University of Copenhagen
Faculty of Science

Asymptotic Symmetries in Gauge Theories

Master's Thesis

Written by Anna Louise Juul Willumsen

August 16th, 2021

Supervised by
Cristian Vergu, Emil Bjerrum-Bohr
Niels Bohr Institute

Abstract

Symmetries are extremely important in modern physics. However, the analysis of their implications, or even identifying the right symmetry group, are rather complicated in theories with gauge symmetries, such as electromagnetism, gravity, or Yang-Mills theory. So far, there is no general procedure for answering these questions. In gravity, the identification of the symmetry group can be done by an analysis of the asymptotic infinity. This thesis will attempt to investigate if some of the techniques, used to describe the symmetry group of gravitational theories, can be applied to the case of electromagnetism, or to Yang-Mills theories.

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1 Introduction and Motivation

1.1 The Importance of Symmetries

Symmetries can be described as translations or transformations of an object or system, which leaves said thing invariant or "untouched".
An example is translation symmetry; moving an object in space from one place to another, and the object being left unchanged, implies translation symmetry (or invariance) for said object. Additionally, this also implies conservation laws in the form of momentum conservation. This type of relation can be derived from Noether's Theorem, which was proven by Emmy Noether in 1915 (published in 1918)[1]: Symmetries imply conservation laws (to state it simply).
Another symmetry can be rotation of an object, e.g. a perfect sphere, which looks the same, no matter which axis you rotate the sphere about. Rotation symmetry implies angular momentum conservation. A last example of symmetry could be transformations in time; this symmetry implies energy conservation of an object or system - if it is unchanged under the time transformation (let 5 minutes pass, for example, and see if the object remains the same, and has the same physical properties).
Thus these symmetries are essential to understanding the physics around us, and makes calculations much more simple, due to the conservation laws.

Looking at the infrared (IR) sector of physics, or physics as seen from far away; within the geometry of asymptotically flat space-time (constructed by conformal rescalings of Minkowski space (MS)); Interesting features are found at the infinity boundary. There has been found a relation from the symmetries - here they are called asymptotic symmetries - and their conservation laws, which implies this 'triangular equivalence relation', explained in the following subsection. Asymptotic symmetries (or asymptotically flat space-time symmetries) can be defined as symmetries or conserved charges of any system with an asymptotic region or boundary.

The asymptotic symmetries have, to some extend, been explored for the gravity case, and recently for quantum electrodynamics (QED) and non-abelian gauge theory. However, they are still a subject of ongoing research, and their implications are still being unravelled.
For the gravity case, Bondi, van der Burg, Metzner, and Sachs (BMS) wanted to recover the Poincaré group of special relativity as the symmetry group of asymptotically flat spacetimes in general relativity (GR). They instead found the infinite-dimensional BMS group.
The thesis will follow the derivation of this symmetry group, by two different approaches - that of Andrew Strominger in [Str], and that of Roger Penrose in [Pen74] - and try to explore how these approaches can be applied in the case of gauge theories, such as electromagnetism (EM) and Yang-Mills (YM) theory.

[1] See article [Wik]

1.2 The Infrared Triangle

The triangular equivalence relation, that governs the IR dynamics/sector of all physical theories with massless particles, is depicted in *Figure 1*.

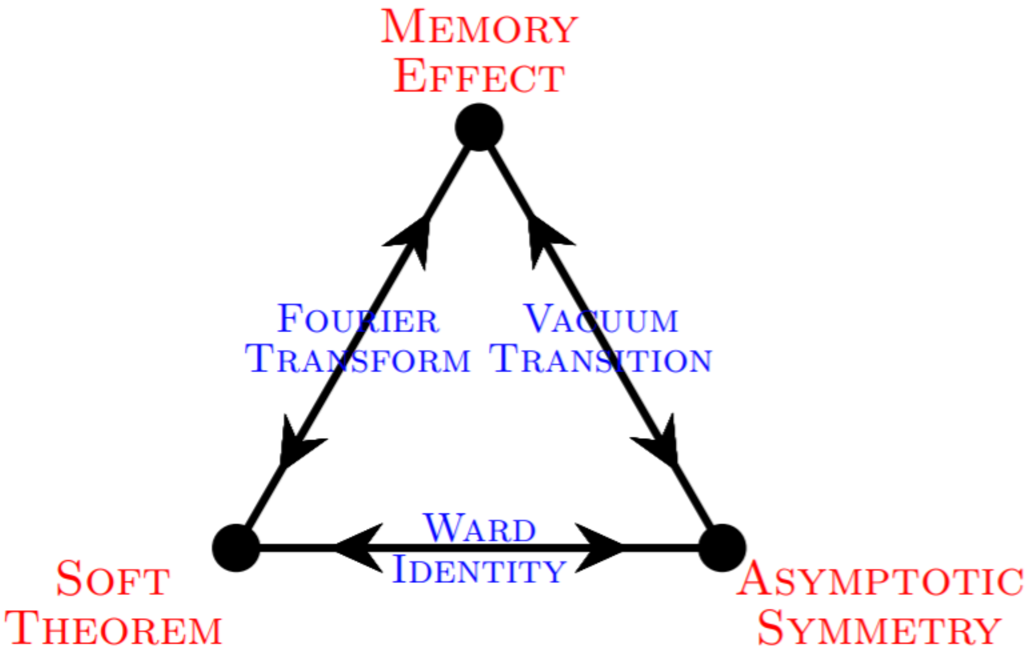


Figure 1: Infrared Triangle. The figure is taken from [Str].

The corners represent different notations, but describes the same subject:

- Soft theorems: Properties of Feynman diagrams and scattering amplitudes at zero or low energy; describing that infinitely many soft (zero or low energy) particles are produced in any physical process.
- $\longleftrightarrow$ Ward identity: Consequences for scattering processes of the symmetry.
- Asymptotic symmetries: Conservation laws (due to conserved charges). They take a simple form, when looking at how they act on the asymptotic region of space-time.
- $\longleftrightarrow$ Vacuum transitions: A step function.
- Memory effect: In the theory of gravity, a gravitational wave produces a shift in the relative position of an inertial detector pair. This shift is caused by the Memory effect.
- $\longleftrightarrow$ Fourier transform in time.

1.3 Relevant Topics

Relevant topics, where this IR triangle application can prove useful:

- **Theories:** QED, YM theory, gravity, massless particles/scalars/fermions (also theories without local symmetry).
- **Order:** Leading IR behaviour of amplitudes (or soft theorems), subleading, subsubleading, etc.
- **Version:** Classical version, quantum version (with anomalies).
- **Geometry:** 4-Dimensional (4D) MS, $D < 4$, $D > 4$, Cosmology.
- **Symmetries:** $N = 0, 1, ..., 8$ (Number of symmetries).

1.4 Motivation/Application

One such triangle, should potentially exist for every type of massless particle. One can use these IR triangles to find or correct anomalies of the theories. Below are reasons for their conveniences:

- Connecting 3 different subjects/theories: Find 1 corner, you can work out the other 2. There are three different ways of characterising the behaviour of the universe around us, at very large distances.
- Understanding holographic structure of quantum gravity in 4D asymptotically flat space-times - by understanding the symmetries. This viewpoint is close to our reality.
- In flat-space quantum gravity, one can make connections of the asymptotic symmetries, with symmetries of conformal field theory acting on a sphere.
- Organisation/improvement of IR/jet/LHC (Large hadron collider) physics, by the study of soft particles. Generically, no IR finite $\mathcal{S}$-matrix exists in gauge theories. Through some kind of understanding of the deep IR symmetries, there might be a way to define the (soft part of the) $\mathcal{S}$-matrix to satisfaction.
- Black hole information paradox: Understanding soft gravitons/photons in evaporation processes in IR, might lead us to an answer. *"All roads lead to black holes"* - from [Str] p. 8.
- Explain miracles in $N = 4$ YM scattering amplitude (cancellations of diagrams).

1.5 Structure

Section 2 will describe the connection of groups and lay a foundation for the notation used further on.
Section 3 will present the conformal rescaling of MS and the geometrical interpretation.
Sections 4 and 5 will go through the equivalence of matching conditions, conserved charges, Ward identities, soft theorems and asymptotic symmetries in QED.
Section 6 will include an alternative definition for the $\mathcal{S}$-matrix, as well as the soft theorems for non-abelian gauge theories and their implications.
Section 7 will present Penrose's approach of finding the BMS group, by a direct analysis of the asymptotic geometry, and explore its implications.
Section 8 compares the group elements of the Poincaré and BMS group.
Section 9 includes Strominger's approach of defining the BMS group, the supertranslations and superrotations, and scattering for gravity theory.
Section 10 is discussing the $\log r/r$ behavior, when calculating the coefficients of the large-r expansion of the gauge parameters in QED; the $\log r/r$ term is needed in order to satisfy the wave equation.
Section 11 will go through the attempts of applying the methods of Strominger and Penrose in gauge theories, the implications and ways around them.
Section 12 includes the a describtion of why a conformal coupling term is needed in the action of a scalar field, when applying Weyl transformations.
Section 13 sets up the energy-momentum tensor for a conformally coupled (charged) scalar.
Section 14 discusses different ways of identifying past- and future null infinity, as well as a way of mapping spatial infinity to $r = 0$, by inversion.
Section 15 discusses the work, [AR92] by Ashtekar and Romano, where they wanted to find a manifestly coordinate independent treatment of spatial infinity, which avoids the awkwardness of the differentiability conditions.

1.6 Sidenote

From here on forward, whole sections or subsections marked with (S) will follow the work (and use equations) described in [Str], and if marked with (P) will follow the work (and use equations) described in [Pen74].
[Str], by Andrew Strominger, is connecting the subjects of soft theorems, the memory effect and asymptotic symmetries in four-dimensional QED, non-abelian gauge theory and gravity.
[Pen74], by Roger Penrose, is discussing the BMS group, by a direct analysis of the geometry and symmetries.

2 Orthogonal and Conformal Groups (P)

2.1 Homomorphisms between Orthogonal and Conformal Groups

In the study of group theory, the following homomorphisms can be defined:

The local isomorphism mapping the group $SL(2,\mathbb{C})$ of complex unimodular (2×2) matrices (giving rise to the algebra of spinors) onto the identity connected component of the Lorentz group $O(1,3)$ in an essentially (2-1) manner is:

$$SL(2,\mathbb{C}) \to O(1,3)$$

'Essentially' referring to the ambiguity of elements A, B mapping onto element Q in $O(1,3)$; they are connected by a curve in $SL(2,\mathbb{C})$, and by a closed curve through Q in $O(1,3)$.

Furthermore the (local isomorphism) mapping, which is in an essentially (1-1) (but inessentially (2-1)) fashion, is defined as:

$$O(1,3) \to C(2)$$

Where $C(2)$ is the conformal group, described in the following subsection.
In a similar fashion, the higher dimensional analogue is the local isomorphism, also in an essential (2-1) manner, with $SU(2,2)$ being the group of unimodular pseudo-unitary $(++--)$ (4×4) matrices (giving rise to the algebra of twistors):

$$SU(2,2) \to O(2,4)$$

And similarly we have the mapping (local isomorphism, essential (2-1) fashion):

$$O(2,4) \to C(1,3)$$

Connecting the two, will give the local isomorphism below, but this mapping is in an essentially (4-1) fashion instead:

$$SU(2,2) \to C(1,3)$$

From this story, a pattern can be seen for the local isomorphisms: $O(p+1, q+1) \to C(p,q)$

2.2 The Conformal Group $C(2)$

The orientation-preserving **local** conformal maps of the plane to itself is represented by:

$$\zeta \to \tilde{\zeta} = f(\zeta), \tag{1}$$

where f is a holomorphic (complex analytic) function, $\zeta = x + iy$, and x, y are standard Cartesian coordinates for the plane. The line-element, or metric, can then be expressed as:

$$\begin{aligned} d\sigma^2 &= dx^2 + dy^2 = d\zeta d\bar{\zeta} \\ d\sigma^2 &\to d\tilde{\sigma}^2 = |f'(\zeta)|^2 d\sigma^2 \end{aligned} \tag{2}$$

This map of the Euclidean plane to itself constitute an infinite-dimensional group.

For a **global** map, we require f and its inverse to be singular over the whole plane, such that f is a linear function:

$$f(\zeta) = \alpha\zeta + \beta \tag{3}$$

The reason it should be linear, is that we want the Weyl factor, $|f'(\zeta)|^2$, to be well-defined (non-singular) everywhere, including at ∞.
Thus the group of orientation-preserving conformal maps of the plane to itself, is described by 4 real parameters. The maps are generated by Euclidean motions ($|\alpha| = 1$) and dilations (α real, $\beta = 0$).

To describe $C(2)$, one must compactify the plane by the addition of a point at infinity, ie. a projection of the unit sphere S^2 to the plane, where S^2 is given by $X^2+Y^2+Z^2 = 1$ (using standard Cartesian coordinates for Euclidean 3-space), see *Figure 2*.

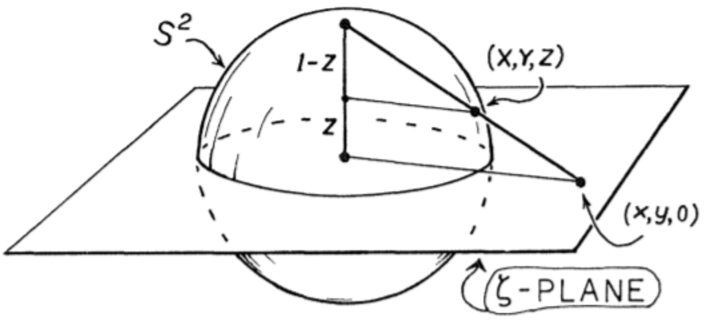


Figure 2: Projection from the unit sphere north pole to the plane. The figure is taken from [Pen74].

Projecting point (X, Y, Z) from $(0, 0, 1)$ on S^2 to the plane at $z = 0$, is a conformal map, where:

$$\zeta = x + iy = \frac{X + iY}{1 - Z}; \quad X + iY = \frac{2\zeta}{1 + \zeta\bar{\zeta}}, \quad Z = \frac{\zeta\bar{\zeta} - 1}{1 + \zeta\bar{\zeta}} \tag{4}$$

One way to see this, is to make the coordinate change $\zeta = e^{i\phi}\cot\left(\frac{\theta}{2}\right)$, and see the metric $d\sigma^2$ for S^2 is given by:

$$dl^2 = d\theta^2 + \sin^2\theta d\phi^2 = \frac{4d\zeta d\bar{\zeta}}{(1 + \zeta\bar{\zeta})^2} = \frac{4d\sigma^2}{(1 + \zeta\bar{\zeta})^2} \tag{5}$$

$d\sigma^2$ being the metric for the plane $z = 0$ (see *Equation 2*). S^2 with the north pole removed is conformally identical with the Euclidean plane. Adding the north pole provides the conformal compactification of the plane.

$C(2)$ is the group of conformal maps of the compactified plane (S^2) to itself. The connected component of identity in $C(2)$ is the orientation-**preserving** conformal maps of S^2, given by:

$$\zeta \to \tilde{\zeta} = f(\zeta) = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}, \tag{6}$$

being regular at $\zeta = \infty$. f thus gives a six-real-parameter group, with the parameters normalized by $\alpha\delta - \beta\gamma = 1$. The unimodular complex (2×2) matrix, $SL(2, \mathbb{C})$ element $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ represent the transformation.

The orientation-**reversing** elements of $C(2)$ are the $\zeta \to \bar{\zeta}$ (complex conjugate) compositions of *Eq. 6*. All elements of $C(2)$ are generated by dilations, Euclidean motions, and the inversion $\zeta \to \tilde{\zeta} = \bar{\zeta}^{-1}$. This inversion interchanges $\zeta = 0$ and $\zeta = \infty$ (the north- and south pole) of S^2.

2.3 The Connection of $C(2)$ with $O(1,3)$

To establish a connection of $C(2)$ with $O(1,3)$, consider the 4D MS with metric $ds^2 = dT^2 - dX^2 - dY^2 - dZ^2$, and null-cone N of origin, described by equation $T^2 - X^2 - Y^2 - Z^2 = 0$, with generators of N being null rays through the origin, given by $T : X : Y : Z = const.$
Consider a unit sphere S^2 to be a space-like 3-plane $T = 1$ of N. This gives a (1-1) correspondence between generators of N and points of S^2 (given by intersections of generators of N w. $T = 1$). S^2 is thus a realisation the space of generators of N, but one could use any other cross-section $\hat{S}^2$ of N for this. See *Figure 3*.

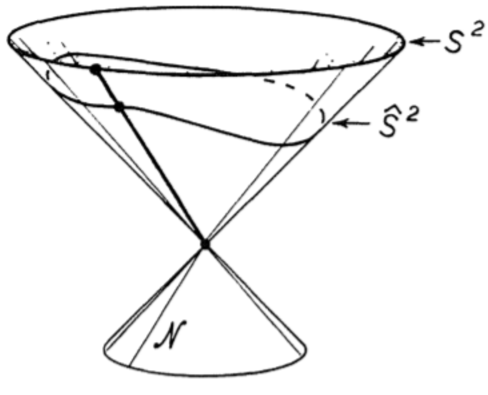


Figure 3: Cross-sections of N. The figure is taken from [Pen74].

The conformal structure of S^2 reflects an intrinsic conformal structure on the space of generators of N.
Furthermore the generators of the null cone N establish a (1-1) conformal map between any two cross sections of N, so the space of generators has a conformal structure (that of any of the cross-sections).
Definition: A map, carrying such cross-section into another, with points on the same

generator of N, corresponding to one another, is a **conformal map**.

To see that the generator map is conformal, the metric can be re-expressed on the form:

$$ds^2 = -r^2\gamma_{\alpha\beta}dx^\alpha dx^\beta + 0.dr^2, \tag{7}$$

with x^α and r being new coordinates on N, and coordinate lines $x^\alpha = \text{const}$ being generators of N (using spherical polar coordinates $ds^2 = dT^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$, where $T = r$ on N). By specifying r to be a function of x^α, a cross-section of N will be given. Any two cross-sections give conformally related metrics, mapped to one another by generators of N ($x^\alpha = \text{const}$).

A Lorentz transformation, L, will send N into itself, and generators of N into other generators of N, and cross-sections S^2 into another with the same intrinsic metric. Since the map from $\hat{S}^2$ to S^2 along generators of N is conformal, L induces a transformation on the space of generators of N, which makes a conformal map of S^2 to itself. This establishes a homomorphism $O(1,3) \to C(2)$, with the same dimensionality of each group (6). Thus the inverse image of elements of $C(2)$ is a discrete (each point is isolated) set of elements of $O(1,3)$. Hence, the mapping is a local isomorphism, and L is defined by its effect on N (up to a space-time reflection in the origin). Thus this homomorphism is inessentially (2-1). For a restriction to the orthocronous $O(1,3)$, L preserves the time-direction (both the orientation and direction), and would yield a global isomorphism between the two groups (1-1).

Instead of considering cross-sections of N by $T = 1$; the intersection of N with the null 3-plane $T = Z + 1$, will give the parabolic section E^2 with intrinsic metric $d\sigma^2 = -ds^2 = dX^2 + dY^2$. E^2 is intrinsically a Euclidean plane - not a cross-section of N, as generators $T - Z = X = Y = 0$ (parallel to the 3-plane) does not give rise to a point on E^2, but corresponds to the point at infinity at E^2, as the relation between E^2 and S^2 is conformal; See *Figure 4*.
The 2-plane meets S^2, E^2, and plane $Z = 0 = T - 1$. The conformal relation of E^2 and S^2 is established via. generators of N, or 2-planes through the parallel generator $T - Z = X = Y = 0$.

2.4 The Connection of $C(1,3)$ with $O(2,4)$

To illustrate the general case of local isomorphisms $O(p+1, q+1) \to C(p,q)$, consider the case $O(2,4) \to C(1,3)$. **Defining** $C(1,3)$ (or $C(p,q)$) as the group of conformal self-transformations of appropriate pseudo-Euclidean space (4D MS), compactified by the addition of extra points at infinity - by going to a null cone of the origin in a 6D pseudo-Euclidean $(++----)$ space, with metric:

$$ds^2 = dT^2 + dV^2 - dW^2 - dX^2 - dY^2 - dZ^2 \tag{8}$$

The null cone K of origin has equation $T^2 + V^2 - W^2 - X^2 - Y^2 - Z^2 = 0$. By analogy with the parabolic intersection E^2 of N, we have an intersection M^4 of K, with null

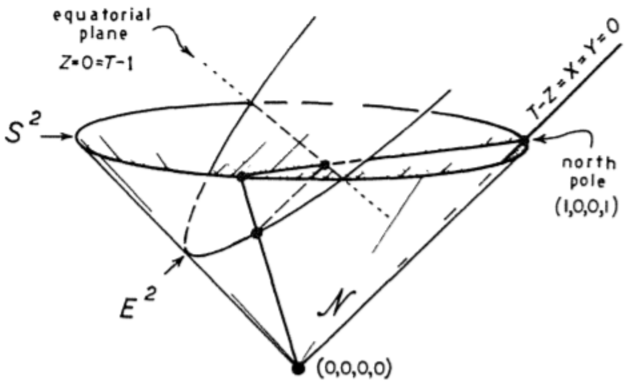


Figure 4: The Euclidean plane being embedded as a parabolic section of N. The figure is taken from [Pen74].

hyperplane $V - W = 1$ (analogue of $T = Z + 1$), and the intrinsic metric of M^4 is given by $ds^2 = dT^2 - dX^2 - dY^2 - dZ^2$ (analogue of $-ds^2 = dX^2 + dY^2$), so M^4 is intrinsically identical to MS.
In 6-space; M^4 has paraboloid form with coordinates defined by:

$$V = \frac{1}{2}(1 - T^2 + X^2 + Y^2 + Z^2) = W + 1 \tag{9}$$

All generators of K (set of points with its coordinates being constant and satisfying equation of K), except for those in the null hyperplane $V = W$ (analogue of $T - Z = X = Y = 0$), meets M^4 in a unique point.
Thus the set G of unoriented generators of compact topological space K, is the compactification of M^4. Members of K, not in $V = W$, is in (1-1) correspondence with M^4, and those in $V = W$ supply extra points necessary for compactification - they represent points at infinity. The generators of K thus establish conformal maps of any two (local) cross-sections of K. G is therefore a conformal manifold, identified as the compactification of M^4.
Definition: $C(1,3)$ (or $C(p,q)$) is a group of self-transformations of G, preserving its conformal structure.

There is no analogue for the infinite set of **local** conformal transformations of the plane, *Equation 1*, in higher dimensions than 2. However, the **global** conformal self-transformations of M^4, is the same as for E^2, generated by pseudo-Euclidean motions (constituting the Poincaré group) and dilations; these constituting the 11-parameter orthochronous group, called the causal group. To compactify M^4, one must adjoin an entire null-cone, and not just a point.
The inversion of M^4 (analogue of $\zeta \to \tilde{\zeta} = \bar{\zeta}^{-1}$), corresponding to the reflection $W \to -W$ in 6-space, expressed as $(T, X, Y, Z) \to -\{T^2 - X^2 - Y^2 - Z^2\}^{-1}(T, X, Y, Z)$, is not well-defined on the null cone $T^2 - X^2 - Y^2 - Z^2 = 0$, but maps the null cone to infinity. This reflection exchanges the null cone at the origin with the null cone at

infinity; thus the elements at infinity must have the structure of a null cone.

For an analogue of S^2, consider intersection of K with hyperplane $T = 1$. It has a structure of a de Sitter space (dS_3), but requires points at infinity (corresponding to generator of K in $T = 0$) to be added, in order to represent the entire compact space G. The section of K with $W = 1$, defining the anti-de Sitter space (AdS_3), also requires points at infinity to become compact.

2.5 Model of Compactified Minkowski Space, G

For an adequate model of compactified MS, consider intersection of K with the 5-sphere, defined by $T^2 + V^2 + W^2 + X^2 + Y^2 + Z^2 = 2$. This gives the compact space-time model H; the topological product of a three-sphere in (W, X, Y, Z)-space (defined by $W^2+X^2+Y^2+Z^2 = 1$) with a 1-sphere (circle) in (T, V)-space (defined by $T^2+V^2 = 1$). This is not a model of G, but a twofold covering. Each point of H is represented twice (as $\pm$) on H; the space-time H is connected so the two-fold nature of the covering is essential. The topology of G is $S^1 \times S^3$.

The pseudo-orthogonal group $O(2, 4)$ acts on the 6-space and K (as K is invariant). Each $O(2, 4)$ transformation induces a conformal map of G to itself, and a homomorphism $O(2, 4) \to C(1, 3)$ is obtained.
The group $O(2, 4)$ has 15 parameters; $C(1, 3)$ cannot have more than 15 (the maximum for local conformal self-transformations of a 4-manifold), ie. the homomorphism is a local isomorphism. It is (2-1), as $O(2, 4)$ transformations can reverse the direction of generators K, where G is the space of unoriented generators. The $O(2, 4)$ transformation is a reflection in the origin, representing the identity on G.

We have an essential (2-1) nature for $O(p + 1, q + 1)$, if p, q are odd (local isomorphism).
We have an inessential (2-1) nature for $O(p + 1, q + 1)$, if p, q are even (local isomorphism).
For $p + q$ being odd, we have a local (1-1), hence global, isomorphism.

A smooth map from one space-time to another, is characterised as conformal, as it takes null cones into null cones, or null geodesics into null geodesics.
All null cones on M^4 arise as intersections with tangent 5-planes to K, ie. null 5-planes through the origin of 6-space.

3 Geometry (P)

3.1 Conformal Rescaling of Minkowski Space

This section will go into detail of the construction of conformal infinity for MS, by a different approach than in *Section 2.5*. First rescale the metric of space-time M from the physical metric ds, to the new 'unphysical' metric $d\hat{s}$, conformally related by $ds = \Omega d\hat{s}$, with Ω being a smooth, positive function defined on M. The metric tensor is rescaled by $g_{ab} \to \hat{g}_{ab} = \Omega^2 g_{ab}, \quad g^{ab} \to \hat{g}^{ab} = \Omega^{-2} g^{ab}$.

Provided suitable asymptotic structure of M, and appropriate Ω, more points can be added to the manifold, so the metric $\hat{g}_{ab}$ extends smoothly to them. Ω could also be extended in the same way, however it becomes zero at these points; implying the metric being infinite (and can thus not be extended). Thus, the new points are infinitely distant from their neighbours, and physically represent points at infinity.
Above conformal rescaling/Weyl transformation is not the same kind of conformal transformation (conformal mapping) as considered in *Section 2*; points of the manifold are not transformed, but the metric is replaced. Conformal rescalings/Weyl transformations form an infinite dimesional Abelian group (multiplicative group of smooth, positive functions on the manifold), and has no element in common with the 15 parameter conformal group ($C(1,3)$) - conformal rescalings have nothing to do with coordinate changes.

Many physical concepts are invariant under conformal rescalings (not those with gravitational interactions, though).
This conformal technique is useful for radiation properties of zero rest-mass fields. Incoming and outgoing radiation fields are defined precisely for asymptotically flat, curved space-times in terms of values of fields at the adjoint points at infinity, ie. on past- and future null infinity $\mathcal{I}^{\pm}$, near spatial infinity i^0 - at $\mathcal{I}^{\pm}_{\pm}$ (see *Section 3.2*).
The conformal technique gives a coordinate-free definition of asymptotic flatness in GR. The asymptotic symmetry group, BMS, has a clear geometrical interpretation, and so has its relation to gravitational radiation, using the conformal technique.

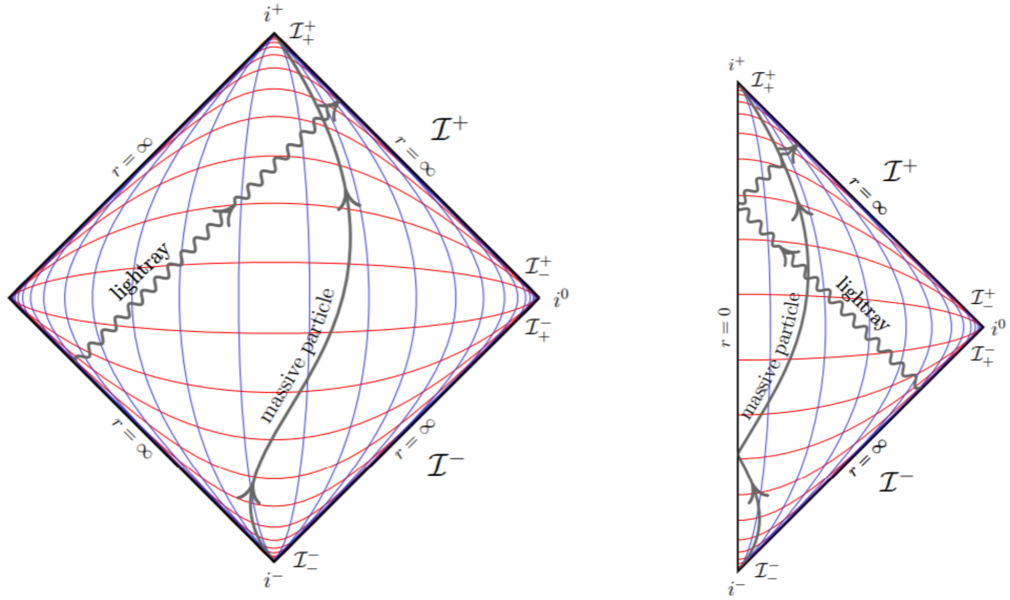
3.2 Minkowski Space Penrose Diagram (S)

To talk about infinity, it is useful to consider the geometry depicted by a Penrose diagram, see *Figure 5*, where all of Minkowski space is pulled into a finite region by a conformal transformation that diverges at the boundaries. The exact technique will be covered in *Section 3.4*.
In the figure, the world-line of a massive particle, moving at constant velocity, from past time-like infinity, i^-, to future time-like infinity, i^+, is illustrated. $\mathcal{I}^{\pm}$ denotes future- and past null infinity, and in-between lies spatial infinity, i^0 (almost anything, say Maxwell's equations, is singular at this point). The boundary components $\mathcal{I}^{\pm}_{\pm}$, which are not equal or Lorentz invariant near i^0, are two-spheres and not points (like $i^{\pm}$).

3.3 Coordinate Change of the Minkowski Space Metric

Now we accompany the conformal rescaling of MS, with a coordinate change, in order to assign finite coordinates to the new adjoint points at infinity. The physical metric in spherical polar coordinates is:

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{10}$$



(a) (b)

Figure 5: Penrose diagrams of Minkowski space, where lightrays always travel at 45° degrees, and where for **(b)**, every point is an S^2, except for $r = 0$, $i^{\pm}$ and i^0, being points. The figure is taken from [Str].

Introducing advanced time and retarded time to better describe $\mathcal{I}^{\mp}$:

$$v = t + r, \qquad u = t - r, \tag{11}$$

so that $\mathcal{I}^{\pm}$ is now parametrized by $(u, \hat{x})$ and $(v, \hat{x})$, respectively.
The metric then becomes:

$$ds^2 = dudv - \frac{1}{4}(v-u)^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{12}$$

For asymptotically simple space-times, the choice of Ω must have properties: $\Omega\lambda \to \text{const}$ as $\lambda \to \pm\infty$, with λ being an affine parameter, and that along any null geodesic: $\Omega \to 0$ at $\pm\infty$ (like the reciprocal of λ).
Each $u = \text{const}$ hypersurface is a future light cone, generated by null geodesics (straight lines in MS) with $\theta, \phi = \text{const}$. Thus v serves as an affine parameter into the future on each of these null geodesics. The same goes for u, being an affine parameter into the past on these radial null-geodesics. One must therefore require that $\Omega v \to \text{const}$ as $v \to \infty$ on u, and $\Omega u \to \text{const}$ as $u \to \infty$ on v (with $\theta, \phi = \text{const}$). We want Ω to be kept smooth over the finite parts of the space-time, so we choose:

$$\Omega = (1+u^2)^{-\frac{1}{2}}(1+v^2)^{-\frac{1}{2}}. \tag{13}$$

Rescaling the metric ($d\hat{s} = \Omega ds$) gives:

$$d\hat{s} = \Omega^2 ds^2 = \frac{dudv}{(1+u^2)(1+v^2)} - \frac{(v-u)^2}{4(1+u^2)(1+v^2)}(d\theta^2 + \sin^2\theta d\phi^2) \tag{14}$$

3.4 Assigning Finite Coordinates to Points at Infinity

To assign finite coordinates to points at infinity, make replacement $u = \tan p, \quad v = \tan q$, thus the metric, *Eq. 14*, takes the form:

$$d\hat{s} = dpdq - \frac{1}{4}\sin^2(q-p)(d\theta^2 + \sin^2\theta d\phi^2) \tag{15}$$

The range of coordinates p, q for MS is illustrated in *Figure 6*, which illustrates the same Penrose diagram as *Figure 5b*.

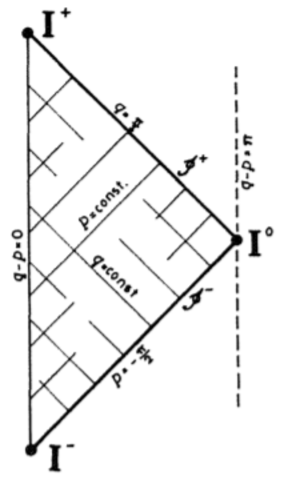


Figure 6: Range of coordinates p, q for MS. The figure is taken from [Pen74].

The metric is regular on $\mathcal{I}^{\pm}$ (future- and past null infinity), and can be extended beyond in a non-singular fashion. For $0 \leq q - p \leq \pi$, the global structure is the product of a space-like 3-sphere with an infinite time-like line (that of an Einstein static universe).

This can be shown by choosing coordinates $T = \frac{1}{2}(p+q)$ and $\rho = q - p$ and obtain metric

$$d\hat{s} = dT^2 - \frac{1}{4}[d\rho^2 + \sin^2\rho(d\theta^2 + \sin^2\theta d\phi^2)], \tag{16}$$

describing space-time ξ, where the term in the brackets is the metric of a unit 3-sphere. The portion of ξ, being conformal to the original MS (that portion lying between the light-cone points of $I^{\pm}$ (or $i^{\pm}$)), wraps around ξ itself, and meets at the point I^0 (or i^0). The 2D case is illustrated in *Figure 7.* (MS being conformally identical to the portion of the Einstein static universe, whoose boundary represent conformal infinity of MS).

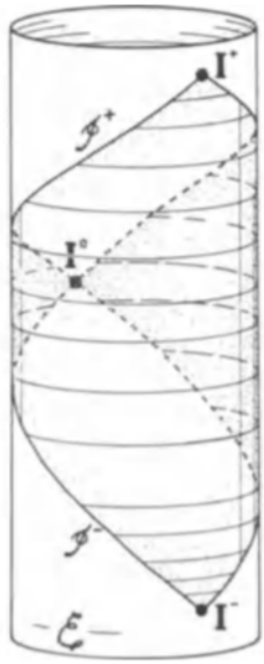


Figure 7: Illustrating MS as a portion of an Einstein static universe. The figure is taken from [Pen74].

How the redundant regions (MS) of the light-cone points $i^{\pm}$, i^0 fit together in this Einstein static universe picture, is illustrated in *Figure 8* - i^0 being spatially the antipode of $i^{\pm}$. The spatial location of i^- correspond to that of i^+. The null geodesic segments connecting $i^{\pm}$ with i^0 (boundary of MS region) is denoted $\mathcal{I}^{\pm}$.

To sum it up:

- $i^{\pm}$ denotes the future- and past temporal (time-like) infinity - being points on the compactified MS.
- $\mathcal{I}^{\pm}$ denotes the future- and past null infinity - being two-spheres times $\mathbb{R}$ on the compactified MS.
- i^0 denotes spatial infinity - being a point on the compactified MS.
- A time-like straight line requires a past- and a future endpoint on $i^{\pm}$ respectively.
- A null straight line requires a past- and a future endpoint on $\mathcal{I}^{\pm}$ respectively.
- A space-like straight line is a closed curve through i^0, when i^0 is adjoined.

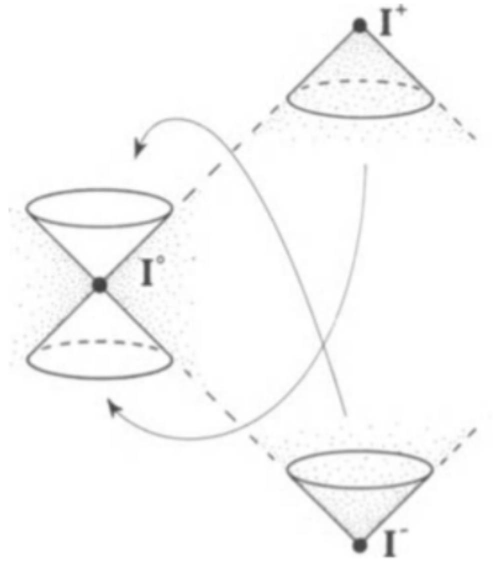


Figure 8: If i^-, i^0, and i^+ are identified as a single point, they fit together as illustrated. The figure is taken from [Pen74].

Null geodesics remain null geodesics after conformal rescaling, so null straight lines become null geodesics with respect to the induced metric - unlike for time- or space-like straight lines.

3.5 Identifying Past- and Future Null Infinity

Thus for Minkowski space: Null geodesics will originate at point a^- on $\mathcal{I}^-$ and pass through the same point a^+ on $\mathcal{I}^+$ (see *Figure 9*).
The future lightcone of a point on $\mathcal{I}^-$ is a null hyperplane in MS (the limit of a lightcone, when the vertex recedes into the past along a null straight line). The past lightcone of a point on $\mathcal{I}^+$ is also a null hyperplane - thus null hyperplanes acquires a past vertex, a^-, on $\mathcal{I}^-$, and a corresponding future vertex, a^+, on $\mathcal{I}^+$. On the Einstein static universe, the future lightcone of a^- will be focused at a point spatially antipodal to a^-, ie. a^+. Thus $\mathcal{I}^\pm$ is identified by the points $a^\pm$.

The points $i^\pm$, i^0 simply become one (as in *Figure 8*). This point, I (or i), becomes a normal interior point of the identified manifold (compactified MS). Since a^- is identified with a^+, each null geodesic becomes closed, with topology of a circle S^1. Regarding conformal structure, every point is on equal footing.

3.6 Curved Asymptotically Flat Space-Times

For curved asymptotically flat space-times (not MS), $\mathcal{I}^\pm$ must be left as distinct boundary hypersurfaces (conformal infinity), as there is no natural association of point a^- on

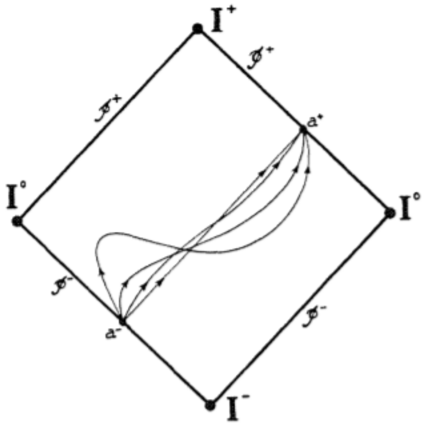


Figure 9: For conformal infinity for MS, every null geodesic originating at some $a^- \in \mathcal{I}^-$, must terminate at $a^+ \in \mathcal{I}^+$. The figure is taken from [Pen74].

$\mathcal{I}^-$ with a unique point a^+ on $\mathcal{I}^+$ - null geodesics from a^- will not focus clearly at point of a^+, but will cross over one another, before $\mathcal{I}^+$ is reached. Any identification of $\mathcal{I}^-$ with $\mathcal{I}^+$ results in singularities in metric $d\hat{s}$ along $\mathcal{I}$, no matter choice of Ω.

3.7 Conformal Infinity of the Schwarzschild Solution

A simple example of an asymptotically flat, curved space-time, is the Schwarzschild solution. To examine its conformal infinity, consider its metric:

$$ds^2 = (1 - \frac{2m}{r})dt^2 - \frac{dr^2}{1 - \frac{2m}{r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{17}$$

Instead of trying to obtain $\mathcal{I}^\pm$ simultaneously (as for MS), we introduce retarded- and advanced time-coordinates: $u = t - r - 2m\log(r - 2m)$, $\quad v = t + r + 2m\log(r - 2m)$, and get the respective metrics:

$$\begin{aligned} ds^2 &= (1 - \frac{2m}{r})du^2 - 2dudr - r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ ds^2 &= (1 - \frac{2m}{r})dv^2 - 2dvdr - r^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned} \tag{18}$$

One can choose $\Omega = r^{-1} = w$, and get the rescaled, unphysical respective metrics:

$$\begin{aligned} d\hat{s}^2 = \Omega^2 ds^2 &= (w^2 - 2mw^3)du^2 - 2dudw - d\theta^2 - \sin^2\theta d\phi^2 \\ d\hat{s}^2 = \Omega^2 ds^2 &= (w^2 - 2mw^3)dv^2 + 2dvdw - d\theta^2 - \sin^2\theta d\phi^2, \end{aligned} \tag{19}$$

which are regular and analytic on their respective hypersurfaces $w = 0$.
The physical space-time (retarded/advanced, $\mp$) is given for $w > 0$ and can be extended to include boundary hypersurfaces $\mathcal{I}^\pm$ (given when $w = 0$), hence the boundary

$\mathcal{I} = \mathcal{I}^- \cup \mathcal{I}^+$ will be adjoined to the space-time. A further extension across $w = 0$ to negative values of w, involves a reversal of the sign of the mass (in the metric). The derivative at $\mathcal{I}$ of the conformal curvature contains information about the mass. Thus by attempting to identify $\mathcal{I}^{\pm}$, with same sign of the mass on the two sides, a discontinuity in the derivative of the curvature across $\mathcal{I}$ occurs (metric $d\hat{s}$ fail to be $\mathscr{C}^3$ at $\mathcal{I}$)[2].

The points $i^{\pm}$, i^0 for the general case, must be singular for the conformal geometry, therefore it is reasonable not to include the points as part of the conformal infinity, as depicted in *Figure 10.*

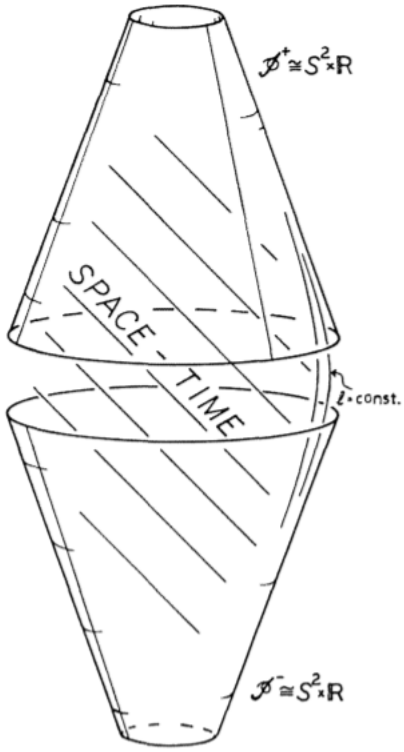


Figure 10: Conformal infinity for asymptotically flat space-time, eg. the Schwarzschild solution. The figure is taken from [Pen74].

Illustrating two disjoint boundary null hypersurfaces $\mathcal{I}^{\pm}$, each a cylinder with topology $S^2 \times \mathbb{R}$ (null, as $d\hat{s}$ at $w = 0$ is degenerate). These hypersurfaces are generated by null geodesics ($\theta, \phi =$ const, $w = 0$), whoose tangents are normals to the hypersurfaces, and they are the "$\mathbb{R}$'s" of the topological product $S^2 \times \mathbb{R}$ (the inverse images of points of S^2 in the natural projection $S^2 \times \mathbb{R} \to S^2$).

[2] A function is $\mathscr{C}^n$ if it is n times differentiable with the continous n'th derivative.

3.8 Other Asymptotically Flat Space-Times

Other asymptotically flat space-times will also give rise to the same structure. Consider the metric

$$ds^2 = r^{-2}Adr^2 + 2B_i dx^i dr + r^2 C_{ij} dx^i dx^j, \tag{20}$$

with coordinates r, x^1, x^2, x^3, and A, B_i, C_{ij} being suitably smooth functions of x^0, x^1, x^2, x^3, with $x^0 = r^{-1}$, and the functions also being smooth at $x^0 = 0$. Setting $\Omega = r^{-1}$, we get:

$$d\hat{s}^2 = \Omega ds^2 = A(dx^0)^2 - 2B_i dx^i dx^0 + C_{ij} dx^i dx^j, \tag{21}$$

which will be perfectly regular at $x^0 = 0$ (provided the determinant formed from A, B_i, C_{ij} doesn't vanish). Thus a conformal infinity will exist for such metric.
Many metrics in the study of gravitational radiation have the form of *Equation 20*, also that of BMS, describing a situation with an isolated source (with asymptotic flatness) and outgoing gravitational radiation.

3.9 Asymptotically Simple Space-Times

Asymptotically **flat** space-times is a subclass of weakly asymptotically **simple** space-times.
The space-time M is asymptotically simple if; **1)** a conformal factor Ω exists so the metric $d\hat{s} = \Omega ds$ remains smooth on the extension $\bar{M}$ of manifold M, including boundary $\mathcal{I}$; **2)** Ω being smooth through $\bar{M} = M \cup \mathcal{I}$, zero on $\mathcal{I}$, and having non-zero gradient at $\mathcal{I}$; **3)** every maximal null geodesic in M has both a past- and future end-point on $\mathcal{I}$ (more weakly; should reach a past- and future conformal infinity, as some geodesics might not escape to conformal infinity, as with a black hole).
Space-time M is **weakly** asymptotically simple if it possesses conformal infinity of asymptotically simple space-times, but can also have other infinities as well; thus for an asymptotically simple M', a neighbourhood K' of $\mathcal{I}'$ in $\bar{M}'$, should have a portion $K' \cap M'$ being isometric with a subset of M.

The boundary $\mathcal{I}$ must be a smooth hypersurface (if null, it has the same topological structure as for the MS and Schwarzschild cases; see *Figure 10*). If Einstein's vacuum equation (of the metric), without the cosmological term, holds near conformal infinity $\mathcal{I}$ (through $K - \mathcal{I}$, K being the neighbourhood of $\mathcal{I}$ in M'), then $\mathcal{I}$ is null everywhere - if the cosmological term is present, then $\mathcal{I}$ is space-like or time-like. These are consequences of the transformation formula of the Ricci tensor, written as:

$$P_{ab} = \hat{P}_{ab} + \Omega^{-1}\hat{\nabla}_a\hat{\nabla}_b\Omega - \frac{1}{2}\Omega^{-2}\hat{g}_{ab}\hat{g}^{cd}\hat{\nabla}_a\Omega\hat{\nabla}_b\Omega \tag{22}$$

with $P_{ab} = \frac{1}{12}Rg_{ab} - \frac{1}{2}R_{ab}$ being a tensor containing same information as the Ricci tensor (R_{ab}) for ds, and $\hat{\nabla}_a$ denotes the covariant derivative for $d\hat{s}$, and $\hat{P}_{ab}$ being a tensor of $d\hat{s}$. Since Einstein's equations (without cosmological term) hold near $\mathcal{I}$, $R_{ab} = 0 \Rightarrow P_{ab} = 0$; multiplying *Equation 22* by Ω^2 and use condition $\Omega = 0$ on $\mathcal{I}$ (or taking the trace, using $\hat{g}_{ab}$), then vector $\hat{n}_a = \mp\hat{\nabla}_a\Omega$ is null at $\mathcal{I}^{\pm}$, which is normal to $\mathcal{I}$; $\mathcal{I}$ is a hypersurface.

In order to prove this, we write Einstein's Equations (In Vacuum):

$$R_{ab} - \frac{1}{2} g_{ab} R = 0.$$

Contract with g^{ab} to obtain $R = 0$, which plugged back implies $R_{ab} = 0$. Meaning that for:

$$P_{ab} = \frac{1}{12} R g_{ab} - \frac{1}{2} R_{ab}.$$

when Einstein's equations in vacuum are satisfied, then $P_{ab} = 0$. We know the transformation property *Eq. 22*. Plug $P_{ab} = 0$ into the equation to obtain:

$$0 = \hat{P}_{ab} + \Omega^{-1} \hat{\nabla}_a \hat{\nabla}_b \Omega - \frac{1}{2} \Omega^{-2} \hat{g}_{ab} \hat{g}^{cd} (\hat{\nabla}_c \Omega)(\hat{\nabla}_d \Omega).$$

Next, multiply by Ω^2 and take the limit to $\mathcal{I}^{\pm}$ (where $\Omega \to 0$). We get:

$$\hat{g}^{cd} (\hat{\nabla}_c \Omega)(\hat{\nabla}_d \Omega) = 0.$$

This means that $\hat{n}_a$ is null, since $\hat{g}^{ab} \hat{n}_a \hat{n}_b = 0$.

Assuming vacuum (or Einstein-Maxwell equations hold) near $\mathcal{I}$, and no cosmological constant; take $P_{ab} = 0$ near $\mathcal{I}$, and consider the trace-free part of *Equation 22* - then multiply by Ω and set $\Omega = 0$ to get $\hat{\nabla}_a \hat{n}_b = (\frac{1}{4} \hat{g}^{cd} \hat{\nabla}_c \hat{n}_d) \hat{g}_{ab}$ at $\mathcal{I}$ - these are **non-rotating** and **shear-free** generators of $\mathcal{I}$. Shear-free as $\hat{\nabla}_a \hat{n}_b$ is trace-free; small shapes are preserved, following the generators along $\mathcal{I}$. Non-rotating as $\mathcal{I}$ is a null hypersurface (due to symmetry of $\hat{\nabla}_a \hat{n}_b$). Take two cross-sections S_1, S_2 of either $\mathcal{I}^{\pm}$; then the correspondence between these, established by the generators, is conformal. The space of generators of $\mathcal{I}^{\pm}$ has conformal structure.

4 Matching Conditions for QED (S)

4.1 Classical Electromagnetism: Liénard-Wiechert Solution

The theory of electromagnetism can be described by the Maxwell-action (coupled to any kind of matter):

$$S = -\frac{1}{4e^2} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} + S_M, \tag{23}$$

with $\mu, \nu = 0, 1, 2, 3$, e is the electron charge, g is the determinant of the metric tensor, $g_{\mu\nu}$, S_M denotes a general matter (hence subscript M) action, and the field strength, F, is related to the gauge field, A, by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
The equations of motion (EOM) are given by:

$$d * F = e^2 * j \qquad \Longrightarrow \qquad \nabla^\mu F_{\mu\nu} = e^2 j_\nu, \tag{24}$$

where the Hodge dual, $*$, of a p-form, V, on an n-dimensional manifold, is a differential form, which is defined as $(*V)_{\mu_1,\mu_2,...,\mu_{n-p}} = \frac{1}{p!}\epsilon_{\mu_1,\mu_2,...,\mu_{n-p}}{}^{\nu_1,\nu_2,...,\nu_p}V_{\nu_1,\nu_2,...,\nu_p}$, an $(n-p)$-form (in Euclidean flat space-time, and ϵ is the Levi-Civita symbol - to get it in a general space-time, one needs to multiply the left-hand side (LHS) by $\sqrt{g}$), ∇ is the Nabla operator, and j, the charge current, is given by $j^\nu = -\frac{\delta S_M}{\delta A_\nu}$.
The electric charge inside a 2-sphere (S^2) at infinity, can be defined as:

$$Q_E = \frac{1}{4e^2}\int_{S^2} *F = \int_\Sigma *j \in \mathbb{Z}, \tag{25}$$

using integration by parts, and with Σ being any slice with the S^2 boundary. The electric charge is the normal component of the charge current integrated over that slice (the flux of the electric field Gauss law).
The theory is gauge invariant - under an infinitesimal gauge transformation with $\varepsilon \sim \varepsilon + 2\pi$, and a finite transformation of the matter field, Φ_k with charge $Q_k \in \mathbb{Z}$, by $\varepsilon \to \varepsilon + 2\pi$; the following fields transforms as:

$$\delta_\varepsilon A = d\varepsilon, \quad \delta_\varepsilon \Phi_k = i\varepsilon Q_k \Phi_k, \qquad \Phi_k \to e^{iQ_k\varepsilon}\Phi_k \tag{26}$$

To solve the EOM, consider a source, j, of n particles (non-displaced), with constant four-velocity, $U_k^\mu = \gamma_k(1, \vec{\beta}_k)$, with $U_k^2 = -1$, $\gamma_k^2 = \frac{1}{1-\beta_k^2}$, and β being the velocity relative to the speed of light. Then each particle is a point source, with a worldline parametrized by τ, so $x_k^\mu(\tau) = U_k^\mu \tau$:

$$j_\mu(x) = \sum_{k=1}^{n} Q_k \int d\tau U_{k\mu}\delta^4(x^\nu - U_k^\nu \tau) \tag{27}$$

The solution for the radial component of the electric field is:

$$F_{rt}(\vec{x}, t) = \frac{e^2}{4\pi}\sum_{k=1}^{n} \frac{Q_k\gamma_k(r - t\hat{x}\cdot\vec{\beta}_k)}{|\gamma_k^2(t - r\hat{x}\cdot\vec{\beta}_k)^2 - t^2 + r^2|^{\frac{3}{2}}} \tag{28}$$

with $r^2 = \vec{x}\cdot\vec{x}$ and $\vec{x} = r\hat{x}$. Hence it is not singled valued at $r = \infty$, but discontinued, which is an important point, when considering fields in compactified MS.

4.2 Antipodal Matching Condition

The Liénard-Wiechert solution in retarded coordinates (*Eq. 11*, $t = u + r$) is:

$$F_{rt} = F_{ru} = \frac{e^2}{4\pi}\sum_{k=1}^{n} \frac{Q_k\gamma_k(r - (u+r)\hat{x}\cdot\vec{\beta}_k)}{|\gamma_k^2(u + r - r\hat{x}\cdot\vec{\beta}_k)^2 - (u+r)^2 + r^2|^{\frac{3}{2}}} \tag{29}$$

In the limit of r going to infinity it becomes:

$$F_{rt}|_{\mathcal{I}^+} = \frac{e^2}{4\pi r^2}\sum_{k=1}^{n} \frac{Q_k}{\gamma_k^2(1 - \hat{x}\cdot\vec{\beta}_k)^2} \tag{30}$$

The electric field at $\mathcal{I}^-$, using advanced coordinates ($t = v - r$), is:

$$F_{rt} = F_{rv} = \frac{e^2}{4\pi} \sum_{k=1}^{n} \frac{Q_k \gamma_k (r - (v-r)\hat{x} \cdot \vec{\beta}_k)}{|\gamma_k^2 (v - r - r\hat{x} \cdot \vec{\beta}_k)^2 - (v-r)^2 + r^2|^{\frac{3}{2}}} \tag{31}$$

In the limit of r going to infinity it becomes:

$$F_{rt}|_{\mathcal{I}^-} = \frac{e^2}{4\pi r^2} \sum_{k=1}^{n} \frac{Q_k}{\gamma_k^2 (1 + \hat{x} \cdot \vec{\beta}_k)^2} \tag{32}$$

The value of the field depends on how you approach the value at i^0; hence the values of the fields do not match. The discontinuity is dictated by Lorentz invariance (which is not smooth near spatial infinity).
However one can find the antipodal relation (matching condition):

$$\lim_{x\to\infty} r^2 F_{ru}(\hat{x})|_{\mathcal{I}^+_-} = \lim_{x\to\infty} r^2 F_{rv}(-\hat{x})|_{\mathcal{I}^-_+} \tag{33}$$

One can map MS onto a cylinder, using conformal rescaling, as was done in *Section 3.4* with *Figure 7*. Looking at a similar figure, *Figure 11*, notice the cross at i^0, spatial infinity; *Eq. 33* states that the fields are continous along the genrators of $\mathcal{I}$, even when they cross i^0, though they can become quite diffucult to describe, due to i^0 being a singularity.

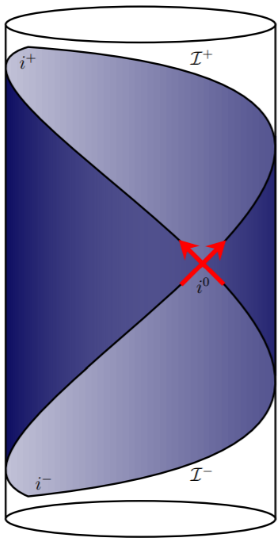


Figure 11: Minkowski space conformally compactified onto the $S^3 \times \mathbb{R}$ Einstein static universe (a cylinder). The figure is taken from [Str].

4.3 Asymptotic Expansion

The Minkowski metric in retarded coordinates (in the neighbourhood of $\mathcal{I}^+$) is:

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}, \qquad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2} \tag{34}$$

$\gamma_{z\bar{z}}$ being the unit sphere metric.
The standard metric $ds^2 = -dt^2 + (d\vec{x})^2$ is related to the metric in retarded coordinates by:

$$(\vec{x})^2 = r^2, \qquad t = u + r, \qquad x^1 + ix^2 = \frac{2rz}{1+z\bar{z}}, \qquad x^3 = r\frac{1-z\bar{z}}{1+z\bar{z}}, \tag{35}$$

where z runs over the complex plane so that; $z = 0$ is the north pole, $z = \infty$ is the south pole, and the equator is at $z\bar{z} = 1$.
The metric in advanced coordinates (for $\mathcal{I}^-$) is given by:

$$ds^2 = -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \tag{36}$$

The two metrics, in retarded- and advanced time, are related so that the z in the $\mathcal{I}^-$ coordinate, is the antipodal of the z in the $\mathcal{I}^+$ coordinate (as described in *Section 3.5*).

Expanding around $\mathcal{I}^+$, given a field, e.g. the z-component of the gauge field, it can be expressed as a sum:

$$A_z(u, r, z, \bar{z}) = \sum_{n=0}^{\infty} \frac{A_z^{(n)}(u, z, \bar{z})}{r^n} \tag{37}$$

Applying such expansion, generalises the matching condition to:

$$F_{ru}^{(2)}(z, \bar{z})|_{\mathcal{I}^+_-} = F_{rv}^{(2)}(z, \bar{z})|_{\mathcal{I}^-_+}, \qquad F_{ru}^{(2)}(z, \bar{z})|_{\mathcal{I}^+_-} = F_{ru}^{(2)}(-\infty, z, \bar{z}) \tag{38}$$

4.4 An Infinity of Conserved Charges

The matching condition implies an infinite number of conserved charges, which will be shown in this section. Consider a function, ε, on MS obeying:

$$\varepsilon(z, \bar{z})|_{\mathcal{I}^+_-} = \varepsilon(z, \bar{z})|_{\mathcal{I}^-_+} \tag{39}$$

Defining future- and past charges as:

$$Q_\varepsilon^+ = \frac{1}{e^2}\int_{\mathcal{I}^+_-} \varepsilon * F, \qquad Q_\varepsilon^- = \frac{1}{e^2}\int_{\mathcal{I}^-_+} \varepsilon * F, \qquad Q_\varepsilon^+ = Q_\varepsilon^- \tag{40}$$

which gives an infinite number of conservation laws, associated to the EM field.
Using the Gauss law to write the surface integral expression of the charge gives;

$$Q_\varepsilon^+ = \frac{1}{e^2}\int_{\mathcal{I}^+} d\varepsilon \wedge *F + \int_{\mathcal{I}^+} \varepsilon * j + \cancelto{0}{\frac{1}{e^2}\int_{\mathcal{I}^+_+} \varepsilon * F} \tag{41}$$

$$Q_\varepsilon^- = \frac{1}{e^2}\int_{\mathcal{I}^-} d\varepsilon \wedge *F + \int_{\mathcal{I}^-} \varepsilon * j + \cancelto{0}{\frac{1}{e^2}\int_{\mathcal{I}^-_-} \varepsilon * F}, \tag{42}$$

with restrictions, $\partial_u \varepsilon = \partial_v \varepsilon = 0$, giving:

$$\sum_{k=1}^{m} Q_k^{in} = \sum_{k=1}^{n} Q_k^{out} \tag{43}$$

It states, that the sum of all incoming charges, must be equal to the sum of all outgoing charges.

The charges in retarded and advanced coordinates are:

$$\begin{aligned} Q_\varepsilon^+ &= \frac{1}{e^2}\int_{\mathcal{I}^+_-} \varepsilon * F = \frac{1}{e^2}\int_{\mathcal{I}^+_-} d^2z \gamma_{z\bar{z}} \varepsilon F_{ru}^{(2)} \\ Q_\varepsilon^- &= \frac{1}{e^2}\int_{\mathcal{I}^-_+} d^2z \gamma_{z\bar{z}} \varepsilon F_{rv}^{(2)} \end{aligned} \tag{44}$$

Use the constraint equations on the null surfaces; the constraint equation near $\mathcal{I}^+$ has the leading term (expansion in powers of $\frac{1}{r}$):

$$\partial_u F_{ru}^{(2)} + D^z F_{uz}^{(0)} + D^{\bar{z}} F_{u\bar{z}}^{(0)} + e^2 j_u^{(2)} = 0, \tag{45}$$

where D is the covariant derivative, such that $D^z = \gamma^{z\bar{z}} D_{\bar{z}}$. We now integrate by parts and use above constraint equation, which gives:

$$Q_\varepsilon^+ = -\frac{1}{e^2}\int_{\mathcal{I}^+} du d^2z (\partial_z \varepsilon F_{u\bar{z}}^{(0)} + \partial_{\bar{z}} \varepsilon F_{uz}^{(0)}) + \int_{\mathcal{I}^+} du d^2z \varepsilon \gamma_{z\bar{z}} j_u^{(2)} = Q_S^+ + Q_H^+ \tag{46}$$

consisting of a soft term (first term), including massless charges only, and a hard term (second term). The soft term involves the term (the zero-mode of the radiative component with respect to retarded time):

$$\begin{aligned} &\int_{-\infty}^{\infty} du F_{uz}^{(0)} e^{i\omega u} \\ &\int_{-\infty}^{\infty} du F_{uz}^{(0)} \equiv N_z \quad as \quad \omega \to 0 \end{aligned} \tag{47}$$

This term creates and annihilates soft particles (zero-energy).
Considering the curl:

$$\begin{aligned} \partial_{\bar{z}} N_z - \partial_z N_{\bar{z}} &= \int_{-\infty}^{\infty} du [\partial_{\bar{z}} F_{uz}^{(0)} - \partial_z F_{u\bar{z}}^{(0)}] \\ &= -\int_{-\infty}^{\infty} du \partial_u F_{z\bar{z}}^{(0)} = -F_{z\bar{z}}^{(0)}\Big|_{\mathcal{I}^+_-}^{\mathcal{I}^+_+}, \end{aligned} \tag{48}$$

by using the Bianchi identity in the second line. Assuming no magnetic monopoles (no asymptotic states with magnetic charges) - and therefore no long-ranged magnetic fields, so $F_{z\bar{z}}|_{\mathcal{I}^+_\pm} = 0$, one can define:

$$N_z \equiv e^2 \partial_z N, \tag{49}$$

with N being a real scalar. Imposing gauge condition $A_u^{(0)} = 0$, it follows that:

$$e^2 \partial_z N = \int_{-\infty}^{\infty} du F_{uz}^{(0)} = A_z^{(0)}|_{\mathcal{I}^+_+} - A_z^{(0)}|_{\mathcal{I}^+_-} \tag{50}$$

For a finite energy shift, one needs the gauge field and shift to be pure gauge (gauge transform of zero) in both the beginning and end of $\mathcal{I}^+$.

5 From Conserved Charges to Soft Theorems in QED (S)

From finding an infinity of conserved charges, one expects to find the (infinitely many) associated symmetries as well.

5.1 Canonical Electrodynamics at $\mathcal{I}$

Noether's theorem establishes a connection between conserved charges and symmetries. And when going from charges to symmetries; symmetries can be determined by the Dirac bracket action of the charges (on the phase space). In a canonical Hamiltonian formalism; having Q (charge) commute with H (Hamiltonian), and defining a phase space Γ, with coordinates $x^I = q^i, p_j$, where I is a continuous index in field theory. The symplectic two-form on this space is:

$$\Omega = \frac{1}{2} \Omega_{IJ} dx^I \wedge dx^J \tag{51}$$

and its quantum commutators are constructed as $[A, B] = i\Omega^{IJ} \partial_I A \partial_J B$.

5.1.1 Symplectic Form

The classical phase spase in EM, can be defined as the allowed initial data on a Cauchy surface. For a Cauchy surface Σ in MS, the symplectic form for free electrodynamics is:

$$\Omega_\Sigma = -\frac{1}{e^2} \int_\Sigma \delta(*F) \wedge \delta A \tag{52}$$

δA is a one-form on the phase space Γ, so the term in the integral is a wedge product of the infinite dimensional manifold, which describes this phase space. Writing out the indices, the symplectic form becomes:

$$\Omega_\Sigma = \frac{1}{e^2} \int_\Sigma d\Sigma^\mu \delta F_{\mu\nu} \wedge \delta A^\nu \tag{53}$$

When choosing $\Sigma = \mathcal{I}^+$, it becomes:

$$\Omega_{\mathcal{I}^+} = \frac{1}{e^2} \int dud^2z(\delta F^{(0)}_{uz} \wedge \delta A^{(0)}_{\bar{z}} + \delta F^{(0)}_{u\bar{z}} \wedge \delta A^{(0)}_{z}) \tag{54}$$

Defining the constant part of $A^{(0)}_z$ as:

$$\begin{aligned} A^{(0)}_z(u,z,\bar{z}) &= \hat{A}_z(u,z,\bar{z}) + \partial_z\phi(z,\bar{z}) \\ \partial_z\phi &\equiv \frac{1}{2}\left[A^{(0)}_z|_{\mathcal{I}^+_+} + A^{(0)}_z|_{\mathcal{I}^+_-}\right] \end{aligned} \tag{55}$$

The first term is dependent on u, the second term is not. Substituting *Eq. 55* into *Eq. 54*, with N as in *Eq. 49*, the symplectic form becomes:

$$\Omega_{\mathcal{I}^+} = \frac{2}{e^2} \int dud^2z\partial_u\delta\hat{A}_z \wedge \delta\hat{A}_{\bar{z}} - 2\int d^2z\partial_z\delta\phi \wedge \partial_{\bar{z}}\delta N \tag{56}$$

The radiative components of the gauge field are paired in the symplectic form, and the soft photon mode is paired with the ϕ field (the sum of the boundary values).

5.1.2 Commutators

Now discussing commutators, one has to invert the symplectic form in the terms separately. The first term implies:

$$\begin{aligned} [\partial_u\hat{A}_z(u,z,\bar{z}), \hat{A}_{\bar{w}}(u',w,\bar{w})] &= -\frac{ie^2}{2}\delta(u-u')\delta^2(z-w) \\ [\hat{A}_z(u,z,\bar{z}), \hat{A}_{\bar{w}}(u',w,\bar{w})] &= -\frac{ie^2}{4}\Theta(u-u')\delta^2(z-w) \\ \Theta(u) &= \frac{1}{\pi i}\int \frac{d\omega}{\omega}e^{i\omega u} \end{aligned} \tag{57}$$

with $\Phi(u < u') = -1$ and $\Phi(u > u') = +1$. For the commutator between ϕ and N (not dependent on u); the commutators are:

$$\begin{aligned} [\phi(z,\bar{z}), N(w,\bar{w})] &= -\frac{i}{4\pi}\log|z-w|^2 + f(z,\bar{z}) + g(w,\bar{w}) \\ [\partial_z\phi(z,\bar{z}), \partial_{\bar{w}}N(w,\bar{w})] &= -\frac{i}{4\pi}\partial_z\partial_{\bar{w}}\log|z-w|^2 = \frac{i}{2}\delta^2(z-w) \\ \partial_{\bar{z}}\frac{1}{z-w} &= 2\pi\delta^2(z-w) \end{aligned} \tag{58}$$

5.2 Large Gauge Symmetry

Now the previous commutators can be used to compute the commutator action of Q^+_ε, which has a matter term and a linear soft photon term (which do not commute with

$A_z^{(0)}$):

$$[Q_\varepsilon^+, A_z^{(0)}(u,z,\bar{z})] = i\partial_z\varepsilon(z,\bar{z})$$
$$[Q_\varepsilon^-, A_z^{(0)}(v,z,\bar{z})] = i\partial_z\varepsilon(z,\bar{z}) \tag{59}$$
$$[Q_\varepsilon^+, N(z,\bar{z})] = 0 \quad [Q_\varepsilon^+, \hat{A}_z(u,z,\bar{z})] = 0 \quad [Q_\varepsilon^+, \phi(z,\bar{z})] = 0$$

Symmetries generated by the conserved charges, Q_ε^+, are 'large' gauge transformations (of A_z) with parameter ε, which does not die off at infinity. ε goes to an u-independent, angle-dependent function at $\mathcal{I}^+$, and at $\mathcal{I}^-$, the gauge parameter approaches the antipodally transformed function, for transformations generated by Q_ε^-.
We need to check that these charges also generates the gauge transformations on the matter fields, here the 'hard' term comes in:

$$\left[\int_{\mathcal{I}^+} \varepsilon * j, \Phi_k(u,z,\bar{z})\right] = -Q_k\varepsilon(z,\bar{z})\Phi_k(u,z,\bar{z}) = i\delta_\varepsilon\Phi_k(u,z,\bar{z})$$
$$[Q_\varepsilon^+, \Phi_k(u,z,\bar{z})] = i\delta_\varepsilon\Phi_k(u,z,\bar{z}) \tag{60}$$

The sum of the hard and soft term, generates local angle dependent gauge transformations (on $\mathcal{I}^+$).

5.3 Ward Identity

Each symmetry in a quantum theory corresponds to a Ward identity, which is the dynamical consequences from the fact that the conserved charges commutes with the $\mathcal{S}$-matrix, or equivalently with the Hamiltonian, as $\mathcal{S} \sim e^{iHT}$ for $T \to \infty$.

5.3.1 Symmetries of the $\mathcal{S}$-Matrix

The $\mathcal{S}$-matrix (scattering matrix) can be written as $\langle f|\,\mathcal{S}\,|i\rangle$, where f stands for final state, and i for initial state. If the charge is conserved, it can be written as

$$\langle f|\,(Q_\varepsilon^+\mathcal{S} - \mathcal{S}Q_\varepsilon^-)\,|i\rangle = 0 \tag{61}$$

The sum of the incoming charges is equal to the sum of the outgoing charges. Furthermore, the action of a charge, acting on the initial and final state, can be split up into terms of the action of the soft- and hard charge respectively, such that; $\langle f|\,(Q_S^+\mathcal{S} - \mathcal{S}Q_S^-)\,|i\rangle = -\langle f|\,(Q_H^+\mathcal{S} - \mathcal{S}Q_H^-)\,|i\rangle$. Thus one can define:

$$Q_\varepsilon^-\,|i\rangle = -2\int d^z\partial_{\bar{z}}\varepsilon\partial_z N^-(z,\bar{z})\,|i\rangle + \sum_{k=1}^{m} Q_k^{in}\varepsilon(z_k^{in},\bar{z}_k^{in})\,|i\rangle \tag{62}$$

for Q_ε^- on the initial state, and:

$$\langle f|\,Q_\varepsilon^+ = 2\int d^z\partial_{\bar{z}}\partial_z\varepsilon\,\langle f|\,N(z,\bar{z}) + \sum_{k=1}^{n} Q_k^{out}\varepsilon(z_k^{out},\bar{z}_k^{out})\,\langle f| \tag{63}$$

for Q_ε^+ on the final state. Finally the Ward identity can be written as

$$
\begin{aligned}
&2\int d^z\partial_{\bar{z}}\partial_z\varepsilon \langle f|\left(N(z,\bar{z})\mathcal{S}-\mathcal{S}N^-(z,\bar{z})\right)|i\rangle \\
&=\left[\sum_{k=1}^{m}Q_k^{in}\varepsilon(z_k^{in},\bar{z}_k^{in})-\sum_{k=1}^{n}Q_k^{out}\varepsilon(z_k^{out},\bar{z}_k^{out})\right]\langle f|\mathcal{S}|i\rangle
\end{aligned}
\tag{64}
$$

The term on the RHS (right hand side) in the bracket denotes incoming/outgoing charges weighted by the value of the gauge-parameter, at the angle they are coming in (going out) from. This equation states an infinite number of ward identities, one for each ε on the sphere. They relate any $\mathcal{S}$-matrix element of initial and final states, to the same $\mathcal{S}$-matrix element with a soft photon inserted.

5.3.2 Mode Expansions

Next it can be shown that the Ward identity (*Eq. 64*), following from conserved charges, is the same as the soft theorem in abelian gauge theory. We have until now defined particles by the points at null infinity, they came in at, and used advanced and retarded coordinates to derive conservation laws from antipodal matching conditions. We need to rewrite the Ward identity in terms of a plane-wave basis, as traditionally used in QFT, using a mode-expansion of A_z.

Going back to standard Cartesian coordinates in MS:

$$ds^2=-dt^2+d\vec{x}\cdot d\vec{x} \tag{65}$$

Near $\mathcal{I}^+$, the gauge field, A_ν, has the on-shell outgoing plane wave mode-expansion, where $q^2=0$:

$$A_\nu(x)=e\sum_{\alpha=\pm}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2\omega}\left[\varepsilon_\nu^{*\alpha}(\vec{q})a_\alpha^{\text{out}}(\vec{q})e^{iq\cdot x}+\varepsilon_\nu^{\alpha}(\vec{q})a_\alpha^{\text{out}}(\vec{q})^\dagger e^{-iq\cdot x}\right] \tag{66}$$

with normalization condition of the polarization vectors $\varepsilon_\alpha^\nu\varepsilon_{\beta\nu}^*=\delta_{\alpha\beta}$ and

$$\left[a_\alpha^{\text{out}}\left(\vec{q}\right),a_\beta^{\text{out}}\left(\vec{q}'\right)^\dagger\right]=\delta_{\alpha\beta}(2\pi)^3\left(2\omega_q\right)\delta^3\left(\vec{q}-\vec{q}'\right) \tag{67}$$

being satisfied. Rewrite this using retarded coordinates, where the metric takes the form (*Eq. 34*):

$$ds^2=-du^2-2dudr+2r^2\gamma_{z\bar{z}}dzd\bar{z} \tag{68}$$

The null vector q^μ, labelled by a point, $w(z,\bar{z})$, on the sphere, and satisfying $q^\mu q_\mu=0$; can be written as:

$$q^\mu=\frac{\omega}{1+z\bar{z}}(1+z\bar{z},z+\bar{z},-i(z-\bar{z}),1-z\bar{z})=\left(\omega,q^1,q^2,q^3\right) \tag{69}$$

At $w=0$, we find $q^\mu = \omega(1,0,0,1)$, a null vector pointing along the x^3-axis. One can also write the polarization vectors, orthogonal to q^μ as:

$$\varepsilon^{+\mu}(\vec{q}) = \frac{1}{\sqrt{2}}(\bar{z},1,-i,-\bar{z}), \quad \varepsilon^{-\mu}(\vec{q}) = \frac{1}{\sqrt{2}}(z,1,i,-z), \quad q_\mu \varepsilon^{\pm\mu}(\vec{q}) = 0, \quad \varepsilon^\mu_\alpha \varepsilon^*_{\beta\mu} = \delta_{\alpha\beta} \tag{70}$$

Considering the gauge field, $A_z^{(0)}$, in the future null infinity, $\mathcal{I}^+$, given by:

$$\begin{aligned} A_z^{(0)}(u,z,\bar{z}) &= \lim_{r\to\infty} A_z(u,r,z,\bar{z}) \\ &= -\frac{i}{8\pi^2}\frac{\sqrt{2}e}{1+z\bar{z}}\int_0^\infty d\omega \left[a_+^{\text{out}}(\omega\hat{x})e^{-i\omega u} - a_-^{\text{out}}(\omega\hat{x})^\dagger e^{i\omega u}\right], \quad \hat{x} = \hat{x}(z,\bar{z}) \end{aligned} \tag{71}$$

$A_z^{(0)}$ is expected to create and annihilate photons at $(z,\bar{z})$, and furthermore rotating about this point, it creates one photon helicity (negative), and annihilates the other (positive), where $A_{\bar{z}}$ does the opposite. $\hat{x}$ being a unit vector pointing at $(z,\bar{z})$ on the sphere.

The Ward identity involves a term $\partial_z N$, so its mode-expansion is needed. First defining (to be precise about the zero-momentum limit):

$$\partial_z N = \frac{1}{2e^2}\lim_{\omega\to 0^+}\int_{-\infty}^\infty du \left(e^{i\omega u} + e^{-i\omega u}\right) F_{uz}^{(0)} \tag{72}$$

ensuring $\partial_z \partial_{\bar{z}} N$ to be Hermitian (referring to the exponential terms). Using the above equation for the gauge field, one finds the expansions:

$$\partial_z N = -\frac{1}{8\pi e}\frac{\sqrt{2}}{1+z\bar{z}}\lim_{\omega\to 0^+}\left[\omega a_+^{\text{out}}(\omega\hat{x}) + \omega a_-^{\text{out}}(\omega\hat{x})^\dagger\right] \tag{73}$$

The similar formula for $\partial_z N^-$ is given by:

$$\partial_z N^- = -\frac{1}{8\pi e}\frac{\sqrt{2}}{1+z\bar{z}}\lim_{\omega\to 0^+}\left[\omega a_+^{\text{in}}(\omega\hat{x}) + \omega a_-^{\text{in}}(\omega\hat{x})^\dagger\right] \tag{74}$$

Considering the special case of which we have

$$\varepsilon(w,\bar{w}) = \frac{1}{z-w} \tag{75}$$

and using

$$\partial_{\bar{z}}\frac{1}{z-w} = 2\pi\delta^2(z-w) \tag{76}$$

to perform the LHS integral of the Ward identity, *Eq. 64*, it can be written as:

$$4\pi \langle f| \left(\partial_z N \mathcal{S} - \mathcal{S}\partial_z N^-\right)|i\rangle = \left[\sum_{k=1}^m \frac{Q_k^{\text{in}}}{z - z_k^{\text{in}}} - \sum_{k=1}^n \frac{Q_k^{\text{out}}}{z - z_k^{\text{out}}}\right]\langle f|\mathcal{S}|i\rangle \tag{77}$$

Finally one can re-express the Ward identity as

$$\lim_{\omega\to 0}\left[\omega \langle f|\left(a_+^{\text{out}}(\omega\hat{x})\mathcal{S} \;\; -\mathcal{S}a_-^{\text{in}}(\omega\hat{x})^\dagger\right)|i\rangle\right]$$

$$= \sqrt{2}e(1+z\bar{z})\left[\sum_{k=1}^{n}\frac{Q_k^{\text{out}}}{z-z_k^{\text{out}}} - \sum_{k=1}^{m}\frac{Q_k^{\text{in}}}{z-z_k^{\text{in}}}\right]\langle f|\mathcal{S}|i\rangle \tag{78}$$

5.3.3 Soft Theorems

To show the final steps of recovering the standard soft theorem, one could rewrite the z's in terms of particle momenta; but the reverse is easier. We begin with the standard soft photon theorem, attempting to recover the newly found Ward identity, *Eq. 78*, by going from momentum space to points on a sphere (from plane waves to z's). The soft photon theorem can be written as:

$$\begin{aligned}\lim_{\omega\to 0}\left[\omega \langle f| a_+^{\text{out}}(\vec{q})\mathcal{S}|i\rangle\right] &= e\lim_{\omega\to 0}\left[\sum_{k=1}^{m}\frac{\omega Q_k^{\text{out}}\, p_k^{\text{out}}\cdot\varepsilon^+}{p_k^{\text{out}}\cdot q} - \sum_{k=1}^{n}\frac{\omega Q_k^{\text{in}}\, p_k^{\text{in}}\cdot\varepsilon^+}{p_k^{\text{in}}\cdot q}\right]\langle f|\mathcal{S}|i\rangle \\ &= -\lim_{\omega\to 0}\left[\omega\langle f|\mathcal{S}a_-^{\text{in}\,\dagger}(\vec{q})|i\rangle\right]\end{aligned} \tag{79}$$

with $|i\rangle = \left|p_1^{\text{in}},\dots,p_n^{\text{in}}\right\rangle, \langle f| = \left\langle p_1^{\text{out}},\dots,p_m^{\text{out}}\right|$. The LHS annihilates a positive helicity photon, and the RHS creates a negative helicity photon. The middle term dictates initial- and final states, described by particles with momenta and charges - the Weinberg soft factor.
$q^\mu = (\omega,\vec{q})$ is the soft photon momentum, and $\omega\to 0$ being the soft limit, where we have a soft/Weinberg pole - and additionally a collinear pole, since q, p are null vectors. Hence the pole-structure of the two equations agrees.
Writing the hard particle photons in terms of energy and a point on the sphere, as for example:

$$\left(p_k^{\text{in}}\right)^\mu = E_k^{\text{in}}\left(1, \frac{z_k^{\text{in}}+\bar{z}_k^{\text{in}}}{1+z_k^{\text{in}}\bar{z}_k^{\text{in}}}, \frac{-i\left(z_k^{\text{in}}-\bar{z}_k^{\text{in}}\right)}{1+z_k^{\text{in}}\bar{z}_k^{\text{in}}}, \frac{1-z_k^{\text{in}}\bar{z}_k^{\text{in}}}{1+z_k^{\text{in}}\bar{z}_k^{\text{in}}}\right) \tag{80}$$

Similarly for incoming q, and outgoing p and q. Substituting this into the soft photon theorem, one reproduces the Ward identity. This result connects the soft photon theorem to the large gauge symmetry of EM.

5.4 Feynman Diagrammatics

This section will review the standard field theory derivation of the leading photon- and graviton soft theorems (in the form given by Weinberg).

5.4.1 Soft Photons

Starting with the soft photon theorem (see *Eq. 79* in comparison), which states that any $\mathcal{S}$-matrix element with a soft photon added, is the original matrix element, multiplied by the soft $q^\mu \to 0$ factor (with added corrections of order q^0):

$$\langle f|\, a_+^{\text{out}}\,(\vec{q})\mathcal{S}\,|i\rangle = e\left[\sum_{k=1}^{m} \frac{Q_k^{\text{out}}\, p_k^{\text{out}} \cdot \varepsilon^+}{p_k^{\text{out}} \cdot q} - \sum_{k=1}^{n} \frac{Q_k^{\text{in}}\, p_k^{\text{in}} \cdot \varepsilon^+}{p_k^{\text{in}} \cdot q}\right] \langle f|\, \mathcal{S}\,|i\rangle + \mathcal{O}\left(q^0\right) \tag{81}$$

Consider a scattering process with incoming and outgoing particles, then add one outgoing photon with momentum q, see *Figure 12*.

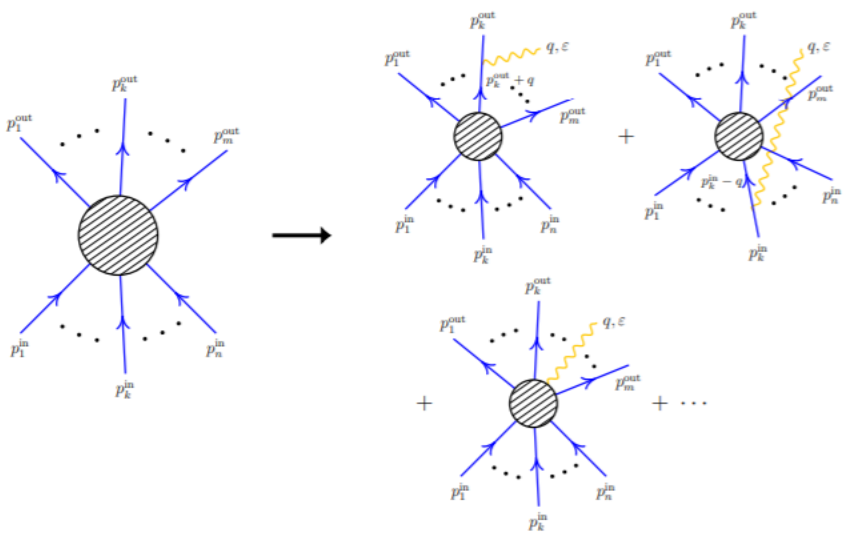


Figure 12: On the left is a Feynman diagram representing $n \to m$ scattering. On the right, the effect of adding an outgoing soft photon (or graviton) with momentum q and polarization ε is illustrated. In the upper diagrams the soft particle attaches to an external propagator, while in the lower one it attaches to an internal propagator. The figure is taken from [Str].

In the soft limit, the amplitude is a sum of terms, where the photon attaches to external or internal lines. To compute this process, using the LSZ method; first compute the time ordered Green's functions, using Feynman $i\epsilon$ prescriptions, and then amputate the external legs. When attaching an additional photon to an external leg, which is amputated, only a vertex and propagator for the (external leg's) particle needs to be added. This is the only difference to the same diagram, but without the added external soft photon.

Consider the interaction: The interaction vertex is $\mathcal{L}_{\text{int}} = -A^\mu j_\mu$. The charge current of a scalar field of charge Q is:

$$j_\mu = iQ\left(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi\right) \tag{82}$$

For a plane wave, the charge current is: $j_\mu \sim 2Qp_\mu$, when the normalization for single particle states is:

$$\langle p \mid p' \rangle = 2\omega_p (2\pi)^3 \delta^3 \left(p - p'\right) \tag{83}$$

The propagator for a (single) scalar particle is:

$$\frac{-i}{(p+q)^2 + m^2} = \frac{-i}{p^2 + 2p \cdot q + q^2 + m^2} = \frac{-i}{2p \cdot q} \tag{84}$$

using the on-shell condition $q^2 = 0$ and $p^2 = -m^2$, and its vertex factor $ie\varepsilon^\mu 2Qp_\mu$. So the total contribution becomes:

$$ie\varepsilon^\mu \left(2Qp_\mu\right) \frac{-i}{(p+q)^2 + m^2} \to \frac{eQ\varepsilon \cdot p}{q \cdot p} \tag{85}$$

Thus for all outgoing and incoming (with minus signs) particles, the contribution is:

$$\sum_{k=1}^{m} \frac{eQ_k^{\text{out}}\, p_k^{\text{out}} \cdot \varepsilon}{p_k^{\text{out}} \cdot q} - \sum_{k=1}^{n} \frac{eQ_k^{\text{in}}\, p_k^{\text{in}} \cdot \varepsilon}{p_k^{\text{in}} \cdot q} \tag{86}$$

Internal propagators are never on-shell, so they do not contribute to the pole in the soft limit. The soft factor shifts by:

$$\sum_{k=1}^{m} eQ_k^{\text{out}} - \sum_{k=1}^{n} eQ_k^{\text{in}} = 0 \tag{87}$$

when shifting ε^μ by q^μ. The **global charge conservation** ensures that the soft factor is gauge invariant. Works in the same way for charged particles, and not only scalars.

5.4.2 Soft Gravitons

To generalise to gravity, the interaction, and equations needed for gravity are:

$$\begin{aligned} &\mathcal{L}_{\text{int}} = \sqrt{8\pi G} h^{\mu\nu} T_{\mu\nu}, \\ &\varepsilon_{\mu\nu} q^\mu = 0, \quad \varepsilon^{\mu\nu} \eta_{\mu\nu} = 0, \\ &T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} \partial^\rho \phi \partial_\rho \phi \end{aligned} \tag{88}$$

$T_{\mu\nu}$ being the stress tensor (for a scalar field), and $\varepsilon_{\mu\nu}$ being the polarization tensor for gravitons. Same propagator as for photons $\frac{-i}{2p \cdot q}$. The scalar interaction is $T_{\mu\nu} \sim 2p_\mu p_\nu$. The product of vertex and propagator becomes:

$$i\sqrt{32\pi G} \varepsilon^{\mu\nu} p_\mu p_\nu \frac{-i}{(p+q)^2 + m^2} \to \sqrt{8\pi G} \frac{\varepsilon^{\mu\nu} p_\mu p_\nu}{p \cdot q} \tag{89}$$

for one external particle. The soft factor is generally:

$$\sqrt{8\pi G} \sum_{k=1}^{m} \frac{\varepsilon^{\mu\nu} p_{k\mu}^{\text{out}}\, p_{k\nu}^{\text{out}}}{p_k^{\text{out}} \cdot q} - \sqrt{8\pi G} \sum_{k=1}^{n} \frac{\varepsilon^{\mu\nu} p_{k\mu}^{\text{in}} p_{k\nu}^{\text{in}}}{p_k^{\text{in}} \cdot q} \tag{90}$$

Shifting $\varepsilon^{\mu\nu} \to \varepsilon^{\mu\nu} + \Lambda^\mu(q)q^\nu$, the analogue of gauge invariance in the gravity case, the soft factor shifts by:

$$\Lambda^\mu \left[\sum_{k=1}^{m} p_{k\mu}^{\text{out}} - \sum_{k=1}^{n} p_{k\mu}^{\text{in}} \right] = 0 \tag{91}$$

due to **global energy-momentum conservation**.

5.5 Asymptotic Symmetries

So far it has been shown, that the derived conserved charges generate large gauge transformations (see *Section 5.2*). This section will describe the asymptotic symmetries.
To find these symmetries, you have set of fields, where you specify how the components fall off, as you go near null infinity. Then make the boundary conditions as strong as possible, but without ruling out anything unreasonable. You often find gauge symmetries, which respect the boundary conditions, but behaves non-trivially at infinity, which might lead to acting non-trivially on the Hilbert space (quantum case). For the gravity case, this analysis lead to discovering the BMS group, which includes supertranslations (retarded time shifted by an arbitrary function on the sphere). *"It is an art and not a science"* - from [Str] p. 34.
An asymptotic symmetry group (ASG) is defined as the allowed gauge/local symmetries, divided by the trivial ones:

$$\text{ASG} = \frac{\text{allowed gauge symmetries}}{\text{trivial gauge symmetries}}. \tag{92}$$

The allowed are the ones, that respect the boundary conditions, where the trivial ones act trivially on (do not change) the physical data.
With the intent of covering the Poincaré group in general relativity, considering asymptotically flat space-time, at spatial infinity, Bondi, van der Burg, Metzner and Sachs showed the ASG to be the BMS (Bondi-Metzner-Sachs) group, containing the Poincaré group as a subgroup, but with an additional infinity of generators, the supertranslations. This implies that GR does not reduce to special relativity for weak fields at long distances.
Another example is Brown and Henneaux, taking a 3D anti-de Sitter space, with weak enough boundary conditions to allow all the black hole solutions. They found that the ASG/asymptotic symmetry algebra were two copies of the Virasoro algebra.

In electrodynamics (gauge theory), one must look at the boundary conditions of $\mathcal{I}^-$ and $\mathcal{I}^+$, and derive the asymptotic behaviour from the field equations. Consider a sphere at large r. Its surface area grows like r^2; thus, for the energy flux at any time to be finite, we must have $T_{uu} \sim \mathcal{O}(\frac{1}{r^2})$, where the $\frac{1}{r^2}$ comes from inverting the metric on the sphere (with radius r). We also have the relation:

$$T_{uu} \sim F_{uz} F_{u\bar{z}} \frac{\gamma^{z\bar{z}}}{r^2} + \dots \tag{93}$$

This, and F_{ur} being the long range electric field, suggests:

$$F_{uz} \sim \mathcal{O}(1), \quad F_{ur}, F_{zr} \sim \mathcal{O}\left(\frac{1}{r^2}\right) \tag{94}$$

Thus, the boundary fall-off conditions for the gauge fields must be:

$$A_z \sim \mathcal{O}(1), \quad A_r \sim \mathcal{O}\left(\frac{1}{r^2}\right), \quad A_u \sim \mathcal{O}\left(\frac{1}{r}\right) \tag{95}$$

which allows all initial data. The allowed gauge transformations are therefore:

$$\delta A_z = \partial_z \varepsilon, \quad \delta A_u = \partial_u \varepsilon, \quad \delta A_r = \partial_r \varepsilon \tag{96}$$

being consistent with:

$$\varepsilon = \varepsilon(z, \bar{z}) + \mathcal{O}\left(\frac{1}{r}\right) \tag{97}$$

This suggests the non-trivial large gauge transformations are generated by functions, ε, dependent on $(z, \bar{z})$, and not u. This is the electrodynamic analogue of the BMS transformations of gravity theory.

5.6 Massive QED

This subsection will generalise the previous results to massive particles. For a massless field, time-like infinity is not important. A massive particle on the other hand, never makes it to null infinity (they go from i^- and asymptote to i^+). How does a gauge transformation/charge act on a particle, that doesn't make it to null infinity? We have found the hard charge action on asymptotic massive states, which then let us transform into momentum space to reproduce the soft theorem.

We need to extend the asymptotic gauge parameter $\varepsilon(z, \bar{z})$ from null-infinity into the bulk interior of MS and $i^{\pm}$. It is convenient to use Lorentz gauge, $\nabla^\mu A_\mu = 0$, implying that the gauge parameter satisfies $\Box \varepsilon = 0$, as $\delta A_\mu = \partial_\mu \varepsilon$. The boundary condition is that the gauge parameter asymptotes to a specific function, $\varepsilon(z, \bar{z})$, on $\mathcal{I}^+$. It has the general solution (that approaches $\varepsilon(z, \bar{z})$):

$$\varepsilon(x) = \int d^2\hat{q}\, G(x, \hat{q}) \varepsilon(\hat{q}) \tag{98}$$

where the point on the asymptotic sphere is parametrized by a unit vector $\hat{q}$, pointing towards $(z, \bar{z})$. The Green's function, G, satisfying properties of the massless scalar wave equation, is:

$$\Box G(x, \hat{q}) = 0, \qquad \lim_{\substack{r \to \infty \\ u \to \text{ fixed}}} G(x, \hat{q}) = \delta^2(\hat{x} - \hat{q}) \tag{99}$$

with solution:

$$G(x, \hat{q}) = -\frac{\sqrt{\gamma(\hat{q})}}{4\pi} \frac{x^\mu x_\mu}{(q \cdot x)^2} \tag{100}$$

where $q^\mu = (1, \hat{q})$ and $\sqrt{-g} = r^3\sqrt{\gamma}$, with γ being the inverse metric on the unit sphere S^2. The properties are showed by:

$$\begin{aligned}
\Box G(x,\hat{q}) &= -\frac{\sqrt{\gamma(\hat{q})}}{4\pi}\partial_\nu\left[\frac{2x^\nu}{(q\cdot x)^2} - \frac{2x^\mu x_\mu}{(q\cdot x)^3}q^\nu\right] \\
&= -\frac{\sqrt{\gamma(\hat{q})}}{4\pi}\left[\frac{8}{(q\cdot x)^2} - \frac{4}{(q\cdot x)^2} - \frac{4}{(q\cdot x)^2} + \frac{6x^\mu x_\mu}{(q\cdot x)^4}q^2\right] = 0, \qquad (101)\\
G(x,\hat{q}) &= \frac{\sqrt{\gamma(\hat{q})}}{4\pi}\frac{u(u+2r)}{[u+r(1-\hat{q}\cdot\hat{x})]^2}
\end{aligned}$$

The second property being G given in retarded coordinates, for which it diverges if $\hat{q} = \hat{x}$ and vanishes if $\hat{q} \neq \hat{x}$, for large r. In the limit of $\mathcal{I}^-$ ($r \to \infty$), keeping $v = u + 2r = \text{const}$, G localises to the point $\hat{q} = -\hat{x}$; implying that the large gauge parameter in the $\mathcal{I}^-$ is antipodally related to one, in the $\mathcal{I}^+$ limit. This antipodal map is required by Lorentz invariance, since the gauge chosen is invariant.

5.6.1 Hyperbolic Slicing

Now we want to find the limit of the bulk gauge parameter onto i^+, by introducing a new set of coordinates. The retarded and advanced coordinates are good near null infinity, but not near past- and future timelike infinity, $i^\mp$. That's why one can use hyperbolic slicing of MS, with slices labelled by the coordinate:

$$\tau^2 = -x^\mu x_\mu = t^2 - r^2 \qquad (102)$$

which is invariant under boosts about the origin (it doesn't change τ). It maps about the origin as showed in *Figure 13*.
So surfaces of fixed τ, are mapped into themselves. These surfaces are Euclidean AdS_3 or $\mathbb{H}_3$ (hyperbolic three-space), for $\tau^2 > 0$, and for $\tau^2 < 0$, the surfaces are 3D dS_3. So to study time-like infinity, one should study these AdS_3 slices, and for spatial infinity, one should study the dS_3 slices. We define the parameter:

$$\rho = \frac{r}{\sqrt{t^2 - r^2}} \qquad (103)$$

as a radial coordinate on the $\mathbb{H}_3$ slices, where τ labels the slices. The Minkowski metric in these new coordinates takes the form:

$$ds^2 = -d\tau^2 + \tau^2\left[\frac{d\rho^2}{1+\rho^2} + \rho^2 d\Omega_2^2\right] \qquad (104)$$

The metric on the $\tau = \text{const}$ hypersurfaces are Euclidean AdS_3 ($\mathbb{H}$), which has symmetry group $SL(2,\mathbb{C})$. These Lorentz transformations thus act on each of these slices, mapping them into themselves. One can rewrite the Green's function, in terms of the new coordinates:

$$G(\tau,\rho,\hat{x};\hat{q}) = \frac{\sqrt{\gamma(\hat{q})}}{4\pi\left[\sqrt{1+\rho^2} - \rho\hat{q}\cdot\hat{x}\right]^2} \qquad (105)$$

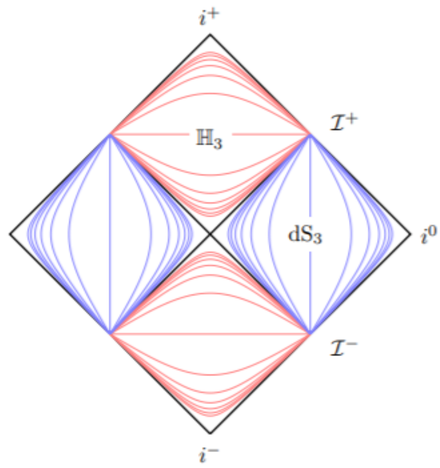


Figure 13: Hyperbolic slicing of Minkowski space. The slices correspond to constant $\tau^2 = t^2 - r^2$ surfaces. The red lines correspond to $\mathbb{H}_3$ slices and have $\tau^2 > 0$, whereas the blue lines correspond to the dS_3 slices with $\tau^2 < 0$. The figure is taken from [Str].

which is known in context of AdS_3, and is called the (3D) bulk-to-boundary propagator (for a massless scalar), relating quantities on boundaries, to those in the bulk. τ has dropped out of the equation ($\partial_\tau G = 0$), implying that the gauge parameter is τ-independent ($\partial_\tau \varepsilon = 0$). Hyperbolic slicing is thus really useful, as we see the holographic structure of AdS_3 in MS.
Considering a particle with constant momentum and which follows a trajectory, $\vec{r} = \frac{1}{E}\vec{p}t + \vec{r}_0$ for a fixed $\vec{r}_0$, as illustrated in *Figure 14*.

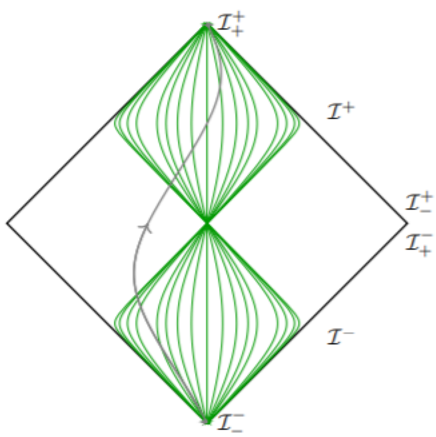


Figure 14: Hypersurfaces of constant ρ are shown in green. The grey line is the worldline of a massive particle moving at constant velocity. The worldline of a particle asymptotes to a surface of constant ρ as $\tau \to \infty$. The figure is taken from [Str].

We are interested in the phase associated to a such particle. Before we took our charges and surface integrals, and extended them as bulk-integrals near time-like infinity, but didn't consider the endpoints (nothing's up there). To have a closer look at the endpoints of time-like infinity, we take a slice at constant τ, and take a limit by letting $\tau \to \infty$. For a massive particle we have $m^2 = E^2 - \vec{p}^2$ and $\vec{x} = \frac{\vec{p}t}{E}$, and ($\tau$, ρ and $\hat{x}$) are given by:

$$\begin{aligned}
&\rho = \tfrac{r}{\sqrt{t^2-r^2}} = \tfrac{\left|\frac{1}{E}\vec{p}t+\vec{r}_0\right|}{\sqrt{t^2-\left(\frac{1}{E}\vec{p}t+\vec{r}_0\right)^2}},\\
&\tau = \sqrt{t^2-r^2} = \sqrt{t^2-\left(\tfrac{1}{E}\vec{p}t+\vec{r}_0\right)^2},\\
&\hat{x} = \tfrac{\vec{r}}{r} = \tfrac{\vec{p}t+E\vec{r}_0}{|\vec{p}t+E\vec{r}_0|},\\
&\rho^2 = \tfrac{r^2}{t^2-r^2} = \tfrac{\frac{|\vec{p}|^2t^2}{E^2}}{t^2-\frac{|p|^2t^2}{E^2}} = \tfrac{\vec{p}^2}{E^2-\vec{p}^2} = \tfrac{\vec{p}^2}{m^2}
\end{aligned} \tag{106}$$

At late times, $t \to \infty$, we have

$$\rho \to \frac{|\vec{p}|}{m}, \quad \tau \to \frac{m}{E}t, \quad \hat{x} \to \hat{p} \tag{107}$$

with $\hat{p} = \frac{\vec{p}}{p}$. In *Figure 14*, is shown the hypersurfaces of constant ρ, and it illustrates that any particle moving with constant velocity, asymptotes to one of these lines. The gauge transformation (parameter), ε, is constant along these lines, so in this gauge, it is known what kind of phase such particle should have; $e^{iQ\varepsilon(\rho,\hat{q})}$. The hard part of the large gauge charge is thus:

$$\begin{aligned}
Q_\varepsilon^{+H}|\vec{p}\rangle &= Q\varepsilon\left(\frac{|\vec{p}|}{m},\hat{p}\right)|\vec{p}\rangle\\
&= Q\varepsilon(z(p_1),\bar{z}(p_1))\,|\vec{p}\rangle
\end{aligned} \tag{108}$$

The action of the large gauge charge on a massive particle is thus proportional to the Lorenz gauge value of the gauge parameter, at the point (of $\mathbb{H}_3$), where the massive particle asymptotes to.

Looking back on *Section 2*, the following will be useful to reflect upon:

The $\mathbb{H}_3$ (Euclidean AdS_3) hypersurfaces, and the 3-dimensional de Sitter space dS_3, from hyperbolic slicing of MS (when discussing massive QED), has isometry group $SL(2,\mathbb{C})$ - a group of Lorentz transformations mapping the slices into themselves.

The 4D $SL(2,\mathbb{C})$ Lorentz invariance acts as the global 2D conformal group on the celestial sphere CS^2 (useful when rewriting the 4D Minkowski $\mathcal{S}$-matrix into a 2D correlator on CS^2).

Other than that, $SL(2,\mathbb{C})$ is the global conformal group of the two-sphere S^2 (representing points on the past- and future null infinity boundary, $\mathcal{I}^\pm$).

5.6.2 Soft Theorem

One can use this result to derive the Ward identity, which turns out to agree with the Weinberg soft theorem. The derivation is similar to the massless case, but the action of the charges on the initial and final states (*Eq. 62 and 63*), now has an extra term from massive states, such that:

$$
\begin{aligned}
\langle f| Q^+_\varepsilon = -2 \int d^2w \partial_{\bar{w}} \varepsilon \langle f| \partial_w N \\
+ \sum_{k\in \text{ massless}} Q_k \varepsilon \left(z_k, \bar{z}_k\right) \langle f| + \sum_{k\in \text{ massive}} Q_k \varepsilon \left(\frac{|\vec{p}_k|}{m_k}, \hat{p}_k\right) \langle f|
\end{aligned}
\tag{109}
$$

and the Ward identity becomes:

$$
\begin{aligned}
-2 \int d^2w \partial_{\bar{w}} \varepsilon \partial_w \langle f| \left[N(w, \bar{w})\mathcal{S} - \mathcal{S}N^-(w, \bar{w})\right] |i\rangle \\
= - \left[\sum_{k\in \text{ massless}} Q_k \varepsilon \left(z_k, \bar{z}_k\right) + \sum_{k\in \text{ massive}} Q_k \varepsilon \left(\frac{|\vec{p}_k|}{m_k}, \hat{p}_k\right)\right] \langle f| \mathcal{S} |i\rangle
\end{aligned}
\tag{110}
$$

Using the same procedure as for *Eq. 77* (set $\varepsilon(w, \bar{w}) = \frac{1}{z-w}$), one will find:

$$
\begin{aligned}
\frac{\sqrt{2}}{1+z\bar{z}} \lim_{\omega\to 0^+} \left[\omega \langle f| a_+(\omega \hat{x}(z, \bar{z}))\mathcal{S} |i\rangle\right] \\
= e \left[\sum_{k\in \text{ massless}} \frac{Q_k}{z - z_k} + \sum_{k\in \text{ massive}} Q_k \varepsilon \left(\frac{|\vec{p}_k|}{m_k}, \hat{p}_k\right)\right] \langle f| \mathcal{S} |i\rangle
\end{aligned}
\tag{111}
$$

Finally using $G\left(\frac{|\overrightarrow{p_k}|}{m}, \hat{p}_k; w, \bar{w}\right) = \frac{1}{2\pi} \partial_{\bar{w}} \left[\frac{\sqrt{2}}{1+w\bar{w}} \frac{p_k\cdot\varepsilon^+}{p_k\cdot\hat{q}}\right]$, one finds:

$$
\varepsilon\left(\frac{|\overrightarrow{p_k}|}{m}, \hat{p}_k\right) = \int d^2w G\left(\frac{|\overrightarrow{p_k}|}{m}, \hat{p}_k; w, \bar{w}\right) \frac{1}{z-w} = \frac{\sqrt{2}}{1+z\bar{z}} \frac{p_k \cdot \varepsilon^+(z, \bar{z})}{p_k \cdot \hat{q}(z, \bar{z})}
\tag{112}
$$

which can then be plugged back into the Ward identity (*Eq. 111*) to obtain the soft photon theorem for massive charged particles in the form of *Eq. 81.*

5.7 Magnetic Charges

A pole occurring, when a soft photon is attached to an external leg in a scattering process, leads to corrections (using LSZ). One can use electromagnetic duality transformations to find these corrections for the magnetic charges.
There are conservation laws for magnetic charges, just as for the electric charges. Something like $\tilde{Q}^+_\varepsilon = \int_{\mathcal{I}^+_-} \varepsilon F = \tilde{Q}^-_\varepsilon$ can be showed for the magnetic charges (by antipodal continuity of the magnetic field). This implies a new set of symmetries. It turns out to be the exact same set, as the infinite set of electric conservation laws; but in the usual electric theory, these symmetries cannot be understood as a nontrivial subgroup of the usual electric gauge symmetry - and vice versa if we write in a magnetic form. Asymptotic symmetries act simply, and are most naturally characterised in terms of

their action on the Hilbert space, described in the asymptotic region at null infinity. Not all asymptotic symmetries arise as a nontrivial subgroup of some manifest gauge symmetry.

One needs a theory that exists non-perturbatively; and QED, reigning over photons and electrons, does not exist non-perturbatively due to the Landau pole. We need a bigger theory, which often contain magnetic monopoles. We hope to see, that the soft photon theorem is corrected, when magnetic monopoles are present in the asymptotic states.
Consider a scattering process, where we create a monopole with magnetic charge g, so there is going to be some coupling to the photon (see *Figure 15*), as moving monopoles radiate their electromagnetic objects. One can use the LSZ method to truncate external lines, to get a propagator $\frac{1}{p\cdot q}$ times a vertex factor as corrections to the soft theorem.

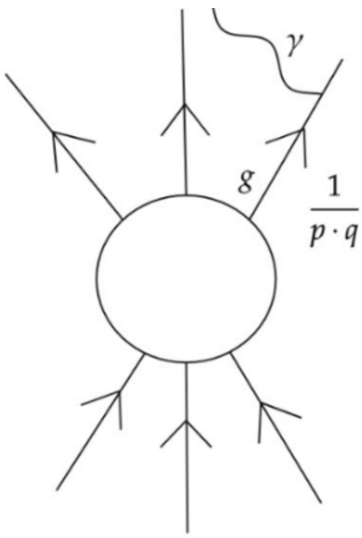


Figure 15: Feynman diagram with photon coupling to an external leg of a scattering process with a magnetic monopole.

Consider the perturbative electric soft factor:

$$S_0^\alpha = \sum_k \frac{eQ_k p_k \cdot \varepsilon^\alpha}{q \cdot p_k} \tag{113}$$

Consider a theory with magnetic monopoles; we have point particles with charges:

$$M_k = \frac{1}{2\pi} \int_{S_k^2} F \tag{114}$$

This is an integral over the two-sphere surrounding the charge. The pole in the soft factor is unchanged, so we only need to determine the vertex factor. Thus we transform to dual variables (field strength, coupling constant, and electric and magnetic charges):

$$\tilde{F} = -\frac{2\pi}{e^2} * F, \quad \tilde{e} = \frac{2\pi}{e}, \quad \tilde{Q}_k = \frac{1}{\tilde{e}^2}\int_{S_k^2} *\tilde{F} = M_k, \quad \tilde{M}_k = \frac{1}{2\pi}\int_{S_k^2} \tilde{F} = -Q_k \tag{115}$$

where the duality interchanges the electric and magnetic charges as $(\tilde{Q}_k, \tilde{M}_k) = (M_k, -Q_k)$. New electric charges are equal to old magnetic charges, and new magnetic charges are equal to minus the old electric charges. The magnetic charges couple to the dual electric field, the same way the electric charges couple to the original electric field. The dual gauge potential couples to magnetic charges as:

$$\tilde{F} = \mathrm{d}\tilde{A} = -\frac{2\pi}{e^2} * \ \mathrm{d}A \tag{116}$$

as the usual gauge potential does to electric charges $F = dA$, $A = e\varepsilon_\alpha e^{iq\cdot x}$. The polarization is related as:

$$A_\mu(q) = e\varepsilon_\mu^\alpha(q), \quad \tilde{A}_\mu(q) = \tilde{e}\tilde{\varepsilon}_\mu^\alpha(q) \tag{117}$$

or $\tilde{A} = \tilde{e}\tilde{\varepsilon}_\alpha e^{iq\cdot x}$. Thus the correction by the magnetic charges, to the soft formula must be:

$$S_0^\alpha = \sum_k \frac{p_k \cdot (Q_k e\varepsilon^\alpha + M_k \tilde{e}\tilde{\varepsilon}^\alpha)}{q \cdot p_k} \tag{118}$$

(for final states). It can be checked that the formula is invariant under electric-magnetic duality. Writing out the duality transformations in retarded coordinates near future null infinity, the expansions become:

$$\tilde{F}_{z\bar{z}}^{(0)} = \frac{2\pi i}{e^2}\gamma_{z\bar{z}} F_{ru}^{(2)}, \quad \tilde{F}_{ru}^{(2)} = \frac{2\pi i}{e^2}\gamma^{z\bar{z}} F_{z\bar{z}}^{(0)}, \quad \tilde{F}_{uz}^{(0)} = \frac{2\pi i}{e^2} F_{uz}^{(0)} \tag{119}$$

implying:

$$\tilde{A}_z^{(0)} = \frac{2\pi i}{e^2} A_z^{(0)} \tag{120}$$

corresponding to the radiative mode of the EM field with one polarization - transversely polarized, near null infinity. This implies that the duality transformation is just a multiplication by i - a phase shift by $\frac{\pi}{2}$, which is equivalent to exchanging the two fields (a 90 degree phase shift).

The soft factor for positive helicity is thus written as:

$$S_0^\alpha = \sum_k \frac{\left(eQ_k + \frac{2\pi i}{e} M_k\right) p_k \cdot \varepsilon^+}{q \cdot p_k} \tag{121}$$

For opposite helicity, a combination of $eQ_k + \frac{2\pi i}{e} M_k$ appears instead.

The outgoing magnetic charge (on $\mathcal{I}^+$) is defined by:

$$\tilde{Q}_\varepsilon^+ = \frac{1}{2\pi}\int_{\mathcal{I}_-^+} \varepsilon F = \frac{i}{2\pi}\int_{\mathcal{I}_-^+} d^2 z \varepsilon F_{z\bar{z}}^{(0)} \tag{122}$$

Using the Lorentz invariant matching condition:

$$F_{z\bar{z}}^{(0)}\Big|_{\mathcal{I}_-^+} = -\, F_{z\bar{z}}^{(0)}\Big|_{\mathcal{I}_+^-} \tag{123}$$

which implies:

$$\tilde{Q}_\varepsilon^+ = \tilde{Q}_\varepsilon^- = -\frac{i}{2\pi}\int_{\mathcal{I}_+^-} d^2 z \varepsilon F_{z\bar{z}}^{(0)} \tag{124}$$

states that outgoing charges (on $\mathcal{I}^+$) are equal to incoming charges on $\mathcal{I}^-$.

Thus there is a second infinity of conserved charges, when adding magnetic charges. What is the associated symmetry to these conserved charges? In the electric case, we constructed Dirac brackets and computed the action of the charge on the fields (assumed that $F_{z\bar{z}}^{(0)}$ vanished at $\mathcal{I}_\pm^+$, which cannot be done when magnetic charged particles are present). Computing the Poisson/Dirac brackets with magnetic charges present, with the same boundary conditions is not canonically shown.
Duality covariance gives an obvious guess for what symmetries these charges should generate; transformations under which $\tilde{A}$ shifts by $\partial_z\varepsilon$. One can figure out the commutator of the magnetic charges with the electric potentials A_z, using *Eq. 120*. These magnetic gauge transformations on the dual gauge fields should thus be:

$$\begin{aligned} \tilde{\delta}_\varepsilon \tilde{A}_z^{(0)} &= \partial_z \varepsilon, \\ \tilde{\delta}_\varepsilon A_z^{(0)} &= -\frac{ie^2}{2\pi}\partial_z\varepsilon \end{aligned} \tag{125}$$

the gauge potential is transforming as the electric gauge field, but with an imaginary gauge parameter $\tilde{\varepsilon} = -\frac{ie^2}{2\pi}\varepsilon$, and the original real $U(1)$ symmetry has been enhanced to a complex $U(1)$ symmetry. This complexification of the gauge group, enables one to find the electric- and magnetic gauge symmetry on the asymptotic fields, simultaniously and locally. The Ward identity of this complexified large gauge symmetry is precisely the full nonperturbative electric and magnetic soft photon theorem.
The large electric transformations/symmetries can be thought of as a non-trivial subgroup of the gauge group symmetry, but the magnetic symmetries can not. There might be a lot of asymptotic symmetries (acting nicely on asymptotic infinity), which could be understood as subgroups of some bulk gauge symmetry - however, it is not true for all.

6 Celestial Correlator and Non-abelian Gauge Theory (S)

6.1 The $\mathcal{S}$-matrix as a Celestial Correlator

So far, the convention used for describing scattering, has been the conventional description of an $\mathcal{S}$-matrix mapping an incoming- (initial) to an outgoing (final) Hilbert space. The Hilbert spaces are described as asymptotic, non-interacting energy-momentum eigenstates. However an alternate description can be used, namely that the scattering is a type of correlation function on the sphere, as illustrated in *Figure 16* - which can be

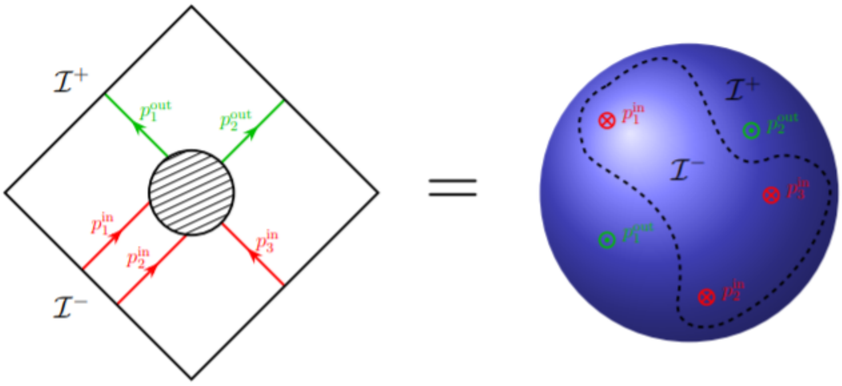


Figure 16: The 4D Minkowski $\mathcal{S}$-matrix, written as a 2D correlator on the celestial sphere. The dashed line on CS^2 separates regions associated with incoming- and outgoing particles. The figure is taken from [Str].

more convenient, both computationally and conceptually.
To be put more explicit; the 4D Minkowski $\mathcal{S}$-matrix for any given theory, can be written as a 2D correlator on the celestial sphere CS^2, parametrized by the asymptotic angle $(z, \bar{z})$ on $\mathcal{I}$.
Incoming and outgoing massless particles are respresented by operators at the location, where they enter $\mathcal{I}^-$ and exit $\mathcal{I}^+$, with the operators being labelled by quantum numbers, such as energy. The angles on past- and future null infinity are antipodally identified, such that free massless particles enters and exits at the same point on the celestial sphere. The 4D $SL(2, \mathbb{C})$ Lorentz invariance acts as the global 2D conformal group on CS^2.

Continuing the massless case, the incoming (in) and outgoing (out) massless particles are labelled by operators:

$$O_k(z, \bar{z}) \tag{126}$$

where the parameter

$$z = \frac{x^1 + ix^2}{r + x^3} \tag{127}$$

denotes a point on the sphere at $\mathcal{I}^\pm$, where a paticle of type k enters or exits the spacetime. Massive particles however, as discussed in *Section 5.6*, enter and exit timelike infinity $i^\pm$, at a definite point on the hyperboloid $\mathbb{H}_3$ (in *Eq. 107*), corresponding to operators that are smeared on CS^2, with weighting given by *Eq. 105*. Alternatively, the boost-eigenstate wave functions for massive particles, can be associated to a point on CS^2, by using the bulk-to-boundary propagator on $\mathbb{H}_3$.
One can thus express the n-particle scattering amplitudes in the form of a celestial

correlator on CS^2 as:

$$\langle f| \mathcal{S} |i\rangle \to \langle \mathcal{O}_1 (z_1, \bar{z}_1) \cdots \mathcal{O}_n (z_n, \bar{z}_n)\rangle . \tag{128}$$

The $\mathcal{S}$-matrix is simply rewritten in a different notation. We have switched to a convention, in which the sum of the four-momentum is zero, rather than the difference between the in- and out four momentum. As we are interested in Lorentz invariant theories, we write down how the 4D Lorentz symmetry, $SL(2,\mathbb{C})$, acts on the celestial sphere:

$$z \to \frac{az+b}{cz+d}, \quad ad - bc = 1 \tag{129}$$

$SL(2,\mathbb{C})$ is the global conformal group of the two-sphere. In QFT, scattering amplitudes must transform covariantly under the Lorentz group. Hence, these correlation functions must transform covariantly under the global conformal group $SL(2,\mathbb{C})$, and must look a lot like the correlation functions in a 2D conformal field theory (CFT) on the sphere. This becomes more clear, when the external particle wave functions, instead of being traditional plane waves, are taken to be $SL(2,\mathbb{C})$ primaries, labelled by their conformal dimensions and location on CS^2.

For quantum gravity, with issues involving IR finite quantum anomalies, the global conformal group gets enhanced to the full local conformal group $z \to w(z)$, appearing in 2D CFT. This could place constraints on the structure of scattering amplitudes. One may also attempt to find a 2D CFT whose correlators on CS^2 reproduce the $\mathcal{S}$-matrix of a four-dimensional quantum theory of gravity, which would provide a microscopic realization of the holographic principle in 4D flat space quantum gravity (or flat space holography).
If the QFT can be coupled to gravity, it suggests the celestial correlators are those of a 2D CFT. As an example, the tree-level celestial correlators involving soft gluons are those of a non-abelian two-dimensional Kac-Moody algebra.

6.2 Non-abelian Gauge Theory

Turning to the soft gluon (an exchange particle/gauge boson for the strong force between quarks) theorem, which applies to theories with a non-abelian gauge group $\mathcal{G}$. This theory has (quantum) corrections at the loop level, that are related to the running of the gauge coupling in the IR, affecting the coupling constants in front of the hard part of the charges. But even in $N = 4$ Yang-Mills theory, where the coupling does not run, there are one-loop exact corrections to the soft gluon theorem.

Soft theorems are important for controlling IR divergences, and getting IR finite inclusive cross sections. The missing uncorrected quantum soft theorem in the non-abelian case, is related to the fact that there is no known unitary $\mathcal{S}$-matrix for quantum non-abelian gauge theories.
One can define finite inclusive cross sections with an IR cutoff, which is acceptable for

most experimental applications, as they have an IR cutoff due to limits on detector sensitivity. It is a problem, e.g. for the (theoretical) studies of $N = 4$ YM scattering amplitudes (which do not exist). Understanding the IR symmetries might enable a construction of a finite unitary $\mathcal{S}$-matrix or a suitable replacement.

In this section we will stick to tree level for the non-abelian case, by using the formalism of celestial correlators. In this formalism, the scattering of a soft gluon becomes the insertion of a current into a correlation function on CS^2.
It will be shown that this current obeys the Ward identities of a $\mathcal{G}$-current algebra, giving an alternative representation of the asymptotic symmetry group. Furthermore, the infinity of non-abelian conserved charges will be constructed.

6.2.1 $\mathcal{G}$-Kac-Moody Algebra

Using said formalism, the asymptotic particles are represented by the operators $\mathcal{O}_k$ (the k'th representation of the non-abelian gauge group $\mathcal{G}$) on the celestial sphere, and the scattering amplitudes expressed by celestial correlators can be written as:

$$\langle f | \mathcal{S} | i \rangle = \langle \mathcal{O}_1 (E_1, z_1, \bar{z}_1) \cdots \mathcal{O}_n (E_n, z_n, \bar{z}_n) \rangle . \tag{130}$$

with E_n being the energy of the n'th particle. The generators of the gauge group, T^a, in the adjoint representation, of $\mathcal{G}$ satisfy:

$$\left[T_k^a, T_k^b \right] = i f^{abc} T_k^c \tag{131}$$

where the f^{abc} are the real, completely antisymmetric structure constants, normalized as:

$$f^{acd} f^{bcd} = \delta^{ab} = \mathrm{tr} \left[T^a T^b \right] \tag{132}$$

where the trace is over the suppressed color index.
The field strength, constructed from the gauge field $\mathcal{A}_\mu = \mathcal{A}_\mu^a T^a$, is written as:

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i \left[\mathcal{A}_\mu, \mathcal{A}_\nu \right] = \mathcal{F}_{\mu\nu}^a T^a \tag{133}$$

and obeys EOM:

$$\nabla^\nu \mathcal{F}_{\nu\mu} - i \left[\mathcal{A}^\nu, \mathcal{F}_{\nu\mu} \right] = g_{YM}^2 j_\mu^M \tag{134}$$

with g_{YM} being the gauge (or Yang-Mills) coupling, and j^M the matter color current. The non-abelian gauge transformations act on the gauge- and matter fields as:

$$\delta_\varepsilon \mathcal{A}_\mu = \partial_\mu \varepsilon - i \left[\mathcal{A}_\mu, \varepsilon \right], \quad \delta_\varepsilon \phi_k = i \varepsilon^a T_k^a \phi_k, \quad \delta_\varepsilon j_\mu^M = -i \left[j_\mu^M, \varepsilon \right] \tag{135}$$

Near future null infinity, the gauge field has the large-r expansion, just as in the abelian case:

$$\begin{aligned} \mathcal{A}_z(u, r, z, \bar{z}) &= A_z(u, z, \bar{z}) + \mathcal{O}\left(\tfrac{1}{r}\right) \\ \mathcal{A}_r(u, r, z, \bar{z}) &= \tfrac{1}{r^2} A_r(u, z, \bar{z}) + \mathcal{O}\left(\tfrac{1}{r^3}\right) \\ \mathcal{A}_u(u, r, z, \bar{z}) &= \tfrac{1}{r} A_u(u, z, \bar{z}) + \mathcal{O}\left(\tfrac{1}{r^2}\right) \end{aligned} \tag{136}$$

where the field strength has the expansion:

$$\mathcal{F}_{ur} = \frac{1}{r^2} F_{ur} + \mathcal{O}\left(\frac{1}{r^3}\right), \quad \mathcal{F}_{uz} = F_{uz} + \mathcal{O}\left(\frac{1}{r}\right), \quad \mathcal{F}_{z\bar{z}} = F_{z\bar{z}} + \mathcal{O}\left(\frac{1}{r}\right), \tag{137}$$

with the leading components given by:

$$F_{ur} = \partial_u A_r + A_u, \quad F_{uz} = \partial_u A_z, \quad F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z - i\left[A_z, A_{\bar{z}}\right] \tag{138}$$

The gauge field expansions (asymptotics) allow large gauge transformations to be infinitesimally generated by:

$$\delta_\varepsilon A_z(u, z, \bar{z}) = D_z \varepsilon(z, \bar{z}) \tag{139}$$

The scattering problem becomes well defined if we impose the antipodal boundary condition:

$$A_z|_{\mathcal{I}^+_-} = A_z|_{\mathcal{I}^-_+} \tag{140}$$

along with $F_{ru}|_{\mathcal{I}^+_-} = F_{rv}|_{\mathcal{I}^-_+}$. This is preserved by large gauge transformations, provided:

$$\varepsilon(z, \bar{z})|_{\mathcal{I}^+_-} = \varepsilon(z, \bar{z})|_{\mathcal{I}^-_+} \tag{141}$$

Hence, given any one solution of the scattering problem, infinitely more may be generated by acting on both the initial and final data with such a transformation. As in the abelian case, the infinity of conservation laws is an immediate consequence of the need for a matching condition.
In this notation, the tree-level non-abelian soft theorem can be written as:

$$\langle \mathcal{O}_1(p_1) \cdots \mathcal{O}_n(p_n)\, \mathcal{O}^a(q, \varepsilon) \rangle_{U=1} = g_{YM} \sum_{k=1}^{n} \frac{p_k \cdot \varepsilon}{p_k \cdot q} \langle \mathcal{O}_1(p_1) \cdots T^a_k \mathcal{O}_k(p_k) \cdots \mathcal{O}_n(p_n) \rangle_{U=1} + \mathcal{O}\left(q^0\right) \tag{142}$$

where the $U = 1$ is for assuming a flat color connection on the sphere, with color connection $A_z = U^{-1}\partial_z U$ with $U \in G$ on the S^2.
We have n incoming (and outgoing) particles, and a gluon on the $(n+1)$ spot, represented by the soft gluon operators $\mathcal{O}^a(q, \varepsilon)$ (with momentum q, color index a, and polarization ε). On the RHS, the k'th particle is acted on by the gauge group generator.
A pair of quarks near $\mathcal{I}^+$, which were initially in a color singlet, will generically not be a singlet at late times, as the color frame changes. This is the "color memory" effect.
The soft gluon operator at $\mathcal{I}^\pm$ is defined as:

$$\begin{aligned} N_z &= \int_{-\infty}^{\infty} du F_{uz} = A_z|_{\mathcal{I}^+_+} - A_z|_{\mathcal{I}^+_-} \\ N^-_z &= \int_{-\infty}^{\infty} dv F^-_{vz} = A^-_z|_{\mathcal{I}^-_+} - A^-_z|_{\mathcal{I}^-_-} \end{aligned} \tag{143}$$

We define soft gluon current as:

$$J_z = -\frac{4\pi}{g^2_{YM}}\left(N_z - N^-_z\right) \tag{144}$$

One can now take these new operators, which involved integrals over time, and rewrite them in terms of creation and annihilation operators to create this current. Then trading the momenta p_k for z_k, and q for z, the soft theorem can be rewritten as:

$$\left\langle J_z^a \mathcal{O}_1\left(z_1, \bar{z}_1\right) \cdots \mathcal{O}_n\left(z_n, \bar{z}_n\right)\right\rangle_{U=1} = \sum_{k=1}^{n} \frac{1}{z-z_k}\left\langle \mathcal{O}_1\left(z_1, \bar{z}_1\right) \cdots T_k^a \mathcal{O}_k\left(z_k, \bar{z}_k\right) \cdots \mathcal{O}_n\left(z_n, \bar{z}_n\right)\right\rangle_{U=1} \tag{145}$$

This is in 2D CFT the formula for a Kac-Moody current into a correlation function, or the Ward identity of a holomorphic Kac-Moody symmetry.
Now we multiply both sides of the formula by ε, and integrate over some arbitrary contour, C in the sphere. Let's consider the weighted integral of the current around said contour on CS^2:

$$J_C[\varepsilon] = \oint_C \frac{dz}{2\pi i} \operatorname{tr}\left[\varepsilon J_z\right] \tag{146}$$

with $\varepsilon(z)$ being a holomorphic function in the interior of the contour. Inserting these operators (which generates gauge transformations ε inside the contour), one obtains:

$$\left\langle J_{\mathcal{C}}[\varepsilon] \mathcal{O}_1 \cdots \mathcal{O}_n\right\rangle_{U=1} = \sum_{k \in C}\left\langle \mathcal{O}_1 \cdots \varepsilon^a\left(z_k\right) T_k^a \mathcal{O}_k \cdots \mathcal{O}_n\right\rangle_{U=1} \tag{147}$$

with $\varepsilon_k = \varepsilon^a T_k^a$. This Ward identity relates $U=1$ amplitudes to the more general case. Consider an infinitesimal change in the flat connection:

$$\delta U(z, \bar{z}) = i\varepsilon(z, \bar{z}) + \ldots \tag{148}$$

choosing $\varepsilon = 0$ outside of the contour. The change in the correlator is thus simply given by a large gauge transformation of the operators themselves:

$$\begin{aligned} \delta_{\varepsilon}\left\langle \mathcal{O}_1 \cdots \mathcal{O}_n\right\rangle_{U=1} &\equiv \left\langle \mathcal{O}_1 \cdots \mathcal{O}_n\right\rangle_{U=1+i\varepsilon} - \left\langle \mathcal{O}_1 \cdots \mathcal{O}_n\right\rangle_{U=1} \\ &= i \sum_{k \in \mathcal{C}}\left\langle \mathcal{O}_1 \cdots \varepsilon^a\left(z_k\right) T_k^a \mathcal{O}_k \cdots \mathcal{O}_n\right\rangle_{U=1} \\ &= i\left\langle J_C[\varepsilon] \mathcal{O}_1 \cdots \mathcal{O}_n\right\rangle_{U=1} \end{aligned} \tag{149}$$

and this relation can be characterized by:

$$J_C[\varepsilon] = \int_{R_C} d^2 z \gamma_{z\bar{z}} \varepsilon^a \frac{\delta}{\delta U^a} \tag{150}$$

Thus, an insertion of J_C will generate locally holomorphic large gauge transformations on the boundary, which are characterized by a general flat connection $A_z = U^{-1}\partial_z U$.

6.2.2 Conserved Charges

We could have also formulated this problem as an infinity of charged conservation laws. Defining the charge:

$$Q_{\varepsilon}^{+} = \frac{1}{g_{YM}^2} \int_{\mathcal{I}_-^+} \operatorname{tr}[\varepsilon * F] \tag{151}$$

integrating it by parts, and writing it as an integral over $\mathcal{I}^+$, the final expression will contain a soft and hard term $Q^+_\varepsilon = Q^-_\varepsilon = Q^{+S}_\varepsilon + Q^{+H}_\varepsilon$, and the soft term will be related to the current by:

$$J_\mathcal{C}[\varepsilon] = Q^{-S}_\varepsilon - Q^{+S}_\varepsilon \tag{152}$$

This operator generates transformations on anything inside the contour.

7 The B.M.S. Group (P)

In this section I will focus on the attempt of finding the BMS group, by the approach of Penrose. In *Section 9*, I will explain, how Strominger went about his method, and then compare the two approaches in *Section 11*.

The generators of $\mathcal{I}^\pm$ has a conformal structure, which is essential for the definition of the BMS group. The definition of asymptotic flatness states, that a conformal infinity $\mathcal{I}$ exists with the described structure as in *Section 3.9*. This is ensured by weak asymptotic simplicity and the Einstein-Maxwell equations (with no cosmological term) holding near $\mathcal{I}$.

7.1 Group of Symmetries of Conformal Infinity

In special relativity (SR), the Poincaré group is the group of symmetries of MS, preserving its metric structure. In general relativity (GR) no such group arises, as the group of symmetries of a general space-time (the 'general coordinate group', being the group of diffemorphisms of the space-time, only preserves smoothness). Restricting to an asymptotically flat space-time, the new concept of an asymptotic symmetry group arises. The asymptotic symmetry group, the BMS group (by Bondi, Metzner, and Sachs), was defined as the group of transformations between asymptotically flat coordinate systems of a certain type.

Consider MS; transformations belonging to the Poincaré group are metric preserving, hence conformal. They thus induce transformations of $\mathcal{I}$ to itself, which also are conformal, as $\mathcal{I}$ remains invariant. Thus the Poincaré group is the group of conformal symmetries, of Minkowski conformal infinity.

Let's take the group of conformal symmetries of conformal infinity $\mathcal{I}$ of an asymptotically flat space-time, as an example.
What structure of $\mathcal{I}$ is to be preserved by transformations of this group? Its inner conformal metric is too weak a structure, as $\mathcal{I}$ is a null hypersurface. Distances along generators of $\mathcal{I}$ are zero, and their ratios cannot be assigned. The group of self-transformations of $\mathcal{I}$, preserving the weak inner conformal metric, have some significance as an asymptotic symmetry group. It is larger than the BMS group, and is referred to as the Newman-Unti group. To aquire the BMS group, one needs to strengthen the structure of $\mathcal{I}$.

7.2 The Inner Conformal Metric of $\mathcal{I}$

First we want to examine the nature of the inner conformal metric of $\mathcal{I}$, and the self-transformations preserving it.
Asymptotically flat space-times, M, described as a disjoint union of two smooth hypersurfaces $\mathcal{I}^{\pm}$; with $\mathcal{I}^{-}$ consisting of past endpoints of null geodesics in M, and $\mathcal{I}^{+}$ of future endpoints. The topology of $\mathcal{I}^{\pm}$ is $S^2 \times \mathbb{R}$, where the $\mathbb{R}$'s are the null geodesic generators of $\mathcal{I}^{\pm}$ - they are **shear-free**, ie. they establish a conformal mapping between any two S^2 cross-sections of either $\mathcal{I}^{-}$ or $\mathcal{I}^{+}$.
Theorem: Any conformal 2-surface with the topology of a sphere S^2, is conformal to the unit 2-sphere in Euclidean 3-space.
One can thus assume Ω to be chosen, such that the cross-section S of $\mathcal{I}^{+}$ has the (unphysical/rescaled) metric $d\hat{s}^2$ of a unit 2-sphere. Given one Ω, one can make the choice $\Omega' = \theta\Omega$, which also vanishes at $\mathcal{I}$; with a non-zero gradient. With θ being an arbitrary smooth, positive function on $\mathcal{I}$ - one can rescale the metric on $\mathcal{I}$ as pleased. One can also use this to scale $d\hat{s}$ along every generator of $\mathcal{I}^{+}$, such the divergence of the generators vanishes. Hence, a continuous succession of cross-section of $\mathcal{I}^{+}$ can have metrics agreeing with S, mapped along the generators. One can thus assign the metric of $\mathcal{I}^{+}$ to be:

$$dl^2 = -d\hat{s}^2 = d\theta^2 + \sin^2\theta d\phi^2 + 0.du^2, \tag{153}$$

with θ, ϕ being spherical polar coordinates for S and are constant along generators of $\mathcal{I}^{+}$. u is the parameter defined along each generator of $\mathcal{I}^{+}$, for where surfaces $u = \text{const}$ are cross-sections of $\mathcal{I}^{+}$. Every cross-section, given by $u = f(\theta, \phi)$ of $\mathcal{I}^{+}$, has the metric of a unit 2-sphere. The same story goes for $\mathcal{I}^{-}$, but with v instead of u.

7.3 Group of Self-Transformations, Preserving its Inner Conformal Metric

Now consider the group of self-transformations of $\mathcal{I}^{+}$, preserving its inner metric. Any smooth transformation of $\mathcal{I}^{+}$ to itself, sending each generator into itself (while preserving the orientation on each generator) is allowed. These are given by:

$$\theta \to \theta, \quad \phi \to \phi, \quad u \to F(u, \theta, \phi), \tag{154}$$

F being smooth, with $\frac{\partial F}{\partial u} > 0$. The metric (*Equation 153*) is unchanged. One can also allow conformal transformations of the (θ, ϕ)-sphere to itself ($C(2)$).
Introducing variable shift $\zeta = e^{i\phi}\cot\left(\frac{\theta}{2}\right)$, the metric becomes (see *Equation 5*):

$$dl^2 = \frac{4d\zeta d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2} + 0.du^2 \tag{155}$$

The conformal maps of the sphere are given by (see *Equation 6*);

$$\zeta \to \tilde{\zeta} = f(\zeta) = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}, \tag{156}$$

Thus the general (Newman-Unti) transformations take the form:

$$\zeta \to \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}, \quad u \to F(u, \theta, \phi), \tag{157}$$

with $\frac{\partial F}{\partial u} > 0$. This is called the Newman-Unti group.

There is a lot of freedom in the function F, that we want to reduce. One can assign more geometrical structure to $\mathcal{I}^+$. The preservation of this additional structure reduces the freedom in F to a function of only θ, ϕ; and one will then obtain the BMS group.

7.4 Geometrical Interpretation of a Point of $\mathcal{I}$

Assume M is MS; each point p of $\mathcal{I}^+$ is associated with a null-hyperplane π in M. Every generator of π is a null geodesic, attaining the same future end-point p on $\mathcal{I}^+$. π can be physically interpreted as the constant phase hypersurface in a plane-wave, parallel to null hyperplanes/phase hypersurfaces belonging to the same plane-wave. They will terminate at other endpoints on $\mathcal{I}^+$, but the totality for these points will constitute the generators γ of $\mathcal{I}^+$ through p (of given plane wave/null direction in MS). Parallel null geodesics will terminate on $\mathcal{I}^+$ at points of one generator of $\mathcal{I}^+$.
Different null geodesics through p (apart from γ) are the generators of π. For fixed π, the space of generators has the structure of an Euclidean plane E^2, as any cross section of π has the intrinsic metric of an Euclidean plane; the projection along generators of π maps these planes isometrically.

In the neighbourhood of p, the null geodesic generates the past light cone of p. One can see the Euclidean structure, if we take a parabolic section of the past null-cone of p, by null 3-plane near p, parallel to γ - see *Figure 17* .
This is essentially the same situation as in *Figure 4*. (E^2 planes are conformally related to the unit sphere S^2, established through generators ν). The $-d\hat{s}^2$ metric is that of a Euclidean plane, conformal to E^2.
One can refer to the E^2 structure of the generator space of π, to tangent space to $\mathcal{I}^+$ at p. Any generators ν of π (null geodesics through p) is associated with a 2-plane element at p, spanned by directions γ, ν at p. γ is normal to $\mathcal{I}^+$ at p, so the orthogonal complement is also a 2-plane element, N, tangent to $\mathcal{I}^+$ (with no direction in γ); and is characterised by it being tangent to $\mathcal{I}^+$ and orthogonal to ν.
Thus the tangent 2-planes to $\mathcal{I}^+$ can be used to represent null geodesics in M (no matter if M is flat, curved or asymptotically flat).

Consider a 3-plane element at p, containing the direction γ; intersections with the null cone at p gives a 1D system of null geodesics in π - with M being MS, this corresponds to a straight line in E^2. The orthogonal complement is a line-element at p, tangent to $\mathcal{I}^+$ (with no direction γ).

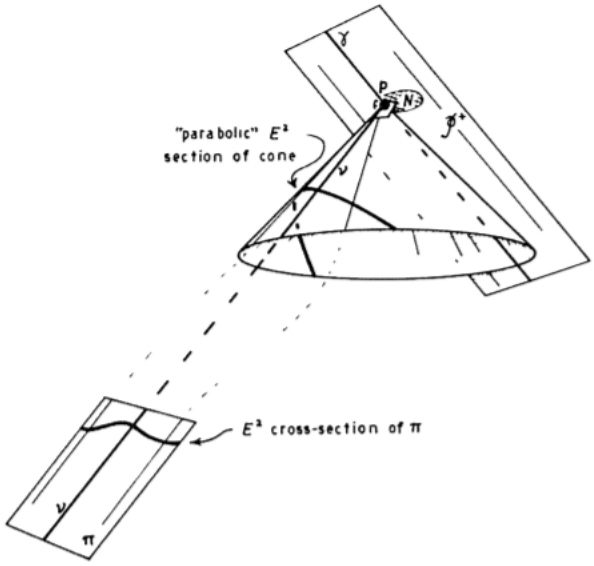


Figure 17: The Euclidean plane being embedded as a parabolic section of a null cone. The null hyperplane, π, becomes the past light cone of p. The generator, ν, is represented by an orthogonal 2-plane, N, tangent to $\mathcal{I}^+$ at p. The figure is taken from [Pen74].

Thus line-elements at p represent straight lines in E^2. Directions tangent to $\mathcal{I}^+$ at p represent oriented straight lines in E^2.

7.5 Geometrical Interpretation for an Asymptotically Flat M

For an asymptotically flat M, the past light cone of a point p on $\mathcal{I}^+$ (the locus of a null geodesic in M, terminating at p) is a null hypersurface. π in M, being an asymptotic plane in the future, is a constant phase hypersurface of an outgoing asymptotic plane wave, where different constant phase hypersurfaces for one wave, will be the past light cones with vertices on one generator of $\mathcal{I}^+$. However cross-sections of π are not Euclidean planes now, and different cross-sections are not isometric with one another. But taking the limit of these cross-sections, as they recede/go to the future, one can recover an exact Euclidean plane E^2 to represent the space of generators of π (see *Figure 17*). Parabolic sections of cones of p are considered to be in the tangent space at p - thus one can get an exact E^2 structure, by taking the cone as close to $\mathcal{I}^+$ as possible/pleased. In tangent space, the **parabolic section** is given by equation $\hat{x}^a\hat{n}^a = -1$, with $\hat{x}^a$ being the position vector of a point on the section (obeying $\hat{x}^a\hat{x}^b\hat{g}_{ab} = 0$, so it lies on the null cone), and the null vector $\hat{n}^a$ (See last part of *Section 3.9*) is given by $\hat{n}^a = -\bar{\nabla}_a\Omega$ at p.

Now consider a point p' on π lying in the remote future along π, and just to the past of p on the light cone of p. Label p' as the position vector $d\hat{x}^a$, relative to p. The conformal factor at p' is thus; $d\Omega = d\hat{x}^a\hat{\nabla}_a\Omega$, as $\Omega = 0$ at p.

To go from physical to unphysical distances at p', one must divide with above factor, which is equivalent to expanding by the factor $\hat{x}^a : d\hat{x}^a$, $\hat{x}^a$ being in the same null direction from p as p'.
To measure the distance between points on π, lying in the remote future (near to p), one can refer to the parabolic section E^2. It is only relevant on which generator of π the point lies, not its distance to said generator.
Thus the E^2 section describes the geometry of the cross-section of π, in the limit of the cross-section receeding to the remote future.

Null geodesics in M is represented by 2-plane elements, tangent to $\mathcal{I}^+$, as for MS. A line-element tangent to $\mathcal{I}^+$ at p (not in direction of γ) represents a 1D system of generators of π, corresponding to a straight line in E^2 - asymptotically they generate a null 2-plane in π. A tangent direction to $\mathcal{I}^+$ at p, correspond to an oriented straight line in E^2.

7.6 The Strong Conformal Geometry

Now consider the relevant structure of $\mathcal{I}^+$. The conformal geometry of $\mathcal{I}^+$ gives rise to the definition of an angle between two line-elements, tangent to $\mathcal{I}^+$ at p. These line-elements are represented by straight lines in E^2; thus the angle corresponds to the Euclidean angle between the two intersecting straight lines in E^2. When the 2-plane, spanned by the line-elements, contains the direction of γ (and is therefore not lying in this direction), the angle is zero. The separation of this null angle (with units of distance/time) describes the distance in E^2 between the parallel straight lines. See *Figure 18*.
The inner conformal metric of $\mathcal{I}^+$ assigns a measure of angle between two (non-null) tangent directions to $\mathcal{I}^+$. When the angle is zero, one needs the strong conformal geometry of $\mathcal{I}^+$ to define the null angle between the directions, which are represented by straight lines in E^2. Thus finite angles are obtained as angles between the lines, and null angles as the distance between them, when the angle is zero.

Thus a physically meaningful 'strong conformal geometry' has been added to $\mathcal{I}^+$. The concept of angle between (non-null) tangent directions at a point of $\mathcal{I}^+$, comes from the content of the inner conformal metric of $\mathcal{I}^+$. The strong conformal geometry assigns a measure of separation, the null angle, if the angle is zero.

Another way of specifying strong conformal geometry of $\mathcal{I}^+$ is to replace Ω by $\Omega' = \theta\Omega$ (θ being smooth and positive on M'), then the normal vector $\hat{n}^a = \hat{g}^{ab}\hat{\nabla}_b\Omega$ becomes:

$$\hat{n}'^a = \hat{g}'^{ab}\hat{\nabla}'_b\Omega' = \theta^{-2}\hat{g}^{ab}\hat{\nabla}_b(\theta\Omega) = \theta^{-1}\hat{n}^a \tag{158}$$

on $\mathcal{I}^+$, as $\Omega = 0$.
Now set $dl^2 = -d\hat{s}^2$, such that: $\hat{n}^a dl = \hat{n}'^a dl'$. The quantity $\hat{n}^a dl$ or $\hat{n}^a\hat{n}^b d\hat{s}^2$ defines the invariant structure on $\mathcal{I}^+$. Vectors $\hat{n}^a$ are tangent to the generators of $\mathcal{I}^+$; and define

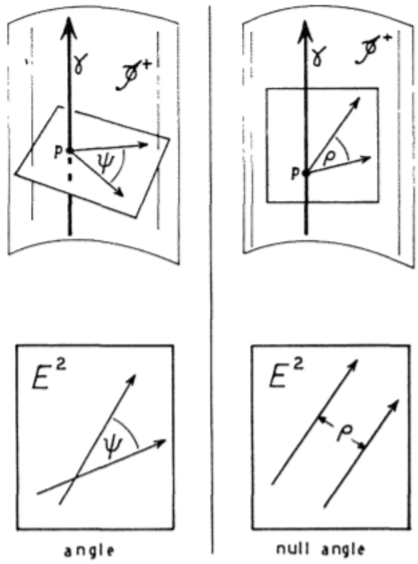


Figure 18: Angles between two tangent directions to $\mathcal{I}^+$. The plane in the top right corner contains the direction of γ, and the angle is therefore zero. The figure is taken from [Pen74].

for the special parameter u, along the generators:

$$\frac{\partial}{\partial u} = \hat{n}^a \hat{\nabla}^a \Rightarrow \hat{n}^a \hat{\nabla}^a u = 1 \tag{159}$$

Thus the inverse invariance of $\hat{n}^a dl$ gives the invariance of the ratio; $\frac{du}{dl}$, the null angle. See *Figure 19.*

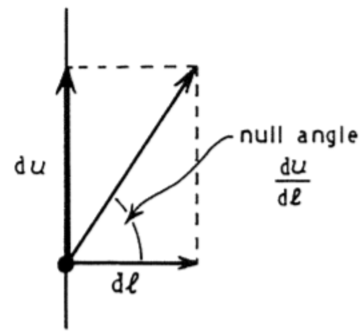


Figure 19: The null angle. The figure is taken from [Pen74].

One can make choice of metric dl for the cross-sections of $\mathcal{I}^+$. This singles out a specific scaling for u. Choosing dl to be that of a unit sphere, the entire metric takes the form (see *Equation 153*):

$$dl^2 = -d\hat{s}^2 = d\theta^2 + \sin^2\theta d\phi^2 + 0.du^2, \tag{160}$$

The scalings for u, defined by the null angle, $\frac{du}{dl}$, are thus fixed. The arbitrariness of u lies in fixing the origin of the u-coordinate on each generator of $\mathcal{I}^+$.

7.7 Transformations of the BMS Group

The transformation of $\mathcal{I}^+$ to itself, preserving $\hat{n}^a dl$, ie. angles and null angles, must comply that any expansion or contraction of spatial distances dl, must be accompanied by an equal expansion/contraction of the u-parameter scaling. Allowed transformations must have the form of *Equation 157*, as they preserve the inner conformal metric; but F must now have a form, allowing invariance of the ratio $\frac{du}{dl}$.
The sphere of the cross-section of $\mathcal{I}^+$ undergoes a conformal mapping $dl \to Kdl$, with K being a positive function of either θ, ϕ or $\zeta, \bar{\zeta}$. This implies $du \to Kdu$.
As K is independent of u, u transforms as $u \to K(u + a(\zeta, \bar{\zeta}))$; a being a real function defined on the ζ-sphere. Using *Eq. 155* and *Eq. 157* (conformal transformations of the sphere), we get:

$$K = \frac{1 + \zeta\bar{\zeta}}{(|\alpha\zeta + \beta|^2 + |\gamma\zeta + \delta|^2)}, \tag{161}$$

the complex parameters $\alpha, \beta, \gamma, \delta$ still upholding $\alpha\delta - \gamma\beta = 1$. The general form of the transformation becomes:

$$\zeta \to \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}, \quad u \to \frac{(1 + \zeta\bar{\zeta})(u + a(\zeta, \bar{\zeta}))}{(|\alpha\zeta + \beta|^2 + |\gamma\zeta + \delta|^2)} \tag{162}$$

with a being suitably smooth on the sphere. These transformations define the BMS group; the group of self-transformations of $\mathcal{I}^+$ which preserves the strong conformal geometry.

Transformations of the form:

$$\zeta \to \zeta, \quad u \to u + a(\zeta, \bar{\zeta}) \tag{163}$$

are the supertranslations, sending each generator of $\mathcal{I}^+$ into itself, shunted along itself by an amount $a(\zeta, \bar{\zeta})$. Among these are the 'normal' translations, where a is composed only of spherical harmonics of order 0 and 1;

$$a(\zeta, \bar{\zeta}) = \frac{A + B\zeta + \bar{B}\zeta + C\zeta\bar{\zeta}}{1 + \zeta\bar{\zeta}}, \tag{164}$$

with A, C being real, and B complex.
Supertranslations form an infinite parameter (largest proper normal) subgroup of the BMS group, and the translations form a four-parameter (normal) subgroup (the only of this kind for the BMS group).

The factor group of the BMS group by the group of supertranslations (the orthochronous Lorentz group), is the conformal group on the sphere S^2 (which is the space of generators of $\mathcal{I}^+$). The translation group is obtained canonically as a subgroup of BMS, where the Lorentz group only is canonically obtained as a factor group (by the infinite-parameter abelian group of supertranslations). They can not be canonically fitted together to obtain the Poincaré group from the BMS group.
For the Poincaré group, the Lorentz group is a factor group, but by the four-parameter abelian group of translations. Thus both the (orthochronous) Poincaré and BMS group is a semi-direct product of the orthochronous Lorentz group with the translation (supertranslation) group. But this does not allow the Poincaré group to be a subgroup of the BMS group in a canonical way.
This discussion will be continued in *Section 8*.

7.8 Twistor Theory

To further investigate the BMS group and its implications, Penrose had also studied Twistor theory, which I will only shortly comment upon, to reflect on its relation to conformal groups and conditions of the BMS group.

This formalism is built upon twistors, rather than space-time points or geometry to express physical properties.
Basic twistors form a complex 4D vector space (for flat-space twistors) on which $SU(2,2)$ matrices act, which corresponds to conformal transformations on (compactified) MS, according to the local isomorphism $SU(2,2) \to C(1,3)$. Such twistor may be physically interpreted in classical theory as a zero rest-mass particle with intrinsic spin. Space-time points are interpreted as subspaces of twistor-space.
In relation to the shear-free condition on congruences of null geodesics; The twistor-space complex structure is related via. a contour integration to the zero rest-mass free-field equations. This leads to a twistor formalism involving complex contour integration, for canculations of quantum scattering amplitudes and massless quantum electrodynamics, which appears to be free of divergences.
The holomorphic nature of functions on twistor space is closely related to the shear-free condition.

8 BMS and Poincaré Group of Minkowski Space

This section will go through, how the Poincare group arises as subgroup of BMS group for MS.

To identify the Poincaré and BMS group, we will make use of following notations, of the five groups involved:

- BMS (BMS group)

- ST (group of supertranslations)
- T (group of translations)
- L (Lorentz group)
- P (Poincaré group)

When considering two groups how can one compare them? How do you express the fact that one of them is "inside" another?
Let's look at these examples in the case of the Poincare group. The Poincare group contains the group of translations and the group of rotations. Mathematically, this means that there are homomorphisms: $L \to P$ and $T \to P$. In other words, one can find a copy of Lorentz transformations and of translations inside the Poincare group.

Next, a natural question is, can we go "the other way"? In other words, given an element of P, can we project it back to L or to T? This is where rotations and translations differ, because we can sort of do this for rotations but not for translations.

A translation of the coordinate x, by an amount a, is given by: $x \to x + a$
A Lorentz transformation is given by: $x \to \Lambda x$

If we take $\Lambda^{-1} a \Lambda$ (an inverse rotation, a translation, and a rotation), we get:

$$x \to \Lambda x \to \Lambda x + a \to \Lambda^{-1}(\Lambda x + a) = x + \Lambda^{-1} a$$

This is a translation.

If we instead take the quantity $a^{-1} \Lambda a$ (an inverse translation, a rotation, and a translation), we get:

$$x \to x + a \to \Lambda(x + a) \to \Lambda(x + a) - a$$

This does not amount to a rotation.

Translations and rotations work quite differently, and is the reason why you can find T as a normal subgroup of P, but you can't do the same for L.

A subgroup, which has the property that T is inside P, is called a normal subgroup. For such groups we can define a quotient between two groups (which is also a group). In our example we have a sequence of group homomorphisms:

$$T \to P \to P/T$$

So T is embedded in P, and when taking the quotient by it we obtain a factor subgroup P/T. This factor subgroup is just the Lorentz group.

However, you can not do the analogous construction with the Lorentz group, L, instead of the group of translations:

$$L \to P \to P/L \quad \text{(does not exist!)}$$

since L is not a normal group in P.

8.1 Group- and Symmetry Diagram

Now back to the five groups we started with. We have that

$$T \to P \to L(= P/T)$$

$$ST \to BMS \to L = (BMS/ST)$$

$T \to P$ means that T is embedded in (or sits in) P. One can get the factor subgroup $L = P/T$, as we can identify a group of T as a normal subgroup of the Poincaré group. We have the same story for $T \to ST$; T is a normal subgroup, and therefore you can take the quotient ST/T.

Similarly, there is a map $P \to BMS$, where you can take the quotient BMS/P, and moreover ST/T should be isomorphic to BMS/P.
However, while we have an arrow $P \to BMS$, there is no arrow in the opposite direction, which is what would be required, to identify the Poincare subgroup of the BMS group (one cannot project it back).

Several of these relations between the BMS and Poincaré group can be seen in *Figure 20.*

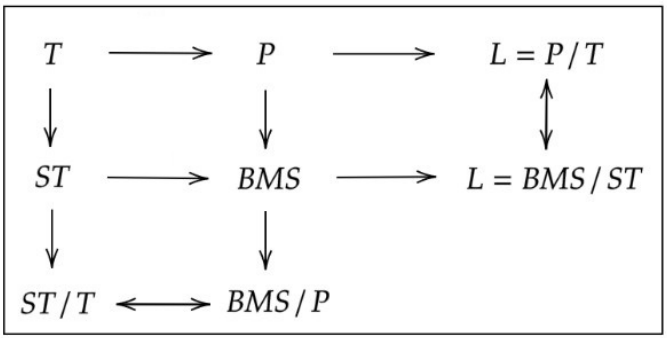


Figure 20: Diagram of relations between group elements of the BMS- and Poincaré group.

The way these groups fit together has some physical implications as well. Penrose dis-ussed this as well, which is that there is a difficulty in defining the angular momentum. One cannot uniquely define the Lorentz transformations of the BMS group.
The problem of identifying a particular subgroup of the BMS group as the (restricted)

Poincare group, lies in deciding which elements of the BMS group are to be regarded as "supertranslation-free" rotations (or Lorentz rotations).
The translations may be distinguished from the remaining supertranslations in a BMS-invariant way. However, Lorentz rotations cannot be so distinguished from the general BMS rotations.

9 The BMS Group and Scattering for Gravity (S)

This section will go through Strominger's approach of finding the BMS group.

9.1 Asymptotically Flat Spacetimes

Flat MS in retarded coordinates near future null infinity is described by the metric:

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \tag{165}$$

Working in Bondi coordinates $(u, r, z, \bar{z})$ with $\Theta^A = (z, \bar{z})$, the metric takes the (asymptotic) form:

$$\begin{aligned} ds^2 &= -Udu^2 - 2e^{2\beta}dudr + g_{AB}\left(d\Theta^A + \frac{1}{2}U^A du\right)\left(d\Theta^B + \frac{1}{2}U^B du\right) \\ &\partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0 \end{aligned} \tag{166}$$

being the Bondi gauge. Asymptotically flat means that, at generic points in the middle of null infinity, the metric should go like:

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &+ \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}\bar{z}^2 + D^zC_{zz}dudz + D^{\bar{z}}C_{\bar{z}\bar{z}}dud\bar{z}, \\ &+ \frac{1}{r}\left(\frac{4}{3}\left(N_z + u\partial_z m_B\right) - \frac{1}{4}\partial_z\left(C_{zz}C^{zz}\right)\right)dudz + \text{ c.c. } + \dots, \end{aligned} \tag{167}$$

(the first three terms being the flat Minkowski metric - and the fourth term containing the Bondi mass aspect, m_B - the rest of the terms are subleading terms), with $U_z = D^zC_{zz}$ (C_{zz} being a Weyl tensor component, see *Section 18.1* in the *Appendix* for the derivation of this constraint) and $N_{zz} = \partial_u C_{zz}$ (Bondi News tensor/time derivative of gravitational waves - the gravitational analouge of the field strength $F_{uz} = \partial_u A_z$), which corresponds to the large-r falloffs (in Bondi gauge of the metric components):

$$\begin{aligned} &g_{uu} = -1 + \mathcal{O}\left(\tfrac{1}{r}\right), \quad g_{ur} = -1 + \mathcal{O}\left(\tfrac{1}{r^2}\right), \quad g_{uz} = \mathcal{O}(1), \\ &g_{zz} = \mathcal{O}(r), \quad g_{z\bar{z}} = r^2\gamma_{z\bar{z}} + \mathcal{O}(1), \quad g_{rr} = g_{rz} = 0 \end{aligned} \tag{168}$$

which comes from the boundary constraints.

9.2 Supertranslations

Asymptotic symmetries are generated by diffemorphisms, that preserve the Bondi gauge and the boundary falloffs. BMS were looking for symmetries, acting in the asymptotic region, where space time is close to flat. They therefore expectied to find the isometries of flat space time, the Poincaré group, such that general relativity would reduce to special relativity in some IR limit (at large distances and weak fields). But what they got was an infinite-dimensional group, containing the finite-dimensional Poincaré group as a sub-group, called the BMS group. In this group the four global translations are evaluated to a function's worth of "supertranslations".

9.2.1 BMS Analysis

Going through this analysis of BMS, we first make a simplifying assumption, such that six Lorentz generators are eliminated. We restrict to diffemorphisms that have large-r falloffs:

$$\xi^u, \xi^r \sim \mathcal{O}(1), \quad \xi^z, \xi^{\bar{z}} \sim \mathcal{O}\left(\frac{1}{r}\right) \tag{169}$$

The Lie derivative of the metric components at large-r are thus:

$$\begin{aligned}
\mathcal{L}_\zeta g_{ur} &= -\partial_u \zeta^u + \mathcal{O}\left(\frac{1}{r}\right) \\
\mathcal{L}_\zeta g_{zr} &= r^2 \gamma_{z\bar{z}} \partial_r \zeta^{\bar{z}} - \partial_z \zeta^u + \mathcal{O}\left(\frac{1}{r}\right) \\
\mathcal{L}_\zeta g_{z\bar{z}} &= r\gamma_{z\bar{z}} \left[2\zeta^r + rD_z\zeta^z + rD_{\bar{z}}\zeta^{\bar{z}}\right] + \mathcal{O}(1), \\
\mathcal{L}_\zeta g_{uu} &= -2\partial_u \zeta^u - 2\partial_u \zeta^r + \mathcal{O}\left(\frac{1}{r}\right)
\end{aligned} \tag{170}$$

(for a derivation and a discussion of the Lie derivatives, see *Section 18.2* in the *Appendix*), where $\mathcal{L}_\zeta g_{ab} = \zeta^c \partial_c g_{ab} + \partial_a \zeta^c g_{cb} + \partial_b \zeta^c g_{ca}$. Requiring that Bondi gauge conditions (the metric in *Eq. 166*) and that its falloffs be preserved at large-r, gives the solution:

$$\zeta = f\partial_u - \frac{1}{r}\left(D^z f \partial_z + D^{\bar{z}} f \partial_{\bar{z}}\right) + D^z D_z f \partial_r + \dots \tag{171}$$

where f is any function of $(z, \bar{z})$, and the last term (which may not be universal) comes from the condition $g_{ur} = -1 + \mathcal{O}(\frac{1}{r^2})$. The transformations generated by this ζ-function are the supertranslations, illustrated in *Figure 21*, and are generalizations of the four translations in MS.

Under a supertranslation, the retarded time u, is shifted independently at every angle on $\mathcal{I}$. For f being constant, generates u-translations. If f is taken to be the $l = 1$ harmonic on the sphere, we retrieve 3 spatial translations.

The action of supertranslations on data (N_{zz}, m_B, C_{zz}) (of $\mathcal{I}^+$), is determined by taking

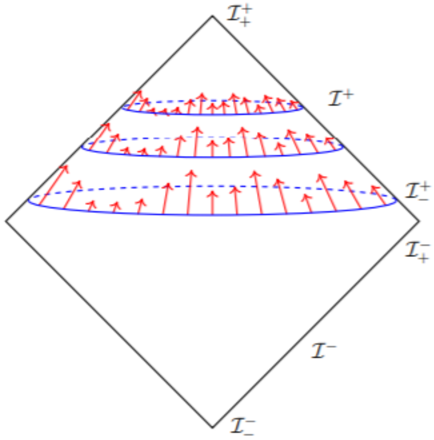


Figure 21: Under a supertranslation, the retarded time u is shifted independently at every angle on $\mathcal{I}$. The figure is taken from [Str].

the Lie derivative of the appropriate component of the metric, in the large-r expansion, which gives:

$$\begin{aligned} \mathcal{L}_f N_{zz} &= f\partial_u N_{zz} \\ \mathcal{L}_f m_B &= f\partial_u m_B + \frac{1}{4}\left[N^{zz}D_z^2 f + 2D_z N^{zz} D_z f + c.c.\right], \\ \mathcal{L}_f C_{zz} &= f\partial_u C_{zz} - 2D_z^2 f \end{aligned} \tag{172}$$

This equation implies, if we supertranslate flat MS described by $m_B = N_{zz} = C_{zz} = 0$, then the supertranslated spacetime will also have these properties, except for a nonzero C_{zz}; but with a vanishing Riemann tensor, which requires:

$$C_{zz} = -2D_z^2 C \tag{173}$$

for a function $C(z, \bar{z})$.
Assuming the geometry is governed by the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T^M_{\mu\nu} \tag{174}$$

also assuming that $T^M_{\mu\nu}$ is a matter stress tensor corresponding to massless modes. One can then plug in *Eq. 167*, and expand in large-r. One will then find that the leading uu component of the Einstein equations is:

$$\partial_u m_B = \frac{1}{4}\left[D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right] - T_{uu} \tag{175}$$

with

$$T_{uu} = \frac{1}{4}N_{zz}N^{zz} + 4\pi G \lim_{r\to\infty}\left[r^2 T^M_{uu}\right] \tag{176}$$

The supertranslations transform one geometry into a new, physically inequivalent geometry, though they are diffeomorphisms. Consider a scenario where an outgoing pulse of gravitational (or electromagnetic) waves crosses the south pole of $\mathcal{I}^+$, with another pulse crossing the north pole of $\mathcal{I}^+$, both at retarded time $u = 100$. If we supertranslate this solution with a function $f(z, \bar{z})$, with the property $f(\text{south pole}) = 100$ and $f(\text{north pole}) = 0$; The scenario now has an outgoing pulse at the north pole at $u = 100$ and one at the south pole at $u = 200$. Hence, the supertranslation changes the outgoing data.
Initial data, or Cauchy data, is specified by the parameters:

$$\left\{ N_{zz}(u, z, \bar{z}),\, C(z, \bar{z})|_{\mathcal{I}^+_-},\, m_B(z, \bar{z})|_{\mathcal{I}^+_-} \right\}. \tag{177}$$

There is a similar story at $\mathcal{I}^-$, with metric:

$$ds^2 = -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + \frac{2m_B}{r}dv^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + \dots \tag{178}$$

with $v = t + r$, and where z is antipodally related to the one on future null infinity, such that $z \to -\frac{1}{\bar{z}}$.
Supertranslations act on past null infinity as:

$$\mathcal{L}_f N_{zz} = f\partial_v N_{zz}, \quad \mathcal{L}_f C_{zz} = f\partial_v C_{zz} + 2D_z^2 f \tag{179}$$

The constraint equation thus takes the form:

$$\partial_v m_B = \frac{1}{4}\left(D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right) + T_{vv}, \quad T_{vv} = \frac{1}{4}N_{zz}N^{zz} + 4\pi G \lim_{r\to\infty}\left[r^2 T^M_{vv}\right] \tag{180}$$

and the analogue of the Cauchy data is:

$$\left\{ N_{zz}(v, z, \bar{z}),\, C(z, \bar{z})|_{\mathcal{I}^-_+},\, m_B(z, \bar{z})|_{\mathcal{I}^-_+} \right\} \tag{181}$$

9.2.2 The Scattering Problem

The scattering problem in classical general relativity is to find the map from Cauchy data on $\mathcal{I}^-$ to that on $\mathcal{I}^+$.
It has been proposed that the data on $\mathcal{I}^+$ at most up to a supertranslation, the BMS$^+$ frame, should be determined by the Lorentz- and CPT- (conjugation, parity, and time) invariant matching conditions:

$$C(z, \bar{z})|_{\mathcal{I}^+_-} = C(z, \bar{z})|_{\mathcal{I}^-_+}, \quad m_B(z, \bar{z})|_{\mathcal{I}^+_-} = m_B(z, \bar{z})|_{\mathcal{I}^-_+} \tag{182}$$

(using *Eq. 173*), and the diagonal subgroup, which preserves these conditions are defined as:

$$f(z, \bar{z})|_{\mathcal{I}^+_-} = f(z, \bar{z})|_{\mathcal{I}^-_+} \tag{183}$$

which fixes the BMS$^+$ frame in terms of the BMS$^-$ frame. This matching condition is equivalent to Weinberg's soft graviton theorem.

9.2.3 Conserved Charges

Infinitely many matching conditions, one for every point on the celestial sphere, implies an infinite number of conserved charges. These supertranslation charges are:

$$\begin{aligned} Q_f^+ &= \frac{1}{4\pi G}\int_{\mathcal{I}^+_-} d^2z\gamma_{z\bar{z}} f m_B \\ Q_f^- &= \frac{1}{4\pi G}\int_{\mathcal{I}^-_+} d^2z\gamma_{z\bar{z}} f m_B \end{aligned} \tag{184}$$

From the matching condition, we get the conservation law:

$$Q_f^+ = Q_f^- \tag{185}$$

As conserved charges commutes with the $\mathcal{S}$-matrix:

$$Q_f^+\mathcal{S} - \mathcal{S}Q_f^- = 0 \tag{186}$$

one can write out an expansion of the charges, by integrating by parts, using the constraint equation, and assuming the Bondi mass decays to zero in the far future, and get:

$$\begin{aligned} Q_f^+ &= \frac{1}{4\pi G}\int_{\mathcal{I}^+} dud^2z\gamma_{z\bar{z}} f\left[T_{uu} - \frac{1}{4}\left(D_z^2N^{zz} + D_{\bar{z}}^2N^{\bar{z}\bar{z}}\right)\right] \\ Q_f^- &= \frac{1}{4\pi G}\int_{\mathcal{I}^-} dvd^2z\gamma_{z\bar{z}} f\left[T_{vv} + \frac{1}{4}\left(D_z^2N^{zz} + D_{\bar{z}}^2N^{\bar{z}\bar{z}}\right)\right] \end{aligned} \tag{187}$$

The conservation law then states:

$$\int_{\mathcal{I}^+} du\gamma_{z\bar{z}}\left[T_{uu} - \frac{1}{4}\left(D_z^2N^{zz} + D_{\bar{z}}^2N^{\bar{z}\bar{z}}\right)\right] = \int_{\mathcal{I}^-} dv\gamma_{z\bar{z}}\left[T_{vv} + \frac{1}{4}\left(D_z^2N^{zz} + D_{\bar{z}}^2N^{\bar{z}\bar{z}}\right)\right] \tag{188}$$

Again we have a hard and a soft (zero mode) contribution, also the energy is conserved at every angle. Sandwiching *Eq. 186* between initial- and final states, the resulting Ward identity is shown to be equivalent (by usual method) to Weinberg's soft graviton theorem:

$$\langle f|\, a_\pm\mathcal{S}\,|i\rangle = \sqrt{8\pi G}\sum_k \frac{\varepsilon^{\pm\mu\nu}p_{k\mu}p_{k\nu}}{q\cdot p_k}\langle f|\,\mathcal{S}\,|i\rangle\,, \tag{189}$$

where $a_\pm$ annihilates a helicity $\pm$ graviton.

9.3 Superrotations

In *Eq. 167*, we before pointed out the Bondi mass, however in the last term we have a parameter N_z, the angular momentum aspect, which bears the same relation to the total angular momentum, which the Bondi mass aspect bears to the total mass. We will see that the matching condition for N_z leads to conserved superrotation charges.

9.3.1 Conserved Charges

For the Bondi mass aspect, we had the constraint equation $G_{uu} = 8\pi G T^M_{uu}$, however N_z is subject to the constraint equation $G_{uz} = 8\pi G T^M_{uz}$. The leading uz component of the Einstein equations is thus:

$$\partial_u N_z = \frac{1}{4}\partial_z \left(D_z^2 C^{zz} - D_{\bar{z}}^2 C^{\bar{z}\bar{z}}\right) - u\partial_u \partial_z m_B - T_{uz} \tag{190}$$

with the momentum density (in the z-direction of the gravitational field) being defined as:

$$T_{uz} \equiv 8\pi G \lim_{r\to\infty} \left[r^2 T^M_{uz}\right] - \frac{1}{4}\partial_z \left(C_{zz} N^{zz}\right) - \frac{1}{2} C_{zz} D_z N^{zz} \tag{191}$$

We can fix N_z by the matching condition:

$$N_z(z,\bar{z})|_{\mathcal{I}^+_-} = N_z(z,\bar{z})|_{\mathcal{I}^-_+} \tag{192}$$

which implies a second infinity of conserved charges, constructed by an arbitrary vector field Y^z on the sphere. Conservation of the superrotation charges is defined as:

$$Q^+_Y = \frac{1}{8\pi G}\int_{\mathcal{I}^+_-} d^2 z \left(Y_{\bar{z}} N_z + Y_z N_{\bar{z}}\right) = \frac{1}{8\pi G}\int_{\mathcal{I}^-_+} d^2 z \left(Y_{\bar{z}} N_z + Y_z N_{\bar{z}}\right) = Q^-_Y \tag{193}$$

9.3.2 Symmetries

In the derivation of supertranslations, an important restriction was imposed; components of the vector field, ζ, required to be bounded in an orthonormal frame, as the latter approaches future null infinity. This is a too strong assumption, as it rules out boosts and rotations. We will see that supertranslations generalizes to superrotations.
Lorentz Killing vectors are of the form:

$$\zeta_Y = \left(1 + \frac{u}{2r}\right) Y^z \partial_z - \frac{u}{2r} D^{\bar{z}} D_z Y^z \partial_{\bar{z}} - \frac{1}{2}(u+r) D_z Y^z \partial_r + \frac{u}{2} D_z Y^z \partial_u + c.c. \tag{194}$$

where $(Y^z, Y^{\bar{z}})$ is a 2D vector field on the celestial sphere. The first three terms are that of MS. At null infinity, it simplifies to:

$$\zeta_Y|_{\mathcal{I}^+} = Y^z \partial_z + \frac{u}{2} D_z Y^z \partial_u + \text{ c.c.} \tag{195}$$

If we take:

$$Y^z = 1, z, z^2, i, iz, iz^2 \tag{196}$$

then the 6 real vector fields ζ_Y generate the Lorentz transformations (rotations and boosts). For a general Y^z, the Lie derivatives with respect to ζ_Y of the metric components are found to be:

$$\begin{aligned}
\mathcal{L}_Y g_{ur} &= \mathcal{O}\left(\frac{1}{r^2}\right) \\
\mathcal{L}_Y g_{zr} &= \mathcal{O}\left(\frac{1}{r}\right) \\
\mathcal{L}_Y g_{z\bar{z}} &= \mathcal{O}(r) \\
\mathcal{L}_Y g_{uu} &= \mathcal{O}\left(\frac{1}{r}\right) \\
\mathcal{L}_Y g_{\bar{z}\bar{z}} &= 2r^2\gamma_{z\bar{z}}\partial_{\bar{z}}Y^z + \mathcal{O}(r)
\end{aligned} \tag{197}$$

The last equation implies that $Y^{\bar{z}}$ is a holomorphic function on the sphere, locally solved by $Y^z = z^n$, but only the restricted ones leads to the globally defined conformal Killing vector fields of the sphere. Choosing $Y^z = \frac{1}{z-w}$, then we get:

$$\partial_{\bar{z}}Y^z = 2\pi\delta^2(z-w) \neq 0 \tag{198}$$

and the falloff condition is violated at $z = w$, implying that the Lie bracket algebra of $Y^z = z^n$ for any n, is the centerless Virasoro algebra, 2D CFT. We learned that 4D quantum gravity transforms under the exact same infinte-dimensional group as for 2D CFT, which is quite useful.

9.3.3 Canonical Formalism

The superrotation charges should generate superrotation symmetries in a canonical formalism, however only at linerized order, as singularities prevent the exponentiation of the infinitesimal transformations. First we investigate, how the boundary data affects the geometry change under superrotations. The Lie derivative with respect to Y of the C_{zz} component of the metric is given by:

$$\delta_Y C_{zz} = \frac{u}{2}D\cdot YN_{zz} + Y\cdot DC_{zz} - \frac{1}{2}D\cdot YC_{zz} + 2D_zY^zC_{zz} - uD_z^3Y^z \tag{199}$$

Taking the u-derivative, one will get:

$$\delta_Y N_{zz} = \frac{u}{2}D\cdot Y\partial_u N_{zz} + Y\cdot DN_{zz} + 2D_zY^zN_{zz} - D_z^3Y^z \tag{200}$$

If we sit at $u = 0$, the last two terms are exactly the infinitesimal transformation law of a stress tenser in 2D CFT (the linerization of the Schwarzian derivative).
Consider the conserved superrotation charge:

$$Q_Y^+ = \frac{1}{8\pi G}\int_{\mathcal{I}_-^+} d^2z\,[Y_{\bar{z}}N_z + Y_zN_{\bar{z}}] \tag{201}$$

and integrating by parts using constraints (*Eq. 190*), we get:

$$\begin{aligned} Q_Y^+ &= Q_H^+ + Q_S^+ \\ Q_S^+ &= -\frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z\left[D_z^3 Y^z u N_{\bar{z}}^z + D_{\bar{z}}^3 Y^{\bar{z}} u N_z^{\bar{z}}\right] \\ Q_H^+ &= \frac{1}{8\pi G}\int_{\mathcal{I}^+} dud^2z\left(Y_{\bar{z}}T_{uz} + Y_z T_{u\bar{z}} + u\partial_z Y_{\bar{z}}T_{uu} + u\partial_{\bar{z}}Y_z T_{uu}\right) \end{aligned} \tag{202}$$

where the soft charges are linear in the metric fluctuation, C_{zz}, while the hard charge is quadratic. The commutators are:

$$\begin{aligned} \left[N_{\bar{z}\bar{z}}(u,z,\bar{z}), C_{ww}\left(u',w,\bar{w}\right)\right] &= 16\pi G i\gamma_{z\bar{z}}\delta^2(z-w)\delta\left(u-u'\right) \\ \left[Q_S^+, C_{zz}\right] &= -iuD_z^3 Y^z \\ \left[Q_H^+, C_{zz}\right] &= \frac{iu}{2}D\cdot Y N_{zz} + iY\cdot DC_{zz} - \frac{i}{2}D\cdot Y C_{zz} + 2iD_z Y^z C_{zz}. \end{aligned} \tag{203}$$

From this, one can conclude that:

$$\left[Q_Y^+, \ldots\right] = i\delta_Y \tag{204}$$

Hence, the conserved charge generates the symmetry as expected.

9.3.4 Subleading Soft Theorem

There must be a second soft theorem in gravity, a new subleading soft theorem, equivalent to the superrotation charge conservation equation *Eq. 193.* Superrotation charge conservation sates the quantum $\mathcal{S}$-matrix:

$$\langle f|\left(Q_Y^+\mathcal{S} - \mathcal{S}Q_Y^-\right)|i\rangle = 0 \tag{205}$$

By using the usual method and: $Y^z = \frac{1}{z-(q^1+\frac{iq^2}{q^0}+q^3)}$, one can re-express the equation in momentum space as:

$$\begin{aligned} \lim_{\omega\to 0}\left(1+\omega\partial_\omega\right)\langle p_{n+1}, p_{n+2}, \ldots|a_-(q)\mathcal{S}|\, p_1, p_2, \ldots\rangle = \\ \sqrt{8\pi G}S^{(1)-}\langle p_{n+1}, p_{n+2}, \ldots|\mathcal{S}|p_1, p_2, \ldots\rangle \end{aligned} \tag{206}$$

where $a_-(q)$ is the annihilation operator for a negative helicity graviton of four-momentum $q = \omega(1,\hat{q})$; and the subleading soft factor is:

$$S^{(1)-} = -i\sum_k \frac{p_{k\mu}\varepsilon^{-\mu\nu}q^\lambda J_{k\lambda\nu}}{p_k\cdot q}, \quad J_{k\mu\nu} \equiv L_{k\mu\nu} + S_{k\mu\nu} \tag{207}$$

with $L_{k\mu\nu}$ and $S_{k\mu\nu}$ being the orbit angular momentum and the helicity of the internal spin of the k'th particle. This being valid at tree level (using diagrammatics), confirms the existence of an infinite number of conserved superrotation charges. The replacement $fT_{uu} \to Y^z T_{uz}$ is akin to $p_\nu \to q^\mu J_{\mu\nu}$.

To check this is true, recall that the Weinberg soft graviton theorem involves the leading soft factor:

$$\sum_k \frac{\varepsilon^{\mu\nu} p_{k\mu} p_{k\nu}}{p_k \cdot q} \tag{208}$$

which should vanish for pure gauge gravitons, ie. when:

$$\varepsilon^{\mu\nu} = \Lambda^\mu q^\nu \tag{209}$$

With this choice, the soft factor becomes:

$$\Lambda^\mu \sum_k \frac{p_{k\mu} p_k \cdot q}{p_k \cdot q} = \Lambda^\mu \sum_k p_{k\mu} = 0 \tag{210}$$

following from energy-momentum conservation. Consistency thus demands a similar property for the subleading soft factor, $S^{(1)}(\epsilon_\Lambda)$. Inserting the following for symmetry:

$$\varepsilon_\Lambda^{\mu\nu} = q^\mu \Lambda^\nu + q^\nu \Lambda^\mu \tag{211}$$

one finds that:

$$iS^{(1)}(\varepsilon_\Lambda) = q^\mu \Lambda^\nu \sum_k J_{k\mu\nu} + \sum_k \frac{p_k \cdot \Lambda q^\mu q^\nu J_{k\mu\nu}}{p_k \cdot q} = 0 \tag{212}$$

where the second term vanishes, due to J being antisymmetric and $q^\mu q^\nu$ being symmetric. The first term vanishes by angular momentum conservation. Angular momentum is to superrotations (sub-leading soft theorem) as energy-momentum is to supertranslations (leading soft theorem).

9.4 The Memory Effect

Lastly I will shortly describe the mechanics of the third corner of the IR triangle; the memory effect.

The gravitational memory effect is the persistent changes in the relative position of pairs of points in space, due to the passing of a gravitational wave.
In 2016 the first gravitational wave was detected by the LIGO detectors; two perpendicular detector arms, where a beam of light would be split into two beams, and reflect at each end of the arms. The beams of light would then meet again at the beam splitter, and cancel by destructive interference if the arms were of equal length. The residual deformation of the detector pair (the lengths of two detector arms not being equal), resulted in a signal being detected, ie. from a gravitational wave.

This corner of the infrared triangle had been predicted before this detection of the gravitational wave. This is due to its equivalence with the soft theorem and asymptotic symmetries of the theory, as described in *Section 1.2*.

10 Discussion of the $\log r/r$ Behaviour

10.1 Subleading Soft Photons and Large Gauge Transformations

This section will follow the work described in [MC].

From *Section 5*, we saw that the classic Weinberg's soft photon theorem can be understood as a Ward identity associated to an infinite dimensional symmetry group of QED - by considering large gauge transformations at null infinity.
This section will show that there exists a class of large $U(1)$ gauge transformations, from which the associated (electric and magnetic) charges can be computed - and their Ward identities are equivalent to Low's theorem.
In the same way, the sub-subleading theorem in gravity in terms of Ward identities, can be shown to be associated to large diffeomorphisms.

10.1.1 Introduction

Low has shown that the factorization of scattering amplitudes applies also to the next order in the photon energy; Low's Theorem is:

$$\lim_{\omega\to 0}\left(1+\omega\partial_\omega\right)\mathcal{M}_{n+1}\left(k_1,\ldots,k_n;\omega\hat{q}\right)=S^{(1)}\mathcal{M}_n\left(k_1,\ldots,k_n\right) \tag{213}$$

$S^{(1)}$ being a sum of differential operators acting on the external momenta, k_i. Lysov, Pasterski and Strominger showed that the theorem is equivalent to Ward identities of infinitely many charges, that are parametrized by vector fields on the sphere; the charges being interpreted as local generalizations of electric and magnetic dipole moments. This section shows that these charges are associated to certain large $U(1)$ gauge transformations.

Consider massless scalar QED, and work in harmonic gauge ($\nabla^\mu A_\mu = 0$). Global symmetries can thus arise from residual, large gauge transformations which are parametrized by solutions of the wave equation:

$$\Box\lambda = 0 \tag{214}$$

In retarded $(u, r, \hat{x})$ coordinates, this equation can be solved in an $r \to \infty$ expansion once the asymptotic behavior of λ is specified. For a given large gauge parameter λ, one can associate charges of electric and magnetic type according to:

$$Q_\lambda = \int_\Sigma d^3V\partial_a\left(\lambda E^a\right), \quad \tilde{Q}_\lambda = \int_\Sigma d^3V\partial_a\left(\lambda B^a\right) \tag{215}$$

E^a and B^a being the electric and magnetic fields with respect to the hypersurface, Σ. Simple Fourier space reasoning suggests one should look at large gauge parameters whose $\mathcal{O}(r^0)$ component is linear in u. For this to be compatible with *Eq. 214*, the gauge parameter must have an $\mathcal{O}(r)$ piece. It is shown that such solutions exist (at least asymptotically) and the corresponding charges at null infinity will be found. These are divergent, but by projecting out a soft photon contribution, the charges are rendered finite. These charges are the same as the ones Low found; ie. there is an equivalence of the (electric and magnetic) Ward identities with Low's subleading soft photon theorem.

10.1.2 $\mathcal{O}(1)$ Large Gauge Transformation and Associated Charges

For charges associated with large gauge transformations with asymptotic behaviour, we have:

$$\lambda(u,r,\hat{x}) = \varepsilon(\hat{x}) + \mathcal{O}\left(r^{-\epsilon}\right) \tag{216}$$

which can be satisfied to $\mathcal{O}(r^{-1})$, determining the asymptotic form of the subleading term ($\ln r/r$) (See *Section 10.1.3*). However only the leading term contributes to the charge ($\varepsilon(\hat{z})$).

The (known) electric-/magnetic-type charges for the gauge parameter (*Eq. 216*) have Ward identities corresponding to Weinberg's soft photon theorem.

10.1.3 Large Gauge Parameters

Calculating the coefficients of the large-r expansion of gauge parameters; we start with the ansatz:

$$\lambda = r^2 \overset{(2)}{\lambda} + r \overset{(1)}{\lambda} + \overset{(0)}{\lambda} + \frac{\log r}{r} \overset{(\log r/r)}{\lambda} + \mathcal{O}\left(r^{-1}\right) \tag{217}$$

The two last terms correspond to small gauge parameters, where they have been taken as $\mathcal{O}(r^{-\epsilon})$ throughout this section - such a fall-off is enough to guarantee a vanishing contribution to the charges. For a solution to the wave equation however, $\Box\lambda = 0$, this form above is needed. $\mathcal{O}(r^{-1})$ parameters behave like regular scalar fields, satisfying the wave equation, and are associated to small gauge parameters with their own 'free data'.

However, the $\log r/r$ term is needed in order to satisfy the wave equation. Applying the wave operator to the ansatz, we get:

$$\begin{aligned} \Box\lambda = r\left[-6\partial_u \overset{(2)}{\lambda}\right] + \left[(\Delta+6)\overset{(2)}{\lambda} - 4\partial_u \overset{(1)}{\lambda}\right] + r^{-1}\left[(\Delta+2)\overset{(1)}{\lambda} - 2\partial_u \overset{(0)}{\lambda}\right] + \\ r^{-2}\left[\Delta \overset{(0)}{\lambda} - 2\partial_u \overset{(\log r/r)}{\lambda}\right] + \mathcal{O}\left(r^{-3}\log r\right) \end{aligned} \tag{218}$$

It is convenient to write the wave operator as $\Box\lambda = r^{-1}(\partial_r^2 - 2\partial_r\partial_u + r^{-2}\Delta)(r\lambda)$. The general solution, setting this to zero is:

$$\begin{aligned}\overset{(2)}{\lambda}(u,\hat{x}) &= \nu(\hat{x}) \\ \overset{(1)}{\lambda}(u,\hat{x}) &= \frac{1}{4}\int_0^u du' \left((\Delta+6)\overset{(2)}{\lambda}\right) + \mu(\hat{x}) \\ \overset{(0)}{\lambda}(u,\hat{x}) &= \frac{1}{2}\int_0^u du'((\Delta+2)\overset{(1)}{\lambda}) + \varepsilon(\hat{x}) \\ \overset{(\log r/r)}{\lambda}(u,\hat{x}) &= \frac{1}{2}\int_0^u du'(\Delta\overset{(0)}{\lambda}) + \eta(\hat{x})\end{aligned} \tag{219}$$

The $\mathcal{O}(1)$ large gauge parameter correspond to setting integration 'constants' (a function on the sphere) $\nu = \mu = 0$, and the $\mathcal{O}(r)$ parameter corresponds to $\nu = \varepsilon = 0$. The $\log r/r$ is crucial, else one would have gotten $\Delta \overset{(0)}{\lambda} = 0$ which would have eliminated the $\mathcal{O}(1)$ large gauge transformation.

10.2 A Note on the Subleading Soft Graviton

This section will follow the work described in [EH].

In harmonic gauge, the soft part of the charge, generating infinitesimal superrotations, can be expressed in terms of the metric components, evaluated at the boundaries of null infinity, that are subleading in a large radius expansion. The spin memory observable can be recast in terms of these boundary values.

10.2.1 Introduction

It is mentioned, how a current, corresponding to Low's subleading soft theorem in EM, can be expressed in terms of boundary values of the gauge potential at null infinity. The purpose here, is then to examine the analogous method/computations for gravity in harmonic gauge - showing that the subleading soft graviton mode, appearing in the superrotation charge, can also be rewritten in terms of a difference between boundary values of the metric.

10.2.2 Setup

Considering linearized gravity in 4D, one can set up the Einstein's equations in harmonic gauge, impose boundary conditions on the metric perturbations, and identify residual symmetries that the boundary conditions allow.

10.2.3 Boundary Conditions

Choosing falloffs of the matter stress tensor, $T_{\mu\nu}$, to be consistent with a massless scalar field gives:

$$\begin{aligned} G_{uu} \sim \mathcal{O}\left(r^{-2}\right), \quad G_{ur} \sim \mathcal{O}\left(r^{-4}\right), \quad G_{rr} \sim \mathcal{O}\left(r^{-4}\right) \\ G_{uA} \sim \mathcal{O}\left(r^{-2}\right), \quad G_{rA} \sim \mathcal{O}\left(r^{-3}\right), \quad G_{AB} \sim \mathcal{O}\left(r^{-1}\right) \end{aligned} \tag{220}$$

while the asymptotics of $T_{\mu\nu}$ is consistent with a metric with boundary behavior:

$$\begin{aligned} h_{uu} \sim \mathcal{O}\left(r^{-1}\log r\right), \quad h_{ur} \sim \mathcal{O}\left(r^{-1}\log r\right), \quad h_{rr} \sim \mathcal{O}\left(r^{-1}\log r\right) \\ h_{uA} \sim \mathcal{O}(\log r), \qquad h_{rA} \sim \mathcal{O}(\log r), \qquad h_{AB} \sim \mathcal{O}(r\log r) \end{aligned} \tag{221}$$

From the previous sections, and in [MC], it is stated that logarithmic r-dependence is needed for a consistent solution of the linearized Einstein equations with matter in 4D, in harmonic gauge. The term in the metric expansion with coefficient $\frac{1}{r^n}$ is denoted by superscript (n), and $\frac{\log r}{r^n}$ by a tilde with superscript (n).
The residual diffemorphisms for harmonic gauge are parametrized by the free data:

$$\left\{\xi^{u(1)}(u,z,\bar{z}), \xi^{r(1)}(u,z,\bar{z}), \xi^{A(2)}(u,z,\bar{z})\right\} \tag{222}$$

These functions can be used to perform residual gauge fixing to arrive at the stronger fall-offs:

$$\begin{aligned} h_{uu} &= \sum_{n=2}^{\infty} \frac{h_{uu}^{(n)}}{r^n} + \sum_{n=1}^{\infty} \frac{\tilde{h}_{uu}^{(n)}\log r}{r^n}, \quad & h_{ur} &= \sum_{n=2}^{\infty} \frac{h_{ur}^{(n)}}{r^n} + \sum_{n=2}^{\infty} \frac{\tilde{h}_{ur}^{(n)}\log r}{r^n}, \\ h_{rr} &= \sum_{n=3}^{\infty} \frac{h_{rr}^{(n)}}{r^n} + \sum_{n=3}^{\infty} \frac{\tilde{h}_{rr}^{(n)}\log r}{r^n}, \quad & h_{uA} &= \sum_{n=1}^{\infty} \frac{h_{uA}^{(n)}}{r^n} + \sum_{n=1}^{\infty} \frac{\tilde{h}_{uA}^{(n)}\log r}{r^n}, \\ h_{rA} &= \sum_{n=1}^{\infty} \frac{h_{rA}^{(n)}}{r^n} + \sum_{n=2}^{\infty} \frac{\tilde{h}_{rA}^{(n)}\log r}{r^n}, \quad & h_{AB} &= \sum_{n=-1}^{\infty} \frac{h_{AB}^{(n)}}{r^n} + \sum_{n=0}^{\infty} \frac{\tilde{h}_{AB}^{(n)}\log r}{r^n}. \end{aligned} \tag{223}$$

10.3 Asymptotic Charges and Coherent States in QCD

This $\log r/r$ behavior is also used in [RG], to split the asymptotic charge for non-abelian large gauge transformations, into a piece linear in the gauge field (soft part), and a piece non-linear in the gauge fields (hard part). Here they have imposed conditions on the falloffs of the non-abelian gauge fields at large-r in accordance with the findings of [MC] and [EH], discussed in this section. From the gauge fields, they get the leading order component for the field strengths, which lead to the equations of motion implying constraint equations on $\mathcal{I}^+$ at $\mathcal{O}(1)$ in the large-r expansion; which can be used to rewrite the charge as described. This linearized piece is then used to find the linearized large gauge transformations in quantum chromodynamics (QCD), amongst other things.

11 Method

The main aspect of the thesis, is the two different approaches for these asymptotic symmetries, that we see for Strominger and Penrose. We want this equivalence to be made more clear.

We want to use this direct analysis of Penrose, to generalise the concept of asymptotic symmetry groups to QED and YM theory, as described for the gravity theory case.

There is a difficulty in defining the boundary conditions, ie. comparing the two null boundaries (as well as analysing points $i^{\pm}$ and i^{0}). Strominger solves this by the use of matching conditions of the fields. Penrose avoids these problem areas all-together, arguing that they can be rightfully left out for his purposes. We will attempt to discuss solutions to these problems as well.

11.1 Approaches of Analysing the BMS Group

It is clear that Penrose and Strominger use two very different methods for their analysis of the BMS group. They can be shortly described by:

Strominger starts with a discussion of asymptotic symmetries in QED, see *Sections 4 and 5*. First he derives conserved charges, following from antipodal matching conditions of the fields. He then show that they, via a canonical formalism, generate asymptotic symmetries; ie. he shows that the soft theorems give rise to Ward identities, corresponding to divergent large gauge transformations (asymptotic symmetries) acting on the celestial sphere at null infinity.

For non-abelian gauge theory (*Section 6*), he used the formalism of the $\mathcal{S}$-matrix as a celestial correlator, where the scattering of a soft gluon becomes the insertion of a current into a correlation function on CS^2. This current obeys the Ward identities of a $\mathcal{G}$-current Kac-Moody algebra, giving an alternative representation of the asymptotic symmetry group. The conserved charges are derived from the matching condition, shown to generate the symmetry and to imply the tree-level non-abelian soft theorem.

For his analysis (in gravity) of the BMS group he defines a large-r geometric constraint of the metric (see *Section 9*). He uses an analysis of asymptotic data (leading metric components) to obtain the matching condition for the Bondi mass m_B, which led to conserved supertranslation charges of past- and future null infinity. The supertranslations are generalizations of the four translations in Minkowski space.
He also derives conserved charges from a matching condition for subleading metric components, the angular momentum aspect N_z. These are the superrotation charges. Angular momentum is to superrotations (sub-leading soft theorem) as energy-momentum is to supertranslations (leading soft theorem).

Penrose on the other hand, focused only on gravity, and used a direct analysis of the geometry and the asymptotic symmetries (see *Section 7*). He assigned structure to $\mathcal{I}$; its weak inner conformal metric. The group of self-transformations of $\mathcal{I}$, preserving this weak inner conformal metric is the Newman-Unti group. Since $\mathcal{I}$ is a null hypersurface, its structure is too weak, and a strong conformal geometry was needed in order to restrict the Newman-Unti group, to the BMS group, and reduce the freedom in F.

The concept of angle between (non-null) tangent directions at a point of $\mathcal{I}^+$, comes from the content of the inner conformal metric of $\mathcal{I}^+$. When this angle is zero, the strong conformal geometry of $\mathcal{I}^+$ is needed to define this null angle between the directions. This way he found quantities $n^a dl$, which have to be preserved under transformations of $\mathcal{I}^+$ to itself, leading to the restrictions of the functions F, and hence the definition of BMS transformations; the group of self-transformations of $\mathcal{I}^+$, which preserves its strong conformal geometry.
He then begins a discussion of how the supertranslations form an infinite parameter (normal) subgroup (and the translations form a four-parameter normal subgroup) of the BMS group. The more elaborate discussion of this and its implication was covered in *Section 8*.

11.2 Gravity and Gauge Theory

To apply methods of gravity theory to gauge theory, it is convenient to first set up the similarities and differences of the two theories. This is just a short and unfinished discussion of this subject.

In gauge theory, the gauge group consists of global gauge transformations, and can be described by a Kac-Moody (Lie) algebra, as we saw in *Section 6*.
In gravity, the analogue of these large gauge transformations, in compactified MS, has been the BMS group.

One can wonder, could the Kac-Moody algebra be the gauge theoretic analogue of the BMS group? Could this be a way to apply the methods of Penrose, to find the asymptotic symmetries of gauge theories by a direct analysis of the algebra?

The method of finding the asymptotic symmetries in these theories could be described by (by using a direct analysis):
Gravity: Start with metric, compute Weyl tensor, impose decay when $r \to \infty$, which leads to constraints on metric components.
Gauge theory: Start with gauge field $A = A_r dr + A_u du + A_z dz + A_{\bar{z}} d\bar{z}$, compute field strength $F = \frac{1}{2} F_{ru} dr \wedge du + \frac{1}{2} F_{rz} dr \wedge dz ...$, set $A_u = 0$ (lightcone gauge) and compute Kac-Moody algebra.

11.3 Funny Gauge Transformations

The purpose of this section is to give a short, not fully explored, mathematical interpretation of how some gauge transformations have weird properties, and can change fields at infinity.

For the vector potential A, a gauge transformation means $\delta A = \partial \lambda$, or $\delta A_i = \partial_i \lambda$. The magnetic field is given by $B_{ij} = \partial_{[i} A_{j]}$, such that:

$$\delta B_{ij} = \partial_{[i} \delta A_{j]} = \partial_{[i} \partial_{j]} \lambda \to 0 \tag{224}$$

However, this is not always the case, for some λ you can create sources (change fields at infinity) by integrating charges;

$$\lambda = \arctan \frac{y}{x}$$

for example, as the derivative with either x or y will not cause the quantity to vanish;

$$\partial_x \lambda = \frac{-y}{x^2 + y^2}, \quad \partial_x \lambda = \frac{x}{x^2 + y^2}$$

.

In 2 dimensions, the gauge transformations are described by $A_z = \partial_z \lambda$ and $A_{\bar{z}} = \partial_{\bar{z}} \lambda$, where:

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$$

and we therefore get:

$$A_z = \partial_z = \frac{1}{2} \frac{-y - ix}{x^2 + y^2} = \frac{-i}{2} \frac{1}{z} = \frac{-i}{2z}$$

In this case the magnetic field is given by:

$$B_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z$$

If we plug in the values for A_z and $A_{\bar{z}}$, we will see that this doesn't become zero, as:

$$\partial_{\bar{z}} \frac{1}{z} \sim \delta^2(z)$$

11.4 Application to Electromagnetism and Yang-Mills Theories

In order to explore the asymptotic symmetries of Electromagnetism and Yang-Mills theories, a good place to start is to set up the equations, describing how particles in such theories interact. Below is a list of things to set up and figure out:

- EOM
- Energy-momentum tensor
- Use Weyl transformations to obtain a non-singular metric on $\mathcal{I}^{\pm}$
- Define an analog of $\hat{n} = \hat{\nabla}\Omega$. Is it null?
- How does $\hat{n}$ transform under a change of Weyl factor ($\Omega \to \Theta\Omega$)?
- Are there other analogs of $\hat{n}$? Meaning quantities which transform covariantly under change of Weyl factor? If yes, then we can potentially construct extra invariant expressions (which should be preserved by asymptotic transformations).
- Asymptotics of gauge fields in r of $\log r$. Can we recover them from this above analysis?

- Yang-Mills vs QED.
- If we solve all these try with fermions or two-form fields instead of scalars. In 4D a two-form field can be (locally) dualized to a scalar.
- Supersymmetry, supergravity...

To begin with, we will set up the energy-momentum tensor for a photon, a charged massless (conformally coupled) scalar, and gravity, all combined. The idea is to put these fields together, and work out how they act near null infinity, and whether any corrections are needed.

12 Conformal Coupling of the Scalar

If we apply a Weyl transformation to the metric $g \to \Omega^2 g$, the omegas cancel for the electromagnetic action. We want to check with scalar fields instead:

Let us start out with a discussion of the invariance of a scalar field theory under Weyl transformations.
Leaving the scalar field invariant under a Weyl transformations, results in the action of the scalar field not being invariant, as the weight from the metric does not cancel.
We can choose that scalar fields transform with weight ± 1 under Weyl transformations, such that:

$$\phi \to \phi' = \pm\Omega\phi$$

This helps cancel the weight, but give us derivatives of Ω in the action.
Adding a term to the action, called the conformal coupling:

$$R\phi^*\phi,$$

where R is the Ricci scalar; and choosing the correct numerical constant for this term ($\frac{1}{6}$ in 4D), makes the action invariant under Weyl transformations.

The action of the scalar field is:

$$S_{scalar} = \int d^4x\sqrt{-g}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi^*)$$

Weyl transformations of the metric are given by:

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

The determinant of the metric transforms as:

$$\det g \to \det\hat{g} = \det\big(\Omega^2 g\big) = \Omega^8 \det g,$$

and the metric with upper indices transforms as:

$$g^{\mu\nu} \to \hat{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}.$$

Thus we have, for the term in the action:

$$\sqrt{-g}g^{\mu\nu} = \Omega^{-2}\sqrt{-\hat{g}}\hat{g}^{\mu\nu}.$$

Lastly, the scalar field transforms as:

$$\phi \to \hat{\phi} = \Omega^{-1}\phi$$

The scalar action can now be written as (using $\phi = \Omega\hat{\phi}$):

$$\begin{aligned} S_{scalar} &= \int d^4x\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\Omega^{-2}\partial_\mu(\Omega\hat{\phi})\partial_\nu(\Omega\hat{\phi}^*) \\ &= \int d^4x\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\left((\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi}^*) + \Omega^{-2}(\partial_\mu\Omega)(\partial_\nu\Omega)|\hat{\phi}|^2 + \Omega^{-1}(\partial_\mu\Omega)(\partial_\nu\hat{\phi}^*)\hat{\phi} + \Omega^{-1}(\partial_\mu\hat{\phi})(\partial_\nu\Omega)\hat{\phi}^*\right) \\ &= \int d^4x\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\left((\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi}^*) + \Omega^{-2}(\partial_\mu\Omega)(\partial_\nu\Omega)|\hat{\phi}|^2 + \Omega^{-1}((\partial_\mu\Omega)(\partial_\nu\hat{\phi}^*)\hat{\phi} + (\partial_\mu\hat{\phi})(\partial_\nu\Omega)\hat{\phi}^*)\right), \end{aligned} \tag{225}$$

where we get this derivative of Ω. So now we will add a coupling to the scalar curvature:

$$S_{conformal-scalar} = \int d^4x\sqrt{-g}(g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi^*) + C\cdot R|\phi|^2), \tag{226}$$

where C is a numerical constant. We have that the Ricci scalar transforms as:

$$R \to \hat{R} = \Omega^{-2}(R + 6\Box\lambda - 6(\partial\lambda)^2),$$

with $\lambda = -\frac{1}{2}\log\Omega$ and $\Box = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}g^{\mu\nu}\partial_\mu)$, so that:

$$R = \Omega^2\hat{R} + 6(\partial\lambda)^2 - 6\Box\lambda$$

$$R = \Omega^2\hat{R} + 6(\partial(-\frac{1}{2}\log\Omega))^2 - 6\Box(-\frac{1}{2}\log\Omega)$$

Now we will show that the constant is $C = \frac{1}{6}$, so that the action becomes invariant under Weyl rescalings. We have as before:

$$\begin{aligned} S &= \int d^4x\sqrt{-\hat{g}}(\hat{g}^{\mu\nu}\left((\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi}^*) + \Omega^{-2}(\partial_\mu\Omega)(\partial_\nu\Omega)|\hat{\phi}|^2 + \Omega^{-1}((\partial_\mu\Omega)(\partial_\nu\hat{\phi}^*)\hat{\phi} + (\partial_\mu\hat{\phi})(\partial_\nu\Omega)\hat{\phi}^*)\right) \\ &\quad + C\cdot\Omega^{-4}R|\phi|^2) \\ &= \int d^4x\sqrt{-\hat{g}}\left(\hat{g}^{\mu\nu}(\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi}^*) + \Omega^{-2}((\partial\lambda)^2 + \Box\lambda)|\hat{\phi}|^2 + C\cdot\Omega^{-4}(\Omega^2\hat{R}|\phi|^2 + 6(\partial\lambda)^2|\phi|^2 - 6\Box\lambda|\phi|^2)\right) \\ &= \int d^4x\sqrt{-\hat{g}}\left(\hat{g}^{\mu\nu}(\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi}^*) + \Omega^{-2}((\partial\lambda)^2 + \Box\lambda)|\hat{\phi}|^2 + C\cdot\Omega^{-4}(\Omega^4\hat{R}|\hat{\phi}|^2 - 6\Omega^2((\partial\lambda)^2 + \Box\lambda)|\hat{\phi}|^2)\right) \\ &= \int d^4x\sqrt{-\hat{g}}\left(\hat{g}^{\mu\nu}(\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi}^*) + \Omega^{-2}((\partial\lambda)^2 + \Box\lambda)|\hat{\phi}|^2 + C\cdot(\hat{R}|\hat{\phi}|^2 - 6\Omega^{-2}((\partial\lambda)^2 + \Box\lambda)|\hat{\phi}|^2)\right) \end{aligned} \tag{227}$$

If we set $C = \frac{1}{6}$, the λ terms will cancel, and we will have the same action left, as before the Weyl transformation:

$$\hat{S} = \int d^4x\sqrt{-\hat{g}}\left(\hat{g}^{\mu\nu}(\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi}^*) + \frac{1}{6}\hat{R}|\hat{\phi}|^2\right) \tag{228}$$

To show the steps in-between the first and second equality in *Equation 227*, by the use of integration by parts, see below:

$$\begin{aligned}
&\hat{g}^{\mu\nu}(\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi}^*) + \Omega^{-2}((\partial\lambda)^2 + \Box\lambda)|\hat{\phi}|^2 \\
&= \hat{g}^{\mu\nu}\left((\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi}^*) + \Omega^{-2}(\partial_\mu\Omega)(\partial_\nu\Omega)|\hat{\phi}|^2 + \Omega^{-1}((\partial_\mu\Omega)(\partial_\nu\hat{\phi}^*)\hat{\phi} + (\partial_\mu\hat{\phi})(\partial_\nu\Omega)\hat{\phi}^*)\right) \\
&= \hat{g}^{\mu\nu}(\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi}^*) + \hat{g}^{\mu\nu}(\partial_\mu\log\Omega)(\partial_\nu\log\Omega)|\hat{\phi}|^2 + \hat{g}^{\mu\nu}\Omega^{-1}((\partial_\mu\Omega)(\partial_\nu\hat{\phi}^*)\hat{\phi} + (\partial_\mu\hat{\phi})(\partial_\nu\Omega)\hat{\phi}^*) \\
&= \hat{g}^{\mu\nu}(\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi}^*) + 4\hat{g}^{\mu\nu}(\partial_\mu\lambda)(\partial_\nu\lambda)|\hat{\phi}|^2 + \hat{g}^{\mu\nu}\Omega^{-1}((\partial_\mu\Omega)(\partial_\nu\hat{\phi}^*)\hat{\phi} + (\partial_\mu\hat{\phi})(\partial_\nu\Omega)\hat{\phi}^*) \\
&= \hat{g}^{\mu\nu}(\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi}^*) + 4(\partial\lambda)^2|\hat{\phi}|^2 - 2\hat{g}^{\mu\nu}(\partial_\mu\lambda)((\partial_\nu\hat{\phi}^*)\hat{\phi} + (\partial_\mu\hat{\phi})\hat{\phi}^*) \\
&= \hat{g}^{\mu\nu}(\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi}^*) + 4(\partial\lambda)^2|\hat{\phi}|^2 - 2\hat{g}^{\mu\nu}(\partial_\mu\lambda)\partial_\nu(\hat{\phi}^*\hat{\phi}) \\
&= \hat{g}^{\mu\nu}(\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi}^*) + 4(\partial\lambda)^2|\hat{\phi}|^2 + 2(\hat{\Box}\lambda)|\hat{\phi}|^2 \\
&= \hat{g}^{\mu\nu}(\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi}^*) + \Omega^{-2}(\partial\lambda)^2|\hat{\phi}|^2 + \Omega^{-2}(\Box\lambda)|\hat{\phi}|^2 \\
&= \hat{g}^{\mu\nu}(\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi}^*) + \Omega^{-2}((\partial\lambda)^2 + \Box\lambda)|\hat{\phi}|^2
\end{aligned} \tag{229}$$

using $\frac{1}{\Omega}\partial\Omega = \partial\log\Omega$ in the third line.

For further investigation, we can try to use Penrose's idea of Weyl rescaling at infinity and see what happens:
One can use the choice of

$$\Omega = \frac{1}{r},$$

as was done to obtain *Equation 21.* So now, for example, we have that the scalar field transforms as:

$$\phi \to \hat{\phi} = \frac{1}{r}\phi$$

under a Weyl transformation.
However, I will not go into more detail with this.

13 Energy-Momentum Tensor

For theories coupled to gravity, we can obtain the energy-momentum (E-M) tensor by taking a variation of the action, with respect to the metric. In this approach, the E-M tensor is automatically symmetric. The action we will be working with is:

$$S = S_{\text{gravity}} + S_{\text{matter}} + S_{\text{interaction}},$$

The canonical approach of defining the E-M tensor, involves using:

$$p\dot{q} - L = \dot{q}\frac{\partial L}{\partial \dot{q}} - L.$$

If you try this for EM, we will get a non-symmetric E-M tensor. In this case, $q \to A_\mu(x)$, $\dot{q} \to \partial_0 A_\mu(x)$, $L \to -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, $p \to \frac{\partial L}{\partial \dot{q}} = -F^{0\mu}$. Therefore we get:

$$T_\mu{}^\nu = \partial_\mu \phi \frac{\delta L}{\delta \phi_\nu} - \delta_\mu^\nu L.$$

which is not even gauge invariant. There is a procedure to fix it (Belinfante; a modification of $T_{\mu\nu}$, so that it is constructed from the canonical E-M tensor and the spin current, to become symmetric and still conserved).
If we instead define the tensor, by coupling to gravity, we obtain:

$$T_\mu{}^\nu = \frac{1}{2}(F_{\mu\rho}F^{\nu\rho} - \frac{1}{4}\delta_\mu^\nu F_{\sigma\rho}F^{\sigma\rho})$$

This is gauge-invariant and symmetric. It is also traceless (due to conformal symmetry).

13.1 Boundary Discussion (Free Scalar Field)

The action of a free scalar field is:

$$S = \int_M d^4x |\partial\phi|^2 \tag{230}$$

Varying the action one will get:

$$\begin{aligned} \partial S &= \int_M d^4x \left(\partial_\mu \delta\phi \partial^\mu \phi^* + \partial_\mu \phi \partial^\mu \delta\phi^*\right) \\ &= \int_M d^4x \left(\partial_\mu(\delta\phi\partial^\mu\phi^*) - \delta\phi(\Box\phi^*) + \partial^\mu(\partial_\mu\phi\delta\phi^*) - (\Box\phi)\delta\phi^*\right) \\ &= \int_M d^4x \left(-\delta\phi(\Box\phi^*) - (\Box\phi)\delta\phi^*\right) + \int_{\partial M} d^3\Sigma^\nu \left(\delta\phi(\partial_\nu\phi^*) + (\partial_\nu\phi)\delta\phi^*\right) \end{aligned} \tag{231}$$

using integration be parts in the first step. In the second step we have now split it into two integrals (Using Stokes/Greens Theorem); one a bulk integral, the other a boundary integral, consisting of the two total derivative terms.

Setting $\delta S = 0$, this must imply for the bulk contribution that $\Box\phi = 0$ (or $\Box\phi^* = 0$), and for the boundary contribution that $\delta\phi(\partial_n\phi^*) = 0$, with $\partial_n\phi^*$ being the normal direction on the boundary, depicted in *Figure 22*.

Boundary Conditions: There are two natural boundary conditions following the above:

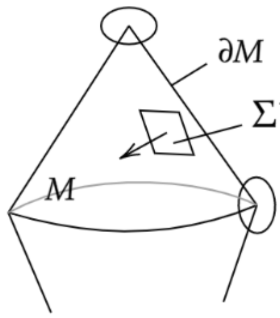


Figure 22: Normal direction to the boundary surface

1. Dirichlet boundary condition: $\delta\phi = 0$ (ϕ is fixed to some $\phi_0(x)$ for $x \in \partial M$)
2. Neumann boundary condition: $\partial_n \phi = 0$

For a conformally coupled scalar (see *Section 12*, and following in next section), the variation of the action can also be split up into two integrals, a bulk and a boundary integral (the boundary integral is the same as for a free scalar):

$$\begin{aligned} S &= \int_M d^4x\sqrt{-g}(|\partial\phi|^2 - \frac{1}{6}R|\phi|^2) \\ \delta S &= \int_M d^4x(...) + \int_{\partial M}(...) \end{aligned} \tag{232}$$

What about for the gravity action, does the boundary integral exist? Take the variation:

$$\delta_g S = \int_M d^4x\sqrt{-g}\delta g^{\mu\nu}T_{\mu\nu} + \int_{\partial M}(...) \tag{233}$$

We start with calculating the difficult part:

$$\delta_g R = \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \tag{234}$$

with $\delta R_{\mu\nu} = \delta R^{\rho}_{\mu\rho\nu}$ and

$$R^{\sigma}_{\mu\nu\rho} = \partial_\mu \Gamma^{\sigma}_{\nu\rho} - \partial_\nu \Gamma^{\sigma}_{\mu\rho} + [\Gamma, \Gamma]^{\sigma}_{\mu\nu\rho} \tag{235}$$

where taking the variation/derivative of the two first terms, followed by integration by parts, leads to contributions to the boundary integral. Worthy of note is also that $\delta\Gamma^{\rho}_{\mu\nu}$ is a tensor, however $\Gamma^{\rho}_{\mu\nu}$ itself is not.

13.2 Energy-Momentum Tensor for a Conformally Coupled Scalar

We want to find the E-M tensor for a conformally coupled scalar, by variation with respect to the metric.

We first write up the E-M tensor with this conformal coupling term, which is a result of the discussion in *Section 12*:

$$T_{\mu\nu} = (\partial_\mu \phi^*)(\partial_\nu \phi) + (\partial_\nu \phi^*)(\partial_\mu \phi) + \frac{1}{6} R|\phi|^2 \tag{236}$$

Now varying the action for a conformally coupled scalar, with respect to the metric:

$$\delta_g S_{conf.coupl.\phi}[g, \phi] = \int d^4x \sqrt{g} \delta g^{\mu\nu} T_{\mu\nu} \tag{237}$$

Varying the metric is the same as letting $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. Since $T_{\mu\nu}$ now has a term with the Ricci scalar (in contrary to the E-M tensor of a scalar, without the conformal coupling term), the variation of this is (using $R = g^{\mu\nu} R_{\mu\nu}$):

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \tag{238}$$

and the variation of the Ricci tensor (using $R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu}$) is:

$$\delta R_{\mu\nu} = \delta g^{\rho\sigma} R_{\rho\mu\sigma\nu} + g^{\rho\sigma} \delta R_{\rho\mu\sigma\nu} \tag{239}$$

The variation of the Riemann curvature tensor is: ($R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}$):

$$\delta R^\rho_{\sigma\mu\nu} = \partial_\mu \delta\Gamma^\rho_{\nu\sigma} - \partial_\nu \delta\Gamma^\rho_{\mu\sigma} + \delta\Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\mu\lambda} \delta\Gamma^\lambda_{\nu\sigma} - \delta\Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\nu\lambda} \delta\Gamma^\lambda_{\mu\sigma} \tag{240}$$

One thing to notice is that Γ is not a tensor, however $\delta\Gamma$ is.
Finally we have that the Christoffel symbol is:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{241}$$

and its variation is:

$$\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2} \delta g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) + \frac{1}{2} g^{\rho\sigma} (\partial_\mu \delta g_{\nu\sigma} + \partial_\nu \delta g_{\mu\sigma} - \partial_\sigma \delta g_{\mu\nu}) \tag{242}$$

This will be continued in *Section 13.2.2*.

13.2.1 Properties of an Energy-Momentum Tensor

We have that the variation of the action, of a conformally coupled scalar, with respect to the metric is (*Eq. 237*):

$$\delta_g S_{ccs} = \int d^4x \sqrt{g} \delta g^{\mu\nu} T_{\mu\nu} \tag{243}$$

The properties of an E-M tensor of a conformal scalar field are, that it is:

- Symmetric (couples to $g_{\mu\nu}$, which is symmetric)

- Conserved
- Traceless (a consequence of conformal symmetry)

So now we write up all possible terms for the E-M tensor, such that these conditions hold, for a conformally coupled scalar:

$$T_{\mu\nu} = (\partial_\mu \phi^*)(\partial_\nu \phi) + (\partial_\nu \phi^*)(\partial_\mu \phi) + c\eta_{\mu\nu}(\partial^\rho \phi^*)(\partial_\rho \phi) + d(\phi^*(\partial_\mu \partial_\nu \phi) + (\partial_\mu \partial_\nu \phi^*)\phi) \quad (244)$$

Clearly this quantity is symmetric, and to check conservation we need to prove:

$$\begin{aligned}
0 &= \partial^\mu T_{\mu\nu} = \eta^{\mu\sigma}\partial_\sigma T_{\mu\nu} \\
&= \eta^{\mu\sigma}(\partial_\sigma \partial_\mu \phi^*)(\partial_\nu \phi) + \eta^{\mu\sigma}(\partial_\mu \phi^*)(\partial_\sigma \partial_\nu \phi) + \eta^{\mu\sigma}(\partial_\sigma \partial_\nu \phi^*)(\partial_\mu \phi) + \eta^{\mu\sigma}(\partial_\nu \phi^*)(\partial_\sigma \partial_\mu \phi) \\
&+ c\eta^{\mu\sigma}\eta_{\mu\nu}(\partial_\sigma \partial^\rho \phi^*)(\partial_\rho \phi) + c\eta^{\mu\sigma}\eta_{\mu\nu}(\partial^\rho \phi^*)(\partial_\sigma \partial_\rho \phi) \\
&+ d\eta^{\mu\sigma}(\partial_\sigma \phi^*(\partial_\mu \partial_\nu \phi) + (\partial_\sigma \partial_\mu \partial_\nu \phi^*)\phi) + d\eta^{\mu\sigma}(\phi^*(\partial_\sigma \partial_\mu \partial_\nu \phi) + (\partial_\mu \partial_\nu \phi^*)\partial_\sigma \phi) \\
&= (\Box \phi^*)(\partial_\nu \phi) + (\partial_\nu \phi^*)(\Box \phi) + d(\phi^*(\partial_\nu \Box \phi) + (\partial_\nu \Box \phi^*)\phi) \\
&+ (1 + c + d)\eta^{\mu\sigma}(\partial_\mu \phi^*)(\partial_\sigma \partial_\nu \phi) + (1 + c + d)\eta^{\mu\sigma}(\partial_\sigma \partial_\nu \phi^*)(\partial_\mu \phi)
\end{aligned} \quad (245)$$

The equations of motion, when the metric is flat, $g = \eta$, are $\Box\phi = 0$ and $\Box\phi^* = 0$. Thus in order for the above to become zero, we must have:

$$1 + c + d = 0 \quad (246)$$

Now we check for tracelessness:

$$0 = \eta^{\mu\nu}T_{\mu\nu} = 2\left(\partial^\mu \phi^*\right)\left(\partial_\mu \phi\right) + 4c\left(\partial^\mu \phi^*\right)\left(\partial_\mu \phi\right) + d\left(\left(\Box \phi^*\right)\phi + \phi^*(\Box \phi)\right) \quad (247)$$

For this to be zero, we must have:

$$2 + 4c = 0 \quad (248)$$

With the two restrictions on the constants, we find that:

$$c = -\frac{1}{2}, \quad d = -\frac{1}{2} \quad (249)$$

Thus the E-M tensor has the final form of:

$$\begin{aligned}
T_{\mu\nu} &= \left(\partial_\mu \phi^*\right)\left(\partial_\nu \phi\right) + \left(\partial_\nu \phi^*\right)\left(\partial_\mu \phi\right) \\
&- \frac{1}{2}\eta_{\mu\nu}\left(\partial^\rho \phi^*\right)\left(\partial_\rho \phi\right) \\
&- \frac{1}{2}\left(\phi^*\left(\partial_\mu \partial_\nu \phi\right) + \left(\partial_\mu \partial_\nu \phi^*\right)\phi\right)
\end{aligned} \quad (250)$$

This will be our reference.

13.2.2 The E-M Tensor of a Conformally Coupled (Charged) Scalar

The action of a conformally coupled scalar is (writing out *Eq. 237*):

$$S_{ccs} = \int d^4x\sqrt{g}\left(\left(\partial_\mu\phi^*\right)\left(\partial_\nu\phi\right)g^{\mu\nu} + \frac{1}{6}R|\phi|^2\right) \tag{251}$$

If ϕ is charged, the partial derivative becomes the covariant derivative $\partial_\mu \to D_\mu = \partial_\mu + iA_\mu$, and the energy-momentum tensor is not symmetric. This can be dealt with by letting $\partial_\mu\partial_\nu \to D_{(\mu}D_{\nu)}$.

Taking the variation gives:

$$\begin{aligned}\delta_g S_{ccs} =& \int d^4x\left[\left(-\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu}\right)\mathcal{L}_{\text{ccs}} + \sqrt{g}\left(\delta g^{\mu\nu}\left(\partial_\mu\phi^*\right)\left(\partial_\nu\phi\right) + \frac{1}{6}|\phi|^2\left(\delta g^{\mu\nu}R_{\mu\nu} + \nabla_\rho A^\rho\right)\right)\right]\\ =& \int d^4x\delta g^{\mu\nu}\sqrt{g}\left(-\frac{1}{2}g_{\mu\nu}\mathcal{L}_{\text{ccs}} + \left(\partial_\mu\phi^*\right)\left(\partial_\nu\phi\right) + \frac{1}{6}|\phi|^2R_{\mu\nu}\right) + \text{tot derivative}\end{aligned} \tag{252}$$

The total derivative is (using $(\nabla_\rho A^\rho) = \frac{1}{\sqrt{g}}\partial_\rho(\sqrt{g}A^\rho)$):

$$\begin{aligned}\text{Total derivative } =& \frac{1}{6}\int d^4x\sqrt{g}|\phi|^2(\nabla_\rho A^\rho)\\ =& \frac{1}{6}\int d^4x|\phi|^2\partial_\rho\left(\sqrt{g}A^\rho\right)\\ =& \frac{1}{6}\int_{\text{boundary}} d\vec{S}\cdot\left(|\phi|^2\sqrt{g}\vec{A}\right) - \frac{1}{6}\int d^4x\left(\partial_\rho|\phi|^2\right)\sqrt{g}A^\rho\end{aligned} \tag{253}$$

where

$$A^\rho = g^{\sigma\nu}\delta\Gamma^\rho_{\nu\sigma} - g^{\sigma\rho}\delta\Gamma^\mu_{\mu\sigma} \tag{254}$$

The variation of the Christoffel symbol, for a flat metric (we want the metric to be flat, such that we can impose the EOM $\Box\phi = 0$ and $\Box\phi^* = 0$, as in the previous section), compared to *Eq. 242*, is:

$$\left(\delta\Gamma^\rho_{\mu\nu}\right)\big|_{g=\eta} = \tfrac{1}{2}\eta^{\rho\sigma}\left(\partial_\mu\delta g_{\sigma\nu} + \partial_\nu\delta g_{\sigma\mu} - \partial_\sigma\delta g_{\mu\nu}\right) \tag{255}$$

We then use this in the equation for the gauge field:

$$\begin{aligned}A^\rho\big|_{g=\eta} =& \eta^{\sigma\nu}\delta\Gamma^\rho_{\nu\sigma}\big|_{g=\eta} - \eta^{\sigma\rho}\delta\Gamma^\mu_{\mu\sigma}\big|_{g=\eta}\\ =& \eta^{\sigma\nu}\frac{1}{2}\eta^{\rho\tau}\left(\partial_\nu\delta g_{\sigma\tau} + \partial_\sigma\delta g_{\nu\tau} - \partial_\tau\delta g_{\nu\sigma}\right) - \eta^{\sigma\rho}\frac{1}{2}\eta^{\mu\tau}\left(\partial_\mu\delta g_{\sigma\tau} + \partial_\sigma\delta g_{\mu\tau} - \partial_\tau\delta g_{\mu\sigma}\right)\\ =& \frac{1}{2}\left(\partial^\sigma(\delta g^\rho_\sigma) + \partial^\nu(\delta g^\rho_\nu) - \partial^\rho(\delta g^\sigma_\sigma) - \partial^\tau(\delta g^\rho_\tau) - \partial^\rho(\delta g^\tau_\tau) + \partial^\mu(\delta g^\rho_\mu)\right)\\ =& \partial^\sigma(\delta g^\rho_\sigma) - \partial^\rho(\delta g^\tau_\tau)\end{aligned} \tag{256}$$

The contribution (of the total derivative) from integration by parts is (see *Eq. 253*):

$$-\frac{1}{6}\int d^4x\sqrt{g}\left(\partial_\rho|\phi|^2\right)A^\rho \tag{257}$$

We then evaluate for $g=\eta$, and obtain:

$$\begin{aligned}&= -\tfrac{1}{6}\int d^4x\left(\partial_\rho|\phi|^2\right)\left(\partial^\sigma\left(\delta g^\rho_\sigma\right)-\partial^\rho(\delta g^\tau_\tau)\right)\\ &= \tfrac{1}{6}\int d^4x\left(\partial_\sigma\partial_\rho|\phi|^2\delta g^{\sigma\rho}-\Box|\phi|^2\eta_{\sigma\rho}\delta g^{\sigma\rho}\right)\\ &= \tfrac{1}{6}\int d^4x\delta g^{\sigma\rho}\left(\partial_\sigma\partial_\rho|\phi|^2-\eta_{\sigma\rho}\Box|\phi|^2\right)\end{aligned} \tag{258}$$

The contribution to $T_{\mu\nu}$, is then equal to:

$$\begin{aligned}&\frac{1}{6}(\partial_\mu\partial_\nu|\phi|^2-\eta_{\mu\nu}\Box|\phi|^2)\\ &=\frac{1}{6}((\partial_\mu\partial_\nu\phi^*)\,\phi+(\partial_\mu\phi^*)\,(\partial_\nu\phi)+(\partial_\nu\phi^*)\,(\partial_\mu\phi)+\phi^*\,(\partial_\mu\partial_\nu\phi)\\ &-\eta_{\mu\nu}\,(\Box\phi^*)\,\phi-\eta_{\mu\nu}\phi^*(\Box\phi)\;-\eta_{\mu\nu}2\,(\partial^\rho\phi^*)\,(\partial_\rho\phi))\end{aligned} \tag{259}$$

The other contributions to $T_{\mu\nu}$ are (see *Eq. 252*):

$$\begin{aligned}&-\frac{1}{2}\eta_{\mu\nu}\mathcal{L}_{ccs}+\frac{1}{2}\left((\partial_\mu\phi^*)\,(\partial_\nu\phi)+(\mu\leftrightarrow\nu)\right)\\ &=-\frac{1}{2}\eta_{\mu\nu}\left((\partial^\rho\phi^*)\,(\partial_\rho\phi)\right)+\frac{1}{2}\,(\partial_\mu\phi^*)\,(\partial_\nu\phi)+\frac{1}{2}\,(\partial_\nu\phi^*)\,(\partial_\mu\phi)\end{aligned} \tag{260}$$

Now subtracting (the minus sign comes from the coefficient $\frac{1}{6}$) *Eq. 259* from *Eq. 260*, we obtain:

$$\begin{aligned}T_{\mu\nu}=&\frac{1}{3}\,(\partial_\mu\phi^*)\,(\partial_\nu\phi)+\frac{1}{3}\,(\partial_\nu\phi^*)\,(\partial_\mu\phi)-\frac{1}{6}\eta_{\mu\nu}\,(\partial^\rho\phi^*)\,(\partial_\rho\phi)\\ &-\frac{1}{6}\,(\partial_\mu\partial_\nu\phi^*)\,\phi-\frac{1}{6}\phi^*\,(\partial_\mu\partial_\nu\phi)+\frac{1}{6}\eta_{\mu\nu}\,(\Box\phi^*)\,\phi+\frac{1}{6}\eta_{\mu\nu}\phi^*(\Box\phi)\end{aligned} \tag{261}$$

which is both symmetric and traceless:

$$0=\eta^{\mu\nu}T_{\mu\nu}=\frac{2}{3}|\partial\phi|^2-\frac{4}{6}|\partial\phi|^2-\frac{1}{6}\,(\Box\phi^*)\,\phi-\frac{1}{6}\phi^*(\Box\phi)+\frac{4}{6}\,(\Box\phi^*)\,\phi+\frac{4}{6}\phi^*(\Box\phi) \tag{262}$$

Where the first two terms cancel, and the other terms become zero by the EOM (vanish on-shell). And checking conservation:

$$\begin{aligned}\partial^\mu T_{\mu\nu}=&\frac{1}{6}\Bigg(2\,(\Box\phi^*)\,(\partial_\nu\phi)+2\,(\partial_\mu\phi^*)\,(\partial_\mu\partial_\nu\phi)+2\,(\partial_\mu\partial_\nu\phi^*)\,(\partial_\mu\phi)+2\,(\partial_\nu\phi^*)\,(\Box\phi)\\ &-(\partial_\nu\partial_\rho\phi^*)\,(\partial_\rho\phi)-(\partial_\rho\phi^*)\,(\partial_\nu\partial_\rho\phi)\\ &-(\partial_\nu\Box\phi^*)\,\phi-(\partial_\mu\partial_\nu\phi^*)\,(\partial_\mu\phi)-(\partial_\mu\phi^*)\,(\partial_\mu\partial_\nu\phi)-\phi^*\,(\partial_\nu\Box\phi)\\ &+(\partial_\nu\Box\phi^*)\,\phi+(\Box\phi^*)\,(\partial_\nu\phi)+(\partial_\nu\phi^*)\,(\Box\phi)+\phi^*\,(\partial_\nu\Box\phi)\Bigg)=0\end{aligned} \tag{263}$$

We see that all terms cancel eachtother.
Thus we have found a symmetric, conserved and tranceless E-M tensor for a **charged** conformally coupled scalar, *Eq. 261*, compared to the version of a non-charged scalar (*Eq. 250*).

14 Using Weyl Transformations to obtain a Non-Singular Metric on $\mathcal{I}^{\pm}$

Focusing on $U(1)$ gauge theory (QED) on MS; a Weyl transformation of the metric is:

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \tag{264}$$

where we will use the choice $\Omega = \frac{1}{r}$, as described in *Section 12*. We study the theory on the compactified MS, which has points at infinity added to it (the boundaries $\mathcal{I}^{\pm}$, and points i^0 and $i^{\pm}$). Before the space were open, not including the boundaries, however now the space is open, boundaries included.
For this compact space, we have the coordinates for Ω:

$$\begin{aligned} &\Omega_+(r, u, \theta, \phi) \to \text{used to add } \mathcal{I}^+ \\ &\Omega_-(r, v, \theta, \phi) \to \text{used to add } \mathcal{I}^- \end{aligned} \tag{265}$$

This leads us to the question of how the two boundaries are connected, which calls for a matching condition of i^0 (antipodal), such that we can relate the BMS_+ group (near future null infinity) to the BMS_- group (at past null infinity).

14.1 Identifying $\mathcal{I}^+$ and $\mathcal{I}^-$

A short discussion of this topic appeared in [Pen74], and in *Section 3.7*, where the Schwarzschild 'unphysical' metric had been written in the form, using retarded- and advanced time coordinates, with the choice of conformal factor $\Omega = \frac{1}{r} = w$ (*Eq. 19*); and where an extension from one boundary to the other, ie. including negative values of w, would involve a reversal of the sign of the mass. The identification of the two boundaries, with the same sign of the mass, would imply a discontinuity in the derivative of the curvature across $\mathcal{I}$.
Thus Penrose chose to only work with the two disjoint boundary hypersurfaces $\mathcal{I}^+$ and $\mathcal{I}^-$.

Consider the metric:

$$ds^2 = dr^2 - dt^2 + r^2 d\Omega^2 = r^2 \left(\frac{dr^2}{r^2} - \frac{dt^2}{r^2} + d\Omega^2 \right) \tag{266}$$

where $d\Omega^2$ is the metric on a unit sphere. Converting to retarded and advanced coordinates, $u = t - r$ and $v = r + t$, give $r = \frac{1}{2}(v - u)$ and $t = \frac{1}{2}(v + u)$, such that:

$$dr^2 - dt^2 = (d(r-t))(d(r+t)) = -du dv \tag{267}$$

$$ds^2 = \frac{1}{4}(v-u)^2 \left(\frac{-du dv}{(v-u)^2/4} + d\Omega^2 \right) \tag{268}$$

where this Ω only depends on angles.

We have a freedom in the choice of Ω (a different Ω, than in the discussion above!), as $\Omega' \to \theta\Omega$, where θ is smooth and positive (const. at ∞), and a function of angles. A matching condition at i^0, imposes a constraint between $\theta|_{\mathcal{I}^+}$ and $\theta|_{\mathcal{I}^-}$. An should be something like:

$$\theta|_{\mathcal{I}^+}(\vec{n}) = \theta|_{\mathcal{I}^-}(-\vec{n}) \tag{269}$$

The sign difference makes this an antipodal matching condition.

14.2 Conformal Invariance (Inversion)

Picture the Einstein static universe (*Figure 11*); an idea is to map infinity to zero; by inversion in flat space.

If we take a vector x^μ, and then apply an inversion, one will get $I(x^y) = \frac{x^\mu}{x^2}$.
A Lorentz transformation, a translation, and an inversion gives a conformal transformation:

$$\begin{aligned} x^\mu \xrightarrow{\text{inv}} \frac{x^\mu}{x^2} \xrightarrow{\text{translation}} \frac{x^\mu}{x^2} + a^\mu \xrightarrow{inv} \frac{\frac{x^\mu}{x^2} + a^\mu}{(\frac{x^\mu}{x^2} + a^\mu)^2} \\ = \frac{\frac{x^\mu}{x^2} + a^\mu}{\frac{1}{x^2} + 2\frac{xa}{x^2} + a^2} = \frac{x^\mu + a^\mu x^2}{1 + 2ax + a^2x^2} \end{aligned} \tag{270}$$

where the last equality is a special conformal transformation.

Consider the inversion ($\infty \to 0$), of metric $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, with $x'^\mu = \frac{x^\mu}{x^2}$ and $dx'^\mu = \frac{dx^\mu}{x^2} - \frac{2x^\mu x_\nu dx^\nu}{(x^2)^2}$, such that:

$$ds'^2 = \eta_{\mu\nu}(dx'^\mu)(dx'^\nu) = \Omega(x) ds^2 \tag{271}$$

We can compute the Ω:

$$\begin{aligned} ds'^2 &= \eta_{\mu\nu} \left(\frac{dx^\mu}{x^2} - \frac{2x^\mu x \cdot dx}{(x^2)^2} \right) \left(\frac{dx^\nu}{x^2} - \frac{2x^\nu x \cdot dx}{(x^2)^2} \right) \\ &= \eta_{\mu\nu} \left(\frac{dx^\mu dx^\nu}{x^4} - \frac{2dx^\mu x^\nu x \cdot dx}{x^6} - \frac{2dx^\nu x^\mu x \cdot dx}{x^6} + \frac{4x^\mu x \cdot dx x^\nu x \cdot dx}{x^8} \right) \\ &= x^{-4} ds^2 - 2x^{-6} \eta_{\mu\nu} \left(dx^\mu x^\nu x \cdot dx + dx^\nu x^\mu x \cdot dx \right) + 4x^{-8} \eta_{\mu\nu} (x^\mu x \cdot dx x^\nu x \cdot dx) \\ &= x^{-4} ds^2 - 4x^{-6} (x \cdot dx)^2 + 4x^{-8} x^2 (x \cdot dx)^2 \\ &= x^{-4} ds^2 \end{aligned} \tag{272}$$

which implies that $\Omega(x) = x^{-4}$. This is just a conformal transformation; the inversion maps infinity to zero.

15 Treating Spatial Infinity as a Boundary of Space Time

In [AR92] by Ashtekar and Romano, they wanted to find a manifestly coordinate independent treatment of spatial infinity, which avoids the awkwardness of the differentiability conditions at i^0. This is due to spatial infinity arising as a single point, where the vertex of the light cone is representing future and past null infinity, $\mathcal{I}^{\pm}$. Spatial infinity is naturally tied to null infinity, and it is possible to establish theorems relating the two regimes. However, all points at spatial infinity are squeezed down to a single point, and one has to use awkward differentiability conditions at this point.

15.1 Asymptotic Structure of Minkowski Space

(t, x, y, z) denotes a Cartesian chart for MS $(\hat{M}, \hat{\eta}_{ab})$, where $\hat{M}$ is a tensor field that refers to the physical spacetime, MS, and $\hat{\eta}_{ab}$ is the physical spacetime metric. Introducing standard hyperbolic coordinates $(\rho, \chi, \theta, \phi)$ as:

$$\begin{aligned} t &= \rho \sinh \chi \\ x &= \rho \cosh \chi \sin \theta \cos \phi \\ y &= \rho \cosh \chi \sin \theta \sin \phi \\ z &= \rho \cosh \chi \cos \theta \end{aligned} \tag{273}$$

The metric takes the form:

$$\begin{aligned} \hat{\eta}_{ab} &= -\nabla_a t \nabla_b t + \nabla_a x \nabla_b x + \nabla_a y \nabla_b y + \nabla_a z \nabla_b z \\ &= \nabla_a \rho \nabla_b \rho + \rho^2 \left[-\nabla_a \chi \nabla_b \chi + \cosh^2 \chi \left(\nabla_a \theta \nabla_b \theta + \sin^2 \theta \nabla_a \phi \nabla_b \phi \right) \right] \\ &= \nabla_a \rho \nabla_b \rho + \rho^2 h_{ab} \end{aligned} \tag{274}$$

with spacetime signature $(- + ++)$, and where h_{ab} is the unit time-like hyperboloid metric. Here spatial infinity is described by the limit $\rho \to \infty$.

This is the equivalent to a flat metric (Euclidean metric tensor) in polar coordinates:

$$ds^2 = dx^2 + dy^2 = d\rho^2 + \rho^2 d\theta^2 \tag{275}$$

where $ds^2 = \hat{\eta}_{ab}$ and $d\theta^2 = h_{ab}$. The extend of the coordinates can be pictured by *Figure 23.*

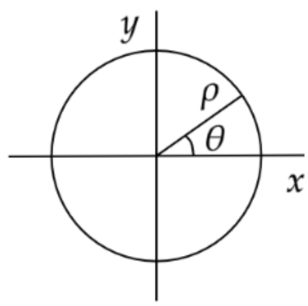


Figure 23: Polar coordinates, ρ being the measure of distance from the origin, and θ the angle.

Thus $\rho = \text{const}$ implies a circle.

Now we define coordinate $\Omega := \frac{1}{\rho}$, and extend $\hat{M}$ to include points, where Ω vanishes. The metric can be rewritten in terms of the new coordinate:

$$\hat{\eta}_{ab} = \Omega^{-4}\nabla_a\Omega\nabla_b\Omega + \Omega^{-2}h_{ab} \tag{276}$$

and is singular at $\Omega = 0$.

The equivalent here, using $\Omega = \frac{1}{\rho} \implies \rho = \frac{1}{\Omega}$ and $d\rho = -\frac{d\Omega}{\Omega^2}$. Thus $d\rho^2 = \Omega^{-4}d\Omega^2$. If we plug this in, the Euclidean metric becomes:

$$ds^2 = \Omega^{-4}d\Omega^2 + \Omega^{-2}d\theta^2 \tag{277}$$

Applying a Weyl transformation to the metric, one obtains:

$$d\hat{s}^2 = \Omega^4 ds^2 = d\Omega^2 + \Omega^2 d\theta^2 \xrightarrow{\Omega\to 0} d\Omega^2 = \Omega^4 d\rho^2 \tag{278}$$

The induced 3-metric, $\hat{q}_{ab}$, defined on the $\Omega = \text{const}(\neq 0)$, 3-surfaces, is:

$$\hat{q}_{ab} = \hat{\eta}_{ab} - l^{-1}\nabla_a\Omega\nabla_b\Omega \tag{279}$$

with $l := \hat{\eta}^{ab}\nabla_a\Omega\nabla_b\Omega\left(= \Omega^4\right)$, and therefore it follows that $\hat{q}_{ab} = \Omega^{-2}h_{ab}$.

To show this, one can simply plug in $\hat{\eta}_{ab}$ and l:

$$\begin{aligned}\hat{q}_{ab} &= \Omega^{-4}\nabla_a\Omega\nabla_b\Omega + \Omega^{-2}h_{ab} - \Omega^{-4}\nabla_a\Omega\nabla_b\Omega \\ \hat{q}_{ab} &= \Omega^{-2}h_{ab}\end{aligned} \tag{280}$$

Furthermore, one can find:

$$\begin{aligned}\hat{q}_{ab} &= \hat{\eta}_{ab} - l^{-1}\nabla_a\Omega\nabla_b\Omega \\ \implies (\hat{\eta}_{ab} - \hat{q}_{ab})l &= \nabla_a\Omega\nabla_b\Omega \\ \implies l &= \frac{\nabla_a\Omega\nabla_b\Omega}{\hat{\eta}_{ab} - \hat{q}_{ab}} \\ l &= \frac{\nabla_a\Omega\nabla_b\Omega}{\Omega^{-4}\nabla_a\Omega\nabla_b\Omega} = \Omega^4\end{aligned} \tag{281}$$

Although $\Omega^2\hat{\eta}_{ab}$ does not admit a smooth extension to the 3-surface corresponding to $\Omega = 0$, the rescaled 3-metric:

$$q_{ab} := h_{ab} = \Omega^2\hat{q}_{ab} = \Omega^2\left(\hat{\eta}_{ab} - l^{-1}\nabla_a\Omega\nabla_b\Omega\right) \tag{282}$$

is well-defined on the boundary.

The analogue here, for the induced metric on $\rho = \text{const}$ (or $\Omega = \text{const}$), from *Eq. 277*, is:

$$\begin{aligned} ds^2|_{\text{induced}} &= \rho^2 d\theta^2 = \Omega^{-2} d\theta^2 \\ \Omega^2 ds^2|_{\text{induced}} &= d\theta^2 \quad (\implies \Omega^2 \hat{q}_{ab} = h_{ab}) \end{aligned} \tag{283}$$

To get information from the $\Omega = const$ surfaces, consider the contravariant normal:

$$\hat{\eta}^{ab} \nabla_b \Omega = \Omega^4 \left(\frac{\partial}{\partial \Omega} \right)^a \tag{284}$$

$\frac{\partial}{\partial \Omega^a}$ is well defined when $\Omega = 0$, and thus:

$$n^a := \Omega^{-4} \hat{\eta}^{ab} \nabla_b \Omega \tag{285}$$

admits a smooth extension to the $\Omega = 0$ boundary.

To picture this, we write up the quantity:

$$\hat{q}_{ab} \hat{\eta}^{bc} \nabla_c \Omega = \nabla_a \Omega - l^{-1} \left(\nabla_a \Omega \right) \cdot l = 0 \tag{286}$$

since $\hat{q}_{ab} = \hat{\eta}_{ab} - l^{-1} (\nabla_a \Omega)(\nabla_b \Omega)$. Thus $\hat{q}_{ab}$ must be a projection orthogonal to $(\nabla_a \Omega)$, and $n_a = \nabla_a \Omega$ is a normal to this 3-surface, as pictured in *Figure 24*.

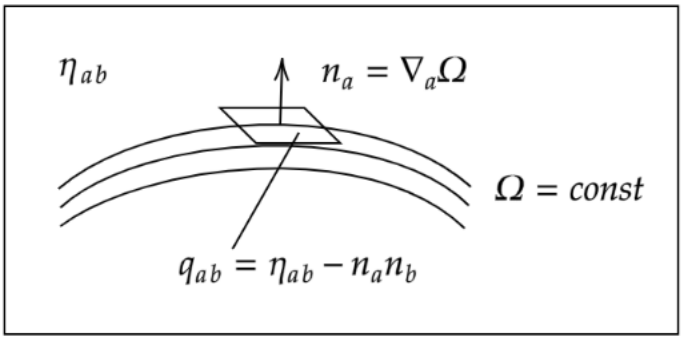


Figure 24: Illustration of the normal to the 3-surface of the induced metric. Note that $l = \eta^{ab} (\nabla_a \Omega)(\nabla_b \Omega) = \eta^{ab} n_a n_b = 1$.

In order to represent spatial infinity by a 3-surface, one must rescale the 3-metric and the normals to the $\Omega = \text{const}$ surfaces by different powers of Ω.

For curved space-times to be asymptotically flat at spatial infinity, they should resemble MS sufficiently to admit fields q_{ab} and n^a (*Eq. 282* and *Eq. 285*) to have smooth limits to the boundary.

15.2 Hyperbolic Slicing II

In comparison to this, Strominger had used (see *Section 5.6.1*), to describe massive QED, the dS_3 slices to resolve the structure of spatial infinity, which Ashtekar and Romano had further described in their paper.

These dS_3 slices ($\tau^2 = $ const surfaces) are labeled by:

$$\tau^2 = t^2 - r^2$$

for $\tau^2 < 0$, and are illustrated in *Figure 13*.
The method was to take a slice of constant τ, and then let $\tau \to \infty$, to extend the slice to the boundary, in order to examine points near spatial infinity, see *Figure 25*.

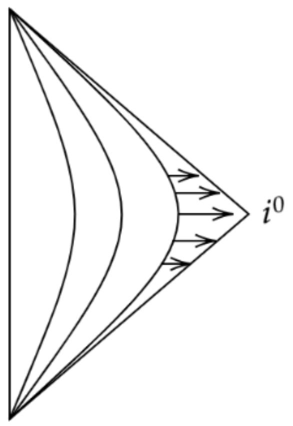


Figure 25: Examining points near spatial infinity by taking a slice of constant τ, and extend it to infinity.

16 Summary and Discussion

Overall it has been made clear just how important symmetries are, and how looking at the null infinity boundary of a compactified MS, opens up these different paths to analysing the asymptotic symmetries, due to the triangular equivalence relation of the IR sector of physics.

The approach of this thesis was to use the direct analysis of the geometry and asymptotic symmetries in gravity, as used by Penrose, and generalise it to gauge theories, in order to make the correspondence, between his' and Strominger's analysis, more clear.

The main points to take from this thesis is (the first part will be a repeat of the discussion in *Section 11.1*)**:**

Strominger starts with a discussion of asymptotic symmetries in QED, see *Sections 4 and 5*. First he derives conserved charges, following from antipodal matching conditions of the fields. He then show that they, via a canonical formalism, generate asymptotic symmetries; ie. he shows that the soft theorems give rise to Ward identities, corresponding to divergent large gauge transformations (asymptotic symmetries) acting on the celestial sphere at null infinity.

For non-abelian gauge theory (*Section 6*), he used the formalism of the $\mathcal{S}$-matrix as a celestial correlator, where the scattering of a soft gluon becomes the insertion of a current into a correlation function on CS^2. This current obeys the Ward identities of a $\mathcal{G}$-current Kac-Moody algebra, giving an alternative representation of the asymptotic symmetry group. The conserved charges are derived from the matching condition, shown to generate the symmetry and to imply the tree-level non-abelian soft theorem.

For his analysis (in gravity) of the BMS group he defines a large-r geometric constraint of the metric (see *Section 9*). He uses an analysis of asymptotic data (leading metric components) to obtain the matching condition for the Bondi mass m_B, which led to conserved supertranslation charges of past- and future null infinity. The supertranslations are generalizations of the four translations in Minkowski space.
He also derives conserved charges from a matching condition for subleading metric components, the angular momentum aspect N_z. These are the superrotation charges. Angular momentum is to superrotations (sub-leading soft theorem) as energy-momentum is to supertranslations (leading soft theorem).

Penrose on the other hand, focused only on gravity, and used a direct analysis of the geometry and the asymptotic symmetries (see *Section 7*). He assigned structure to $\mathcal{I}$; its weak inner conformal metric. The group of self-transformations of $\mathcal{I}$, preserving this weak inner conformal metric is the Newman-Unti group. Since $\mathcal{I}$ is a null hypersurface, its structure is too weak, and a strong conformal geometry was needed in order to restrict the Newman-Unti group, to the BMS group, and reduce the freedom in F.
The concept of angle between (non-null) tangent directions at a point of $\mathcal{I}^+$, comes from the content of the inner conformal metric of $\mathcal{I}^+$. When this angle is zero, the strong conformal geometry of $\mathcal{I}^+$ is needed to define this null angle between the directions. This way he found quantities $n^a dl$, which have to be preserved under transformations of $\mathcal{I}^+$ to itself, leading to the restrictions of the functions F, and hence the definition of BMS transformations; the group of self-transformations of $\mathcal{I}^+$, which preserves its strong conformal geometry.
He then begins a discussion of how the supertranslations form an infinite parameter (normal) subgroup (and the translations form a four-parameter normal subgroup) of the BMS group. The more elaborate discussion of this and its implication was covered in *Section 8*.

For the analysis of the components of the BMS and Poincaré group (see *Section 8*), we had that the Poincaré group cannot be found as a subgroup of the BMS group in a canonical way, which is related to the physical implications of how Lorentz rotations cannot be distinguished from the general BMS rotations (in a BMS-invariant way).

For massless particles it was important to distinguish the past- and future null infinity boundaries, in order to establish matching conditions of fields. A problem area was also when we got closer to spatial infinity, i^0, a singularity, which Penrose left as an exercise for the reader.
For massive particles however, the difficulty lied in the particles now ending at time-like infinity, i^+.
For spatial infinity, we shortly discussed how it could be related to $r = 0$ by inversion, in *Section 14*. This lead up to the discussed in *Section 15*, with an attempt to find a manifestly coordinate independent treatment of spatial infinity, which avoids the awkwardness of the differentiability conditions, by representing i^0 by a 3-surface. This was closely related to *Section 5.6*, where we discussed a way to analyse both of these points (i^0, i^+), by hyperbolic slicing of MS. By looking at slices of constant τ, which correspond to AdS_3 (at i^+) for $\tau^2 > 0$, and dS_3 (at i^0) for $\tau^2 < 0$, and then letting $\tau \to \infty$, one can analyse these endpoints.

The take on the asymptotic symmetries of magnetic charges in QED (*Section 5.7*), is that a second infinity of conserved charges is generated, when adding magnetic charges. These can be associated to symmetries, interpreted as electric gauge transformations but with an imaginary gauge parameter. The large electric transformations can be thought of as a non-trivial subgroup of the (electric) gauge group symmetry, but the magnetic ones can not. Not all asymptotic symmetries arise as a nontrivial subgroup of some manifest gauge symmetry.

What we would have wanted to figure out is:

A lot of ideas was in play, in order to make the connection of the approach of Penrose and Strominger more clear, though they all have very loose ends:

We started out writing the E-M tensor for a (conformally coupled) scalar (see *Section 13*). The purpose of this, was to unite this with the energy-momentum tensor for gravity and for a photon. By putting all these fields together, we wanted to see how they act near null infinity and what correction are needed, for example when applying a Weyl transformation. Applying a Weyl transformation to the action of a scalar (see *Section 12*), resulted in the action not being invariant, and was the reason why we had to add the conformal coupling term. We did not get any further than setting up the E-M tensor for the conformally coupled scalar, and this story therefore is not finished.

Furthermore the challenges with the boundary conditions, when setting up an E-M tensor, has also been briefly discussed in *Section 13.1.* We saw that the variation of the action can be split into two integrals, one of which is a boundary integral, and must therefore follow certain boundary conditions. However, this is also a very unfinished story.

Discussing the $\log r/r$ behaviour, as described in *Section 10*, when calculating the coefficients of the large-r expansion of the gauge parameters in QED; the $\log r/r$ term is needed in order to satisfy the wave equation. A thing to consider, is whether this story could be recovered in our analysis, had we gotten further. This was just one of many things to analyse; the rest of them is listed in *Section 11.4*, where we only got started on the third step (in *Section 14*), and can therefore not really conclude anything from it.

Another small thought, was whether one could use the formalism of the $\mathcal{S}$-matrix as a celestial correlator and the Kac-Moody algebra, for a direct analysis of asymptotic symmetries in gauge theories, as shortly mentioned in *Section 11.2.* This is also very incomplete.

17 Acknowledgements

First of all, I would like to thank my supervisor, Cristian Vergu, for all his help and guidance throughout the project. It has been a interesting and valuable journey, and I have learned many new concepts and methods that will surely help me in the future. Also thanks to Emil Bjerrum-Bohr for taking the role as supervisor.

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18 Appendix

18.1 Covariant Derivative and Restrictions on U

We define the vector U with components $U_z(z,\bar{z})$ and $U_{\bar{z}}(z,\bar{z})$, where $(z,\bar{z})$ are coordinates on the two-sphere with metric $g_{z\bar{z}} = 2\gamma_{z\bar{z}}dzd\bar{z}$, where $\gamma = (0, \gamma_{z\bar{z}}, \gamma_{z\bar{z}}, 0)$.

The covariant derivative of U is given by

$$D_\mu U = \partial_\mu U^\nu + \Gamma^\nu_{\mu\rho}U^\rho \tag{287}$$

with the Christoffel symbol defined as:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{288}$$

For a round metric, the non-vanishing components are $g_{z\bar{z}}$, so we have the non-vanishing component:

$$\Gamma^z_{zz} = \frac{1}{2}g^{z\sigma}(2\partial_z g_{z\sigma} - \partial_\sigma g_{zz}) = g^{z\bar{z}}\partial_z g_{z\bar{z}} = \partial_z \log(g_{z\bar{z}}) \tag{289}$$

Using $g^{z\bar{z}} = (g_{z\bar{z}})^{-1}$ in the last step. Another possible component:

$$\Gamma^z_{z\bar{z}} = \frac{1}{2}g^{z\bar{z}}(\partial_z g_{\bar{z}\bar{z}} + \partial_{\bar{z}} g_{z\bar{z}} - \partial_{\bar{z}} g_{z\bar{z}}) = 0 \tag{290}$$

The same goes for $\Gamma^{\bar{z}}_{z\bar{z}}$. The final non-vanishing component is:

$$\Gamma^{\bar{z}}_{zz} = \frac{1}{2}g^{\bar{z}z}(\partial_z g_{zz}) = 0 \tag{291}$$

Taking the covariant derivative with respect to vectors U_z and $U_{\bar{z}}$:

$$\begin{aligned}(D_zU)^z &= \partial_z U^z + \Gamma^z_{zz}U^z + \Gamma^z_{z\bar{z}}U^{\bar{z}} = \partial_z U^z + \Gamma^z_{zz}U^z \\ (D_zU)^{\bar{z}} &= \partial_z U^{\bar{z}} + \Gamma^{\bar{z}}_{zz}U^z + \Gamma^{\bar{z}}_{z\bar{z}}U^{\bar{z}} = \partial_z U^{\bar{z}}\end{aligned} \tag{292}$$

Taking the metric to be on the form (*Eq. 167*)

$$ds^2 = -du^2 - dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + U_z dudz + rC_{\bar{z}\bar{z}}d\bar{z}^2 + U_{\bar{z}}dud\bar{z} + ... \tag{293}$$

The Weyl tensor is given by:

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{2}(g_{\nu\rho}R_{\sigma\mu} + g_{\mu\sigma}R_{\rho\nu} - g_{\nu\sigma}R_{\rho\mu} - g_{\mu\rho}R_{\sigma\nu}) + \frac{1}{6}R(g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\rho\nu}) \tag{294}$$

where the Riemann tensor is:

$$R_{\rho\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma_{\rho\mu\lambda}\Gamma_{\lambda\nu\sigma} - \Gamma_{\rho\nu\lambda}\Gamma_{\lambda\mu\sigma} \tag{295}$$

One can now compute the components (here done with Sagemath):

18.1.1 The Weyl Tensor Components

The components of the Weyl tensor with two r indices.

$$C_{ruru} = (\frac{U_{\bar{z}}\partial_u\partial_z\gamma_{z\bar{z}}}{8\,\gamma_{z\bar{z}}^2} + \frac{U_z\partial_u\partial_{\bar{z}}\gamma_{z\bar{z}}}{8\,\gamma_{z\bar{z}}^2} - \frac{U_{\bar{z}}\partial_u\gamma_{z\bar{z}}\partial_z\gamma_{z\bar{z}}}{8\,\gamma_{z\bar{z}}^3} - \frac{U_z\partial_u\gamma_{z\bar{z}}\partial_{\bar{z}}\gamma_{z\bar{z}}}{8\,\gamma_{z\bar{z}}^3} + \frac{\partial_z\partial_{\bar{z}}\gamma_{z\bar{z}}}{3\,\gamma_{z\bar{z}}^2} - \frac{\partial_z\gamma_{z\bar{z}}\partial_{\bar{z}}\gamma_{z\bar{z}}}{3\,\gamma_{z\bar{z}}^3} + \frac{1}{3})r^{-2} + \mathcal{O}\left(r^{-3}\right) \tag{296}$$

$$C_{rurz} = (\frac{\partial_{\bar{z}}C_{zz}}{4\,\gamma_{z\bar{z}}} + \frac{U_z\partial_z\partial_{\bar{z}}\gamma_{z\bar{z}}}{12\,\gamma_{z\bar{z}}^2} - \frac{U_z\partial_z\gamma_{z\bar{z}}\partial_{\bar{z}}\gamma_{z\bar{z}}}{12\,\gamma_{z\bar{z}}^3} - \frac{1}{6}\,U_z)r^{-2} + \mathcal{O}\left(r^{-3}\right) \tag{297}$$

$$\begin{aligned} C_{rzrz} =&(\frac{C_{zz}\partial_z\partial_{\bar{z}}\gamma_{z\bar{z}}}{6\,\gamma_{z\bar{z}}^2} - \frac{C_{zz}\partial_z\gamma_{z\bar{z}}\partial_{\bar{z}}\gamma_{z\bar{z}}}{6\,\gamma_{z\bar{z}}^3} + \frac{1}{6}\,C_{zz})r^{-1} + (-C_{zz}\mathrm{m_B} + \frac{C_{zz}^2\partial_u C_{\bar{z}\bar{z}}}{12\,\gamma_{z\bar{z}}^2} + \frac{C_{\bar{z}\bar{z}}C_{zz}\partial_u C_{zz}}{12\,\gamma_{z\bar{z}}^2} \\ &+ \frac{C_{zz}\partial_{\bar{z}}U_z}{12\,\gamma_{z\bar{z}}} + \frac{C_{zz}\partial_z U_{\bar{z}}}{12\,\gamma_{z\bar{z}}} - \frac{C_{\bar{z}\bar{z}}C_{zz}^2\partial_u\gamma_{z\bar{z}}}{6\,\gamma_{z\bar{z}}^3} - \frac{C_{zz}\partial_z^2 C_{\bar{z}\bar{z}}}{12\,\gamma_{z\bar{z}}^2} - \frac{C_{zz}\partial_{\bar{z}}^2 C_{zz}}{12\,\gamma_{z\bar{z}}^2} \\ &+ \frac{C_{zz}\partial_z C_{\bar{z}\bar{z}}\partial_z\gamma_{z\bar{z}}}{12\,\gamma_{z\bar{z}}^3} + \frac{C_{zz}\partial_{\bar{z}}C_{zz}\partial_{\bar{z}}\gamma_{z\bar{z}}}{12\,\gamma_{z\bar{z}}^3})r^{-2} + \mathcal{O}\left(r^{-3}\right) \end{aligned} \tag{298}$$

$$\begin{aligned} C_{rzr\bar{z}} =&(\frac{\partial_z\partial_{\bar{z}}\gamma_{z\bar{z}}}{6\,\gamma_{z\bar{z}}} - \frac{\partial_z\gamma_{z\bar{z}}\partial_{\bar{z}}\gamma_{z\bar{z}}}{6\,\gamma_{z\bar{z}}^2} + \frac{1}{6}\,\gamma_{z\bar{z}}) + (-\gamma_{z\bar{z}}\mathrm{m_B} + \frac{5\,C_{zz}\partial_u C_{\bar{z}\bar{z}}}{24\,\gamma_{z\bar{z}}} - \frac{C_{\bar{z}\bar{z}}\partial_u C_{zz}}{24\,\gamma_{z\bar{z}}} \\ &- \frac{C_{\bar{z}\bar{z}}C_{zz}\partial_u\gamma_{z\bar{z}}}{6\,\gamma_{z\bar{z}}^2} - \frac{\partial_z^2 C_{\bar{z}\bar{z}}}{12\,\gamma_{z\bar{z}}} - \frac{\partial_{\bar{z}}^2 C_{zz}}{12\,\gamma_{z\bar{z}}} + \frac{\partial_z C_{\bar{z}\bar{z}}\partial_z\gamma_{z\bar{z}}}{12\,\gamma_{z\bar{z}}^2} + \frac{\partial_{\bar{z}}C_{zz}\partial_{\bar{z}}\gamma_{z\bar{z}}}{12\,\gamma_{z\bar{z}}^2} - \frac{1}{6}\,\partial_{\bar{z}}U_z \\ &+ \frac{1}{3}\,\partial_z U_{\bar{z}})r^{-1} + (\frac{1}{6}\,U_z U_{\bar{z}} + \frac{C_{\bar{z}\bar{z}}C_{zz}}{6\,\gamma_{z\bar{z}}} - \frac{U_z\partial_z C_{\bar{z}\bar{z}}}{4\,\gamma_{z\bar{z}}} - \frac{C_{\bar{z}\bar{z}}\partial_z U_z}{24\,\gamma_{z\bar{z}}} - \frac{C_{zz}\partial_{\bar{z}}U_{\bar{z}}}{24\,\gamma_{z\bar{z}}} + \frac{C_{\bar{z}\bar{z}}U_z\partial_z\gamma_{z\bar{z}}}{24\,\gamma_{z\bar{z}}^2} \\ &+ \frac{C_{zz}U_{\bar{z}}\partial_{\bar{z}}\gamma_{z\bar{z}}}{24\,\gamma_{z\bar{z}}^2} - \frac{\partial_{\bar{z}}C_{\bar{z}\bar{z}}\partial_z C_{zz}}{24\,\gamma_{z\bar{z}}^2} + \frac{\partial_z C_{\bar{z}\bar{z}}\partial_{\bar{z}}C_{zz}}{24\,\gamma_{z\bar{z}}^2} + \frac{C_{zz}\partial_{\bar{z}}C_{\bar{z}\bar{z}}\partial_z\gamma_{z\bar{z}}}{12\,\gamma_{z\bar{z}}^3} + \frac{C_{\bar{z}\bar{z}}C_{zz}\partial_z\partial_{\bar{z}}\gamma_{z\bar{z}}}{6\,\gamma_{z\bar{z}}^3} \\ &+ \frac{C_{\bar{z}\bar{z}}\partial_z C_{zz}\partial_{\bar{z}}\gamma_{z\bar{z}}}{12\,\gamma_{z\bar{z}}^3} - \frac{C_{\bar{z}\bar{z}}C_{zz}\partial_z\gamma_{z\bar{z}}\partial_{\bar{z}}\gamma_{z\bar{z}}}{3\,\gamma_{z\bar{z}}^4})r^{-2} + \mathcal{O}\Big(r^{-3}\Big) \end{aligned} \tag{299}$$

The component C_{rurz} can also be written as:

$$C_{rurz} = R_{rurz} + \frac{1}{2}(g_{ur}R_{rz} - g_{uz}R_{rr}) = -\frac{1}{4r^2}(U_z - D^z C_{zz}) + \mathcal{O}(r^{-3}) \tag{300}$$

And since

$$C_{rzrz} = R_{rzrz} - \frac{1}{2}g_{zz}R_{rr} = \mathcal{O}(r^{-3}), \tag{301}$$

we must have the constraint condition, in order to have an asymptotically flat spacetime:

$$U_z = D^z C_{zz} \tag{302}$$

18.2 Lie Derivative and Calculation of Super Translations

The Lie derivative, $\mathcal{L}_\zeta$, is a derivative along a vector, $\zeta = \zeta^\mu \partial_\mu$, being a vector field. They act on scalar fields like:

$$\mathcal{L}_\zeta \phi = \zeta^\mu \partial_\mu \phi \tag{303}$$

On one forms, $\Omega = \Omega_\nu dx^\nu$, like:

$$\mathcal{L}_\zeta \Omega = (\mathcal{L}_\zeta \Omega_\nu) dx^\nu + \Omega_\nu d(\mathcal{L}_\zeta x^\nu), \tag{304}$$

where $\mathcal{L}_\zeta x \to \zeta^\mu \partial_\mu x^\nu = \zeta^\nu$

Killing vectors are vectors, ζ, such that $\mathcal{L}_\zeta g = 0$, where g is the metric. The Lie derivative acts like the Leibnitz formula ($(fg)' = f'g + fg'$). $\mathcal{L}_\zeta$ commutes with the differential d ($\mathcal{L}_\zeta d\rho = d(\mathcal{L}_\zeta \rho)$).

18.2.1 Calculation of Supertranslations

We now focus on the Lie derivative, $\mathcal{L}_\zeta$, with vector

$$\zeta = \zeta^u \partial_u + \zeta^r \partial_r + \zeta^z \partial_z + \zeta^{\bar{z}} \partial_{\bar{z}} \tag{305}$$

We will start using the two first components to get g_{uu} and g_{ur}. The metric will only involve the first three terms:

$$ds^2 = -du^2 - dudr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z} + ... \tag{306}$$

Again we focus on the two first terms for the components of interest. Applying the Lie derivative to ds^2 to get the metric components at large-r:

$$\begin{aligned} \mathcal{L}_\zeta ds^2 &= (\mathcal{L}_{\zeta^u \partial_u} + \mathcal{L}_{\zeta^r \partial_r})(-du^2 - dudr) \\ &= -d\zeta^u du - dud\zeta^u - 2d\zeta^u dr - 2dud\zeta^r \\ &= -2d\zeta^u du - 2d\zeta^u dr - 2dud\zeta^r \end{aligned} \tag{307}$$

Now expanding by using $d\zeta^b = \partial_a \zeta^b dx^a$, which is $d\zeta^u = (\partial_u \zeta^u)du + (\partial_r \zeta^u)dr + (\partial_z \zeta^u)dz + (\partial_{\bar{z}} \zeta^u)d\bar{z}$. The three terms respectively give

$$\begin{aligned} &-2[(\partial_u \zeta^u)du + (\partial_r \zeta^u)dr]du \\ &-2[(\partial_u \zeta^u)du + (\partial_r \zeta^u)dr]dr - 2(\partial_z \zeta^u)dzdr \\ &-2[(\partial_u \zeta^r)du + (\partial_r \zeta^r)dr]du \end{aligned} \tag{308}$$

The second term also gives a contribution to the g_{zr} metric component. Thus we have

$$\mathcal{L}_\zeta ds^2 = -2\left[(\partial_u \zeta^u)du^2 + (\partial_u \zeta^r)du^2 + (\partial_u \zeta^u)dudr + (\partial_z \zeta^u)dzdr + (\partial_r \zeta^u)dr^2 + (\partial_r \zeta^u)drdu + (\partial_r \zeta^r)drdu\right] \tag{309}$$

Thus from looking at the first three terms, and the properties of the metric:

$$g_{\mu\nu} = \begin{pmatrix} g_{uu} & g_{ur} \\ g_{ru} & g_{rr} \end{pmatrix} \tag{310}$$

$$\begin{aligned} g &= \begin{pmatrix} du & dr \end{pmatrix} \begin{pmatrix} g_{uu} & g_{ur} \\ g_{ru} & g_{rr} \end{pmatrix} \begin{pmatrix} du \\ dr \end{pmatrix} \\ &= g_{uu} du du + 2 g_{ur} du dr + g_{rr} dr dr + ... \end{aligned} \tag{311}$$

one can write the Lie derivative with respect to the metric components as:

$$\begin{aligned} \mathcal{L}_\zeta g_{uu} &= -2\partial_u \zeta^u - 2\partial_u \zeta^r + \mathcal{O}\left(\frac{1}{r}\right) \\ \mathcal{L}_\zeta g_{ur} &= -\partial_u \zeta^u + \mathcal{O}\left(\frac{1}{r}\right) \end{aligned} \tag{312}$$

Now to get the metric components g_{zr} and g_{zz}, we will do the rest of the computation (and remember we need to add contribution $-2(\partial_z \zeta^u) dz dr$):

$$\mathcal{L}_\zeta ds^2 = (\mathcal{L}_{\zeta^r \partial_r} + \mathcal{L}_{\zeta^z \partial_z} + \mathcal{L}_{\zeta^{\bar{z}} \partial_{\bar{z}}})(2r^2 \gamma_{z\bar{z}} dz d\bar{z}) \tag{313}$$

Which gives us three terms:

$$\begin{aligned} &+ 2(\mathcal{L}_{\zeta^r \partial_r} r^2)\gamma_{z\bar{z}} dz d\bar{z} \\ &+ 2r^2(\mathcal{L}_{\zeta^z \partial_z} \gamma_{z\bar{z}} + \mathcal{L}_{\zeta^{\bar{z}} \partial_{\bar{z}}} \gamma_{z\bar{z}}) dz d\bar{z} \\ &+ 2r^2 \gamma_{z\bar{z}}((\mathcal{L}_{\zeta^z \partial_z} dz) d\bar{z} + dz(\mathcal{L}_{\zeta^{\bar{z}} \partial_{\bar{z}}} d\bar{z})) \end{aligned} \tag{314}$$

Furthermore we have:

$$\mathcal{L}_{\zeta^r \partial_r} r^2 = \zeta^r 2r \tag{315}$$

which is contributing to $g_{z\bar{z}}$.

$$\mathcal{L}_{\zeta^z \partial_z} dz = (\partial_z \zeta^z) dz \tag{316}$$

which is also contributing to $g_{z\bar{z}}$.

$$\mathcal{L}_{\zeta^{\bar{z}} \partial_{\bar{z}}} d\bar{z} = [(\partial_r \zeta^{\bar{z}}) dr + (\partial_{\bar{z}} \zeta^{\bar{z}}) d\bar{z}] \tag{317}$$

where the first term is contributing to g_{zr}, and the second to $g_{z\bar{z}}$. Thus we have:

$$\mathcal{L}_\zeta g_{zr} = r^2 \gamma_{z\bar{z}} \partial_r \zeta^{\bar{z}} - \partial_z \zeta^u + \mathcal{O}\left(\frac{1}{r}\right) \tag{318}$$

For the final component, we need to take into account:

$$\mathcal{L}_\zeta \gamma_{z\bar{z}} = \zeta^\mu \partial_\mu \gamma_{z\bar{z}} + (\partial_z \zeta^z) \gamma_{\mu\bar{z}} + (\partial_{\bar{z}} \zeta^{\bar{z}}) \gamma_{z\mu} \tag{319}$$

where you can only plug in z or $\bar{z}$. This, together with the definition of the covariant derivative, finally lead to the last component:

$$\mathcal{L}_\zeta g_{z\bar{z}} = r\gamma_{z\bar{z}}[2\zeta^r + rD_z \zeta^z + rD_{\bar{z}} \zeta^{\bar{z}}] + \mathcal{O}(1) \tag{320}$$

These Lie derivative of the metric components, together with the Bondi gauge conditions and large-r falloffs, lead to the expression for the generator of supertranslations, ζ (*Eq. 171*).

List of Abbreviations

ASG: Asymptotic symmetry group

BMS: Bondi, Metzner and Sachs

CFT: Conformal field theory

(N)D: N-dimensional

EM: Electromagnetism

E-M Energy-momentum

EOM: Equations of motion

GR: Genral relativity

IR: Infrared

LHS/RHS: Left-hand side/right-hand side

MS: Minkowski space

QCD: Quantum chromodynamics

QED: Quantum electrodynamics

SR: Special relativity

YM: Yang-Mills