Master Thesis

On-shell methods for anomalous dimensions and beta functions

Written by *Alicia Astorga Elcarte*

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Supervised by

Matthias Wilhelm and Emil Bjerrum-Bohr

University of Copenhagen

UNIVERSITY OF COPENHAGEN

FACULTY:	Faculty of Science
INSTITUTE:	Niels Bohr Institute
AUTHOR(S):	Alicia Astorga Elcarte
EMAIL:	wvs890@alumni.ku.dk
TITLE:	Master Thesis
SUPERVISOR(S):	Matthias Wilhelm and Emil Bjerrum-Bohr
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NAME ______________________

SIGNATURE ______________________

DATE ______________________

Abstract

On-shell methods have proven to be a powerful alternative to Feynman diagrams for the calculation of scattering amplitudes. They can also be applied to partially off-shell quantities, such as form factors of gauge invariant local composite operators. The discontinuities of these quantities can be computed via phase-space integrals that are related to the anomalous dimensions of these operators as well as the β-functions of the corresponding couplings. These integrals are computed by means of a parametrization of the spinors. Based on the method in which the dilatation operator, which measures the anomalous dimensions, is given by minus the phase of the scattering matrix, divided by π, we study these parametrizations at one- and for a special type of two-loop order. We also study the parametrization of the triple cut in the three-particle channel that would allow to fully extend these methods to the second order in perturbation theory.

Contents

1 Introduction

There exists four well established fundamental interactions in nature; the gravitational, electromagnetic, weak and strong interactions. Up to this moment, we are able to explain at the quantum level all except the gravitational one. Hence it is one of the long-standing challenges of fundamental physics research activities to find a quantum theory for gravity and to unify these four interactions. The other interactions are explained by the so-called Standard Model (SM) of particle physics, which gives an explanation to a large amount of physical phenomena in nature. In the Standard Model, matter fields are classified into fermions and bosons. Fermions are the constituents of matter and they exist as quarks and leptons.

Each of them is composed by six particles that can be organized in three generations (pairs of particles). The first generation corresponds to the lightest particles. Quarks (u, d, c, s, b, t) carry also color charge, which is a defining feature of the strong interactions and they couple into colourless bound states. Leptons are also arranged in three generations, each of them formed by a charged particle, (e^-, μ^-, τ^-), and their corresponding neutrino $(\nu_e, \nu_\mu, \nu_\tau)$. They obey Fermi statistics which leads to the Pauli Exclusion Principle. Vector bosons are the carriers of the forces and in fact, fermions interact by exchanging bosons. Each fundamental force has its own vector boson: photons γ mediates electromagnetism, Z and $W^\pm$ weak interactions, and gluons strong interactions. Each of them carries spin 1. There is also a scalar boson with spin zero, the Higgs boson.

However, the Standard Model fails to explain some physical phenomena in Nature, such as the anomalous magnetic moment of the muon, whose measurement disagrees with the predictions from the Standard Model in 4.2σ, result that was obtained from the experiment at Fermilab [1]. This measurement is one of the evidences that there should be physics beyond the Standard Model. However, the theoretical direction is still unclear. One of the candidates would be to extend the Standard Model to a supersymmetric theory leading to the so-called Minimal Supersymmetric Standard Model.

Moreover, due to the arguments presented above, it is particularly useful to use the simplest four dimensional quantum field theory, the so-called maximally supersymmetric $\mathcal{N} = 4$ SYM theory. This theory, composed by massless fields, can be used as a toy model to develop calculational techniques that can be applied to other theories such as the Standard Model. Like the Standard Model, it is a non-Abelian gauge theory, but with many more symmetries. More concretely, as already mentioned the Standard Model is a gauge theory with the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ while $\mathcal{N} = 4$ SYM theory is a gauge theory with gauge group $SU(3)$. It is composed of one gauge field, four fermions, four antifermions and six scalars, all related by supersymmetry. One of the main particularities of this theory is that it is conformally invariant, which when applied to a sypersymmetric theory leads to superconformal invariance.

In particular this symmetry is exact in the sense that it is still preserved when quantizing the theory. This implies that the strength of the interactions is independent of the scale, i.e., that its β-function is zero up to all orders. All the symmetries present in this theory form the so-called $PSU(2,2|4)$ group, which is composed by translations, super translations, Lorentz rotations, internal rotations, conformal and superconformal transformations, and the scaling or dilatation operator, each of them with its own generator. Note that this group comes from the irreducible superconformal algebra $\mathfrak{psu}(2,2|4)$, and that by adding the hypercharge $\mathfrak{B}$ and the central charge $\mathfrak{C}$ one gets the $\sqcap(2,2|4)$ algebra. Although conformal invariance is an exact symmetry, some other symmetries suffer anomalies when quantizing the theory, this means that its generators do receive quantum corrections, this is the case of the dilatation operator.

Particularly, $\mathcal{N}=4$ SYM theory is not only useful for calculations in gauge theories. Due to the Anti-de-Sitter/Conformal Field Theory (AdS/CFT) correspondence, $\mathcal{N}=4$ SYM theory is dual to a type of string theory, the so-called type II B string theory in the $\mathrm{AdS}^5 \times \mathrm{S}^5$. More precisely, the 10-dimensional type II B superstring theory on the product space $\mathrm{AdS}^5 \times \mathrm{S}^5$ is equivalent to $\mathcal{N}=4$ SYM theory with gauge group $SU(N)$living on the flat 4-dimensional boundary of AdS^5 [28]. This equivalence means that there is a one-to-one correspondence between the symmetries, observables and correlation functions of the theories. Therefore, this correspondence is significant in terms of a better understanding of string theory. Moreover, in the so-called planar limit [24], both theories simplified and acquire the property of integrability.

Due to integrability one can obtain some observables of $\mathcal{N}=4$ SYM theory in an analytical way. Moreover, this supersymmetric theory matches some regimes of well-defined theories present in Nature. In particular, computations of gauge boson amplitudes in $\mathcal{N}=4$ SYM theory are really useful for gluon scattering at high energies where asymptotic freedom occur [23].

In the perturbative expansion, different powers of the number of colors occur. In the planar limit, the number of colors $N \to \infty$. This implies that the Feynman diagrams of the corresponding interactions are associated to 2-dimensional surfaces, i.e., that these diagrams are planar in their structure. Remarkably, planarity can in general refer to any space, but in this work the focus will be on color and momentum space.

Although there are some examples in which planarity occurs in both spaces (such as Feynman diagrams containing elementary interactions), in general one can have diagrams that are planar in color space but not in momentum space (gauge invariant local composite operators for instance), and viceversa, see fig. 1. Moreover, one can also have planar contributions made of non-planar components. For instance the contraction of a diagram with double-trace color structure and a composite operators (both non-planar in color space) gives a planar diagram.

The integrability property shown in the planar limit of $\mathcal{N}=4$ SYM theory was found in the spectrum of anomalous dimensions of gauge-invariant local composite operators, or equivalently in the energy spectrum of strings in the dual $\mathrm{AdS}^5 \times \mathrm{S}^5$ string theory. The solutions in this limit are given by the Bethe ansatz equations, [29], with some corrections coming from the so-called finite-size effects. This effect arise from diagrams that are non-planar in momentum space but can still be planar in their color structure and hence contribute to the 't Hooft limit.

They appear by the wrapping mechanism, which stems from interactions wrapping around the operator. The structure of gauge-invariant local composite operators stems from traces (or

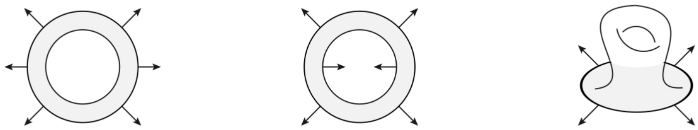


Figure 1: Three examples of planar and non planar topologies. Figure (a) represents a single trace one-loop planar graph, (b) double-trace one-loop diagram and (c) single trace non-planar diagram. Only the diagram in (a) contributes in the planar limit [35].

products of traces) of irreducible fields in the same spacetime point. Single trace operators are considered the constituents of the spectrum of more general operators. In particular, it was found that the action of the one-loop dilatation operator of $\mathcal{N} = 4$ SYM theory on single-trace operators maps to the action of the Hamiltonian of an integrable spin chain.

Remarkably important in this work are on-shell methods due to the fact that they are based on general principles of quantum field theory, and consequently the techniques reviewed here are expected to be applicable to general gauge theories. Observables obtained using on-shell methods are based on physical quantities, since only physical degrees of freedom appear on-shell. Some examples of on-shell methods are generalized unitarity [11, 8, 5], tree-level recursion relations [12]. The on-shell methodology used throughout corresponds to the method derived in [14] in terms of the spinor helicity formalism, which is particularly useful for theories based on massless particles with spin such as $\mathcal{N} = 4$ SYM theory. Since, by means of spinor helicity formalism, one can focus on the physical helicities of the particles to build the corresponding observables.

The main quantity computed via on-shell methods is the scattering matrix, which describe the interaction between incoming and outgoing on-shell external states. The definition of on-shell particles imply that they fulfill the condition $p_i = m^2$, and for massless particles $m^2 = 0$. From scattering matrices, one can compute cross-sections, that correspond to the physical quantities detected at colliders.

In this work we will be interested in analyze to which extent on-shell methods can be used to compute off-shell quantities such as correlation functions of gauge invariant local composite operators. One way to do so is by the computation of form factors. In particular, form factors provide a bridge between the purely on-shell amplitudes and the purely off-shell correlation functions, figure 1. They act as matrix elements between on-shell asymptotic states and gauge invariant operators:

$$\mathcal{F}_{\mathcal{O},n}(1, \ldots, n; x) = \langle 1, \ldots, n \mid \mathcal{O}(x) \mid 0 \rangle \,. \tag{1}$$

These operators are local and therefore it becomes particularly interesting to work in momentum space in order to remove the position dependence. In momentum space, this local operator is an off-shell quantity with a momentum $q \neq 0$ associated. Additionally, in momentum space a delta function arises that ensures momentum conservation throughout the process. Hence, each particle will be defined by simply specifying its momentum and its helicity. Moreover, form factors are important quantities when studying some mechanisms such as the Higgs production at the LHC via its decaying to 3 gluons [19].

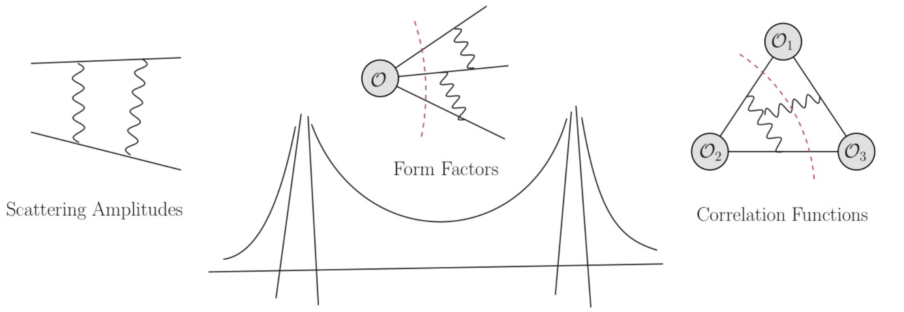


Figure 2: Form factors (center) as bridge between on-shell scattering amplitudes (on the left) and purely off-shell correlation functions between gauge invariant local composite operators (on the right) [35].

The method used in this work is based on the connection between the high-energy behavior of the scattering matrix and the running of the coupling as well as the renormalisation of local operators. This relation is equivalent to find the connection between the phase and the energy dependence of form factors. The computations made throughout this work are done in perturbation theory. Due to the analyticity of form factors in this regime,

$$F\left(p_1,\ldots,p_n\right) \to F\left(p_1\mathrm{e}^{i\alpha},\ldots,p_n\mathrm{e}^{i\alpha}\right) = \mathrm{e}^{i\alpha D}F\left(p_1,\ldots,p_n\right), \quad \text{where} \quad D \equiv \sum_i p_i^\mu \frac{\partial}{\partial p_i^\mu} \approx -\mu\partial_\mu. \tag{2}$$

Combining this expression with the formal definition of a form factor as a small perturbation to the scattering matrix, one obtains the central relation of this work:

$$e^{-i\pi D}F^* = SF^*. \tag{3}$$

This equation states that the dilatation operator is minus the phase of the S-matrix divided by π. The right-hand side of this equation represents a unitarity cut. In the planar limit of $\mathcal{N}=4$ SYM theory, the dilatation generator acts as a generator of the algebra $\mathfrak{psu}(2,2\mid 4)$. Moreover, this generator receives quantum corrections and hence one can express the renormalization group equation as,

$$DF = -\mu\partial_\mu F = \left(\beta\left(g^2\right)\frac{\partial}{\partial g^2} + \gamma_{\mathcal{O}} - \gamma_{\mathrm{IR}}\right)F, \tag{4}$$

From this equation, we know that the action of the dilatation operator on a form factor is given in terms of ultraviolet and infrared anomalous dimensions as well as the β-function.

Relating both definitions of the dilatation operator, we can obtain anomalous dimensions directly from the scattering matrix. Based on both definitions of the dilatation operator, and performing a perturbative expansion of the relation, one can obtain the anomalous dimensions at different loop orders from the product of lower-loop scattering matrices and form factors.

The appearance of anomalous dimensions is a crucial aspect of interpreting results in the Standard Model Effective Field Theory (SMEFT). The basic idea behind the SMEFT definition is to use effective field theories that extend the Standard Model Lagrangian by adding higher dimension operators [22]. Anomalous dimensions, through renormalisation, induces the mixing of these operators, explaining how the constraints applied to one operator affect the coefficients of other operators.

This work is structured as follows. We can divide this work in two parts with different structure. The first part include chapters 2-4 and it contains theoretical grounds for the computations in the consecutive chapters. In chapter 2, we give an introduction to $\mathcal{N} = 4$ SYM theory, explaining its symmetries, field content and important limits. In chapter 3, form factors are introduced particularizing to minimal form factors, since they will play an important role throughout this work. Also, spinor helicity formalism is reviewed, as well as the corresponding Feynman rules. Chapter 4 will be a review of two different methods for computing anomalous dimensions. Last section will be dedicated to the explanation of some selection rules that will precisely set the motivation for obtaining non-vanishing anomalous dimensions.

The second part comprises different cases to which anomalous dimensions can be computed by means of the method in [14]. In particular, in chapter 5 we will compute the spinor parametrization for the computation of one-loop anomalous dimensions through $2 \to 2$ amplitudes. Chapter 6 studies the one-loop case with the $2 \to 3$ amplitude case. Chapter 7 introduces a new parametrization for the computation of two-loop anomalous dimensions from the $3 \to 2$ amplitude. In the last chapter, we study the parametrization for the $3 \to 3$ amplitude.

2 $\mathcal{N} = 4$ SYM theory

In this chapter we will review the maximally supersymmetric $\mathcal{N} = 4$ SYM theory. In the first section we will give a short introduction to supersymmetry, explaining its main motivation and its basic principles. We will afterwards define the $\mathcal{N} = 4$ SYM theory, specifying its particle content as well as its symmetries. Then we will take the so-called planar limit of the theory, and define single trace operators and the dilatation operator. In particular, we will study the correspondence of the planar $\mathcal{N} = 4$ SYM theory and the spin chain representation and how to obtain the dilatation operator.

2.1 Introduction to supersymmetry

Supersymmetry maps particles and fields of integer spin (bosons) into particles and fields of half odd integer spin (fermions) and viceversa. Supersymmetry is generated by a supercharge Q_α, which is a fermionic space-time spinor operator of spin $1/2$ that commutes with the local and global internal symmetries. Therefore, the spectrum comes in pairs of a boson and a fermion where they have the same internal quantum numbers and the same mass.

All the particle physics we know is described by the Standard Model of particle physics, which is built upon the local $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge symmetry. These symmetries are directly related to the fundamental interactions. The strength of each interaction is controlled by its coupling constant. Only from the particle content of the Standard Model, we see that there is no candidate for a supersymmetric pair of particles and therefore the basic characteristic of a supersymmetric theory is not fulfilled in the Standard Model. However, symmetries break spontaneously, which means that although the laws of physics are invariant under this symmetry, the equations of motions are no longer invariant under this symmetry. In particular, this broken symmetries occur for each subsector of the Standard Model. For instance, in the $SU(3)_c$ sector the spontaneous breaking of approximate chiral symmetry for the up and down quarks is responsible for the comparatively small masses of the pions, as well as for the comparatively large masses of the nucleons [32]. Moreover, the spontaneous breaking of $SU(2)$ is responsible for the masses of quarks, leptons, $W^{\pm}$ and Z particles.

This suggests that there may be a supersymmetry spontaneously broken in Nature. The breaking of the sypersymmetry would be in the spectrum where fermions and bosons would have the same quantum numbers, but different masses in each pair. The problem is that the Standard Model does not fulfill this spontaneous breaking of supersymmetry either. However, extending the particle content of the Standard Model via the introduction of the smallest number of supplementary fields would give the o-called *Minimal Supersymmetric Standard Model* [16]. The idea behind it is that for each known non-supersymmetric particle, we add a hypothetical supersymmetric partner, see figure 3.

The full significance of supersymmetry occurs when it is combined with the principle of the unification of the strong, electromagnetic and weak forces (most likely also gravity). Remarkably,

Non-Susy Sector	Susy Partners	Spin Susy Partners
2 Higgs (required)	Higgsinos	1/2
Leptons (ν_e e^-) $\cdots$	sLeptons ($\tilde{\nu}_e$ $\tilde{e}^-$) $\cdots$	0
Quarks (u d) $\cdots$	sQuarks ($\tilde{u}$ $\tilde{d}$) $\cdots$	0
Gluons g	Gluinos $\tilde{g}$	
Photon γ	photino $\tilde{\gamma}$	1/2
$W^{\pm}$, Z	Winos $\tilde{W}^{\pm}$, Zino $\tilde{Z}$	

Figure 3: Particle contents of the *Minimal Supersymmetric Standard Model* [16].

the three couplings of the Standard Model $g_i(\mu)$ approximately intersect in one point around $M_U \sim 10^{15} GeV$. This scale is not to far from the Planck scale $M_P \sim 10^{19} GeV$ at which quantum gravity is supposed to play a central role. Therefore we can interpret the scale M_U, where the three couplings approximately meet considering the existence of a supersymmetric theory, as the scale where the three forces are unified into a single one. The smallest such group which contains the gauge group of the Standard Model as a subgroup is $SU(5)$, but the inclusion of a massive neutrino can be realized only in the larger $SO(10)$.

$$SU(3)_c \times SU(2)_L \times U(1)_Y \subset SU(5) \subset SO(10) \tag{5}$$

Starting at the highest energy scale, just below M_P, and running down to lower energy scales, there will initially be only a single force, governed by the unified simple gauge group. The strong, electromagnetic and weak interactions will emerge as the result of a phase transition at the unification scales M_U, where the unified gauge symmetry is broken down to the $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge group. This unification scheme supposes a suitable theory to remove some problems that appears in our theory. For example, the Abelian gauge interactions $U(1)_Y$ and $U(1)_{em}$ are not asymptotically free, which means that they would increase indefinitely as the energy scale increases. This will lead to a Landau pole and a breakdown of unitarity at high energies.

With the unification scheme this problem is automatically avoided since the $U(1)$ gauge group arises from the breaking of an asymptotically free gauge theory with simple gauge group that it is the one that remains at large energies. But there are some other problems that arises with this unification scheme (like the value of the proton life-time that comes 4 orders of magnitude lower). But these problems can be solved by combining the unification scheme with the Supersymmetric Extension of the Standard Model.

2.2 Maximally Supersymmetric Yang-Mills theory in 4 dimensions

The maximally supersymmetric Yang-Mills (SYM) theory in four dimensions was first considered by Brink, Scherk and Schwarz who constructed its Lagrangian by dimensional reduction of $\mathcal{N} = 1$

SYM in ten dimensions [10]. It is a simplified toy theory based on Yang-Mills theory that does not describe reality but gives solution to problems that are a good approximation to the problems in the real theory. Moreover, it also serves as a test ground to develop techniques that can be used for the Standard Model. One of its remarkable properties is that it is conformal, which means that there is no inherent mass scale in the theory, even at quantum level. The main consequence is that its β-function is zero to all orders in perturbation theory.

In particular, in $\mathcal{N} = 4$ SYM theory fermion and boson fields are related by four supersymmetries (i.e, exchanging vector boson, fermion and scalar fields in a certain way leaves the theory invariant). It is one of the simplest field theories in 4 dimensions and it can be thought of as the most symmetric field theory that does not involve gravity. The importance of its study relies on several reasons. More concretely, this theory becomes really important when we refer to the maximal transcendentality principle [18], which relates some results of $\mathcal{N} = 4$ SYM theory to the maximally transcendental part of their counterparts in pure Yang-Mills theory.

Due to its symmetries, the Anti-de-Sitter/Conformal Field Theory (AdS/CFT) correspondence conjectures that $\mathcal{N} = 4$ SYM theory is dual to type IIB string theory on $AdS_5 \times S^5$ [28]. Although this correspondence is not proven yet, it is a conjecture that can be checked via integrability. More concretely, integrability was shown in the planar limit of $\mathcal{N} = 4$ SYM theory in the spectrum of anomalous dimensions of gauge-invariant local composite operators, as well as in the energy spectrum of strings in the dual $AdS^5 \times S^5$ string theory.

In principle, in theories such as QCD that has a running coupling constant, there exists a mass scale in which the interaction runs from weak to strong coupling. This scale determines where the confinement sets. However, in $\mathcal{N} = 4$ SYM where conformal invariance is present, there is no confinement, which means that none of the physical particles of QCD could be formed. But, although we know that QCD is not conformal, the behaviour it presents at high energies is asymptotically free and is precisely at these high energies where it can be considered close to being conformal. In fact, many essential features of high energy gluon scattering, relevant for the LHC, can be learned by studying gauge boson amplitudes in $\mathcal{N} = 4$ SYM theory.

2.3 Supersymmetric algebra

Fields in supersymmetric theories where the spin is not greater than one are formed by gauge fields of spin 1, Weyl fermion fields of spin 1/2 and scalars of spin 0. These fields are restricted to be organised in multiplets of the corresponding symmetry algebras. Consequently, the field content of $\mathcal{N} = 4$ SYM theory consists of one gauge field A_μ, four fermions ψ_α^A transforming in the anti-fundamental representation of $SU(4)$, four antifermions $\overline{\psi}_{\dot{\alpha}A}$ transforming in the fundamental representation of $SU(4)$ (where $A = 1, 2, 3, 4$), and six real scalars ϕ_I with $I = 1, ..., 6$ transforming in the fundamental representation of $SO(6)$ in order to match the number of bosonic and fermionic on-shell degrees of freedom.

Remarkably, there is a global symmetry $SU(4) = SO(6)$, called R-symmetry, that exploits the isomorphism between the algebras of $SO(6)$ and $SU(4)$. This means that there exists some matrices σ^I_{AB} that are the chiral projections of the gamma matrices in six dimensions, and obey $\sigma^I_{AB} = -\sigma^I_{BA}$. In particular we can define the scalar field as $\phi_{AB} = \sigma^I_{AB}\phi_I$. The Lorentz spacetime index $\mu = 0, 1, 2, 3$, $\alpha, \dot{\alpha} = 1, 2$ represent the spinor indices of the $SU(2)$ and $\overline{SU}(2)$ algebras, respectively. These two independent $SU(2)$ algebras make up the 4-dimensional Lorentz algebra, which means that there exists some matrices $\sigma^\mu_{\alpha,\dot{\alpha}} = (1, \sigma_1, \sigma_2, \sigma_3)_{\alpha,\dot{\alpha}}$ that allows us to exchange a Lorentz index μ for a pair of spinor indices $\alpha, \dot{\alpha}$. All fields transform in the adjoint representation of the $SU(N)$ group.

We can define the covariant derivative as

$$D_\mu = d_\mu - ig_{YM}A_\mu. \tag{6}$$

Although the covariant derivative is not a field, it has to act on other field. Additionally, one can construct the so-called field strength from the covariant derivative,

$$F_{\mu\nu} = \frac{i}{g_{YM}}[D_\mu, D_\nu]. \tag{7}$$

We assume the gauge group to be either $SU(N)$ or $U(N)$, where all the adjoint fields are represented by (traceless) hermitian $N \times N$ matrices. The action of $N = 4$ SYM is then [34],

$$\begin{aligned} S = \int \mathrm{d}^4x \, \mathrm{tr}(&-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \left(\mathrm{D}^\mu\bar{\phi}^j\right)\mathrm{D}_\mu\phi_j + i\bar{\psi}^{\dot{\alpha}A}\mathrm{D}^\alpha_{\dot{\alpha}}\psi_{\alpha A} \\ &+ g_{\mathrm{YM}}\left(\frac{i}{2}\epsilon^{ijk}\phi_i\left\{\psi^\alpha_j, \psi_{\alpha k}\right\} + \phi_j\left\{\bar{\psi}^{\dot{\alpha}4}, \bar{\psi}^j_{\dot{\alpha}}\right\} + \text{h.c.}\right) \\ &- \frac{g^2_{\mathrm{YM}}}{4}\left[\bar{\phi}^j, \phi_j\right]\left[\bar{\phi}^k, \phi_k\right] + \frac{g^2_{\mathrm{YM}}}{2}\left[\bar{\phi}^j, \bar{\phi}^k\right]\left[\phi_j, \phi_k\right]\Big), \end{aligned} \tag{8}$$

where we have defined the complex scalars $\phi_j = \phi_{j4}$, $\bar{\phi}_j = (\phi_j)^*$ with $with i = 1, 2, 3$, which transforms in the fundamental and anti-fundamental representations of $SU(3) \subset SU(4)$, respectively. This action is invariant under the $\mathcal{N} = 4$ super Poincaré algebra. Furthermore, $\mathcal{N} = 4$ SYM theory is conformally invariant, that together with supersymmetry, leads to superconformal invariance. Remarkably, this symmetry is exact in the sense that it remains invariant when quantizing the theory, contrary to what is expected from a conformal symmetry where anomalies occur at quantum level. In this cases, it becomes necessary to renormalise the theory. One of the coeffciients associated to renormalization is the β-function, which signals the dependence of the coupling constant g on the energy scale,

$$\beta = \mu\frac{dg}{d\mu}. \tag{9}$$

In the case of $\mathcal{N} = 4$ SYM theory, since superconformal symmetry is preserved even at quantum level, it is believed that the β-function vanishes to all orders both in perturbation theory and non-perturbatively,

$$\beta = 0. \tag{10}$$

This fact makes the $\mathcal{N} = 4$ SYM theory a more treatable theory in which performing some computations becomes easier.

The super Poincaré algebra consists of translations and super translations $\mathfrak{P}_\mu$, Lorentz rotations $\mathfrak{L}_{\mu\nu}$, and internal rotations $\mathfrak{R}$. Furthermore, $\mathcal{N} = 4$ SYM also include conformal and superconformal transformations $\mathfrak{K}, \mathfrak{S}, \mathfrak{Q}$ and the scaling or dilatation operator $\mathfrak{D}$. The commutators of this symmetries are:

$$\begin{array}{ll}
\left[\mathfrak{L}^\alpha_\beta, \mathfrak{J}_\gamma\right] = \delta^\alpha_\gamma \mathfrak{J}_\beta - \frac{1}{2}\delta^\alpha_\beta \mathfrak{J}_\gamma, & [\mathfrak{L}^\alpha, \mathfrak{J}^\gamma] = -\delta^\gamma_\beta \mathfrak{J}^\alpha + \frac{1}{2}\delta^\alpha_\beta \mathfrak{J}^\gamma \\
[\mathfrak{R}^a_b, \mathfrak{J}_c] = \delta^a_c \mathfrak{J}_b - \frac{1}{4}\delta^a_b \mathfrak{J}_c, & [\mathfrak{R}^a_b, \mathfrak{J}^c] = -\delta^c_b \mathfrak{J}^a + \frac{1}{4}\delta^a_b \mathfrak{J}^c \\
\left[\dot{\mathfrak{L}}^{\dot\alpha}_{\dot\beta}, \mathfrak{J}_{\dot\gamma}\right] = \delta^{\dot\alpha}_{\dot\gamma} \mathfrak{J}_{\dot\beta} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta} \mathfrak{J}_{\dot\gamma}, & \left[\dot{\mathfrak{L}}^{\dot\alpha}_{\dot\beta}, \mathfrak{J}^{\dot\gamma}\right] = -\delta^{\dot\gamma}_{\dot\beta} \mathfrak{J}^{\dot\alpha} + \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta} \mathfrak{J}^{\dot\gamma} \\
\left[\mathfrak{S}^\alpha, \mathfrak{P}_{\dot\beta\gamma}\right] = \delta^\alpha_\gamma \dot{\mathfrak{Q}}_{\dot\beta a}, & \left[\mathfrak{K}^{\alpha\dot\beta}, \dot{\mathfrak{Q}}_{\dot\gamma c}\right] = \delta^{\dot\beta}_{\dot\gamma} \mathfrak{S}^\alpha{}_c \\
\left[\dot{\mathfrak{S}}^{a\dot\alpha}, \mathfrak{P}_{\dot\beta\gamma}\right] = \delta^{\dot\alpha}_{\dot\beta} \mathfrak{Q}^a{}_\gamma, & \left[\mathfrak{K}^{\alpha\dot\beta}, \mathfrak{Q}^c{}_\gamma\right] = \delta^\alpha_\gamma \dot{\mathfrak{S}}^{c\dot\beta} \\
\left\{\dot{\mathfrak{Q}}_{\dot\alpha a}, \mathfrak{Q}^b\right\} = \delta^b_a \mathfrak{P}_{\dot\alpha\beta}, & \left\{\dot{\mathfrak{S}}^{a\dot\alpha}, \mathfrak{S}^\beta{}_b\right\} = \delta^a_b \mathfrak{K}^{\beta\dot\alpha}
\end{array} \tag{11}$$

$$\begin{aligned}
\left[\mathfrak{K}^{\alpha\dot\beta}, \mathfrak{P}_{\dot\gamma\delta}\right] &= \delta^{\dot\beta}_{\dot\gamma} \mathfrak{L}^\alpha{}_\delta + \delta^\alpha_\gamma \dot{\mathfrak{L}}^{\dot\beta}{}_{\dot\delta} + \delta^\alpha_\gamma \delta^{\dot\beta}_{\dot\delta} \mathfrak{D} \\
\left\{\mathfrak{S}^\alpha{}_a, \mathfrak{Q}^b{}_\beta\right\} &= \delta^b_a \mathfrak{L}^\alpha{}_\beta + \delta^\alpha_\beta \mathfrak{R}^b{}_a + \frac{1}{2}\delta^b_a \delta^\alpha_\beta (\mathfrak{D} - \mathfrak{C}) \\
\left\{\dot{\mathfrak{S}}^{a\dot\alpha}, \dot{\mathfrak{Q}}_{\dot\beta b}\right\} &= \delta^a_b \dot{\mathfrak{L}}^{\dot\alpha}{}_{\dot\beta} - \delta^{\dot\alpha}_{\dot\beta} \mathfrak{R}^a{}_b + \frac{1}{2}\delta^a_b \delta^{\dot\alpha}_{\dot\beta} (\mathfrak{D} + \mathfrak{C})
\end{aligned} \tag{12}$$

All of these generators constitute the superconformal group $U(2,2|4)$. Note that this group comes from the irreducible superconformal algebra $\mathfrak{psu}(2,2|4)$, and that by adding the hypercharge $\mathfrak{B}$ and the central charge $\mathfrak{C}$ one gets the group $U(2,2|4)$. Only Lorentz and internal symmetries ($SU(2) \times SU(2)$ and $SU(4)$) do not depend on the coupling constant of the theory, the rest of the generators receive radiative corrections. In particular, the dilatation generator $\mathfrak{D}$ receive loop corrections.

2.4 't Hooft limit

For a $U(N)$ gauge group, different powers of N appear in the perturbative expansion. Some interesting properties arise when treating N as an additional coupling constant. In particular, 't Hooft realised that in the large N limit, there is an association between a Feynman graph and a 2-dimensional surface. This is the so-called 't Hooft or planar limit $N \to \infty$, $g_{YM} \to 0$ with the 't Hooft coupling $\lambda = g^2_{YM} N$ fixed. This association between a Feynman graph and a 2-dimensional surface means that the graph should be planar in its structure. We can study this structure in terms of the gauge group. This can be done by the use of fat graphs, also called double line notation, see figure 4.

Let us consider a gauge invariant Feynman graph. We define every $U(N)$ vector index of a field as a black or a white dot depending on the position of the index (black $\to$ upper, white $\to$ lower). In this way, all propagators of a Feynman graph are represented by two parallel lines.In this representation, all dots have one incoming and one outgoing arrow, these arrows form closed circles that are known as index loops and they do provide a power of the color number δ^a_a =Tr1 $= N$. Therefore it is easy to perform the contractions of indices. For a gauge invariant graph all indices must be contracted, which means that there are no unconnected dots, i.e., the graph is planar.

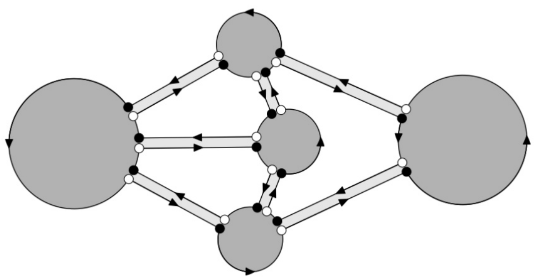


Figure 4: Fat graph. The circles are U(N) traces of the vertices and operators and the fat lines are propagators. For each face of the graph there is a closed empty trace Tr 1 = N [4].

In particular, the notion of planarity can be applied to different spaces, in this work we will consider both color and momentum spaces. Remarkably, the notion of planarity is not necessarily fulfilled in both spaces. In particular, there are some diagrams that can be planar in color but not in momentum space. This is the case of gauge-invariant local composite operators, which are colour singlets but have a non-vanishing momentum. Moreover, there exists diagrams that are not planar individually due to a double-trace structure, but becomes planar when contracted with composite operators, see figure 5. The reason is that the contraction of a composite operator with a double-trace diagram gives an extra power of the number of colors with respect to the contraction with a single-trace diagram.

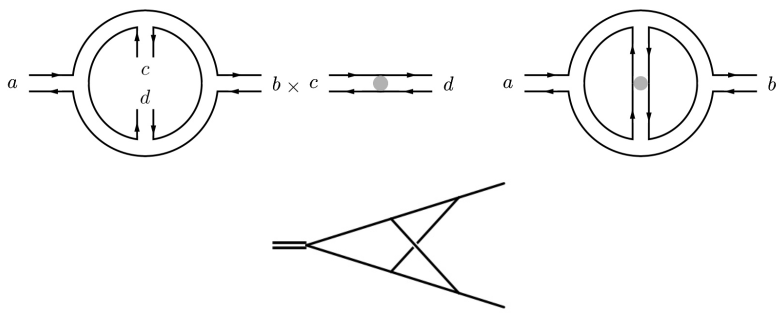


Figure 5: Example of a graph (left) that is not planar due to its double-trace structure which becomes planar (right) in color space when contracted with a composite operator but (below) it is still non-planar in momentum space [33].

2.5 Single trace operators

In this section we will explain which type of operators we will use as well as their field content and symmetries. In particular, we will study in more detail the algebra of this theory as well as its generators and the relations between them. Our work will be based on the so-called gauge invariant local composite operators. This type of operators are formed by traces of fields or products of fields that transform covariantly. Therefore, these operators are formed by both scalar ϕ^I and fermionic (antifermionic) fields ψ^A_α ($\overline{\psi}_{A\dot{\alpha}}$) and by gauge-invariant combinations of the gauge field, as is the case for both the covariant derivative D_μ and the field strength $F_{\mu\nu}$. The general structure of a single trace operators is [30],

$$\mathcal{O}(x) = \mathrm{Tr}\left[\chi_1(x)\chi_2(x)\ldots\chi_L(x)\right], \tag{13}$$

where $\chi_i(x)$ refers to a covariant field. We see that, taking or not the derivatives, this operator is gauge invariant. Other operators can be formed by taking the product of single trace ones. Note that the field strength tensor is composed by antisymmetric products of covariant derivatives. Henceforth we will only consider symmetric products in the structure of these operators. Moreover, by means of the Bianchi identity,

$$\mathcal{D} \wedge \mathcal{F} = 0, \tag{14}$$

together with the equations of motion of the fields, we do not need to consider all the combinations of the partial derivatives. Putting together all the constraints, one can define the set of irreducible fields that will take part in the structure of gauge invariant local composite operators, see figure 6:

field	D_0	$\mathfrak{su}(2)\times\mathfrak{su}(2)$	$\mathfrak{su}(4)$	B	L
$\mathcal{D}^k\mathcal{F}$	$k+2$	$[k+2, k]$	$[0,0,0]$	$+1$	1
$\mathcal{D}^k\Psi$	$k+\frac{3}{2}$	$[k+1, k]$	$[0,0,1]$	$+\frac{1}{2}$	1
$\mathcal{D}^k\Phi$	$k+1$	$[k, k]$	$[0,1,0]$	± 0	1
$\mathcal{D}^k\dot{\Psi}$	$k+\frac{3}{2}$	$[k, k+1]$	$[1,0,0]$	$-\frac{1}{2}$	1
$\mathcal{D}^k\dot{\mathcal{F}}$	$k+2$	$[k, k+2]$	$[0,0,0]$	-1	1

Figure 6: Irreducible fields [4].

Once, we have the algebra of $\mathcal{N}=4$ SYM theory, it is necessary to find its representation. In field theories, we will mainly deal with non-compact or infinite-dimensional highest-weight representations. The vector space on which the representation acts is characterised by its $highest-weight$ or *primary state*, which corresponds to a field or a local operator. The primary state is defined by means of the action of the Cartan generators. For the particular case of the $\mathfrak{psu}(2,2|4)$ algebra, the generators of the Cartan subalgebra are given by [4],

$$\begin{aligned}
\mathfrak{J}^+ &\in \left\{\mathfrak{K}^{\alpha\dot\beta}, \mathfrak{S}^\alpha_b, \dot{\mathfrak{S}}^{a\dot\beta}, \mathfrak{L}^\alpha_\beta(\alpha<\beta), \dot{\mathfrak{L}}^{\dot\alpha}(\dot\alpha<\dot\beta), \mathfrak{R}^a_b(a<b)\right\}\\
\mathfrak{J}^0 &\in \left\{\mathfrak{L}^\alpha{}_\beta(\alpha=\beta), \dot{\mathfrak{L}}^{\dot\alpha}_{\dot\beta}(\dot\alpha=\dot\beta), \mathfrak{R}^a_b(a=b), \mathfrak{D}, \mathfrak{B}, \mathfrak{C}\right\},\\
\mathfrak{J}^- &\in \left\{\mathfrak{P}_{\dot\alpha\beta}, \mathfrak{Q}^a{}_\beta, \dot{\mathfrak{Q}}_{\dot\alpha b}, \mathfrak{L}^\alpha_\beta(\alpha>\beta), \dot{\mathfrak{L}}^{\dot\alpha}_{\dot\beta}(\dot\alpha>\dot\beta), \mathfrak{R}^a_b(a>b)\right\}.
\end{aligned} \tag{15}$$

All the elements of the Cartan subalgebra commute among each other. The primary state is then defined as the state annihilated by the raising operator $\mathfrak{J}^+$. The action of the Cartan generator $\mathfrak{J}^0$ gives the weight of the primary operator, and the application of lowering operators $\mathfrak{J}^-$ leads to new states called *descendants*. In general, the action of the lowering operator will create an infinite multiplet of states. These states create a space that corresponds to a module of $\mathfrak{psu}(2,2|4)$. In general, the module obtained is irreducible. However, there will be special highest weight for which one finds a reducible multiple. In that case, the irreducible module is called *short*.

In the case of the maximal compact subalgebra $\mathfrak{su}(2)\times\mathfrak{su}(2)\times\mathfrak{su}(4)$ of $\mathfrak{psu}(2,2|4)$, the modules will split into finite-dimensional modules of the subalgebra. Note that the indices of the fields associated with either of the $SU(2)$ Lorentz group are symmetrized, whereas the indices associated to $SU(4)$ are antisymmetrized. For this reason, it can be conveniently expressed in terms of bosonic and fermionic creation operators. The irreducible fields acting on the traces of single trace operators transform in the so-called singleton representation $\mathcal{V}_\mathrm{S}$ of $PSU(2,2\mid 4)$ and form the spin chain of $\mathcal{N}=4$ SYM theory. This representation can be constructed by two sets of bosonic oscillators $\mathbf{a}_{i,\alpha}, \mathbf{a}^{\dagger\beta}_j$, $\mathbf{b}_{i,\dot\alpha}, \mathbf{b}^{\dagger\dot\beta}_j$ and one set of fermionic oscillator $\mathbf{d}_{i,A}, \mathbf{d}^{\dagger B}_j$ satisfying the following commutation relations [33]:

$$\left[\mathbf{a}_{i,\alpha}, \mathbf{a}^{\dagger\beta}_j\right] = \delta^\beta_\alpha\delta_{i,j}, \quad \left[\mathbf{b}_{i,\dot\alpha}, \mathbf{b}^{\dagger\dot\beta}_j\right] = \delta^{\dot\beta}_{\dot\alpha}\delta_{i,j}, \quad \left\{\mathbf{d}_{i,A}, \mathbf{d}^{\dagger B}_j\right\} = \delta^B_A\delta_{i,j}. \tag{16}$$

The indices i,j refers to the sites in which the oscillators act. The fields are then obtained by acting with the creation operator on a Fock vacuum $|0\rangle$:

$$\begin{aligned}
\mathrm{D}^k F &\mathrel{\widehat{=}} \left(\mathbf{a}^\dagger\right)^{k+2}\left(\mathbf{b}^\dagger\right)^k \mathbf{d}^{\dagger 1}\mathbf{d}^{\dagger 2}\mathbf{d}^{\dagger 3}\mathbf{d}^{\dagger 4}|0\rangle,\\
\mathrm{D}^k \psi_{ABC} &\mathrel{\widehat{=}} \left(\mathbf{a}^\dagger\right)^{k+1}\left(\mathbf{b}^\dagger\right)^k \mathbf{d}^{\dagger A}\mathbf{d}^{\dagger B}\mathbf{d}^{\dagger C}|0\rangle,\\
\mathrm{D}^k \phi_{AB} &\mathrel{\widehat{=}} \left(\mathbf{a}^\dagger\right)^{k}\left(\mathbf{b}^\dagger\right)^k \mathbf{d}^{\dagger A}\mathbf{d}^{\dagger B}|0\rangle,\\
\mathrm{D}^k \bar\psi_{A} &\mathrel{\widehat{=}} \left(\mathbf{a}^\dagger\right)^{k}\left(\mathbf{b}^\dagger\right)^{k+1} \mathbf{d}^{\dagger A}|0\rangle,\\
\mathrm{D}^k \bar F &\mathrel{\widehat{=}} \left(\mathbf{a}^\dagger\right)^{k}\left(\mathbf{b}^\dagger\right)^{k+2} |0\rangle,
\end{aligned} \tag{17}$$

where $\psi_{ABC} = \epsilon_{ABCD}\psi^D$ represents fermion fields and $\bar\psi_A$ antifermion fields. In terms of the set of oscillators, the generators of $U(2,2|4)$ can be expressed as:

$$
\begin{aligned}
&\mathfrak{L}^{\alpha}_{i,\beta} = \mathbf{a}^{\dagger\alpha}_i \mathbf{a}_{i,\beta} - \tfrac{1}{2}\delta^{\alpha}_{\beta}\mathbf{a}^{\dagger\gamma}_i \mathbf{a}_{i,\gamma}, && \mathfrak{Q}^{\alpha A}_i = \mathbf{a}^{\dagger\alpha}_i \mathbf{d}^{\dagger A}_i,\\
&\dot{\mathfrak{L}}^{\dot\alpha}_{i,\dot\beta} = \mathbf{b}^{\dagger\dot\alpha}_i \mathbf{b}_{i,\dot\beta} - \tfrac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\mathbf{b}^{\dagger\dot\gamma}_i \mathbf{b}_{i,\dot\gamma}, && \mathfrak{S}_{i,\alpha A} = \mathbf{a}_{i,\alpha}\mathbf{d}_{i,A},\\
&\mathfrak{R}^{A}_{i,B} = \mathbf{d}^{\dagger A}_i \mathbf{d}_{i,B} - \tfrac{1}{4}\delta^{A}_{B}\mathbf{d}^{\dagger C}_i \mathbf{d}_{i,C}, && \dot{\mathfrak{Q}}^{\dot\alpha}_{i,A} = \mathbf{b}^{\dagger\dot\alpha}_i \mathbf{d}_{i,A},\\
&\mathfrak{D}_i = \tfrac{1}{2}\left(\mathbf{a}^{\dagger\gamma}_i \mathbf{a}_{i,\gamma} + \mathbf{b}^{\dagger\dot\gamma}_i \mathbf{b}_{i,\dot\gamma} + 2\right), && \dot{\mathfrak{S}}^{A}_{i,\dot\alpha} = \mathbf{b}_{i,\dot\alpha}\mathbf{d}^{\dagger A}_i,\\
&\mathfrak{P}^{\alpha\dot\alpha}_i = \mathbf{a}^{\dagger\alpha}_i \mathbf{b}^{\dagger\dot\alpha}_i, && \mathfrak{K}_{i,\alpha\dot\alpha} = \mathbf{a}_{i,\alpha}\mathbf{b}_{i,\dot\alpha},\\
&\mathfrak{C}_i = \tfrac{1}{2}\left(\mathbf{a}^{\dagger\gamma}_i \mathbf{a}_{i,\gamma} - \mathbf{b}^{\dagger\dot\gamma}_i \mathbf{b}_{i,\dot\gamma} - \mathbf{d}^{\dagger C}_i \mathbf{d}_{i,C} + 2\right), && \mathfrak{B}_i = \mathbf{d}^{\dagger C}_i \mathbf{d}_{i,C}.
\end{aligned}
\tag{18}
$$

These are the generators of $U(2,2|4)$, which correspond to the generators of $PSU(2,2|4)$ in addition to the central charge $\mathfrak{C}_i$ and the hypercharge $\mathfrak{B}_i$. In particular, the central charge vanishes in all the physical fields. Single trace operators are then obtained by taking tensor product of the singleton representation,

$$
\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \ldots \otimes \mathcal{V}_L. \tag{19}
$$

In the spin chain representation, each index correspond to a site and the total number L refers to the length of the chain. Moreover, single trace operators are invariant under cyclic permutations. This means that remains invariant if a bosonic field changes its position from the first to the last position of the traces, and acquires a minus sign if the field is fermionic.

2.6 One-loop dilatation generator

In this section we will show how to obtain the one-loop dilatation operator. First we will show the result in terms of harmonic, to afterwards derive the more treatable results in terms of the super harmonic variables. section is mainly based on [3].

In perturbation theory, the full generators of the algebra $\mathfrak{psu}(2,2|4)$ can be expressed as a series expansion in the coupling constant:

$$
\mathfrak{J}(g) = \sum_{k=0}^{\infty} g^k \mathfrak{J}_k, \tag{20}
$$

.

where the sum runs over the different quantum corrections, the term $k = 0$ correspond to the generator in the classical theory. The one-loop dilatation generator corresponds, $\mathfrak{D}^{(2)}$ to the first quantum correction the classical dilatation generator receives, and hence it should be invariant under the classical generators,

$$
\left[\mathfrak{J}_0, \mathfrak{D}^{(2)}\right] = 0. \tag{21}
$$

Therefore, under the classical algebra $\mathfrak{psu}(2,2|4)$, the one-loop anomalous dilatation generator will be considered as an independent object. Consequently, the one-loop dilatation generator can act as a Hamiltonian of the one-dimensional $\mathfrak{psu}(2,2|4)$ spin chain representation, see figure [**:fig2.5**]. In perturbation theory,

$$\mathfrak{J}(g) = \mathfrak{J} + \mathcal{O}(g), \quad \mathfrak{D}(g) = \mathfrak{D} + g^2\mathfrak{D}^{(2)} + \mathcal{O}\left(g^3\right), \quad [\mathfrak{J}, \mathfrak{D}^{(\in)}] = 0, \tag{22}$$

planar $\mathcal{N} = 4$ SYM	$\mathfrak{psu}(2,2|4)$ spin chain
single trace operator	cyclic spin chain
field	spin site
anomalous dilatation operator $g^{-2}\delta\mathfrak{D}$	Hamiltonian $\mathcal{H}$
anomalous dimension $g^{-2}\delta D$	energy E

Figure 7: Correspondence between $N = 4$ SYM and the spin chain picture

The irreducible modules of $PSU(2,2|4)$ can be obtained as a tensor product of individual singleton representations,

$$\mathcal{V}_{\mathrm{F}} \otimes \mathcal{V}_{\mathrm{F}} = \sum_{j=0}^{\infty} \mathcal{V}_j, \tag{23}$$

Denoting the projection operator to the subspace $\mathcal{V}_j$,

$$\mathbb{P}_j : \mathcal{V}_{\mathrm{F}} \otimes \mathcal{V}_{\mathrm{F}} \to \mathcal{V}_j, \tag{24}$$

the one-loop dilatation operator is:

$$\mathfrak{D}^{(2)} = 2\sum_{i=1}^{L}\sum_{j=0}^{\infty} h(j)\left(\mathbb{P}_j\right)_{ii+1}, \tag{25}$$

where $h(j) = \sum_{i+1}^{i} \frac{1}{i}$ is the j^{th} harmonic number. Let us obtain the one-loop dilatation operator in terms of the harmonic oscillator rather than the projection operator. In particular, this expression is more handy and will be more useful for general computations. We use $\mathbf{A}_i^\dagger$ with $a = 1, \ldots, 8$ to represent the eight oscillators,

$$\mathbf{A}_i^\dagger = \left(\mathbf{a}_i^{\dagger 1}, \mathbf{a}_i^{\dagger 2}, \mathbf{b}_i^{\dagger 1}, \mathbf{b}_i^{\dagger 2}, \mathbf{d}_i^{\dagger 1}, \mathbf{d}_i^{\dagger 2}, \mathbf{d}_i^{\dagger 3}, \mathbf{d}_i^{\dagger 4}\right), \tag{26}$$

Hence, we can define a general state in $\mathcal{V}_F \otimes \mathcal{V}_F$ as:

$$|s_1, \ldots, s_n; A\rangle = \mathbf{A}_{s_1,A_1}^\dagger \ldots \mathbf{A}_{s_n,A_n}^\dagger |00\rangle. \tag{27}$$

The label $s_k = 1, 2$ determines the site on which the k^{th} oscillator acts. The action of the one-loop operator on one of these states can be written as a weighted sum over all the reorderings,

$$\left(\mathfrak{D}^{(1)}\right)_{12} \mathbf{A}_{s_1}^{\dagger A_1} \cdots \mathbf{A}_{s_n}^{\dagger A_n} |0\rangle = \sum_{s'_1,\ldots,s'_n=1}^{2} \delta_{C_2,0} c\left(n, n_{12}, n_{21}\right) \mathbf{A}_{s'_1}^{\dagger A_1} \cdots \mathbf{A}_{s'_n}^{\dagger A_n} |0\rangle. \tag{28}$$

The sum goes over the sites 1,2, $\delta_{C_2,0}$ project to states where the central charge zero, n denote the number of oscillators that are at both sites and n_{ij} the number of oscillators that change site from i to j. From this we see that the action of the one-loop dilatation generator does not depend on the type of oscillators, but only on the number of oscillators that change site. In particular, the coefficients read:

$$c\left(n, n_{12}, n_{21}\right) = \begin{cases} 2h\left(\frac{1}{2}n\right) & \text{if } n_{12} = n_{21} = 0, \\ 2(-1)^{1+n_{12}n_{21}}\ \mathrm{B}\left(\frac{1}{2}\left(n_{12}+n_{21}\right), 1+\frac{1}{2}\left(n-n_{12}-n_{21}\right)\right) & \text{else}. \end{cases} \tag{29}$$

Using an integral representation by defining [36],

$$\left(\mathbf{A}_i^\dagger\right)^{\vec{n}_i} = \left(\mathbf{a}_i^{\dagger 1}\right)^{a_i^1} \left(\mathbf{a}_i^{\dagger 2}\right)^{a_i^2} \left(\mathbf{b}_i^{\dagger \dot{1}}\right)^{b_i^{\dot{1}}} \left(\mathbf{b}_i^{\dagger \dot{2}}\right)^{b_i^{\dot{2}}} \left(\mathbf{d}_i^{\dagger 1}\right)^{d_i^1} \left(\mathbf{d}_i^{\dagger 2}\right)^{d_i^2} \left(\mathbf{d}_i^{\dagger 3}\right)^{d_i^3} \left(\mathbf{d}_i^{\dagger 4}\right)^{d_i^4}, \tag{30}$$

one obtains:

$$\left(\mathfrak{D}^{(1)}\right)_{12} \left(\mathbf{A}_1^\dagger\right)^{\vec{n}_1} \left(\mathbf{A}_2^\dagger\right)^{\vec{n}_2} |0\rangle = 4\delta_{C_2,0} \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \cot\theta \left(\left(\mathbf{A}_1^\dagger\right)^{\vec{n}_1} \left(\mathbf{A}_2^\dagger\right)^{\vec{n}_2} - \left(\mathbf{A}_1'^\dagger\right)^{\vec{n}_1} \left(\mathbf{A}_2'^\dagger\right)^{\vec{n}_2} \right) |0\rangle, \tag{31}$$

where

$$\begin{pmatrix} \mathbf{A}_1'^\dagger \\ \mathbf{A}_2'^\dagger \end{pmatrix} = V(\theta) \begin{pmatrix} \mathbf{A}_1^\dagger \\ \mathbf{A}_2^\dagger \end{pmatrix}, \quad V(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{32}$$

This last expression is more suitable when making the connection with scattering amplitudes.

3 Form factors

In this chapter we will study form factors. These quantities are of great importance since they provide the connection between on-shell methods and off-shell quantities. Hence, they can be used to test until which extent do on-shell methods apply to off-shell quantities, such as correlation functions of gauge invariant local composite operators. We will afterwards define the so-called minimal form factors. Finally, we will give some comments on the structure of operators in different subsectors.

3.1 Introduction to form factors

In this section we will give a short introduction to form factors. We will also introduce the variables we will be using throughout this work as well as the notation used.

Scattering amplitudes are of great importance since from them we can obtain cross-sections, which represent the physical quantity measured in particle colliders. On-shell methods have been a powerful method for studying these quantities. In particular, they represent the overlap of an n-particle on-shell state with the vacuum,

$$\mathcal{A}_n(1,\ldots,n) = \langle 1,\ldots,n \mid 0\rangle. \tag{33}$$

Correlation functions of gauge-invariant local composite operators $\mathcal{O}$ are off-shell quantities that represent the products of these operators at positions x_i between two vacuum states. Form factors of gauge-invariant local composite operators are defined as the overlap between a state created by the operator $\mathcal{O}(x)$ from the vacuum with an n particle out-going on-shell state. They form a bridge between the fully on-shell scattering amplitudes and the fully off-shell correlation functions.

On-shell methods require to work in momentum space. The Fourier transformation of form factors change the x-dependence of the operator to the off-shell momentum q which does not satisfy the mass-shell condition,

$$\mathcal{F}_{\mathcal{O}}(1,\ldots,n;q) = \int \mathrm{d}^4x \mathrm{e}^{-iqx}\langle 1,\ldots,n|\mathcal{O}(x)|0\rangle = (2\pi)^4\delta^4\left(q-\sum_{i=1}^{n}p_i\right)\langle 1,\ldots,n|\mathcal{O}(0)|0\rangle, \tag{34}$$

where the delta function ensure momentum conservation.

On-shell particle are defined as the ones satisfying the mass-shell condition $p^2 = m^2$. In the case of masslss particles this means $p^2 = 0$. Hence, it is useful to express the momenta of the external on-shell particles $i = 1, ..., n$ in terms of the spinor-helicity variables [17],

$$p_j^{\alpha\dot{\alpha}} \equiv p_j^{\mu}\sigma_{\mu}^{\alpha\dot{\alpha}} = \lambda_j^{\alpha}\tilde{\lambda}_j^{\dot{\alpha}}, \tag{35}$$

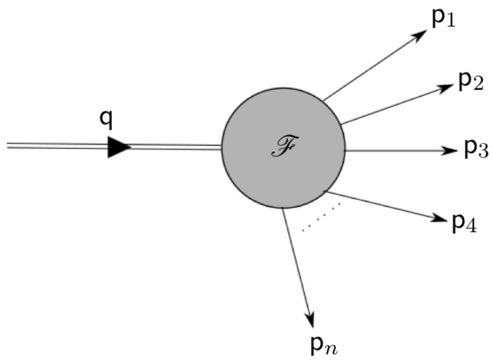


Figure 8: Pictorical representation of a form factor. The double line arrow represent the off-shell inserted, whereas the outgoing arrows represent the on-shell n particles [25].

where $(\sigma_\mu)^{\alpha\dot{\alpha}}$ represent the four-dimensional Pauli matrices. In particular, the spinor variables are two dimensional and they do transform under $SU(2)$ and $\overline{SU}(2)$. Hence, each on-shell massless spin s particle is characterised by its helicity $h = \pm|s|$ and a pair of spinors $\lambda_j^\alpha, \tilde{\lambda}_j^{\dot{\alpha}}$.In particular, these variables are really useful because physics remains invariant when multiplying spinor and antispinor by opposite phases. This is the so-called little-group scaling because the same phases would arise from a rotation along the propagation axis of the particle i. Contractions of spinor helicity variables are written as:

$$\begin{aligned}
&\langle ij\rangle \equiv \varepsilon^{\alpha\beta} \left(\lambda_i\right)_\alpha \left(\lambda_j\right)_\beta , \\
&[ij] \equiv \varepsilon^{\dot{\alpha}\dot{\beta}} \left(\tilde{\lambda}_i\right)_{\dot{\alpha}} \left(\lambda_j\right)_{\dot{\beta}} , \\
&s_{ij} = \langle ij\rangle[ji].
\end{aligned} \tag{36}$$

where we have introduced the angle and square bracket notation. Another important quantity is the so-called Mandelstam variables $(p_i + p_j)^2 = \langle ij\rangle[ji] = s_{ij}$. They also satisfy the Schouten identities,

$$\langle ij\rangle\langle kl\rangle + \langle ik\rangle\langle lj\rangle + \langle il\rangle\langle jk\rangle = 0, \quad [ij][kl] + [ik][lj] + [il][jk] = 0. \tag{37}$$

For real momenta we will have $[ij] = \langle ij\rangle^*$ and hence $\bar{\lambda}_i^{\dot{\alpha}} = (\lambda_i^\alpha)^*$. Moreover, the spinor products are complex square roots of the momentum invariants,

$$\langle ij\rangle = \sqrt{s_{ij}} e^{i\phi_{ij}}, \quad [ij] = \sqrt{s_{ij}} e^{-i\phi_{ij}}. \tag{38}$$

In $\mathcal{N} = 4$ SYM theory, one can package all scattering amplitudes of n different particles with fixed helicity into a superamplitude, making manifest the supersymmetries of the theory.

This can be done by making use of the $\mathcal{N}=4$ Nair's on-shell superfield [31],

$$\Phi\left(p_i,\tilde{\eta}_i\right)=g_+\left(p_i\right)+\tilde{\eta}_i^A\bar{\psi}_A\left(p_i\right)+\frac{\tilde{\eta}_i^A\tilde{\eta}_i^B}{2!}\phi_{AB}\left(p_i\right)+\frac{\epsilon_{ABCD}\tilde{\eta}_i^A\tilde{\eta}_i^B\tilde{\eta}_i^C}{3!}\psi^D\left(p_i\right)+\tilde{\eta}_i^1\tilde{\eta}_i^2\tilde{\eta}_i^3\tilde{\eta}_i^4 g_-\left(p_i\right), \tag{39}$$

where A acts as an anti-fundamental $SU(4)$ index. This expression depends on fermionic variables $\tilde{\eta}$, and the external states are characterised by different powers of these variables. A state with a helicity h_i corresponds to $2-2h_i$ powers of η_i. By means of Nair's on-shell superfield we can combine also all the form factors into one super form factor.

Therefore, any external field Φ_i can be completely characterised by its super momentum $\Lambda=(\lambda_i,\tilde{\lambda}_i,\tilde{\eta})$. The superfield can also be expanded in the fermionic variables η, which are related to the variables $\tilde{\eta}$ via:

$$\begin{aligned} 1=\eta_i^1\eta_i^2\eta_i^3\eta_i^4,\quad \tilde{\eta}_{iA}=\frac{1}{3!}\epsilon_{ABCD}\eta_i^B\eta_i^C\eta_i^D,\quad \tilde{\eta}_{iA}\tilde{\eta}_{iB}=\frac{1}{2!}\epsilon_{ABCD}\eta_i^C\eta_i^D,\\ \tilde{\eta}_{iA}\tilde{\eta}_{iB}\tilde{\eta}_{iC}=\epsilon_{ABCD}\eta_i^D,\tilde{\eta}_{i1}\qquad\qquad \tilde{\eta}_{i2}\tilde{\eta}_{i3},\tilde{\eta}_{i4}=1.\end{aligned} \tag{40}$$

We can then combine component form factors into one super form factor. The component form factors can be extracted from the super form factors by taking derivatives with respect to the η_i variables:

$$\mathcal{F}_{\mathcal{O}}\left(1^{g^+},2^{g^-},\ldots,n^{\phi_{12}};q\right)=1\left(\frac{\partial}{\partial\eta_2^1}\frac{\partial}{\partial\eta_2^2}\frac{\partial}{\partial\eta_2^3}\frac{\partial}{\partial\eta_2^4}\right)\cdots\left(-\frac{\partial}{\partial\eta_n^1}\frac{\partial}{\partial\eta_n^2}\right)\mathcal{F}_{\mathcal{O}}(1,2,\ldots,n;q)\bigg|_{\eta_i^A=0}, \tag{41}$$

where the superscripts specify the helicities and flavours of the respective fields and the sign takes into account that the $\tilde{\eta}_i$ variables anticommute. Amplitudes and form factors also depend on the color degrees of freedom of each field. We can then split off the dependence on the gauge group generators by defining color-ordered amplitudes $\hat{\mathcal{A}}_n$,

$$\mathcal{A}_n(1,\ldots,n)=g_{\mathrm{YM}}^{n-2}\sum_{\sigma\in\mathbb{S}_n/\mathbb{Z}_n}\mathrm{tr}\left(\mathrm{T}^{a_{\sigma(1)}}\cdots\mathrm{T}^{a_{\sigma(n)}}\right)\hat{\mathcal{A}}_n(\sigma(1),\ldots,\sigma(n))+\text{multi-trace terms}, \tag{42}$$

and color-ordered form factors $\hat{\mathcal{F}}_{O,n}$

$$\mathcal{F}_{\mathcal{O},n}(1,\ldots,n;q)=g_{\mathrm{YM}}^{n-L}\sum_{\sigma\in\mathbb{S}_n/\mathbb{Z}_n}\mathrm{tr}\left(\mathrm{T}^{a_{\sigma(1)}}\ldots\mathrm{T}^{a_{\sigma(n)}}\right)\hat{\mathcal{F}}_{\mathcal{O},n}(\sigma(1),\ldots,\sigma(n);q)+\text{multi-trace terms}, \tag{43}$$

where T^a with $a=1,...,N^2-1$ are the generators of the gauge group $SU(N)$ and the sum is over all non-cyclic permutations. We normalize the generator in the standard amplitudes convention by $Tr[T^aT^b]=\delta^{ab}$ and we define $f^{abc}=-i\,\mathrm{tr}[[T^a,T^b]T^c]$ as the structure constants. The only difference between both expressions is that the momentum q in the operator does not take part in the color ordering, because the operator is a color singlet.

Multi-trace terms start to appear at one-loop order, but they will be suppressed in the planar limit (subleading in the 't Hooft limit, they appear in powers of $1/N$). Therefore, double-trace terms part of amplitudes and form factors will be suppressed in single-trace parts of amplitudes and form factors, respectively. However, the double-trace part of amplitudes can contribute to single-trace form factors when contracted with composite operators, as was explained in the previous chapter when defining the 't Hooft limit.

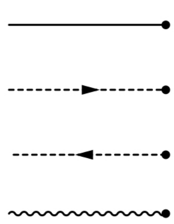
3.2 Minimal form factors

The length of an operator $\mathcal{O}$ is defined as the number of irreducible fields in the operator. The form factor of an operator $\mathcal{O}$ is minimal if the number of external legs is equal to the length of the operator, i.e., it is the form factor with the lowest number of external legs. Equivalently, we can define a minimal form factor $\mathcal{F}_{\mathcal{O},L(\mathcal{O})}$, such that

$$\mathcal{F}^{(0)}_{\mathcal{O},L(\mathcal{O})} \neq 0 \;\text{ while }\; \mathcal{F}^{(0)}_{\mathcal{O},n} = 0 \;\; when \;\; n < L(\mathcal{O}). \tag{44}$$

Consequently, in minimal form factors there is a one-to-one correspondence between on-shell external particles and the fields in the operator. Form factors can now be easily computed via Feynman rules. In particular, in the free theory no interactions occur and the Feynman rules reduce to vertices of the composite operator and the outgoing fields.

Moreover, form factors in the free theory vanish unless the number of external fields equals the number of irreducible fields in the gauge-invariant local composite operator $\mathcal{O}$. As every occurrence of the Yang-Mills coupling constant g_{YM} either increases the number of legs or the number of loops, the form factor in the free theory equals the minimal tree-level form factor in the interacting theory.



p	$= 1$	for an outgoing scalar ϕ_{AB}
$p,\pm$	$= \bar{u}_{\pm}(p)$	for an outgoing fermion ψ^A of helicity $\pm\frac{1}{2}$
$p,\pm$	$= v_{\pm}(p)$	for an outgoing antifermion $\bar{\psi}_A$ of helicity $\pm\frac{1}{2}$
p,μ	$= \epsilon_{\mu,\pm}(p,r)$	for an outgoing gluon of helicity ± 1

Figure 9: Feynman rules [34].

For example, for a scalar ϕ_{AB} the Feynman rule is simply 1. But to this we have to add the factor $\tilde{\eta}_A\tilde{\eta}_B$ in agreement with the Nair's on-shell superfield definition. The same occurs for fermion and antifermions, each of them have different powers of the fermionic variables depending on its helicity. In particular an outgoing antifermion of positive helicity has to be dressed with one power $\tilde{\eta}_A$ whereas an outgoing fermion of positive helicity carries a factor $\frac{1}{3!}\epsilon_{ABCD}\tilde{\eta}_A\tilde{\eta}_B\tilde{\eta}_C\tilde{\eta}_D$.

In the free theory, the covariant derivative $D_{\alpha\dot{\alpha}}$ reduces to the partial derivative $\partial_{\alpha\dot{\alpha}}$. In momentum space the partial derivative translates simply to the on-shell momentum of the external particles $p_{\alpha\dot{\alpha}} = \lambda_\alpha\tilde{\lambda}_{\dot{\alpha}}$ on which the covariant derivative acts. The Feynman rules for an outgoing gauge field of positive or negative helicity in spinor helicity variables look

$$\epsilon_+^{\alpha\dot{\alpha}}(p_i;r_i) = \sqrt{2}\frac{\lambda_{r_i}^\alpha\tilde{\lambda}_{p_i}^{\dot{\alpha}}}{\langle r_i p_i\rangle}, \quad \epsilon_-^{\alpha\dot{\alpha}}(p_i;r_i) = \sqrt{2}\frac{\lambda_{p_i}^\alpha\tilde{\lambda}_{r_i}^{\dot{\alpha}}}{[p_i r_i]}, \tag{45}$$

where r is an arbitrary reference vector. Recalling that gauge-invariant local composite operators contain gauge fields only in the gauge-covariant and irreducible combinations of the self-dual and anti-self-dual field strengths. For the free theory,

$$F_{\alpha\beta} = \frac{1}{2\sqrt{2}}\epsilon^{\dot{\alpha}\dot{\beta}}\left(\partial_{\alpha\dot{\alpha}}A_{\beta\dot{\beta}} - \partial_{\beta\dot{\beta}}A_{\alpha\dot{\alpha}}\right), \quad \bar{F}_{\dot{\alpha}\dot{\beta}} = \frac{1}{2\sqrt{2}}\epsilon^{\alpha\beta}\left(\partial_{\alpha\dot{\alpha}}A_{\beta\dot{\beta}} - \partial_{\beta\dot{\beta}}A_{\alpha\dot{\alpha}}\right). \tag{46}$$

Replacing the fields by the polarization vectors and the derivatives for each spinor helicity variables we obtain:

$$\begin{aligned}
F_{\alpha\beta} &\xrightarrow{\epsilon_+} -\tfrac{1}{2\sqrt{2}}\epsilon_{\dot{\alpha}\dot{\beta}}\left(\lambda_{p_i}^\alpha\tilde{\lambda}_{p_i}^{\dot{\alpha}}\epsilon_+^{\beta\dot{\beta}} - \lambda_{p_i}^\beta\tilde{\lambda}_{p_i}^{\dot{\beta}}\epsilon_+^{\alpha\dot{\alpha}}\right) = 0\\
F_{\alpha\beta} &\xrightarrow{\epsilon_-} -\tfrac{1}{2\sqrt{2}}\epsilon_{\dot{\alpha}\dot{\beta}}\left(\lambda_{p_i}^\alpha\tilde{\lambda}_{p_i}^{\dot{\alpha}}\epsilon_-^{\beta\dot{\beta}} - \lambda_{p_i}^\beta\tilde{\lambda}_{p_i}^{\dot{\beta}}\epsilon_-^{\alpha\dot{\alpha}}\right) = \lambda_{p_i}^\alpha\lambda_{p_i}^\beta\\
\bar{F}_{\dot{\alpha}\dot{\beta}} &\xrightarrow{\epsilon_+} -\tfrac{1}{2\sqrt{2}}\epsilon_{\alpha\beta}\left(\lambda_{p_i}^\alpha\tilde{\lambda}_{p_i}^{\dot{\alpha}}\epsilon_+^{\beta\dot{\beta}} - \lambda_{p_i}^\beta\tilde{\lambda}_{p_i}^{\dot{\beta}}\epsilon_+^{\alpha\dot{\alpha}}\right) = \tilde{\lambda}_{p_i}^{\dot{\alpha}}\tilde{\lambda}_{p_i}^{\dot{\beta}}\\
\bar{F}_{\dot{\alpha}\dot{\beta}} &\xrightarrow{\epsilon_-} -\tfrac{1}{2\sqrt{2}}\epsilon_{\alpha\beta}\left(\lambda_{p_i}^\alpha\tilde{\lambda}_{p_i}^{\dot{\alpha}}\epsilon_-^{\beta\dot{\beta}} - \lambda_{p_i}^\beta\tilde{\lambda}_{p_i}^{\dot{\beta}}\epsilon_-^{\alpha\dot{\alpha}}\right) = 0.
\end{aligned} \tag{47}$$

In the super form factor we thus have $\lambda_i^\alpha\lambda_i^\beta\tilde{\eta}^1\tilde{\eta}^2\tilde{\eta}^3\tilde{\eta}^4$ and $\tilde{\lambda}_i^{\dot{\alpha}}\tilde{\lambda}_i^{\dot{\beta}}$ for the self-dual and anti-self-dual parts of the fields strength, respectively. These comment are summarised in the following table

Fields in local operators		On-shell legs in minimal form factors
$\bar{F}^{\dot{\alpha}\dot{\beta}}$	$\xrightarrow{g_+}$	$\tilde{\lambda}^{\dot{\alpha}}\tilde{\lambda}^{\dot{\beta}}$
$\bar{\Psi}^{\dot{\alpha}A}$	$\xrightarrow{\bar{\psi}_{\dot{\alpha}A}}$	$\tilde{\lambda}^{\dot{\alpha}}\eta^A$
Φ^{AB}	$\xrightarrow{\phi_{AB}}$	$\eta^A\eta^B$
$\Psi^{\alpha ABC}$	$\xrightarrow{\psi_{\alpha ABC}}$	$\lambda^\alpha\eta^A\eta^B\eta^C$
$F^{\alpha\beta}$	$\xrightarrow{g_-}$	$\lambda^\alpha\lambda^\beta\eta^1\eta^2\eta^3\eta^4$
$D^{\alpha\dot{\alpha}}$	$\xrightarrow{p_{\alpha\dot{\alpha}}}$	$\lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}$

Figure 10: Correspondence between local operators and minimal form factors [35].

Therefore, through the minimal form factor, the local operator is naturally translated in terms of spinor helicity variables:

$$\text{off-shell local field} \xrightarrow{\text{minimal form factor}} \text{on-shell spinor helicity quantity},$$

as was summarize in figure 10. These expressions are both symmetric under $SU(2)$ and

$\overline{SU}(2)$ indices, and antisymmetric under $SU(4)$ indices as it is necessary since the theory should obey these symmetries due to the $PSU(2,2|4)$ group representation.

Consequently, for a gauge-invariant local composite operator $\mathcal{O}$ characterised by its oscillator representation $\{\vec{n}_i\}_{i=1,\ldots,L} = \left\{\left(a_i^1, a_i^2, b_i^{\dot{1}}, b_i^{\dot{2}}, d_i^1, d_i^2, d_i^3, d_i^4\right)\right\}_{i=1,\ldots,L}$, the minimal color-ordered tree-level super form factor is:

$$\begin{aligned}\hat{\mathcal{F}}_{\mathcal{O},L}\left(\Lambda_1,\ldots,\Lambda_L;q\right) = (2\pi)^4\delta^4\left(q-\sum_{i=1}^{L}p_i\right)\sum_{\sigma\in\mathbb{Z}_L}\\ \prod_{i=1}^{L}\left(\lambda^1_{\sigma(i)}\right)^{a_i^1}\left(\lambda^2_{\sigma(i)}\right)^{a_i^2}\left(\tilde{\lambda}^{\dot{1}}_{\sigma(i)}\right)^{b_i^{\dot{1}}}\left(\tilde{\lambda}^{\dot{2}}_{\sigma(i)}\right)^{b_i^{\dot{2}}}\left(\tilde{\eta}^{1}_{\sigma(i)}\right)^{d_i^1}\left(\tilde{\eta}^{2}_{\sigma(i)}\right)^{d_i^2}\left(\tilde{\eta}^{3}_{\sigma(i)}\right)^{d_i^3}\left(\tilde{\eta}^{4}_{\sigma(i)}\right)^{d_i^4}.\end{aligned} \tag{48}$$

The sum in the expression reflects the cyclic invariance of the single-trace operator. The product refers to the grading, i.e., the order with respect to i. At tree-level, form factors of composite operators that are linear combinations of the eight oscillators n are given by the respective linear combinations. This is due to the fact that at tree-level there is no operator-mixing. However, at higher loops, operator-mixing occurs and the considerations above cannot take place.

The expression of the super form factor above is just the one we would obtain by replacing the oscillators in the oscillator representation according to (apart from the momentum-conserving delta function and the normalisation factor L):

$$\begin{array}{lll}\mathbf{a}_i^{\dagger\alpha}\to\lambda_i^\alpha, & \mathbf{b}_i^{\dagger\dot{\alpha}}\to\tilde{\lambda}_i^{\dot{\alpha}}, & \mathbf{d}_i^{\dagger A}\to\tilde{\eta}_i^A,\\ \mathbf{a}_{i,\alpha}\to\partial_{i,\alpha}=\frac{\partial}{\partial\lambda_i^\alpha}, & \mathbf{b}_{i,\dot{\alpha}}\to\partial_{i,\dot{\alpha}}=\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{\alpha}}}, & \mathbf{d}_{i,A}\to\partial_{i,A}=\frac{\partial}{\partial\tilde{\eta}_i^A}.\end{array} \tag{49}$$

Therefore, there is a direct correspondence between the super minimal tree-level form factor in terms of super spinor helicity variables and the normalised and graded cyclically invariant state in the spin-chain picture in terms of the oscillators:

$$\hat{\mathcal{F}}_{\mathcal{O},L}\left(\Lambda_1,\ldots,\Lambda_L;q\right) = L(2\pi)^4\delta^4\left(q-\sum_{i=1}^{L}p_i\right)\times\mathcal{O}\ \Big|\ \begin{array}{l}\mathbf{a}_i^{\dagger\alpha}\to\lambda_i^\alpha\\ \mathbf{b}_i^{\dagger\dot{\alpha}}\to\tilde{\lambda}_i^{\dot{\alpha}}\\ \mathbf{d}_i^{\dagger A}\to\eta_i^A\end{array} \tag{50}$$

Moreover, if we replace the oscillators according to eq. 49 in the generators in eq. 18 we obtain the representation of the centrally extended algebra of $PSU(2,2|4)$ in terms of on-shell superfields,

$$\begin{aligned}
\mathfrak{L}^{\alpha}_{i,\beta} &= \lambda^{\alpha}_i \partial_{i,\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}\lambda^{\gamma}_i\partial_{i,\gamma}, & \mathfrak{Q}^{\alpha A}_i &= \lambda^{\alpha}_i \tilde{\eta}^A_i, \\
\dot{\mathfrak{L}}^{\dot\alpha}_{i,\dot\beta} &= \tilde\lambda^{\dot\alpha}_i \partial_{i,\dot\beta} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\tilde\lambda^{\dot\gamma}_i\partial_{i,\dot\gamma}, & \mathfrak{S}_{i,\alpha A} &= \partial_{i,\alpha}\partial_{i,A}, \\
\mathfrak{R}^{A}_{i,B} &= \tilde\eta^{A}_i \partial_{i,B} - \frac{1}{4}\delta^{A}_{B}\tilde\eta^{C}_i\partial_{i,C}, & \mathfrak{Q}^{\dot\alpha}_{i,A} &= \tilde\lambda^{\dot\alpha}_i \partial_{i,A}, \\
\mathfrak{D}_i &= \frac{1}{2}\left(\lambda^{\gamma}_i\partial_{i,\gamma} + \tilde\lambda^{\dot\gamma}_i\partial_{i,\dot\gamma} + 2\right), & \dot{\mathfrak{S}}^{A}_{i,\dot\alpha} &= \partial_{i,\dot\alpha}\tilde\eta^A_i, \\
\mathfrak{C}_i &= \frac{1}{2}\left(\lambda^{\gamma}_i\partial_{i,\gamma} - \tilde\lambda^{\dot\gamma}_i\partial_{i,\dot\gamma} - \tilde\eta^{C}_i\partial_{i,C} + 2\right), & \mathfrak{P}^{\alpha\dot\alpha}_i &= \lambda^{\alpha}_i\tilde\lambda^{\dot\alpha}_i, \\
\mathfrak{B} &= \tilde\eta^{C}_i\partial_{i,C}, & \mathfrak{K}_{i,\alpha\dot\alpha} &= \partial_{i,\alpha}\partial_{i,\dot\alpha}.
\end{aligned} \tag{51}$$

Once we have defined both the form factors and the generators in the oscillator as well as in the spinor helicity formalism, we are interested in knowing the action of any generator of $PSU(2,2|4)$ on the on-shell part of the form factor,

$$\sum_{i=1}^{n} \mathfrak{J}_i \hat{\mathcal{F}}_{\mathcal{O},n}(1,\ldots,n;q). \tag{52}$$

As can be seen from eq. 51, some of the generators contain differential operators that can act on both the momentum-conservation delta function as well as in the polynomial in terms of the super spinor helicity variables. The action on the momentum conserving delta function will give some spacetime dependent terms that will be dropped out when performing the Fourier transformation.

On the other hand, the action of the generators on the polynomials correspond to the action of these generators on the fields that contain the composite operator in the oscillator representation. Therefore, the action of the generators over the form factor corresponds to the form factor of the generators acting directly on the operators,

$$\sum_{i=1}^{L} \mathfrak{J}_i \hat{\mathcal{F}}_{\mathcal{O},L}(1,\ldots,L;q) = \hat{\mathcal{F}}_{\mathfrak{J}\mathcal{O},L}(1,\ldots,L;q). \tag{53}$$

From this last section, we can conclude that color-ordered minimal tree-level form factors of any operator can be obtained by the replacement of the oscillators in the operator's oscillator representation by spinor helicity variables (apart from the normalization factor and the momentum-conserving delta function). The generators of $PSU(2,2|4)$ are related by the same replacement. Therefore, minimal tree-level form factors translate the spin-chain of free $N=4$ SYM theory into the language of scattering amplitudes.

In fact, this is a special case of a superconformal Ward identity for form factors, that in principle should also hold for the interacting theory. In practice, however, in the interacting theory quantum corrections appear for generators both in composite operators as well as scattering amplitudes. Indeed, form factors receive quantum corrections for the scattering amplitudes as well as for the composite operators. Later in this work, we will calculate the corrections to the action of a generator, the dilatation operator, on composite operators via form factors.

3.3 Sectors of operators

Let us now give some comments on the different operators depending on the sector we are considering. The simplest example we can have is an operator formed by one type of scalar field with no covariant derivatives, as $\mathrm{tr}((\phi^{12})^2)$, that belongs to the stress tensor supermultiplet. In particular, this operator is protected in the sense that it does not have UV divergences and therefore no corrections are needed. Some of the simplest sectors that are not protected are summarized as follows:

$$\begin{aligned} SU(2) &: \{\Phi^{12}, \Phi^{13}\} \\ SO(6) &: \{\Phi^{AB};\ \text{for all } A, B\} \\ SU(2 \mid 3) &: \{\Phi^{12}, \Phi^{23}, \Phi^{13}, \Psi^{123,\alpha}\} \ \text{ for } \ \alpha = 1, 2 \\ SL(2) &: \{\Phi^{12}, D^{\alpha\dot{\alpha}}\} \ \text{ for } \ \alpha = \dot{\alpha} = 1 \end{aligned} \tag{54}$$

In particular, the operators of the $SU(2)$ sector consist on two types of scalars and no covariant derivatives, in the $SL(2)$ operators carry only one scalar field but there is no restriction in the number of covariant derivatives. A generalization of the $SU(2)$ sector is the $SO(6)$ where all six scalars are allowed to appear in the operator. In particular this sector is only closed at one-loop order. Finally the $SU(2|3)$ also represents a generalization of $SU(2)$ where operators contain three scalars and a single fermion. The operators in these sectors are called non-BPS operators and have non-trivial anomalous dimensions due to quantum corrections.

4 Computation of anomalous dimensions

In four-dimensional theories with massless particles, the computation of anomalous dimensions can be derived by means of different methods. In this chapter we will review two methodologies for computing anomalous dimensions either from unitarity cuts or from generalized unitarity. In the first section, we will give an overview of a method for obtaining anomalous dimensions from the scattering matrix. In the second section we will introduce another method in which anomalous dimensions are obtained from generalized unitarity, and by comparing them see the advantages of the first method. In the last section we will particularize to the SMEFT, and study the selection rules that provide trivial zeros for the anomalous dimensions, and therefore see the motivation for obtaining methods for analysing non-vanishing entries of the anomalous dimension matrix.

4.1 Anomalous dimensions from the S-matrix

The Lagrangian of a theory describes its particle content as well as its interactions. Therefore, it is the central point around which all the physics arise inside a certain theory. On the other hand, on-shell techniques are known to be a powerful tool based on physical grounds, since on-shell particles are defined as real particles obeying the equations of motion. Our interest is to approach via on-shell methods the physics that arise from the Lagrangian. The key idea is to consider long distance propagation states, since they can be considered effectively on-shell in the appropriate metric.

In particular, we will focus on the fact that large logarithms signaling the running of couplings originate from these effectively on-shell states. This is achieved by the use of form factors, which represent a bridge between on-shell external particle states and off-shell correlation functions of local operators. Consequently, from form factors, we can obtain the scale dependence of a local operator in terms of the propagation of on-shell external particles.

This method will be based on the connection between the high-energy behavior of the scattering matrix of a theory (propagation of on-shell particles) and the running of the coupling as well as the renormalization of local operators. This relation is equivalent to find the connection between the phase and the energy dependence of form factors. Our study will be focused on weakly-coupled field theories, and hence we will work in perturbation theory.

In this frame the analyticity of form factors can be made explicit since complex conjugation requires to replace the time-ordered propagators by anti-time-ordered propagators. Due to the Feynman prescription, $s_{ij} \to s_{ij} + i\epsilon$, and analyticity then translates to $s_{ij} + i\epsilon \to s_{ij} - i\epsilon$. This corresponds to a rotation in the s_{ij} complex plane and it is generated by a complex dilatation of the momenta,

$$F\left(p_1, \ldots, p_n\right) \to F\left(p_1 \mathrm{e}^{i\alpha}, \ldots, p_n \mathrm{e}^{i\alpha}\right) = \mathrm{e}^{i\alpha D} F\left(p_1, \ldots, p_n\right), \quad \text{where} \quad D \equiv \sum_i p_i^\mu \frac{\partial}{\partial p_i^\mu} \approx -\mu \partial_\mu. \tag{55}$$

The second relation in the expression of the dilatation operator comes from working in dimensional regularisation and from the fact that there is a single renormalisation scale $\mu = \mu_{\mathrm{UV}} = \mu_{\mathrm{IR}}$. From this picture, one can see that the form factor is related to its complex conjugate when the rotation angle reaches the value $\alpha = \pi$, that corresponds to the form factor using anti-time-ordered propagators. By substituting this value of the angle in the previous expression one obtains,

$$F = e^{-i\pi D} F^*. \tag{56}$$

Moreover, a form factor can in general be defined as an amplitude where the operator injects zero momentum $q = 0$,

$$A_i\left(1^{h_1}, \ldots, n^{h_n}\right) = \left\langle k_1^{h_1}, \ldots, k_n^{h_n} \left|\mathcal{O}_i(0)\right| 0\right\rangle. \tag{57}$$

In particular, when the operator is set to the identity one recovers the usual scattering amplitude. On the other hand, the optical theorem expresses the unitarity of the scattering matrix $SS^\dagger = 1$. Recalling that the form factor (when acting as an operator) is formally defined as a small perturbation to the scattering matrix, $\delta S = i\mathcal{F}$, and together with the unitarity of the scattering matrix leads to $\mathcal{F} = S\mathcal{F}^\dagger S$. This means that form factors also obey unitarity. Particularizing for vacuum initial states, this relation reads,

$$F = SF^*, \tag{58}$$

where the product contains a phase-space integral over intermediate n-particle states summed over all n, and in diagrammatic language this means that a cut goes through form factors diagrams. Writing explicitly the initial and final states, this equation can analogously be expressed as:

$$F_{\mathcal{O}}(\vec{n}) \equiv {}_{\text{out}}\left\langle \vec{n}|\mathcal{O}(0)|0\right\rangle = \sum_{\vec{m}} {}_{\text{out}}\langle \vec{n} \mid \vec{m}\rangle_{\text{in}}\, {}_{\text{in}}\langle \vec{m}|\mathcal{O}(0)|0\rangle = \sum_{\vec{m}} S_{nm}\, {}_{\text{in}}\langle \vec{m}|\mathcal{O}(0)|0\rangle, \tag{59}$$

where in the last step we used the definition of the scattering matrix between two states, $S_{nm} = {}_{\text{out}}\left\langle \vec{n} \mid \vec{m}\right\rangle_{\text{in}}$. From this equation a CPT transformation is required [21]. This transformation relates in and out states in the inner product. More concretely, for a local operator we have $\langle \vec{n}|\mathcal{O}(0)|0\rangle = \left\langle 0 \left|\mathcal{O}^\dagger(0)\right| \vec{n}\right\rangle_{\text{in}}$. From this considerations we can derive the relation between the anomalous dimensions and the scattering matrix [14]:

$$e^{-i\pi D} F^* = SF^*. \tag{60}$$

This equation states that the dilatation operator is minus the phase of the S-matrix divided by π. As already mentioned, the right-hand side of this equation represents a unitarity cut. From this expression, we see precisely that the scale dependence of the form factor is encoded in the coefficients of its logarithms.

In the definitions implemented so far we have not particularized to any theory. However, in this work we particularize to the planar limit of $\mathcal{N}=4$ SYM theory, where the dilatation generator act as a generator of the algebra $\mathfrak{psu}(2,2\mid 4)$. Moreover, this generator receives quantum corrections and hence one can express the renormalization group equation as,

$$DF=-\mu\partial_{\mu}F=\left(\beta\left(g^{2}\right)\frac{\partial}{\partial g^{2}}+\gamma_{\mathcal{O}}-\gamma_{\mathrm{IR}}\right)F, \tag{61}$$

where $\gamma_{\mathcal{O}}$ is the anomalous dimension of the local operator, γ_{IR} is the infrared anomalous dimension that arises from soft and/or collinear divergences and β is the beta function that signals the running of the coupling constant with respect to the renormalisation scale. From this equation, we know that the action of the dilatation operator on a form factor is given in terms of ultraviolet and infrared anomalous dimensions as well as the β-function. Relating both definitions of the dilatation operator, we can obtain anomalous dimensions directly from the scattering matrix.

Moreover, let us focus on the leading order approximation to this expressions by taking the leading order approximation of the scattering matrix $S=1+i\mathcal{M}$. Then eq.59 reads,

$$(e^{-i\pi D}-1)F^{*}=i\mathcal{M}F^{*}, \tag{62}$$

Since the imaginary part does not affect our calculations, we can drop out the complex conjugation of the form factor. Combining both definitions of the dilatation operator, one obtain, at the leading order:

$$\left(\beta\left(g^{2}\right)\frac{\partial}{\partial g^{2}}+\gamma_{\mathcal{O}}-\gamma_{\mathrm{IR}}\right)F=-\frac{1}{\pi}\left(\mathcal{M}F\right). \tag{63}$$

The right hand side represents the convolution of the leading order scattering matrix and the minimal form factor, represented by a phase-space integral of lower loop on-shell form factors and leading order amplitudes. In particular, these equations are valid at any loop order, but since our study focuses o weakly coupled field theories, we will consider perturbative expansions of this expression at every loop order. In particular, at first loop order we obtain,

$$\left(\gamma_{\mathcal{O}}^{(1)}-\gamma_{\mathrm{IR}}^{(1)}\right)\langle p_{1},\ldots,p_{n}|\mathcal{O}|0\rangle^{(0)}=-\frac{1}{\pi}\langle p_{1},\ldots,p_{n}|\mathcal{M}\otimes\mathcal{O}|0\rangle^{(0)}, \tag{64}$$

where $\otimes$ is the convolution at this loop order, represented by the phase-space integral over intermediate two particle states in the product MF. The convolution then represents the cut through form factor diagrams that are then joined to the particles that end up in the final external states by the tree-level scattering matrix M, see figure 11.

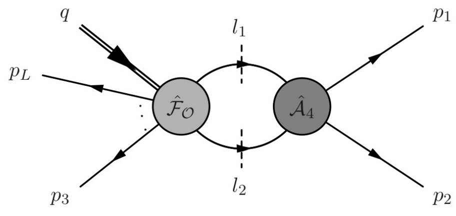


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Figure 11: Unitarity cut relvevant at one-loop [34]

4.2 Anomalous dimensions from UV divergences

In this section we will give an overview of another a method that is traditionally used for computing anomalous dimensions. In particular, in this method one obtain anomalous dimensions from on-shell one-loop amplitudes. One can obtain the UV divergences by first calculating the coefficients of scalar bubble integrals by unitarity cuts, and the subtracting the IR divergences. This section is mainly based [7].

In theories such as the SMEFT, one studies the renormalisation of an operator $\mathcal{O}_i$ by another operator $\mathcal{O}_j$. In particular, the external particles correspond to the operator $\mathcal{O}_i$ with an insertion of a higher dimension operator $\mathcal{O}_j$. The Lagrangian of the operators also depend on the so-called Wilson coefficients c_j of the form $\Delta\mathcal{L} = \sum_j c_j \mathcal{O}_j$. The Wilson coefficients associated to the higher dimension operators are suppressed by powers of a high-energy scale that depends on the dimension of the operator. Then the renormalization group equation in terms of the Wilson coefficients takes the following expression [6]:

$$16\pi^2 \frac{\partial c_i}{\partial \log \mu} = \gamma_{ij} c_j, \tag{65}$$

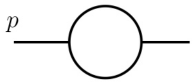
where γ_{ij} is the anomalous dimension matrix governing the renormalisation group running and μ is the renormalisation scale. In massless four dimensional theories, one-loop on-shell amplitudes can be decomposed in terms of a basis of scalar integrals and rational terms,

$$A_i^{(1)} = \sum_s a_{4,i}^s I_{4,s} + \sum_s a_{3,i}^s I_{3,s} + \sum_s a_{2,i}^s I_{2,s} + \text{rational terms}, \tag{66}$$

where $I_{4,s}$ refers to boxes, $I_{3,s}$ to triangles and $I_{2,s}$ to bubbles. The indices $a_{4,i}^s$, $a_{3,i}^s$ and $a_{2,i}^s$ are the gauge-invariant coefficients of the boxes, triangles and bubbles, respectively. In general, the coefficients depend on color and the dimensional regularization parameter ϵ.

The integrals are expanded in the regularization parameter producing UV and IR divergences. These integrals capture the branch cuts of the loop amplitudes and their coefficients can be obtained from the tree level amplitudes by generalized unitarity. Since only the scalar bubble integrals contain UV divergences, we will look for a formula that relates the UV divergences to the scalar bubble coefficients $a_{2,i}^s$.

Bubble coefficients can come from massive bubbles, in which the momentum is greater than zero) or from massless bubbles (where the momentum is simply zero). Coefficients of massive bubbles can be obtained through unitarity cuts. However, massless bubbles, that also contain UV divergences, vanish in dimensional regularization due to the cancellation between UV and collinear IR divergences.



$$\Big|_{p^2=0} \propto \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} + \log\frac{\mu_{UV}^2}{\mu_{IR}^2}\,. \tag{67}$$

Collinear factors depend only on the external legs. Therefore, the collinear divergences of the tree amplitudes should be canceled by collinear loop IR divergences. Consequently, the UV divergence in massless bubbles are obtained by the collinear divergences of the amplitudes,

$$-\frac{1}{2\epsilon}\gamma_{\mathrm{c}}^{\mathrm{IR}(1)}A_i^{(0)} := -\frac{1}{2\epsilon}\sum_p \gamma_{\mathrm{c},p}^{\mathrm{IR}}A_i^{(0)}. \tag{68}$$

where the sum runs over all external legs and and $\gamma_{c,p}^{\mathrm{IR}}$ is the collinear factor for each particle p.

Finally, we can also find contributions to the $1/\epsilon$ UV pole from the one-loop β-function:

$$\frac{1}{2\epsilon}\left(n - L_i\right)\widetilde{\beta}^{(1)}A_i^{(0)}, \tag{69}$$

where $\widetilde{\beta}^{(1)} = \beta^{(1)}/g^{(4)}$, and L_i is the length of the operator $\mathcal{O}_i$. From this we can obtain the UV anomalous dimensions from the bubble coefficients,

$$\frac{1}{(4\pi)^2}\sum_s a_{2,i}^s = -\frac{1}{2}\left[\gamma_{ij}^{\mathrm{UV}} - \gamma_{\mathrm{c}}^{\mathrm{IR}}\delta_{ij} + \left(n - L_i\right)\widetilde{\beta}^{(1)}\delta_{ij}\right]A_j^{(0)}. \tag{70}$$

This method is really powerful at one-loop. However, at higher loops it becomes more and more complicated. At higher loops there is no decomposition of loop amplitudes in terms of simple scalar integrals. Therefore, one has to construct the full amplitudes by means of unitarity and then extracting the UV divergences by computing the loop integration.

The dilatation operator can also be obtained from UV divergences by means of loop corrections. Considering the planar $\mathcal{N} = 4$ SYM theory, loop corrections to scattering matrix can be written as:

$$\hat{\mathcal{A}}_n\left(\tilde{g}^2, \varepsilon\right) = \tilde{\mathcal{I}}\left(\tilde{g}^2, \varepsilon\right)\hat{\mathcal{A}}_n^{(0)} = \left(1 + \sum_{\ell=1}^{\infty}\tilde{g}^{2\ell}\tilde{\mathcal{I}}^{(\ell)}(\varepsilon)\right)\hat{\mathcal{A}}_n^{(0)}, \tag{71}$$

where $\tilde{\mathcal{I}}^{(l)}$ is the ratio between the l-loop and tree-level amplitude and $\tilde{g}^2$ is the modified effective planar coupling constant. The IR divergences from loop corrections are well known:

$$\log\left(\tilde{\mathcal{I}}\left(\tilde{g}^2, \varepsilon\right)\right) = \sum_{\ell=1}^{\infty}\tilde{g}^{2\ell}\left[-\frac{\gamma_{\mathrm{cusp}}^{(\ell)}}{8(\ell\varepsilon)^2} - \frac{\mathcal{G}_0^{(\ell)}}{4\ell\varepsilon}\right]\sum_{i=1}^{n}\left(-\frac{s_{ii+1}}{\mu^2}\right)^{-\ell\varepsilon} + \mathrm{Fin}\left(\tilde{g}^2\right) + \mathcal{O}(\varepsilon), \tag{72}$$

where γ_{cusp} is the cusp anomalous dimension and $\mathcal{G}_0$ is the collinear anomalous dimension. $\text{Fin}(\tilde{g}^2)$ refers to a finite part in the ε expansion. This expression is also valid for form factors, but in this case renormalisation is required since local composite operators are not UV finite. More concretely, for operators that are eigenstates under renormalization, loop corrections are simply the ratio between the l-loop and the tree-level form factor.

Nevertheless, for operators that are not eigenstates under renormalization, i.e., that renormalise non-diagonally, loop corrections acts as operators over a tree-level form factor and giving another tree-level form factor. The renormalized operators are defined in terms of the bare operators and the renormalization constant $\mathcal{Z}$,

$$\mathcal{O}_{\text{ren}} = \mathcal{Z}\mathcal{O}_{\text{bare}}\,, \quad \mathcal{Z} = \mathbf{1} + g^2\mathcal{Z}^{(1)} + g^4\mathcal{Z}^{(2)} + \mathcal{O}\left(g^6\right). \tag{73}$$

Moreover, the anomalous dilatation operator is defined as:

$$\delta\mathfrak{D} = 2\varepsilon g^2 \frac{\partial}{\partial g^2}\log\mathcal{Z} = \sum_{\ell=1}^{\infty} g^{2\ell}\mathfrak{D}^{(\ell)}. \tag{74}$$

The renormalised form factor is just the form factor of the renormalised operator,

$$\hat{\mathcal{F}}_{\mathcal{O}^a_{\text{ren}},n} = \mathcal{Z}\hat{\mathcal{F}}_{\mathcal{O}^a_{\text{bare}},L}(1,\ldots,L;q), \tag{75}$$

where on the right hand side, the renormalization constant $\mathcal{Z}$ acts as an operator on the tree-level form factor. On the other hand, loop corrections to form factors can be written as:

$$\hat{\mathcal{F}}_{\mathcal{O},L}(1,\ldots,L;q) = \mathcal{I}\hat{\mathcal{F}}^{(0)}_{\mathcal{O},L}(1,\ldots,L;q) = \left(1 + \sum_{\ell=1}^{\infty}\tilde{g}^{2\ell}\mathcal{I}^{(\ell)}\right)\hat{\mathcal{F}}^{(0)}_{\mathcal{O},L}(1,\ldots,L;q). \tag{76}$$

Hence, the renormalized form factors can be obtained by acting with the renormalized interaction operators:

$$\underline{\mathcal{I}} = \mathcal{I}\mathcal{Z}, \quad \underline{\mathcal{I}}^{(\ell)} = \sum_{l=0}^{\ell}\mathcal{I}^{(l)}\mathcal{Z}^{(\ell-l)}. \tag{77}$$

Since the IR divergences are universal, we can express the bare loop correction as;

$$\log(\mathcal{I}) = \sum_{\ell=1}^{\infty}\tilde{g}^{2\ell}\left[-\frac{\gamma^{(\ell)}_{\text{cusp}}}{8(\ell\varepsilon)^2} - \frac{\mathcal{G}^{(\ell)}_0}{4\ell\varepsilon}\right]\sum_{i=1}^{L}\left(-\frac{s_{ii+1}}{\mu^2}\right)^{-\ell\varepsilon} - \sum_{\ell=1}^{\infty}\tilde{g}^{2\ell}\frac{\mathfrak{D}^{(\ell)}}{2\ell\varepsilon} + \text{Fin}\left(\tilde{g}^2\right) + \mathcal{O}(\varepsilon) \tag{78}$$

Hence, we can obtain the dilatation operator form the loop corrections in the planar limit. In particular, these loop corrections are computed via on-shell methods.

Moreover, when extracting anomalous dimensions from UV divergences in dimensional regularization evanescent subdivergences occur. In these subdivergences, the counterterm vanishes in four dimensions, but cannot be ignored in dimensional regularization. Their operators then vanish only for four-dimensional external states, but not for general D dimensions. An example of an evanescent operator is:

$$\left(\bar{\psi}\gamma_{[\alpha}\gamma_{\mu}\gamma_{\nu}\gamma_{\sigma}\gamma_{\rho]}\psi\right)\left(\bar{\psi}\gamma^{[\alpha}\gamma^{\mu}\gamma^{\nu}\gamma^{\sigma}\gamma^{\rho]}\psi\right). \tag{79}$$

At one-loop, they vanish for four dimensions. However, at higher loops in dimensional regularization they can generate UV divergences. The effect of these operators must then be taking into account in order to extract two-loop UV divergences and their anomalous dimension. In fact, they will always appear in dimensional regularization and they get more and more complicated when increasing the loop order. Since, they are not physical, it becomes necessary to find a method where the evanescent operators are sidestepped.

Actually, on-shell methods defined in the previous section completely sidestep the evanescent operators at any loop order, as demonstrated in [9] by focusing on renormalization scale dependence instead of divergences.

4.3 Selection rules

In this section we will analyze vanishing entries of the anomalous dimension matrix at one and higher loops. We will see that there exists some selection rules that allows to identify clearly and faster these vanishing entries. The properties will be based only on the assumption of working in massless field theories.

The zeros of the anomalous dimension relies on the supersymmetry of the theory [20]. Since one-loop amplitudes can be expressed in terms of tree-level amplitudes as follows from unitarity, vanishing of tree-level amplitudes directly implies vanishing of certain logarithms and their associated anomalous dimensions.

What makes them nontrivial is that Feynman diagrams exist but do not generate the appropriate logarithms and therefore its anomalous dimension vanishes. Since Feynman diagrams are not gauge invariant, at higher loop it is more convenient to work with on-shell quantities as they use gauge-invariant quantities as inputs, and hence the vanishing o f anomalous dimensions can be directly seen from them.

4.3.1 Length selection rules

In the language of operators, form factors can be seen as the renormalisation of an operator whose field content overlap with the on-shell external states (lower dimension operator) by an operator which inserts off-shell momentum (higher dimension operator). Therefore, confirmation of the existence of a diagram is made by means of the renormalisation of operators. In other words, in order to check at which order do diagrams exist one can check at which order renormalisation of operators is possible.

The existence of diagrams is necessary for the computation of renormalisation coefficients such as the anomalous dimensions. Hence, one would like to derive a theorem that states under which conditions a lower dimension operator $\mathcal{O}_s$ can be renormalised by a higher dimension operator $\mathcal{O}_l$.

The use of minimal form factors makes locality manifest which means that there is no dependence on the coupling constant and therefore the β-function vanishes. By definition, the first order at which renormalization can occur means deriving the first order at which valid diagrams exist. This means that lower loop amplitudes do not exist and hence infrared singularities vanish at that order, $\gamma_{IR} = 0$. With all of this constraints, it is straightforward to see that one can analyze anomalous dimensions by means of sums of cuts.

$$\left(\gamma_{sl}^{\mathrm{UV}}\right)^{(0)} {}_s\left\langle p_1, \ldots, p_n \left|\mathcal{O}_s\right| 0\right\rangle^{(0)} = -\frac{1}{\pi} {}_s\left\langle p_1, \ldots, p_n \left|\mathcal{M} \otimes \mathcal{O}_l\right| 0\right\rangle . \tag{80}$$

This equation makes straightforward that analyzing non-renormalisation of operators correspond to study the possible unitarity cuts, and that this lead directly to the entries of the anomalous dimension matrix. The method will be based on the length and field content of operators. More concretely, the total number of external states correspond to the length of the operator $\mathcal{O}_s$:

$$n_{\mathcal{M}} + n_F - 2k = l\left(\mathcal{O}_s\right), \tag{81}$$

where $n_{\mathcal{M}}$ refers to the number of particles in the scattering amplitude, n_F to the number of particles in the form factor and k to the number of particles crossing the cut. The factor 2 comes from the contribution of k particles crossing the cut coming form the amplitude and another k particles crossing the cut coming from the form factor. In order to derive a non-renormalisation condition that only depends on the length of both operators, one should put some constraints. The number of particles crossing the amplitudes is bound from below by kinematic arguments, since one needs at least two external particles in the amplitude and hence $n_{\mathcal{M}} \geq k + 2$.

The second constraint stems from the fact that all particles are on-shell and then scaleless bubbles evaluate to zero since we are considering dimensional regularization. Hence, the number of particles in the form factor at one-loop (including the ones crossing the cut) should be at least equal to the length of the operator $\mathcal{O}_l$, i.e.,, $n_F \geq l(\mathcal{O}_l)$. This relation holds also at tree-level. However, for the same operator, when the number of loops in the form factor increases, the number of external legs decreases, $n_F \geq l(\mathcal{O}_l) - (L_F - 1) - \delta_{L_F,0}$, where L_F represents the number of loops contained in the form factor.

However, at higher loops the number of external particles relative to the length of the operator should decrease and the delta function states that the number of particles is the same at tree-level and one-loop. The number of loops in the form factor L_F is related to the total number of loops L by means of the number of particles crossing the cut $L \geq L_F + (k-1)$. This relations sets the minimum value of loops. One can then express n_F in terms of the number of loops L, and substituting the value of $n_{\mathcal{M}}$ leads to the following equation:

$$l\left(\mathcal{O}_l\right) - L + 2 - \delta_{L_F,0} \leq l\left(\mathcal{O}_s\right). \tag{82}$$

From this statement, one can see that the general condition relating the lengths of both operators is:

$$L > l\left(\mathcal{O}_l\right) - l\left(\mathcal{O}_s\right). \tag{83}$$

Therefore, if the number of loops is equal or less than the difference between the length of both operators, renormalisation cannot be given. This means that there are no valid diagrams in which unitarity cuts are allowed and then that the anomalous dimension entry vanishes, $\gamma^{UV} = 0$. This condition only depends on the length of the operators, but one wants to define a condition in terms of the field content since depending on the particle content of the operators unitarity cuts can appear even at higher loops that the one predicted with the condition above. Consequently, the generalization of the theorem due to the particle content of the operators states that whenever the only valid diagrams involve scaleless bubble integrals, then there will be no renormalisation of $\mathcal{O}_s$ by $\mathcal{O}_l$. This is due to the lack of nonzero cuts encapsulating all the loops.

An important consequence if this theorem is that at loop order $L = l\left(\mathcal{O}_l\right) - l\left(\mathcal{O}_s\right) + 1$, the only contribution will be the $(L+1)$-particle cut. Therefore, the contributions to this cut will only be by means of four dimensional tree-level amplitudes. In general, whenever an entry is not vanishing at a loop order using length selection rules, one can evaluate this entry at the next loop order since only tree-level quantities will enter the cut. One can see that this theorem has been derived by means only of the particle content as well as the number of fields of both operators. Hence, one can generalize this theorem to massless theories.

4.3.2 Zeros from vanishing one-loop amplitudes

In theories where the β-function is non-zero at one-loop, two-loop anomalous dimensions are scheme dependent. This means that they can vanish depending on the scheme choice. This selection rule does not apply to the maximally supersymmetric $\mathcal{N} = 4$ SYM theory, since in this case the β-function is zero.

However, we will briefly give some comments since they suppose another motivation for finding a way to extract non-vanishing anomalous dimensions. As was seen in previous chapters, the contributions to two-loop anomalous dimensions can come from different contributions: the three particle cut between a tree-level amplitude and a tree-level form factor, two particle cut between one-loop amplitude and tree-level form factor, and double cut between tree-level amplitude a one-loop form factor.

Apart from the non-renormalisation theorem already described, zeros of anomalous dimensions can come from other contributions. More concretely, there can be vanishing entries from the vanishing of one-loop amplitudes, where the two-loop anomalous dimensions will depend on finite rational terms that can be set to zero by particularizing to another scheme.

In particular, this sometimes involves a cancellation between different contributions to the logarithms from one-loop terms in the cut. The cancellation of divergences comes from the fact that two-loop anomalous dimensions should not have any kinematic dependence (that appears in the logarithms) and therefore this implies that logarithmic terms resulting from the cut should cancel against logarithmic terms in the one-loop amplitudes.

Although one-loop amplitudes are zero, the two-loop anomalous dimensions can depend on finite remainder terms, but since these terms are local, they can be redefined by picking an appropriate choice of scheme and in that way eliminate the anomalous dimensions. This choice of a particular scheme is equivalent to perform a finite renormalization of the operators [7]:

$$\widetilde{\mathcal{O}}_i = Z^{\text{fin}}_{ij} O_j, \quad \text{where} \quad Z^{\text{fin}}_{ij} = \delta_{ij} + f_{ij}\left(g^{(4)}\right), \tag{84}$$

where $(g^{(4)})$ refers to the dimensional-four coupling. Since we are studying a perturbation expansion of weakly coupled theories, the coefficient f_{ij} has a perturbative expansion that starts at one-loop, since at tree-level this problems do not arise. The form factor of that renormalised operator starts changing at one-loop, that corresponds to when the constant f_{ij} starts acting,

$$\tilde{F}^{(1)}_i = F^{(1)}_i + f^{(1)}_{ij} F^{(0)}_j. \tag{85}$$

Anomalous dimensions at one-loop are unaffected by this redefinition since they do only depend on tree-level form factors that do not change under this renormalisation. However, at two-loop, there exists dependence on one-loop form factors that lead to the appearance of the constant f_{ij}. This, together with the fact that IR divergences are independent of the scheme choice and recalling that one-loop anomalous dimensions are also independent of this scheme redefinition, lead to the expression of the two-loop anomalous dimensions of the renormalised operator,

$$\widetilde{\gamma}^{\text{UV}(2)}_{ij} = \gamma^{\text{UV}(2)}_{ij} + f^{(1)}_{ik}\gamma^{\text{UV}(1)}_{kj} - \gamma^{\text{UV}(1)}_{ik} f^{(1)}_{kj} - \beta^{(1)}\partial f^{(1)}_{ij}. \tag{86}$$

Note that the anomalous dimension at one-loop do not have the tilde since they are independent of this redefinition. However, one can see that there is an additional running of the two-loop anomalous dimension matrix coming from the last term. One is interested in finding the value of f_{ij} that will cause as many vanishings in the two-loop anomalous dimension matrix as possible. The last term just corresponds to a mixing between operators at one-loop that were already mixing. Consequently, the new scheme choice cause the mixing between operators at one-loop that we already mixed in the original scheme but prevents these operators to mix at two-loops.

4.3.3 Zeros from color selection rules

Another way to set some of the entries of the two-loop anomalous dimension matrix to zero is by a mismatch between the color structure of the cuts and the operators involved. The general procedure will be to study the color structure of the amplitudes that compose the cuts and determine whether a given operator can yield a nonzero contribution.

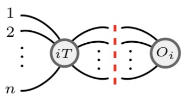
5 $2 \to 2$ case

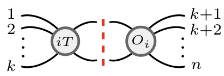
In this chapter we will use the methodology introduced in the first section of the previous chapter to review the computation of the one-loop anomalous dimension. This case is the simplest one in which the method can be applied. In the first section we will derive the parametrization of the spinors crossing the cut in terms of the external spinors. In the next section we will derive the full one-loop anomalous dimension in any gauge theory, and more particularly in the planar limit of $\mathcal{N} = 4$ SYM theory. The following two sections constitute examples of this methodology, in which we will derive the twist-two anomalous dimensions and the one-loop β-function from the anomalous dimension.

5.1 Parametrization of spinors

In this section we will derive the full one-loop parametrization of spinors. The motivation of this parametrization stems from the fact that the integrals computed in the convolution turn out to be simpler if one uses the correct parametrization.

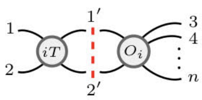
More concretely, we are interested in finding an angular dependence which sets the integral dependence to be over the azimuthal and the polar angles. Since our theory is composed by massless particles with spin, we work in the spinor helicity formalism, as was already described in chapter 3. The main goal is then to describe the intermediate spinors (referring to the particles crossing the cut) in terms of the external spinors (referring to the external particles), based only on an angular dependence. At one-loop the convolution corresponds to the contraction between the tree-level scattering matrix and the tree-level form factor with a two-particle phase space integral [21]:

$$\langle 1, \ldots, n| \, O_j \, |0\rangle \, (\gamma_{ji} - \gamma^i_{\mathrm{IR}}\delta_{ij}) =$$ 

$$= \sum_{k=2}^{n}$$ 

$$+ \cdots \,, \tag{87}$$

where the second equality stems from the fact that the scattering matrix could also act between disconnected pieces. Therefore, the second equality means that the particles to the right of the form factor are the disconnected particles of the scattering matrix, but are connected to the form factor. This means that the disconnected pieces of the scattering matrix are absorbed and connected to the form factor. Hence, both the scattering matrix and the form factors are connected by two-particle cuts. At one-loop order, the cut in the figure is the only one contributing, and, in terms of the spinor helicity formalism, the phase-space integral reads [21],



$$= \frac{1}{2}\int \prod_{i=1}^{2} \frac{d^3 p'_i}{(2\pi)^3 2p'^0_i} (2\pi)^4 \delta^{(4)}\left(p_1 + p_2 - p'_1 - p'_2\right) \langle 12 \,| M_{2\to 2}|\, 1'2'\rangle \, \langle 1'2'3 \ldots n|\mathcal{O}|0\rangle \,, \tag{88}$$

where the momentum conservation delta function arises from the interaction between particles in the tree-level scattering matrix, and the sum runs over all possible intermediate helicity states. The integration measure corresponds to the Lorentz Invariant Phase Space integral. It will in general depend on the intermediate states labelled by spinor helicity variables. The momentum conservation delta function fixes $p_1 + p_2 = p_1' + p_2'$. Particularizing the final states to vacuum states, and expressing the measure in terms of spinor helicity variables we obtain the following expression:

$$\frac{2}{(2\pi)^3}\int d^2\lambda_1' d^2\tilde{\lambda}_1' d^2\lambda_2' d^2\tilde{\lambda}_2' \delta^4\left(p_1 + p_2 - p_1' - p_2'\right)\langle 12\left|M_{2\to 2}\right| 1'2'\rangle \langle 1'2'|\mathcal{O}|0\rangle\,. \tag{89}$$

The differential means the integration over spinor and antispinor variables, for instance $d^2\lambda_1^1 = d\lambda_1^1 d\tilde{\lambda}_1^1$. Hence, it is straightforward to look for a parametrization that relates the intermediate spinors with the external spinors.

In particular, we know that the external spinors in the scattering matrix, λ_1 and λ_2, form a complete basis under which every intermediate spinor can be expressed as:

$$\left(\begin{array}{c}\lambda_1' \\ \lambda_2'\end{array}\right) = A\left(\begin{array}{c}\lambda_1 \\ \lambda_2\end{array}\right),\quad A = \left(\begin{array}{cc}a_{11} & a_{12} \\ a_{21} & a_{22}\end{array}\right), \tag{90}$$

where A is the transformation matrix. The intermediate spinors will rotate in terms of the external spinors through the polar and the azimuthal variables. The polar angle dependence will result in a rotation matrix over the angle θ, whereas the azimuthal dependence will be in terms of complex phases. The most general parametrization can be expressed in terms of 8 parameters, in a one-to-one correspondence with the number of degrees of freedom. Hence, the most general parametrization for the $2 \to 2$ case is:

$$\begin{aligned}\left(\begin{array}{c}\lambda_1'^1 \\ \lambda_2'^1\end{array}\right) &= r_1 e^{i\sigma_1} U\left(\begin{array}{c}\lambda_1^1 \\ \lambda_2^1\end{array}\right),\\ \left(\begin{array}{c}\lambda_1'^2 \\ \lambda_2'^2\end{array}\right) &= r_2 U\left(\begin{array}{cc}\cos\sigma_2 & -\sin\sigma_2 \\ \sin\sigma_2 & \cos\sigma_2,\end{array}\right)\left(\begin{array}{c}\lambda_1^2 \\ \lambda_2^2\end{array}\right),\end{aligned} \tag{91}$$

where

$$U = \left(\begin{array}{cc}\mathrm{e}^{i\phi_2}\cos\theta & -\mathrm{e}^{i(\phi_1+\phi_2)}\sin\theta \\ \mathrm{e}^{i\phi_3}\sin\theta & \mathrm{e}^{i(\phi_1+\phi_3)}\cos\theta,\end{array}\right), \tag{92}$$

where r_1 and r_2 run from 0 to ∞, $0 \leq \theta, \sigma_2 \leq \pi/2$ and $0 \leq \phi, \sigma_1 \leq 2\pi$. We assume all the parameters to be real. Replacing the complex phases by its complex conjugates, we obtain the expressions for the antispinors,

$$
\begin{aligned}
\begin{pmatrix} \tilde{\lambda}_1^{\prime 1} \\ \tilde{\lambda}_2^{\prime 1} \end{pmatrix} &= r_1 e^{-i\sigma_1} U^* \begin{pmatrix} \tilde{\lambda}_1^1 \\ \tilde{\lambda}_2^1 \end{pmatrix}, \\
\begin{pmatrix} \tilde{\lambda}_1^{\prime 2} \\ \tilde{\lambda}_2^{\prime 2} \end{pmatrix} &= r_2 U^* \begin{pmatrix} \cos\sigma_2 & -\sin\sigma_2 \\ \sin\sigma_2 & \cos\sigma_2, \end{pmatrix} \begin{pmatrix} \tilde{\lambda}_1^2 \\ \tilde{\lambda}_2^2 \end{pmatrix},
\end{aligned}
\tag{93}
$$

Hence, by imposing momentum conservation one fixes the value of some parameters[36]:

$$
\begin{aligned}
\delta^4(P) = \delta^4\left(\sum_{i=1}^{4} P_i\right) &= \delta^4\left(-p_1 - p_2 + p_1' + p_2'\right) = \prod_{\alpha=1}^{2} \prod_{\dot{\beta}=1}^{2} \delta\left(-\lambda_1^\alpha \tilde{\lambda}_1^{\dot{\beta}} - \lambda_2^\alpha \tilde{\lambda}_2^{\dot{\beta}} + \lambda_1^{\prime\alpha} \tilde{\lambda}_1^{\prime\dot{\beta}} + \lambda_2^{\prime\alpha} \tilde{\lambda}_2^{\prime\dot{\beta}}\right) \\
&= \frac{i\delta\left(r_1 - 1\right)\delta\left(r_2 - 1\right)\delta\left(\sigma_1\right)\delta\left(\sigma_2\right)}{4\left(\sum_{i=1}^{2} \lambda_i^1 \bar{\lambda}_i^{\mathrm{i}}\right)\left(\sum_{i=1}^{2} \lambda_i^2 \bar{\lambda}_i^2\right)\left(\langle 12\rangle\left(\bar{\lambda}_1^{\mathrm{i}} \bar{\lambda}_1^2 + \bar{\lambda}_2^{\mathrm{i}} \bar{\lambda}_2^{\dot{2}}\right) + \left(\lambda_1^1 \lambda_1^2 + \lambda_2^1 \lambda_2^2\right)[12]\right)}.
\end{aligned}
\tag{94}
$$

Substituting the values of each parameters in both expressions below, we arrive at the following parametrization:

$$
\begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix} = U \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{i\phi_2}\cos\theta & -\mathrm{e}^{i(\phi_1+\phi_2)}\sin\theta \\ \mathrm{e}^{i\phi_3}\sin\theta & \mathrm{e}^{i(\phi_1+\phi_3)}\cos\theta \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}. \tag{95}
$$

In the spinor helicity formalism, the momentum of the particles remains imvariant when multiplying spinor and antispinor by opposite phases. This is known as little group scaling, which states that there exists a global phase for each particle. We can easily convince ourselves that the measure and the momentum conserving delta functions are also independent of these phases. More concretely, also the integrand does not depend on the global phase. Consequently, the integrals over two phases (one for each particle) are trivial and can be dropped out. Finally, one gets the final $2 \to 2$ parametrization:

$$
\begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta\mathrm{e}^{i\phi} \\ \sin\theta\mathrm{e}^{-i\phi} & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}. \tag{96}
$$

This parametrization represents a rotation, in which the integration over spinor helicity variables has been translated to the integration over angular variables. The complex conjugate of this expression holds for the antispinors. This parametrization has been computed in the center of mass frame, where $p_1' + p_2' = p_1 + p_2$, and it is clear that the parametrization only depends on the polar angle θ and the azimuthal angle ϕ. Moreover, this parametrization is covariant and hence it is valid for any frame.

Once we have the intermediate spinors parametrized in terms of external spinors, the integration is over the azimuthal and the polar angle. In order to obtain the new expression, we have to perform the change of variables in eq. 89. The change of variables is made through the computation of the Jacobian, which is just the derivative of each of the 8 spinor-helicity variables with respect to each of the 8 parameters that are present in the most general expression of the parametrization,

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \boldsymbol{\lambda}}{\partial \theta} & \cdots & \frac{\partial \boldsymbol{\lambda}}{\partial r_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial \lambda_1^1}{\partial \theta} & \cdots & \frac{\partial \lambda_1^1}{\partial r_2} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{\lambda}_2^2}{\partial \theta} & \cdots & \frac{\partial \tilde{\lambda}_2^2}{\partial r_2} \end{bmatrix}. \tag{97}$$

In order to express the measure in terms of the new variables, one should compute the determinant of the Jacobian. The result I got is,

$$-8(\lambda_1^1\tilde{\lambda}_1^1 + \lambda_2^1\tilde{\lambda}_2^1)(\lambda_1^2\tilde{\lambda}_1^2 + \lambda_2^2\tilde{\lambda}_2^2)\sin\theta\cos\theta((\tilde{\lambda}_1^1\tilde{\lambda}_1^2 + \tilde{\lambda}_2^1\tilde{\lambda}_2^2)\langle 12\rangle + (\lambda_1^1\lambda_1^2 + \lambda_2^1\lambda_2^2)[12]), \tag{98}$$

which is precisely the same expression as in the denominator of the momentum conservation delta function in Eq. 94 times a factor that depends only on the polar angle and a minus sign. Hence, the change of variables reads,

$$d^2\lambda_1' d^2\tilde{\lambda}_1' d^2\lambda_2' d^2\tilde{\lambda}_2' \delta^4(P) \longrightarrow d\phi_1 d\phi_2 d\phi_3 d\theta(-2i)\sin\theta\cos\theta. \tag{99}$$

However, we still have to consider the little group scaling. The fact that spinor and antispinor are defined only up to a phase, allows to drop out the integration of ϕ_2 and ϕ_3, since different values of these phases lead to the same momenta. Finally, the full integral in the $2 \to 2$ is,

$$I_{2\to 2} = \frac{1}{16\pi}\int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^{\pi/2} 2\sin\theta\cos\theta d\theta \left\langle 12\left|\mathcal{M}_{2\to 2}\right|1'2'\right\rangle \langle 1'2'|\mathcal{O}|0\rangle\,. \tag{100}$$

This equation plays the central role of this chapter. We can then compute anomalous dimensions by means of minimal form factors, and its contraction with tree-level scattering amplitudes. In general, every operator acting in an on-shell state would lead to its anomalous dimension, as can be seen from the previous equation. Therefore, this expression is valid for every theory. However, the main focus in this work will be on gauge theories, since it is the mathematical frame in which interactions between particles occur.

5.2 One-loop β-function in Yang-Mills theory

In this section we show the computation of the one-loop β-function in Yang-Mills theory by means of the parametrization derived in the last section. This section is mainly based on [14].

Let us consider the Lagrangian density $\mathcal{L} = -G^a_{\mu\nu}G^{\mu\nu\,a}/(4g^2)$. Before using the expression in eq. 100, IR divergences must be subtracted. In this particular case, where number of initial states equals two, one can construct IR-safe ratios by putting the stress-tensor $T^{\mu\nu}$ in the denominator leading to the one-loop equation,

$$\gamma_{\mathcal{L}}^{(1)} \langle p_1, p_2|\mathcal{O}|0\rangle^{(0)} = -\frac{1}{\pi}\langle p_1, p_2|\mathcal{M}\otimes\mathcal{L}|0\rangle^{(0)} + \frac{1}{\pi}\langle p_1, p_2|\mathcal{L}|0\rangle^{(0)} \frac{\langle p_1, p_2\,|\mathcal{M}\otimes T^{\mu\nu}|\,0\rangle^{(0)}}{\langle p_1, p_2\,|T^{\mu\nu}|\,0\rangle^{(0)}}. \tag{101}$$

In order to obtain the one-loop anomalous dimension, one should study how the four-point tree amplitude acts on the form factors for the Lagrangian and the stress-tensor. The on-shell four-gluon amplitude can be computed following the Parke-Taylor expression and considering the initial state as a color-singlet:

$$\mathcal{M}^{abcd}_{1^-2^-3^+4^+}\delta^{cd} = -2g^2 C_A \delta^{ab}\frac{\langle 12\rangle^4}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle}, \tag{102}$$

using the parametrization of eq. 96 the amplitude evaluates to,

$$\left\langle 1^a_- 2^b_- \left|\mathcal{M}^{(0)}\right| 1'^c_- 2'^d_- \right\rangle \delta^{cd} = 2g^2 C_A \delta^{ab}\frac{1}{\cos^2\theta\sin^2\theta} \tag{103}$$

Therefore, convolution of the tree amplitude with both the Lagrangian density and the stress-tensor correspond to the following integrals:

$$\left\langle 1^a_- 2^b_- |\mathcal{M}\otimes\mathcal{L}|0\right\rangle^{(0)} = \frac{2g^2 C_A}{16\pi}\int \frac{\mathrm{d}\Omega}{4\pi}\frac{1}{\cos^2\theta\sin^2\theta}\left(\frac{1}{2}\delta^{ab}\langle 1'2'\rangle^2\right), \tag{104}$$

$$\begin{aligned}\left\langle 1^a_- 2^b_+ \left|\mathcal{M}\otimes T^{\alpha\beta,\dot\alpha\dot\beta}\right| 0\right\rangle^{(0)} = \frac{2g^2 C_A}{16\pi}\int \frac{\mathrm{d}\Omega}{4\pi}\frac{1}{\cos^2\theta\sin^2\theta}&\Big(2\delta^{ab}\lambda_1'^{\alpha}\lambda_1'^{\beta}\tilde\lambda_2'^{\dot\alpha}\tilde\lambda_2'^{\dot\beta}\cos^4\theta \\ &+2\delta^{ab}\tilde\lambda_1'^{\dot\alpha}\tilde\lambda_1'^{\dot\beta}\lambda_2'^{\alpha}\lambda_2'^{\beta}\sin^4\theta e^{4i\phi}\Big).\end{aligned} \tag{105}$$

In terms of the intermediate spinors. Using the parametrization in eq. 96, and substituting the integrals below in eq. 101 the anomalous dimensions is,

$$\begin{aligned}\gamma_{\mathcal{L}}^{(1)} &\equiv -\frac{1}{\pi}\left(\frac{\left\langle 1^a_- 2^b_- |\mathcal{M}\otimes\mathcal{L}|0\right\rangle^{(0)}}{\left\langle 1^a_- 2^b_- |\mathcal{L}|0\right\rangle^{(0)}} - \frac{\left\langle 1^a_- 2^b_+ \left|\mathcal{M}\otimes T^{\alpha\beta,\dot\alpha\dot\beta}\right| 0\right\rangle^{(0)}}{\left\langle 1^a_- 2^b_+ \left|T^{\alpha\beta,\dot\alpha\dot\beta}\right| 0\right\rangle^{(0)}}\right) \\ &= -\frac{2g^2 C_A}{16\pi^2}\int_0^{\frac{\pi}{2}} 2\sin\theta\cos\theta \mathrm{d}\theta\left(\frac{1}{\cos^2\theta\sin^2\theta} - \frac{\cos^8\theta+\sin^8\theta}{\cos^2\theta\sin^2\theta}\right) \\ &= -\frac{2g^2}{16\pi^2}\times\frac{11C_A}{3}\end{aligned} \tag{106}$$

In particular, the anomalous dimension is directly related to the β-function of the running coupling. This anomalous dimension is related to the β-function,

$$\gamma_{\mathcal{L}} = g^2\frac{\partial}{\partial g^2}\left(\frac{\beta\left(g^2\right)}{g^2}\right). \tag{107}$$

Hence, the anomalous dimension of the Lagrangian density in Yang-Mills theory is simply the derivative of the β-function. Substituting the value of the anomalous dimension in eq.106, the one-loop β-function is:

$$\beta\left(g^2\right) = -\frac{2g^4}{16\pi^2} \times \frac{11C_A}{3}, \qquad \text{where} \qquad \beta\left(g^2\right) \equiv \mu\partial_\mu g^2(\mu). \tag{108}$$

5.3 Full one-loop anomalous dimension

In this section, we will review the computation of the full one-loop anomalous dimension for general operators in any gauge theory. This derivation constitutes one of the main applications of the method derived in [14]. As in the previous section, the goal is to apply eq.100. But, before we give some comments on the infrared divergences.

Although physical observables are IR finite, as proven in the KLN theorem [26], its constituents can be IR divergent. This is the case of form factors and amplitudes. In general, IR divergences can be either soft divergences, arising when the momentum of massless particles has vanishing components; or collinear divergences, pointing out that the massless propagator is (anti)-parallel to an external state.

In gauge theories, infrared divergences are universal, as it was first predicted by Catani [15]. A generalization for arbitrary number of loops and legs in a general $SU(N)$ gauge theory using soft collinear effective theory (SCET) was given by Becher and Neubert [2]. The procedure to obtain the IR divergences was to express UV renormalized amplitudes up to two loop order in terms of universal subtraction operators $I^{(n)}(\epsilon)$. These subtraction operators are defined as:

$$\begin{aligned}\boldsymbol{I}^{(1)}(\epsilon) &= \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \sum_i \left(\frac{1}{\epsilon^2} + \frac{g_i}{\boldsymbol{T}_i^2}\frac{1}{\epsilon}\right) \sum_{j\neq i} \frac{T_i^a T_j^a}{2} \left(\frac{\mu^2}{-s_{ij}}\right)^{\epsilon}, \\ \boldsymbol{I}^{(2)}(\epsilon) &= \frac{e^{-\epsilon\gamma_E}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left(K + \frac{\beta_0}{2\epsilon}\right) \boldsymbol{I}^{(1)}(2\epsilon) - \frac{1}{2}\boldsymbol{I}^{(1)}(\epsilon) \left(\boldsymbol{I}^{(1)}(\epsilon) + \frac{\beta_0}{\epsilon}\right) + \boldsymbol{H}^{(2)}_{\text{R.S.}}(\epsilon),\end{aligned} \tag{109}$$

where T_i^a refers to the gauge group generator acting on the particle i, μ is the 't Hooft renormalisation scale of dimensional regularization. K is given by $K = \gamma_1^{cusp}/(2\gamma_0^{cusp}$ and $g_i = -\gamma_0^i/2$ and $C_i = T_i^2$ corresponds to the Casimir operator of the corresponding color representation $C_q = C_{\bar{q}} = C_F$ and $C_g = C_A$. Due to color conservation, the following relation between gauge group operators and scattering amplitudes is fulfilled $\sum_i \boldsymbol{T_i}|\mathcal{M}_n(\epsilon,p)\rangle = 0$, where $p \equiv p_1, ..., p_n$. This in fact corresponds to a generalization of Catani's result valid to all orders in perturbation theory.

The exact expression for the full one-loop IR anomalous dimension in color space formalism is:

$$\gamma_{\text{IR}}^{(1)}\left(\{p_i\};\mu\right) = \sum_{i<j} T_i^a T_j^a \gamma_{\text{cusp}} \log\frac{\mu^2}{-s_{ij}} + \sum_i \gamma_i^{\text{coll.}}, \tag{110}$$

where the cusp anomalous dimension takes the following form $\gamma_{cusp} = g^2/(4\pi)^2$. The collinear contributions take different expressions depending on the particles considered [2]:

$$\gamma_g^{\text{coll}} = -b_0 \frac{g^2}{16\pi^2}, \quad \gamma_\psi^{\text{coll}} = -3C_F \frac{g^2}{16\pi^2}, \tag{111}$$

where b_0 is the one-loop beta function coefficient. The definition of the IR divergent parts carries some arbitrariness as to which finite pieces are included, but this arbitrariness cancels in physical quantities between real emission and virtual contributions. The first term in eq. 110, that depends on $\log \mu^2$ comes from soft radiation and can be identified with the integral in the previous section, eq.104, which corresponds to an integral over real radiation. Therefore, the general one-loop dilatation operator in an arbitrary gauge theory is,

$$\begin{aligned}\gamma_{\mathcal{O}}^{(1)} \langle p_1, \ldots, p_n|\mathcal{O}|0\rangle^{(0)} = &-\frac{1}{\pi}\left\langle p_1, \ldots, p_n \left| \sum_{i<j}\left(\mathcal{M}_{ij}^{2\to 2} + \frac{2g^2 T_i^a T_j^a}{\sin^2\theta\cos^2\theta}\right)\otimes\mathcal{O}\right| 0\right\rangle^{(0)} \\ &+ \langle p_1, \ldots, p_n|\mathcal{O}|0\rangle^{(0)} \times \sum_{i=1}^{n}\gamma_i^{\text{coll.}} ,\end{aligned} \tag{112}$$

where $\mathcal{M}_{ij}$ denotes the $2 \to 2$ amplitude acting on the particles i and j. Hence, the full on-loop anomalous dimension is composed by two terms, the first one is a sum over unitarity cuts and the soft contribution to the anomalous dimension and second correspond to a sum over hard-collinear divergences.

5.3.1 Limit in the planar $\mathcal{N} = 4$ SYM theory

In the planar limit of $\mathcal{N} = 4$ SYM theory, interactions occur between neighbouring fields. At one-loop order, this means that interactions can occur at two neighbouring fields at the same time. At leading order in the $1/N_c$ expansion, $T_i \times T_j \to -\frac{N_c}{2}\mathbf{1}$ for neighbouring legs and zero otherwise. In $\mathcal{N} = 4$ SYM theory each field is characterised by its super-momentum $\lambda_i = (\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i)$. Therefore, in $\mathcal{N} = 4$ SYM, apart from the usual momentum conserving delta function, there exists a super-momentum conserving delta function associated to the fermionic $\tilde{\eta}_i$ variables. It is defined as:

$$\delta^8(Q) = \delta^8(\sum_{i=1}^{4}\lambda_i\tilde{\eta}_i) = \prod_{A=1}^{4}\sum_{j=2}^{4}\sum_{i=1}^{j-1}\langle ij\rangle\tilde{\eta}_i^A\tilde{\eta}_j^A \operatorname{sign}(i,j), \quad \operatorname{sign}(i,j) = (-1)^{\theta(2-i)+\theta(2-j)} \tag{113}$$

In particular, for the $2 \to 2$ case, this equation reads:

$$\begin{aligned}\delta^8(Q) = &\prod_{A=1}^{4}\left(\langle 12\rangle\tilde{\eta}_1^A\tilde{\eta}_2^A - \langle 1l_1\rangle\,\tilde{\eta}_1^A\tilde{\eta}_{l_1}^A - \langle 1l_2\rangle\,\tilde{\eta}_1^A\tilde{\eta}_{l_2}^A - \langle 2l_1\rangle\,\tilde{\eta}_2^A\tilde{\eta}_{l_1}^A - \langle 2l_2\rangle\,\tilde{\eta}_2^A\tilde{\eta}_{l_2}^A + \langle l_1 l_2\rangle\,\tilde{\eta}_{l_1}^A\tilde{\eta}_{l_2}^A\right) \\ = &\langle 12\rangle^4 \mathrm{e}^{4i(\phi_1+\phi_2+\phi_3)}\prod_{A=1}^{4}\Big(\mathrm{e}^{-i(\phi_1+\phi_2+\phi_3)}\tilde{\eta}_1^A\tilde{\eta}_2^A + \mathrm{e}^{-i\phi_3}\left(\sin\theta\tilde{\eta}_1^A + \mathrm{e}^{-i\phi_1}\cos\theta\tilde{\eta}_2^A\right)\tilde{\eta}_{l_1}^A \\ &+\mathrm{e}^{-i\phi_2}\left(\mathrm{e}^{-i\phi_1}\sin\theta\tilde{\eta}_2^A - \cos\theta\tilde{\eta}_1^A\right)\tilde{\eta}_{l_2}^A + \tilde{\eta}_{l_1}^A\tilde{\eta}_{l_2}^A\Big)\end{aligned} \tag{114}$$

From this expression, one can directly see that the $\tilde{\eta}_i$ variables follow a rotation of the form:

$$\begin{pmatrix} \tilde{\eta}_l^A \\ \tilde{\eta}_{l_2}^A \end{pmatrix} = U^* \begin{pmatrix} \tilde{\eta}_1^A \\ \tilde{\eta}_2^A \end{pmatrix}, \tag{115}$$

which corresponds to the same rotation as the antispinors, see eq. 96. Hence, due to the super-momentum conservation delta function, the $\tilde{\eta}_i$ transform analogously as the $\tilde{\lambda}_i$. Furthermore, the amplitude in the planar limit generalizes the expression in eq. 102 by using Nair's on-shell superfield Eq. 39. In this expression, for the case of a gluon, one extracts four powers of $\tilde{\eta}_i$. In terms of the current parametrization, it corresponds to an extra $\cos^4\theta$ factor,

$$\left\langle 1_-^a 2_+^b |\mathcal{M}| 1'_\Phi 2'_{\bar{\Phi}} \right\rangle = -2g^2 C_A \delta^{ab} \frac{\langle 11'\rangle^2 \langle 12'\rangle^2}{\langle 11'\rangle \langle 12'\rangle \langle 21'\rangle \langle 22'\rangle} = -2g^2 C_A \delta^{ab} \frac{\cos^4\theta}{\cos^2\theta \sin^2\theta}. \tag{116}$$

Finally, substituting this amplitude and the expression for the generators in eq. 112, one obtains [14]:

$$\gamma_{\mathcal{O}}^{(1)} \langle 1,\dots,n|\mathcal{O}|0\rangle^{(0)} = \frac{4g^2 N_c}{16\pi^2} \sum_{i=1}^{n} \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \cot\theta \begin{pmatrix} \langle 1,\dots,i,i+1,\dots,n|\mathcal{O}|0\rangle^{(0)} \\ -\langle 1,\dots,i',(i+1)',\dots,n|\mathcal{O}|0\rangle^{(0)} \end{pmatrix}. \tag{117}$$

This expression corresponds to the one-loop dilatation operator in the planar limit of $\mathcal{N}=4$ SYM theory. In this limit, local operators represent spin chain states, and the dilatation generator represents the spin chain Hamiltonian. Hence, by using the superoscillator representation of the algebra $\mathfrak{psu}(2,2\mid 4)$, Zwiebel derived the same expression for the spin chain representation in [36].

The correspondence between superoscillator variables and the super spinor helicity variables used in this section relies on the fact that they do fulfill the same commutation relations, $[a_\alpha, a^{\dagger\beta}] = \delta_\alpha^\beta = [\partial_\alpha, \lambda^\beta]$. Moreover, this framework can also be used to study this methodology in presence of matter fields. In the computations of the previous section, in terms of the Lagrangian density, there is no couple to matter. However, the infrared structure of the theory changes when considering matter contributions, detected by the stress tensor.

The way the stress tensor couples to matter fields, in particular to fermions and scalars, can be seen from the following equations,

$$\begin{aligned} \left\langle 1_{\bar{\Phi}} 2_\Phi \left| T^{\alpha\beta,\dot{\alpha}\dot{\beta}} \right| 0 \right\rangle &= \frac{1}{3}\left(p_1^{\alpha\dot{\alpha}} p_1^{\beta\dot{\beta}} + p_2^{\alpha\dot{\alpha}} p_2^{\beta\dot{\beta}} - p_1^{\alpha\dot{\alpha}} p_2^{\beta\dot{\beta}} - p_1^{\beta\dot{\alpha}} p_2^{\alpha\dot{\beta}} - p_1^{\alpha\dot{\beta}} p_2^{\beta\dot{\alpha}} - p_1^{\beta\dot{\beta}} p_2^{\alpha\dot{\alpha}}\right), \\ \left\langle 1_{\bar{\psi}} 2_\psi \left| T^{\alpha\beta,\dot{\alpha}\dot{\beta}} \right| 0 \right\rangle &= \frac{1}{2}\left(\lambda_1^\alpha \lambda_1^\beta \tilde{\lambda}_1^{\dot{\alpha}} \tilde{\lambda}_2^{\dot{\beta}} + \lambda_1^\alpha \lambda_1^\beta \tilde{\lambda}_1^{\dot{\beta}} \tilde{\lambda}_2^{\dot{\alpha}} - \lambda_1^\alpha \lambda_2^\beta \tilde{\lambda}_2^{\dot{\alpha}} \tilde{\lambda}_2^{\dot{\beta}} - \lambda_1^\beta \lambda_2^\alpha \tilde{\lambda}_2^{\dot{\alpha}} \tilde{\lambda}_2^{\dot{\beta}}\right). \end{aligned} \tag{118}$$

The derivation was made by requiring that the expectation value should return the momentum of the particles, and that the stress-tensor should be conserved. The computation of the tree-level amplitude is analogous to the case below, eq. 116, but extracting different powers of $\tilde{\eta}_i$ depending on which particle we consider.

In this case, three contributions appear corresponding to fermions, scalars and antifermions, for which we should extract three, two and one powers of $\tilde{\eta}_i$, respectively. In terms of the parametrization in eq. 96, this dependence will be reflected in terms of angular variables. After dropping out the azimuthal integration, the general expression is,

$$\frac{\left\langle 1_-^a 2_+^b \left| \mathcal{M} \otimes T^{\alpha\beta,\dot{\alpha}\dot{\beta}} \right| 0 \right\rangle^{(0)}}{\left\langle 1_-^a 2_+^b \left| T^{\alpha\beta,\dot{\alpha}\dot{\beta}} \right| 0 \right\rangle^{(0)}} \simeq \frac{2g^2}{16\pi} \int_0^{\frac{\pi}{2}} \frac{2\sin\theta\cos\theta \mathrm{d}\theta}{\cos^2\theta\sin^2\theta} \left[C_A \left(\cos^8\theta + \sin^8\theta\right) \right. \\ \left. + 2n_f T_f \left(\cos^6\theta\sin^2\theta + \sin^6\theta\cos^2\theta\right) + 2n_s T_s \cos^4\theta\sin^4\theta \right], \tag{119}$$

where n_s, n_f correspond to the number of complex scalars and Dirac fermions ($2n_f$ Weyl fermions), respectively. C_A denotes the Casimir operator in the adjoint representation, which corresponds to N_c for a general gauge group $SU(N_c)$, as occurs in eq. 116. Therefore, one can see that depending on the particle content of the theory, the infrared structure will be modified following this equation.

In order to check the validity of this expression, one can substitute the particle content of $\mathcal{N} = 4$ SYM theory and see that the only contribution in the brackets reduces to C_A. Consequently, substituting this expression in the anomalous dimensions, one obtain the vanishing of the β-function by means of eq. 107, as was expected due to the superconformal invariance of $\mathcal{N} = 4$ SYM theory.

5.4 Anomalous dimensions in twist-two operators

Another interesting application due to its structure is the computation of anomalous dimensions for twist-two operators decaying to two external states. Leading twist-two operators are very important objects in Yang-Mills theory. They naturally appear in QCD for describing deep inelastic scattering. In general, twist-two operators are particularly useful for theories involving massless particles with spin.

The generic form of twist-two operators for spin j partilces is $\mathcal{O}_j^{\mu_1,\ldots,\mu_j} = \prod_j [\bar{q}\gamma^{\mu_1} D^{\mu_2} ... D^{\mu_n} q]$ where $\prod_j$ refers to the projector on a representation with spin j. Let us now consider an example of a twist-two operator in a two-scalar interaction, i.e., with no spin interaction. The general form of this operator is,

$$\mathcal{O}_m = i^n \bar{\Phi} \partial^{\mu_1} \cdots \partial^{\mu_m} \Phi. \tag{120}$$

Therefore, the contraction with external states will return polynomials in its momentum. For the scalars particles in this example, the contraction takes the following form,

$$\left\langle 1_\Phi \bar{1}_{\bar{\Phi}} \left| \mathcal{O}_m \right| 0 \right\rangle = p_1^{\mu_1} \cdots p_1^{\mu_m}, \tag{121}$$

when contracted with the external particles. In general form factors are also contracted with the intermediate states crossing the cut. However, in this concrete example, the contraction with intermediate states returns a momentum proportional to the external momenta, which means that the angular dependence can be treated independently,

$$p_1'^{\alpha\dot{\alpha}} \equiv \lambda_1'^{\alpha}\tilde{\lambda}_1'^{\dot{\alpha}} = \lambda_1^{\alpha}\tilde{\lambda}_1^{\dot{\alpha}} \left(\cos\theta - \sin\theta \mathrm{e}^{i\phi}\right)\left(\cos\theta + \sin\theta \mathrm{e}^{-i\phi}\right) = p_1^{\alpha\dot{\alpha}}(\cos(2\theta) - i\sin(2\theta)\sin\phi). \tag{122}$$

Hence, from the structure of these form factors whose dependence resides only on the external momentum, form factors of intermediate states are proportional to form factors of external states. The second quantity to be computed is the tree-level scattering matrix, which for scalar fields does not depend on the azimuthal angle.

Hence, the only dependence resides in eq. 122. In this way, one can already drop out the integration over the azimuthal angle to simplify the computations. In this particular case, by rescaling the polar angle from $2\theta \to \theta$, eq. 122 rescales as $(\cos(2\theta) - i\sin(2\theta)\sin\phi) \to (\cos(\theta) - i\sin(\theta)\sin\phi)$, and the integration over the azimuthal angle is:

$$\int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi}(\cos\theta - i\sin\theta\sin\phi)^m = P_m(\cos\theta), \tag{123}$$

leads to the Legendre polynomials in terms of the polar angle. The remaining integration over the polar angle depends on the tree-level amplitude, and it takes the following expression,

$$\gamma^{(1)}_{\mathcal{O}_m} - \gamma^{(1)}_{\mathrm{IR}} = -\frac{1}{16\pi^2}\int_0^{\pi} \frac{\sin\theta \mathrm{d}\theta}{2}\mathcal{M}^{(0)}(\cos\theta)P_m(\cos\theta) \equiv -\frac{1}{\pi}a_m^{(0)}, \tag{124}$$

where a_m is identified with the partial-wave amplitude with angular momentum m. This term can also be identified with the phase of the leading approximation of the scattering matrix with definite angular momentum $S_m = 1 + i\mathcal{M}_m = \mathrm{e}^{ia_m}$. Hence, from this results, one can interprets two different conclusions.

The first one shows the expected role of the anomalous dimension acting as minus the phase of the scattering matrix divided by π, which is the central argument of this work. The second stems from the fact that the phase corresponds also to the partial-wave amplitude with definite angular momentum, which reflects that two-particle states with definite angular momentum map to twist-two operators and that from twist-two operators, one can obtain the angular decomposition of the scattering matrix.

As a final crosscheck, let us give one example of an application of this equation. Let us consider the case of two scalar in the planar limit of $\mathcal{N} = 4$ SYM theory, where the twist-two operator takes the following form: $\mathcal{O}_m = \mathrm{tr}[Z\partial_m^+ Z]$, where m is even. Substituting the value of the operator in eq. 117, one simply obtains:

$$\gamma^{(1)}_{\mathcal{O}_m} = -\frac{2g^2N_c}{16\pi^2}\int_0^{\pi} \frac{2\ \mathrm{d}\theta}{\sin\theta}\left(P_m(\cos\theta) - P_m(1)\right) = \frac{g^2N_c}{16\pi^2} \times 8S_1(m), \tag{125}$$

where $S_1(m) = \sum_{i=1}^{m} \frac{1}{i}$ denote the sum of harmonic numbers. This dependence in the harmonic numbers occurs in the computation of anomalous dimensions of operators in the $SL(2)$ sector, as stated for example in [27], which reflects the fact that twist-two operators constitute a particular example of length-two operators in the $SL(2)$ sector.

6 $2 \to 3$ case

In the previous chapter, we derived the full one-loop anomalous dimension from $2 \to 2$ amplitudes. However, in principle there are more diagrams that can contribute to the one-loop anomalous dimension. In general, the requirement of having two-particle cuts leaves some freedom in the number of external particles. In this section, we review the computations of the one-loop anomalous dimensions from $2 \to 3$ amplitudes.

These amplitudes correspond to length-changing effects. In particular, they occur when different length operators mix under renormalisation. In order to illustrate this case, let us consider a theory in which two different length operators interact. One suitable example for this purpose corresponds to the Yukawa interaction.

6.1 Yukawa interaction

In this section we review the computations in [14], studying the Yukawa interaction between one real scalar and one Weyl fermion. The interaction Lagrangian takes the following form,

$$\mathcal{L}_{\text{int}} = -\lambda \mathcal{O}_\lambda - y \mathcal{O}_y \quad \text{with} \quad \mathcal{O}_\lambda = \frac{1}{4!}\phi^4 \quad \text{and} \quad \mathcal{O}_y = \frac{1}{2}(\psi\psi\phi + \text{h.c.}). \tag{126}$$

Due to the presence of operators of different lengths, we will have different contributions for the anomalous dimension matrix. One will have in general for different contributions to the anomalous dimension matrix. The diagonal entries correspond to the length-preserving entries which lead to the one-loop anomalous dimension with no operator mixing. These two examples are analogous to the computations made in the last chapter. Moreover, infrared divergences are diagonal in its structure and consequently they will only appear in the diagonal entries.

The other two contributions come from operator mixing. On the one hand, one case corresponds to length-decreasing mixing which involves the study of the $3 \to 2$ amplitude over a three-particle cut. This case will be studied in the next chapter, since it involves computations at two-loop and a new parametrization is required.

On the other hand, the last contribution corresponds to length-increasing mixing, whose anomalous dimension can be obtained form the sum of two contributions: a $2 \to 2$ amplitude acting on the non-minimal form factor and the $2 \to 3$ amplitude acting on the minimal form factor. The latter is the interesting case of study in this chapter. However, we will briefly derive all the entries except for the two-loop contribution. Then, the matrix form of the renormalization equation is:

$$\left(\mu\frac{\partial}{\partial\mu} + \sum_{a=y,\lambda} \beta(a)\frac{\partial}{\partial a}\right)\begin{pmatrix} \mathcal{O}_y \\ \mathcal{O}_\lambda \end{pmatrix} = \begin{pmatrix} \gamma_{yy} & \gamma_{y\lambda} \\ \gamma_{\lambda y} & \gamma_{\lambda\lambda} \end{pmatrix}\begin{pmatrix} \mathcal{O}_y \\ \mathcal{O}_\lambda \end{pmatrix}. \tag{127}$$

As was already discussed, the form factor can be defined as a small perturbation to the scattering matrix, and hence they represent the derivative with respect to the coupling, $\mathcal{F}_a = -\frac{\partial}{\partial a}\mathcal{M}$. With this definition we can separate the renormalization group equations in terms of the presence or not of the interaction,

$$\left(\mu_{\rm UV}\frac{\partial}{\partial\mu_{\rm UV}} + \sum_{b=y,\lambda}\beta(b)\frac{\partial}{\partial b}\right)\mathcal{M} = 0, \quad \left(\mu_{\rm UV}\frac{\partial}{\partial\mu_{\rm UV}} + \sum_{b=y,\lambda}\beta(b)\frac{\partial}{\partial b}\right)\mathcal{F}_a = -\sum_{b=y,\lambda}\gamma_{ab}\mathcal{F}_b, \tag{128}$$

where in the second equation we can see explicitly the interaction between the form factors of the different operators. In this case, we have only considered UV renomalization, since infrared divergences will only appear in diagonal elements where there is no mixing of operators, and they can be computed independently (see example of the previous chapter **??**). Analogously to eq. 107, but including operator mixing:

$$\frac{\partial}{\partial a}\beta(b) = \gamma_{ab}, \quad a, b = \lambda \text{ or } y. \tag{129}$$

In order to make the computations consistent and easy to follow, we will first briefly review the calculations of the diagonal anomalous dimensions corresponding to acting with the tree $2 \to 2$ amplitude over the minimal form factor, to straightforwardly show to the interesting case of study in this chapter, i.e., the length decreasing mixing case involving the $2 \to 3$ amplitude.

The computation of the diagonal entries in the anomalous dimension matrix involves infrared divergences that can be computed through the stress tensor. The minimal form factors of the corresponding operators are,

$$\langle 1_\phi 2_\phi 3_\phi 4_\phi\, |\mathcal{O}_\lambda|\, 0\rangle = 1, \quad \left\langle 1_{\bar\psi} 2_{\bar\psi} 3_\phi\, |\mathcal{O}_y|\, 0\right\rangle = \langle 12\rangle, \quad \langle 1_\psi 2_\psi 3_\phi\, |\mathcal{O}_y|\, 0\rangle = [12], \tag{130}$$

with the corresponding elemental scattering amplitudes,

$$\mathcal{M}_{1_\phi 2_\phi 3_\phi 4_\phi} = -\lambda, \quad \mathcal{M}_{1_{\bar\psi} 2_{\bar\psi} 3_\phi} = -y\langle 12\rangle, \quad \mathcal{M}_{1_\psi 2_\psi 3_\phi} = -y[12]. \tag{131}$$

Hence, the different four point amplitudes for this case will be the ones for which the final states are either two (anti-)fermions, two scalars or one scalar and one fermion. Following the parametrization in eq.96, the four point amplitudes reads:

$$\begin{aligned}
&\left\langle 1_\phi 2_\phi|\mathcal{M}|1'_{\bar\psi}2'_\psi\right\rangle = -\left\langle 1_{\bar\psi} 2_\psi|\mathcal{M}|1'_\phi 2'_\phi\right\rangle^* = y^2\left(\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}\right)\mathrm{e}^{-i\phi},\\
&\left\langle 1_{\bar\psi} 2_{\bar\psi}|\mathcal{M}|1'_{\bar\psi}2'_{\bar\psi}\right\rangle = \left\langle 1_{\bar\psi} 2_\psi|\mathcal{M}|1'_{\bar\psi}2'_\psi\right\rangle = -\left\langle 1_{\bar\psi} 2_\psi|\mathcal{M}|1'_\psi 2'_{\bar\psi}\right\rangle = -y^2,\\
&\left\langle 1_{\bar\psi} 2_{\bar\psi}|\mathcal{M}|1'_\psi 2'_\psi\right\rangle = -3y^2, \quad \left\langle 1_{\bar\psi} 2_\phi|\mathcal{M}|1'_{\bar\psi}2'_\phi\right\rangle = -y^2\frac{1+\cos^2\theta}{\cos\theta}.
\end{aligned} \tag{132}$$

The computation of these amplitudes was made by the factorization in terms of the elemental particles of eq. [**:eq6.6**]. Les us illustrate it with one example:

$$
\begin{aligned}
\mathcal{M}_{1_\psi 2_\psi 1'_{\bar{\psi}} 2'_{\bar{\psi}}} &= \left\langle 1_\psi 2_\psi \mid \mathcal{M} \mid 1'_\psi 2'_\psi \right\rangle = \left\langle 1_\psi 2_\psi \mid \mathcal{M} \mid i_\phi \right\rangle \frac{1}{\langle 12\rangle\,[12]} \left\langle i_\phi \mid \mathcal{M} \mid 1'_\psi 2'_\psi \right\rangle \\
&= (-y[12]) \frac{1}{\langle 12\rangle\,[12]} (-y\,\langle 34\rangle) = y^2 \frac{\langle 34\rangle}{\langle 12\rangle},
\end{aligned}
\tag{133}
$$

where the first equality stems from crossing symmetry, and in the last step we have made use of the parametrization. In the diagonal elements, the infrared divergences acting in the diagonal elements will come form the coupling of the stress tensor to either scalars and fermions. Then, we will have two different collinear anomalous dimensions. Acting with the tree-level scattering matrix over the stress-tensor in the same way as in eqs. 104, 105 of the previous chapter, we obtain:

$$
\begin{aligned}
2\gamma_\phi^{\text{coll.}} &\equiv \frac{1}{\pi} \frac{\left\langle 1_\phi 2_\phi \left| \mathcal{M} \otimes T^{\alpha\beta,\dot\alpha\beta} \right| 0 \right\rangle^{(0)}}{\left\langle 1_\phi 2_\phi \left| T^{\alpha\beta,\dot\alpha\dot\beta} \right| 0 \right\rangle^{(0)}} \\
&= \frac{1}{16\pi^2} \int_0^{\frac{\pi}{2}} 2\cos\theta \sin\theta \mathrm{d}\theta \left(-\frac{1}{4}\lambda(1+\cos(4\theta)) + 6y^2\cos^2(2\theta) \right) = \frac{2y^2}{16\pi^2}, \\
2\gamma_\psi^{\text{coll.}} &= \frac{1}{16\pi^2} \int_0^{\frac{\pi}{2}} 2\cos\theta \sin\theta \mathrm{d}\theta \left(2y^2\cos^2(2\theta) - y^2\cos(4\theta) \right) = \frac{y^2}{16\pi^2}.
\end{aligned}
\tag{134}
$$

Both expressions do not depend on the λ term, since this corresponds to the pure 4-scalar interaction, and in that case infrared divergences are zero. The extra two comes from the contribution of fermions and antifermions.

Hence, the diagonal anomalous dimension in the scalar case comes from the sum of the infrared divergences and the contribution of all the scalar four-point amplitudes. More concretely, there will be a contribution for every possible combination of the final states. In this case, since all particles are scalars, all possible combinations will contribute analogously. Finally the expression of the anomalous dimension,

$$
\begin{aligned}
\gamma_{\lambda\lambda}^{(1)} &= 4\gamma_\phi^{\text{coll.}} - \frac{1}{\pi} \frac{\left\langle 1_\phi 2_\phi 3_\phi 4_\phi \left| (\mathcal{M}_{12} + \mathcal{M}_{13} + \mathcal{M}_{14} + \mathcal{M}_{23} + \mathcal{M}_{24} + \mathcal{M}_{34}) \otimes \mathcal{O}_\lambda \right| 0 \right\rangle^{(0)}}{\left\langle 1_\phi 2_\phi 3_\phi 4_\phi \left| \mathcal{O}_\lambda \right| 0 \right\rangle^{(0)}} \\
&= \frac{4y^2}{16\pi^2} + \frac{6\lambda}{16\pi^2},
\end{aligned}
\tag{135}
$$

and the corresponding β-function:

$$
\beta(\lambda) = \frac{3\lambda^2}{16\pi^2}. \tag{136}
$$

The other anomalous dimension entry corresponds to the Yukawa vertex, where the amplitudes contributing correspond to the ones having either two (anti)fermions, or one (anti)fermion with one scalar as final states, as they are the only ones the minimal form factor can be contracted with. The anomalous dimension and the β-function are:

$$\gamma^{(1)}_{yy} = 2\gamma^{\text{coll.}}_{\bar{\psi}} + \gamma^{\text{coll.}}_{\phi} - \frac{1}{\pi}\frac{\left\langle 1_{\bar{\psi}}2_{\bar{\psi}}3_{\phi}\left|(\mathcal{M}_{12}+\mathcal{M}_{13}+\mathcal{M}_{23})\otimes\mathcal{O}_y\right|0\right\rangle^{(0)}}{\left\langle 1_{\bar{\psi}}2_{\bar{\psi}}3_{\phi}\left|\mathcal{O}_y\right|0\right\rangle^{(0)}}$$
$$= \frac{2y^2}{16\pi^2} - \frac{1}{16\pi^2}\int_0^{\frac{\pi}{2}} 2\cos\theta\sin\theta \mathrm{d}\theta\left(-4y^2 - 2y^2\left(1+\cos^2\theta\right) - 2y^2\left(1+\sin^2\theta\right)\right) = \frac{12y^2}{16\pi^2}, \tag{137}$$

and the corresponding β-function:

$$\beta(\lambda) = \frac{4y^3}{16\pi^2}. \tag{138}$$

After having reviewed the $2\to 2$ amplitude, it is straightforward to continue with the $2\to 3$ case. This case corresponds to the length increasing mixing between the operators $\mathcal{O}_\lambda$ and $\mathcal{O}_y$. In this case there will be no infrared anomalous dimensions, since, as already commented below, they are diagonal in their structure.

The anomalous dimension will then be computed from two contributions. The first one corresponds to acting with the $2\to 2$ on the non-minimal form factor $\mathcal{O}_y$, which will carry an extra scalar field. The second contribution will come from $2\to 3$ scattering acting on the minimal form factor,

$$\gamma^{(1)}_{y\lambda} = -\frac{1}{\pi}\frac{\left\langle 1_\phi 2_\phi 3_\phi 4_\phi\left|\left(\mathcal{M}^{2\leftarrow 2}_{12}+\mathcal{M}^{2\leftarrow 2}_{13}+\mathcal{M}^{2\leftarrow 2}_{14}+\mathcal{M}^{2\leftarrow 2}_{23}+\mathcal{M}^{2\leftarrow 2}_{24}+\mathcal{M}^{2\leftarrow 2}_{34}\right)\otimes\mathcal{O}_y\right|0\right\rangle^{(0)}}{\langle 1_\phi 2_\phi 3_\phi 4_\phi\left|\mathcal{O}_\lambda\right|0\rangle^{(0)}}$$
$$-\frac{1}{\pi}\frac{\left\langle 1_\phi 2_\phi 3_\phi 4_\phi\left|\left(\mathcal{M}^{3\leftarrow 2}_{123}+\mathcal{M}^{3\leftarrow 2}_{124}+\mathcal{M}^{3\leftarrow 2}_{134}+\mathcal{M}^{3\leftarrow 2}_{234}\right)\otimes\mathcal{O}_y\right|0\right\rangle^{(0)}}{\langle 1_\phi 2_\phi 3_\phi 4_\phi\left|\mathcal{O}_\lambda\right|0\rangle^{(0)}}. \tag{139}$$

Contributions from the first line will carry a factor of y^3 coming from the y^2 in the amplitudes and the y term in the non-minimal form factor. The amplitudes in the second line will come from amplitudes of the form,

$$\mathcal{M}_{1_\psi 2_\psi 3_\phi 4_\phi 5_\phi} = y\lambda\frac{1}{\langle 12\rangle} - y^3\left(\frac{\langle 35\rangle}{\langle 13\rangle\langle 25\rangle} + 5 \text{ permutations of } (345)\right). \tag{140}$$

Therefore, contributions coming from the second line will carry both a λy factor from the first term and y^3 from the rest, leading to an interplay between both lines. Let us focus on the first contribution. Let us illustrate the computation of anomalous dimensions by computing the contributions of the three diagrams in figure 12, that will lead to some cancellations.

The first two diagrams correspond to the integral between four-point amplitudes and the non-minimal form factor, whereas the third one to the five-point amplitude with the minimal form factor. The first diagram corresponds to:

$$\left\langle 1_\phi 2_\phi|\mathcal{M}|1'_\psi 2'_{\bar{\psi}}\right\rangle_{\text{first term}}\left\langle 1'_\psi 2'_{\bar{\psi}} 3_\phi 4_\phi \mid \mathcal{O}_y \mid 0\right\rangle = y^3\frac{\langle 2'1\rangle\langle 1'3\rangle}{\langle 1'1\rangle\langle 2'3\rangle} = y^3\frac{\cos\theta}{\sin\theta e^{i\phi}}\left(\frac{\langle 13\rangle\cos\theta - \langle 23\rangle\sin\theta e^{i\phi}}{\langle 13\rangle\sin\theta e^{-i\phi} + \langle 23\rangle\cos\theta}\right), \tag{141}$$

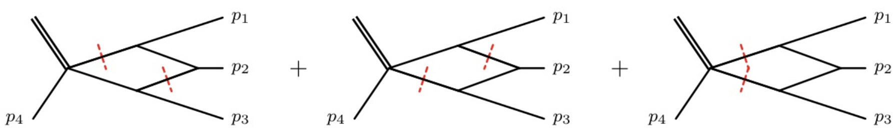


Figure 12: Three different box diagrams whose sum leads to logarithm cancellations [14].

where in the second line we have particularized to the parametrization in eq.96. The azimuthal integration can be done by the change of variables $z = \mathrm{e}^{i\phi}$ that leads to a contour integral along the unit circle that is solved by means of Cauchy's theorem,

$$\int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \int_0^{\frac{\pi}{2}} \mathrm{d}\theta 2\cos^2\theta \frac{\langle 13\rangle \cos\theta - \mathrm{e}^{i\phi}\langle 23\rangle \sin\theta}{\langle 13\rangle \sin\theta + \mathrm{e}^{i\phi}\langle 23\rangle \cos\theta} = \int_0^1 \frac{\mathrm{d}z}{z} \int_0^{\frac{\pi}{2}} \mathrm{d}\theta 2\cos^2\theta \frac{\langle 13\rangle \cos\theta - z\langle 23\rangle \sin\theta}{\langle 13\rangle \sin\theta + z\langle 23\rangle \cos\theta}$$

$$= -\int_0^{\frac{\pi}{2}} \mathrm{d}\theta 2\cos^2\theta \left(\frac{\cos\theta}{\sin\theta} - \frac{1}{\cos\theta\sin\theta}\Theta\left(1 - \left|\frac{\langle 13\rangle \sin\theta}{\langle 23\rangle \cos\theta}\right|\right)\right) = 1 + \log\frac{s_{23}}{s_{13}+s_{23}}. \tag{142}$$

The second diagram corresponds to the exchange between particles 1 and 3. Therefore, by simply replacing $1 \leftrightarrow 3$ one obtains the contribution for the second diagram,

$$\left\langle 2_\phi 3_\phi |\mathcal{M}| 1'_\psi 2'_{\bar\psi}\right\rangle_{\text{first term}} \left\langle 1'_\psi 2'_{\bar\psi} 1_\phi 4_\phi \mid \mathcal{O}_y \mid 0\right\rangle \longrightarrow 1 + \log\frac{s_{12}}{s_{12}+s_{13}}. \tag{143}$$

In the third diagram participates the five-point amplitude acting on the minimal form factor. The parametrization in eq. 96 parametrizes two intermediate spinors in terms of two external spinors. Hence, it becomes necessary to express the three external particles in terms of two particles. Focusing only in the y^3 term in the five point amplitude, the third diagram corresponds to:

$$\begin{aligned}\left\langle 1_\phi 2_\phi 3_\phi |\mathcal{M}| 1'_{\bar\psi} 2'\bar\psi\right\rangle \left\langle 1'_{\bar\psi} 2'_{\bar\psi} 4_\phi \,|\mathcal{O}_y|\, 0\right\rangle = \frac{\langle 13\rangle}{\langle 1'1\rangle\,\langle 2'3\rangle}\,\langle 1'2'\rangle = \\ = \int_0^{\frac{\pi}{2}} \mathrm{d}\theta 2\frac{\cos\theta}{\sin\theta}\Theta\left(1 - \left|\frac{[12]\langle 23\rangle \cos\theta}{\langle 13\rangle\sqrt{s_{123}}\sin\theta}\right|\right) = \log\left(\frac{(s_{12}+s_{13})\,(s_{13}+s_{23})}{s_{12}s_{23}}\right).\end{aligned} \tag{144}$$

In order to make use of the parametrization in eq.96 we should express the three external particles as a two particle by means of,

$$\begin{aligned} p_a &= p_1\frac{s_{123}}{s_{12}+s_{13}}, \qquad p_b = p_2 + p_3 - p_1\frac{s_{23}}{s_{12}+s_{13}}, \\ \lambda_a &= \lambda_1\sqrt{\frac{s_{123}}{s_{12}+s_{13}}}, \quad \lambda_b = ([12]\lambda_2 + [13]\lambda_3)\,\frac{1}{\sqrt{s_{12}+s_{13}}}, \end{aligned} \tag{145}$$

where we impose the particles a and b as on-shell particles fulfilling $p_a + p_b = p_1 + p_2 + p_3$.

Analogously to the previous cases, the integration then takes the explicit form,

$$\int \frac{\mathrm{d}\Omega}{4\pi} \frac{\langle 1'2' \rangle \langle 13 \rangle}{\langle 1'1 \rangle \langle 2'3 \rangle} = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \frac{2\cos\theta}{\sin\theta + \mathrm{e}^{i\phi}\cos\theta \frac{[12]\langle 23 \rangle}{\langle 13 \rangle \sqrt{s_{123}}}}. \tag{146}$$

These three diagrams are then an example of the non-trivial mixing of the two types of contributions to the anomalous dimension, since the sum of them leads to the cancellation of the logarithm terms,

$$\left(1 + \log \frac{s_{23}}{s_{13} + s_{23}}\right) + \left(1 + \log \frac{s_{12}}{s_{12} + s_{13}}\right) + \log\left(\frac{(s_{12} + s_{13})(s_{13} + s_{23})}{s_{12} s_{23}}\right) = 2. \tag{147}$$

Finally, from the other term in the amplitude λy, we get the full anomalous dimension,

$$\gamma_{y\lambda} = -\frac{96y^3}{16\pi^2} + \frac{8y\lambda}{16\pi^2}. \tag{148}$$

7 $3 \to 2$ case

In this chapter we will use the unitarity method in order to compute the two-loop anomalous dimension. Just as in the one-loop case, there are in general different cuts contributing to the two-loop anomalous dimension. More concretely, in this chapter we will study the simplest two-loop cut which corresponds to convolution between the $3 \to 2$ amplitude and the minimal form factor form factor.

In the first section we will give the new parametrization corresponded to this cut. In the next two sections, we will review two applications to this case. More concretely, the first application, particularizing to the Yukawa theory, we will compute the final entry to the anomalous dimension and, consequently, give the expression of the full anomalous dimension matrix with its corresponding total β-function. In the second application we review another example of the computation of the dilatation operator by means of the parametrization derived in the first section.

7.1 Parametrization of spinors

In this section we will derive the full parametrization for the $2 \to 3$ amplitude. Starting with the most general expression, we analyze which constrains should be applied in order to obtain the final expression suitable for the computations of the next sections. The most general parametrization will be of the form,

$$\begin{pmatrix} \lambda_1' \\ \lambda_2' \\ \lambda_3' \end{pmatrix} = A \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}. \tag{149}$$

The first step is to express the parametrization in terms of as many parameters as variables have the theory. The goal is to express the three intermediate spinors crossing the cut in terms of the two external spinors. Taking into account the first and second components of both spinors and antispinors one amounts to a total of 12 spinor variables. Therefore, the parametrization will in principle be constitute of 12 different parameters. We do show in order to obtain a square matrix for the Jacobian, which simplifies some computations such as the determinant.

In particular, there will be 10 different contributions corresponding to angles and 2 radial variables. Furthermore, from the previous equation we see that the transformation matrix is a 3×2 matrix, formed in general by the product of a unitary 3×3 matrix U times a 3×2 rectangular matrix U_0. This choice in not unique in the sense that the same total transformation matrix holds by applying the following transformation to each of the matrices:

$$U \to U \,\mathrm{diag}\left\{e^{i\phi_a}, e^{i\phi_b}, e^{i\phi_c}\right\}, \quad U_0 \to U_0 \,\mathrm{diag}\left\{e^{i\phi_a}, e^{i\phi_b}\right\}. \tag{150}$$

With this freedom we can then define both matrices as,

$$U = \text{diag}\left(e^{i\phi_2}, e^{i\phi_3}, e^{i\phi_4}\right) R_1(\theta_3) R_3(\theta_2) \,\text{diag}\left\{1, 1, e^{i\rho}\right) U_0 \,\text{diag}\left\{1, e^{i\phi_1}\right\}, \tag{151}$$

where,

$$U_0 = \begin{pmatrix} 1 & 0 \\ 0 & \cos\theta_1 \\ 0 & \sin\theta_1 \end{pmatrix}. \tag{152}$$

The matrix $R_i(x)$ refers to the rotation matrix along the i axis by an angle x. All in all, the change of variables for this case will be of the form [36],

$$\begin{aligned} \begin{pmatrix} \lambda_1'^1 \\ \lambda_2'^1 \\ \lambda_3'^1 \end{pmatrix} &= r_1 e^{i\sigma_1} U \begin{pmatrix} \lambda_1^1 \\ \lambda_2^1 \end{pmatrix}, \\ \begin{pmatrix} \lambda_1'^2 \\ \lambda_2'^2 \\ \lambda_3'^2 \end{pmatrix} &= r_2 U R_3(\sigma_2) \begin{pmatrix} \lambda_1^2 \\ \lambda_2^2 \end{pmatrix}, \end{aligned} \tag{153}$$

where r_1 and r_2 run from 0 to ∞, $0 \leq \theta, \sigma_2 \leq \pi/2$ and $0 \leq \phi, \sigma_1, \rho \leq 2\pi$. We assume all the parameters to be real. The complex conjugate of this expression leads to the parametrization of antispinors. Imposing momentum conservation from the delta function fixes the values of some of the parameters. Computing the whole expression equivalents to,

$$\begin{aligned} \delta^4(P) = \delta^4\left(-p_1 - p_2 + p_1' + p_2' + p_3'\right) &= \prod_{\alpha=1}^{2} \prod_{\dot\beta=1}^{2} \delta\left(-\lambda_1^\alpha \tilde\lambda_1^{\dot\beta} - \lambda_2^\alpha \tilde\lambda_2^{\dot\beta} + \lambda_1'^\alpha \tilde\lambda_1'^{\dot\beta} + \lambda_2'^\alpha \tilde\lambda_2'^{\dot\beta} + \lambda_3'^\alpha \tilde\lambda_3'^{\dot\beta}\right) \\ &= \frac{i\delta(r_1 - 1)\,\delta(r_2 - 1)\,\delta(\sigma_1)\,\delta(\sigma_2)}{4\left(\sum_{i=1}^2 \lambda_i^1 \bar\lambda_i^{\dot 1}\right)\left(\sum_{i=1}^2 \lambda_i^2 \bar\lambda_i^{\dot 2}\right)\left(\langle 12\rangle \left(\bar\lambda_1^{\dot 1} \bar\lambda_1^{\dot 2} + \bar\lambda_2^{\dot 1} \bar\lambda_2^{\dot 2}\right) + \left(\lambda_1^1 \lambda_1^2 + \lambda_2^1 \lambda_2^2\right)[12]\right)}. \end{aligned} \tag{154}$$

Just as in the $2 \to 2$ parametrization, momentum conservation fixes $r_i = 1$ and $\sigma_i = 0$. Replacing these values in the transformation matrix leads to,

$$A = \begin{pmatrix} e^{i\phi_2} c_2 & -e^{i(\phi_1+\phi_2)} c_1 s_2 \\ e^{i\phi_3} c_3 s_2 & e^{i(\phi_1+\phi_3)}\left(c_1 c_2 c_3 - e^{i\rho} s_1 s_3\right) \\ e^{i\phi_4} s_2 s_3 & e^{i(\phi_1+\phi_4)}\left(c_1 c_2 s_3 + e^{i\rho} c_3 s_1\right) \end{pmatrix}. \tag{155}$$

The little group scaling condition sets three of the azimuthal angles to zero due to the three intermediate spinors. In particular, the vanishing phases will be the ones that lead to the same momenta. Hence, setting $\phi_2, \phi_3, \phi_4 \to 0$ and $\phi_1 \to \phi$, the final parametrization,

$$\begin{pmatrix} \lambda'_1 \\ \lambda'_2 \\ \lambda'_3 \end{pmatrix} = \begin{pmatrix} c_2 & -e^{i\phi} c_1 s_2 \\ c_3 s_2 & e^{i\phi}\left(c_1 c_2 c_3 - e^{i\rho} s_1 s_3\right) \\ s_2 s_3 & e^{i\phi}\left(c_1 c_2 s_3 + e^{i\rho} c_3 s_1\right) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{156}$$

The complex conjugation of this expression holds for the antispinors. Hence, through this parametrization, the relation between intermediate and external spinors relies only on angular variables. The integration now should be done over the polar angle θ and the azimuthal angle ϕ.

The change of variables for the measure is done through the determinant of the Jacobian. The computation of the Jacobian represents the partial derivative of each of the external spinors with respect to each of the 12 parameters. Only after its computation, together with the momentum conservation delta function, one sets $r_i = 1$ and $\sigma_i = 0$. The Jacobian for this case is then a square 12×12 matrix. The expression of the determinant that we obtained is:

$$32 \cos\theta_1 \cos\theta_2 \cos\theta_3 \sin\theta_1 \sin^3\theta_2 \sin\theta_3 \langle 12\rangle [21] f(\lambda, \bar\lambda) \tag{157}$$

where $f(\lambda, \bar\lambda)$ represents the spinor dependence. We saw that after some calculations its expression is:

$$\begin{aligned} f(\lambda, \bar\lambda) = &\left((\lambda_1^2)^2(\lambda_2^1)^2 - (\lambda_1^1)^2(\lambda_2^2)^2\right) \bar\lambda_1^{\dot 1} \bar\lambda_2^{\dot 1} \left((\bar\lambda_1^{\dot 2})^{\dot 2} + (\bar\lambda_2^{\dot 2})^2\right) - \left(\lambda_1^1 \lambda_1^2 (\bar\lambda_1^{\dot 1})^2 + \lambda_2^1 \lambda_2^2 (\bar\lambda_2^{\dot 1})^2\right) \\ &\left((\bar\lambda_1^{\dot 2})^2 + (\bar\lambda_2^{\dot 2})^2\right) \langle 12\rangle + \left((\lambda_1^2)^2 + (\lambda_2^2)^2\right) (\lambda_1^1 \bar\lambda_1^{\dot 1} + \lambda_2^1 \bar\lambda_2^{\dot 1})(\lambda_1^1 \bar\lambda_1^{\dot 2} + \lambda_2^1 \bar\lambda_2^{\dot 2})[21]. \end{aligned} \tag{158}$$

After some computations, this function leads to the following expression:

$$f(\lambda, \bar\lambda) = 4\left(\lambda_1^1 \bar\lambda_1^{\dot 1} + \lambda_2^1 \bar\lambda_2^{\dot 1}\right)\left(\lambda_1^2 \bar\lambda_1^{\dot 2} + \lambda_2^2 \bar\lambda_2^{\dot 2}\right)\left(\langle 12\rangle \left(\bar\lambda_1^{\dot 1} \bar\lambda_1^{\dot 2} + \bar\lambda_2^{\dot 1} \bar\lambda_2^{\dot 2}\right) + \left(\lambda_1^1 \lambda_1^2 + \lambda_2^1 \lambda_2^2\right)[12]\right), \tag{159}$$

which is precisely the same that appears in the denominator of the momentum conserving delta function, and hence when contracting the Jacobian with the momentum delta function, one obtains the integration measure in terms of the angular variables,

$$d^2\lambda'_1 d^2\tilde\lambda'_1 d^2\lambda'_2 d^2\tilde\lambda'_2 d^2\lambda'_3 d^2\tilde\lambda'_3 \delta^4(P) \longrightarrow d\rho d\phi_1 d\phi_2 d\phi_3 d\phi_4 d\theta 2\cos\theta_1 \sin\theta_1 2\cos\theta_3 \sin\theta_3 4\cos\theta_2 \sin^3\theta_2. \tag{160}$$

Furthermore, the integration of three of the phases drops out due to little group scaling in the sense that different values of the phases ϕ_2, ϕ_3 and ϕ_4 lead to the same momenta. Renaming $\phi_1 \to \phi$, the full integral in the $3 \to 2$ case is,

$$I_{3\to 2} = \frac{\langle 12\rangle[21]}{4^4 \pi^3 3!} \int d\Omega_3 \left\langle 12 \left|\mathcal{M}_{3\to 2}\right| 1'2'3'\right\rangle \langle 1'2'3'|\mathcal{O}|0\rangle, \tag{161}$$

with the measure given by,

$$d\Omega_3 = 4\cos\theta_3 \sin^3\theta_3 d\theta_3 2\cos\theta_2 \sin\theta_2 d\theta_2 2\cos\theta_1 \sin\theta_1 d\theta_1 \frac{d\rho}{2\pi}\frac{d\phi}{2\pi}. \tag{162}$$

This equation will be the starting point of the next sections. By means of this equation one can compute two-loop anomalous dimensions.

7.2 Yukawa interaction

Considering the Yukawa interaction introduced in the last chapter, in this section we will review the computation of the anomalous dimension from the $3 \to 2$ amplitude acting on the minimal form factor. Moreover, being the only anomalous dimension entry not computed yet, the calculation will lead to the complete anomalous dimension matrix and the complete β-function. This section is mainly based on [14].

In the previous chapter, we review the computation of the length-preserving and the length-increasing mixing anomalous dimension. Therefore, in this case we will compute the length-decreasing mixing of $\mathcal{O}_\lambda$ into $\mathcal{O}_y$. The novelty of this method is that the computations involve a three-particle cut, and this will lead to the two-loop anomalous dimension. Therefore the anomalous dimension will take the form,

$$\gamma_{\lambda y} = -\frac{1}{\pi}\frac{\left\langle 1_\psi 2_\psi 3_\phi \left| \left(\mathcal{M}_{12}^{2\leftarrow 3}\right) \otimes \mathcal{O}_\lambda \right| 0 \right\rangle^{(0)}}{\left\langle 1_\psi 2_\psi 3_\phi \left| \mathcal{O}_y \right| 0 \right\rangle^{(0)}}. \tag{163}$$

The expression of the five point amplitude in eq. 140 leads to seven contributions, six of them proportional to y^3 and one proportional to λy. In particular the λy contribution does not depend on the intermediate spinors and consequently the angular integration will be trivial. Substituting this amplitude in the integral of eq. 160,

$$\frac{\langle 12\rangle[21]}{4^4\pi^3 3!}\int d\Omega_3 \left\langle 1_\psi 2_\psi \left|\mathcal{M}_{2\to 3}\right| 1'_\phi 2'_\phi 3'_\phi \right\rangle_{\lambda y\text{ term}} \left\langle 1'_\phi 2'_\phi 3'_\phi 3_\phi |\mathcal{O}_\lambda| 0\right\rangle = -\frac{1}{6}\frac{\lambda y}{4^4\pi^3}\left\langle 1_\psi 2_\psi 3_\phi \left|\mathcal{O}_y\right| 0\right\rangle . \tag{164}$$

Let us now consider the first term proportional to y^3 in the five-point amplitude. Applying the parametrization in eq. 156 leads to a contribution that only depends on elementary trigonometric functions,

$$\left\langle 1_\psi 2_\psi \left|\mathcal{M}^{2\leftarrow 3}\right| 1'_\phi 2'_\phi 3'_\phi \right\rangle\Big|_{\text{first } y^3\text{ term}} = -\frac{y^3}{\langle 12\rangle}\left(\mathrm{e}^{i\rho}\tan\theta_1 \cot\theta_2 \csc\theta_2 \cot\theta_3 + \cot^2\theta_2 + 1\right). \tag{165}$$

Then the integration reads,

$$\frac{s_{12}}{4^4\pi^3}\frac{1}{3!}\int \mathrm{d}\mu \left\langle 1_\psi 2_\psi \left|\mathcal{M}^{2\leftarrow 3}\right| 1'_\phi 2'_\phi 3'_\phi \right\rangle\Big|_{\text{first } y^3\text{ term}} \left\langle 1'_\phi 2'_\phi 3'_\phi 3_\phi \left|\mathcal{O}_\lambda\right| 0\right\rangle = \frac{2y^3}{3!(4\pi)^4}\left\langle 1_\psi 2_\psi 3_\phi \left|\mathcal{O}_y\right| 0\right\rangle . \tag{166}$$

From eq. 163, that only depends on trigonometric functions and the external spinors, one can see that the contributions from the other five permutations lead to the same integral. Consequently, the anomalous dimension is:

$$\gamma_{\lambda y} = -\frac{2y^3}{(4\pi)^4} + \frac{1}{6}\frac{y\lambda}{(4\pi)^4}, \tag{167}$$

and the corresponding two-loop β-function,

$$\beta^2(y) = -\frac{2y^3\lambda}{(16\pi^2)^2} + \frac{1}{12}\frac{y\lambda^2}{(16\pi^2)^2}. \tag{168}$$

Recalling the results from the previous chapter, the total anomalous dimension matrix,

$$\begin{pmatrix} \gamma_{yy} & \gamma_{y\lambda} \\ \gamma_{\lambda y} & \gamma_{\lambda\lambda} \end{pmatrix} = \frac{1}{16\pi^2}\begin{pmatrix} 12y^2 + O\left(y^4\right) & -96y^3 + 8y\lambda + O\left(y^5\right) \\ -\frac{2y^3}{16\pi^2} + \frac{1}{6}\frac{y\lambda}{16\pi^2} + O\left(y^5\right) & 6\lambda + 4y^2 + O\left(y^4\right) \end{pmatrix}, \tag{169}$$

with the one-loop β-function,

$$\beta^{(1)}(y) = \frac{1}{16\pi^2}\left(4y^3\right), \quad \beta^{(1)}(\lambda) = \frac{1}{16\pi^2}\left(-24y^4 + 4y^2\lambda + 3\lambda^2\right) \tag{170}$$

7.3 Computation of $\mathfrak{D}_{3\to 2}$

As another application of the parametrization derived in the first section, in this section we will derive the dilatation operator in the planar limit of $\mathcal{N} = \triangle$ SYM theory. In [36] Zwiebel derived a relation between the dilatation generator $\mathfrak{D}_{L\to 2}$ and tree-level scattering amplitudes. The general form of this relation is,

$$\begin{aligned} &\langle \Lambda_1, \Lambda_2 \left|\mathfrak{D}_{L\to 2}\right| \mathcal{P}_L\rangle = \\ &2(2\pi)^{1-2L}\int \mathrm{d}\Lambda_3\, \mathrm{d}\Lambda_4 \ldots \mathrm{d}\Lambda_{L+2} A_{L+2}\left(1^-, 2^-, 3, \ldots L+2\right) \mathcal{P}_L((L+2), (L+1), \ldots 3), \end{aligned} \tag{171}$$

where $\Lambda_i = (\lambda_i, \bar{\lambda}_i, \eta_i)$ refers to the supermomentum of the particle i. In this section we will use this expression particularizing to $L = 3$, that takes the following form,

$$\begin{aligned} \left\langle \bar{\Lambda}_1\bar{\Lambda}_2 \left|\mathfrak{D}^{[0]}_{3\to 2}\right| \mathcal{P}_3\right\rangle &= 2(2\pi)^{-5}\int \mathrm{d}\bar{\Lambda}_3\, \mathrm{d}\bar{\Lambda}_4\, \mathrm{d}\bar{\Lambda}_5 A_5^{\mathrm{MHV}}\left(\bar{1}^-, \bar{2}^-, \bar{3}, \bar{4}, \bar{5}\right) \mathcal{P}_3(\bar{5}, \bar{4}, \bar{3}) \\ &= 2(2\pi)^{-5}\int \mathrm{d}\bar{\Lambda}_3\, \mathrm{d}\bar{\Lambda}_4\, \mathrm{d}\bar{\Lambda}_5 \frac{\delta^4(P)\delta^8(Q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\mathcal{P}_3(\bar{5}, \bar{4}, \bar{3}), \end{aligned} \tag{172}$$

where the $\bar{\Lambda}_i$ stems from the fact that we will use the variable $\bar{\eta}$ instead of η in the supermomentum definition. The particles $3, 4, 5$ are the intermediate particles crossing the cut. As in the $2 \to 2$ case, the supermomentum conservation imposes that the $\bar{\eta}$ variables rotate in the same way as the λ.

By means of the parametrization in eq. 155, the ϕ_i integrations can be dropped out and then the spinors $\lambda, \bar{\lambda}$ and $\bar{\eta}$ rotate as:

$$\begin{pmatrix} \lambda'_1 \\ \lambda'_2 \\ \lambda'_3 \end{pmatrix} = \begin{pmatrix} c_2 & -c_1 s_2 \\ c_3 s_2 & c_1 c_2 c_3 - e^{i\rho} s_1 s_3 \\ s_2 s_3 & c_1 c_2 s_3 + e^{i\rho} c_3 s_1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} \bar{\lambda}'_1 \\ \bar{\lambda}'_2 \\ \bar{\lambda}'_3 \end{pmatrix} = \begin{pmatrix} c_2 & -c_1 s_2 \\ c_3 s_2 & c_1 c_2 c_3 - e^{-i\rho} s_1 s_3 \\ s_2 s_3 & c_1 c_2 s_3 + e^{-i\rho} c_3 s_1 \end{pmatrix} \begin{pmatrix} \bar{\lambda}_1 \\ \bar{\lambda}_2 \end{pmatrix},$$

$$\begin{pmatrix} \bar{\eta}'_1 \\ \bar{\eta}'_2 \\ \bar{\eta}'_3 \end{pmatrix} = \begin{pmatrix} c_2 & -c_1 s_2 \\ c_3 s_2 & c_1 c_2 c_3 - e^{i\rho} s_1 s_3 \\ s_2 s_3 & c_1 c_2 s_3 + e^{i\rho} c_3 s_1 \end{pmatrix} \begin{pmatrix} \bar{\eta}_1 \\ \bar{\eta}_2 \end{pmatrix}, \tag{173}$$

where the only difference resides in the complex conjugate of the ρ phase in the antispinor parametrization.

Expressing now the intermediate spinors by means of this parametrization, and computing the denominator of the amplitude, one obtains simply an expression that depends only on the angular variables. Performing the change of variables in eqs. 157, 158, the integral takes the form,

$$\left\langle \bar{\Lambda}_1 \bar{\Lambda}_2 \middle| \mathfrak{D}^{[0]}_{3\to 2} \middle| \mathcal{P}_3 \right\rangle = \frac{16}{2\pi i} \int \mathrm{d}\rho \prod_{i=1}^{3} \mathrm{d}\theta_i \frac{c_2}{c_1} \frac{e^{-i\rho}}{1 - e^{i\rho} \frac{s_1 c_2 s_3}{c_1 c_3}} [12] \mathcal{P}_3 \left(\bar{1}', \bar{2}', \bar{3}' \right). \tag{174}$$

Expressing $\mathcal{P}_3 \left(\bar{1}', \bar{2}', \bar{3}' \right)$ by means of eq. 172, and performing the integral over ρ, yields to different types of integrals,

$$\begin{aligned} I_1(i,j,k) &= 8 \int_0^{\pi/2} \mathrm{d}\theta_1 \int_0^{\pi/2} \mathrm{d}\theta_2 \int_0^{\pi/2} \mathrm{d}\theta_3 \theta \left(\frac{c_1 c_3}{s_1 s_3} - c_2 \right) \left(s_1^{2i+1} c_1 \right) \left(s_2^{2j+1} c_2 \right) \left(s_3^{2k+1} c_3 \right), \\ I_2(i,j,k) &= 8 \int_0^{\pi/2} \mathrm{d}\theta_1 \int_0^{\pi/2} \mathrm{d}\theta_2 \int_0^{\pi/2} \mathrm{d}\theta_3 \theta \left(c_2 - \frac{c_1 c_3}{s_1 s_3} \right) \left(s_1^{2i+1} c_1 \right) \left(s_2^{2j+1} c_2 \right) \left(s_3^{2k+1} c_3 \right), \end{aligned} \tag{175}$$

where θ is the Heaviside function. Both integrals, when evaluated analytically, depend on the harmonic numbers $S_k(n)$. Moreover, the integral in eq.174 is finite since it is expressed as a finite sum of both I_1 and I_1.

8 $3 \to 3$ case

In this chapter, we will focus on the parametrization for the three-particle cut in the three-particle channel. In particular, we will describe the computations derived towards this goal and the result we obtained. We will also comment on what should be the ideal form of this parametrization. We will briefly comment on what applications would the correct parametrization has.

The external spinors $\lambda_1, \lambda_2, \lambda_3$ form a complete basis. Every intermediate spinor can be expressed in terms of this basis through a transformation matrix A,

$$\begin{pmatrix} \lambda'_1 \\ \lambda'_2 \\ \lambda'_3 \end{pmatrix} = A \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{176}$$

where A represents a 3×3 square matrix, that will depend on the angular variables θ and ϕ. This matrix belongs to the $SU(3)$ group. Matrices contained in this group correspond to square 3×3 unitary matrices with determinant 1. Consequently the matrix A should fulfill both requirements: $AA^\dagger = 1$ and $\mid A \mid = 1$. Analogously to the cases explained in previous chapters, the most general parametrization of the spinors takes the following expression,

$$\begin{pmatrix} \lambda'^1_1 \\ \lambda'^1_2 \\ \lambda'^1_3 \end{pmatrix} = r_1 e^{i\sigma_1} U \begin{pmatrix} \lambda^1_1 \\ \lambda^1_2 \\ \lambda^1_3 \end{pmatrix}, \qquad \begin{pmatrix} \lambda'^2_1 \\ \lambda'^2_2 \\ \lambda'^2_3 \end{pmatrix} = r_2 U R_3 \left(\sigma_2\right) \begin{pmatrix} \lambda^2_1 \\ \lambda^2_2 \\ \lambda^2_3 \end{pmatrix}. \tag{177}$$

The complex conjugate of this expression leads to the expression for the antispinors,

$$\begin{pmatrix} \bar\lambda'^1_1 \\ \bar\lambda'^1_2 \\ \bar\lambda'^1_3 \end{pmatrix} = r_1 e^{-i\sigma_1} U^* \begin{pmatrix} \bar\lambda^1_1 \\ \bar\lambda^1_2 \\ \bar\lambda^1_3 \end{pmatrix}, \qquad \begin{pmatrix} \bar\lambda'^2_1 \\ \bar\lambda'^2_2 \\ \bar\lambda'^2_3 \end{pmatrix} = r_2 U^* R_3 \left(\sigma_2\right) \begin{pmatrix} \bar\lambda^2_1 \\ \bar\lambda^2_2 \\ \bar\lambda^2_3 \end{pmatrix}. \tag{178}$$

The focus of this chapter will be on finding the correct expression for the matrix U. In this case, there exists 12 variables corresponding to the six spinor variables as well as six antispinor variables. Hence, the matrix A should in principle be expressed in terms of 12 parameters, in a one-to-one correspondence with the spinor variables. From the expression above, we have already four of them corresponding to $0 \le r_1, r_2 < \infty$, $0 \le \sigma_1 \le 2\pi$ and $0 \le \sigma_2 \le \pi/2$. Therefore, we should find the matrix U in terms of the eight remaining parameters.

In particular, due to the dependence in the $r_1, r_2, \sigma_1, \sigma_2$ parameters, the matrix U have to fulfill the same requirements as the matrix A, which implies unitarity $UU^\dagger = 1$ and having determinant one $\mid U \mid = 1$. The following transformation of U leaves the transformation matrix A unchanged [21]:

$$U \to U \operatorname{diag}\left\{e^{i\phi_a}, e^{i\phi_b}, e^{i\phi_c}\right\}, \tag{179}$$

which simply corresponds to the multiplication by a diagonal matrix containing phases. Hence, the general dependence of the matrix U will be on three polar angles $0 \leq \theta_1, \theta_2, \theta_3 \leq \pi/2$ and five azimuthal ones $0 \leq \phi_1, \phi_2, \phi_3, \phi_4, \phi_5 \leq 2\pi$. Further constraints can be applied to the matrix U in terms of the phase space integral associated with this cut. In particular, they come from the angular dependence of the determinant of the Jacobian. For this case the Jacobian takes the following expression:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \lambda_1'^1}{\partial \theta_1} & \cdots & \frac{\partial \lambda_1'^1}{\partial r_2} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{\lambda}_3'^2}{\partial \theta_1} & \cdots & \frac{\partial \tilde{\lambda}_3'^2}{\partial r_2} \end{bmatrix}. \tag{180}$$

It represents the partial derivative of the 12 intermediate spinor and antispinor variables with respect to the 12 parameters contained in the matrix A. In general, the determinant of the Jacobian will depend on these parameters as well as on the external spinors. However, the dependence on the external spinors is the same as the one in the denominator of the momentum conservation delta function, which also fixes the value of four parameters $r_1 = r_2 = 1$ and $\sigma_1 = \sigma_2 = 0$. Hence, the determinant of the Jacobian will depend on the remaining eight angular parameters. Moreover, through this determinant one obtains the measure of the phase space integral in terms of the θ_i and ϕ_i variables [13],

$$\frac{1}{2\pi^5} \sin\theta_1 \cos^3\theta_1 \; \mathrm{d}\theta_1 \sin\theta_2 \cos\theta_2 \; \mathrm{d}\theta_2 \sin\theta_3 \cos\theta_3 \; \mathrm{d}\theta_3 \; \mathrm{d}\phi_1 \; \mathrm{d}\phi_2 \; \mathrm{d}\phi_3 \; \mathrm{d}\phi_4 \; \mathrm{d}\phi_5. \tag{181}$$

Furthermore, due to little group scaling, the momentum of each of the three particles is defined up to a phase, and hence three of the phase integrals should be trivial, leaving an parametrization that depends only on five parameters. This is in accordance to what one would expect after imposing the momentum conserving delta function. The three on-shell particles have each one three degrees of freedom, which amounts to a total of nine degrees of freedom. The momentum conserving delta function then fixes four of them obtaining the remaining five degrees of freedom.

This means that the matrix U should be a unitarity matrix with determinant equals one. The determinant of the Jacobian should depend only on five parameters: two phases and the three polar angles, and the dependence on the latter should as the one in eq. 181. Imposing these conditions we obtained the following expression for the matrix U,

$$\begin{pmatrix} \mathrm{e}^{i(\phi_3+\phi_5)} c_1 c_2 & -\mathrm{e}^{i(\phi_1+\phi_2)} s_1 & -\mathrm{e}^{i(\phi 2+\phi_3+\phi_5)} c_1 s_2 \\ \mathrm{e}^{-i(\phi_1+\phi_2-\phi_4)} (c_2 c_3 s_1 - \mathrm{e}^{i(\phi_1+\phi_2)} s_2 s_3) & \mathrm{e}^{-i(\phi_2-\phi_4-\phi_5)} (c_1 c_3 & \mathrm{e}^{-i(\phi_1-\phi_4)} (-c_3 s_1 s_2 - \mathrm{e}^{i(\phi_1+\phi_2)} c_2 s_3) \\ \mathrm{e}^{-i(\phi_1+\phi_2+\phi_3+\phi_4)} (\mathrm{e}^{i(\phi_1+\phi_2)} c_3 s_2 + c_2 s_1 s_3) & \mathrm{e}^{-i(\phi_2+\phi_3+\phi_4+\phi_5)} c_1 s_3 & \mathrm{e}^{-i(\phi_1+\phi_3+\phi_4)} (\mathrm{e}^{i(\phi_1+\phi_2)} c_2 c_3 - s_1 s_2 s_3) \end{pmatrix} \tag{182}$$

In particular, this matrix is unitary $UU^\dagger = 1$ and its determinant $\mid U \mid = 1$. By means of eqs. 177, 178 we can obtain the spinor and antispinor variables and consequently compute the Jacobian, eq. 180, and its determinant. The momentum conserving delta function cancels the dependence on the external spinors and fixes the value of $r_1, r_2, \sigma_1, \sigma_2$.

The dependence with respect to each of the angular variables will be done by giving random values to all the parameters except one, and representing in terms of the non-fixed parameter. By doing so, we obtained that, accordingly to what should be found, the Jacobian depends only on five parameters. More concretely, on the three polar angles and on two phases ϕ_4 and ϕ_5.

The dependence on the polar angles is also as expected, as shown in the figures A.3. This dependence was made through a numerical approach. We know that the result has a simple dependence in terms of trigonometric functions of the polar angles, see eq. 181. Hence, we can just plot the determinant as a function of the parameters we are interested in, for numeric values of all other parameters.

However, this parametrization still needs to be improved. The problem relies on the dependence on the two phases. Let us illustrate it with one specific example. Consider simplest example which corresponds to the minimal two-loop form factor for a generic operator in the $SU(2)$ sector. Operators in the $SU(2)$ sector are formed from two types of scalar fields that share one $SU(4)$ index, as defined in eq. 54. Let us now consider the six-point scattering amplitude between the same type of scalar field, [34]:

$$A_6^{(0)} = \frac{s_{23}s_{1'2'}}{s_{11'}s_{33'}s_{233'}} + \frac{s_{12}s_{2'3'}}{s_{11'}s_{33'}s_{33'2'}} - \frac{s_{123}}{s_{11'}s_{33'}}, \tag{183}$$

and let us pick one of the Mandelstam variables and express it by means of this parametrization,

$$\begin{aligned} s_{33'2'} = s_{121'} = {} & (1+\cos^2\theta_1\cos^2\theta_2+\sin^2\theta_1)s_{12} + \mathrm{e}^{i(\phi_2-\phi_1)}\cos\theta_1\sin\theta_1\sin\theta_2\,\langle p_1 \mid p_3\rangle\,[p_2 \mid p_1] \\ & + \mathrm{e}^{i\phi_2}\cos^2\theta_1\cos\theta_2\sin\theta_2\,\langle p_2 \mid p_3\rangle\,[p_2 \mid p_1] + \mathrm{e}^{i(\phi_1-\phi_2)}\cos\theta_1\sin\theta_1\sin\theta_2\,\langle p_1 \mid p_2\rangle\,[p_3 \mid p_1] \\ & + \cos^2\theta_1\sin^2\theta_2 s_{13} + \mathrm{e}^{-i\phi_2}\cos^2\theta_1\cos\theta_2\sin\theta_2\,\langle p_1 \mid p_2\rangle\,[p_3 \mid p_2] + \cos^2\theta_1\sin^2\theta_2 s_{23}. \end{aligned} \tag{184}$$

Finding the residues of this expression already involves complicated expressions in which changes of integration should be done repeatedly. Moreover, the denominator of that term also contains another two Mandelstam variables $s_{11'}$ and $s_{33'}$, that when expressed in terms of the parametrization leads to more complicated expressions and to square roots in the denominator that needs to be rationalized.

Therefore, the necessity of finding a better parametrization is evident. Then, the ideal parametrization will depend only on the three polar angles. The fact that the phase dependence drops out has direct consequences from a computational point of view. In this case, the integrand would take more handy expression and the square roots in the denominator would be probably be solvable more easily.

From a more physical point of view, the fact that the number of free parameters equals three can also be explained. Let us know briefly comment on the $2 \to 2$ parametrization. In that case, two particles carrying three degrees of freedom each, sum up to six free parameters. Four of them are fixed by the momentum conserving delta function, leading to two remaining parameters. Of the two remaining parameters, the phase ϕ fixes the central charge to zero, eq.28, and since the incoming particles have zero central charge, this fixes the central charge constraint for the remaining particle.

The θ integration will then yields to transcendentality one functions, as desired for the free parameter of the theory. This is in accordance with the result obtained of the anomalous dimension from twist-two operators in 125, where the dependence relies on the harmonic number $S_1(m)$, which represents a function of transcendentality 1.

Considering now the $3 \to 3$ case, three particles which has three degrees of freedom amounts to nine free parameters. Again, momentum conservation fixes four, yielding to five parameters. This is in accordance to the parametrization we found in eq.182. However, analogously to the $2 \to 2$ case, one would expect that two of the phases fix the central charge of two of the three particles, leading to the central charge constraint of the third particle automatically fulfilled (incoming particles already carry zero central charge).

Hence, one would expect the ideal $3 \to 3$ parametrization to depend only on the three polar angles. This would lead transcendentality 3 functions. In two-loop integrals, one can have functions of transcendentality 4. For instance, two-loop remainder functions contains a universal piece of maximal transcendentality four [27]. Furthermore, integrals of transcendentality 4 functions, when computing the cuts, lowers the transcendentality by 1, leading a result of transcendentality 3, as one would expect. The final result should be a rational term, which can be obtained through the cancellation of transcendentality 3 functions, see eq. 147.

This parametrization would have a direct application in the SMEFT context, which will allow to compute more entries in the two-loop anomalous dimension regarding the the three particle cut in the three particle channel, that would extend the computations derived in [7, 21]. Moreover, another application would be to compute the $\mathfrak{D}_{3\to 3}$ as an extension of the work derived in [36].

9 Conclusions

The main goal of this work was to apply the methodology derived in [14], which states that the dilatation operator is minus the phase of the scattering matrix, divided by π, to compute the anomalous dimensions and β-functions of the associated couplings. We gave an overview about the mathematical framework we will be working in, the maximally supersymmetric $\mathcal{N} = 4$ SYM theory. In particular, we saw some of their important applications such as integrability in the planar limit and its analogy with the spin chain in $\mathfrak{psu}(2, 2 \mid 4)$.

We have afterwards studied form factors due to its role as a bridge between the purely on-shell scattering matrices and the purely off-shell correlation functions in terms of the spinor helicity variables. In particular, we have particularised to minimal tree-level form factors, since they translate the spin chain picture of $\mathcal{N} = 4$ SYM theory to the language of scattering amplitudes.

More concretely, we have seen that, up to a normalisation factor and a momentum conserving delta function, the minimal color-ordered tree-level form factor of any operator can be obtained by replacing the oscillators in the oscillator representation by spinor helicity variables. In particular, the generators of $PSU(2, 2|4)$ are related by the same replacement. Therefore, minimal tree-level form factors translate the spin-chain of free $\mathcal{N} = 4$ SYM theory into the language of scattering amplitudes.

Hence, through form factors, one can apply on-shell methods to quantities that contains composite operators. Moreover, the analyticity of form factors makes explicit its relation with the dilatation operator. Recalling the formal definition of form factors as perturbations, we saw the relation between the dilatation operator and the scattering matrix. Moreover, since the dilatation operator acts also as a generator of the $\mathfrak{psu}(2, 2 \mid 4)$ algebra, it does receive quantum corrections when quantizing the theory and consequently is subject to renormalisation.

On the one hand, we obtained the dilatation operator as the phase of the scattering matrix divided by π. On the other hand, we obtained the dilatation generator as a coefficient in the renormalisation group equation, making explicit its relation with other coefficients in the equation such as anomalous dimensions and β-functions. Therefore, the relation between both definitions give the anomalous dimensions from the high-energy regime of the scattering matrix.

Particularizing to weakly coupled field theories, one can perform a perturbative expansion and compute anomalous dimensions at every loop order from products of lower-loop amplitudes and lower-loop form factors. In particular, trying to set a motivation for finding methods to compute anomalous dimensions, we briefly commented on some of the trivial vanishing entries of the anomalous dimensions via selection rules.

The applications of the method in [14] were reviewed in different cases. The first case of study considered was the $2 \to 2$ tree amplitude. We derived the full parametrization from which intermediate spinors can be expressed in terms of the external spinors through an angular dependence. The applications of this derivation are multiple. For instance, this method provides an efficient way yo calculate the β-function of Yang-Mills theory by simply computing the tree-level four-gluon amplitude and integrating over the two-body phase space integral.

Furthermore, through the subtraction of the one-loop infrared divergences, that are universal in gauge theories [15], we obtained the full expression of the one-loop anomalous dimension in any gauge theory. Moreover, a similar expression can be taken in the planar limit of $\mathcal{N} = 4$ SYM theory. The particularity of this limit is that particles are now expressed in terms of their supermomentum $(\lambda_i, \bar{\lambda}_i, \bar{\eta}_i)$ and that new constraints will come from the supermomentum conservation delta function.

The coupling of Yang-Mills to masses was also reviewed. In the last section, anomalous dimensions from twist-two operators was obtained, and its correspondence to the partial-wave amplitude with definite angular momentum. This equivalence reflects that two-particle states with definite angular momentum map to twist-two operators.

In the next chapter, we continue giving examples of the computation of one-loop anomalous dimensions as well as its β-functions by studying the Yukawa interaction from $2 \to 2$ as well as the $2 \to 3$ amplitude. In particular, the latter correspond to a case that can in general appear at every loop order, in the sense, that at every loop, contributions from more external particles can in general appear.

In the next chapter, we derived a new parametrization for the $3 \to 2$ amplitude from which two-loop anomalous dimensions and β-functions are computed. Furthermore, two applications were presented, the length-increasing mixing of the Yukawa interaction from the previous chapter, as well as the dilatation operator $\mathfrak{D}_{3\to 2}$.

In the last chapter we described the computations made in order to derive a parametrization for the $3 \to 3$ case. We described our results and discuss that its limitations. In particular, we analyzed what should be the ideal dependence relying on the central charge constraint and on transcendentality.

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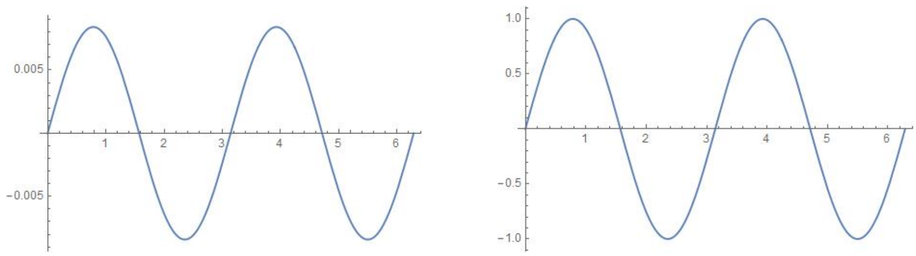
A Appendix

In this appendix, we show the numeric plots of the Jacobian in terms of the polar angles for each of the three parametrizations. In the numeric approach, one gives numeric real random values between 0 and 1 to all the parameters except the one represented. The plots for the Jacobian are scale due to the random values given to the other parameters. For all the figures, we have two plots. On the left, the plot of the Jacobian with respect to one of the parameters. On the right the plot corresponding to the angular dependence. All the plots have been made with Mathematica.

A.1 $2 \to 2$ parametrization

In this section the Jacobian only depends on one plar angle θ. The computation in this case was pretty simple and no numeric approach was needed. However, we represent it here for consistency, and as a crosscheck.

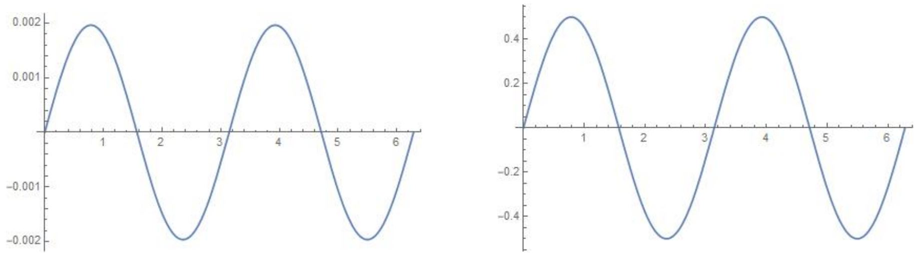


(185)

Figure 13: Checking the θ dependence. On the left the plot for the Jacobian. On the right expected dependence $\sin(2\theta)$.

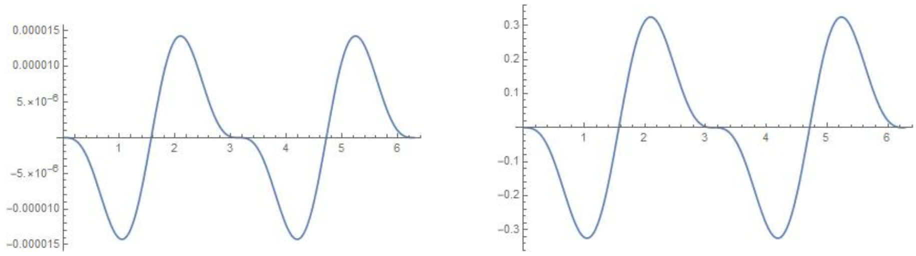
A.2 $3 \to 2$ parametrization

The numeric approach was necessary for this case due to the large expression of the Jacobian. It was also computed analytically by the use of minors.



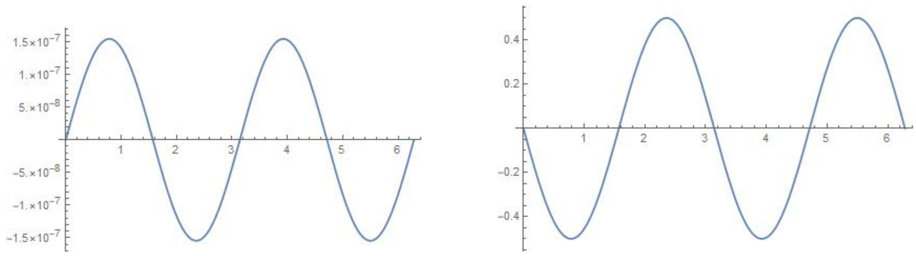
(186)

Figure 14: Checking the θ_1 dependence. On the left the plot for the Jacobian. On the right expected dependence $\sin(2\theta_1)$.



(187)

Figure 15: Checking the θ_2 dependence. On the left the plot for the Jacobian. On the right expected dependence $-\sin^3(\theta_2)\cos(\theta_2)$.

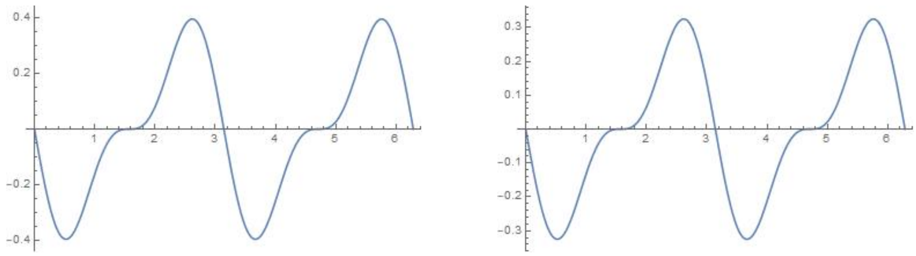


(188)

Figure 16: Checking the θ_3 dependence. On the left the plot for the Jacobian. On the right expected dependence $\sin(2\theta_3)$.

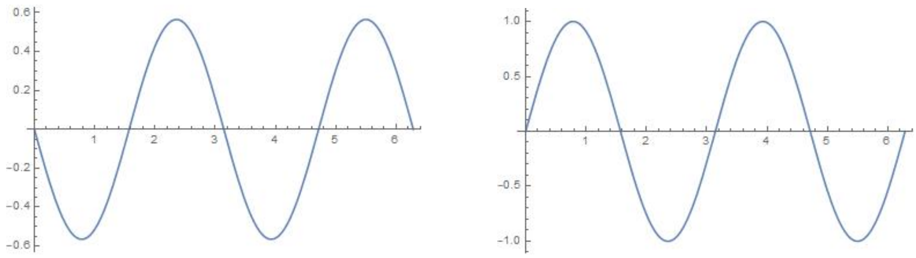
A.3 $3 \to 3$ parametrization

In this section, the way of computing the Jacobian was by the numeric approach.



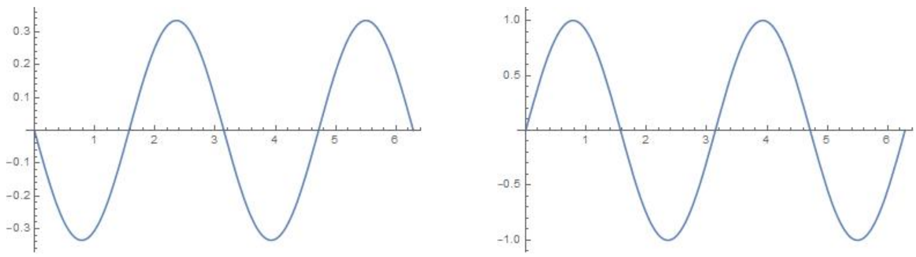
(189)

Figure 17: Checking the θ dependence. On the left the plot for the Jacobian. On the right expected dependence $-\sin^3(\theta_1)\cos(\theta_1)$.



(190)

Figure 18: Checking the θ_2 dependence. On the left the plot for the Jacobian. On the right expected dependence $\sin(2\theta_2)$.



(191)

Figure 19: Checking the θ_3 dependence. On the left the plot for the Jacobian. On the right expected dependence $\sin(2\theta_3)$.