{(0)} = \frac{2}{r_0^4}\frac{1}{\Gamma^2}\frac{I_0}{\Lambda}\left[\frac{\partial_\theta[(2+r_0^2\Omega_1)\Lambda\psi_0']}{I_0} + 2r_0\Lambda\Gamma^2\left(\frac{1}{\Gamma^2\Lambda^2}\frac{\Gamma'}{\Gamma} - \frac{I_0'}{I_0}\right) + r_0^2\frac{\psi_1'}{\psi_0'}\left(\frac{I_0'}{I_0} - \frac{\psi_0''}{\psi_0'} + \frac{\psi_1''}{\psi_1'}\right)\right], \tag{3.3.3a}$$

$$j^R_{(0)} = \frac{2}{r_0^3}\frac{1}{\Gamma^2}\left(\frac{2}{\Gamma}\frac{I_0'}{\Lambda} - r_0\frac{I_1'}{\Lambda}\right), \tag{3.3.3b}$$

$$j^\theta_{(0)} = \frac{2}{r_0^2}\frac{1}{\Gamma^2}\frac{I_1}{\Lambda}, \tag{3.3.3c}$$

$$j^\Phi_{(0)} = -j^T_{(0)} + \frac{2}{r_0^4}\frac{1}{\Gamma^2}\frac{I_0}{\Lambda}\left[\frac{2}{\Lambda}\frac{\psi_0'}{I_0}\left(\frac{\psi_0''}{\psi_0'} - \frac{\Lambda'}{\Lambda}\right) + r_0\Gamma^2\Lambda^2\left(\frac{\Lambda'}{\Lambda} + \frac{\Gamma'}{\Gamma} + \frac{1}{2}\frac{I_0'}{I_0}\right) - r_0^2\frac{\psi_1}{\psi_0'}\right]. \tag{3.3.3d}$$

Both of the Znajek conditions Eq. (2.5.2) and Eq. (2.5.3) inherently enforce the FF condition on the field while ensuring that $(F^{(0)})^2 = 0$. However, it is still interesting to see how the field variables I_1, ψ_1, Ω_1 are found such that we may establish a general course of action regarding later calculations. I will thus linger at this post-NHEK order before moving on to the second post-NHEK order corrections. In order to determine the field variables we need to consider the first order correction to the Bianchi identity as well as the FF condition as mentioned. The Bianchi identity $(dF)^{(0)} = 0$ yields

$$\Omega_1 = \frac{\psi_1}{\psi_0'}\Omega_0', \tag{3.3.4}$$

while the FF condition $(F \cdot j)^{(-1)} = 0$ yields

$$\psi_1 = \frac{I_1}{I_0'}\psi_0', \tag{3.3.5a}$$

$$I_1 = \frac{\Lambda}{r_0}\left[\partial_\theta[(r_0\Omega_0 - 1)\psi_1] - \frac{\Lambda^2\Gamma}{r_0}\left(r_0\Omega_0 - \frac{2}{\Gamma^2\Lambda^2}\right)\psi_0'\right]. \tag{3.3.5b}$$

Eq. (3.3.5b) is recognised as the second Znajek condition which we recall is also needed to ensure regularity at the event horizon at extremality. From Eq. (3.3.5) combined with Eq. (3.3.4) it is possible to construct a first-order linear differential equation describing ψ_1, namely

$$\psi_1' - \left(\frac{\Lambda'}{\Lambda} + \frac{\psi_0''}{\psi_0'}\right)\psi_1 - \frac{\Lambda^2\Gamma}{r_0}\frac{\psi_0'}{r_0\Omega_0 - 1}\left(r_0\Omega_0 - \frac{2}{\Gamma^2\Lambda^2}\right) = 0 \tag{3.3.6}$$

from which it is evident that indeed $\psi_1 = \psi_1(\psi_0, \Omega_0)$. The solution to the differential equation yields a result which is presented using the function $\mathcal{G}$ defined as

$$\mathcal{G}' := \frac{\Lambda\Gamma}{r_0\Omega_0 - 1}\left(r_0\Omega_0 - \frac{2}{\Lambda^2\Gamma^2}\right), \tag{3.3.7}$$

such that the first post-NHEK order correction to the field variables reads

$$\psi_1 = \frac{\Lambda \mathcal{G}}{r_0}\psi_0', \tag{3.3.8a}$$

$$\Omega_1 = \frac{\Lambda \mathcal{G}}{r_0}\Omega_0', \tag{3.3.8b}$$

$$I_1 = \frac{\Lambda \mathcal{G}}{r_0} I_0'. \tag{3.3.8c}$$

The relations in Eq. (3.3.8) reduce the field strength to

$$\begin{aligned} F^{(0)} =& \frac{r_0}{2}\mathcal{G} I_0 dT \wedge dR + \frac{r_0}{2}\mathcal{G}\left[I_0\left(\frac{\mathcal{G}'}{\mathcal{G}} + \frac{\Lambda'}{\Lambda} + \frac{\psi_0''}{\psi_0'}\right) + \Lambda\psi_0'\Omega_0'\right] R\, dT \wedge d\theta \\ & \frac{r_0}{2}\left(\frac{2I_0}{\Gamma\Lambda} + \mathcal{G} I_0'\right)\frac{dR}{R} \wedge d\theta + \psi_0' d\theta \wedge d\Phi. \end{aligned} \tag{3.3.9}$$

We have consequently seen that it is indeed possible to find the unknown field variables in terms of ψ_0, I_0, and Ω_0.

3.3.2 2nd order correction

The second post-NHEK order correction to the field strength with $n = 1$ is

$$\begin{aligned} F^{(1)} =& \frac{r_0^3}{4}\left(\frac{I_0}{\Lambda}\frac{\psi_2}{\psi_0'} + \psi_1\Omega_1\right) R\, dT \wedge dR + \frac{r_0^3}{8}\left(\frac{I_0}{\Lambda}\frac{\psi_2}{\psi_0'} + \psi_0'\Omega_2 + 2\psi_1'\Omega_1\right) R^2 dT \wedge d\theta \\ & + \frac{r_0}{8}\left(\frac{4}{\Gamma}\frac{I_0 + r_0 I_1}{\Lambda} + r_0^2\frac{I_2}{\Lambda}\right) dR \wedge d\theta + \frac{r_0}{2}\psi_1\, dR \wedge d\Phi + \frac{r_0}{2}\psi_1' R\, d\theta \wedge d\Phi \end{aligned} \tag{3.3.10}$$

with the unknown variables ψ_2, Ω_2, and I_2 that likewise must be found in terms of ψ_0, I_0 and Ω_0, and also ψ_1, Ω_1, I_1 given by Eq. (3.3.8). At this order, the four-current is quite involved and of the form

$$j^{(1)} = j^T_{(1)}\partial_T + R^2 j^R_{(1)}\partial_R + R j^\theta_{(1)}\partial_\theta + R j^\Phi_{(1)}\partial_\Phi \tag{3.3.11}$$

with components that can be found in Appendix A.

From the Bianchi identity $(dF)^{(1)} = 0$ the field variable Ω_2 is found to be

$$\Omega_2 = \frac{\psi_2}{\psi_0'}\Omega_0' + \frac{\psi_1}{\psi_0'}\Omega_1' - \frac{\psi_1'}{\psi_0'}\Omega_1 = \left[\frac{\psi_2}{\psi_0'} + \frac{\mathcal{G}\Lambda^2}{r_0^2}\left(\frac{\Omega_0''}{\Omega_0'} - \frac{\psi_0''}{\psi_0'}\right)\right]\Omega_0', \tag{3.3.12}$$

while the FF condition $(F \cdot j)^{(0)} = 0$ yields the following expression for I_2

$$I_2 = \frac{\psi_2}{\psi_0'} I_0' + \frac{\psi_1}{\psi_0'} I_1' + \frac{\psi_1'}{\psi_0'} I_1 = \left[\frac{\psi_2}{\psi_0'} + \frac{\mathcal{G}\Lambda^2}{r_0^2}\left(\frac{I_0''}{I_0'} - \frac{\psi_0''}{\psi_0'}\right)\right] I_0' \tag{3.3.13}$$

with $\mathcal{G}$ given by Eq. (3.3.7). From the FF condition it is also possible to construct a second-order linear differential equation for ψ_2 which is quite lengthy

$$\psi_2'' + a(\theta)\psi_2' + b(\theta)\psi_2 + c(\theta) = 0, \tag{3.3.14}$$

with coefficients $a(\theta)$, $b(\theta)$, and $c(\theta)$ given by

$$a(\theta) = 2\frac{I_0'}{I_0} - \frac{\Lambda'}{\Lambda} - 2\frac{\psi_0''}{\psi_0'}, \tag{3.3.15a}$$

$$b(\theta) = 2 - \frac{\Lambda''}{\Lambda} - \frac{\psi_0^{(3)}}{\psi_0'} + 2\left(\frac{\psi_0''}{\psi_0'}\right)^2 - \left(2\frac{I_0'}{I_0} - \frac{\Lambda'}{\Lambda}\right)\left(\frac{\Lambda'}{\Lambda} + \frac{\psi_0''}{\psi_0'}\right), \tag{3.3.15b}$$

$$c(\theta) = -\frac{\Lambda^2\psi_0'}{r_0^4}\left(A(\theta) + \mathcal{G}(\theta)B(\theta) + \mathcal{G}^2(\theta)C(\theta)\right), \tag{3.3.15c}$$

where the functions $A(\theta)$, $B(\theta)$, and $C(\theta)$ are presented in Appendix A. The field strength $(F^{(1)})^2$ can be written quite compactly as

$$(F^{(1)})^2 = \frac{2}{r_0^4}\left(\frac{\psi_0'}{\Gamma\Lambda}\right)^2 (D(\theta) + \mathcal{G}(\theta)E(\theta)) \tag{3.3.16}$$

using the functions $D(\theta)$ and $E(\theta)$ which are presented in Appendix A. Before I move on to analysing whether the second order correction to the field strength is magnetically-dominated or not, let us first consider the Menon-Dermer solution. Given that the expansion of the field is consistent, it should be possible to recover one of the only known exact results for the Kerr metric; the Menon-Dermer class of solutions.

3.4 Menon-Dermer solution from NHEK

As we by now know, one of the only known exact solutions that is stationary and axisymmetric in the FFE Kerr background is the Menon-Dermer (MD) class of solutions, represented by a set of field variables $(\psi_{\text{MD}}(\theta), \Omega_{\text{MD}}(\theta), I_{\text{MD}}(\theta))$ with no radial dependence. The angular velocity is fixed as

$$r_0\Omega_{\text{MD}} = \frac{2}{\sin^2\theta}, \tag{3.4.1}$$

while I_{MD} is given by the Znajek condition and ψ_{MD} is arbitrary. Since the current associated with this solution flows along null geodesics, the solution itself is also null, $F_{\text{MD}}^2 = 0$, everywhere in the magnetosphere. It is possible to show that the MD class of solutions follows from the NHEK order given that all (ψ_n, Ω_n, I_n) $\forall\, n \geq 1$ vanish, i.e. it is found by removing all of the post-NHEK orders in the expansion Eq. (3.2.2). In order to find the field's angular velocity we demand that the first order field variables in Eq. (3.3.8) are zero. For this to be realised $\mathcal{G}$ must vanish, yielding

$$r_0\Omega_0 = \frac{2}{\Lambda^2\Gamma^2}, \tag{3.4.2}$$

where $\Omega_{\text{MD}} = \Omega_0$, which means that from the first Znajek condition I find $I_{\text{MD}} = I_0$

$$I_0 = \frac{\Lambda}{r_0}(r_0\Omega_0 - 1)\psi_0' = \frac{\Lambda}{r_0}\left(\frac{2}{\Lambda^2\Gamma^2} - 1\right)\psi_0'$$

$$
\begin{aligned}
&= \frac{2\sin\theta}{r_0(1+\cos^2\theta)}\left(\frac{2}{\sin^2\theta}-1\right)\psi_0' \\
&= \frac{2\psi_0'}{r_0\sin\theta}\left(\frac{1}{1+\cos^2\theta}(2-\sin^2\theta)\right) \\
&= \frac{2}{r_0}\frac{\psi_0'}{\sin\theta} = \frac{2}{r_0}\frac{\psi_0'}{\Lambda\Gamma},
\end{aligned} \tag{3.4.3}
$$

and $\psi_{\text{MD}} = \psi_0$ is still arbitrary. Thus without the post-NHEK orders, the Kerr field strength Eq. (2.4.15) is found using Eq. (3.4.2) and Eq. (3.4.3)

$$
\begin{aligned}
F_{\text{MD}} &= -\frac{\Sigma I_0}{\Delta\Gamma\Lambda} d\theta\wedge dr + \partial_\alpha\psi_0 dx^\alpha \wedge (d\phi - \Omega_0 dt) \\
&= -\frac{\Sigma I_0}{\Delta\Gamma\Lambda} d\theta\wedge dr + \psi_0' d\theta \wedge (d\phi - \Omega_0 dt) \\
&= -\frac{\Sigma I_0}{\Delta\Gamma\Lambda} d\theta\wedge dr + \frac{r_0}{2} I_0\Gamma\Lambda d\theta \wedge (d\phi - \Omega_0 dt) \\
&= -\frac{\Sigma I_0}{\Delta\Gamma\Lambda} d\theta\wedge dr + \frac{r_0}{2} I_0\Gamma\Lambda d\theta \wedge d\phi - \frac{I_0}{\Gamma\Lambda} d\theta\wedge dt \\
&= -\frac{I_0}{\Gamma\Lambda} d\theta\wedge\left[dt + \frac{\Sigma}{\Delta}dr\right] + \frac{r_0}{2} I_0\Gamma\Lambda d\theta\wedge d\phi \\
&= -\frac{I_0}{\Gamma\Lambda} d\theta\wedge\left[dt + \frac{r^2 + (r_0/2)^2(1+\Gamma^2\Lambda^2)}{\Delta}dr\right] + \frac{r_0}{2} I_0\Gamma\Lambda d\theta\wedge d\phi \\
&= -\frac{I_0}{\Gamma\Lambda} d\theta\wedge\left[dt + \frac{r^2 + (r_0/2)^2}{\Delta}dr\right] + \frac{r_0}{2} I_0\Gamma\Lambda d\theta\wedge d\phi - \frac{I_0(r_0/2)^2\Gamma\Lambda}{\Delta} dr\wedge d\theta \\
&= -\frac{I_0}{\Gamma\Lambda} d\theta\wedge\left[dt + \frac{r^2 + (r_0/2)^2}{\Delta}dr\right] + \frac{r_0}{2} I_0\Gamma\Lambda d\theta\wedge \left(d\phi + \frac{r_0}{2\Delta}dr\right),
\end{aligned} \tag{3.4.4}
$$

which along with its associated current appear to be singular on the rotational axis but it is possible to fix this issue by choosing the appropriate ψ_0. The associated current can be written as follows

$$
j^\mu_{\text{MD}} = \frac{I_0'}{\Sigma\Gamma\Lambda} n^\mu, \tag{3.4.5}
$$

where

$$
n = n^\mu\partial_\mu = \left(\frac{r^2+a^2}{\Delta}\partial_t - \partial_r + \frac{a}{\Delta}\partial_\phi\right)\Big|_{a=r_0/2}, \tag{3.4.6}
$$

and this current along with the MD field strength will reduce to the NHEK attractor solution in the NHEK regime when all post-NHEK orders are ignored. This emphasises that one always ends up with the null and self-similar NHEK attractor solution in the NHEK region of an arbitrary stationary, axisymmetric and regular FF magnetosphere further cementing the result found by Camilloni et al. [3].

3.5 Sign of F^2

In order to solve Eq. (3.3.14) analytically (such that in turn $(F^{(1)})^2$ can be found analytically) you need an ansatz for ψ_0. In [3], Camilloni et al. present the following educated guess

$$\psi_0(\theta) = k_0 \int d\theta \, (1 - r_0\Omega_0)^{-1}\Lambda^{-3/2}, \quad \psi_0(0) = 0 \tag{3.5.1}$$

where k_0 is an integration constant. This produces the general solution

$$\psi_2(\theta) = \frac{1}{(1 - r_0\Omega_0)} \left[\psi_2^h(\theta) + \psi_2^{nh}(\theta)\right] \tag{3.5.2}$$

with the non-homogeneous ψ_2^{nh} and homogeneous ψ_2^h solutions

$$\psi_2^h(\theta) = c_1 \cos\left(\sqrt{2}\theta\right) + c_2 \sin\left(\sqrt{2}\theta\right), \quad c_1, c_2 \in \mathbb{R}, \tag{3.5.3a}$$

$$\begin{aligned} \psi_2^{nh}(\theta) = &+ \cos\left(\sqrt{2}\theta\right) \int d\theta \, c(\theta)(1 - r_0\Omega_0)\Lambda^{1/2} \frac{\sin\left(\sqrt{2}\theta\right)}{\sqrt{2}} \\ &- \sin\left(\sqrt{2}\theta\right) \int d\theta \, c(\theta)(1 - r_0\Omega_0)\Lambda^{1/2} \frac{\cos\left(\sqrt{2}\theta\right)}{\sqrt{2}}, \end{aligned} \tag{3.5.3b}$$

with $c(\theta)$ given by Eq. (3.3.15c). Since $\psi_2 = \mathcal{O}(\theta^2)$ it is regular on the rotation axis. The paper shows that it is indeed possible to construct a magnetically-dominated magnetosphere with $F^2 > 0$ (at least in some regions of the magnetosphere), however, with the drawback that the field strength is irregular on the rotation axis. In order to show this explicitly by constructing novel perturbative solutions, the arbitrary angular velocity Ω_0 is chosen to be

$$r_0\Omega_0 = 1 + \frac{\beta}{2}\left(1 - \frac{2}{\Gamma^2\Lambda^2}\right), \qquad \beta \in \mathbb{R}_{\neq 0} \tag{3.5.4}$$

which in turn from Eq. (3.3.7) yields

$$\mathcal{G}(\theta) = g - \left(1 + \frac{2}{\beta}\right)\cos\theta \tag{3.5.5}$$

with the integration constant g. The choice $\beta = -2$ returns the Menon-Dermer angular velocity Ω_{MD} while $g = 0$ results in $(F^{(1)})^2 = 0$, meaning that the NHEK radial corrections to the MD solution are only taken into account for $g \neq 0$. In fact, selecting $(\beta = -2, g = 0)$ will result in ψ_2^{nh} and the first post-NHEK order vanishing while one can ensure that ψ_2^h and the second post-NHEK order vanishes by fixing the coefficients c_1, c_2 appropriately.

Due to the integral in the non-homogeneous part of ψ_2 in Eq. (3.5.3b) it is difficult to analytically determine the sign of $(F^{(1)})^2$ everywhere. In order to circumvent this problem one can consider a Taylor expansion of the field around the rotation axis, i.e. θ, by inserting the ansatz for ψ_0 as well as Eq. (3.5.2) into Eq. (3.3.16) before letting $\theta \to 0$

$$(F^{(1)})^2 = -2\sqrt{2}\frac{k_0}{r_0^2}\frac{c_2}{\theta^2} + 2\frac{k_0}{r_0^2}\left[2c_1 - 5\frac{k_0}{r_0^2}\left(7 + \frac{10}{\beta}\right)g - \frac{k_0}{r_0^2}\frac{(2+\beta)(6+11\beta)}{\beta^2}\right]\frac{1}{\theta} + \frac{4\sqrt{2}}{3}\frac{k_0}{r_0^2}c_2 + \mathcal{O}(\theta). \tag{3.5.6}$$

It is evident from Eq. (3.5.6) that $(F^{(0)})^2$ diverges at the rotation axis so in order to ensure regularity, the coefficients c_1, c_2 are fixed to be

$$c_1 = \frac{1}{2}\frac{k_0}{r_0^2}\left[5g^2 - 2\left(7 + \frac{10}{\beta}\right)g + \frac{(2+\beta)(6+11\beta)}{\beta^2}\right], \quad c_2 = 0, \tag{3.5.7}$$

which with the choice of $k_0 = 2$, $r_0 = 1$ yields for small polar angles

$$\begin{aligned}(F^{(1)})^2 = &+\left[-8g^2 + \frac{32}{3}\frac{g}{\beta} + \frac{16}{3}\frac{(2+\beta)(2+3\beta)}{\beta^2}\right]\theta \\ &-\left[\frac{34}{15}g^2 + \left(4 + \frac{40}{9}\frac{1}{\beta}\right)g - \frac{8}{45}\frac{(2+\beta)(11+12\beta)}{\beta^2}\right]\theta^3 \\ &+\left[\frac{g^2}{21} - \frac{8}{105}\frac{43+25\beta}{\beta}g + \frac{8}{105}\frac{(2+\beta)(6+\beta)}{\beta^2}\right]\theta^5 \\ &+\left[\frac{349}{1400}g^2 - \frac{3770+1767\beta}{5670}\frac{g}{\beta} - \frac{(2+\beta)(187794+100163\beta)}{1247400\beta^2}\right]\theta^7 \\ &+\left[\frac{203537}{2494800}g^2 + \frac{3398+2348\beta}{31185}\frac{g}{\beta} - \frac{(2+\beta)(187794+100163\beta)}{1247400\beta^2}\right]\theta^9 + \mathcal{O}(\theta^{11}).\end{aligned} \tag{3.5.8}$$

Indeed it was found in [3] that it is possible to construct a magnetically-dominated magnetosphere given $\beta = -2$, $0.67 \lessapprox g \leq 1$ as I have plotted in Figure 3.1.

The energy and momentum outflow presented in Eq. (2.5.4) are found at the leading order in λ using the ansatz for ψ_0, Eq. (3.5.1), and Eq. (3.5.4) for Ω_0 yielding

$$\begin{aligned}\frac{1}{2\pi}\frac{dE}{dt} &= \int d\theta\,\Omega_0\left(\frac{1}{r_0} - \Omega_0\right)(\psi_0')^2\Lambda \\ &= \frac{k_0^2}{8\beta r_0^2}[(6+7\beta)\theta + \beta(8\cot\theta + \cos\theta\sin\theta) + \sin(2\theta)],\end{aligned} \tag{3.5.9a}$$

$$\begin{aligned}\frac{1}{2\pi}\frac{dJ}{dt} &= \int d\theta\left(\frac{1}{r_0} - \Omega_0\right)(\psi_0')^2\Lambda \\ &= \frac{k_0^2}{8\beta r_0}[6\theta + \sin(2\theta)].\end{aligned} \tag{3.5.9b}$$

For $\theta \to 0$ then $dE/dt \to \infty$ while $dL/dt \to 0$ meaning that at the rotation axis, the energy outflow diverges whereas the angular momentum outflow is finite and negative given $\beta < 0$. The divergence of the energy outflow is probably due to the ansatz for ψ_0 and Ω_0 and could

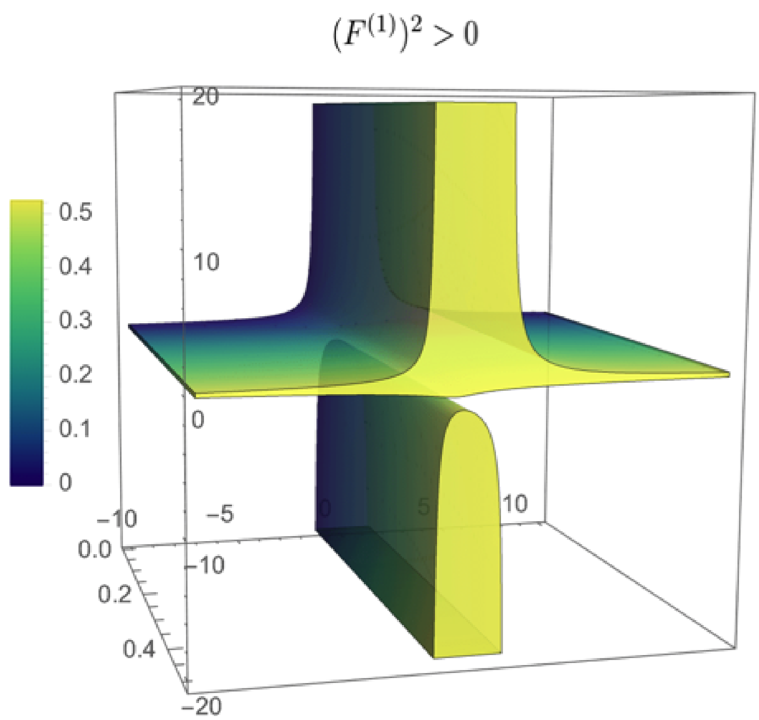

Figure 3.1: Plot showing the positive, i.e. magnetically-dominated regions of $(F^{(1)})^2$ from Eq. (3.5.8) in the range $g \in (-10, +10)$ and $\beta \in (-20, +20)$ with $k_0 = 2$, $r_0 = 1$ and coefficients c_1, c_2 given by Eq. (3.5.7).

perhaps be fixed given other ansatzes are used or if the critical surface known as the inner light surface had been taken into account. Seeing as my thesis focuses more on the lesser well-established near-NHEK limit I will not be investigating these features further but I will briefly return to the subject of light surfaces in connection with the near-NHEK limit in Chapter 5.

With the procedure of constructing a magnetically-dominated magnetosphere in place in the NHEK region (that, however, isn't entirely regular), it is now fitting to move on to the near-NHEK limit. Here I will utilise the same method established in the NHEK limit in order to find the near-NHEK attractor solution. This will in turn allow me determine whether it is possible to find magnetically-dominated solutions for the field strength or not.

Chapter 4

The near-NHEK limit

The extreme Kerr black hole and its NHEK region is a useful tool for understanding the magnetosphere of a rotating black hole but it is in fact generally accepted that the black hole angular momentum has an astrophysical upper limit lower than $J/M^2 = 1$. In 1974, Thorne released a paper [34] in which he concludes that while the accretion of matter onto the black hole will spin it up to extremality, the radiation from the disc absorbed by the horizon will limit the spin-up to $J/M^2 = 0.998$. Thus it is natural to turn our attention to the near-extreme Kerr black hole in order to, hopefully, find a more physically suitable solution to the problem at hand.

In this Chapter I will be presenting the near-NHEK geometry in order to find the appertaining near-NHEK attractor solution which serves as a starting point for the perturbative analysis of the near-NHEK limit of the Kerr metric. This will be followed by the attempts at computing the first two post-near-NHEK order corrections and the struggles of finding a consistent expansion that entailed.

4.1 Near-NHEK geometry

In order to realise the near-NHEK limit we once again zoom into the region near the Kerr event horizon $r = r_+$ using the following scaling coordinates [12]

$$a = \frac{r_0}{2}\sqrt{1-\sigma^2\lambda^2}, \tag{4.1.1a}$$

$$\tilde{T} = \lambda\frac{t}{r_0}, \quad \tilde{R} = 2\frac{r-r_+}{\lambda r_0}, \quad \tilde{\Phi} = \phi - \frac{t}{r_0} = \phi - \frac{\tilde{T}}{\lambda}. \tag{4.1.1b}$$

The spin parameter a now depends on the scaling parameter λ as well as $0 < \sigma \ll 1$ which characterises the deviation from extremality ($\sigma = 0$). The spin parameter is expanded as $a = \frac{r_0}{2} - \left(\frac{r_0}{2}\sigma\lambda\right)^2 + \mathcal{O}(\lambda^3)$ meaning that the scaling limit $\lambda \to 0$ is not a coordinate limit unlike the NHEK limit, and the near-NHEK geometry is thus only valid for near-extreme Kerr black holes [12]. As the Kerr black hole approaches extremality its near horizon region will develop a throat governed by the NHEK spacetime and by the near-NHEK spacetime as seen in Figure 4.1. The far region (extreme Kerr) metric fails to accurately portray the

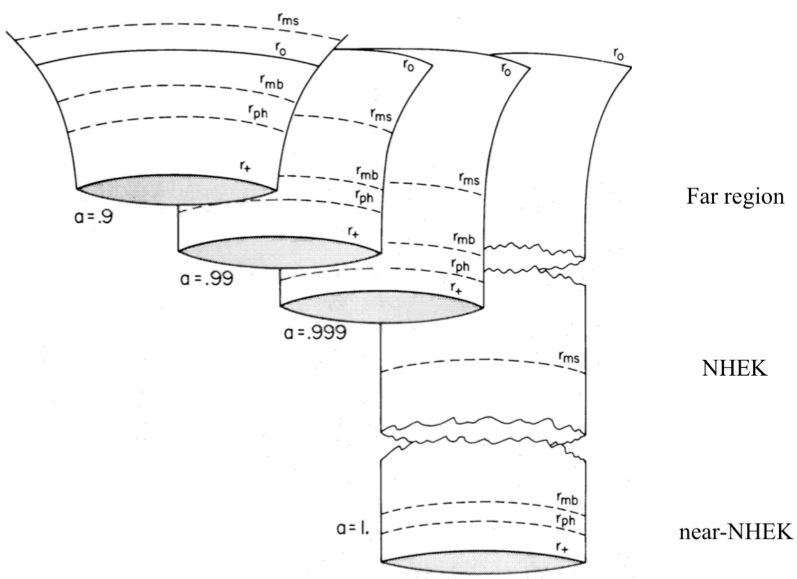

Figure 4.1: Illustration of the geometry of a (near-)extreme Kerr black hole with spin parameter $a = \frac{r_0}{2}\sqrt{1-\sigma^2\lambda^2}$. The far region (extreme Kerr) expands into the throat region consisting of the NHEK and near-NHEK regions as the black hole approaches extremality for $\lambda \to 0$. Modified Figure 1 from [16].

throat region while the NHEK and near-NHEK metrics are not asymptotically flat and thus cannot describe the far region.

The near-NHEK geometry is obtained by letting $\lambda \to 0$. The coordinates $\tilde{T}$, $\tilde{R}$, and σ are held fixed as the radial component approaches the horizon and a approaches extremality yielding

$$d\tilde{s}^2 = \frac{r_0}{2}\Gamma\left[-\tilde{R}(\tilde{R}+2\sigma)d\tilde{T}^2 + \frac{d\tilde{R}^2}{\tilde{R}(\tilde{R}+2\sigma)} + d\theta^2 + \Lambda^2\left(d\tilde{\Phi} + (\tilde{R}+\sigma)d\tilde{T}\right)^2\right], \tag{4.1.2}$$

with the event horizon located at $\tilde{R}=0$. The near-NHEK geometry is embedded with the same isometry group as the NHEK geometry, i.e. $\mathrm{SO}(2,1)\times U(1)$, again allowing for the possibility of constructing analytical solutions using the FFE approach. The near-NHEK metric admits the following Killing vectors [16]

$$R=\partial_{\tilde{\Phi}},\quad H_0=\frac{1}{\sigma}\partial_{\tilde{T}},\quad H_\pm=\frac{e^{\mp\sigma\tilde{T}}}{\sqrt{\tilde{R}^2-\sigma^2}}\left[\frac{\tilde{R}}{\sigma}\partial_{\tilde{T}}\pm(\tilde{R}^2-\sigma^2)\partial_{\tilde{R}}-\sigma\partial_{\tilde{\Phi}}\right]. \tag{4.1.3}$$

In [16] it was found that there in fact is an infinite amount of physically distinct near-horizon limits which describe the near-NHEK throat physics but at different scales for

$$a=\frac{r_0}{2}\sqrt{1-\sigma^2\lambda^2},\quad \tilde{T}=\lambda^p\frac{t}{r_0},\quad \tilde{R}=\frac{2r-r_0}{\lambda^p r_0},\quad \tilde{\Phi}=\phi-\frac{t}{r_0},\quad 0\leq p\leq 1. \tag{4.1.4}$$

The choice of $p=1$ ensures that one zooms into the near-horizon region while simultaneously letting the black hole spin approach extremality. Note that the above equations with $p=1$ are almost equivalent to the choice of scaling coordinates in this thesis, apart from $\tilde{R}=(2r-r_0)/(\lambda r_0)$ compared to $\tilde{R}=2(r-r_0)/(\lambda r_0)$ in Eq. (4.1.1b).

4.1.1 The near-NHEK attractor solution

The leading order of the field strength in the near-NHEK limit is found using the same expansion of F in λ as before Eq. (3.2.1a), imposed on the general Kerr field strength Eq. (2.4.15) yielding

$$\tilde{F}^{(-1)}=\frac{r_0 I_0}{\Lambda}d\left[\tilde{T}-\frac{1}{2\sigma}\log\left(1+\frac{2\sigma}{\tilde{R}}\right)\right]\wedge d\theta. \tag{4.1.5}$$

This is the so-called near-NHEK attractor solution found by Camilloni et al. [12]. The four-current associated with the field strength is found to be

$$\tilde{j}^{(-1)}=\frac{4}{r_0^3}\frac{1}{\Gamma^2\Lambda}\left[\partial_\theta\Big(\frac{\Lambda}{r_0}(r_0\Omega_0-1)\psi_0'\Big)\frac{\partial_{\tilde{T}}-(\tilde{R}+\sigma)\partial_{\tilde{\Phi}}}{\tilde{R}(\tilde{R}+2\sigma)}-I_0'\partial_{\tilde{R}}\right], \tag{4.1.6}$$

with which $\tilde{F}^{(-1)}$ satisfies the equations of FFE, i.e. including the FF condition. It is also self-similar under scalings $\tilde{T}\to\tilde{T}/c$, $\tilde{R}\to\tilde{R}c$, $\sigma\to\sigma c$, and it is inherently axisymmetric and stationary. Furthermore, the near-NHEK attractor solution is null, $(\tilde{F}^{(-1)})^2=0$, as can be seen from

$$(\tilde{F}^{(-1)})^2=\frac{8}{r_0^2\,\Gamma^2\Lambda^2}\frac{1}{\tilde{R}(\tilde{R}+2\sigma)}\left[\frac{\Lambda^2}{r_0^2}(r_0\Omega_0-1)^2(\psi_0')^2-I_0^2\right]=\frac{R^2}{\tilde{R}(\tilde{R}+2\sigma)}(F^{(-1)})^2 \tag{4.1.7}$$

due to the first Znajek condition.

Conclusively, just as with the NHEK attractor solution in the NHEK limit, any magnetosphere which is regular, axisymmetric and stationary is governed by Eq. (4.1.5) in the near-NHEK limit.

4.2 From near-NHEK to NHEK

By now it becomes increasingly evident that the NHEK and near-NHEK attractor solutions are very similar in form which makes it interesting to compare the two. There are several ways to recover the NHEK metric from the near-NHEK metric, and these consist of the following [12]

- The most straightforward way to recover the NHEK metric is by setting $\sigma = 0$ in Eq. (4.1.2) which is the same as letting the black hole reach extremality as seen by Eq. (4.1.1a). In this case, the near-NHEK attractor solution reduces to

$$\tilde{F}^{(-1)} = F^{(-1)} + \mathcal{O}(\sigma). \tag{4.2.1}$$

- By letting $\sigma \ll R$ such that the near-NHEK metric becomes asymptotically NHEK. This again illustrates how the spacetime of the near-extreme black hole can be considered to consist of the three different patches or regions; Kerr, NHEK, and near-NHEK as illustrated in Figure 4.1. In this case, the near-NHEK attractor has the same tensorial structure as the NHEK attractor

$$\tilde{F}^{(-1)} \sim \frac{r_0 I_0}{\Lambda} d\left(\tilde{T} - \frac{1}{\tilde{R}}\right) \wedge d\theta. \tag{4.2.2}$$

- Using the local diffeomorphism from near-NHEK coordinates $(\tilde{T}, \tilde{R}, \theta, \tilde{\Phi})$ to NHEK coordinates (T, R, θ, Φ) that is given by

$$\tilde{T} = -\frac{1}{2\sigma}\log\left(T^2 - \frac{1}{R^2}\right), \quad \tilde{R} = -\sigma\left(T + \frac{1}{R}\right)$$
$$\tilde{\Phi} = \Phi + \frac{1}{2}\log\left(\frac{T + 1/R}{T - 1/R}\right), \tag{4.2.3}$$

  the NHEK attractor solution is

$$\tilde{F}^{(-1)} = -\frac{1}{\sigma}\left(T - \frac{1}{R}\right)^{-1} F^{(-1)}, \tag{4.2.4}$$

  with

$$\tilde{j}^{(-1)} = -\frac{1}{\sigma}\left(T - \frac{1}{R}\right)^{-1} j^{(-1)}, \tag{4.2.5}$$

  thus allowing for $\tilde{F}^{(-1)}$ and $F^{(-1)}$ to be superposed to creative new FF solutions.

4.3 Near-NHEK post order corrections

In order to find the near-NHEK post order corrections to see whether a magnetically-dominated magnetosphere can be recovered in higher orders of λ, I will initially be following (what I dub) the "naive approach". I will assume that the expansion of the field variables ψ_n, Ω_n, and I_n for $0 < n \leq 2$ is identical to the expansion used in the NHEK limit Eq. (3.2.2). However, as will soon become apparent, this expansion is insufficient in recovering a physically acceptable non-zero field strength and thus other measures must be sought out.

4.3.1 The naive approach

Using the same expansion of the metric, field, current and so forth in λ as for the NHEK limit (Eq. (3.2.1) and Eq. (3.2.6)) and the same Taylor expansion of the field variables, the *first order correction* to the near-NHEK field strength is found to be

$$\begin{aligned}\tilde{F}^{(0)} = &\frac{r_0^2}{2}\frac{I_0}{\Lambda}\frac{\psi_1}{\psi_0'}\,d\tilde{T}\wedge d\tilde{R} + \frac{r_0^2}{2}(\tilde{R}+\sigma)\left(\frac{I_0}{\Lambda}\frac{\psi_1'}{\psi_0'} + \psi_0'\Omega_1\right)d\tilde{T}\wedge d\theta \\ &+ \frac{r_0}{2}\frac{\tilde{R}+\sigma}{\tilde{R}(\tilde{R}+2\sigma)}\left(\frac{2}{\Gamma}\frac{I_0}{\Lambda} + r_0\frac{I_1}{\Lambda}\right)d\tilde{R}\wedge d\theta + \psi_0'\,d\theta\wedge d\tilde{\Phi},\end{aligned} \tag{4.3.1}$$

and the four-current is of the form

$$\tilde{j}^{(0)} = \frac{1}{r_0^4\Gamma^4\Lambda^2\tilde{R}(\tilde{R}+2\sigma)\psi_0'}\left[\tilde{j}^{\tilde{T}}_{(0)}\partial_{\tilde{T}} + \frac{1}{\Lambda}\tilde{j}^{\tilde{\Phi}}_{(0)}\partial_{\tilde{\Phi}}\right] + \tilde{j}^{\tilde{R}}_{(0)}\partial_{\tilde{R}} + \tilde{j}^{\theta}_{(0)}\partial_\theta \tag{4.3.2}$$

with the following components

$$\begin{aligned}\tilde{j}^{\tilde{T}}_{(0)} = &\,2(\tilde{R}+\sigma)\Big(r_0I_0(\Lambda(2\Gamma'\psi_0' - 4\Gamma\psi_0'' + r_0\Gamma^2\psi_1'') + r_0\Gamma^2\Lambda'\psi_1') \\ &+ \Gamma\Lambda\psi_0'\Big(\Lambda'\psi_0'(\Gamma(r_0^2\Omega_1 + 4r_0\Omega_0 - 2) - 4r_0\Omega_0 + 4) + \Lambda\Big(r_0(\Omega_0'(4(\Gamma-1)\psi_0' + r_0\Gamma\psi_1') \\ &+ r_0\Gamma\psi_0'\Omega_1') + \Gamma\psi_0''(r_0^2\Omega_1 + 4r_0\Omega_0 - 2)\Big)\Big)\Big),\end{aligned} \tag{4.3.3a}$$

$$\tilde{j}^{\tilde{R}}_{(0)} = -\frac{2(\tilde{R}+\sigma)\,(r_0\Gamma I_1' - 2I_0')}{r_0^3\Gamma^3\Lambda}, \tag{4.3.3b}$$

$$\tilde{j}^{\theta}_{(0)} = \frac{2I_1}{r_0^2\Gamma^2\Lambda}, \tag{4.3.3c}$$

$$\begin{aligned}\tilde{j}^{\tilde{\Phi}}_{(0)} = &\,\Gamma\psi_0'\Big(\Big(r_0\Gamma^3\psi_0'\Omega_0'\Lambda^5 + 8r_0\psi_0'\Omega_0'\Lambda^3 \\ &- 2\Gamma(r_0(r_0\psi_1'\Omega_0' + \psi_0'(4\Omega_0' + r_0\Omega_1'))\Lambda^3 + (\Lambda^2(\Omega_1r_0^2 + 4\Omega_0r_0 - 2) - 2)\psi_0''\Lambda \\ &+ ((\Omega_1r_0^2 + 4\Omega_0r_0 - 2)\Lambda^2 + 2)\Lambda'\psi_0')\Big)\tilde{R}^2 + 2\sigma\Big(r_0\Gamma^3\psi_0'\Omega_0'\Lambda^5 + 8r_0\psi_0'\Omega_0'\Lambda^3 \\ &- 2\Gamma\Big(r_0(r_0\psi_1'\Omega_0' + \psi_0'(4\Omega_0' + r_0\Omega_1'))\Lambda^3 + (\Lambda^2(\Omega_1r_0^2 + 4\Omega_0r_0 - 2) - 2)\psi_0''\Lambda\end{aligned}$$

$$+ ((\Omega_1 r_0^2 + 4\Omega_0 r_0 - 2)\Lambda^2 + 2)\Lambda'\psi_0'\Big)\Big)\tilde{R} - 2\sigma^2\Lambda^2\Big((-4r_0\Omega_0 + \Gamma(\Omega_1 r_0^2 + 3\Omega_0 r_0 - 1) + 4)\Lambda'\psi_0'$$
$$+ \Lambda\Big(r_0\psi_0'((3\Gamma - 4)\Omega_0' + r_0\Gamma\Omega_1') + \Gamma(\psi_1'\Omega_0' r_0^2 + (\Omega_1 r_0^2 + 3\Omega_0 r_0 - 1)\psi_0'')\Big)\Big)\Big)$$
$$+ r_0\Lambda I_0\Big(-2\Big(r_0\Lambda'\psi_1'\Gamma^2 + \Lambda(r_0\psi_1''\Gamma^2 - 4\psi_0''\Gamma + 2\Gamma'\psi_0')\Big)\sigma^2 + 2\tilde{R}(\Lambda^2(3\Lambda'\psi_0' + \Lambda\psi_0'')\Gamma^4$$
$$+ 2\Lambda^3\Gamma'\psi_0'\Gamma^3 - 2r_0(\Lambda'\psi_1' + \Lambda(\psi_1 + \psi_1''))\Gamma^2 + 8(\Lambda'\psi_0' + \Lambda\psi_0'')\Gamma - 4\Lambda\Gamma'\psi_0')\sigma$$
$$+ \tilde{R}^2\Big(\Lambda^2(3\Lambda'\psi_0' + \Lambda\psi_0'')\Gamma^4 + 2\Lambda^3\Gamma'\psi_0'\Gamma^3 - 2r_0(\Lambda'\psi_1' + \Lambda(\psi_1 + \psi_1''))\Gamma^2$$
$$+ 8(\Lambda'\psi_0' + \Lambda\psi_0'')\Gamma - 4\Lambda\Gamma'\psi_0'\Big)\Big). \tag{4.3.3d}$$

The correction to the Bianchi identity $(d\tilde{F})^{(0)} = 0$ produces an equation from which Ω_1 is found to be identical to Eq. (3.3.4), and again, similarly to what was found in the NHEK limit, the correction to the FF condition $(\tilde{F} \cdot \tilde{j})^{(-1)} = 0$ yields ψ_1 equal to Eq. (3.3.5a) with I_1 given by the second Znajek condition. Together with ψ_1 this allows one to rewrite ψ_1, I_1, and Ω_1 with the exact same structure as for the first post-NHEK corrections, i.e. Eq. (3.3.8). The correction to $(\tilde{F}^{(0)})^2$ is found to be

$$(\tilde{F}^{(0)})^2 = -\frac{8(\tilde{R} + \sigma)}{r_0^4\,\Gamma^3\Lambda^2\,\tilde{R}(\tilde{R} + 2\sigma)}\big(r_0^3\,\Gamma\,I_0 I_1 - \Lambda^2\psi_0'(r_0\Omega_0 - 1)\big)$$
$$\times\big(\psi_0'\big(r_0^2\,\Gamma\,\Omega_1 + 2r_0(\Gamma - 1)\Omega_0 + 2\big) + r_0\,\Gamma\,\psi_1'(r_0\Omega_0 - 1)\big) \tag{4.3.4}$$

which is null when the two Znajek conditions are implemented. Thus, so far no problems regarding the results or method have surfaced, and we can safely move on to the next order of corrections.

The *second order correction* to the field strength is found as

$$\tilde{F}^{(1)} = \frac{r_0^3(\tilde{R} + \sigma)}{4}\left(\frac{\psi_2}{\psi_0'}\frac{I_0}{\Lambda} + \psi_1\Omega_1\right) d\tilde{T} \wedge d\tilde{R}$$
$$+ \frac{r_0^3(\tilde{R} + \sigma)^2}{8}\left(\Omega_2\psi_0' + 2\Omega_1\psi_1' + \frac{I_0}{\Lambda}\frac{\psi_2'}{\psi_0'}\right) d\tilde{T} \wedge d\theta$$
$$+ \frac{r_0\Big(4I_0(\tilde{R}(\tilde{R} + 2\sigma) - 2\sigma^2(\Gamma - 1)) + r_0(\tilde{R} + \sigma)^2\,(4I_1 + r_0 I_2\Gamma)\Big)}{8\tilde{R}\,(\tilde{R} + 2\sigma)\,\Gamma\,\Lambda} d\tilde{R} \wedge d\theta$$
$$+ \frac{r_0}{3}\psi_1\,d\tilde{R} \wedge d\tilde{\Phi} + \frac{r_0}{2}(\tilde{R} + \sigma)\psi_1'\,d\theta \wedge d\tilde{\Phi}, \tag{4.3.5}$$

with a rather lengthy current

$$\tilde{j}^{(1)} = \frac{1}{2r_0^4\Gamma^5\Lambda\tilde{R}(\tilde{R} + 2\sigma)}\left[\partial_{\tilde{T}} - \frac{1}{\Lambda^3\psi_0'}\partial_{\tilde{\Phi}}\right] + \tilde{j}^{\tilde{R}}_{(1)}\partial_{\tilde{R}} + \tilde{j}^{\theta}_{(1)}\partial_\theta, \tag{4.3.6}$$

where the components can be found in Appendix B. The Bianchi identity $(d\tilde{F})^{(1)} = 0$ yields Ω_2 identical to Eq. (3.3.12), while the FF condition $(\tilde{F} \cdot \tilde{j})^{(0)} = 0$ produces a set of equations that must be reduced to zero. All but one, are solved by the first Znajek condition, the first order corrections to the field variables as well as I_2 which is found to be identical to Eq. (3.3.13). The remaining equation which is yet to be reduced to zero is used to find a second-order linear differential equation for ψ_2 of the same form as Eq. (3.3.14), i.e.

$$\psi_2'' + \tilde{a}(\theta)\psi_2' + \tilde{b}(\theta)\psi_2 + \tilde{c}(\theta) = 0, \tag{4.3.7}$$

where $\tilde{a}(\theta)$, $\tilde{b}(\theta)$, and $\tilde{c}(\theta)$ are constants of similar form to $a(\theta)$, $b(\theta)$, and $c(\theta)$. In fact $\tilde{a}(\theta) = a(\theta)$, while the constant $\tilde{b}(\theta)$ is evidently dependent on $\tilde{R}$

$$\begin{aligned}\tilde{b}(\theta) = &\frac{1}{\Lambda^2(\tilde{R}+\sigma)^2\psi_0'(r_0\Omega_0 - 1)}\Big[2\sigma\tilde{R}\Big(\Lambda\Big(\psi_0'(\Lambda'' - r_0(2\Lambda'\Omega_0' + \Omega_0\Lambda'')) - 3\Lambda'\psi_0''(r_0\Omega_0 - 1)\Big)\\
&+ \Lambda'^2\psi_0'(1 - r_0\Omega_0) + \Lambda^2(-2r_0\psi_0''\Omega_0' + 2\psi_0'(r_0\Omega_0 - 1) + {\psi_0}^{(3)}(1 - r_0\Omega_0))\Big)\\
&+ \tilde{R}^2\Big(\Lambda(\psi_0'(\Lambda'' - r_0(2\Lambda'\Omega_0' + \Omega_0\Lambda'')) - 3\Lambda'\psi_0''(r_0\Omega_0 - 1)) + \Lambda'^2\psi_0'(1 - r_0\Omega_0)\\
&+ \Lambda^2\Big(-2r_0\psi_0''\Omega_0' + 2\psi_0'(r_0\Omega_0 - 1) + {\psi_0}^{(3)}(1 - r_0\Omega_0)\Big)\Big) + \sigma^2\Big(\Lambda\Lambda'(-2r_0\psi_0'\Omega_0'\\
&- 3\psi_0''(r_0\Omega_0 - 1)) + \Lambda\Big(\Lambda''\psi_0'(1 - r_0\Omega_0) + \Lambda({\psi_0}^{(3)}(1 - r_0\Omega_0) - 2r_0\psi_0''\Omega_0')\Big)\\
&+ \Lambda'^2\psi_0'(1 - r_0\Omega_0)\Big)\Big].\end{aligned} \tag{4.3.8}$$

The constant $\tilde{c}(\theta)$ is very lengthy and can be found in Appendix B. From the Taylor expansion of the field variables Eq. (3.2.2) as well as the MD class of solutions, we know that all field variables ψ_n, Ω_n, I_n $\forall$ $n \geq 0$ can only depend on the variable θ and must not contain a radial dependence in order to ensure regularity at the horizon. However, from the differential equation for ψ_2 in the near-NHEK limit, specifically Eq. (4.3.8), it becomes evident that the only way to ensure that all radial dependence is eliminated is by demanding that $\sigma = 0$, which would result in a return to where we started; the NHEK limit. This is not the desired outcome as we stand to learn nothing about the behaviour of the magnetosphere in the near-NHEK limit, and it can therefore be concluded that the "naive approach" is inadequate. As a consequence thereof, it is rather pointless to calculate $(\tilde{F}^{(1)})^2$ as we now know that the above method is wrong. This leaves us in a bit of a pickle as alternative methods for producing a magnetically-dominated magnetosphere in the near-NHEK limit of the Kerr metric must be found.

4.3.2 A brief history of a failed attempt

On the path to singling out an appropriate expansion which allows for the construction of a FF, stationary and axisymmetric magnetically-dominated magnetosphere I had to work through different suggestions through trial and error. In this Section I will present another attempt which was unsuccessful in recovering a physically acceptable solution as it is equally important to delve into the failed attempts as the successful ones, since they provide us with hints about how to proceed and assist us in the process of learning.

My first attempt aside from the "naive approach" was made using the following suggestion. Similarly to the NHEK limit expansion it is assumed that the field variables follow this expansion around the horizon

$$\psi(r,\theta) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n \psi}{\partial r^n}(r - r_+)^n, \tag{4.3.9}$$

however, with

$$\begin{aligned} r - r_+ &= \lambda \tilde{R} r_+ \\ &= \frac{r_0}{2}\lambda \tilde{R}\left(1 + \sqrt{1 - \frac{4a^2}{r_0^2}}\right) \end{aligned} \tag{4.3.10}$$

such that for $a = (r_0/2)\sqrt{1-\sigma^2\lambda^2}$ I find

$$r - r_+ = \frac{r_0}{2}\lambda \tilde{R}(1 + \sigma\lambda) \tag{4.3.11}$$

resulting in the following expansion written explicitly

$$\psi(r,\theta) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^n \psi}{\partial r^n}\left(\frac{r_0}{2}\lambda\tilde{R}\right)^n (1+\sigma\lambda)^n \tag{4.3.12a}$$

$$= \psi_0 + \left(\frac{r_0}{2}\tilde{R}\psi_1\right)\lambda + \frac{r_0}{8}\tilde{R}\left(4\sigma\psi_1 + r_0\tilde{R}\psi_2\right)\lambda^2 + \mathcal{O}(\lambda^3), \tag{4.3.12b}$$

i.e. an extra $\sigma\lambda$-term compared to the NHEK limit Taylor expansion and a "mixing" of field variables at the second order post-near-NHEK correction. The Taylor expansion Eq. (4.3.12) successfully recovers the near-NHEK attractor solution presented in Section 4.1.1 that is FF, self-similar, axisymmetric and stationary. When looking at the first post-near-NHEK order corrections I find the following field strength

$$\begin{aligned} \tilde{F}^{(0)} = \frac{r_0}{2}\psi_1(r_0\Omega_0 - 1)d\tilde{T}\wedge d\tilde{R} + d\theta \wedge \Bigg[&\psi_1' d\tilde{\Phi} - \frac{r_0}{2}\tilde{R}\left(r_0\Omega_1\psi_0' + \psi_1'(r_0\Omega_0 - 1)\right)d\tilde{T} \\ &- \frac{r_0\big(2I_0(-\sigma\tilde{R}(\Gamma-3) + \tilde{R}^2 + 2\sigma^2) + r_0\Gamma I_1\tilde{R}(\tilde{R}+2\sigma)\big)}{2\Gamma\Lambda\tilde{R}(\tilde{R}+2\sigma)^2}d\tilde{R}\Bigg] \end{aligned} \tag{4.3.13}$$

and current

$$\tilde{j}^{(0)} = \frac{1}{r_0^4\tilde{R}(\tilde{R}+2\sigma)^2\Gamma^4\Lambda}\left[2\tilde{j}^{\tilde{T}}_{(0)}\partial_{\tilde{T}} - \frac{1}{\Lambda^2}\tilde{j}^{\tilde{\Phi}}_{(0)}\partial_{\tilde{\Phi}}\right] + \tilde{j}^{\tilde{R}}_{(0)}\partial_{\tilde{R}} + \tilde{j}^{\theta}_{(0)}\partial_\theta, \tag{4.3.14}$$

where the components are rather messy and given by

$$\begin{aligned}
\tilde{j}^{\tilde{T}}_{(0)} &= \Lambda\Big(2(\tilde{R}+\sigma)(\tilde{R}+2\sigma)\Gamma'\psi_0'(r_0\Omega_0-1)+\Gamma\Big(2\sigma\tilde{R}(\Gamma(r_0^2\psi_1'\Omega_0'+r_0\psi_1''(r_0\Omega_0-1)\\
&+\psi_0''(r_0^2\Omega_1+4r_0\Omega_0-1))-6\psi_0''(r_0\Omega_0-1))+r_0\psi_0'\Big(r_0\tilde{R}\Gamma(\tilde{R}+2\sigma)\Omega_1'\\
&+4(\tilde{R}+\sigma)\Omega_0'(\tilde{R}(\Gamma-1)+\sigma(\Gamma-2))\Big)+\tilde{R}^2(\Gamma\Big(r_0^2\psi_1'\Omega_0'+r_0\psi_1''(r_0\Omega_0-1)\\
&+\psi_0''(r_0^2\Omega_1+4r_0\Omega_0-2)\Big)-4\psi_0''(r_0\Omega_0-1))+4\sigma^2\psi_0''(r_0(\Gamma-2)\Omega_0+2)\Big)\Big)\\
&+\Gamma\Lambda'\Big(\psi_0'\Big(r_0^2\tilde{R}\Gamma\Omega_1(\tilde{R}+2\sigma)+2(\tilde{R}+\sigma)\Big(\tilde{R}(-\Gamma+2r_0(\Gamma-1)\Omega_0+2)+2\sigma(r_0(\Gamma-2)\Omega_0+2)\Big)\Big)\\
&+r_0\tilde{R}\Gamma(\tilde{R}+2\sigma)\psi_1'(r_0\Omega_0-1)\Big), \qquad (4.3.15)\\
\tilde{j}^{\tilde{R}}_{(0)} &= \frac{4I_0'(\tilde{R}+\sigma\Gamma+\sigma)-2r_0\tilde{R}\Gamma I_1'}{r_0^3\Gamma^3\Lambda}, \qquad (4.3.16)\\
\tilde{j}^{\theta}_{(0)} &= \frac{2I_1}{r_0^2\Gamma^2\Lambda}, \qquad (4.3.17)\\
\tilde{j}^{\tilde{\Phi}}_{(0)} &= 4\Lambda^2\Big(\Gamma(\Gamma+r_0(\Gamma-4)\Omega_0+4)\Lambda'\psi_0'+\Lambda(\psi_0'(2(r_0\Omega_0-1)\Gamma'+r_0(\Gamma-4)\Gamma\Omega_0')\\
&+\Gamma(\Gamma+r_0(\Gamma-4)\Omega_0+4)\psi_0'')\Big)\sigma^3+2\tilde{R}\Big(-4\Gamma^3(r_0\Omega_0-1)\Gamma'\psi_0'\Lambda^5\\
&-2\Gamma^4\Big(3(r_0\Omega_0-1)\Lambda'\psi_0'+\Lambda(r_0\psi_0'\Omega_0'+(r_0\Omega_0-1)\psi_0'')\Big)\Lambda^4+10(r_0\Omega_0-1)\Gamma'\psi_0'\Lambda^3\\
&-20\Gamma((r_0\Omega_0-1)\Lambda'\psi_0'+\Lambda(r_0\psi_0'\Omega_0'+(r_0\Omega_0-1)\psi_0''))\Lambda^2+\Gamma^2\Big(8\Lambda'\psi_0'\\
&+\Lambda\Big(\Big(4r_0\psi_1(r_0\Omega_0-1)-5\psi_0''+r_0\Big(2r_0\psi_1'\Omega_0'+\psi_0'(15\Omega_0'+2r_0\Omega_1')+(15\Omega_0+2r_0\Omega_1)\psi_0''\\
&+2(r_0\Omega_0-1)\psi_1''\Big)\Big)\Lambda^2+\Lambda'((2\Omega_1r_0^2+15\Omega_0r_0-5)\psi_0'+2r_0(r_0\Omega_0-1)\psi_1')\Lambda-8\psi_0''\Big)\Big)\Big)\sigma^2\\
&+2\tilde{R}^2\Big(-4\Gamma^3(r_0\Omega_0-1)\Gamma'\psi_0'\Lambda^5-2\Gamma^4(3(r_0\Omega_0-1)\Lambda'\psi_0'+\Lambda(r_0\psi_0'\Omega_0'+(r_0\Omega_0-1)\psi_0''))\Lambda^4\\
&+8(r_0\Omega_0-1)\Gamma'\psi_0'\Lambda^3-16\Gamma((r_0\Omega_0-1)\Lambda'\psi_0'+\Lambda(r_0\psi_0'\Omega_0'+(r_0\Omega_0-1)\psi_0''))\Lambda^2\\
&+\Gamma^2\Big(8\Lambda'\psi_0'+\Lambda((4r_0\psi_1(r_0\Omega_0-1)-6\psi_0''+r_0(3r_0\psi_1'\Omega_0'+\psi_0'(14\Omega_0'+3r_0\Omega_1')\\
&+(14\Omega_0+3r_0\Omega_1)\psi_0''+3(r_0\Omega_0-1)\psi_1''))\Lambda^2+\Lambda'((3\Omega_1r_0^2+14\Omega_0r_0-6)\psi_0'\\
&+3r_0(r_0\Omega_0-1)\psi_1')\Lambda-8\psi_0'')\Big)\Big)\sigma+\tilde{R}^3\Big(-2\Gamma^3(r_0\Omega_0-1)\Gamma'\psi_0'\Lambda^5-\Gamma^4\Big(3(r_0\Omega_0-1)\Lambda'\psi_0'\\
&+\Lambda(r_0\psi_0'\Omega_0'+(r_0\Omega_0-1)\psi_0'')\Big)\Lambda^4+4(r_0\Omega_0-1)\Gamma'\psi_0'\Lambda^3-8\Gamma\Big((r_0\Omega_0-1)\Lambda'\psi_0'\\
&+\Lambda(r_0\psi_0'\Omega_0'+(r_0\Omega_0-1)\psi_0'')\Big)\Lambda^2+2\Gamma^2\Big(2\Lambda'\psi_0'+\Lambda\Big((r_0\psi_1(r_0\Omega_0-1)-2\psi_0''\\
&+r_0(r_0\psi_1'\Omega_0'+\psi_0'(4\Omega_0'+r_0\Omega_1')+(4\Omega_0+r_0\Omega_1)\psi_0''+(r_0\Omega_0-1)\psi_1''))\Lambda^2\\
&+\Lambda'((\Omega_1r_0^2+4\Omega_0r_0-2)\psi_0'+r_0(r_0\Omega_0-1)\psi_1')\Lambda-2\psi_0''\Big)\Big)\Big). \qquad (4.3.18)
\end{aligned}$$

The Bianchi identity $(d\tilde{F})^{(0)}=0$ yields the same Ω_1 as in the NHEK limit, while ψ_1 found from the FF condition $(\tilde{F}\cdot\tilde{j})^{(-1)}=0$ is also identical to the first post-NHEK order variable.

However, the second Znajek condition fails to reduce one of the equations produced by the FF condition to null and instead the following field variable I_1 is found

$$I_1 = \frac{1}{r_0^3 I_0 \tilde{R}(\tilde{R}+2\sigma)} \Bigg[2I_0^2 r_0^2 \sigma(\tilde{R}+\sigma) + \frac{\Lambda^2}{\Gamma}(r_0\Omega_0 - 1)\psi_0' \bigg(\Big(2(\tilde{R}+\sigma)\Big(\tilde{R} + r_0\tilde{R}(\Gamma+1)\Omega_0 + \sigma(2+\Gamma+r_0(\Gamma-2)\Omega_0)\Big) + r_0^2\tilde{R}(\tilde{R}+2\sigma)\Gamma\Omega_1 \Big)\psi_0' + r_0\tilde{R}(\tilde{R}+2\sigma)\Gamma(r_0\Omega_0-1)\psi_1' \bigg) \Bigg]. \tag{4.3.19}$$

Already we notice a problem since we only allow $I_1 = I_1(\theta)$ while Eq. (4.3.19) clearly also contains a radial dependence. The expansion is thus inconsistent at the first post order correction and, therefore, cannot be used to recover a magnetically-dominated magnetosphere in the near-NHEK limit. I must yet again turn my attention towards other options.

Chapter 5

Novel general expansion

In this Chapter I will be presenting an expansion of the field variables with respect to not only the scaling parameter λ but also the parameter κ (or σ) which is introduced in order to present the most general Taylor expansion of the field variables. I will examine the regularity condition using said expansion and confirm whether it is possible to recover the Menon-Dermer class of solutions given the exclusion of the purely radial near-NHEK contributions before presenting the leading order of the near-NHEK limit of the Kerr metric as well as the post-near-NHEK order corrections up to the second order in λ.

5.1 Expansion of the field variables

Consider the spin parameter parametrised by κ

$$a = \frac{1}{2}\sqrt{1-\kappa^2}, \tag{5.1.1}$$

and coordinates

$$\hat{T} = \frac{t}{r_0}, \quad \hat{R} = 2\frac{r-r_+}{r_0}, \quad \Phi = \phi - \frac{t}{r_0}. \tag{5.1.2}$$

In the near-NHEK limit the field variables are expanded for small $\hat{R} = \lambda\tilde{R}$ and $\kappa = \lambda\sigma$ yielding the following

$$\begin{aligned}\psi(\hat{R},\theta,\kappa) &= \psi|_{\hat{R}=\kappa=0} + \lambda\left(\sigma\partial_\kappa\psi|_{\hat{R}=\kappa=0} + \tilde{R}\partial_{\hat{R}}\psi|_{\hat{R}=\kappa=0}\right) \\ &+ \frac{1}{2}\lambda^2\left(\sigma^2\partial_\kappa^2\psi|_{\hat{R}=\kappa=0} + 2\tilde{R}\sigma\partial_\kappa\partial_{\hat{R}}\psi|_{\hat{R}=\kappa=0} + \hat{R}^2\partial_{\hat{R}}^2\psi|_{\hat{R}=\kappa=0}\right) + \mathcal{O}(\lambda^3).\end{aligned} \tag{5.1.3}$$

In order to make the above expression more digestible and present it in a similar way to earlier Taylor expansions of the field variables at the horizon, I define

$$\psi|_{\hat{R}=\kappa=0} := \psi_{00}, \quad \partial_\kappa \psi|_{\hat{R}=\kappa=0} := \psi_{01}, \quad \partial_{\hat{R}}\psi|_{\hat{R}=\kappa=0} := \psi_{10},$$
$$\partial_\kappa^2 \psi|_{\hat{R}=\kappa=0} := \psi_{02}, \quad \partial_\kappa \partial_{\hat{R}}\psi|_{\hat{R}=\kappa=0} := \psi_{11}, \quad \partial_{\hat{R}}^2\psi|_{\hat{R}=\kappa=0} := \psi_{20},$$

with the subscript "00" pertaining to the leading order, "01" and "10" to the first order post-near-NHEK correction, and "02", "20", and "11" to the second order post-near-NHEK correction. The θ dependence is omitted explicitly but is implied as we know from the MD class of solutions that the field variables cannot contain a radial dependence.

Thus in Boyer-Lindquist coordinates Eq. (5.1.3) reads

$$\psi(r,\theta) = \psi_{00} + \kappa\psi_{01} + 2\frac{r-r_+}{r_0}\left(\psi_{10} + \kappa\psi_{11}\right) + \frac{1}{2}\left(\left(2\frac{r-r_+}{r_0}\right)^2 \psi_{20} + \kappa^2\psi_{02}\right), \quad (5.1.4a)$$

$$I(r,\theta) = I_{00} + \kappa I_{01} + 2\frac{r-r_+}{r_0}\left(I_{10} + \kappa I_{11}\right) + \frac{1}{2}\left(\left(2\frac{r-r_+}{r_0}\right)^2 I_{20} + \kappa^2 I_{02}\right), \quad (5.1.4b)$$

$$\Omega(r,\theta) = \Omega_{00} + \kappa\Omega_{01} + 2\frac{r-r_+}{r_0}\left(\Omega_{10} + \kappa\Omega_{11}\right) + \frac{1}{2}\left(\left(2\frac{r-r_+}{r_0}\right)^2 \Omega_{20} + \kappa^2\Omega_{02}\right), \quad (5.1.4c)$$

where I have included the expressions for I and Ω for completeness.

5.2 Expansion of the Znajek condition

Before I move on to the explicit perturbation of the field it is fitting to consider the Znajek condition at near-extremality with the expansion of field variables presented in the previous Section. The Znajek condition on the event horizon is, as before, given by Eq. (2.5.1). The scaling coordinates Eq. (4.1.1b) as well as Eq. (5.1.4) are implemented in the condition while letting $\lambda \to 0$, yielding to the leading order the first Znajek condition[1]

$$I_{00} = \frac{\Lambda}{r_0}(r_0\Omega_{00} - 1)\psi'_{00} \quad (5.2.1)$$

as expected. Expanding the left-hand-side of Eq. (2.5.1) to the first order in λ I find

$$r_0 I_{00}(\tilde{R} + \sigma) + \Gamma\Big[r_0\sigma I_{01} + r_0\tilde{R}I_{10} - \Lambda\Big(r_0\sigma\psi'_{00}(\Omega_{00} + \Omega_{01})$$
$$+ \sigma(r_0\Omega_{00} - 1)\psi'_{01} + \tilde{R}\big((r_0(\Omega_{00} + \Omega_{10}) - 1)\psi'_{00} + (r_0\Omega_{00} - 1)\psi'_{10}\big)\Big)\Big] \quad (5.2.2)$$

[1]Note how this is equivalent to the first Znajek condition, just written using the new convention.

which when evaluated at $\tilde{R} = 0$, i.e. $r = r_+$ yields the following correction to the Znajek condition

$$I_{01} = \frac{\Lambda}{r_0 \Gamma} \left((1 + r_0(\Gamma - 1)\Omega_{00} + r_0 \Gamma \Omega_{01}) \, \psi'_{00} + \Gamma (r_0 \Omega_{00} - 1) \psi'_{01} \right), \tag{5.2.3}$$

while the Znajek condition at the second order in λ is found to be

$$I_{02} = \frac{1}{r_0 \Gamma} \Bigg[-2r_0 I_{01} + \Lambda \bigg(\Big(2 - 2r_0 \Omega_{00} + \Gamma (2r_0 \Omega_{00} - 1 + 2r_0 \Omega_{01} + r_0 \Omega_{02}) \Big) \psi'_{00} \\ + \Gamma \Big(2r_0 \Omega_{01} \psi'_{01} - \psi'_{02} + r_0 \Omega_{00} (2\psi'_{01} + \psi'_{02}) \Big) \bigg) \Bigg]. \tag{5.2.4}$$

Note how the corrections to the Znajek condition only produce equations pertaining to the κ-derivatives of the field variables, that is I_{01}, I_{02}, and so forth. This is because the Znajek condition is evaluated at $\tilde{R} = 0$ where the $\tilde{R}$-derivatives, naturally, are zero. However, the second Znajek condition Eq. (2.5.3) is still valid for $\kappa = 0$, i.e. for I_{10}.

While we are on the subject of regularity I find it fitting to introduce the concept of *light surfaces* which, if physically relevant, can impose further regularity upon the solution.

5.3 Light surfaces

In this Section I will be considering two critical surfaces which I have not yet properly introduced. The two surfaces appearing in the Kerr metric are known as light surfaces and I will presently investigate their relevance to the main objective of this thesis.

It is customary to consider the light surfaces when looking at the field strength in the NHEK and near-NHEK limit since these critical surfaces are singular points of the stream equation and they can thus have an impact on the final result. However, before I delve into how the light surfaces affect, or if they affect, the magnetosphere in the near-NHEK limit of the Kerr metric, it is appropriate to explore the surfaces in greater detail. The Kerr background actually contains four critical surfaces one of which is the event horizon and the second is the asymptotic region at infinity. The remaining two surfaces are found using the velocity vector χ from Eq. (1.1.4), or more precisely by looking at the surfaces where χ of an observer co-rotating with the magnetosphere becomes null

$$\chi^\mu \chi_\mu = g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi} = 0. \tag{5.3.1}$$

This allows one to solve for $r_{\text{LS}}(\theta)$ which reveals the position of the light surfaces. On these surfaces one would have to move with the speed of light to follow the plasma on the magnetic field lines. The two solutions will correspond to the inner light surface (ILS) at $r = r_{\text{ILS}}(\theta)$ and the outer light surface (OLS) at $r = r_{\text{OLS}}(\theta)$, where the OLS can be considered the analogue of the pulsar light cylinder while the ILS is a purely GR occurrence. By considering small but constant Ω it is possible to see that the the radius is of the order $1/\Omega$, i.e. cylindrical similar to the pulsar light cylinder. By instead setting $\Omega = 0$, corresponding to $g_{tt} = 0$, one can infer the existence of the ILS from the existence of the ergosphere [35]. The ILS will lie inside of the ergosphere where the Killing vector χ is spacelike, while χ is timelike for $r_{\text{ILS}}(\theta) < r < r_{\text{OLS}}(\theta)$ and again spacelike for $r_{\text{OLS}}(\theta) < r$. The location of the light surfaces of a Kerr black hole is illustrated in Figure 5.1.

In order to consider the norm of χ we need to make an ansatz for how $\tilde{R}$ scales with λ, since we are solving for $\tilde{R}$ while also expanding in powers of λ. I found that for

$$\tilde{R} = \sum_{n=0}^{2} \lambda^n \tilde{R}_{\text{LS},n} \tag{5.3.2}$$

the leading order is not consistent, so instead the following expansion was used

$$\tilde{R} = \sum_{n=-1}^{1} \lambda^n \tilde{R}_{\text{LS},n}. \tag{5.3.3}$$

The leading order is now consistent but it is evident that since $\tilde{R}_{\text{LS},-1}$ scales as λ^{-1} then the ILS will approach infinity as $\lambda \to 0$. Thus just as how the ergoregion is pushed towards infinity as we zoom into the near-NHEK region, the ILS is also pushed towards infinity. This means that we cannot gain any information by looking at the field lines crossing the ILS as this will not happen in the near-NHEK region of the Kerr black hole. In fact, the ILS will only extend into the near-NHEK region at the magnetic poles. The ILS will presumably appear in orders higher than λ^2 which is outside the scope of this thesis.

5.4 Recovering the Menon-Dermer solution

In order to check the validity of the expansion Eq. (5.1.3) let us consider whether it is possible to show that the Menon-Dermer class of solutions follows from the near-NHEK order. If not then we know that the direction in which we are heading is wrong. By removing all terms proportional to $\tilde{R}$ in the expansion Eq. (5.1.3) it should be possible to recover the Menon-Dermer solution since we know that this class of solutions purely depends on θ and not $\tilde{R}$. This results in the following expansion to the second order in λ (contained in κ)

$$\psi(r,\theta) = \psi_{00} + \kappa\psi_{01} + \frac{1}{2}\kappa^2\psi_{02} + \mathcal{O}(\kappa^3), \tag{5.4.1a}$$

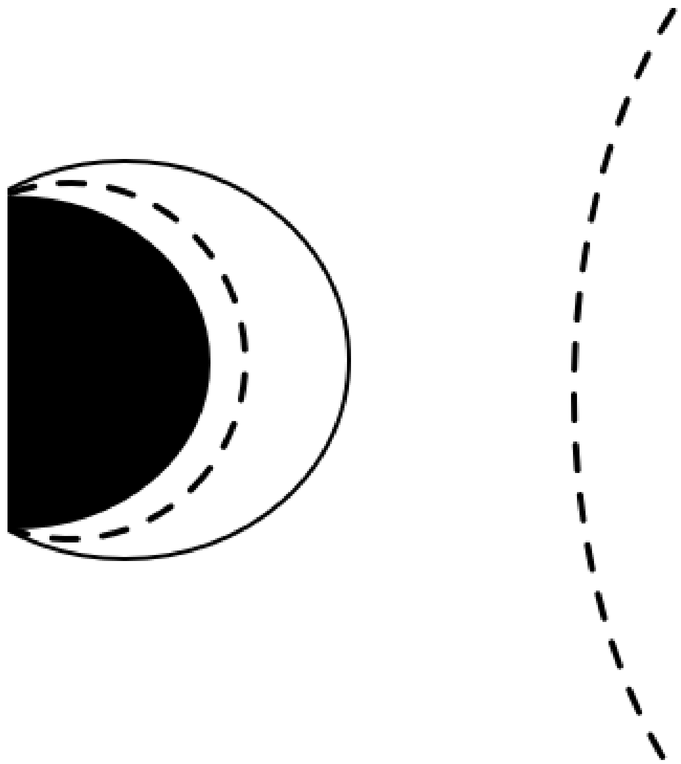

Figure 5.1: Sketch of a Kerr black hole with the inner light surface (dashed line) within the ergosphere (solid thin line), and outer light surface (dashed line) to the right of the black hole.

$$I(r,\theta) = I_{00} + \kappa I_{01} + \frac{1}{2}\kappa^2 I_{02} + \mathcal{O}(\kappa^3), \tag{5.4.1b}$$

$$\Omega(r,\theta) = \Omega_{00} + \kappa\Omega_{01} + \frac{1}{2}\kappa^2 \Omega_{02} + \mathcal{O}(\kappa^3), \tag{5.4.1c}$$

which recovers the near-NHEK attractor solution in the *leading order* as expected. When expanded to the *first post-near-NHEK order* the following correction to the field strength is found

$$\tilde{F}^{(0)}_{\rm MD} = \left[\sigma\left(r_0\Omega_{01}\psi'_{00} + (r_0\Omega_{00} - 1)\psi'_{01}\right) d\tilde{T} + r_0 \frac{I_{00}(\tilde{R}+\sigma) + \sigma I_{01}\Gamma}{\tilde{R}(\tilde{R}+2\sigma)\Gamma\Lambda}\, d\tilde{R} + \psi'_{00}\, d\tilde{\Phi}\right] \wedge d\theta, \tag{5.4.2}$$

with the four-current $\tilde{j}^{(0)}_{\rm MD}$ of the form

$$\tilde{j}^{(0)}_{\rm MD} = \tilde{j}^{\tilde{T}}_{(0)}\partial_{\tilde{T}} + \tilde{j}^{\tilde{R}}_{(0)}\partial_{\tilde{R}} + \tilde{j}^{\tilde{\Phi}}_{(0)}\partial_{\tilde{\Phi}} \tag{5.4.3}$$

and components

$$\tilde{j}^{\tilde{T}}_{(0)} = \frac{1}{r_0^4 \tilde{R}(\tilde{r}+2\sigma)(\Gamma-1)\Gamma^4}\Big[2\Gamma'\Big\{\Big((\tilde{R}+\sigma)(6-6\Gamma+\Gamma^2-2r_0(\Gamma-3)(\Gamma-1)\Omega_{00})$$
$$-r_0\sigma(\Gamma-2)\Gamma\Omega_{01}\Big)\psi'_{00}-\sigma(\Gamma-2)\Gamma(r_0\Omega_{00}-1)\psi'_{01}\Big\}$$
$$+4(\Gamma-1)\Gamma\Big\{r_0\psi''_{00}(2(\tilde{R}+\sigma)(\Gamma-1)\Omega'_{00}+\sigma\Gamma\Omega'_{01})-2(\tilde{R}+\sigma)(r_0\Omega_{00}-1)\psi''_{00}$$
$$+\Gamma\Big(r_0\sigma\psi'_{01}\Omega'_{00}+((\tilde{R}+\sigma)(2r_0\Omega_{00}-1)+r_0\sigma\Omega_{01})\psi''_{00}+\sigma(r_0\Omega_{00}-1)\psi''_{01}\Big)\Big\}\Big], \tag{5.4.4a}$$

$$\tilde{j}^{\tilde{R}}_{(0)} = \frac{2\sqrt{2}\left((\tilde{R}+\sigma)I'_{00}-\sigma\Gamma I'_{01}\right)}{r_0^3\sqrt{1-\Gamma}\,\Gamma^2}, \tag{5.4.4b}$$

$$\tilde{j}^{\tilde{\Phi}}_{(0)} = \frac{1}{r_0^4\tilde{R}(\tilde{r}+2\sigma)(\Gamma-1)^2\Gamma^4}\Big[\Gamma'\Big\{\Big(\tilde{R}^2(12-12r_0\Omega_{00}-(\Gamma-2)\Gamma(-11+r_0(13+(\Gamma-2)\Gamma)\Omega_{00}))$$
$$+\sigma^2(\Gamma-1)(12(r_0\Omega_{00}-1)+\Gamma(10-14r_0\Omega_{00}+\Gamma(3r_0\Omega_{00}-1)+2r_0(\Gamma-2)\Omega_{01}))$$
$$+2\sigma\tilde{R}(12-12r_0\Omega_{00}-(\Gamma-2)\Gamma(-11+r_0(13+(\Gamma-2)\Gamma)\Omega_{00}-r_0(\Gamma-1)\Omega_{01}))\Big)\psi'_{00}$$
$$+2\sigma(\tilde{R}+\sigma)(\Gamma-2)(\Gamma-1)\Gamma(r_0\Omega_{00}-1)\psi'_{01}\Big\}-2(\Gamma-1)\Gamma\Big(r_0(\Gamma-1)\psi'_{00}((-4(\tilde{R}+\sigma)^2$$
$$+3(\tilde{R}+\sigma)^2\Gamma+\tilde{R}(\tilde{R}+2\sigma)\Gamma^2)\Omega'_{00}2\sigma(\tilde{R}+\sigma)\Gamma\Omega'_{01})+r_0\tilde{R}(\tilde{R}+2\sigma)\Gamma^3\Omega_{00}\psi''_{00}$$
$$+4(\tilde{R}+\sigma)^2(r_0\Omega_{00}-1)\psi''_{00}-(\tilde{R}+\sigma)\Gamma(2r_0\sigma\psi'_{01}\Omega'_{00}+((\tilde{R}+\sigma)(7r_0\Omega_{00}-5)+2r_0\sigma\Omega_{01})\psi''_{00}$$
$$+2\sigma(r_0\Omega_{00}-1)\psi''_{01})+\Gamma^2(2r_0\sigma(\tilde{R}+\sigma)\psi'_{01}\Omega'_{00}$$
$$+(-\sigma^2+r_0(3\sigma^2+2\tilde{R}(\tilde{R}+2\sigma))\Omega_{00}+2r_0\sigma(\tilde{R}+\sigma)\Omega_{01})\psi''_{00}+2\sigma(\tilde{R}+\sigma)(r_0\Omega_{00}-1)\psi''_{01})\Big)\Big]. \tag{5.4.4c}$$

Here Λ has been substituted by

$$\Lambda = \frac{\sqrt{2-2\Gamma}}{\Gamma} \tag{5.4.5}$$

for simplicity[2]. The Bianchi identity is automatically satisfied meaning that I cannot learn anything about Ω_{01} from it, and the FF condition only produces one equation which is of the form

$$0 = \tilde{R}(\ldots)+\sigma(\ldots), \tag{5.4.6}$$

meaning that the component proportional to $\tilde{R}$ and the component proportional to σ must be set equal to zero independently. The σ-term is reduced to zero given the modified first Znajek condition Eq. (5.2.1) for I_{00} and the Znajek condition Eq. (5.2.3) for I_{01} while the $\tilde{R}$-term is solved for

$$\Omega_{00} = \frac{2}{r_0\Gamma^2\Lambda^2} \tag{5.4.7}$$

[2]It is easy to verify this expression by substituting Γ and Λ by Eq. (3.1.5).

which is recognised as the leading term of the MD angular velocity Ω_{MD}. This allows one to show that

$$\psi_{01} = c_1 + \frac{r_0}{2}\int d\theta\, I_{01}\Gamma\Lambda, \tag{5.4.8}$$

where c_1 is an arbitrary integration constant and this is exactly the Menon-Dermer $\psi_{\rm MD}$ with $a = r_0/2$, i.e. at extreme Kerr, when $c_1 = 0$. Using Eq. (5.4.7) and the Znajek condition for I_{00} and I_{01} it is now also possible to show that the field strength $(\tilde{F}^{(0)}_{\rm MD})^2$ is null.

Continuing to the *second post-near-NHEK order* I find the field strength to be

$$\begin{aligned}\tilde{F}^{(1)}_{\rm MD} = \Bigg[&\frac{1}{2}\sigma^2\left(r_0\Omega_{02}\psi'_{00} + 2r_0\Omega_{01}\psi'_{01} + (r_0\Omega_{00}-1)\psi'_{02}\right) d\tilde{T} \\ &+ \frac{r_0\left(I_{00}\left(2\sigma\tilde{R} + \tilde{R}^2 - 2\sigma^2(\Gamma-1)\right) + \sigma\left(2I_{01}(\tilde{R}+\sigma) + \sigma I_{02}\Gamma\right)\right)}{2\tilde{R}(\tilde{R}+2\sigma)\Gamma\Lambda}\, d\tilde{R} - \sigma\psi'_{00}\, d\tilde{\Phi}\Bigg] \wedge d\theta,\end{aligned} \tag{5.4.9}$$

with $\tilde{j}^{(1)}_{\rm MD}$ of the following form (and components in Appendix B)

$$\tilde{j}^{(1)}_{\rm MD} = \frac{1}{r_0^4(\Gamma-1)\Gamma^5\tilde{R}(\tilde{R}+2\sigma)}\left[\tilde{j}^{\tilde{T}}_{(1)}\partial_{\tilde{T}} + \frac{1}{\Gamma-1}\tilde{j}^{\tilde{\Phi}}_{(1)}\right] - \frac{1}{r_0^3\sqrt{1-\Gamma}\,\Gamma^3}\tilde{j}^{\tilde{R}}_{(1)}. \tag{5.4.10}$$

The Bianchi identity is again automatically satisfied and the FF condition is solved using the Znajek conditions for I_{00}, I_{01}, and I_{02} as well as Eq. (5.4.7) yielding

$$\Omega_{01} = 0 \tag{5.4.11}$$

which in fact is the first order correction to $\Omega_{\rm MD}$ since

$$\Omega_{\rm MD} = \frac{2}{r_0\Gamma^2\Lambda^2} + \frac{1}{2}\frac{2}{r_0\Gamma^2\Lambda^2}\sigma^2\lambda^2 + \mathcal{O}(\lambda^3). \tag{5.4.12}$$

As expected, the field strength $(\tilde{F}^{(1)}_{\rm MD})^2$ is found to be null when implementing the Znajek conditions and the expressions found for Ω_{00} and Ω_{01} which is consistent with the MD solution. Knowing that it is indeed possible to recover the MD class of solutions from the expansion it is finally time to consider the near-NHEK limit such that it can be determined if a magnetically-dominated magnetosphere can be constructed.

5.5 The near-NHEK limit

In this Section I will present a general perturbation of the Kerr metric in the near-NHEK limit in which the variables ψ_{00}, Ω_{00} are left arbitrary and ψ_{01} and Ω_{01} will be kept unconstrained for the sake of generality.

The expansion Eq. (5.1.3), or Eq. (5.1.4), successfully recovers the near-NHEK geometry

$$d\tilde{s}^2 = \frac{r_0}{2}\Gamma\left[-\tilde{R}(\tilde{R}+2\sigma)d\tilde{T}^2 + \frac{d\tilde{R}^2}{\tilde{R}(\tilde{R}+2\sigma)} + d\theta^2 + \Lambda^2\left(d\tilde{\Phi} + (\tilde{R}+\sigma)d\tilde{T}\right)^2\right] \tag{5.5.1}$$

as well as the near-NHEK attractor

$$\tilde{F}^{(-1)} = \frac{r_0 I_0}{\Lambda} d\left[\tilde{T} - \frac{1}{2\sigma}\log\left(1+\frac{2\sigma}{\tilde{R}}\right)\right]\wedge d\theta, \tag{5.5.2}$$

which as known is null, FF and self-similar. The interesting part, however, is the post-near-NHEK corrections and I will thus quickly move on to these.

The first order correction to the near-NHEK metric is somewhat cumbersome to look at and is found to be, explicitly

$$\begin{aligned}(d\tilde{s}^2)^{(1)} = \frac{r_0^2}{2}\Bigg[&\frac{1}{2}(\tilde{R}+\sigma)\Big(2\tilde{R}(\tilde{R}+2\sigma)(2\Gamma-1) + 2(-(\tilde{R}+\sigma)^2 + (\sigma^2 + 2\tilde{R}(\tilde{R}+2\sigma))\Gamma)\Lambda^2 \\ &- \tilde{R}(\tilde{R}+2\sigma)\Gamma^3\Lambda^4\Big)\, d\tilde{T}^2 + \frac{\Lambda^2}{4}\Big(- \tilde{R}\Gamma^3\Lambda^2(\tilde{R}+2\sigma) + 2\Gamma(2\tilde{R}+\sigma)(2\tilde{R}+3\sigma) \\ &- 4(\tilde{R}+\sigma)^2\Big)\, d\tilde{T}d\tilde{\Phi} + \frac{(\tilde{R}+\sigma)}{\tilde{R}(\tilde{R}+2\sigma)}\, d\tilde{R}^2 + (\tilde{R}+\sigma)\, d\theta^2 + (\tilde{R}+\sigma)(2\Gamma-1)\Lambda^2\, d\tilde{\Phi}^2\Bigg],\end{aligned} \tag{5.5.3}$$

while the correction to the field strength is found to be

$$\begin{aligned}\tilde{F}^{(0)} = d\theta\wedge\Bigg[&\psi_{00}'\, d\tilde{\Phi} - \Big(r_0\sigma\Omega_{01}\psi_{00}' + \sigma(r_0\Omega_{00}-1)\psi_{01}' \\ &+ \tilde{R}\,(r_0\Omega_{10}\psi_{00}' + (r_0\Omega_{00}-1)\psi_{10}')\Big)d\tilde{T} - \frac{r_0(I_{00}(\tilde{R}+\sigma) + (\sigma I_0 + I_{10}\tilde{R})\Gamma)}{\tilde{R}(\tilde{R}+2\sigma)\Gamma\Lambda}\, d\tilde{R}\Bigg] \\ &+ \psi_{10}(r_0\Omega_{00}-1)\, d\tilde{T}\wedge d\tilde{R}.\end{aligned} \tag{5.5.4}$$

The current $\tilde{j}^{(0)}$ is quite lengthy

$$\tilde{j}^{(0)} = \frac{1}{r_0^4\tilde{R}(\tilde{R}+2\sigma)\Gamma^4\Lambda}\left[4\,\tilde{j}^{\tilde{T}}_{(0)}\partial_{\tilde{T}} + \frac{1}{\Lambda^2}\,\tilde{j}^{\tilde{\Phi}}_{(0)}\partial_{\tilde{\Phi}}\right] + \tilde{j}^{\tilde{R}}_{(0)}\partial_{\tilde{R}} + \tilde{j}^{\theta}_{(0)}\partial_\theta \tag{5.5.5}$$

with components

$$\begin{aligned}\tilde{j}^{\tilde{T}}_{(0)} = \Gamma\Lambda'\Big(&\Big\{\sigma(2 - 2r_0\Omega_{00} + \Gamma(-1 + 2r_0\Omega_{00} + r_0\Omega_{01})) \\ &+ \tilde{R}(2 - 2r_0\Omega_{00} + \Gamma(-1 + 2r_0\Omega_{00} + r_0\Omega_{10}))\Big)\psi_{00}' + \Gamma(r_0\Omega_{00}-1)(\sigma\psi_{01}'\tilde{R}\psi_{10}')\Big\} \\ &+ \Lambda\Big\{(\tilde{R}+\sigma)(r_0\Omega_{00}-1)\Gamma'\psi_{00}' + \Gamma\Big(r_0\psi_{00}'\Big((\tilde{R}+\sigma)(\Gamma-1)\Omega_{00}' + \Gamma(\sigma\Omega_{01}' + \tilde{R}\Omega_{10}')\Big) \\ &- 2(\tilde{R}+\sigma)(r_0\Omega_{00}-1)\psi_{00}'' + \Gamma\big(r_0\sigma\psi_{01}'\Omega_{00}' + \sigma(-1 + r_0\Omega_{00} + r_0\Omega_{01})\psi_{00}''\end{aligned}$$

$$+ \sigma(r_0\Omega_{00} - 1)\psi''_{01} + \tilde{R}\Big(r_0\psi'_{10}\Omega'_{00} + (-1 + 2r_0\Omega_{00} + r_0\Omega_{10})\psi''_{00} + (r_0\Omega_{00} - 1)\psi''_{00}\Big)\Big)\Big)\Big\}, \tag{5.5.6a}$$

$$\tilde{j}^{\tilde{R}}_{(0)} = \frac{4(\tilde{R} + \sigma)I'_{00} - 4\Gamma(\sigma I'_{01} + \tilde{R}I'_{10})}{r_0^3\Gamma^3\Lambda}, \tag{5.5.6b}$$

$$\tilde{j}^{\theta}_{(0)} = \frac{4I_{00}}{r_0\Gamma^2\Lambda}, \tag{5.5.6c}$$

$$\begin{aligned}
\tilde{j}^{\tilde{\Phi}}_{(0)} = {} & \frac{r_0\Lambda I_{00}}{\psi'_{00}}\Big\{2\sigma\tilde{R}\Big(2\Gamma^3\Lambda^3\Gamma'\psi'_{00} - 4\Lambda\Gamma'\psi'_{00} + \Gamma^4\Lambda^2(3\Lambda'\psi'_{00} + \Lambda\psi''_{00}) + 8\Gamma(\Lambda'\psi'_{00} + \Lambda\psi''_{00}) \\
& - 2\Gamma^2(\Lambda'(\psi'_{01} + \psi'_{10}) + 2\Lambda\psi_{10})\Big) + \tilde{R}^2\Big(2\Gamma^3\Lambda^3\Gamma'\psi'_{00} - 4\Lambda\Gamma'\psi'_{00} + \Gamma^4\Lambda^2(3\Lambda'\psi'_{00} + \Lambda\psi''_{00}) \\
& + 8\Gamma(\Lambda'\psi'_{00} + \Lambda\psi''_{00}) - 4\Gamma^2(\Lambda'\psi'_{10} + \Lambda\psi_{10})\Big) - 4\sigma^2(\Lambda(\Gamma'\psi'_{00} - 2\Gamma\psi''_{00} + \Gamma^2\psi''_{01}) + \Gamma^2\Lambda'\psi'_{01})\Big\} \\
& + \Gamma\Big\{2\sigma\tilde{R}\Big(- 2\Gamma\Big(- 2\Lambda\psi''_{00} + \Lambda'\psi'_{00}(\Lambda^2(r_0(4\Omega_{00} + \Omega_{01} + \Omega_{10}) - 2) + 2) \\
& + \Lambda^3\Big(- 2\psi''_{00} - \psi''_{01} - \psi''_{10} + r_0\Big(\Omega_{00}(4\psi''_{00} + \psi''_{01} + \psi''_{10}) + \psi'_{00}(4\Omega'_{00} + \Omega'_{01} + \Omega'_{10}) + \psi''_{00}(\Omega_{01} + \Omega_{10}) \\
& + \Omega'_{00}(\psi'_{01} + \psi'_{10})\Big)\Big)\Big) + r_0\Gamma^3\Lambda^5\psi'_{00}\Omega'_{00} + 8r_0\Lambda^3\psi'_{00}\Omega'_{00}\Big) \\
& + \tilde{R}^2\Big(- 4\Gamma\Big(- \Lambda\psi''_{00} + \Lambda'\psi'_{00}(\Lambda^2(2r_0\Omega_{00} + r_0\Omega_{10} - 1) + 1) \\
& + \Lambda^3\Big(r_0\psi'_{00}(2\Omega'_{00} + \Omega'_{10}) + \psi''_{00}(2r_0\Omega_{00} + r_0\Omega_{10} - 1) + r_0\psi'_{10}\Omega'_{00} + \psi''_{10}(r_0\Omega_{00} - 1)\Big)\Big) \\
& + r_0\Gamma^3\Lambda^5\psi'_{00}\Omega'_{00} + 8r_0\Lambda^3\psi'_{00}\Omega'_{00}\Big) - 2\sigma^2\Lambda^2\Big(\Lambda'\psi'_{00}(\Gamma(3r_0\Omega_{00} + 2r_0\Omega_{01} - 1) - 4r_0\Omega_{00} + 4) \\
& + \Lambda\Big(\Gamma(\psi''_{00}(3r_0\Omega_{00} + 2r_0\Omega_{01} - 1) + 2r_0\psi'_{01}\Omega'_{00}) + r_0\psi'_{00}((3\Gamma - 4)\Omega'_{00} + 2\Gamma\Omega'_{01})\Big)\Big)\Big\}.
\end{aligned} \tag{5.5.6d}$$

The Bianchi identity $(d\tilde{F})^{(0)} = 0$ yields the same Ω_{10} as in the NHEK limit (corresponding to Ω_1)

$$\Omega_{10} = \frac{\psi_{10}\Omega_{00'}}{\psi'_{00}}, \tag{5.5.7}$$

and the FF condition $(\tilde{F}\cdot\tilde{j})^{(-1)} = 0$ also yields ψ_{10} similar to what was found in the NHEK limit

$$\psi_{10} = \frac{I_{10}\psi'_{00}}{I'_{00}}. \tag{5.5.8}$$

The second Znajek condition is valid when $\kappa = 0$, i.e. for I_{10} and this paired with the Znajek condition for I_{01} Eq. (5.2.3), satisfies the remaining equation obtained from the FF condition. It is possible to show that the NHEK solution found at the first order solves the first order correction in the near-NHEK limit. Using the NHEK result Eqs. (3.3.7)-(3.3.8) as well as the Znajek conditions for I_{01} and I_{00} one ensures that the field is FF and that the field strength $(\tilde{F}^{(0)})^2$

$$(\tilde{F}^{(0)})^2 = \frac{1}{r_0^4\tilde{R}\Gamma^3\Lambda^2(\tilde{R} + 2\sigma)}\Big[16\Lambda^2\psi'_{00}(r_0\Omega_{00} - 1)(\psi'_{00}(\tilde{R}(r_0(\Gamma - 1)\Omega_{00} + r_0\Gamma\Omega_{10} + 1)$$

$$
\begin{aligned}
&+ r_0\sigma\Gamma(\Omega_{00}+\Omega_{01}) - r_0\sigma\Omega_{00}+\sigma) + \Gamma(r_0\Omega_{00}-1)(\tilde{R}\psi'_{10}+\sigma\psi'_{01}))\\
&- 16r_0^2\Gamma I_{00}(\tilde{R}I_{10}+\sigma I_{01})\Big]
\end{aligned} \tag{5.5.9}
$$

is null, i.e. $(\tilde{F}^{(-1)})^2 = 0$.

The field strength at *the second order* is quite messy but I will present it for good measure

$$
\begin{aligned}
\tilde{F}^{(1)} = &\Big(\tilde{R}(r_0\psi_{10}\Omega_{10} + \psi_{20}(r_0\Omega_{00}-1)) + r_0\sigma\psi_{10}\Omega_{01} + \sigma\psi_{11}(r_0\Omega_{00}-1)\Big)\, d\tilde{R}\wedge d\tilde{T}\\
&+ \Bigg[\Bigg\{\frac{1}{2}\Big(2\sigma\tilde{R}\Big(r_0(\Omega_{11}\psi'_{00} + \Omega_{10}\psi'_{01} + \Omega_{01}\psi'_{10}) + \psi'_{11}(r_0\Omega_{00}-1)\Big)\\
&+ \tilde{R}^2\Big(r_0\Omega_{20}\psi'_{00} + 2r_0\Omega_{10}\psi'_{10} + \psi'_{20}(r_0\Omega_{00}-1)\Big) + r_0\sigma^2\Omega_{02}\psi'_{00}\\
&+ \sigma^2(2r_0\Omega_{01}\psi'_{01} + \psi'_{02}(r_0\Omega_{00}-1))\Big)\Bigg\}\, d\tilde{T}\\
&+ \Bigg\{\frac{1}{2\tilde{R}\Gamma\Lambda(\tilde{R}+2\sigma)}\Big(r_0(I_{00}(2\sigma\tilde{R}+\tilde{R}^2 - 2\sigma^2(\Gamma-1)) + 2(\tilde{R}+\sigma)(\tilde{R}I_{10}+\sigma I_{01})\\
&+ \Gamma(\tilde{R}(\tilde{R}I_{20} + 2\sigma I_{11}) + \sigma^2 I_{02}))\Big)\Bigg\}\, d\tilde{R} + \Big(\sigma\psi'_{01} + \tilde{R}\psi'_{10}\Big)\, d\tilde{\Phi}\Bigg]\wedge d\theta + \psi_{10}\, d\tilde{R}\wedge d\tilde{\Phi},
\end{aligned} \tag{5.5.10}
$$

and the four-current $\tilde{j}^{(1)}$ is even worse to look at. As it is very lengthy I will only present its form while omitting the components that can be found in Appendix B

$$
\begin{aligned}
\tilde{j}^{(1)} = &\frac{1}{r_0^4\Gamma^5\Lambda\tilde{R}(\tilde{R}+2\sigma)}\Big[\tilde{j}^{\tilde{T}}_{(1)}\partial_{\tilde{T}} + r_0\Gamma\tilde{R}(\tilde{R}+2\sigma)\,\tilde{j}^{\tilde{R}}_{(1)}\partial_{\tilde{R}}\\
&+ r_0\Gamma^2\tilde{R}(\tilde{R}+2\sigma)\,\tilde{j}^{\theta}_{(1)}\partial_\theta + \frac{1}{\Lambda^2}\,\tilde{j}^{\tilde{\Phi}}_{(1)}\partial_{\tilde{\Phi}}\Big].
\end{aligned} \tag{5.5.11}
$$

The Bianchi identity $(d\tilde{F})^{(1)} = 0$ contains a term proportional to $\tilde{R}$ and one proportional to σ which must be set equal to zero independently, allowing me to solve for Ω_{20} and Ω_{11}, yielding the following expressions

$$
\Omega_{20} = \frac{\mathcal{G}^2\Lambda^2\,(\psi'_{00}\Omega''_{00} - \psi''_{00}\Omega'_{00}) + 4\psi_{20}\Omega'_{00}}{4\psi'_{00}}, \tag{5.5.12a}
$$

$$
\Omega_{11} = \frac{1}{2}\left(\frac{\Omega'_{00}\,(2\psi_{11} - \mathcal{G}\Lambda\psi'_{01})}{\psi'_{00}} + \mathcal{G}\Lambda\Omega'_{01}\right) \tag{5.5.12b}
$$

where $\mathcal{G}$ is defined in Eq. (3.3.7). From the FF condition $(\tilde{F}\cdot\tilde{j})^{(0)} = 0$, which produces several equations, I again find one equation with a term proportional to $\tilde{R}$ and one proportional to σ that are solved independently yielding

$$
I_{20} = \frac{\mathcal{G}^2\Lambda^2\,(\psi'_{00}I''_{00} - \psi''_{00}I'_{00}) + 4\psi_{20}I'_{00}}{4\psi'_{00}}, \tag{5.5.13a}
$$

$$
\psi_{11} = \frac{1}{2I'_{00}}\Big[2I_{11}\psi'_{00} - \frac{1}{r_0\Gamma^2}\Big\{\mathcal{G}\Lambda\Big(\Gamma(\Lambda'\psi'_{00}(\psi'_{00}(r_0(\Gamma-1)\Omega_{00} + r_0\Gamma\Omega_{01} + 1)
$$

$$+ \Gamma\psi'_{01}(r_0\Omega_{00} - 1)) - r_0\Gamma\psi'_{01}I'_{00}) + \Lambda\psi'_{00}\Big(\Gamma'\psi'_{00}(r_0\Omega_{00} - 1)$$
$$+ \Gamma\Big(\psi''_{00} + r_0\psi'_{00}((\Gamma - 1)\Omega'_{00} + \Gamma\Omega'_{01}) + r_0\psi''_{00}((\Gamma - 1)\Omega_{00} + \Gamma\Omega_{01})$$
$$+ r_0\Gamma\psi'_{01}\Omega'_{00} + \Gamma\psi''_{01}(r_0\Omega_{00} - 1)\Big)\Big)\Big\}\Big]. \tag{5.5.13b}$$

The other equation that I obtained from the FF condition proved to be much more complicated while also being rather long. However, it should still be somewhat similar to the equation from which I found the second-order linear differential equation for ψ_2 in the NHEK limit, just with some additional terms proportional to σ. While the equation also contains terms proportional to σ^2 these are easily eliminated by imposing the Znajek conditions for I_{02} and I_{00}. Thus, the equation should be of the form

$$\psi''_{20} + a(\theta)\psi'_{20} + b(\theta)\psi_{20} + c(\theta) + \sigma(\ldots) = 0, \tag{5.5.14}$$

and in fact when $\sigma \to 0$ I find that $a(\theta)$ and $b(\theta)$ are identical to the constants in the NHEK limit presented in Eqs. (3.3.15a)-(3.3.15b), while the constant $c(\theta)$ differs slightly from the one found in Eq. (3.3.15c). This discrepancy is likely due to the difference of a factor of 2 in the definitions of the NHEK and near-NHEK expansions concerning the terms containing ψ_{n0}, Ω_{n0}, and I_{n0}[3]. Since the constant $c(\theta)$ that I have found in the near-NHEK limit does not contain $\tilde{R}$ it should not be a problem and I will just leave the constant as it is. The term proportional to σ also contains the variable ψ_{20} such that it should be possible to integrate the associated coefficient into the constant $b(\theta)$. The residual with no ψ_{20}-dependence can be used to redefine the NHEK $c(\theta)$ in order to recover the following differential equation for ψ_{20}

$$\psi''_{20} + a(\theta)\psi'_{20} + b^*(\theta)\psi_{20} + c^*(\theta) = 0, \tag{5.5.15}$$

where $b^*(\theta)$ and $c^*(\theta)$ are the revised constants containing σ. The constant $b^*(\theta)$ now reads

$$b^*(\theta) = b(\theta) + \sigma\frac{4\Gamma^4\Lambda I_{00}^2 r_0^3}{\psi'_{00}} \tag{5.5.16}$$

while the constant $c^*(\theta)$ is of the form

$$c^*(\theta) = (\ldots) + \sigma(\ldots), \tag{5.5.17}$$

and can be found in Appendix B as it is extremely long.

The variables ψ_{00}, Ω_{00} are still left arbitrary, while six others remain unfixed. These are Ω_{01}, ψ_{01}, Ω_{02}, I_{11}, ψ_{02}, and of course ψ_{20}, allowing for quite some room to play around

[3] Here $n = [0, 2]$.

with. Since this result is very general it is difficult to analyse the sign of the invariant $(\tilde{F}^{(1)})^2$ which can be found in Appendix B due to its length. Thus in order to make some concrete comments about the field strength, I will compare the novel general expansion to a known result.

Chapter 6

Special case of novel near-NHEK expansion

In 2020 Pompili presented in his Master's thesis[1] [4] an expansion of the field variables in the near-NHEK limit with which it was possible to recover not only the near-NHEK attractor solution but also the NHEK second order differential equation in the second post-near-NHEK order. I will presently argue that the expansion which was found is a special case of our general near-NHEK expansion Eq. (5.1.4), and that it is possible to recreate their results using Eq. (5.1.4) and the scaling coordinates Eq. (4.1.1b) with the appropriate conversion. Let us therefore consider the expansion and relevant results found in [4].

The following expansion of the field variables ψ, Ω, and I was used (written using the same convention as in Chapter 5)

$$\psi(r,\theta) = \psi_{00} + \lambda\left(\frac{r-r_p}{\lambda} + \frac{r_0}{2}\sigma + \sigma^2\lambda s_1(\theta)\right)\psi_{10P} + \lambda^2\psi_{2\tilde{R}}(\tilde{R})\psi_{20P} + \mathcal{O}(\lambda^3), \quad (6.0.1a)$$

$$\Omega(r,\theta) = \Omega_{00} + \lambda\left(\frac{r-r_p}{\lambda} + \frac{r_0}{2}\sigma + \sigma^2\lambda\omega_1(\theta)\right)\Omega_{10P} + \lambda^2\Omega_{2\tilde{R}}(\tilde{R})\Omega_{20P} + \mathcal{O}(\lambda^3), \quad (6.0.1b)$$

$$I(r,\theta) = I_{00} + \lambda\left(\frac{r-r_p}{\lambda} + \frac{r_0}{2}\sigma + \sigma^2\lambda i_1(\theta)\right)I_{10P} + \lambda^2 I_{2\tilde{R}}(\tilde{R})I_{20P} + \mathcal{O}(\lambda^3), \quad (6.0.1c)$$

where I have added a "P" to the name of the first and second order field variables in order to distinguish them from the ones in our general near-NHEK expansion. The functions s_1, i_1, and ω_1 are used to ensure that the second order linear differential equation for ψ_{20} is consistent, with all of the field variables only depending on θ and not $\tilde{R}$. It is, furthermore,

[1] The thesis is available at: `https://www.dropbox.com/sh/ap4co4v4qd91d6z/AADl7xn4mTfmA5wJ108otQ_na?dl=0`.

found that the second order field variables ψ_{20}, Ω_{20}, and I_{20} are consistent when the following expressions for $\psi_{2\tilde{R}}$, $\Omega_{2\tilde{R}}$, and $I_{2\tilde{R}}$ are chosen

$$\psi_{2\tilde{R}} = \frac{1}{2}\left(\frac{r_0}{2}\right)^2 (\tilde{R}(\tilde{R}+2\sigma)), \tag{6.0.2a}$$

$$\Omega_{2\tilde{R}} = \frac{1}{2}\left(\frac{r_0}{2}\right)^2 (\tilde{R}(\tilde{R}+2\sigma)), \tag{6.0.2b}$$

$$I_{2\tilde{R}} = \frac{1}{2}\left(\frac{r_0}{2}\right)^2 (\tilde{R}(\tilde{R}+2\sigma)). \tag{6.0.2c}$$

Our coordinate transformation from r to $\tilde{R}$ is slightly different from the one used in [4], which is

$$\tilde{R} = \frac{2r - r_+}{\lambda r_0}, \tag{6.0.3}$$

compared to our $\tilde{R} = 2(r - r_+)/(\lambda r_0)$. However, as will soon become evident, I am still able to recover the same solution using our expansion and scaling coordinates. Let us consider the expansion Eq. (6.0.1) using our scaling coordinates while letting $\lambda \to 0$ up to the second order in λ

$$\psi(r,\theta) = \psi_{00} + \frac{1}{2}\lambda r_0 \psi_{10P}(\tilde{R}+\sigma) + \lambda^2\left(\frac{1}{8}r_0^2\psi_{20P}\tilde{R}(\tilde{R}+2\sigma) + \sigma^2\psi_{10P}s_1\right), \tag{6.0.4a}$$

$$\Omega(r,\theta) = \Omega_{00} + \frac{1}{2}\lambda r_0 \Omega_{10P}(\tilde{R}+\sigma) + \lambda^2\left(\frac{1}{8}r_0^2\Omega_{20P}\tilde{R}(\tilde{R}+2\sigma) + \sigma^2\Omega_{10P}\omega_1\right), \tag{6.0.4b}$$

$$I(r,\theta) = I_{00} + \frac{1}{2}\lambda r_0 I_{10P}(\tilde{R}+\sigma) + \lambda^2\left(\frac{1}{8}r_0^2 I_{20P}\tilde{R}(\tilde{R}+2\sigma) + \sigma^2 I_{10P}i_1\right), \tag{6.0.4c}$$

and compare it with the expansion of Eq. (5.1.4) using the same scaling coordinates

$$\psi(r,\theta) = \psi_{00} + \lambda(\tilde{R}\psi_{10} + \sigma\psi_{01}) + \frac{1}{2}\lambda^2\left(\tilde{R}^2\psi_{20} + \sigma^2\psi_{02} + 2\sigma\tilde{R}\psi_{11}\right), \tag{6.0.5a}$$

$$\Omega(r,\theta) = \Omega_{00} + \lambda(\tilde{R}\Omega_{10} + \sigma\Omega_{01}) + \frac{1}{2}\lambda^2\left(\tilde{R}^2\Omega_{20} + \sigma^2\Omega_{02} + 2\sigma\tilde{R}\Omega_{11}\right), \tag{6.0.5b}$$

$$I(r,\theta) = I_{00} + \lambda(\tilde{R}I_{10} + \sigma I_{01}) + \frac{1}{2}\lambda^2\left(\tilde{R}^2 I_{20} + \sigma^2 I_{02} + 2\sigma\tilde{R}I_{11}\right). \tag{6.0.5c}$$

From this it is possible to infer that the expansions are identical given

$$\psi_{10} = \psi_{01} = \frac{r_0}{2}\psi_{10P}, \quad \psi_{02} = 2s_1\psi_{10P}, \quad \psi_{11} = \psi_{20} = \left(\frac{r_0}{2}\right)^2\psi_{20P}, \tag{6.0.6a}$$

$$\Omega_{10} = \Omega_{01} = \frac{r_0}{2}\Omega_{10P}, \quad \Omega_{02} = 2\omega_1\Omega_{10P}, \quad \Omega_{11} = \Omega_{20} = \left(\frac{r_0}{2}\right)^2\Omega_{20P}, \tag{6.0.6b}$$

$$I_{10} = I_{01} = \frac{r_0}{2}I_{10P}, \quad I_{02} = 2i_1 I_{10P}, \quad I_{11} = I_{20} = \left(\frac{r_0}{2}\right)^2 I_{20P}. \tag{6.0.6c}$$

Using the expansion Eq. (5.1.4) and scaling coordinates Eq. (4.1.1b) with Eq. (6.0.6) it is possible to show that the first order corrections to the field variables are identical to the NHEK variables (now omitting the subscript "P")

$$\psi_{10} = \frac{\Lambda\mathcal{G}}{r_0}\psi'_{00}, \quad \Omega_{10} = \frac{\Lambda\mathcal{G}}{r_0}\Omega'_{00}, \quad I_{10} = \frac{\Lambda\mathcal{G}}{r_0}I'_{00}, \tag{6.0.7}$$

with

$$\mathcal{G}' := \frac{\Lambda\Gamma}{r_0\Omega_{00}-1}\left(r_0\Omega_{00} - \frac{2}{\Lambda^2\Gamma^2}\right) \tag{6.0.8}$$

just as in [4]. This of course means that the field strength is null, FF, and self-similar at this order. At the second order in the near-NHEK limit the expansion yields

$$\Omega_{20} = \left[\frac{\psi_{20}}{\psi'_{00}} + \frac{\mathcal{G}\Lambda^2}{r_0^2}\left(\frac{\Omega''_{00}}{\Omega'_{00}} - \frac{\psi''_{00}}{\psi'_{00}}\right)\right]\Omega'_{00}, \quad I_{20} = \left[\frac{\psi_{20}}{\psi'_{00}} + \frac{\mathcal{G}\Lambda^2}{r_0^2}\left(\frac{I''_{00}}{I'_{00}} - \frac{\psi''_{00}}{\psi'_{00}}\right)\right]I'_{00}, \tag{6.0.9}$$

i.e. the same as the second order corrections to Ω and I in the NHEK limit and the same as was found in [4]. This allows one to write the second order differential equation for ψ_{20} in the near-NHEK limit as

$$\psi''_{20} + a(\theta)\psi'_{20} + b(\theta)\psi_{20} + c(\tilde{R},\theta) = 0 \tag{6.0.10}$$

with $a(\theta)$, $b(\theta)$ given by Eqs. (3.3.15a)-(3.3.15b), while the radial dependence of $c(\tilde{R},\theta)$ is removed by setting

$$i_1 = s_1 + \frac{r_0}{2}\left[r_0\left(\Omega_{00} + \frac{\mathcal{G}\Lambda\Omega'_{00}}{2}\right) - \frac{1}{\Gamma}\right], \tag{6.0.11a}$$

$$\begin{aligned}\omega_1 = s_1 - s_1'\frac{r_0\Omega_{00}-1}{r_0\Omega'_{00}} + 2s_1\frac{r_0\Omega_{00}(\Gamma-1)+1}{r_0\mathcal{G}\Gamma\Lambda\Omega'_{00}} - \frac{1}{\mathcal{G}\Gamma\Lambda\Omega'_{00}} - \frac{3r_0}{4\Gamma}\\ + \frac{1}{r_0\Omega_{00}-1}\left[\mathcal{G}\frac{r_0^2}{4}\Lambda\Omega'_{00} + \frac{r_0}{4}(3r_0\Omega_{00}+2) + \frac{3r_0\Omega_{00}-1}{2\mathcal{G}\Lambda\Omega'_{00}}\right],\end{aligned} \tag{6.0.11b}$$

where the function s_1 can remain arbitrary while still ensuring that $c(\tilde{R},\theta)$ is reduced to $c(\theta)$ given by Eq. (3.3.15c). This is again in agreement with the results found by [4]. The field strength $\tilde{F}^{(1)}$ will be different from its NHEK counterpart since it depends explicitly on σ and the function s_1 but at second order in λ, $(\tilde{F}^{(1)})^2$ is identical to $(F^{(1)})^2$.

Hence, I have successfully shown that the result found in the thesis [4] is a special case of the more general expansion Eq. (5.1.4) evaluated in the scaling coordinates Eq. (4.1.1b) up to the second order in λ.

Chapter 7

Discussion

In this Chapter I will return to certain points from throughout the thesis which demand further discussion.

Although the validity of the BZ process by now is quite clear there is still much left to be investigated about the transport of energy from the rotating black hole. The earlier setbacks of the theory have been attributed to the lack of understanding of the physical processes behind the mechanism. However, as mentioned, many numerical GRMHD simulations suggest that the BZ process is a good candidate for describing the energy extraction of Kerr black holes, and that the presence of an ergosphere is an important factor in the mechanism. Thus, I accepted the general consensus that this process is in fact the driving factor behind the energy extraction of rotating black holes, such that I was able to start my analysis of the magnetically-dominated magnetosphere. Since the main goal of my thesis was to perturbatively move away from the near-horizon spacetime towards the far region, I did not spend more time familiarising myself with the BZ process and its possible shortcomings.

Throughout the thesis and the articles reviewed within this project, primarily [3, 12], it was assumed that since the only stable and physically acceptable solution in flat spacetime had to be magnetically-dominated with $F^2 > 0$, then this also had to be true in the NHEK and near-NHEK limit of the Kerr solution. However, neither the NHEK nor the near-NHEK spacetimes are asymptotically flat, and it is thus not entirely clear whether the electrically-dominated solution, $F^2 < 0$, or null solution, $F^2 = 0$, could have any physical meaning in this regime, instead of the sought out magnetically-dominated solution. As mentioned in [32], electrically-dominated solutions are unstable with regard to acceleration of charged particles in flat spacetime where it is possible for the magnetic field to locally vanish. This would allow for an infinite acceleration of particles. The instability could possibly be resolved due to the strong gravitational fields in the NHEK and near-NHEK region but as far as I can tell at this date, not much (if any) work concerning this has been published.

The magnetically-dominated NHEK solution had the disadvantage of lacking a regular energy outflow (or negative infall) at the rotation axis, $\sin\theta = 0$, as well as F itself being irregular at the rotational axis. The latter problem was countered by performing a small angle expansion around $\theta = 0$. In [3] it is postulated that this might be due to the choice of ansatz for ψ_0 used to solve the second order linear differential equation for ψ_2. A choice which was made specifically with the intention of finding an exact solution such that the sign of F^2 could be investigated. It is, therefore, natural to search for other ansatzes which could ensure regularity, however, this was not the main goal of my thesis, and I did not have the time necessary to consider other possibilities.

Finding a novel perturbation which allowed for a consistent expansion of the near-NHEK attractor proved to be somewhat tricky. I quickly realised that using the same Taylor expansion of the field variables which led to the NHEK attractor solution as well as the magnetically-dominated second post-NHEK order failed to satisfy the condition of consistency in the near-NHEK limit. I dubbed this the "naive approach" since it did not take the deviation from extremality characterised by σ into account and in turn ignored the contribution from the near-NHEK terms. In an attempt to correct this, my next course of action was to introduce a $\sigma\lambda$-term in the expansion which resulted in a mixing of the field variables ψ_1 and ψ_2 at the second order in λ. This, however, also resulted in an inconsistent expansion of the field variables, specifically yielding a radially dependent polar current which of course was not permitted. The expansion which I settled on did not only permit an expansion of one variable, λ, but in fact an expansion of two variables, i.e. λ and the introduced quantity $\kappa = \lambda\sigma$ to the second order in both λ and κ (or σ). This meant that the purely NHEK terms ψ_{n0} and near-NHEK terms ψ_{0n} were introduced at both the first and second post-near-NHEK order corrections, while also taking the possibility of mixed terms ψ_{nn} into account[1]. Without much loss of generality this expansion made it possible to find that the first order correction to the near-NHEK limit was solved by the same equations and field variables found at the first order in NHEK. I also found that it was possible to construct a consistent second order correction to near-NHEK limit. While perhaps being both the weakness of the expansion as well as the benefit of having constructed such a general perturbation, the transition from having only one of each field variable at each post-near-NHEK

[1] Here $1 \leq n \leq 2$.

order to work with, to having ψ_{10} and ψ_{01} at the first order in λ and ψ_{20}, ψ_{02}, and ψ_{11} at the second order in λ, allows one to play around with, or tweak, the solution much more. The setback of this is that we are left with more variables than equations which means that one has to get creative in order to explicitly determine the unknown variables that of course must be known in terms of ψ_{00}, I_{00}, and Ω_{00}. Only then will it be possible to determine the sign of $\tilde{F}^2$ which, hopefully, is found to be positive given the appropriate ansatz for ψ_{00} and Ω_{00}. Another, arguably less prominent, setback of the expansion in Section 5.1 is that due to the many field variables, some of the first order corrections are rather lengthy and perhaps overwhelming to look at, while the corrections at second order in λ are even worse. Especially concerning the second order differential equation for ψ_{20} where the constant $c^*(\theta)$ from Eq. (5.5.17) is huge, a problem which should be possible to alleviate by using parts of the constant to define some functions that would allow one to write $c^*(\theta)$ much more compactly. I did not have enough time available to perform this algebraic manipulation, but it would definitely help make the results more digestible to the reader.

Ideally the presence of the light surfaces in the black hole magnetosphere should also be taken into account in the NHEK region as the ILS, possibly, will lie within the ergosphere. In [3] they were able to perturbatively construct a magnetically-dominated magnetosphere in the second order of λ without considering the ILS and since the goal was to review the paper, finding the location of the ILS in the NHEK limit was outside the scope of this thesis. However, it could still be interesting to locate the ILS in order to see if this, like in the near-NHEK case, is pushed towards infinity or if it actually contributes to the field strength found in the magnetosphere.

In Section 5.5 I argue that it should be possible to manipulate one of the two equations (the one which did not yield I_{20} and ψ_{11}) obtained from the FF condition at second order in λ. The goal is to write it of the same form as the NHEK second order differential equation for ψ_2 plus some new σ-dependent terms that constitute the deviation from NHEK, i.e. the near-NHEK terms. While I do find that it is indeed possible to recover a differential equation for ψ_{20} with $a(\theta)$ identical to the NHEK constant, $b^*(\theta)$ that is also the same as in the NHEK case plus some small σ deviation, I did not recover a constant $c^*(\theta)$ that was entirely equal to the NHEK $c(\theta)$ plus an additional σ-term. As mentioned, the divergence from the desired result is likely to have arisen due to the discrepancy in the Taylor expansion of the field variables used in the NHEK limit versus the near-NHEK limit. Of course it would be ideal to discover the exact transformation which renders $c^*(\theta) = c(\theta) + \sigma(\dots)$ but at this point I was running out of the time required to perform the necessary calculation. Since the constant did not contain a radial dependence I concluded that the overall result would not differ greatly from if I had indeed succeeded in showing that $c^*(\theta)$ is the same as the NHEK constant with an additional σ-proportional term. Therefore, I accepted that while not ideal, this would probably not affect my calculations.

Chapter 8

Conclusion

In this thesis I have reached the conclusion that the FFE equations are deterministic in the magnetically-dominated magnetosphere. They are thus useful tool for describing the energy extraction from the rotating black hole ergosphere which is governed by the BZ process. I determined that the extreme Kerr background is not able to correctly resolve the near-horizon spacetime which instead is governed by the near-horizon extreme Kerr geometry. The geometry was found to be a spacetime in its own right with co-rotating coordinates (T, R, θ, Φ). I found that the enhanced symmetry of the NHEK regime allows for the possibility of solving the non-linear FFE equations, specifically the stream equation Eq. (2.4.14) by constructing the field strength using the field variables ($\psi(\theta)$, $I(\theta)$, $\Omega(\theta)$). I was able to derive the NHEK geometry Eq. (3.1.4) by considering the scaling coordinates Eq. (3.1.2) and by performing an expansion in λ. Starting from the NHEK geometry it is possible to recover the null (given the Znajek condition of regularity) and self-similar NHEK attractor solution Eq. (3.2.3b) at leading order in λ for $\lambda \to 0$. This is in fact the solution that one finds at leading order for all stationary, axisymmetric field strengths F in the NHEK limit of the Kerr metric. In the first post-NHEK order I recreated the results found in [3] where I found the field variables to be given by Eq. (3.3.8) with the function $\mathcal{G}$ given by Eq. (3.3.7), while the second post-NHEK order field variables Ω_2 and I_2 were found to be Eqs. (3.3.12)-(3.3.13). The magnetic flux ψ_2 was a bit more tricky and had to be determined by the second order linear differential equation Eq. (3.3.14). With appropriate ansatzes for ψ_0 and Ω_0 the general solution Eq. (3.5.2), which consisted of a homogeneous and a non-homogeneous term, was found. This allowed for the expansion of F^2 around $\theta = 0$ such that an explicit equation for the field strength Eq. (3.5.8) could be used to create

Figure 3.1. This illutrates the positive regions of F^2 in the range $\beta = -2$, $0.67 \lessapprox g \leq 1$. I concluded that while the extraction of angular momentum L is finite (and negative) at the rotation axis, the extraction of energy E was irregular for $\theta \to 0$, Eq. (3.5.9). Lastly in the NHEK limit I found that you recover the MD class of solutions Eq. (3.4.4) from the NHEK order as long as the post-NHEK orders vanish.

I concluded that while the NHEK limit of the Kerr solution is a nice introduction to the the near-horizon geometry, observational evidence suggests that black holes are more likely to rotate at a near-extremal rate rather than at extremality with $J = M^2$. Consequently, the near-NHEK limit of the Kerr solution is probably more astrophysically realistic and was introduced. I found the near-NHEK geometry using a new set of scaling coordinates ($\tilde{T}$, $\tilde{R}$, θ, $\tilde{\Phi}$) and an expansion in the deviation of extremality σ given by Eq. (4.1.1b). I find that the near-NHEK spacetime co-exists with the NHEK spacetime, and far region (the extreme Kerr limit). It was concluded that the far region does not resolve the throat region (see Figure 4.1) while neither the near-NHEK spacetime nor the NHEK spacetime are asymptotically flat and thus cannot resolve the far region. From the near-NHEK geometry, the near-NHEK attractor solution was found Eq. (4.1.5) which, just as the NHEK attractor solution, is null and self similar. It is also the solution one ends up with for any stationary, axisymmetric Kerr field strength in the near-NHEK limit. I concluded that the NHEK attractor and near-NHEK attractor solutions are very similar and that it is indeed possible to recover the NHEK attractor solution from the near-NHEK attractor solution in three ways; i) by setting $\sigma = 0$ in Eq. (4.1.2), ii) by considering the limit $\sigma \ll \tilde{R}$, and lastly iii) by using the local diffeomorphism from near-NHEK coordinates to NHEK coordinates Eq. (4.2.3). This also showed that one can superpose $\tilde{F}^{(-1)}$ and $F^{(-1)}$ to create new FF solutions. In my attempts to find a consistent post-near-NHEK expansion, where the field variables only depend on the quantity θ as dictated by the MD case of solutions, I concluded that what I dubbed the "naive approach" failed to produce a consistent second order differential equation for the second order field variable ψ_2. I, furthermore, found that the expansion Eq. (4.3.9) presented in Section 4.3.2 with the extra $\sigma\lambda$-term also could not be used to find consistent field variables, failing at the first order in λ. I concluded that the expansion which is consistent in all post-near-NHEK orders was not only an expansion in λ but in σ too, Eq. (5.1.3). Using this expansion, the Znajek condition of regularity at the horizon now yields conditions for I_{01}, and I_{02} Eqs. (5.2.3)-(5.2.4) while the second Znajek condition was still valid for $\kappa = 0$, i.e. I_{10}. I found that the critical surface, the ILS, is pushed towards infinity in the near-NHEK limit, like the ergoregion, as seen by Eq. (5.3.3). Thus we cannot use this surface to say anything about the field strength inside the magnetosphere. Instead I demonstrated that by removing all radially dependent terms from the novel near-NHEK expansion, you recover the MD class of solutions with corrections to the field strength $\tilde{F}^2_{\rm MD}$ in both the first and second post-near-NHEK order. Both orders of corrections to the field strength are null and self-similar. This is a necessary condition that must be satisfied since the MD solution is one of the only known analytical solutions in the Kerr spacetime. I then found that the first post-near-NHEK order (now including radial dependent terms) is solved by the NHEK field variables[1] found at the first order in λ.[2] Hence, the field strength is null and self-similar at this order. At the second order in λ I was able to determine the variables Ω_{20}, Ω_{11} found in Eq. (5.5.12), and variables I_{20}, I_{11} found in Eq. (5.5.13). I also constructed a second order differential equation for ψ_{20}, Eq. (5.5.15), which approximately contained the NHEK differential equation for ψ_{20} (with $\sigma = 0$) plus some extra σ-proportional near-NHEK terms that were "packed" into the constants $b^*(\theta)$ and $c^*(\theta)$ given by Eqs. (5.5.16)-(5.5.17). This leaves you with the following eight unknown variables: ψ_{00}, Ω_{00}, ψ_{01}, Ω_{01}, Ω_{02}, ψ_{02}, ψ_{11}, and lastly ψ_{20}. I conclude that it is difficult to analytically examine the sign of $\tilde{F}^2$ due to the many unknowns. Finally, I concluded that the expansion and results found by Pompili [4] is a special case of our novel near-NHEK perturbation with the conversion between our and their field variables found to be given by Eq. (6.0.6).

Despite the fact that the NHEK limit of the Kerr solution is less physical than the near-NHEK limit, it still is interesting to explore the possibility of making the energy outflow inside the magnetosphere regular. There is ample opportunity to play around with the ansatz for ψ_0 (or Ω_0), which could be the key to ensuring regularity at the rotation axis. Perhaps it would be interesting to consider the ILS to this purpose since this could produce a regularity condition which could be relevant in the NHEK regime.

The novel near-NHEK expansion and the eight unknown field variables that have yet to be explicitly determined allow for plenty to be discovered. Given that one is creative in terms of finding equations that open for the potential of determining the field variables, it should be possible to make some concrete statements about the positive, or negative, regions of the field strength $\tilde{F}^2$ at second order in λ. Furthermore, it would be interesting to investigate the possibility that the electrically-dominated solution has physical meaning in the near-NHEK region which isn't asymptotically flat, as I already touched upon in

[1] Of course one has to rewrite the variables such that they follow the convention $\psi_0 = \psi_{00}$ and so forth.

[2] The expansion is also technically in σ but I have not written this explicitly since it disrupts the flow of the text.

Chapter 7. Lastly it could be interesting to study the perturbative approach numerically in order to reach higher orders in λ than λ^2 and investigate whether this helps to recover a magnetically-dominated field strength or reveal other fascinating features which can be explored.

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Appendix A

Post-NHEK expressions

The 2nd post-NHEK order correction to the four current is [3]

$$j^{(1)} = j^T_{(1)}\partial_T + R^2 j^R_{(1)}\partial_R + R j^\theta_{(1)}\partial_\theta + R j^\Phi_{(1)}\partial_\Phi, \tag{A.0.1}$$

with the following components

$$\begin{aligned}
j^T_{(1)} = \partial_\theta & \left[\frac{\Gamma(2+r_0^2\Omega_1) - 4(1-\Gamma)(r_0\Omega_0 - 1)}{\Gamma^3}\psi_1' - \frac{4(1-\Gamma)r_0\psi_0'}{r_0^3\Gamma^3}\Omega_1\right] \\
&+ \frac{\Gamma(2+r_0\Omega_1) - 2(1-2\Gamma)(r_0\Omega_0-1)}{r_0^3\Gamma^3}\left(\psi_1 + 2\frac{\Gamma'\psi_1'}{\Gamma}\right) \\
&+ \frac{8\psi_0'}{r_0^4\Gamma^2}\left(\frac{\Lambda'}{\Lambda} + \frac{\psi_0''}{\psi_0'} + \frac{r_0 I_0'}{\Lambda\psi_0'}\right)\left(1 - \frac{1}{\Gamma}\right) + \frac{\partial_\theta(\Lambda\psi_0'\Omega_2)}{2r_0\Gamma^2\Lambda} \\
&+ \frac{\psi_1'}{r_0^2\Gamma^2}\left[\frac{(2+r_0^2\Omega_1)\Lambda'}{r_0\Lambda} - \frac{2I_0}{\Gamma\Lambda\psi_0'}\left(3\frac{\Gamma'}{\Gamma} + 2(1-\Gamma)\frac{\Lambda'}{\Lambda}\right)\right] - \frac{12}{r_0^3\Gamma^3\Lambda}\partial_\theta\left[I_0\left(1-\frac{1}{\Gamma}\right)\right] \\
&- \frac{2\Omega_1\psi_0'}{r_0^2\Gamma^3}\left[2(1-\Gamma)\frac{\Lambda'}{\Lambda} + (5-4\Gamma)\frac{\Gamma'}{\Gamma}\right] + \frac{I_0}{2r_0\Gamma^2\Lambda\psi_0'}\left[\psi_2'' + \psi_1'\left(\frac{I_0'}{I_0} - \frac{\psi_0''}{\psi_0'}\right) + 2\psi_2\right] \\
&- \frac{2I_0\Lambda}{r_0^3}\left(\frac{\Lambda'}{\Lambda} + \frac{I_0'}{2I_0} + \frac{\Gamma'}{\Gamma}\right) + \frac{\Lambda^2\psi_0'}{r_0^4}\left(\frac{\psi_0''}{\psi_0'} + 3\frac{\Lambda'}{\Lambda}\right) - \frac{2\Gamma'\psi_0'}{r_0^4\Gamma^4}(3 + \Gamma^3\Lambda^2 - 5r_0\Omega_0),
\end{aligned} \tag{A.0.2a}$$

$$j^R_{(1)} = \frac{r_0\Gamma(4I_1' - r_0\Gamma I_2') - 4(2-\Gamma)I_0'}{2r_0^3\Gamma^4\Lambda}, \tag{A.0.2b}$$

$$j^\theta_{(1)} = \frac{r_0 I_2\Gamma - 2I_1}{r_0^2\Gamma^3\Lambda}, \tag{A.0.2c}$$

$$\begin{aligned}
j^\Phi_{(1)} = -j^T_{(1)} + \frac{\psi_0'}{2r_0^4\Gamma^3\Lambda^2}\Bigg\{ & -8\frac{\Gamma'}{\Gamma} + 2\Gamma^2\Lambda^3\left[(2+r_0^2\Omega_1)\Gamma'\Lambda - 2r_0\frac{I_0'}{\psi_0'}\right] \\
&+ \Gamma^3\Lambda^3\left[r_0\frac{I_0'}{\psi_0'}\left(r_0\frac{\psi_1'}{\psi_0'} - 2\right) + r_0^2\Lambda\Omega_1' + \Lambda\left(3\frac{\Lambda'}{\Lambda} + \frac{\psi_0''}{\psi_0'} - \frac{2r_0}{\Gamma^2\Lambda^2}\frac{\psi_1}{\psi_0'}\right)(2+r_0^2\Omega_1)\right]
\end{aligned}$$

$$
\begin{aligned}
&+4\Gamma\left[2r_0\Gamma\frac{I_0'}{\psi_0'}+\frac{\Lambda'}{\Lambda}\left(4-r_0\frac{\psi_1'}{\psi_0'}\right)+\left(2r_0\frac{\psi_1}{\psi_0'}-4\frac{\psi_0''}{\psi_0'}+r_0\frac{\psi_1''}{\psi_0'}\right)\right]\\
&+2r_0\frac{I_0}{\psi_0'}\Gamma^3\Lambda^3\left[2\frac{(2+\Gamma)}{\Gamma}\frac{\Lambda'}{\Lambda}+\frac{(3+2\Gamma)}{\Gamma}\frac{\Gamma'}{\Gamma}\right]-r_0^2I_0\frac{\psi_1'}{\psi_0'^2}\Gamma^3\Lambda^3\left(\frac{\psi_0''}{\psi_0'}-2\frac{\Lambda'}{\Lambda}-2\frac{\Gamma'}{\Gamma}-\frac{\psi_1''}{\psi_1'}\right)\\
&+2r_0I_0\frac{\psi_1}{\psi_0'^2}\Gamma^2\Lambda^3\left(1+\frac{2}{\Gamma}-\frac{r_0}{\Gamma^2\Lambda^2}\frac{\psi_2}{\psi_1}\right)\Bigg\}. \qquad \text{(A.0.2d)}
\end{aligned}
$$

The functions in the $c(\theta)$ coefficient of the second-order linear differential equation for ψ_2 are [3]

$$
\begin{aligned}
A(\theta)=&\,8\left(\frac{\psi_0'}{I_0}\right)^2\left(\frac{4\Lambda^2-1}{\Lambda^2}\frac{\Lambda'}{\Lambda}+\frac{4\Lambda^2+1}{\Lambda^2}\frac{\psi_0''}{\psi_0'}-\frac{I_0'}{I_0}\right)\\
&+2r_0\Lambda\Gamma^2\frac{\psi_0'}{I_0}\left[4\left(1+\frac{1}{\Gamma^3\Lambda^2}\right)\frac{\Gamma'}{\Gamma}+5\frac{\Lambda'}{\Lambda}+\frac{\psi_0''}{\psi_0'}+\frac{I_0'}{I_0}-\frac{8}{\Gamma}\left(\frac{\Lambda'}{\Lambda}+\frac{\psi_0''}{\psi_0'}\right)\right]\\
&+4r_0^2\Gamma^2\left[\left(1+\frac{1}{\Gamma^3\Lambda^2}\right)\frac{\Gamma'}{\Gamma}+\frac{\Lambda'}{\Lambda}+\left(\frac{1}{2}-\frac{1}{\Gamma}\right)\frac{I_0'}{I_0}+\frac{\Lambda^2}{2}\left(\frac{\Lambda'}{\Lambda}+\frac{\psi_0''}{\psi_0'}\right)\right], \qquad \text{(A.0.3a)}\\
B(\theta)=&-4r_0\frac{\psi_0'}{I_0}\left[\frac{\Lambda'}{\Lambda}\left(2\frac{\Lambda'}{\Lambda}+3\frac{\Lambda''}{\Lambda'}+7\frac{\psi_0''}{\psi_0'}\right)+2-\left(\frac{\psi_0''}{\psi_0'}\right)^2+3\frac{\psi_0^{(3)}}{\psi_0}+\frac{I_0'}{I_0}\left(\frac{\psi_0''}{\psi_0'}-\frac{I_0''}{I_0'}\right)\right]\\
&+4r_0^2\Lambda\Gamma\left[1-\frac{1}{\Gamma^2\Lambda^2}-\left(\frac{1}{\Gamma^2\Lambda^2}\frac{\Gamma'}{\Gamma}-\frac{I_0'}{I_0}\right)\left(\frac{\Lambda'}{\Lambda'}+\frac{\psi_0''}{\psi_0'}\right)+\frac{\Lambda'}{\Lambda}\left(\frac{\Lambda''}{\Lambda'}+\frac{\psi_0''}{\psi_0'}\right)-\left(\frac{\psi_0''}{\psi_0'}\right)^2+\frac{\psi_0^{(3)}}{\psi_0}{}'\right],\\
&\qquad \text{(A.0.3b)}\\
C(\theta)=&\,r_0^2\Bigg\{\frac{\Lambda''}{\Lambda}\left(\frac{\Lambda^{(3)}}{\Lambda''}+\frac{\psi_0''}{\psi_0'}\right)+\frac{\Lambda'}{\Lambda}\left[\left(\frac{\Lambda'}{\Lambda}+2\frac{I_0'}{I_0}\right)\left(\frac{\Lambda''}{\Lambda'}+\frac{\psi_0''}{\psi_0'}\right)-5\left(\frac{\psi_0''}{\psi_0'}\right)^2+6\frac{\psi_0^{(3)}}{\psi_0'}\right]\\
&+\left[2+2\left(\frac{\psi_0''}{\psi_0'}\right)^2-3\frac{\psi_0^{(3)}}{\psi_0'}\right]\left(\frac{\Lambda'}{\Lambda}+\frac{\psi_0''}{\psi_0'}-\frac{I_0'}{I_0}\right)-\frac{I_0'}{I_0}\frac{\psi_0^{(3)}}{\psi_0'}+\frac{\psi_0^{(4)}}{\psi_0'}\Bigg\}. \qquad \text{(A.0.3c)}
\end{aligned}
$$

The coefficients of Eq. (3.3.16) are given by

$$
\begin{aligned}
D(\theta)=&\,4(1-\Lambda^2)+2r_0(\Gamma+4)\Gamma\Lambda^3\frac{I_0}{\psi_0'}+r_0\left(-\frac{12}{\Gamma^2}+\frac{20}{\Gamma}+\Gamma^2\Lambda^2-8\right)\left(\frac{I_0}{\psi_0'}\right)^2\\
&+r_0^4\frac{\psi_2}{\psi_0'}\left(\frac{\Lambda'}{\Lambda}+\frac{\psi_0''}{\psi_0'}-\frac{\psi_2'}{\psi_2}\right)\left(\frac{I_0}{\psi_0'}\right)^2, \qquad \text{(A.0.4a)}\\
E(\theta)=&\,r_0\Lambda^2\frac{I_0}{\psi_0'}\Bigg\{-4\left(\frac{\Lambda'}{\Lambda}+\frac{\psi_0''}{\psi_0'}+\frac{I_0'}{I_0}+2\frac{\mathcal{G}'}{\mathcal{G}}\right)\\
&+2r_0\mathcal{G}'\frac{I_0}{\psi_0'}\left[\frac{\Lambda'}{\Lambda}+\frac{\psi_0''}{\psi_0'}-2\frac{I_0'}{I_0}+2\left(\frac{I_0'}{I_0}+\frac{\mathcal{G}'}{\mathcal{G}}\right)\frac{\Gamma\Lambda}{\mathcal{G}'}-\frac{1}{2}\frac{\mathcal{G}'}{\mathcal{G}}\right]\\
&+r_0\mathcal{G}\frac{I_0}{\psi_0'}\left[\frac{\Lambda'}{\Lambda}\left(\frac{\Lambda''}{\Lambda'}+\frac{\psi_0''}{\psi_0'}\right)-1-\left(\frac{\psi_0''}{\psi_0'}\right)^2+\frac{\psi_0^{(3)}}{\psi_0'}\right]\Bigg\}. \qquad \text{(A.0.4b)}
\end{aligned}
$$

Appendix B

Post-near-NHEK expressions

The components of the second order correction to the four-current $\tilde{j}^{(1)}$ in Section 4.3.1 are

$$
\begin{aligned}
\tilde{j}^{\tilde{T}}_{(1)} = &\Big(\Gamma\Lambda'(\Gamma^2(\frac{I_0\psi_2'}{\Lambda\psi_0'} + \Omega_2\psi_0' + 2\Omega_1\psi_1')r_0^3 + 8(\Gamma-1)\Gamma(\Omega_1\psi_0' + \Omega_0\psi_1')r_0^2 + 4(6(\Gamma-1)^2\Omega_0\psi_0' \\
&- (\Gamma-2)\Gamma\psi_1')r_0 - 4(\Gamma(3\Gamma-8)+6)\psi_0') + \Lambda\Big(4(-6r_0\Omega_0 + \Gamma(\Omega_1 r_0^2 + 6\Omega_0 r_0 - 4) + 6)\Gamma'\psi_0' \\
&+ \Gamma\Big(\Gamma^2(\Omega_2'\psi_0' + \psi_2'\Omega_0' + 2\psi_1'\Omega_1' + \Omega_2\psi_0'' + 2\Omega_1\psi_1'')r_0^3 + 8\Gamma((\Gamma-1)(\psi_1'\Omega_0' + \psi_0'\Omega_1' + \Omega_1\psi_0'') + \Gamma\Omega_0\psi_1'')r_0^2 \\
&+ 4(6\psi_0'\Omega_0'(\Gamma-1)^2 + 6(\Gamma-2)\Gamma\Omega_0\psi_0'' - \Gamma^2\psi_1'')r_0 + 4\Gamma(8-3\Gamma)\psi_0''\Big)\Big) \\
&+ \frac{r_0\Gamma I_0\Big(24\psi_0'' + r_0\Big(4\Gamma'\psi_1' + \Gamma(r_0\Gamma\psi_2'' - 8\psi_1'')\Big)\Big)}{\psi_0'}\Big)\sigma^2 + 2\tilde{R}(-2r_0\Lambda^2(\psi_0'(3\Omega_0\Lambda' + \Lambda\Omega_0') + \Lambda\Omega_0\psi_0'')\Gamma^5 \\
&- 4r_0\Lambda^3\Omega_0\Gamma'\psi_0'\Gamma^4 + r_0\Big(\Big(\frac{I_0(2\psi_2 + \psi_2'')}{\psi_0'} + \Lambda\Big(\Omega_2'\psi_0' + \psi_2'\Omega_0' + 2\psi_1'\Omega_1' + \Omega_2\psi_0'' + 2\Omega_1(\psi_1 + \psi_1'')\Big)\Big)r_0^2 \\
&+ 8\Lambda(\psi_1'\Omega_0' + \psi_0'\Omega_1' + \Omega_1\psi_0'' + \Omega_0(\psi_1 + \psi_1''))r_0 + \Lambda'\Big(-4\psi_1' + \Omega_0(\psi_2' r_0^2 + 8\psi_1' r_0 + 16\psi_0') \\
&+ r_0(-\psi_2' + (r_0\Omega_2 + 8\Omega_1)\psi_0' + 2r_0\Omega_1\psi_1')\Big) - 4\Lambda(\psi_1 - 4\psi_0'\Omega_0' - 4\Omega_0\psi_0'' + \psi_1'')\Big)\Gamma^3 \\
&- 4(10(\psi_0'(\Omega_0\Lambda' + \Lambda\Omega_0') + \Lambda\Omega_0\psi_0'')r_0 - 6(\Lambda'\psi_0' + \Lambda\psi_0'') \\
&+ \frac{r_0^2(2\psi_0'(\psi_1'\Omega_0' + \psi_0'\Omega_1' + \Omega_1\psi_0'')\Lambda^2 + (2\Omega_1\Lambda'\psi_0'^2 + I_0(\psi_1 + 2\psi_1''))\Lambda + 2I_0\Lambda'\psi_1')}{\Lambda\psi_0'})\Gamma^2 + 4\Big(\frac{I_0\Gamma'\psi_1' r_0^2}{\psi_0'} \\
&+ \frac{6I_0\Lambda' r_0}{\Lambda} + 6\Lambda\psi_0'\Omega_0' r_0 + \frac{6I_0\psi_0'' r_0}{\psi_0'} + \Lambda(\Omega_1 r_0^2 + 5\Omega_0 r_0 - 3)\Gamma'\psi_0'\Big)\Gamma - 24r_0 I_0\Gamma')\sigma \\
&+ \tilde{R}^2\Big(-2r_0\Lambda^2(\psi_0'(3\Omega_0\Lambda' + \Lambda\Omega_0') + \Lambda\Omega_0\psi_0'')\Gamma^5 - 4r_0\Lambda^3\Omega_0\Gamma'\psi_0'\Gamma^4 + r_0\Big(\Big(\frac{I_0(2\psi_2 + \psi_2'')}{\psi_0'} \\
&+ \Lambda(\Omega_2'\psi_0' + \psi_2'\Omega_0' + 2\psi_1'\Omega_1' + \Omega_2\psi_0'' + 2\Omega_1(\psi_1 + \psi_1''))\Big)r_0^2 + 8\Lambda(\psi_1'\Omega_0' + \psi_0'\Omega_1' + \Omega_1\psi_0'' + \Omega_0(\psi_1 + \psi_1''))r_0 \\
&+ \Lambda'(-4\psi_1' + \Omega_0(\psi_2' r_0^2 + 8\psi_1' r_0 + 16\psi_0') + r_0(-\psi_2' + (r_0\Omega_2 + 8\Omega_1)\psi_0' + 2r_0\Omega_1\psi_1'))
\end{aligned}
$$

$$
\begin{aligned}
&- 4\Lambda(\psi_1 - 4\psi_0'\Omega_0' - 4\Omega_0\psi_0'' + \psi_1'')\Big)\Gamma^3 - 4\Big(10(\psi_0'(\Omega_0\Lambda' + \Lambda\Omega_0') + \Lambda\Omega_0\psi_0'')r_0 - 6(\Lambda'\psi_0' + \Lambda\psi_0'') \\
&+ \frac{r_0^2(2\psi_0'(\psi_1'\Omega_0' + \psi_0'\Omega_1' + \Omega_1\psi_0'')\Lambda^2 + (2\Omega_1\Lambda'\psi_0'^2 + I_0(\psi_1 + 2\psi_1''))\Lambda + 2I_0\Lambda'\psi_1')}{\Lambda\psi_0'}\Big)\Gamma^2 \\
&+ 4\Big(\frac{I_0\Gamma'\psi_1' r_0^2}{\psi_0'} + \frac{6I_0\Lambda' r_0}{\Lambda} + 6\Lambda\psi_0'\Omega_0' r_0 + \frac{6I_0\psi_0'' r_0}{\psi_0'} + \Lambda(\Omega_1 r_0^2 + 5\Omega_0 r_0 - 3)\Gamma'\psi_0'\Big)\Gamma - 24r_0 I_0\Gamma'\Big),
\end{aligned}
\tag{B.0.1a}
$$

$$
\tilde{j}^{\tilde{R}}_{(1)} = \frac{r_0\Gamma(\tilde{R}+\sigma)^2\,(4I_1' - r_0\Gamma I_2') + 4I_0'\left(2\sigma\tilde{R}(\Gamma-2) + \tilde{R}^2(\Gamma-2) - 2\sigma^2((\Gamma-1)\Gamma+1)\right)}{2r_0^3\Gamma^4\Lambda},
\tag{B.0.1b}
$$

$$
\tilde{j}^{\theta}_{(1)} = \frac{(\tilde{R}+\sigma)\,(r_0\Gamma I_2 - 2I_1)}{r_0^2\Gamma^3\Lambda},
\tag{B.0.1c}
$$

$$
\begin{aligned}
\tilde{j}^{\tilde{\Phi}}_{(1)} &= (\sigma+\tilde{R})\Big[\Lambda\Big(r_0^3\Big(I_0(\Lambda'\psi_2' + \Lambda\psi_2'') + \Lambda\psi_0'\Big(2\Omega_1\Lambda'\psi_1' + \Omega_2(\Lambda'\psi_0' + \Lambda\psi_0'') + \Lambda(\Omega_2'\psi_0' \\
&+ \psi_2'\Omega_0' + 2\psi_1'\Omega_1' + 2\Omega_1\psi_1'')\Big)\Big)\Gamma^3 + 2r_0^2\Lambda\Big(2I_0\Gamma'\psi_1' + \Gamma((3\Gamma-4)\Lambda'\psi_0'(\Omega_1\psi_0' + \Omega_0\psi_1') - 4I_0\psi_1'') \\
&+ \Lambda\psi_0'\Big(\Omega_1(2\Gamma'\psi_0' + \Gamma(3\Gamma-4)\psi_0'') + \Gamma((3\Gamma-4)(\psi_1'\Omega_0' + \psi_0'\Omega_1') + 3\Gamma\Omega_0\psi_1'')\Big)\Big)\Gamma \\
&- 12(\Gamma-2)\Lambda\psi_0'(\Lambda\psi_0''\Gamma^2 + (\Lambda'\Gamma^2 + \Lambda\Gamma')\psi_0') + r_0\Big(2\Lambda\psi_0'(-\Lambda'\psi_1' + 10\Lambda\psi_0'\Omega_0' + 10\Omega_0(\Lambda'\psi_0' + \Lambda\psi_0'') \\
&- \Lambda\psi_1'')\Gamma^3 - 8\Lambda\psi_0'(-\Lambda'\psi_1' + 5\Lambda\psi_0'\Omega_0' + 5\Omega_0(\Lambda'\psi_0' + \Lambda\psi_0''))\Gamma^2 + 4(\Lambda^2(5\Omega_0\Gamma' + 6\Omega_0')\psi_0'^2 \\
&+ 6I_0(\Lambda'\psi_0' + \Lambda\psi_0''))\Gamma - 24\Lambda^2\Omega_0\Gamma'\psi_0'^2)\Big)\sigma^2 + 2\tilde{R}\Big(r_0^3\Lambda\Big(\psi_0'\Big(\Omega_2'\psi_0' + \psi_2'\Omega_0' + 2\psi_1'\Omega_1' + \Omega_2\psi_0'' \\
&+ 2\Omega_1(2\psi_1 + \psi_1'')\Big)\Lambda^2 + \Big(\Lambda'\psi_0'(\Omega_2\psi_0' + 2\Omega_1\psi_1') + I_0(4\psi_2 + \psi_2'')\Big)\Lambda + I_0\Lambda'\psi_2'\Big)\Gamma^3 \\
&+ 4\psi_0'\Big(\Big(\Lambda(-2\Gamma^3\Lambda^4 - 3\Lambda^2 + 2)\Gamma' + \Gamma(-3\Gamma^3\Lambda^4 + 2(\Gamma+3)\Lambda^2 - 4\Gamma)\Lambda'\Big)\psi_0' \\
&+ \Gamma\Lambda(-\Gamma^3\Lambda^4 + 2(\Gamma+3)\Lambda^2 + 4\Gamma)\psi_0''\Big)\Gamma - r_0^2\Lambda\Big(\Lambda^2\Big(\psi_0'(\psi_1'\Omega_0' + \psi_0'\Omega_1' + \Omega_1\psi_0'')\Lambda^2 \\
&+ (3\Omega_1\Lambda'\psi_0'^2 + I_0(2\psi_1 + \psi_1''))\Lambda + 3I_0\Lambda'\psi_1'\Big)\Gamma^4 + 2\Lambda^3\Gamma'(\Lambda\Omega_1\psi_0'^2 + I_0\psi_1')\Gamma^3 \\
&- 8\Lambda\psi_0'\Big(\Lambda'(\Omega_1\psi_0' + \Omega_0\psi_1') + \Lambda(\psi_1'\Omega_0' + \psi_0'\Omega_1' + \Omega_1\psi_0'' + \Omega_0(2\psi_1 + \psi_1''))\Big)\Gamma^2 \\
&+ 4(2\psi_0'(\psi_1'\Omega_0' + \psi_0'\Omega_1' + \Omega_1\psi_0'')\Lambda^2 + (2\Omega_1\Lambda'\psi_0'^2 + I_0(3\psi_1 + 2\psi_1''))\Lambda + 2I_0\Lambda'\psi_1')\Gamma \\
&- 4\Lambda\Gamma'(\Lambda\Omega_1\psi_0'^2 + I_0\psi_1')\Big)\Gamma + 2r_0\Big(2\Lambda^3(\Lambda^2\Omega_0'\psi_0'^2 + I_0(3\Lambda'\psi_0' + \Lambda\psi_0''))\Gamma^4 \\
&+ \psi_0'\Big(3I_0\Gamma'\Lambda^4 - 2(2\psi_1 - 2\psi_0'\Omega_0' - 2\Omega_0\psi_0'' + \psi_1'')\Lambda^3 + 2\Lambda'(2\Omega_0\psi_0' - \psi_1')\Lambda^2 \\
&- 2(2\psi_1 + \psi_1'')\Lambda + 2\Lambda'\psi_1'\Big)\Gamma^3 - 20\Lambda^2\psi_0'(\psi_0'(\Omega_0\Lambda' + \Lambda\Omega_0') + \Lambda\Omega_0\psi_0'')\Gamma^2 \\
&+ 2\Lambda(\Lambda^2(5\Omega_0\Gamma' + 6\Omega_0')\psi_0'^2 + 6I_0(\Lambda'\psi_0' + \Lambda\psi_0''))\Gamma - 12\Lambda^2 I_0\Gamma'\psi_0'\Big)\Big)\sigma \\
&+ \tilde{R}^2\Big(r_0^3\Lambda(\psi_0'\Big(\Omega_2'\psi_0' + \psi_2'\Omega_0' + 2\psi_1'\Omega_1' + \Omega_2\psi_0'' + 2\Omega_1(2\psi_1 + \psi_1'')\Big)\Lambda^2 + (\Lambda'\psi_0'(\Omega_2\psi_0' + 2\Omega_1\psi_1') \\
&+ I_0(4\psi_2 + \psi_2''))\Lambda + I_0\Lambda'\psi_2')\Gamma^3 + 4\psi_0'\Big(\Big(\Lambda(-2\Gamma^3\Lambda^4 - 3\Lambda^2 + 2)\Gamma' + \Gamma(-3\Gamma^3\Lambda^4 + 2(\Gamma+3)\Lambda^2 - 4\Gamma)\Lambda'\Big)\psi_0'
\end{aligned}
$$

$$+ \Gamma\Lambda(-\Gamma^3\Lambda^4 + 2(\Gamma+3)\Lambda^2 + 4\Gamma)\psi_0''\Big)\Gamma - r_0^2\Lambda\Big(\Lambda^2\Big(\psi_0'(\psi_1'\Omega_0' + \psi_0'\Omega_1' + \Omega_1\psi_0'')\Lambda^2$$
$$+ (3\Omega_1\Lambda'\psi_0'^2 + I_0(2\psi_1 + \psi_1''))\Lambda + 3I_0\Lambda'\psi_1'\Big)\Gamma^4 + 2\Lambda^3\Gamma'(\Lambda\Omega_1\psi_0'^2 + I_0\psi_1')\Gamma^3$$
$$- 8\Lambda\psi_0'\Big(\Lambda'(\Omega_1\psi_0' + \Omega_0\psi_1') + \Lambda\Big(\psi_1'\Omega_0' + \psi_0'\Omega_1' + \Omega_1\psi_0'' + \Omega_0(2\psi_1 + \psi_1'')\Big)\Big)\Gamma^2$$
$$+ 4(2\psi_0'(\psi_1'\Omega_0' + \psi_0'\Omega_1' + \Omega_1\psi_0'')\Lambda^2 + (2\Omega_1\Lambda'\psi_0'^2 + I_0(3\psi_1 + 2\psi_1''))\Lambda + 2I_0\Lambda'\psi_1')\Gamma$$
$$- 4\Lambda\Gamma'(\Lambda\Omega_1\psi_0'^2 + I_0\psi_1')\Big)\Gamma + 2r_0\Big(2\Lambda^3(\Lambda^2\Omega_0'\psi_0'^2 + I_0(3\Lambda'\psi_0' + \Lambda\psi_0''))\Gamma^4 + \psi_0'(3I_0\Gamma'\Lambda^4$$
$$- 2(2\psi_1 - 2\psi_0'\Omega_0' - 2\Omega_0\psi_0'' + \psi_1'')\Lambda^3 + 2\Lambda'(2\Omega_0\psi_0' - \psi_1')\Lambda^2 - 2(2\psi_1 + \psi_1'')\Lambda + 2\Lambda'\psi_1')\Gamma^3$$
$$- 20\Lambda^2\psi_0'(\psi_0'(\Omega_0\Lambda' + \Lambda\Omega_0') + \Lambda\Omega_0\psi_0'')\Gamma^2 + 2\Lambda(\Lambda^2(5\Omega_0\Gamma' + 6\Omega_0')\psi_0'^2 + 6I_0(\Lambda'\psi_0' + \Lambda\psi_0''))\Gamma$$
$$- 12\Lambda^2 I_0\Gamma'\psi_0'\Big)\Big)\Big]. \tag{B.0.1d}$$

The coefficients of the second order linear differential equation for ψ_2 in Section 4.3.1 are

$$\tilde{a}(\theta) = a(\theta), \tag{B.0.2a}$$
$$\tilde{b}(\theta) = \frac{1}{\Lambda^2(\tilde{R}+\sigma)^2\psi_0'(r_0\Omega_0 - 1)}\Big[2\sigma\tilde{R}\Big(\Lambda\Big(\psi_0'(\Lambda'' - r_0(2\Lambda'\Omega_0' + \Omega_0\Lambda'')) - 3\Lambda'\psi_0''(r_0\Omega_0 - 1)\Big)$$
$$+ \Lambda'^2\psi_0'(1 - r_0\Omega_0) + \Lambda^2(-2r_0\psi_0''\Omega_0' + 2\psi_0'(r_0\Omega_0 - 1) + {\psi_0}^{(3)}(1 - r_0\Omega_0))\Big)$$
$$+ \tilde{R}^2\Big(\Lambda(\psi_0'(\Lambda'' - r_0(2\Lambda'\Omega_0' + \Omega_0\Lambda'')) - 3\Lambda'\psi_0''(r_0\Omega_0 - 1)) + \Lambda'^2\psi_0'(1 - r_0\Omega_0)$$
$$+ \Lambda^2\Big(- 2r_0\psi_0''\Omega_0' + 2\psi_0'(r_0\Omega_0 - 1) + {\psi_0}^{(3)}(1 - r_0\Omega_0)\Big)\Big) + \sigma^2\Big(\Lambda\Lambda'(-2r_0\psi_0'\Omega_0'$$
$$- 3\psi_0''(r_0\Omega_0 - 1)) + \Lambda\Big(\Lambda''\psi_0'(1 - r_0\Omega_0) + \Lambda({\psi_0}^{(3)}(1 - r_0\Omega_0) - 2r_0\psi_0''\Omega_0')\Big)$$
$$+ \Lambda'^2\psi_0'(1 - r_0\Omega_0)\Big)\Big], \tag{B.0.2b}$$

with $a(\theta)$ which can be found in Eq. (3.3.15a). The expression for $\tilde{c}(\theta)$ is very long and I have thus imported it directly from Mathematica below. The function $G(\theta)$ is to be read as $\mathcal{G}$ from Eq. (3.3.7).

$$(6\sigma^2 r_0^4 \Gamma[\theta]^2 \Lambda[\theta]^3 \psi2[\theta] \Omega_0[\theta]^2 \psi_0'[\theta] - 2\sigma^2 r_0^5 \Gamma[\theta]^2 \Lambda[\theta]^3 \psi2[\theta] \Omega_0[\theta]^3 \psi_0'[\theta] -$$

$$2\psi_0'[\theta] (4\tilde{R}(2\sigma+\tilde{R})\Gamma[\theta]^2 \Lambda'[\theta]\psi_0'[\theta] + 4(\sigma^2(-1+(-2+\Gamma[\theta])\Gamma[\theta]) - 2\sigma\tilde{R}(1+\Gamma[\theta]+\Gamma[\theta]^2) - \tilde{R}^2(1+\Gamma[\theta]+\Gamma[\theta]^2))\Lambda[\theta]^2\Lambda'[\theta]\psi_0'[\theta] +$$

$$3\tilde{R}(2\sigma+\tilde{R})\Gamma[\theta]^4\Lambda[\theta]^4\Lambda'[\theta]\psi_0'[\theta] - 4\tilde{R}(2\sigma+\tilde{R})\Gamma[\theta]^2\Lambda[\theta]\psi_0''[\theta] + \tilde{R}(2\sigma+\tilde{R})\Gamma[\theta]^3\Lambda[\theta]^5(2\Gamma'[\theta]\psi_0'[\theta]+\Gamma[\theta]\psi_0''[\theta]) +$$

$$2\Lambda[\theta]^3((2\sigma^2+2\sigma\tilde{R}+\tilde{R}^2)\Gamma'[\theta]\psi_0'[\theta] + 2(\sigma^2(-1+(-2+\Gamma[\theta])\Gamma[\theta]) - 2\sigma\tilde{R}(1+\Gamma[\theta]+\Gamma[\theta]^2) - \tilde{R}^2(1+\Gamma[\theta]+\Gamma[\theta]^2))\psi_0''[\theta])) + 4G[\theta]\Lambda[\theta]^2$$

$$\psi_0'[\theta](\sigma^2(2\Gamma[\theta]\Lambda'[\theta]^2\psi_0'[\theta] + \Lambda[\theta](\psi_0'[\theta](-\Gamma'[\theta]\Lambda'[\theta]+2\Gamma[\theta]\Lambda''[\theta]) + 6\Gamma[\theta]\Lambda'[\theta]\psi_0''[\theta]) + \Lambda[\theta]^2(-\Gamma'[\theta]\psi_0''[\theta] + 2\Gamma[\theta]\psi_0^{(3)}[\theta])) +$$

$$2\sigma\tilde{R}(-\Lambda[\theta]\Gamma'[\theta](\Lambda'[\theta]\psi_0'[\theta]+\Lambda[\theta]\psi_0''[\theta]) + \Gamma[\theta](2\Lambda'[\theta]^2\psi_0'[\theta] + 2\Lambda[\theta](\psi_0'[\theta]\Lambda''[\theta]+3\Lambda'[\theta]\psi_0''[\theta]) + \Lambda[\theta]^2(\psi_0'[\theta]+2\psi_0^{(3)}[\theta]))) +$$

$$\tilde{R}^2(-\Lambda[\theta]\Gamma'[\theta](\Lambda'[\theta]\psi_0'[\theta]+\Lambda[\theta]\psi_0''[\theta]) + \Gamma[\theta](2\Lambda'[\theta]^2\psi_0'[\theta] + 2\Lambda[\theta](\psi_0'[\theta]\Lambda''[\theta]+3\Lambda'[\theta]\psi_0''[\theta]) + \Lambda[\theta]^2(\psi_0'[\theta]+2\psi_0^{(3)}[\theta])))) +$$

$$G[\theta]^2(\sigma+\tilde{R})^2\Gamma[\theta]^2\Lambda[\theta]^3(3\Lambda'[\theta]^2\psi_0'[\theta]\psi_0''[\theta] + \Lambda'[\theta](3\psi_0'[\theta]^2\Lambda''[\theta] - 3\Lambda[\theta]\psi_0''[\theta]^2 + 5\Lambda[\theta]\psi_0'[\theta]\psi_0^{(3)}[\theta]) +$$

$$\Lambda[\theta](\psi_0'[\theta]^2\Lambda^{(3)}[\theta] - \Lambda[\theta]\psi_0''[\theta]\psi_0^{(3)}[\theta] + \psi_0'[\theta](3\Lambda''[\theta]\psi_0''[\theta]+\Lambda[\theta]\psi_0^{(4)}[\theta])))) - r_0^3\Lambda[\theta]^2\Omega_0[\theta]$$

$$(4\tilde{R}(2\sigma+\tilde{R})\Gamma[\theta]^3\Lambda[\theta]^3\Omega_0[\theta]^2\Gamma'[\theta]\psi_0'[\theta]^2 + 2\tilde{R}(2\sigma+\tilde{R})\Gamma[\theta]^4\Lambda[\theta]^2\Omega_0[\theta]\psi_0'[\theta](\psi_0'[\theta](3\Omega_0[\theta]\Lambda'[\theta]+\Lambda[\theta]\Omega_0'[\theta]) + \Lambda[\theta]\Omega_0[\theta]\psi_0''[\theta]) +$$

$$4\Omega_0[\theta]^2\psi_0'[\theta]((\tilde{R}(2\sigma+\tilde{R})\Lambda[\theta]\Gamma'[\theta] - (\sigma+\tilde{R})^2(-2+G[\theta]\times\Lambda[\theta]\Gamma'[\theta])\Lambda'[\theta])\psi_0'[\theta] - (\sigma+\tilde{R})^2\Lambda[\theta](-2+G[\theta]\times\Lambda[\theta]\Gamma'[\theta])\psi_0''[\theta]) +$$

$$4\Gamma[\theta]\Omega_0[\theta]\psi_0'[\theta](-2\tilde{R}(2\sigma+\tilde{R})\Lambda[\theta]\psi_0'[\theta]\Omega_0'[\theta] - 2(2\sigma^2+3\tilde{R}(2\sigma+\tilde{R}))\Omega_0[\theta](\Lambda'[\theta]\psi_0'[\theta]+\Lambda[\theta]\psi_0''[\theta]) +$$

$$G[\theta](2\sigma^2(\Lambda[\theta]\Omega_0'[\theta](\Lambda'[\theta]\psi_0'[\theta]+\Lambda[\theta]\psi_0''[\theta]) + \Omega_0[\theta](\Lambda'[\theta]^2\psi_0'[\theta] + \Lambda[\theta]\psi_0'[\theta]\Lambda''[\theta] + 3\Lambda[\theta]\Lambda'[\theta]\psi_0''[\theta] + \Lambda[\theta]^2\psi_0^{(3)}[\theta])) +$$

$$2\sigma\tilde{R}(2\Omega_0[\theta]\Lambda'[\theta]^2\psi_0'[\theta] + 2\Lambda[\theta](\psi_0'[\theta](\Lambda'[\theta]\Omega_0'[\theta]+\Omega_0[\theta]\Lambda''[\theta]) + 3\Omega_0[\theta]\Lambda'[\theta]\psi_0''[\theta]) + \Lambda[\theta]^2(2\Omega_0'[\theta]\psi_0''[\theta] +$$

$$\Omega_0[\theta](\psi_0'[\theta]+2\psi_0^{(3)}[\theta]))) + \tilde{R}^2(2\Omega_0[\theta]\Lambda'[\theta]^2\psi_0'[\theta] + 2\Lambda[\theta](\psi_0'[\theta](\Lambda'[\theta]\Omega_0'[\theta]+\Omega_0[\theta]\Lambda''[\theta]) + 3\Omega_0[\theta]\Lambda'[\theta]\psi_0''[\theta]) +$$

$$\Lambda[\theta]^2(2\Omega_0'[\theta]\psi_0''[\theta] + \Omega_0[\theta](\psi_0'[\theta]+2\psi_0^{(3)}[\theta]))))) + \Gamma[\theta]^2(-8\Omega_0[\theta]^2\Lambda'[\theta](-\sigma^2 - 2\tilde{R}(2\sigma+\tilde{R}) + G[\theta](\sigma+\tilde{R})^2\Lambda'[\theta])\psi_0'[\theta]^2 +$$

$$\Lambda[\theta]\psi_0'[\theta](6\sigma^2\psi2[\theta] + \Omega_0[\theta](\psi_0'[\theta](-8(-\tilde{R}(2\sigma+\tilde{R}) + G[\theta](\sigma+\tilde{R})^2\Lambda'[\theta])\Omega_0'[\theta] + G[\theta](\sigma+\tilde{R})^2\Omega_0[\theta](-8+3G[\theta]\Lambda'[\theta])\Lambda''[\theta]) +$$

$$\Omega_0[\theta](8(\sigma^2+2\tilde{R}(2\sigma+\tilde{R})) - 24G[\theta](\sigma+\tilde{R})^2\Lambda'[\theta] + 3G[\theta]^2(\sigma+\tilde{R})^2\Lambda'[\theta]^2)\psi_0''[\theta])) +$$

$$G[\theta]\Lambda[\theta]^2\Omega_0[\theta](\sigma^2(-8\psi_0'[\theta](\Omega_0'[\theta]\psi_0''[\theta] + \Omega_0[\theta]\psi_0^{(3)}[\theta]) + G[\theta](-3\Omega_0[\theta]\Lambda'[\theta]\psi_0''[\theta]^2 + \psi_0'[\theta]^2$$

$$(2\Omega_0'[\theta]\Lambda''[\theta] + \Omega_0[\theta]\Lambda^{(3)}[\theta]) + \psi_0'[\theta]((2\Lambda'[\theta]\Omega_0'[\theta]+3\Omega_0[\theta]\Lambda''[\theta])\psi_0''[\theta] + 5\Omega_0[\theta]\Lambda'[\theta]\psi_0^{(3)}[\theta]))) + 2\sigma\tilde{R}$$

$$(2\psi_0'[\theta]\Omega_0'[\theta](G[\theta]\psi_0'[\theta]\Lambda''[\theta] + (-4+G[\theta]\Lambda'[\theta])\psi_0''[\theta]) + \Omega_0[\theta](-3G[\theta]\Lambda'[\theta]\psi_0''[\theta]^2 + \psi_0'[\theta]^2(-8+G[\theta]\Lambda^{(3)}[\theta]) + \psi_0'[\theta]$$

$$(3G[\theta]\Lambda''[\theta]\psi_0''[\theta] + (-8+5G[\theta]\Lambda'[\theta])\psi_0^{(3)}[\theta]))) + \tilde{R}^2(2\psi_0'[\theta]\Omega_0'[\theta](G[\theta]\psi_0'[\theta]\Lambda''[\theta] + (-4+G[\theta]\Lambda'[\theta])\psi_0''[\theta]) +$$

$$\Omega_0[\theta](-3G[\theta]\Lambda'[\theta]\psi_0''[\theta]^2 + \psi_0'[\theta]^2(-8+G[\theta]\Lambda^{(3)}[\theta]) + \psi_0'[\theta](3G[\theta]\Lambda''[\theta]\psi_0''[\theta] + (-8+5G[\theta]\Lambda'[\theta])\psi_0^{(3)}[\theta])))) +$$

$$G[\theta]^2\Lambda[\theta]^3\Omega_0[\theta](-2\sigma\tilde{R}(2\psi_0'[\theta]^2\Omega_0'[\theta] + \psi_0''[\theta](2\Omega_0'[\theta]\psi_0''[\theta] + \Omega_0[\theta]\psi_0^{(3)}[\theta]) - \psi_0'[\theta](2\Omega_0'[\theta]\psi_0^{(3)}[\theta] + \Omega_0[\theta]\psi_0^{(4)}[\theta])) +$$

$$\tilde{R}^2(-2\psi_0'[\theta]^2\Omega_0'[\theta] - \psi_0''[\theta](2\Omega_0'[\theta]\psi_0''[\theta] + \Omega_0[\theta]\psi_0^{(3)}[\theta]) + \psi_0'[\theta](2\Omega_0'[\theta]\psi_0^{(3)}[\theta] + \Omega_0[\theta]\psi_0^{(4)}[\theta])) +$$

$$\sigma^2(-2\Omega_0'[\theta](\psi_0''[\theta]^2 - \psi_0'[\theta]\psi_0^{(3)}[\theta]) + \Omega_0[\theta](-\psi_0''[\theta]\psi_0^{(3)}[\theta] + \psi_0'[\theta]\psi_0^{(4)}[\theta]))))) +$$

$$r_0^2\Lambda[\theta]^2(4\tilde{R}(2\sigma+\tilde{R})\Gamma[\theta]^3\Lambda[\theta]^3\Omega_0[\theta]^2\Gamma'[\theta]\psi_0'[\theta]^2 + 2\tilde{R}(2\sigma+\tilde{R})\Gamma[\theta]^4\Lambda[\theta]^2\Omega_0[\theta]\psi_0'[\theta]$$

$$(\psi_0'[\theta](3\Omega_0[\theta]\Lambda'[\theta] + \Lambda[\theta]\Omega_0'[\theta]) + \Lambda[\theta]\Omega_0[\theta]\psi_0''[\theta]) -$$

$$4\Omega_0[\theta]^2\psi_0'[\theta](-6(\sigma+\tilde{R})^2\Lambda'[\theta]\psi_0'[\theta] + 3G[\theta](\sigma+\tilde{R})^2\Lambda[\theta]^2\Gamma'[\theta]\psi_0''[\theta] +$$

$$\Lambda[\theta](\Gamma'[\theta](2\sigma^2 - \tilde{R}(2\sigma+\tilde{R}) + 3G[\theta](\sigma+\tilde{R})^2\Lambda'[\theta])\psi_0'[\theta] - 6(\sigma+\tilde{R})^2\psi_0''[\theta])) +$$

$$4\Gamma[\theta]\Omega_0[\theta]\psi_0'[\theta](-4\tilde{R}(2\sigma+\tilde{R})\Lambda[\theta]\psi_0'[\theta]\Omega_0'[\theta] - 2(2\sigma^2+5\tilde{R}(2\sigma+\tilde{R}))\Omega_0[\theta](\Lambda'[\theta]\psi_0'[\theta]+\Lambda[\theta]\psi_0''[\theta]) +$$

$$G[\theta](2\sigma^2(2\Lambda[\theta]\Omega_0'[\theta](\Lambda'[\theta]\psi_0'[\theta]+\Lambda[\theta]\psi_0''[\theta]) + 3\Omega_0[\theta](\Lambda'[\theta]^2\psi_0'[\theta] + \Lambda[\theta]\psi_0'[\theta]\Lambda''[\theta] + 3\Lambda[\theta]\Lambda'[\theta]\psi_0''[\theta] + \Lambda[\theta]^2\psi_0^{(3)}[\theta])) +$$

$$2\sigma\tilde{R}(6\Omega_0[\theta]\Lambda'[\theta]^2\psi_0'[\theta] + 2\Lambda[\theta](\psi_0'[\theta](2\Lambda'[\theta]\Omega_0'[\theta]+3\Omega_0[\theta]\Lambda''[\theta]) + 9\Omega_0[\theta]\Lambda'[\theta]\psi_0''[\theta]) + \Lambda[\theta]^2$$

$$(4\Omega_0'[\theta]\psi_0''[\theta] + 3\Omega_0[\theta](\psi_0'[\theta]+2\psi_0^{(3)}[\theta]))) + \tilde{R}^2(6\Omega_0[\theta]\Lambda'[\theta]^2\psi_0'[\theta] + 2\Lambda[\theta]$$

$$(\psi_0'[\theta](2\Lambda'[\theta]\Omega_0'[\theta]+3\Omega_0[\theta]\Lambda''[\theta]) + 9\Omega_0[\theta]\Lambda'[\theta]\psi_0''[\theta]) + \Lambda[\theta]^2(4\Omega_0'[\theta]\psi_0''[\theta] + 3\Omega_0[\theta](\psi_0'[\theta]+2\psi_0^{(3)}[\theta]))))) +$$

$$\Gamma[\theta]^2(-16\Omega_0[\theta]^2\Lambda'[\theta](\sigma^2 - \tilde{R}(2\sigma+\tilde{R}) + G[\theta](\sigma+\tilde{R})^2\Lambda'[\theta])\psi_0'[\theta]^2 + \Lambda[\theta]\psi_0'[\theta](2\sigma^2\psi2[\theta] +$$

$$\Omega_0[\theta](\psi_0'[\theta](-4(\sigma^2-4\tilde{R}(2\sigma+\tilde{R}) + 6G[\theta](\sigma+\tilde{R})^2\Lambda'[\theta])\Omega_0'[\theta] + G[\theta](\sigma+\tilde{R})^2\Omega_0[\theta](-16+9G[\theta]\Lambda'[\theta])\Lambda''[\theta]) +$$

$$\Omega_0[\theta](-16\sigma^2 + 16\tilde{R}(2\sigma+\tilde{R}) - 48G[\theta](\sigma+\tilde{R})^2\Lambda'[\theta] + 9G[\theta]^2(\sigma+\tilde{R})^2\Lambda'[\theta]^2)\psi_0''[\theta])) +$$

$$G[\theta]\Lambda[\theta]^2\Omega_0[\theta](\sigma^2(-4\psi_0'[\theta](5\Omega_0'[\theta]\psi_0''[\theta] + \psi_0'[\theta]\Omega_0''[\theta] + 4\Omega_0[\theta]\psi_0^{(3)}[\theta]) + G[\theta](-9\Omega_0[\theta]\Lambda'[\theta]\psi_0''[\theta]^2 +$$

$$\psi_0'[\theta]^2(4\Omega_0'[\theta]\Lambda''[\theta] + 3\Omega_0[\theta]\Lambda^{(3)}[\theta]) + \psi_0'[\theta]((4\Lambda'[\theta]\Omega_0'[\theta]+9\Omega_0[\theta]\Lambda''[\theta])\psi_0''[\theta] + 15\Omega_0[\theta]\Lambda'[\theta]\psi_0^{(3)}[\theta]))) +$$

$$2\sigma\tilde{R}(4\psi_0'[\theta](\Omega_0'[\theta](G[\theta]\psi_0'[\theta]\Lambda''[\theta] + (-5+G[\theta]\Lambda'[\theta])\psi_0''[\theta]) - \psi_0'[\theta]\Omega_0''[\theta]) + \Omega_0[\theta]$$

$$(-9G[\theta]\Lambda'[\theta]\psi_0''[\theta]^2 + \psi_0'[\theta]^2(-16+3G[\theta]\Lambda^{(3)}[\theta]) + \psi_0'[\theta](9G[\theta]\Lambda''[\theta]\psi_0''[\theta] + (-16+15G[\theta]\Lambda'[\theta])\psi_0^{(3)}[\theta]))) +$$

$$\tilde{R}^2(4\psi_0'[\theta](\Omega_0'[\theta](G[\theta]\psi_0'[\theta]\Lambda''[\theta] + (-5+G[\theta]\Lambda'[\theta])\psi_0''[\theta]) - \psi_0'[\theta]\Omega_0''[\theta]) + \Omega_0[\theta]$$

$$(-9G[\theta]\Lambda'[\theta]\psi_0''[\theta]^2 + \psi_0'[\theta]^2(-16+3G[\theta]\Lambda^{(3)}[\theta]) + \psi_0'[\theta](9G[\theta]\Lambda''[\theta]\psi_0''[\theta] + (-16+15G[\theta]\Lambda'[\theta])\psi_0^{(3)}[\theta])))) +$$

$$G[\theta]^2\Lambda[\theta]^3\Omega_0[\theta](-2\sigma\tilde{R}(4\psi_0'[\theta]^2\Omega_0'[\theta] + \psi_0''[\theta](4\Omega_0'[\theta]\psi_0''[\theta] + 3\Omega_0[\theta]\psi_0^{(3)}[\theta]) - \psi_0'[\theta](4\Omega_0'[\theta]\psi_0^{(3)}[\theta] + 3\Omega_0[\theta]\psi_0^{(4)}[\theta])) +$$

$$\tilde{R}^2(-4\psi_0'[\theta]^2\Omega_0'[\theta] - \psi_0''[\theta](4\Omega_0'[\theta]\psi_0''[\theta] + 3\Omega_0[\theta]\psi_0^{(3)}[\theta]) + \psi_0'[\theta](4\Omega_0'[\theta]\psi_0^{(3)}[\theta] + 3\Omega_0[\theta]\psi_0^{(4)}[\theta])) +$$

$$\sigma^2(-4\Omega_0'[\theta](\psi_0''[\theta]^2 - \psi_0'[\theta]\psi_0^{(3)}[\theta]) + 3\Omega_0[\theta](-\psi_0''[\theta]\psi_0^{(3)}[\theta] + \psi_0'[\theta]\psi_0^{(4)}[\theta]))))) +$$

$$r_0(2\psi_0'[\theta](4\tilde{R}(2\sigma+\tilde{R})\Gamma[\theta]^2\Omega_0[\theta]\Lambda'[\theta]\psi_0'[\theta] + 4(-3(\sigma+\tilde{R})^2 + (-2\sigma^2+2\sigma\tilde{R}+\tilde{R}^2)\Gamma[\theta] + (4\sigma^2-\tilde{R}(2\sigma+\tilde{R}))\Gamma[\theta]^2)\Lambda[\theta]^2\Omega_0[\theta]\Lambda'[\theta]\psi_0'[\theta] +$$

$$3\tilde{R}(2\sigma+\tilde{R})\Gamma[\theta]^4\Lambda[\theta]^4\Omega_0[\theta]\Lambda'[\theta]\psi_0'[\theta] - 4\tilde{R}(2\sigma+\tilde{R})\Gamma[\theta]^2\Lambda[\theta]\Omega_0[\theta]\psi_0''[\theta] + \tilde{R}(2\sigma+\tilde{R})\Gamma[\theta]^3\Lambda[\theta]^5\Omega_0[\theta]$$

$$(2\Gamma'[\theta]\psi_0'[\theta] + \Gamma[\theta]\psi_0''[\theta]) + 2\Lambda[\theta]^3(\psi_0'[\theta]((4\sigma^2+2\sigma\tilde{R}+\tilde{R}^2)\Omega_0[\theta]\Gamma'[\theta] + \Gamma[\theta](2\tilde{R}(2\sigma+\tilde{R}) + 3\sigma^2\Gamma[\theta])\Omega_0'[\theta]) +$$

$$2(-3(\sigma+\tilde{R})^2 + (-2\sigma^2+2\sigma\tilde{R}+\tilde{R}^2)\Gamma[\theta] + (4\sigma^2-\tilde{R}(2\sigma+\tilde{R}))\Gamma[\theta]^2)\Omega_0[\theta]\psi_0''[\theta])) +$$

$$4G[\theta]\Lambda[\theta]^2\psi_0'[\theta](\sigma^2(2(-3+\Gamma[\theta])\Gamma[\theta]\Omega_0[\theta]\Lambda'[\theta]^2\psi_0'[\theta] + \Lambda[\theta](\psi_0'[\theta](2\Gamma[\theta](-1+2\Gamma[\theta])\Lambda'[\theta]\Omega_0'[\theta] +$$

$$\Omega_0[\theta](3\Gamma'[\theta]\Lambda'[\theta] + 2(-3+\Gamma[\theta])\Gamma[\theta]\Lambda''[\theta])) + 6(-3+\Gamma[\theta])\Gamma[\theta]\Omega_0[\theta]\Lambda'[\theta]\psi_0''[\theta]) +$$

$$\Lambda[\theta]^2(\Gamma[\theta]((-2+3\Gamma[\theta])\Omega_0'[\theta]\psi_0''[\theta] + \Gamma[\theta]\psi_0'[\theta]\Omega_0''[\theta]) + \Omega_0[\theta](3\Gamma'[\theta]\psi_0''[\theta] + 2(-3+\Gamma[\theta])\Gamma[\theta]\psi_0^{(3)}[\theta]))) +$$

$$2\sigma\tilde{R}(3\Lambda[\theta]\Omega_0[\theta]\Gamma'[\theta](\Lambda'[\theta]\psi_0'[\theta]+\Lambda[\theta]\psi_0''[\theta]) + \Gamma[\theta]^2(2\Omega_0[\theta]\Lambda'[\theta]^2\psi_0'[\theta] + 2\Lambda[\theta](\psi_0'[\theta](2\Lambda'[\theta]\Omega_0'[\theta]+\Omega_0[\theta]\Lambda''[\theta]) +$$

$$3\Omega_0[\theta]\Lambda'[\theta]\psi_0''[\theta]) + \Lambda[\theta]^2(3\Omega_0'[\theta]\psi_0''[\theta] + \psi_0'[\theta](2\Omega_0[\theta]+\Omega_0''[\theta]) + 2\Omega_0[\theta]\psi_0^{(3)}[\theta])) - \Gamma[\theta](6\Omega_0[\theta]\Lambda'[\theta]^2\psi_0'[\theta] + 2$$

$$\Lambda[\theta](\psi_0'[\theta](\Lambda'[\theta]\Omega_0'[\theta]+3\Omega_0[\theta]\Lambda''[\theta]) + 9\Omega_0[\theta]\Lambda'[\theta]\psi_0''[\theta]) + \Lambda[\theta]^2(2\Omega_0'[\theta]\psi_0''[\theta] + 3\Omega_0[\theta](\psi_0'[\theta]+2\psi_0^{(3)}[\theta])))) +$$

$$\tilde{R}^2(3\Lambda[\theta]\Omega_0[\theta]\Gamma'[\theta](\Lambda'[\theta]\psi_0'[\theta]+\Lambda[\theta]\psi_0''[\theta]) + \Gamma[\theta]^2(2\Omega_0[\theta]\Lambda'[\theta]^2\psi_0'[\theta] + 2\Lambda[\theta](\psi_0'[\theta](2\Lambda'[\theta]\Omega_0'[\theta]+\Omega_0[\theta]\Lambda''[\theta]) +$$

$$3\Omega_0[\theta]\Lambda'[\theta]\psi_0''[\theta]) + \Lambda[\theta]^2(3\Omega_0'[\theta]\psi_0''[\theta] + \psi_0'[\theta](2\Omega_0[\theta]+\Omega_0''[\theta]) + 2\Omega_0[\theta]\psi_0^{(3)}[\theta])) - \Gamma[\theta](6\Omega_0[\theta]\Lambda'[\theta]^2\psi_0'[\theta] + 2$$

$$\Lambda[\theta](\psi_0'[\theta](\Lambda'[\theta]\Omega_0'[\theta]+3\Omega_0[\theta]\Lambda''[\theta]) + 9\Omega_0[\theta]\Lambda'[\theta]\psi_0''[\theta]) + \Lambda[\theta]^2(2\Omega_0'[\theta]\psi_0''[\theta] + 3\Omega_0[\theta](\psi_0'[\theta]+2\psi_0^{(3)}[\theta]))))) -$$

$$G[\theta]^2\Gamma[\theta]^2\Lambda[\theta]^3(2\sigma\tilde{R}(9\Omega_0[\theta]\Lambda'[\theta]\psi_0'[\theta](\psi_0'[\theta]\Lambda''[\theta]+\Lambda'[\theta]\psi_0''[\theta]) + \Lambda[\theta](-9\Omega_0[\theta]\Lambda'[\theta]\psi_0''[\theta]^2 + \psi_0'[\theta]^2$$

$$(2\Omega_0'[\theta]\Lambda''[\theta] + 3\Omega_0[\theta]\Lambda^{(3)}[\theta]) + \psi_0'[\theta]((2\Lambda'[\theta]\Omega_0'[\theta]+9\Omega_0[\theta]\Lambda''[\theta])\psi_0''[\theta] + 15\Omega_0[\theta]\Lambda'[\theta]\psi_0^{(3)}[\theta])) +$$

$$\Lambda[\theta]^2(-2\psi_0'[\theta]^2\Omega_0'[\theta] - \psi_0''[\theta](2\Omega_0'[\theta]\psi_0''[\theta] + 3\Omega_0[\theta]\psi_0^{(3)}[\theta]) + \psi_0'[\theta](2\Omega_0'[\theta]\psi_0^{(3)}[\theta] + 3\Omega_0[\theta]\psi_0^{(4)}[\theta]))) +$$

$$\tilde{R}^2(9\Omega_0[\theta]\Lambda'[\theta]\psi_0'[\theta](\psi_0'[\theta]\Lambda''[\theta]+\Lambda'[\theta]\psi_0''[\theta]) + \Lambda[\theta](-9\Omega_0[\theta]\Lambda'[\theta]\psi_0''[\theta]^2 + \psi_0'[\theta]^2$$

$$(2\Omega_0'[\theta]\Lambda''[\theta] + 3\Omega_0[\theta]\Lambda^{(3)}[\theta]) + \psi_0'[\theta]((2\Lambda'[\theta]\Omega_0'[\theta]+9\Omega_0[\theta]\Lambda''[\theta])\psi_0''[\theta] + 15\Omega_0[\theta]\Lambda'[\theta]\psi_0^{(3)}[\theta])) +$$

$$\Lambda[\theta]^2(-2\psi_0'[\theta]^2\Omega_0'[\theta] - \psi_0''[\theta](2\Omega_0'[\theta]\psi_0''[\theta] + 3\Omega_0[\theta]\psi_0^{(3)}[\theta]) + \psi_0'[\theta](2\Omega_0'[\theta]\psi_0^{(3)}[\theta] + 3\Omega_0[\theta]\psi_0^{(4)}[\theta]))) +$$

$$\sigma^2(2\Lambda[\theta]\Omega_0'[\theta](\psi_0'[\theta]^2\Lambda''[\theta] - \Lambda[\theta]\psi_0''[\theta]^2 + \psi_0'[\theta](\Lambda'[\theta]\psi_0''[\theta] + \Lambda[\theta]\psi_0^{(3)}[\theta])) +$$

$$3\Omega_0[\theta](3\Lambda'[\theta]^2\psi_0'[\theta]\psi_0''[\theta] + \Lambda'[\theta](3\psi_0'[\theta]^2\Lambda''[\theta] - 3\Lambda[\theta]\psi_0''[\theta]^2 + 5\Lambda[\theta]\psi_0'[\theta]\psi_0^{(3)}[\theta]) + \Lambda[\theta](\psi_0'[\theta]^2\Lambda^{(3)}[\theta] -$$

$$\Lambda[\theta]\psi_0''[\theta]\psi_0^{(3)}[\theta] + \psi_0'[\theta](3\Lambda''[\theta]\psi_0''[\theta] + \Lambda[\theta]\psi_0^{(4)}[\theta]))))))) \Big/ ((\sigma+\tilde{R})^2 r_0^2 \Gamma[\theta]^2\Lambda[\theta]^3(-1+r_0\Omega_0[\theta])^3\psi_0'[\theta])$$

The components of the second order correction to the MD four-current in Section 5.4 are

$$
\begin{aligned}
\tilde{j}^{\tilde{T}}_{(1)} &= 2(\Gamma-1)\Gamma\Big(\Gamma^2\Big(-2\sigma\tilde{R}\psi''_{01} + 6r_0\sigma\tilde{R}\Omega_{00}\psi''_{00} + 4r_0\sigma\tilde{R}\Omega_{01}\psi''_{00} \\
&+ 2r_0\sigma\psi'_{01}(2(\tilde{R}+\sigma)\Omega'_{00} + \sigma\Omega'_{01}) + 4r_0\sigma\tilde{R}\Omega_{00}\psi''_{01} + 3r_0\tilde{R}^2\Omega_{00}\psi''_{00} - 3\sigma^2\psi''_{00} - 2\sigma^2\psi''_{01} \\
&- \sigma^2\psi''_{02} + 6r_0\sigma^2\Omega_{00}\psi''_{00} + 4r_0\sigma^2\Omega_{01}\psi''_{00} + r_0\sigma^2\Omega_{02}\psi''_{00} + 4r_0\sigma^2\Omega_{00}\psi''_{01} + 2r_0\sigma^2\Omega_{01}\psi''_{01} \\
&+ r_0\sigma^2\psi'_{02}\Omega'_{00} + r_0\sigma^2\Omega_{00}\psi''_{02}\Big) - 2\Gamma\Big(\psi''_{00}\Big(2\sigma\tilde{R}(5r_0\Omega_{00} + r_0\Omega_{01} - 3) + \tilde{R}^2(5r_0\Omega_{00} - 3) \\
&+ 2\sigma^2(3r_0\Omega_{00} + r_0\Omega_{01} - 2)\Big) + 2r_0\sigma(\tilde{R}+\sigma)\psi'_{01}\Omega'_{00} + 2\sigma(\tilde{R}+\sigma)\psi''_{01}(r_0\Omega_{00} - 1)\Big) \\
&+ r_0\psi'_{00}\Big((\Gamma-1)\Omega'_{00}(2\sigma\tilde{R}(\Gamma^2 + 4\Gamma - 6) + \tilde{R}^2(\Gamma^2 + 4\Gamma - 6) + 6\sigma^2(\Gamma - 1)) \\
&+ \sigma\Gamma(4(\Gamma-1)(\tilde{R}+\sigma)\Omega'_{01} + \sigma\Gamma\Omega'_{02})\Big) + r_0\tilde{R}\Gamma^3\Omega_{00}(\tilde{R}+2\sigma)\psi''_{00} + 6(\tilde{R}+\sigma)^2\psi''_{00}(r_0\Omega_{00} - 1)\Big) \\
&- \Gamma'\Big(\psi'_{00}\Big(-2\sigma\tilde{R}(\Gamma-1)\Big(-2r_0\Gamma^2(\Omega_{00} + \Omega_{01}) + 6\Gamma(4r_0\Omega_{00} + r_0\Omega_{01} - 2) + r_0\Gamma^3\Omega_{00} \\
&- 24r_0\Omega_{00} + 24\Big) - \tilde{R}^2(\Gamma-1)(r_0\Gamma^3\Omega_{00} - 2r_0\Gamma^2\Omega_{00} + 12\Gamma(2r_0\Omega_{00} - 1) - 24r_0\Omega_{00} + 24) \\
&+ \sigma^2(\Gamma^3(6r_0\Omega_{00} + 4r_0\Omega_{01} + r_0\Omega_{02} - 3) - 2\Gamma^2(18r_0\Omega_{00} + 8r_0\Omega_{01} + r_0\Omega_{02} - 11) \\
&+ 6\Gamma(9r_0\Omega_{00} + 2r_0\Omega_{01} - 7) - 24r_0\Omega_{00} + 24)\Big) + \sigma\Gamma(2\psi'_{01}\Big(\tilde{R}(\Gamma^2(2r_0\Omega_{00} - 1) + \Gamma(6 - 8r_0\Omega_{00}) \\
&+ 6r_0\Omega_{00} - 6) + \sigma(\Gamma^2(2r_0\Omega_{00} + r_0\Omega_{01} - 1) - 2\Gamma(4r_0\Omega_{00} + r_0\Omega_{01} - 3) + 6r_0\Omega_{00} - 6)\Big) \\
&+ \sigma(\Gamma-2)\Gamma\psi'_{02}(r_0\Omega_{00} - 1))\Big), \qquad \text{(B.0.3)}
\end{aligned}
$$

$$
\tilde{j}^{\tilde{R}}_{(1)} = \sqrt{2}\left(I'_{00}(-2\sigma\tilde{R}(\Gamma-2) - \tilde{R}^2(\Gamma-2) + 2\sigma^2(\Gamma^2 - \Gamma + 1)) + \sigma\Gamma(\sigma\Gamma I'_{02} - 2(\tilde{R}+\sigma)I'_{01})\right), \qquad \text{(B.0.4)}
$$

$$
\begin{aligned}
\tilde{j}^{\tilde{\Phi}}_{(1)} &= \Gamma'\Big(\Big((\Gamma-1)((5r_0\Omega_{00} + 3r_0\Omega_{01} + r_0\Omega_{02} - 3)\Gamma^3 - 2(15r_0\Omega_{00} + 7r_0\Omega_{01} + r_0\Omega_{02} - 9)\Gamma^2 \\
&+ 12(4r_0\Omega_{00} + r_0\Omega_{01} - 3)\Gamma - 24r_0\Omega_{00} + 24)\sigma^3 + \tilde{R}\Big(-2r_0\Omega_{01}\Gamma^5 + (9r_0\Omega_{00} + 8r_0\Omega_{01} \\
&+ r_0\Omega_{02} - 3)\Gamma^4 - (75r_0\Omega_{00} + 34r_0\Omega_{01} + 3r_0\Omega_{02} - 29)\Gamma^3 + 2(99r_0\Omega_{00} + 26r_0\Omega_{01} + r_0\Omega_{02} \\
&- 63)\Gamma^2 - 12(17r_0\Omega_{00} + 2r_0\Omega_{01} - 14)\Gamma + 72(r_0\Omega_{00} - 1)\Big)\sigma^2 + \tilde{R}^2\Big(-r_0\Omega_{01}\Gamma^5 + (6r_0\Omega_{00} \\
&+ 4r_0\Omega_{01})\Gamma^4 + (-60r_0\Omega_{00} - 17r_0\Omega_{01} + 12)\Gamma^3 + 2(90r_0\Omega_{00} + 13r_0\Omega_{01} - 54)\Gamma^2 - 6(33r_0\Omega_{00} \\
&+ 2r_0\Omega_{01} - 27)\Gamma + 72(r_0\Omega_{00} - 1)\Big)\sigma + 2\tilde{R}^3\Big(r_0\Omega_{00}\Gamma^4 + (2 - 10r_0\Omega_{00})\Gamma^3 + 6(5r_0\Omega_{00} - 3)\Gamma^2 \\
&+ (27 - 33r_0\Omega_{00})\Gamma + 12(r_0\Omega_{00} - 1)\Big)\Big)\psi'_{00} + \sigma\Gamma\Big(\Big((\Gamma-1)((3r_0\Omega_{00} + 2r_0\Omega_{01} - 1)\Gamma^2 \\
&- 2(7r_0\Omega_{00} + 2r_0\Omega_{01} - 5)\Gamma + 12(r_0\Omega_{00} - 1))\sigma^2 + 2\tilde{R}(-r_0\Omega_{00}\Gamma^4 + r_0(4\Omega_{00} + \Omega_{01})\Gamma^3 \\
&+ (-17r_0\Omega_{00} - 3r_0\Omega_{01} + 11)\Gamma^2 + 2(13r_0\Omega_{00} + r_0\Omega_{01} - 11)\Gamma - 12r_0\Omega_{00} + 12)\sigma \\
&+ \tilde{R}^2(-r_0\Omega_{00}\Gamma^4 + 4r_0\Omega_{00}\Gamma^3 + (11 - 17r_0\Omega_{00})\Gamma^2 + (26r_0\Omega_{00} - 22)\Gamma - 12r_0\Omega_{00} + 12)\Big)\psi'_{01}
\end{aligned}
$$

$$
+ \sigma(\sigma + \tilde{R})\Gamma(\Gamma^2 - 3\Gamma + 2)(r_0\Omega_{00} - 1)\psi'_{02}\Big)\Big) - 2(\Gamma - 1)\Gamma\Big(r_0\sigma\tilde{R}(2\sigma + \tilde{R})(\psi'_{01}\Omega'_{00} + \Omega_{01}\psi''_{00}
$$

$$
+ \Omega_{00}\psi''_{01})\Gamma^4 + \sigma\Big(5r_0\Omega_{00}\psi''_{00}\sigma^2 + 3r_0\Omega_{01}\psi''_{00}\sigma^2 + r_0\Omega_{02}\psi''_{00}\sigma^2 - 3\psi''_{00}\sigma^2 + 3r_0\Omega_{00}\psi''_{01}\sigma^2
$$

$$
+ 2r_0\Omega_{01}\psi''_{01}\sigma^2 - \psi''_{01}\sigma^2 + r_0\Omega_{00}\psi''_{02}\sigma^2 - \psi''_{02}\sigma^2 + r_0(\sigma + \tilde{R})\psi'_{02}\Omega'_{00}\sigma - 3\tilde{R}\psi''_{00}\sigma
$$

$$
+ 5r_0\tilde{R}\Omega_{00}\psi''_{00}\sigma + 4r_0\tilde{R}\Omega_{01}\psi''_{00}\sigma + r_0\tilde{R}\Omega_{02}\psi''_{00}\sigma + 4r_0\tilde{R}\Omega_{00}\psi''_{01}\sigma + 2r_0\tilde{R}\Omega_{01}\psi''_{01}\sigma - \tilde{R}\psi''_{02}\sigma
$$

$$
+ r_0\tilde{R}\Omega_{00}\psi''_{02}\sigma + r_0\psi'_{01}((3\sigma^2 + 4\tilde{R}\sigma + 2\tilde{R}^2)\Omega'_{00} + 2\sigma(\sigma + \tilde{R})\Omega'_{01}) + 2r_0\tilde{R}^2\Omega_{01}\psi''_{00} + 2r_0\tilde{R}^2\Omega_{00}\psi''_{01}\Big)\Gamma^3
$$

$$
- (\sigma + \tilde{R})(r_0\psi'_{02}\Omega'_{00}\sigma^2 + 15r_0\Omega_{00}\psi''_{00}\sigma^2 + 7r_0\Omega_{01}\psi''_{00}\sigma^2 + r_0\Omega_{02}\psi''_{00}\sigma^2 - 9\psi''_{00}\sigma^2 + 7r_0\Omega_{00}\psi''_{01}\sigma^2
$$

$$
+ 2r_0\Omega_{01}\psi''_{01}\sigma^2 - 5\psi''_{01}\sigma^2 + r_0\Omega_{00}\psi''_{02}\sigma^2 - \psi''_{02}\sigma^2 + r_0\psi'_{01}(7(\sigma + \tilde{R})\Omega'_{00} + 2\sigma\Omega'_{01})\sigma - 4\tilde{R}\psi''_{00}\sigma
$$

$$
+ 16r_0\tilde{R}\Omega_{00}\psi''_{00}\sigma + 7r_0\tilde{R}\Omega_{01}\psi''_{00}\sigma - 5\tilde{R}\psi''_{01}\sigma + 7r_0\tilde{R}\Omega_{00}\psi''_{01}\sigma - 2\tilde{R}^2\psi''_{00} + 8r_0\tilde{R}^2\Omega_{00}\psi''_{00})\Gamma^2
$$

$$
+ 2(\sigma + \tilde{R})(2r_0\sigma(\sigma + \tilde{R})\psi'_{01}\Omega'_{00} + (2(4r_0\Omega_{00} + r_0\Omega_{01} - 3)\sigma^2
$$

$$
+ 2\tilde{R}(7r_0\Omega_{00} + r_0\Omega_{01} - 5)\sigma + \tilde{R}^2(7r_0\Omega_{00} - 5))\psi''_{00} + 2\sigma(\sigma + \tilde{R})(r_0\Omega_{00} - 1)\psi''_{01})\Gamma
$$

$$
+ r_0(\Gamma - 1)\psi'_{00}\Big((\sigma + \tilde{R})((5\Gamma^2 - 10\Gamma + 6)\sigma^2 - 4\tilde{R}(4\Gamma - 3)\sigma + \tilde{R}^2(6 - 8\Gamma))\Omega'_{00} + \sigma\Gamma(((3\Gamma - 4)\sigma^2
$$

$$
+ 2\tilde{R}(\Gamma^2 + 3\Gamma - 4)\sigma + \tilde{R}^2(\Gamma^2 + 3\Gamma - 4))\Omega'_{01} + \sigma(\sigma + \tilde{R})\Gamma\Omega'_{02})\Big) - 6(\sigma + \tilde{R})^3(r_0\Omega_{00} - 1)\psi''_{00}\Big). \tag{B.0.5}
$$

The components of the second order correction to the four-current $\tilde{j}^{(1)}$ in Section 5.5 are

$$
\tilde{j}^{\tilde{T}}_{(1)} = 2\Big(\Gamma\Lambda'\Big(\Big(6r_0\Omega_{00} + \Gamma(-12r_0\Omega_{00} - 4r_0\Omega_{01} + \Gamma(r_0(6\Omega_{00} + 4\Omega_{01} + \Omega_{02}) - 3) + 8) - 6\Big)\psi'_{00}
$$

$$
+ \Gamma\Big(2(-2r_0\Omega_{00} + \Gamma(2r_0\Omega_{00} + r_0\Omega_{01} - 1) + 2)\psi'_{01} + \Gamma(r_0\Omega_{00} - 1)\psi'_{02}\Big)\Big) + \Lambda\Big(2\Gamma'((-3r_0\Omega_{00}
$$

$$
+ \Gamma(3r_0\Omega_{00} + r_0\Omega_{01} - 2) + 3)\psi'_{00} + \Gamma(r_0\Omega_{00} - 1)\psi'_{01}) + \Gamma\Big(\Big(- 3\psi''_{00} - 2\psi''_{01} - \psi''_{02}
$$

$$
+ r_0\Big(\psi'_{02}\Omega'_{00} + 2\psi'_{01}(2\Omega'_{00} + \Omega'_{01}) + (4\Omega_{01} + \Omega_{02})\psi''_{00} + 2\Omega_{01}\psi''_{01} + \Omega_{00}(6\psi''_{00} + 4\psi''_{01} + \psi''_{02})\Big)\Big)\Gamma^2
$$

$$
- 4(r_0\psi'_{01}\Omega'_{00} + (3r_0\Omega_{00} + r_0\Omega_{01} - 2)\psi''_{00} + (r_0\Omega_{00} - 1)\psi''_{01})\Gamma + r_0\psi'_{00}(6\Omega'_{00}(\Gamma - 1)^2
$$

$$
+ \Gamma(4(\Gamma - 1)\Omega'_{01} + \Gamma\Omega'_{02})) + 6(r_0\Omega_{00} - 1)\psi''_{00}\Big)\Big)\Big)\sigma^2 + 2\tilde{R}\Big(- r_0\Lambda^2\Big(\psi'_{00}(3\Omega_{00}\Lambda' + \Lambda\Omega'_{00})
$$

$$
+ \Lambda\Omega_{00}\psi''_{00}\Big)\Gamma^5 - 2r_0\Lambda^3\Omega_{00}\Gamma'\psi'_{00}\Gamma^4 + 2\Big(\Lambda'\Big(- \psi'_{01} - \psi'_{10} - \psi'_{11} + r_0\Big((2\Omega_{10} + \Omega_{11})\psi'_{00}
$$

$$
+ \Omega_{10}\psi'_{01} + \Omega_{01}(2\psi'_{00} + \psi'_{10}) + \Omega_{00}(4\psi'_{00} + 2(\psi'_{01} + \psi'_{10}) + \psi'_{11})\Big)\Big) + \Lambda\Big(2\psi_{20}(r_0\Omega_{00} - 1)
$$

$$
+ 2\psi_{10}(2r_0\Omega_{00} + r_0\Omega_{10} - 1) - \psi''_{01} - \psi''_{10} - \psi''_{11} + r_0\Big((2\psi'_{10} + \psi'_{11})\Omega'_{00} + \psi'_{10}\Omega'_{01} + \psi'_{01}(2\Omega'_{00} + \Omega'_{10})
$$

$$
+ \psi'_{00}(4\Omega'_{00} + 2(\Omega'_{01} + \Omega'_{10}) + \Omega'_{11}) + (2\Omega_{10} + \Omega_{11})\psi''_{00} + \Omega_{10}\psi''_{01} + \Omega_{01}(2\psi''_{00} + \psi''_{10})
$$

$$
+ \Omega_{00}(4\psi''_{00} + 2(\psi''_{01} + \psi''_{10}) + \psi''_{11})\Big)\Big)\Big)\Gamma^3 - 4\Big(\Lambda'((r_0(5\Omega_{00} + \Omega_{01} + \Omega_{10}) - 3)\psi'_{00}
$$

$$
+ (r_0\Omega_{00} - 1)(\psi'_{01} + \psi'_{10})) + \Lambda\Big(\psi_{10}(r_0\Omega_{00} - 1) - 3\psi''_{00} - \psi''_{01} - \psi''_{10} + r_0\Big((\psi'_{01} + \psi'_{10})\Omega'_{00}
$$

$$
\begin{aligned}
&+\psi'_{00}(5\Omega'_{00}+\Omega'_{01}+\Omega'_{10})+(\Omega_{01}+\Omega_{10})\psi''_{00}+\Omega_{00}(5\psi''_{00}+\psi''_{01}+\psi''_{10})\Big)\Big)\Big)\Gamma^2 \\
&+2\Big(6(r_0\Omega_{00}-1)\Lambda'\psi'_{00}+\Lambda(\Gamma'((r_0(5\Omega_{00}+\Omega_{01}+\Omega_{10})-3)\psi'_{00}+(r_0\Omega_{00}-1)(\psi'_{01}+\psi'_{10})) \\
&+6r_0\psi'_{00}\Omega'_{00}+6(r_0\Omega_{00}-1)\psi''_{00})\Big)\Gamma-12\Lambda(r_0\Omega_{00}-1)\Gamma'\psi'_{00}\Big)\sigma+\tilde{R}^2\Big(-r_0\Lambda^2(\psi'_{00}(3\Omega_{00}\Lambda' \\
&+\Lambda\Omega'_{00})+\Lambda\Omega_{00}\psi''_{00})\Gamma^5-2r_0\Lambda^3\Omega_{00}\Gamma'\psi'_{00}\Gamma^4+2\Big(\Lambda'\Big(r_0(4\Omega_{10}+\Omega_{20})\psi'_{00}+2r_0\Omega_{10}\psi'_{10} \\
&-2\psi'_{10}-\psi'_{20}+r_0\Omega_{00}(4(\psi'_{00}+\psi'_{10})+\psi'_{20})\Big)+\Lambda\Big(2\psi_{10}(2r_0\Omega_{00}+r_0\Omega_{10}-1)-2(\psi_{20}+\psi''_{10}) \\
&-\psi''_{20}+r_0\Big(2\psi_{20}\Omega_{00}+(4(\psi''_{00}+\psi''_{10})+\psi''_{20})\Omega_{00}+(4\psi'_{10}+\psi'_{20})\Omega'_{00}+2\psi'_{10}\Omega'_{10} \\
&+\psi'_{00}(4(\Omega'_{00}+\Omega'_{10})+\Omega'_{20})+(4\Omega_{10}+\Omega_{20})\psi''_{00}+2\Omega_{10}\psi''_{10}\Big)\Big)\Big)\Gamma^3-4\Big(\Lambda'((5r_0\Omega_{00} \\
&+2r_0\Omega_{10}-3)\psi'_{00}+2(r_0\Omega_{00}-1)\psi'_{10})+\Lambda\Big(\psi_{10}(r_0\Omega_{00}-1)-3\psi''_{00}-2\psi''_{10}+r_0((5\psi'_{00} \\
&+2\psi'_{10})\Omega'_{00}+2\psi'_{00}\Omega'_{10}+(5\Omega_{00}+2\Omega_{10})\psi''_{00}+2\Omega_{00}\psi''_{10})\Big)\Big)\Gamma^2+2\Big(6(r_0\Omega_{00}-1)\Lambda'\psi'_{00} \\
&+\Lambda(\Gamma'((5r_0\Omega_{00}+2r_0\Omega_{10}-3)\psi'_{00}+2(r_0\Omega_{00}-1)\psi'_{10})+6r_0\psi'_{00}\Omega'_{00}+6(r_0\Omega_{00}-1)\psi''_{00})\Big)\Gamma \\
&-12\Lambda(r_0\Omega_{00}-1)\Gamma'\psi'_{00}\Big), \qquad \text{(B.0.6a)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{j}^{\tilde{R}}_{(1)} &= I'_0\Big(4\sigma\tilde{R}(\Gamma-2)+2\tilde{R}^2(\Gamma-2)-4\sigma^2((\Gamma-1)\Gamma+1)\Big)-2\Gamma\Big(\Gamma(2\sigma\tilde{R}I'_{11}+\tilde{R}^2I'_{20}+\sigma^2I'_{02}) \\
&-2\sigma(\tilde{R}+\sigma)I'_1-2\tilde{R}(\tilde{R}+\sigma)I'_1\Big) \qquad \text{(B.0.6b)}
\end{aligned}
$$

$$
\tilde{j}^{\theta}_{(1)} = 4\Gamma(\tilde{R}I_{20}+\sigma I_{11})-4I_1(\tilde{R}+\sigma). \qquad \text{(B.0.6c)}
$$

The $\tilde{\Phi}$-component of the four-current was imported directly from Mathematica due to its length and is presented below.

$$-2\,\sigma^3\,\Lambda[\theta]^2\,\big(\Gamma[\theta]\,\Lambda'[\theta]\,((-6+6\,r0\,\Omega00[\theta]+\Gamma[\theta]\,(6-10\,r0\,\Omega00[\theta]-4\,r0\,\Omega01[\theta]+\Gamma[\theta]\,(-3+r0\,(5\,\Omega00[\theta]+3\,\Omega01[\theta]+\Omega02[\theta]))))\,\psi00'[\theta]+$$
$$\Gamma[\theta]\,((4-4\,r0\,\Omega00[\theta]+\Gamma[\theta]\,(-1+3\,r0\,\Omega00[\theta]+2\,r0\,\Omega01[\theta]))\,\psi01'[\theta]+\Gamma[\theta]\,(-1+r0\,\Omega00[\theta])\,\psi02'[\theta]))+$$
$$\Lambda[\theta]\,\big(\Gamma'[\theta]\,((6-6\,r0\,\Omega00[\theta]+\Gamma[\theta]\,(-3+5\,r0\,\Omega00[\theta]+2\,r0\,\Omega01[\theta]))\,\psi00'[\theta]+2\,\Gamma[\theta]\,(-1+r0\,\Omega00[\theta])\,\psi01'[\theta])+$$
$$\Gamma[\theta]\,\big(r0\,\psi00'[\theta]\,((6+5\,(-2+\Gamma[\theta])\,\Gamma[\theta])\,\Omega00'[\theta]+\Gamma[\theta]\,((-4+3\,\Gamma[\theta])\,\Omega01'[\theta]+\Gamma[\theta]\,\Omega02'[\theta]))+6\,(-1+r0\,\Omega00[\theta])\,\psi00''[\theta]-$$
$$2\,\Gamma[\theta]\,(2\,r0\,\psi01'[\theta]\,\Omega00'[\theta]+(-3+5\,r0\,\Omega00[\theta]+2\,r0\,\Omega01[\theta])\,\psi00''[\theta]+2\,(-1+r0\,\Omega00[\theta])\,\psi01''[\theta])+$$
$$\Gamma[\theta]^2\,(-3\,\psi00''[\theta]-\psi01''[\theta]-\psi02''[\theta]+r0\,((3\,\psi01'[\theta]+\psi02'[\theta])\,\Omega00'[\theta]+2\,\psi01'[\theta]\,\Omega01'[\theta]+(3\,\Omega01[\theta]+\Omega02[\theta])\,\psi00''[\theta]+$$
$$2\,\Omega01[\theta]\,\psi01''[\theta]+\Omega00[\theta]\,(5\,\psi00''[\theta]+3\,\psi01''[\theta]+\psi02''[\theta]))))\big)\big)\big)+2\,\sigma^2\,\tilde{R}\,\big(18\,\Lambda[\theta]^3\,(-1+r0\,\Omega00[\theta])\,\Gamma'[\theta]\,\psi00'[\theta]-$$
$$\Gamma[\theta]\times\Lambda[\theta]\,\big(\Gamma'[\theta]\,\big(\big(4+\Lambda[\theta]^2\,(-9+15\,r0\,\Omega00[\theta]+4\,r0\,\Omega01[\theta]+2\,r0\,\Omega10[\theta])\big)\,\psi00'[\theta]+2\,\Lambda[\theta]^2\,(-1+r0\,\Omega00[\theta])\,(2\,\psi01'[\theta]+\psi10'[\theta])\big)+$$
$$18\,\Lambda[\theta]\,(\psi00'[\theta]\,((-1+r0\,\Omega00[\theta])\,\Lambda'[\theta]+r0\,\Lambda[\theta]\,\Omega00'[\theta])+\Lambda[\theta]\,(-1+r0\,\Omega00[\theta])\,\psi00''[\theta])\big)+2\,\Gamma[\theta]^4\,\Lambda[\theta]^4$$
$$(-3\,(-1+r0\,\Omega00[\theta])\,\Lambda'[\theta]\,\psi00'[\theta]+\Lambda[\theta]\,(\Gamma'[\theta]\,((2+r0\,\Omega01[\theta])\,\psi00'[\theta]+(-1+r0\,\Omega00[\theta])\,\psi01'[\theta])+\psi00''[\theta]-r0\,(\psi00'[\theta]\,\Omega00'[\theta]+\Omega00[\theta]\,\psi00''[\theta])))+$$
$$\Gamma[\theta]^5\,\Lambda[\theta]^4\,(3\,\Lambda'[\theta]\,((2+r0\,\Omega01[\theta])\,\psi00'[\theta]+(-1+r0\,\Omega00[\theta])\,\psi01'[\theta])+$$
$$\Lambda[\theta]\,(2\,\psi10[\theta]\,(-1+r0\,\Omega00[\theta])+r0\,\psi01'[\theta]\,\Omega00'[\theta]+r0\,\psi00'[\theta]\,\Omega01'[\theta]+2\,\psi00''[\theta]+r0\,\Omega01[\theta]\,\psi00''[\theta]+(-1+r0\,\Omega00[\theta])\,\psi01''[\theta]))+$$
$$2\,\Gamma[\theta]^2\,\Lambda[\theta]^2\,(\Lambda'[\theta]\,((-9+15\,r0\,\Omega00[\theta]+4\,r0\,\Omega01[\theta]+2\,r0\,\Omega10[\theta])\,\psi00'[\theta]+2\,(-1+r0\,\Omega00[\theta])\,(2\,\psi01'[\theta]+\psi10'[\theta]))+$$
$$\Lambda[\theta]\,(6\,\psi10[\theta]\,(-1+r0\,\Omega00[\theta])-9\,\psi00''[\theta]-4\,\psi01''[\theta]-2\,\psi10''[\theta]+r0\,(2\,(2\,\psi01'[\theta]+\psi10'[\theta])\,\Omega00'[\theta]+$$
$$\psi00'[\theta]\,(15\,\Omega00'[\theta]+4\,\Omega01'[\theta]+2\,\Omega10'[\theta])+2\,(2\,\Omega01[\theta]+\Omega10[\theta])\,\psi00''[\theta]+\Omega00[\theta]\,(15\,\psi00''[\theta]+4\,\psi01''[\theta]+2\,\psi10''[\theta]))))-$$
$$\Gamma[\theta]^3\,\big(3\,\Lambda[\theta]^5\,(-1+r0\,\Omega00[\theta])\,\Gamma'[\theta]\,\psi00'[\theta]+4\,\Lambda'[\theta]\,(-2\,\psi00'[\theta]+\psi01'[\theta])+\Lambda[\theta]^2\,\Lambda'[\theta]$$
$$((1+r0\,(9\,\Omega00[\theta]+8\,\Omega01[\theta]+\Omega02[\theta]+3\,\Omega10[\theta]+2\,\Omega11[\theta]))\,\psi00'[\theta]+2\,(-2+r0\,(4\,\Omega00[\theta]+\Omega01[\theta]+\Omega10[\theta]))\,\psi01'[\theta]-\psi02'[\theta]-$$
$$\psi10'[\theta]-2\,\psi11'[\theta]+r0\,(2\,\Omega01[\theta]\,\psi10'[\theta]+\Omega00[\theta]\,(\psi02'[\theta]+3\,\psi10'[\theta]+2\,\psi11'[\theta])))-4\,\Lambda[\theta]\,(2\,\psi10[\theta]-2\,\psi00''[\theta]+\psi01''[\theta])+$$
$$\Lambda[\theta]^3\,(-4\,\psi20[\theta]+4\,\psi11[\theta]\,(-1+r0\,\Omega00[\theta])+4\,\psi10[\theta]\,(-2+r0\,(4\,\Omega00[\theta]+\Omega01[\theta]+\Omega10[\theta]))+\psi00''[\theta]-4\,\psi01''[\theta]-\psi02''[\theta]-\psi10''[\theta]-$$
$$2\,\psi11''[\theta]+r0\,(4\,\psi20[\theta]\times\Omega00[\theta]+(\psi02'[\theta]+3\,\psi10'[\theta]+2\,\psi11'[\theta])\,\Omega00'[\theta]+2\,\psi10'[\theta]\,\Omega01'[\theta]+2\,\psi01'[\theta]\,(4\,\Omega00'[\theta]+\Omega01'[\theta]+\Omega10'[\theta])+$$
$$\psi00'[\theta]\,(9\,\Omega00'[\theta]+8\,\Omega01'[\theta]+\Omega02'[\theta]+3\,\Omega10'[\theta]+2\,\Omega11'[\theta])+8\,\Omega01[\theta]\,\psi00''[\theta]+\Omega02[\theta]\,\psi00''[\theta]+3\,\Omega10[\theta]\,\psi00''[\theta]+2\,\Omega11[\theta]\,\psi00''[\theta]+$$
$$2\,\Omega01[\theta]\,\psi01''[\theta]+2\,\Omega10[\theta]\,\psi01''[\theta]+2\,\Omega01[\theta]\,\psi10''[\theta]+\Omega00[\theta]\,(9\,\psi00''[\theta]+8\,\psi01''[\theta]+\psi02''[\theta]+3\,\psi10''[\theta]+2\,\psi11''[\theta]))))\big)\big)+$$
$$\tilde{R}^3\,\big(12\,\Lambda[\theta]^3\,(-1+r0\,\Omega00[\theta])\,\Gamma'[\theta]\,\psi00'[\theta]-2\,\Gamma[\theta]\times\Lambda[\theta]\,\big(\Gamma'[\theta]\,\big(\big(2+\Lambda[\theta]^2\,(-3+5\,r0\,\Omega00[\theta]+2\,r0\,\Omega10[\theta])\big)\,\psi00'[\theta]+2\,\Lambda[\theta]^2\,(-1+r0\,\Omega00[\theta])\,\psi10'[\theta]\big)+$$
$$6\,\Lambda[\theta]\,(\psi00'[\theta]\,((-1+r0\,\Omega00[\theta])\,\Lambda'[\theta]+r0\,\Lambda[\theta]\,\Omega00'[\theta])+\Lambda[\theta]\,(-1+r0\,\Omega00[\theta])\,\psi00''[\theta])\big)+2\,\Gamma[\theta]^4\,\Lambda[\theta]^4$$
$$(-3\,(-1+r0\,\Omega00[\theta])\,\Lambda'[\theta]\,\psi00'[\theta]+\Lambda[\theta]\,(\Gamma'[\theta]\,((2+r0\,\Omega10[\theta])\,\psi00'[\theta]+(-1+r0\,\Omega00[\theta])\,\psi10'[\theta])+\psi00''[\theta]-r0\,(\psi00'[\theta]\,\Omega00'[\theta]+\Omega00[\theta]\,\psi00''[\theta])))+$$
$$\Gamma[\theta]^5\,\Lambda[\theta]^4\,(3\,\Lambda'[\theta]\,((2+r0\,\Omega10[\theta])\,\psi00'[\theta]+(-1+r0\,\Omega00[\theta])\,\psi10'[\theta])+$$
$$\Lambda[\theta]\,(2\,\psi10[\theta]\,(-1+r0\,\Omega00[\theta])+r0\,\psi10'[\theta]\,\Omega00'[\theta]+r0\,\psi00'[\theta]\,\Omega10'[\theta]+2\,\psi00''[\theta]+r0\,\Omega10[\theta]\,\psi00''[\theta]+(-1+r0\,\Omega00[\theta])\,\psi10''[\theta]))+$$
$$4\,\Gamma[\theta]^2\,\Lambda[\theta]^2\,(\Lambda'[\theta]\,((-3+5\,r0\,\Omega00[\theta]+2\,r0\,\Omega10[\theta])\,\psi00'[\theta]+2\,(-1+r0\,\Omega00[\theta])\,\psi10'[\theta])+\Lambda[\theta]\,(3\,\psi10[\theta]\,(-1+r0\,\Omega00[\theta])-3\,\psi00''[\theta]-$$
$$2\,\psi10''[\theta]+r0\,((5\,\psi00'[\theta]+2\,\psi10'[\theta])\,\Omega00'[\theta]+2\,\psi00'[\theta]\,\Omega10'[\theta]+(5\,\Omega00[\theta]+2\,\Omega10[\theta])\,\psi00''[\theta]+2\,\Omega00[\theta]\,\psi10''[\theta])))-$$
$$\Gamma[\theta]^3\,\big(3\,\Lambda[\theta]^5\,(-1+r0\,\Omega00[\theta])\,\Gamma'[\theta]\,\psi00'[\theta]+4\,\Lambda'[\theta]\,(-2\,\psi00'[\theta]+\psi10'[\theta])+2\,\Lambda[\theta]^2\,\Lambda'[\theta]\,((2+r0\,(2\,\Omega00[\theta]+4\,\Omega10[\theta]+\Omega20[\theta]))\,\psi00'[\theta]+$$
$$2\,(-1+2\,r0\,\Omega00[\theta]+r0\,\Omega10[\theta])\,\psi10'[\theta]+(-1+r0\,\Omega00[\theta])\,\psi20'[\theta])-4\,\Lambda[\theta]\,(2\,\psi10[\theta]-2\,\psi00''[\theta]+\psi10''[\theta])+2\,\Lambda[\theta]^3$$
$$(4\,\psi20[\theta]\,(-1+r0\,\Omega00[\theta])+4\,\psi10[\theta]\,(-1+2\,r0\,\Omega00[\theta]+r0\,\Omega10[\theta])+2\,\psi00''[\theta]-2\,\psi10''[\theta]-\psi20''[\theta]+r0\,((4\,\psi10'[\theta]+\psi20'[\theta])\,\Omega00'[\theta]+2\,\psi10'[\theta]\,\Omega10'[$$
$$\theta]+\psi00'[\theta]\,(2\,\Omega00'[\theta]+4\,\Omega10'[\theta]+\Omega20'[\theta])+(4\,\Omega10[\theta]+\Omega20[\theta])\,\psi00''[\theta]+2\,\Omega10[\theta]\,\psi10''[\theta]+\Omega00[\theta]\,(2\,\psi00''[\theta]+4\,\psi10''[\theta]+\psi20''[\theta]))))\big)\big)+$$
$$\sigma\,\tilde{R}^2\,\big(36\,\Lambda[\theta]^3\,(-1+r0\,\Omega00[\theta])\,\Gamma'[\theta]\,\psi00'[\theta]+2\,\Gamma[\theta]^4\,\Lambda[\theta]^4\,(-9\,(-1+r0\,\Omega00[\theta])\,\Lambda'[\theta]\,\psi00'[\theta]+$$
$$\Lambda[\theta]\,(\Gamma'[\theta]\,((6+r0\,\Omega01[\theta]+2\,r0\,\Omega10[\theta])\,\psi00'[\theta]+(-1+r0\,\Omega00[\theta])\,(\psi01'[\theta]+2\,\psi10'[\theta]))-3\,r0\,\psi00'[\theta]\,\Omega00'[\theta]-3\,(-1+r0\,\Omega00[\theta])\,\psi00''[\theta]))-$$
$$2\,\Gamma[\theta]\times\Lambda[\theta]\,\big(\Gamma'[\theta]\,\big(\big(6+\Lambda[\theta]^2\,(-9+15\,r0\,\Omega00[\theta]+2\,r0\,\Omega01[\theta]+4\,r0\,\Omega10[\theta])\big)\,\psi00'[\theta]+2\,\Lambda[\theta]^2\,(-1+r0\,\Omega00[\theta])\,(\psi01'[\theta]+2\,\psi10'[\theta])\big)+$$
$$18\,\Lambda[\theta]\,(\psi00'[\theta]\,((-1+r0\,\Omega00[\theta])\,\Lambda'[\theta]+r0\,\Lambda[\theta]\,\Omega00'[\theta])+\Lambda[\theta]\,(-1+r0\,\Omega00[\theta])\,\psi00''[\theta])\big)+$$
$$\Gamma[\theta]^5\,\Lambda[\theta]^4\,(3\,\Lambda'[\theta]\,((6+r0\,\Omega01[\theta]+2\,r0\,\Omega10[\theta])\,\psi00'[\theta]+(-1+r0\,\Omega00[\theta])\,(\psi01'[\theta]+2\,\psi10'[\theta]))+\Lambda[\theta]\,(6\,\psi10[\theta]\,(-1+r0\,\Omega00[\theta])+6\,\psi00''[\theta]-$$
$$\psi01''[\theta]-2\,\psi10''[\theta]+r0\,((\psi01'[\theta]+2\,\psi10'[\theta])\,\Omega00'[\theta]+\psi00'[\theta]\,(\Omega01'[\theta]+2\,\Omega10'[\theta])+(\Omega01[\theta]+2\,\Omega10[\theta])\,\psi00''[\theta]+\Omega00[\theta]\,(\psi01''[\theta]+2\,\psi10''[\theta]))))+$$
$$4\,\Gamma[\theta]^2\,\Lambda[\theta]^2\,(\Lambda'[\theta]\,((-9+15\,r0\,\Omega00[\theta]+2\,r0\,\Omega01[\theta]+4\,r0\,\Omega10[\theta])\,\psi00'[\theta]+2\,(-1+r0\,\Omega00[\theta])\,(\psi01'[\theta]+2\,\psi10'[\theta]))+$$
$$\Lambda[\theta]\,(9\,\psi10[\theta]\,(-1+r0\,\Omega00[\theta])-9\,\psi00''[\theta]-2\,\psi01''[\theta]-4\,\psi10''[\theta]+r0\,(2\,(\psi01'[\theta]+2\,\psi10'[\theta])\,\Omega00'[\theta]+$$
$$\psi00'[\theta]\,(15\,\Omega00'[\theta]+2\,\Omega01'[\theta]+4\,\Omega10'[\theta])+2\,(\Omega01[\theta]+2\,\Omega10[\theta])\,\psi00''[\theta]+\Omega00[\theta]\,(15\,\psi00''[\theta]+2\,\psi01''[\theta]+4\,\psi10''[\theta]))))-$$
$$\Gamma[\theta]^3\,\big(9\,\Lambda[\theta]^5\,(-1+r0\,\Omega00[\theta])\,\Gamma'[\theta]\,\psi00'[\theta]+4\,\Lambda'[\theta]\,(-6\,\psi00'[\theta]+\psi01'[\theta]+2\,\psi10'[\theta])+2\,\Lambda[\theta]^2\,\Lambda'[\theta]$$
$$((6+r0\,(6\,\Omega00[\theta]+4\,\Omega01[\theta]+8\,\Omega10[\theta]+2\,\Omega11[\theta]+\Omega20[\theta]))\,\psi00'[\theta]+2\,(-1+2\,r0\,\Omega00[\theta]+r0\,\Omega10[\theta])\,\psi01'[\theta]+2\,(-2+r0\,(4\,\Omega00[\theta]+\Omega01[\theta]+\Omega10[\theta]))$$
$$\psi10'[\theta]+2\,(-1+r0\,\Omega00[\theta])\,\psi11'[\theta]+(-1+r0\,\Omega00[\theta])\,\psi20'[\theta])-4\,\Lambda[\theta]\,(6\,\psi10[\theta]-6\,\psi00''[\theta]+\psi01''[\theta]+2\,\psi10''[\theta])+$$
$$2\,\Lambda[\theta]^3\,(2\,\psi11[\theta]\,(-1+r0\,\Omega00[\theta])+2\,\psi10[\theta]\,(-6+r0\,(12\,\Omega00[\theta]+\Omega01[\theta]+5\,\Omega10[\theta]))-2\,(5\,\psi20[\theta]-3\,\psi00''[\theta]+\psi01''[\theta]+2\,\psi10''[\theta]+\psi11''[\theta])-$$
$$\psi20''[\theta]+r0\,(10\,\psi20[\theta]\times\Omega00[\theta]+4\,\psi01'[\theta]\,\Omega00'[\theta]+8\,\psi10'[\theta]\,\Omega00'[\theta]+2\,\psi11'[\theta]\,\Omega00'[\theta]+\psi20'[\theta]\,\Omega00'[\theta]+2\,\psi10'[\theta]\,\Omega01'[\theta]+2\,(\psi01'[\theta]+\psi10'[\theta])$$
$$\Omega10'[\theta]+\psi00'[\theta]\,(6\,\Omega00'[\theta]+4\,\Omega01'[\theta]+8\,\Omega10'[\theta]+2\,\Omega11'[\theta]+\Omega20'[\theta])+4\,\Omega01[\theta]\,\psi00''[\theta]+8\,\Omega10[\theta]\,\psi00''[\theta]+2\,\Omega11[\theta]\,\psi00''[\theta]+$$
$$\Omega20[\theta]\,\psi00''[\theta]+2\,\Omega10[\theta]\,\psi01''[\theta]+2\,(\Omega01[\theta]+\Omega10[\theta])\,\psi10''[\theta]+\Omega00[\theta]\,(6\,\psi00''[\theta]+4\,\psi01''[\theta]+8\,\psi10''[\theta]+2\,\psi11''[\theta]+\psi20''[\theta]))))\big)\big)$$

The part of the constant $c^*(\theta)$ in Section 5.5 with no σ-dependence is given by

$$
\begin{aligned}
c^* = {}& \frac{1}{4r_0^3\Gamma^2\Lambda I_{00}^3\psi'_{00}}\Big[-4\mathcal{G}I_{00}\psi'_{00}\Big(\Gamma^2\psi'^2_{00}(\Lambda\Big(-2r_0(\psi'_{00}+\psi^{(3)}_{00})\Omega^2_{00}+2(\psi'_{00}-r_0\Omega'_{00}\psi''_{00} \\
&+\psi^{(3)}_{00})\Omega_{00}+3\Omega'_{00}\psi''_{00}+\psi'_{00}\Omega''_{00}\Big)-2(r_0\Omega_{00}-2)\Lambda'\psi'_{00}\Omega'_{00})\Lambda^3-2r_0I_{00}\Gamma\psi'_{00}\Big(\Gamma\Omega_{00}(\psi'_{00}\Lambda'^2 \\
&+3\Lambda\psi''_{00}\Lambda'+\Lambda\psi'_{00}\Lambda'')-\Lambda\Omega'_{00}(\Lambda'\psi'_{00}+\Lambda\psi''_{00})\Big)\Lambda+r_0I^2_{00}\Big(\Gamma\Big((\psi'_{00}+2\psi^{(3)}_{00})\Lambda^2+2(\psi'_{00}\Lambda'' \\
&+3\Lambda'\psi''_{00})\Lambda+2\Lambda'^2\psi'_{00}\Big)-\Lambda\Gamma'(\Lambda'\psi'_{00}+\Lambda\psi''_{00})\Big)\Big)r_0^2+\mathcal{G}^2I^2_{00}\Gamma^2\Lambda\Big(\psi'_{00}(2r_0\Omega'_{00}\psi'^2_{00} \\
&+((1-r_0\Omega_{00})\psi^{(4)}_{00}-2r_0\Omega'_{00}\psi^{(3)}_{00})\psi'_{00}+2r_0\Omega'_{00}\psi''^2_{00})\Lambda^3+r_0(-2\Omega'_{00}\Lambda''\psi'^3_{00}-2\Lambda'\Omega'_{00}\psi''_{00}\psi'^2_{00} \\
&+I_{00}\psi''_{00}\psi^{(3)}_{00})\Lambda^2-r_0I_{00}(\Lambda^{(3)}\psi'^2_{00}+(3\Lambda''\psi''_{00}+5\Lambda'\psi^{(3)}_{00})\psi'_{00}-3\Lambda'\psi''^2_{00})\Lambda \\
&-3r_0I_{00}\Lambda'\psi'_{00}(\psi'_{00}\Lambda''+\Lambda'\psi''_{00})\Big)r_0^2-2\psi'_{00}\Big(4\Gamma^2\psi'^3_{00}(2r_0^3\psi''_{00}\Omega^3_{00}+r_0^2(r_0\psi'_{00}\Omega'_{00}-2\psi''_{00})\Omega^2_{00} \\
&+r_0(\psi''_{00}-2r_0\psi'_{00}\Omega'_{00})\Omega_{00}-\psi''_{00})\Lambda^4+r_0^2I^2_{00}\psi'_{00}(3\Lambda(r_0\Omega_{00}+1)\Lambda'\psi'_{00}\Gamma^4-4(r_0\psi'_{00}\Omega'_{00} \\
&+(3r_0\Omega_{00}+1)\psi''_{00})\Gamma+2(r_0\Omega_{00}+1)\Gamma'\psi'_{00})\Lambda^2+4r_0^3I^3_{00}(\Lambda'\psi'_{00}+\Lambda\psi''_{00}) \\
&+r_0I_{00}\Gamma\psi'^2_{00}\Big(2\Gamma^2(r_0^2\Omega^2_{00}-1)\Gamma'\psi'_{00}\Lambda^5+\Gamma^3(\Omega_{00}\psi'_{00}\Omega'_{00}r_0^2+\Omega^2_{00}\psi''_{00}r_0^2-\psi''_{00})\Lambda^5 \\
&+4(-3r_0^2\Omega^2_{00}+2r_0\Omega_{00}+1)\Lambda'\psi'_{00}\Lambda^2+4\Gamma((\Lambda^2(2r_0^2\Omega^2_{00}+1)-1)\Lambda'\psi'_{00}+\Lambda\psi''_{00})\Big)\Big)\Big],
\end{aligned}
\tag{B.0.7}
$$

while the part of the constant $c^*(\theta)$ with a σ-dependence is extremely lengthy and can thus be found in a drop box that I have created: `https://www.dropbox.com/sh/ap4co4v4qd91d6z/AADl7xn4mTfmA5wJ108otQ_na?dl=0`. Again the function $G(\theta)$ is equal to $\mathcal{G}$, Eq. (3.3.7), and the variables $\psi00$, $\psi10$ are to be read as ψ_{00}, ψ_{10} and so forth.

The second order correction to the field strength in Section 5.5 is

$$
\begin{aligned}
(\tilde{F}^{(1)})^2 = {}& \frac{2}{r_0^4\tilde{R}(2\sigma+\tilde{R})\Gamma^4\Lambda^2}\Big[\tilde{R}^2\Gamma^4\psi'^2_{00}\Lambda^4+2\sigma\tilde{R}\Gamma^4\psi'^2_{00}\Lambda^4-r_0^2\tilde{R}^2\Gamma^4\Omega^2_{00}\psi'^2_{00}\Lambda^4 \\
&-2r_0^2\sigma\tilde{R}\Gamma^4\Omega^2_{00}\psi'^2_{00}\Lambda^4+4\tilde{R}^2\Gamma^2\psi^2_{10}\Lambda^2+8\sigma\tilde{R}\Gamma^2\psi^2_{10}\Lambda^2+4r_0^2\tilde{R}^2\Gamma^2\psi^2_{10}\Omega^2_{00}\Lambda^2 \\
&+8r_0^2\sigma\tilde{R}\Gamma^2\psi^2_{10}\Omega^2_{00}\Lambda^2+12\sigma^2\psi'^2_{00}\Lambda^2+12\tilde{R}^2\psi'^2_{00}\Lambda^2+4\sigma^2\Gamma^2\psi'^2_{00}\Lambda^2-4\tilde{R}^2\Gamma^2\psi'^2_{00}\Lambda^2 \\
&-8\sigma\tilde{R}\Gamma^2\psi'^2_{00}\Lambda^2+12r_0^2\sigma^2\Omega^2_{00}\psi'^2_{00}\Lambda^2+12r_0^2\tilde{R}^2\Omega^2_{00}\psi'^2_{00}\Lambda^2+12r_0^2\sigma^2\Gamma^2\Omega^2_{00}\psi'^2_{00}\Lambda^2 \\
&+8r_0^2\tilde{R}^2\Gamma^2\Omega^2_{00}\psi'^2_{00}\Lambda^2+16r_0^2\sigma\tilde{R}\Gamma^2\Omega^2_{00}\psi'^2_{00}\Lambda^2+24r_0^2\sigma\tilde{R}\Omega^2_{00}\psi'^2_{00}\Lambda^2-24r_0^2\sigma^2\Gamma\Omega^2_{00}\psi'^2_{00}\Lambda^2 \\
&-20r_0^2\tilde{R}^2\Gamma\Omega^2_{00}\psi'^2_{00}\Lambda^2-40r_0^2\sigma\tilde{R}\Gamma\Omega^2_{00}\psi'^2_{00}\Lambda^2+4r_0^2\sigma^2\Gamma^2\Omega^2_{01}\psi'^2_{00}\Lambda^2+4r_0^2\tilde{R}^2\Gamma^2\Omega^2_{10}\psi'^2_{00}\Lambda^2 \\
&+24\sigma\tilde{R}\psi'^2_{00}\Lambda^2-8\sigma^2\Gamma\psi'^2_{00}\Lambda^2-4\tilde{R}^2\Gamma\psi'^2_{00}\Lambda^2-8\sigma\tilde{R}\Gamma\psi'^2_{00}\Lambda^2-24r_0\sigma^2\Omega_{00}\psi'^2_{00}\Lambda^2 \\
&-24r_0\tilde{R}^2\Omega_{00}\psi'^2_{00}\Lambda^2-12r_0\sigma^2\Gamma^2\Omega_{00}\psi'^2_{00}\Lambda^2-48r_0\sigma\tilde{R}\Omega_{00}\psi'^2_{00}\Lambda^2+32r_0\sigma^2\Gamma\Omega_{00}\psi'^2_{00}\Lambda^2 \\
&+24r_0\tilde{R}^2\Gamma\Omega_{00}\psi'^2_{00}\Lambda^2+48r_0\sigma\tilde{R}\Gamma\Omega_{00}\psi'^2_{00}\Lambda^2-8r_0\sigma^2\Gamma^2\Omega_{01}\psi'^2_{00}\Lambda^2-8r_0\sigma\tilde{R}\Gamma^2\Omega_{01}\psi'^2_{00}\Lambda^2 \\
&+16r_0\sigma^2\Gamma\Omega_{01}\psi'^2_{00}\Lambda^2+16r_0\sigma\tilde{R}\Gamma\Omega_{01}\psi'^2_{00}\Lambda^2+16r_0^2\sigma^2\Gamma^2\Omega_{00}\Omega_{01}\psi'^2_{00}\Lambda^2+16r_0^2\sigma\tilde{R}\Gamma^2\Omega_{00}\Omega_{01}\psi'^2_{00}\Lambda^2
\end{aligned}
$$

$$
\begin{aligned}
&- 16r_0^2\sigma^2\Gamma\Omega_{00}\Omega_{01}\psi_{00}'^2\Lambda^2 - 16r_0^2\sigma\tilde{R}\Gamma\Omega_{00}\Omega_{01}\psi_{00}'^2\Lambda^2 - 4r_0\sigma^2\Gamma^2\Omega_{02}\psi_{00}'^2\Lambda^2 + 4r_0^2\sigma^2\Gamma^2\Omega_{00}\Omega_{02}\psi_{00}'^2\Lambda^2 \\
&- 8r_0\tilde{R}^2\Gamma^2\Omega_{10}\psi_{00}'^2\Lambda^2 - 8r_0\sigma\tilde{R}\Gamma^2\Omega_{10}\psi_{00}'^2\Lambda^2 + 16r_0\tilde{R}^2\Gamma\Omega_{10}\psi_{00}'^2\Lambda^2 + 16r_0\sigma\tilde{R}\Gamma\Omega_{10}\psi_{00}'^2\Lambda^2 \\
&+ 16r_0^2\tilde{R}^2\Gamma^2\Omega_{00}\Omega_{10}\psi_{00}'^2\Lambda^2 + 16r_0^2\sigma\tilde{R}\Gamma^2\Omega_{00}\Omega_{10}\psi_{00}'^2\Lambda^2 - 16r_0^2\tilde{R}^2\Gamma\Omega_{00}\Omega_{10}\psi_{00}'^2\Lambda^2 \\
&- 16r_0^2\sigma\tilde{R}\Gamma\Omega_{00}\Omega_{10}\psi_{00}'^2\Lambda^2 + 8r_0^2\sigma\tilde{R}\Gamma^2\Omega_{01}\Omega_{10}\psi_{00}'^2\Lambda^2 - 8r_0\sigma\tilde{R}\Gamma^2\Omega_{11}\psi_{00}'^2\Lambda^2 + 8r_0^2\sigma\tilde{R}\Gamma^2\Omega_{00}\Omega_{11}\psi_{00}'^2\Lambda^2 \\
&- 4r_0\tilde{R}^2\Gamma^2\Omega_{20}\psi_{00}'^2\Lambda^2 + 4r_0^2\tilde{R}^2\Gamma^2\Omega_{00}\Omega_{20}\psi_{00}'^2\Lambda^2 + 4\sigma^2\Gamma^2\psi_{01}'^2\Lambda^2 + 4r_0^2\sigma^2\Gamma^2\Omega_{00}^2\psi_{01}'^2\Lambda^2 \\
&- 8r_0\sigma^2\Gamma^2\Omega_{00}\psi_{01}'^2\Lambda^2 + 4\tilde{R}^2\Gamma^2\psi_{10}'^2\Lambda^2 + 4r_0^2\tilde{R}^2\Gamma^2\Omega_{00}^2\psi_{10}'^2\Lambda^2 - 8r_0\tilde{R}^2\Gamma^2\Omega_{00}\psi_{10}'^2\Lambda^2 \\
&- 8r_0\tilde{R}^2\Gamma^2\psi_{10}^2\Omega_{00}\Lambda^2 - 16r_0\sigma\tilde{R}\Gamma^2\psi_{10}^2\Omega_{00}\Lambda^2 + 16r_0^2\sigma^2\Gamma^2\Omega_{00}^2\psi_{00}'\psi_{01}'\Lambda^2 + 16r_0^2\sigma\tilde{R}\Gamma^2\Omega_{00}^2\psi_{00}'\psi_{01}'\Lambda^2 \\
&- 16r_0^2\sigma^2\Gamma\Omega_{00}^2\psi_{00}'\psi_{01}'\Lambda^2 - 16r_0^2\sigma\tilde{R}\Gamma\Omega_{00}^2\psi_{00}'\psi_{01}'\Lambda^2 - 16\sigma^2\Gamma\psi_{00}'\psi_{01}'\Lambda^2 \\
&- 16\sigma\tilde{R}\Gamma\psi_{00}'\psi_{01}'\Lambda^2 - 16r_0\sigma^2\Gamma^2\Omega_{00}\psi_{00}'\psi_{01}'\Lambda^2 - 16r_0\sigma\tilde{R}\Gamma^2\Omega_{00}\psi_{00}'\psi_{01}'\Lambda^2 \\
&+ 32r_0\sigma^2\Gamma\Omega_{00}\psi_{00}'\psi_{01}'\Lambda^2 + 32r_0\sigma\tilde{R}\Gamma\Omega_{00}\psi_{00}'\psi_{01}'\Lambda^2 - 16r_0\sigma^2\Gamma^2\Omega_{01}\psi_{00}'\psi_{01}'\Lambda^2 \\
&+ 16r_0^2\sigma^2\Gamma^2\Omega_{00}\Omega_{01}\psi_{00}'\psi_{01}'\Lambda^2 - 16r_0\sigma\tilde{R}\Gamma^2\Omega_{10}\psi_{00}'\psi_{01}'\Lambda^2 + 16r_0^2\sigma\tilde{R}\Gamma^2\Omega_{00}\Omega_{10}\psi_{00}'\psi_{01}'\Lambda^2 \\
&+ 4\sigma^2\Gamma^2\psi_{00}'\psi_{02}'\Lambda^2 + 4r_0^2\sigma^2\Gamma^2\Omega_{00}^2\psi_{00}'\psi_{02}'\Lambda^2 - 8r_0\sigma^2\Gamma^2\Omega_{00}\psi_{00}'\psi_{02}'\Lambda^2 + 16r_0^2\tilde{R}^2\Gamma^2\Omega_{00}^2\psi_{00}'\psi_{10}'\Lambda^2 \\
&+ 16r_0^2\sigma\tilde{R}\Gamma^2\Omega_{00}^2\psi_{00}'\psi_{10}'\Lambda^2 - 16r_0^2\tilde{R}^2\Gamma\Omega_{00}^2\psi_{00}'\psi_{10}'\Lambda^2 - 16r_0^2\sigma\tilde{R}\Gamma\Omega_{00}^2\psi_{00}'\psi_{10}'\Lambda^2 - 16\tilde{R}^2\Gamma\psi_{00}'\psi_{10}'\Lambda^2 \\
&- 16\sigma\tilde{R}\Gamma\psi_{00}'\psi_{10}'\Lambda^2 - 16r_0\tilde{R}^2\Gamma^2\Omega_{00}\psi_{00}'\psi_{10}'\Lambda^2 - 16r_0\sigma\tilde{R}\Gamma^2\Omega_{00}\psi_{00}'\psi_{10}'\Lambda^2 \\
&+ 32r_0\tilde{R}^2\Gamma\Omega_{00}\psi_{00}'\psi_{10}'\Lambda^2 + 32r_0\sigma\tilde{R}\Gamma\Omega_{00}\psi_{00}'\psi_{10}'\Lambda^2 - 16r_0\sigma\tilde{R}\Gamma^2\Omega_{01}\psi_{00}'\psi_{10}'\Lambda^2 \\
&+ 16r_0^2\sigma\tilde{R}\Gamma^2\Omega_{00}\Omega_{01}\psi_{00}'\psi_{10}'\Lambda^2 - 16r_0\tilde{R}^2\Gamma^2\Omega_{10}\psi_{00}'\psi_{10}'\Lambda^2 + 16r_0^2\tilde{R}^2\Gamma^2\Omega_{00}\Omega_{10}\psi_{00}'\psi_{10}'\Lambda^2 \\
&+ 8\sigma\tilde{R}\Gamma^2\psi_{01}'\psi_{10}'\Lambda^2 + 8r_0^2\sigma\tilde{R}\Gamma^2\Omega_{00}^2\psi_{01}'\psi_{10}'\Lambda^2 - 16r_0\sigma\tilde{R}\Gamma^2\Omega_{00}\psi_{01}'\psi_{10}'\Lambda^2 \\
&+ 8\sigma\tilde{R}\Gamma^2\psi_{00}'\psi_{11}'\Lambda^2 + 8r_0^2\sigma\tilde{R}\Gamma^2\Omega_{00}^2\psi_{00}'\psi_{11}'\Lambda^2 - 16r_0\sigma\tilde{R}\Gamma^2\Omega_{00}\psi_{00}'\psi_{11}'\Lambda^2 \\
&+ 4\tilde{R}^2\Gamma^2(r_0\Omega_{00} - 1)^2\psi_{00}'\psi_{20}'\Lambda^2 - 4r_0^2\sigma^2 I_{01}^2\Gamma^2 - 4r_0^2 I_{10}^2\tilde{R}^2\Gamma^2 - 8r_0^2\sigma I_{01}I_{10}\tilde{R}\Gamma^2 \\
&- 4r_0^2 I_{00}(I_{02}\sigma^2 + 2I_{11}\tilde{R}\sigma + I_{20}\tilde{R}^2)\Gamma^2 - 4\tilde{R}^2\Gamma^2\psi_{00}'^2 - 8\sigma\tilde{R}\Gamma^2\psi_{00}'^2\Big]. \qquad \text{(B.0.8)}
\end{aligned}
$$

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