FACULTY OF SCIENCE
UNIVERSITY OF COPENHAGEN

Master Thesis

Alexander Karlberg

Space-Cone Gauge and Scattering Amplitudes

Academic advisor: Prof. Poul Henrik Damgaard

Academic co-advisor: Dr. N. Emil J. Bjerrum-Bohr

June 1, 2012

Abstract

This master thesis is about an unusual gauge called the space-cone gauge and how to relate the BCFW recursion to fundamental results of quantum field theory through this gauge. The first step is to introduce the spinor helicity formalism and show how to colour-order scattering amplitudes. The spinor helicity formalism can be identified with a particular gauge choice, the space-cone gauge. With this gauge we gauge fix the Yang-Mills lagrangian as well as the QCD lagrangian. We then proceed to derive properly normalized colour-ordered Feynman rules for Yang-Mills theory and QCD in the space-cone gauge and show how to apply these in calculations of scattering amplitudes. We present the BCFW recursion and show how it can be related to Veltman's largest time equation by exploiting the vertices of the space-cone gauge fixed Yang-Mills lagrangian. We also make a comment regarding the factorization of a six-point $NMHV$ amplitude in this framework. From the largest time equation we can derive a set of cutting equations which at the one-loop level obtains the imaginary part of a particular diagram by cutting it into tree-level diagrams. A similar result known as Feynman's tree theorem exists from which a full diagram can be obtained from cutting it into a different set of trees than the aforementioned cutting equations result in. However, it seems that the two results should be related and we end the thesis by further discussing the largest time equation and investigate how it is related to Feynman's Tree Theorem.

Resumé

Dette speciale handler om et noget ualmindeligt gaugevalg, kaldet space-cone gauge, og om, hvordan man kan relatere BCFW rekursionsrelationen til fundamentale resultater fra kvantefeltteorien via denne gauge. Først introduceres spinorhelicitetsformalismen og vi viser, hvordan man kan farveordne spredningsamplituder. Spinorhelicitetsformalismen kan identificeres med et særligt gaugevalg, space-cone gaugen. Vi bruger denne gauge til at gaugefiksere Yang-Mills langrangen samt QCD lagrangen. Herfra udleder vi korrekt normaliserede farveordnede Feynman regler for Yang-Mills teori og QCD i space-cone gaugen og viser, hvorledes man bruger disse i beregning af spredningsamplituder. Vi præsenterer BCFW rekursionen og viser, hvordan den er relateret til Veltmans largeste time equation ved at udnytte knuderne i den space-cone gaugefikserede Yang-Mills lagrange. Vi kommentere også faktoriseringen af en sekspunkts $NMHV$-amplitude inden for disse rammer. Ud fra largest time equation kan man udlede et sæt af klipligninger som på et-loops niveau giver den imaginære del af et vilkårligt diagram ved at klippe det ud i trædiagrammer. Et lignende resultat, kendt som Feynman's Tree Theorem, opnår at udtrykke et helt loopdiagram som en sum af trædiagrammer og altså ikke kun imaginærdelen. Det virker rimeligt at antage, at disse to resultater er relaterede of vi slutter specialet med at undersøge denne relation.

List of publications

1. A. Karlberg and T. Søndergaard, “Feynman rules for QCD from Space-Cone gauge”, arXiv:1201.1441 [hep-th]. *Under review with Phys. Lett. B*.

Contents

— 1 —

Introduction

At this very moment our current understanding of what makes up nature at the most fundamental level is being tested at the LHC. At the end of this year experimentalists hope to once and for all either to have discovered or excluded the very elusive Higgs boson of the Standard Model. The Higgs boson is the last missing piece of the Standard Model and its possible discovery will be one of many great successes of this theory through more than three decades. No matter whether the Higgs boson really exists or not there will still be work to do when this year is over. As a matter of fact things are only really starting to become interesting and the years to come could possible change our description of nature in ways that we have not yet thought of. We know that the Standard Model has to break down in some way beyond the 1 TeV scale and even though we have many candidates to patch up the Standard Model we still need experimental evidence to conclude how nature really behaves. Now these experiments wouldn't be much worth, if it wasn't for our ability to make predictions based on all the different theories that competes to be the most fundamental theory of our Universe. Most importantly our predictions needs to be so precise that we may distinguish them from each other. In particular it is important to have good control over the Standard Model as it is the benchmark and competitor of any other theory.

The Standard Model and any theory that becomes its successor are quantum theories. This means that all interesting physical quantities are probabilistic by nature and it also explains why we are interested in performing scattering experiments. One of the properties of quantum mechanics is, that a state born as a proton does not have to end its days as a proton. Now particles cannot just decay to another particle just because it exists, it must also not violate any conservation laws. The conservation of energy is the reason we need large accelerators as all the energy that goes into a collision can be converted into all types of particles with a mass lower than the available energy. In that way we may turn stable protons into more exotic particles. Theoretically scattering experiments are connected to *scattering amplitudes* as they provide the probability for a certain process to take place. That is in essence what we are interested in: "Given two protons collide what are the possible outcomes and their probabilities?"

Quantum mechanics gives us a simple way to address this question. Imaging two particles that are initially free and which we intend to collide. We describe these two particles by a state $|p_1p_2\rangle_{in}$ that lives in some Hilbert space. The state of course has many more quantum numbers than just momentum, but we omit these for simplicity. After the particles have collided, we ask ourselves what the probability is to have produced the state $|q_1q_2\dots q_n\rangle_{out}$, or formulated in terms of expectation values

$$\left|{}_{out}\langle q_1q_2\dots q_{n-2}|\,|p_1p_2\rangle_{in}\right|^2 = \left|\langle q_1q_2\cdots q_{n-2}|\,S\,|p_1p_2\rangle\right|^2. \tag{1.1}$$

Here we have introduced the S-matrix which it is customary to decompose into

$$S = 1 + iT \tag{1.2}$$

since the matrix should reduce to the identity whenever no interactions take place. The scattering has to meet momentum conservation and therefore we find a δ-function of momentum conservation inside both the S- and the T-matrix. We now define the scattering amplitude, or just the *amplitude*, $\mathcal{A}_n$ by

$$|\langle q_1 q_2 \dots q_{n-2}| \, iT \, |p_1 p_2\rangle|^2 = \left|(2\pi)^4 \delta(p_1 + p_2 - q_1 - \dots - q_{n-1}) i\mathcal{A}_n\right|^2 . \tag{1.3}$$

How to obtain the scattering amplitude is the main object of concern in this thesis.

Space-Cone Gauge

Determining scattering amplitudes is in general a complicated affair. Almost always the strategy is to deploy perturbation theory and make an expansion in the coupling constant of the theory under consideration. For a long time the canonical approach was through Feynman diagrams. In gauge theories like QCD the different Feynman diagrams that contribute to a particular scattering amplitude is not unique, as the diagrams themselves are not gauge invariant. In ordinary gauges like Lorentz-Feynman or Gervais-Neuveu the number of terms that goes into calculating even the lowest order contributions quickly become unmanageable. There exists a class of gauges called axial gauges of which a particular one seems to be more natural to work in when dealing with Yang-Mills gauge theories. In a series of papers [1–3] in 1999 Chalmers and Siegel discussed methods for simplifying Feynman diagram calculations in various theories. Most interesting was the introduction of the space-cone gauge [1] which is an unconventional gauge in the sense that it introduces complex terms in the Yang-Mills lagrangian by specifying a spatial null-direction. Feynman rules simplify greatly and the amount of algebra needed to calculate specific diagrams become manageable. The space-cone gauge has never been studied very thoroughly and although the article by Chalmers and Siegel is correct, details regarding normalization and explicit Feynman rules have never been discussed. One of the aims of this thesis is to study the space-cone gauge and present the results of [1] in details. In particular we derive explicit Feynman rules and show how to apply these rules for four-, five- and six-point amplitude examples. We also extend the space-cone formalism to also include fermions, something that was not considered by Chalmers and Siegel originally.

For some time now people have stopped looking at conventional Feynman diagrams. This has to do with a breakthrough in the field of scattering amplitudes that started with Witten proposing a duality between tree-level amplitudes and string theory in the now seminal paper [4]. This was quickly followed by the CSW formalism [5] which has some reminiscence of the Feynman diagram method, but now the building blocks are no longer interaction vertices but MHV-amplitudes. Especially the BCFW recursion [6, 7] which relates higher order tree-level amplitudes to lower order ones has been a very powerful tool. So with all these recent developments why do we look at the space-cone formalism? One astonishing feature with the space-cone gauge is that the BCFW recursion relation is almost manifest. With the BCFW recursion we are able to factorize amplitudes into lower order amplitudes by setting internal propagators on-shell. If we try to set propagators on-shell in Feynman diagrams this usually changes the vertices as they are functions of momenta and hence the entire diagram changes. This is not the case in the space-cone gauge as the vertices are independent of one particular momentum component. The only thing that changes is the internal propagators. Vaman and Yao showed in [8] that the propagators shift in exactly the right way as to reproduce the original diagram. Therefore the diagrams of an amplitude themselves factorize into lower order diagrams exactly equal to the lower order amplitudes in the BCFW recursion. The proof of the propagator identity is rooted in Veltmans largest time equation [9] which is deeply connected to unitarity in quantum field theories. Following Vaman and Yao we derive this connection between the largest time equation and the BCFW recursion. We also briefly comment on the factorization of a six-point $NMHV$ amplitude as it introduces a non-trivial four-point interaction.

From Loops to Trees

That the largest time equation and the BCFW recursion is related is not apparent right away. The BCFW recursion relates tree-level amplitudes to other tree-level amplitudes whereas the largest time equation can be used to obtain a set of cutting equations which yields information about the imaginary part of a loop amplitude. Historically though, the BCFW recursion was discovered as Britto, Cachazo and Feng [10] were investigating one-loop amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory and it is therefore not so surprising that the BCFW recursion is connected to a result usually related to loops. In fact a connection between tree-amplitudes and loop-amplitudes have been know to exist for a very long time now. The cutting equations derived from the largest time equation gives a prescription of cutting a given loop-diagram into lower-order loop- and tree-diagrams to obtain the imaginary part of the original loop-diagram. This relates trees and loops indirectly for general loop amplitudes. Another connection between loops and trees goes back to Feynman in 1972 [11, 12] and is now known as Feynman's Tree Theorem. The tree theorem states that any loop-diagram can be expressed as a sum of phase-space integrated tree diagrams and it therefore directly relates trees and loops. For one-loop diagrams Veltman's cutting equations cut the diagrams into tree diagrams. It therefore seems plausible that there is a connection between the tree theorem and Veltman's cutting equations at least at the one-loop level. Such a connection would indirectly connect the tree theorem to the BCFW recursion further rooting the recursion in more fundamental results of quantum field theory. The later part of this thesis is devoted to establishing this connection.

Thesis Outline

In chapter 2 we review some modern techniques of amplitude calculations. We focus on the spinor helicity formalism and the colour-decomposition of tree-level amplitudes. We also derive some results regarding δ-functions and step functions. In chapter 3 we introduce the space-cone gauge and derive in detail the space-cone gauge fixed Yang-Mills lagrangian. We then proceed to derive Feynman rules from this lagrangian and give a number of examples of the use of these Feynman rules. After that we extend the Yang-Mills lagrangian to the QCD lagrangian. We gauge fix this lagrangian and derive Feynman rules for fermion vertices. We give an example of how to use the rules. In chapter 4 we derive the BCFW recursion relation and show in a simple example how it is used to calculate amplitudes. Then we show how the BCFW recursion can be obtained diagrammatically from the space-cone gauge with only the use of a simple propagator identity. We proceed to introduce the largest time equation and show how the propagator identity is rooted in it. The propagator identity is then generalized and we end the chapter by commenting on the factorization of a $NMHV$ six-point amplitude. Chapter 5 shows how the largest time equation can be used to derive a set of cutting equations. Then we introduce Feynman's Tree Theorem and discuss how it is connected to the cutting equations.

Part I

Setting the stage

— 2 —

Amplitudes, a Modern View

Amplitude calculations are at the core of perturbative quantum field theory. They provide a way to calculate various physical quantities, like cross sections, that can be measured by scattering experiments. Although many advances in amplitude calculations have been made over the past decades, problems still emerge at the one-loop level. For tree-level amplitudes there exists a number of both older an more recent results which enables us to present them in beautiful and compact ways. The two main reasons for these compact expressions are colour-ordering of amplitudes[13–17] and the spinor helicity formalism [13, 15–23] which we will review here. Even though we have a compact way of expressing the amplitudes, we still need to be able to calculate the actual amplitudes. Today this is mostly done through the CSW formalism or the BCFW recursion. We will not go discuss the former as we will instead derive a set of Feynman rules to apply in calculations. These Feynman rules and the BCFW recursion will be presented in the two chapters after this one. In the end of this chapter we give some general results regarding δ-functions and step functions that will come in handy when discussing loops.

2.1 Spinor Helicity Formalism

The spinor helicity formalism that we review here provides very compact expressions given by Lorentz invariant *spinor products* for amplitudes involving massless particles. It might seem problematic that we only consider massless particles but since gluons are theoretically massless and the lightest quarks are very light this is not at all a bad approximation in QCD. When dealing with scattering experiments where the energy involved is much larger than the masses of the individual particles the approximation becomes even better. The idea is to write objects that goes into amplitudes in terms of Weyl spinors. Weyl spinors belong to the fundamental $(0, 1/2)$ or $(1/2, 0)$ representations of the Lorentz group and as such they build up the representations of vectors $(1/2, 0) \otimes (0, 1/2)$ and the Dirac spinors $(1/2, 0) \oplus (0, 1/2)$. The decompositions of Dirac spinors into left- and right-handed Weyl spinors are known from most textbooks but we want to make this decomposition manifest for vectors as well and introduce some neat notation.

Weyl Spinors

Consider a left-handed and a right-handed Weyl spinor and denote them

$$\lambda_\alpha, \quad \bar{\lambda}^{\dot{\alpha}}. \tag{2.1}$$

We raise and lower undotted and dotted spinor indices by the matrix

$$C^{\alpha\beta} = C^{\dot{\alpha}\dot{\beta}} = -C_{\alpha\beta} = -C_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.2}$$

such that

$$\lambda^{\alpha} = \lambda_{\beta}C^{\beta\alpha}, \quad \lambda_{\alpha} = C_{\alpha\beta}\lambda^{\beta}, \quad \bar{\lambda}^{\dot{\alpha}} = \bar{\lambda}_{\dot{\beta}}C^{\dot{\beta}\dot{\alpha}}, \quad \bar{\lambda}_{\dot{\alpha}} = C_{\dot{\alpha}\dot{\beta}}\bar{\lambda}^{\dot{\beta}}. \tag{2.3}$$

We define Lorentz invariant spinor products by

$$\langle\lambda\rho\rangle = \lambda^{\alpha}\rho_{\alpha} = \lambda_1\rho_2 - \lambda_2\rho_1 \tag{2.4}$$

$$[\lambda\rho] = \bar{\lambda}_{\dot{\alpha}}\bar{\rho}^{\dot{\alpha}} = \bar{\lambda}^{\dot{1}}\bar{\rho}^{\dot{2}} - \bar{\lambda}^{\dot{2}}\bar{\rho}^{\dot{1}} \tag{2.5}$$

It is natural to introduce angle- and squarebrakets defined as

$$\lambda^{\alpha} = \langle\lambda| \,, \quad \lambda_{\alpha} = |\lambda\rangle \,, \quad \bar{\lambda}_{\dot{\alpha}} = [\lambda| \,, \quad \bar{\lambda}^{\dot{\alpha}} = |\lambda] \,. \tag{2.6}$$

The spinor products are obviously antisymmetric

$$\langle\lambda\rho\rangle = -\langle\rho\lambda\rangle \,, \quad [\lambda\rho] = -[\rho\lambda] \tag{2.7}$$

and therefore $\langle\lambda\lambda\rangle = [\lambda\lambda] = 0$. Since the two Weyl spinors are related to each other by complex conjugation we also have that

$$\langle\lambda\rho\rangle^* = [\rho\lambda] \,. \tag{2.8}$$

There are other useful relations for the spinor products but we will wait until the end of the section with writing them down.

Light-Cone Coordinates

Before we write down vectors in terms of Weyl spinors it is instructive to introduce light-cone coordinates. The transformation of a four-vector, $V^{\mu} = (v^0, v^1, v^2, v^3)$, into light-cone coordinates

$$V^u = (v, \bar{v}, v^+, v^-), \tag{2.9}$$

is given by the unitary transformation

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

such that

$$v = \frac{1}{\sqrt{2}}(v^1 + iv^2), \quad \bar{v} = \frac{1}{\sqrt{2}}(v^1 - iv^2), \quad v^+ = \frac{1}{\sqrt{2}}(v^0 + v^3), \quad v^- = \frac{1}{\sqrt{2}}(v^0 - v^3).$$

We use Latin letters $u, v, ...$ to indicate when an expression is to be evaluated in light-cone coordinates and $\mu, \nu, ...$ for ordinary Minkowski coordinates. For real four-vectors the component $\bar{v}$ is really the complex conjugated of v. Using this transformation, we find that the

Minkowski metric $(+ - --)$ in the new coordinates is given by

$$g_{uv} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.10}$$

or written in terms of the scalar product

$$V_u W^u = -v\bar{w} - \bar{v}w + v^+ w^- + v^- w^+. \tag{2.11}$$

When working with higher rank tensors like F_{uv} the notation of (2.9) is not very helpful. We therefore introduce indices in light-cone coordinates by $\{t, \bar{t}, +, -\}$ such that

$$F_{uv}F^{uw} = F_{+v}F^{-w} + F_{-v}F^{+w} - F_{tv}F^{\bar{t}w} - F_{\bar{t}v}F^{tw}$$

An index l[1] denotes $\{+, -\}$ such that

$$V_l W^l = v^+ w^- + v^- w^+.$$

The light-cone coordinates are in no way special besides the fact that we have explicitly introduced two complex components. These coordinates make calculations in the space-cone much more natural and are well-suited for expressing vectors in terms of Weyl spinors.

Vectors

To relate vectors to Weyl spinors we start by defining the Pauli matrices by

$$\sigma^0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\sigma^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \sigma^3 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.12}$$

with $\sigma^\mu_{\alpha\dot{\beta}} = (\sigma^0_{\alpha\dot{\beta}}, \sigma^1_{\alpha\dot{\beta}}, \sigma^2_{\alpha\dot{\beta}}, \sigma^3_{\alpha\dot{\beta}})$. When we are using index free notation we will also need the usual $\sigma^\mu = (\sigma^0, \sigma^1, \sigma^2, \sigma^3)$ and $\bar{\sigma}^\mu = (\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3)$ Then *any* four-vector, P^μ, can be written as

$$P_{\alpha\dot{\beta}} = P_\mu \sigma^\mu_{\alpha\dot{\beta}} = \frac{1}{\sqrt{2}}\begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} = \begin{pmatrix} p^+ & \bar{p} \\ p & p^- \end{pmatrix} \tag{2.13}$$

The matrix is hermitian for real vectors and it can therefore be decomposed into eigenvalues and eigenvectors. For massless (null) vectors, we know that the four-momentum must satisfy $P^2 = 0$. Since also $P^2 = 2\det(P_{\alpha\dot{\beta}})$ it must hold that $P_{\alpha\dot{\beta}}$ has a zero eigenvalue, and we may therefore express it in terms of its eigenvector with the non-trivial eigenvalue, as described in most textbooks on linear algebra. We therefore have

$$P_{\alpha\dot{\beta}} = \pm p_\alpha p_{\dot{\beta}}.$$

[1]The letter l stands for the light-cone directions.

Now p_α and $p_{\dot\beta}$ are Weyl spinors and complex conjugate of each other. The inner product between two vectors now takes the form

$$P_{\alpha\dot\beta}K^{\alpha\dot\beta} = p_\alpha p_{\dot\beta} k^\alpha k^{\dot\beta} = \langle kp\rangle\,[pk]\,. \tag{2.14}$$

Equivalently we have

$$P_{\alpha\dot\beta}K^{\alpha\dot\beta} = P^\mu \sigma_{\mu\alpha\dot\beta}\sigma_\nu^{\alpha\dot\beta}K^\nu = P^\mu g_{\mu\nu}K^\nu = P\cdot K \tag{2.15}$$

From which it follows that

$$\langle kp\rangle\,[pk] = P\cdot K = \frac{1}{2}(P+K)^2 \tag{2.16}$$

Dirac Spinors

Now consider a massless fermion obeying the Dirac equation

$$P_\mu\gamma^\mu U(P) = \not{P}U(P) = 0 \tag{2.17}$$

A solution to the Dirac equation can be decomposed into two Weyl spinors like

$$U = \begin{pmatrix} u_\alpha \\ u^{\dot\alpha} \end{pmatrix} \tag{2.18}$$

Decomposing the Dirac spinor into two Weyl spinors we have already specified that we want to work in the chiral representation of the γ-matrices. Here they take the simple block matrix form

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma^\mu} & 0 \end{pmatrix} \tag{2.19}$$

Inserting both into the Dirac equation we obtain

$$P_\mu\gamma^\mu U(p) = \begin{pmatrix} 0 & P_\mu\sigma^\mu_{\alpha\dot\beta} \\ P_\mu\sigma^{\mu\dot\alpha\beta} & 0 \end{pmatrix}\begin{pmatrix} u_\beta \\ u^{\dot\beta} \end{pmatrix} = \begin{pmatrix} p_\alpha p_{\dot\beta}u^{\dot\beta} \\ p^{\dot\alpha}p^\beta u_\beta \end{pmatrix} = 0 \tag{2.20}$$

If we choose $u_\beta = p_\beta$ and $u^{\dot\beta} = p^{\dot\beta}$ we see that the equation is satisfied, since $\langle pp\rangle = [pp] = 0$. The solutions to the massless Dirac equation is spanned by $\pm$ helicity solutions. We will relate these to the angle- and squarebrakets by

$$u^+(p) = p_\alpha = |p\rangle\,,\quad u^-(p) = p^{\dot\alpha} = |p] \tag{2.21}$$

and their complex conjugates as

$$\bar{u}^-(p) = p^\alpha = \langle p|\,,\quad \bar{u}^+(p) = p_{\dot\alpha} = [p| \tag{2.22}$$

where $\bar{u} = u^\dagger\gamma^0$. In principle the brackets should correspond to four component spinors, but since they are always of the form, where either the two first or the two last components are zero e.g.

$$u^+(p) = \begin{pmatrix} p_\alpha \\ 0 \end{pmatrix},\qquad u^-(p) = \begin{pmatrix} 0 \\ p_{\dot\alpha} \end{pmatrix} \tag{2.23}$$

the spinor products are just can just as well be defined from (2.21) and (2.22) to mimic vectors. There is one case where this definition is bad, and that is in expressions like $\langle p|\,\gamma^\mu\,|q]$ where a γ-matrix is found between two spinors. In such expression it is understood that the bracket are to be taken as four component spinors given by

$$\langle p| = \begin{pmatrix} p^\alpha & 0 \end{pmatrix}, \quad |p\rangle = \begin{pmatrix} p_\alpha \\ 0 \end{pmatrix}, \quad [p| = \begin{pmatrix} 0 & p_{\dot\alpha} \end{pmatrix}, \quad |p] = \begin{pmatrix} 0 \\ p^{\dot\alpha} \end{pmatrix} \tag{2.24}$$

which is consistent with (2.6) whenever we consider pure spinor products (like $\langle pq\rangle$ or $|p\rangle\,[q|$). Therefore, in our notation, $\langle p|\,\gamma^\mu\,|q] \to \langle p|\,\sigma^\mu{}_{\alpha\dot\beta}\,|q]$ since

$$\begin{pmatrix} p^\alpha & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot\beta} \\ \sigma^{\mu\dot\alpha\beta} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ q^{\dot\beta} \end{pmatrix} = p^\alpha \sigma^\mu_{\alpha\dot\beta} q^{\dot\beta} = \langle p|\,\bar\sigma^\mu\,|q]\,. \tag{2.25}$$

Lastly we should note, that for the massless Dirac equation the two different solutions for particles and antiparticles u and v can be taken to be the same but with opposite spin. We will therefore only work with the u spinor. Lastly we will need to know the γ-matrices in light-cone coordinates. They are given by

$$\gamma^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \bar\gamma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^+ = \begin{pmatrix} 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.26}$$

Polarization Vectors

Anticipating Yang-Mills theory we also want to describe external polarization states in terms of spinor products. In general there are a state for both positive and negative helicity and for in- and outgoing particles. Since we will always assume external particles to be outgoing we will only concern ourselves with the two different helicity states. They are given by

$$\epsilon_+(p)^\mu = -\frac{\langle r_+|\,\gamma^\mu\,|p]}{\langle r_+ p\rangle}, \qquad \epsilon_-(p)^\mu = -\frac{[r_-|\,\gamma^\mu\,|p\rangle}{[r_- p]} \tag{2.27}$$

Here $r_\pm$ is an arbitrary reference vector not collinear with p. This freedom to chose a reference vector corresponds to an on-shell gauge freedom. We will show that this is true in a moment. First we want to determine $\epsilon_{+\alpha\dot\beta}$ by calculating $\langle r_+|\,\sigma^\mu{}_{\gamma\dot\sigma}\,|p]\,\sigma_{\mu\alpha\dot\beta} = \langle r_+|\,\sigma^\mu{}_{\gamma\dot\sigma}\sigma_{\mu\alpha\dot\beta}\,|p]$ according to (2.13). To do so we will need the Fierz identity

$$\sigma^\mu_{\gamma\dot\sigma}\sigma_{\mu\alpha\dot\beta} = -C_{\alpha\gamma}C_{\dot\beta\dot\sigma} \tag{2.28}$$

The derivation of this Fierz' identity can be found in the literature. With this identity at hand, the calculation is trivial

$$\begin{aligned}\langle r_+| \sigma^\mu |p] \sigma_\mu &= \langle r_+| \sigma^\mu \sigma_\mu |p] \\ &= -r_+^\gamma C_{\alpha\gamma} C_{\dot\beta\dot\sigma} p^{\dot\sigma} \\ &= - |r_+\rangle [p|\end{aligned} \tag{2.29}$$

A similar calculation can be done for $[r_-| \gamma^\mu |p\rangle$, from which it follows that

$$\epsilon_+(p)_{\alpha\dot\beta} = \frac{|r_+\rangle [p|}{\langle r_+ p\rangle}, \qquad \epsilon_-(p)_{\alpha\dot\beta} = \frac{|p\rangle [r_-|}{[p r_-]}. \tag{2.30}$$

As is customary in the literature we will not argue that (2.30) are really the polarization vectors we are looking for. Instead we show that they fulfil the common properties of polarization vectors. The two reference vectors are each others complex conjugate if we choose the same reference vector for each of them. They are also transverse to P

$$\begin{aligned}\epsilon_+(p)_{\alpha\dot\beta} P^{\alpha\dot\beta} &= \frac{|r_+\rangle [p|}{\langle r_+ p\rangle} |p] \langle p| \propto [pp] = 0 \\ \epsilon_-(p)_{\alpha\dot\beta} P^{\alpha\dot\beta} &= \frac{|p\rangle [r_-|}{[p r_-]} |p] \langle p| \propto \langle pp\rangle = 0\end{aligned} \tag{2.31}$$

and they are normalized according to

$$\begin{aligned}\epsilon_+(p)_{\alpha\dot\beta} \epsilon_-(p)^{\alpha\dot\beta} &= \frac{|r_+\rangle [p|}{\langle r_+ p\rangle} \frac{|r_-] \langle p|}{[p r_-]} = \frac{\langle p r_+\rangle [p r_-]}{\langle r_+ p\rangle [p r_-]} = -1 \\ \epsilon_+(p)_{\alpha\dot\beta} \epsilon_+(p)^{\alpha\dot\beta} &= \frac{|r_+\rangle [p|}{\langle r_+ p\rangle} \frac{|p] \langle r_+|}{\langle r_+ p\rangle} = \frac{\langle pp\rangle [r_+ r_+]}{\langle r_+ p\rangle [p r_-]} = 0\end{aligned} \tag{2.32}$$

That the choice of reference vector is just a gauge choice can be seen by looking at what happens if we shift the reference vector

$$\begin{aligned}\epsilon_+(p)_{\alpha\dot\beta} - \epsilon'_+(p)_{\alpha\dot\beta} &= \frac{|r_+\rangle [p|}{\langle r_+ p\rangle} - \frac{|r'_+\rangle [p|}{\langle r'_+ p\rangle} \\ &= \frac{|r_+\rangle \langle r'_+ p\rangle [p| - |r'_+\rangle \langle r_+ p\rangle [p|}{\langle r_+ p\rangle \langle r'_+ p\rangle} \\ &= \frac{2 \langle r'_+ r_+\rangle}{\langle r_+ p\rangle \langle r'_+ p\rangle} P_{\alpha\dot\beta}.\end{aligned} \tag{2.33}$$

The reference vector is shifted by an amount proportional to the momentum. When dotted into an on-shell amplitude it will give zero and a shift in reference momentum is therefore the same as an on-shell gauge transformation. In general we are free to choose different reference vectors for different polarization states. However, it is always a good idea to choose one reference vector for all positive helicity states and another for all negative helicity states. This observation will become crucial for our discussion of the space-cone gauge. In section 3.1 we will construct explicit reference vectors to form a spinor basis in which all quantities that we have derived here can be expressed. This spinor basis will make a gauge choice apparent for us.

Spinor Helicity Identities

We end this section by giving the most important identities involving spinors. We will omit those that are not used in the thesis and we will not state identities that have already been given. When dealing with many particles it is a good idea to change notation such that $\langle p_i| = \langle i|$ and similarly for the square bracket. We will do this throughout the thesis. Two of the most important identities are the two Schouten identities

$$[ij]\,[kl] + [ik]\,[lj] + [il]\,[jk] = 0 \tag{2.34}$$

and

$$\langle ij\rangle\,\langle kl\rangle + \langle ik\rangle\,\langle lj\rangle + \langle il\rangle\,\langle jk\rangle = 0 \tag{2.35}$$

where of course the second is just the complex conjugate of the first. The identities follow from the antisymmetry of the spinor products. For a set of momenta that satisfies

$$\sum_i p_i = \sum_i |i\rangle\,[i| = 0 \tag{2.36}$$

we also have the following form of momentum conservation which is obtained by dotting the above equation with $|l]\,\langle k|$

$$\sum_i \langle ki\rangle\,[il] = 0. \tag{2.37}$$

Notice that the two terms where $i = k, l$ are zero and hence can be omitted from the sum. We will also introduce the following notation

$$\langle i|\sum p\,|j] = \sum \langle ip\rangle\,[pj] \tag{2.38}$$

and

$$P_{ij}^2 = P_i \cdot P_j \qquad P_{ijk}^2 = P_{ij}^2 + P_{ik}^2 + P_{jk}^2. \tag{2.39}$$

Lastly the relation that

$$\langle ij\rangle^* = [ji]\,. \tag{2.40}$$

is only true in real Minkowski space. If momenta are taken to be complex the two different spinors become independent of each other.

2.2 Colour-Ordered Amplitudes

In a non-abelian gauge theory scattering amplitudes tend to become very difficult to calculate even at tree-level due to a high amount of algebra dealing with structure constants and generators of the underlying Lie group. The purpose of this section is to introduce *colour-ordered* tree-level amplitudes. The idea is to decompose the full amplitude $\mathcal{A}_n$ into gauge invariant subamplitudes A_n independent of colour indices. The gauge group we will be interested in is $SU(N_c)$ where the special case $N_c = 3$ is QCD if we couple it to fermions. We will in general keep the nomenclature of QCD and refer to quarks and gluons. The generators of $SU(N_c)$ are $N_c \times N_c$ traceless matrices $(T^a)_i^{\bar{j}}$ satisfying

$$[T^a, T^b] = i f^{abc} T^c, \qquad \mathrm{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab} \tag{2.41}$$

where f^{abc} are the structure constants of the theory. Notice the unusual definition with $\frac{1}{2}\delta^{ab}$. In all literature on colour-ordered amplitudes the convention is to take δ^{ab}. This is a bit annoying as most modern literature on Yang-Mills theory has the definition with a factor $\frac{1}{2}$. As we will be working with this convention for Yang-Mills we have worked out all results in terms of this. Therefore our derived results will have a number of extra factors 2 to compensate and one should be aware of this if comparing with other literature in the field. The indices can be raised and lowered as we like, and we will therefore always understand a sum if an index is repeated. The structure constants are given in terms of the generators as

$$f^{abc} = -\frac{i}{2}\left(\mathrm{Tr}[T^a T^b T^c] - \mathrm{Tr}[T^a T^c T^b]\right) \tag{2.42}$$

The generators also obey the following Fierz identity

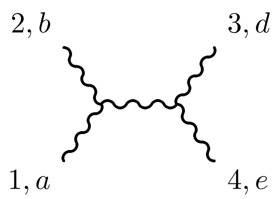
$$(T^a)_{i_1}{}^{\bar{j}_1}(T^a)_{i_2}{}^{\bar{j}_2} = \frac{1}{2}\delta_{i_1}{}^{\bar{j}_2}\delta_{i_2}{}^{\bar{j}_1} - \frac{1}{2N_c}\delta_{i_1}{}^{\bar{j}_1}\delta_{i_2}{}^{\bar{j}_2} \tag{2.43}$$

The Colour-Decomposition of Theories with Feynman Diagrams

To derive the colour-decomposition of an amplitude we will assume that the amplitude itself can be obtained from a set of Feynman rules and diagrams which, anticipating Yang-Mills theory, is a very sane assumption. Particles living in the adjoint representation of $SU(N_c)$ carries an adjoint index $a = 1, 2\ldots, N_c^2-1$ and will contribute with a T^a to a Feynman diagram. Particles living in the fundamental representation carries a fundamental index i or $\bar{j} = 1, 2\ldots, N_c$. In particular a generic Feynman diagram will have a factor of $(T^a)_i{}^{\bar{j}}$ for each gluon-quark-antiquark vertex, a factor of f^{abc} for each gluon three-vertex and a contraction $f^{abc}f^{cde}$ for each gluon four-vertex. When we derive Feynman rules later on we will see that this is true. The important observation is that all structure constants can be expressed in terms of single traces over the generators. We have already established this for a single structure constant (2.42) but it is also true for products of more structure constants. The colour-decomposition of a purely gluonic amplitude takes the following form

$$\mathcal{A}_n^{\mathrm{tree}}(1^{\lambda_1}, 2^{\lambda_2}, \ldots, n^{\lambda_n}) = \left(\frac{g}{2}\right)^{n-2} \sum_{\sigma(2,3,\ldots,n)} \mathrm{Tr}[T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}\cdots T^{a_{\sigma(n)}}]A_n(1^{\lambda_1}, \sigma(2^{\lambda_2}), \ldots, \sigma(n^{\lambda_n})) \tag{2.44}$$

where λ_i is the helicity of particle i and we use $j = 1, 2, \ldots, n$ as short-hand notation for p_j. g is the gauge coupling and the sum is over all permutations of $(2, 3, \ldots, n)$. Notice, that since the trace is cyclic this is the same as all non-cyclic permutations of the legs $(1, 2, \ldots, n)$. The object A_n is called the partial amplitude or the subamplitude. We will not always distinguish between what we call an amplitude or a subamplitude since from now on we are only concerned with calculating subamplitudes. To show how the colour-decomposition works out we look at the four-gluon diagram which is proportional to $f^{abc}f^{cde}$



(2.45)

Evaluating yields

$$\begin{aligned} f^{abc}f^{cde} &= -\frac{1}{4}\left(\mathrm{Tr}[T^aT^bT^c]-\mathrm{Tr}[T^aT^cT^b]\right)\left(\mathrm{Tr}[T^cT^dT^e]-\mathrm{Tr}[T^cT^eT^d]\right) \\ &= -\frac{1}{4}\Big[\mathrm{Tr}[T^aT^bT^c]\mathrm{Tr}[T^cT^dT^e]-\mathrm{Tr}[T^aT^bT^c]\mathrm{Tr}[T^cT^eT^d] \\ &\quad -\mathrm{Tr}[T^aT^cT^b]\mathrm{Tr}[T^cT^dT^e]+\mathrm{Tr}[T^aT^cT^b]\mathrm{Tr}[T^cT^eT^d]\Big] \end{aligned} \tag{2.46}$$

The traces can be written out using (2.43) and (2.41)

$$\begin{aligned} \mathrm{Tr}[T^aT^bT^c]\mathrm{Tr}[T^cT^eT^d] &= (T^a)_{i_1}{}^{\bar{j}_1}(T^b)_{\bar{j}_1}{}^{\bar{j}_2}(T^c)_{\bar{j}_2}{}^{i_1}(T^c)_{i_2}{}^{\bar{j}_3}(T^e)_{\bar{j}_3}{}^{\bar{j}_4}(T^d)_{\bar{j}_4}{}^{i_2} \\ &= ((T^c)_{\bar{j}_2}{}^{i_1}(T^c)_{i_2}{}^{\bar{j}_3})(T^a)_{i_1}{}^{\bar{j}_1}(T^b)_{\bar{j}_1}{}^{\bar{j}_2}(T^e)_{\bar{j}_3}{}^{\bar{j}_4}(T^d)_{\bar{j}_4}{}^{i_2} \\ &= \frac{1}{2}\mathrm{Tr}[T^aT^bT^eT^d]-\frac{1}{2N_c}\mathrm{Tr}[T^aT^b]\mathrm{Tr}[T^eT^d] \\ &= \frac{1}{2}\mathrm{Tr}[T^aT^bT^eT^d]-\frac{1}{8N_c}\delta^{ab}\delta^{de}. \end{aligned} \tag{2.47}$$

The three other terms can be evaluated in a similar way obtaining

$$\mathrm{Tr}[T^aT^bT^c]\mathrm{Tr}[T^cT^dT^e] = \frac{1}{2}\mathrm{Tr}[T^aT^bT^dT^e]-\frac{1}{8N_c}\delta^{ab}\delta^{de} \tag{2.48}$$

$$\mathrm{Tr}[T^aT^cT^b]\mathrm{Tr}[T^cT^dT^e] = \frac{1}{2}\mathrm{Tr}[T^bT^aT^dT^e]-\frac{1}{8N_c}\delta^{ab}\delta^{de} \tag{2.49}$$

$$\mathrm{Tr}[T^aT^cT^b]\mathrm{Tr}[T^cT^eT^d] = \frac{1}{2}\mathrm{Tr}[T^bT^aT^eT^d]-\frac{1}{8N_c}\delta^{ab}\delta^{de} \tag{2.50}$$

(2.51)

Altogether we see that the product can be written out in terms of single traces only

$$f^{abc}f^{cde} = -\frac{1}{8}\left[\mathrm{Tr}[T^aT^bT^dT^e]-\mathrm{Tr}[T^aT^bT^eT^d]-\mathrm{Tr}[T^bT^aT^dT^e]+\mathrm{Tr}[T^bT^aT^eT^d]\right] \tag{2.52}$$

It should be obvious that this procedure can be generalized to any gluonic tree-amplitude to yield single traces as in(2.44). If an amplitude contains a quark and an antiquark the colour-ordered amplitude is given by

$$\mathcal{A}_n^{\mathrm{tree}} = 2\left(\frac{g}{2}\right)^{n-2}\sum_{\sigma(2,3,\ldots,n)}[T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}\cdots T^{a_{\sigma(n)}}]_{\bar{j}_1}{}^{i_2}A_n(1_{\bar{q}}^{\lambda_1},2_q^{\lambda_2},\sigma(3^{\lambda_3})\ldots,\sigma(n^{\lambda_n})) \tag{2.53}$$

For amplitudes with more quarks there exists similar expressions that can be found in [15]. We will not be looking at other types of amplitudes than those with only gluons or one quark/antiquark pair. With the colour-ordered amplitudes follows a set of colour-ordered Feynman rules to obtain the subamplitude. We will derive space-cone colour-ordered Feynman rules for Yang-Mills theory and QCD in the next chapter. The rules for Lorentz-Feynman gauge can be found in some modern textbooks and the reader is encouraged to look in the almost classical set of lecture notes by Lance Dixon [16].

One might ask if the price we pay for the colour-decomposition isn't quiet high? It seems as if for each amplitude $\mathcal{A}_n$ we want to calculate, we have to calculate $(n-1)!$ subamplitudes. What saves us is a number of relations among the subamplitudes. Two of the simplest relations are that the subamplitudes are cyclic invariant and have reflection symmetry

$$A_n(1,2,3,\ldots,n) = A_n(2,3,\ldots,n,1) \tag{2.54}$$

$$A_n(1,2,3,\ldots,n) = (-1)^n A_n(n,n-1,\ldots,2,1). \tag{2.55}$$

Other non-trivial identities exist to further reduce the number of independent subamplitudes. Two important types are the KK-relations [24, 25] and the BCJ-relations [26, 27] which together reduce the number of independent subamplitudes to $(n-3)!$.

MHV Amplitudes

We promised at the beginning of this chapter, that the spinor helicity formalism and the colour-ordered amplitudes would allow very simple and compact expressions to be formed. Usually we denote the different kinds of subamplitudes by their helicity states. It can be shown that the gluonic amplitudes with all $+$ $(-)$ and the amplitudes with one $+$ $(-)$ are zero

$$A_n(1^\pm,2^\pm,\ldots,n^\pm) = 0 \tag{2.56}$$

$$A_n(1^\pm,2^\pm,\ldots,i^\mp,\ldots,n^\pm) = 0. \tag{2.57}$$

The first non-zero amplitude is the amplitude with exactly two negative helicity gluons. We call this the *maximally helicity violating* (MHV) amplitude. Parke and Taylor conjectured in 1986 [28] that this amplitude has a very simple form. The expression was proven to be correct in 1988 by Berends and Giele [29]

$$A_n(1^+,2^+,\ldots,i^-,\ldots,j^-,\ldots,n^+) = 2^{n-2}\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle} \tag{2.58}$$

There exists a similar form for the gluonic amplitude with exactly two positive helicity gluons.

$$A_n(1^-,2^-,\ldots,i^+,\ldots,j^+,\ldots,n^-) = (-1)^n 2^{n-2}\frac{[ij]^4}{[12]\,[23]\cdots[(n-1)n]\,[n1]} \tag{2.59}$$

These amplitudes are called *googly-MHV*-amplitudes and in real Minkowski space it is just the complex conjugated of the MHV-amplitude. Amplitudes with exactly three negative helicity gluons are called next-to maximally helicity violating amplitudes, abbreviated $NMHV$. In general we will speak of N^mMHV amplitudes or googly-N^mMHV amplitudes. Unfortunately there exists no compact way of representing these amplitudes all at once. For the MHV amplitudes with an quark/antiquark pair we have the following two compact expressions for which a proof can be found in [17, 30]. These are given by

$$A_n(1_q^-,2^+,\ldots,i^-,\ldots,(n-1)^+,n_{\bar q}^+) = 2^{n-3}\frac{\langle 1i\rangle^3\langle ni\rangle}{\langle 12\rangle\langle 23\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle} \tag{2.60}$$

and

$$A_n(1_q^-,2^-,\ldots,i^+,\ldots,(n-1)^-,n_{\bar q}^+) = (-1)^{n-1}2^{n-3}\frac{[1i]\,[ni]^3}{[12]\,[23]\cdots[(n-1)n]\,[n1]} \tag{2.61}$$

This ends our short review of modern amplitudes.

2.3 Some Useful Results

We end this chapter by proving two δ-function and step function identities. We will need these throughout the thesis and instead of introducing them as we go, it seems more elegant to present them here. The results can be found in many textbooks but since they are so important to us we will derive them here. We define the δ-function by

$$\int_{-\infty}^{+\infty} f(x)\delta(x-x')\mathrm{d}x = \int f(x)\delta(x-x')\mathrm{d}x = f(x') \tag{2.62}$$

Let $g(x)$ be a function that satisfies the equation

$$g(x_i) = 0$$

for a set of N points $\{x_i\}$. We want to prove that

$$\delta(g(x)) = \sum_{i=1}^{N} \frac{\delta(x-x_i)}{|g'(x_i)|}. \tag{2.63}$$

If g is a increasing piecewise bijective function the following is true

$$\begin{aligned}
\int f(x)\delta(g(x))\mathrm{d}x &= \int f(x)\delta(g(x))\frac{\mathrm{d}x}{\mathrm{d}g(x)}\mathrm{d}g(x) \\
&= \int f(x)\frac{\delta(g(x))}{g'(x)}\mathrm{d}g(x) \\
&= \frac{f(x_1)}{g'(x_1)} + \frac{f(x_2)}{g'(x_2)} + \dots \\
&= \int f(x)\sum_{i=1}^{N}\frac{\delta(x-x_i)}{g'(x_i)}\mathrm{d}x
\end{aligned} \tag{2.64}$$

In the case were $g(x)$ is decreasing, we pick up a sign from the substitution, and hence the absolute value in (2.63).

The step function can be defined as the integral of a δ-function where we now express the δ-function by its Fourier transform.

$$\theta(x) = \int_{-\infty}^{x} \mathrm{d}\tau\delta(\tau), \qquad \delta(\tau) = \int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\mathrm{e}^{-ik\tau}. \tag{2.65}$$

Inserting and performing the τ-integral yields

$$\begin{aligned}
\theta(x) = \int_{-\infty}^{x}\mathrm{d}\tau\delta(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k\int_{-\infty}^{x}\mathrm{d}\tau\mathrm{e}^{-ik\tau}, \qquad x>0 \\
&= \frac{-1}{2\pi i}\int_{-\infty}^{\infty}\mathrm{d}k\left[\frac{\mathrm{e}^{-ik\tau}}{k}\right]_{-\infty}^{x} \\
&= \frac{-1}{2\pi i}\int_{-\infty}^{\infty}\mathrm{d}k\frac{\mathrm{e}^{-ikx}}{k}.
\end{aligned} \tag{2.66}$$

For positive x this expression should be 1 and 0 for negative. We could specify a contour but in the spirit of Feynman we choose an ϵ-prescription.

$$\theta(x) = \lim_{\epsilon \to 0^+} \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \mathrm{d}k \frac{\mathrm{e}^{-ikx}}{k + i\epsilon} \tag{2.67}$$

For $\theta(-x)$ we may shift the sign of the exponential or pick the negative pole. Closing the contour in the negative half-plane we pick up a minus sign as well.

$$\theta(-x) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathrm{d}k \frac{\mathrm{e}^{-ikx}}{k - i\epsilon} \tag{2.68}$$

For all further calculations we will let the ϵ-limit be understood.

Part II

Trees and Space-Cone

— 3 —

Space-Cone Gauge for QCD

When working with gauge-theories it is well-known that different gauges are better suited for some calculations that others. Some gauges keep the lagrangian manifestly Lorentz invariant whereas others provide simpler Feynman rules. Most gauges are what we call covariant gauges but there also exists a class of non-covariant gauges of which the axial gauges are a special type. These non-covariant gauges are discussed in detail in [31]. The most famous of the axial gauges is the light-cone gauge which one obtains by setting either one of $a^{\pm}$ equal to zero, where $a^{\pm}$ are the light-cone components of the gauge field A_{μ}. What we will do in this chapter is to mimic the light-cone gauge by setting $a = 0$ (or equivalently $\bar{a}$). This means that we are going away from real Minkowski space as a is complex valued. This direction in space is, as opposed to the $x^{\pm}$ direction, space-like but since it is a complex direction we will see that it is still a null-direction just as $x^{\pm}$.

This gauge was first introduced by Chalmers and Siegel [1] where they dubbed it *space-cone* gauge because of its apparent parallel to the light-cone gauge. Our goal is to gauge fix the Yang-Mills lagrangian with the space-cone gauge and obtain colour-ordered Feynman rules from it following [1]. From there we extend our results to include fermionic matter thereby describing the full theory of QCD in space-cone gauge. We will motivate why the space-cone gauge is a natural gauge choice in the spinor helicity formalism by constructing an explicit spinor basis. With this basis it will be clear that the space-cone gauge is almost manifest in the spinor helicity formalism.

3.1 The Setup

Before deriving the space-cone lagrangian for Yang-Mills we very briefly review the most important properties of Yang-Mills theory. Let the Yang-Mills lagrangian be given by

$$\mathcal{L}_{YM} = -\frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu a} = \frac{1}{2g^2} \mathrm{Tr}\left[\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right] \tag{3.1}$$

where

$$\mathcal{F}_{\mu\nu} = -igT^a F^a_{\mu\nu}, \qquad [T^a, T^b] = i f_{abc} T^c, \qquad \mathrm{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab} \tag{3.2}$$

and

$$F^c_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g f_{abc} A^a_\mu A^b_\nu.$$

We define the covariant derivative, ∇_μ, in the fundamental representation by

$$\nabla_\mu = \partial_\mu - igA^a_\mu T^a$$

and in the adjoint representation we hence have

$$\nabla_\mu \phi_a = \partial_\mu \phi_a - igA^b_\mu (T^b)_{ac}\phi_c = \partial_\mu \phi_a + gf_{abc}A^b_\mu \phi_c.$$

The classical equations of motion for this lagrangian is

$$\partial^\mu F^a_{\mu\nu} + gf_{abc}A^{\mu b}F^c_{\mu\nu} = \nabla^\mu F^a_{\mu\nu} = 0$$

when no sources are present. This is of course no more than a simple rewriting of the Euler-Lagrange equations yielding

$$\frac{\partial \mathcal{L}}{\partial A^a_\nu} - \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^a_\nu)} = 0. \tag{3.3}$$

Defining the differential operator

$$\mathcal{D}^{\nu a} = \frac{\partial}{\partial A^a_\nu} - \partial^\mu \frac{\partial}{\partial(\partial^\mu A^a_\nu)}$$

we have

$$\begin{aligned}
\mathcal{D}^{\nu a}\mathcal{L}_{YM} &= -\frac{1}{4}\mathcal{D}^{\nu a}(F^c_{\sigma\rho}F^{\sigma\rho c}) \\
&= -\frac{1}{2}\mathcal{D}^{\nu a}(F^{\sigma\rho c})F^c_{\sigma\rho} \\
&= -\frac{1}{2}(gf^{dbc}A^{\sigma d}\delta^{\nu\rho}\delta^{ab} + gf^{dbc}A^{\rho b}\delta^{\nu\sigma}\delta^{ad} - \partial^\mu(\delta^{\mu\sigma}\delta^{\nu\rho} - \delta^{\mu\rho}\delta^{\nu\sigma})\delta^{ac})F^c_{\sigma\rho} \\
&= (gf^{abc}A^{\mu b} + \partial^\mu \delta^{ac})F^c_{\mu\nu} = \nabla^\mu F^a_{\mu\nu} = 0.
\end{aligned} \tag{3.4}$$

This form of the equations of motions will prove useful when eliminating degrees of freedom in the gauge field A_μ.

Constructing a Spinor Basis

When we introduced polarization vectors in terms of spinor helicity, we saw that it was necessary to specify a reference vector for both the positive and negative helicity polarization vector. In principle we do not have to choose the same two every time, but in practice this turns out to be smart. We therefore construct these two reference momenta explicitly, and use them to write a properly normalized spinor basis for any vector in light-cone coordinates. Having done so, we will see that we can turn our two reference momenta into a gauge choice for the gauge field A_μ. We already know that the choosing reference vectors amounts to a gauge choice, but what is not apparent is, that this gauge choice can also be applied directly into the lagrangian. The reference momenta is chosen such, that they correspond to two particles travelling along the $i = 3$ axis but in opposite directions. We take the negative helicity particle to travel in the positive direction and the positive helicity particle to be travelling along the negative direction. This amount to only having momentum component along the "+"- or "−"-light-cone direction respectively. Explicitly we have

$$P_\ominus = \begin{pmatrix} p^+_\ominus & 0 \\ 0 & 0 \end{pmatrix} = |+\rangle\,[+| = \begin{pmatrix} \sqrt{p^+_\ominus} \\ 0 \end{pmatrix} \begin{pmatrix} \sqrt{p^+_\ominus} & 0 \end{pmatrix} \tag{3.5}$$

if it has negative helicity and

$$P_\oplus = \begin{pmatrix} 0 & 0 \\ 0 & p_\oplus^- \end{pmatrix} = |-\rangle\,[-| = \begin{pmatrix} 0 \\ \sqrt{p_\oplus^-} \end{pmatrix} \begin{pmatrix} 0 & \sqrt{p_\oplus^-} \end{pmatrix} \tag{3.6}$$

if it has positive helicity in accordance with (2.13). Note here the counter-intuitive definition that we take the positive helicity momentum to be in the negative light-cone direction and vice versa. The reason for doing so is, that the positive helicity reference spinor is to be used with a negative helicity gluon. With this choice, the external polarization factors become easier to remember since we will use $|+\rangle$ with positive helicity gluons and $|-\rangle$ negative helicity gluons. Since $P_\oplus \cdot P_\ominus = p_\oplus^- p_\ominus^+ = \langle -+\rangle\,[+-]$ we may chose the following normalized basis for any matrix of the form (2.13)

$$\frac{|+\rangle\,[+|}{p_\ominus^+}, \quad \frac{|-\rangle\,[-|}{p_\oplus^-}, \quad \frac{|+\rangle\,[-|}{\sqrt{\langle -+\rangle\,[+-]}}, \quad \frac{|-\rangle\,[+|}{\sqrt{\langle -+\rangle\,[+-]}} \tag{3.7}$$

in which four-vectors take the form

$$P = p^+ \frac{|+\rangle\,[+|}{p_\ominus^+} + p^- \frac{|-\rangle\,[-|}{p_\oplus^-} + \bar{p}\frac{|+\rangle\,[-|}{\sqrt{\langle -+\rangle\,[+-]}} + p\frac{|-\rangle\,[+|}{\sqrt{\langle -+\rangle\,[+-]}}. \tag{3.8}$$

From this we can explicitly calculate the light-cone components of any four-vector. Using the definition $P = |p\rangle\,[p|$ and the property that $\langle qq\rangle = [qq] = 0$ we arrive at

$$p^+ = \frac{\langle -p\rangle\,[p-]}{p_\oplus^-}, \quad p^- = \frac{\langle +p\rangle\,[p+]}{p_\ominus^+}, \quad \bar{p} = \frac{\langle p-\rangle\,[p+]}{\sqrt{\langle -+\rangle\,[+-]}}, \quad p = \frac{\langle p+\rangle\,[p-]}{\sqrt{\langle -+\rangle\,[+-]}}. \tag{3.9}$$

Now to the polarization vectors. The two helicity vectors that we have constructed may be used as reference vectors in (2.30). If we set $|r_+\rangle = +$ and $[r_-| = [-|$ we have

$$\epsilon_+(p)_{\alpha\dot\beta} = \frac{|+\rangle\,[p|}{\langle +p\rangle}, \qquad \epsilon_-(p)_{\alpha\dot\beta} = \frac{|p\rangle\,[-|}{[p-]}. \tag{3.10}$$

We will find later on, that we are only concerned with the $(\epsilon)^\pm$ components of the polarization vector and we therefore already state these here. Using (3.9) we get

$$(\epsilon_+)^+ = \frac{\langle -+\rangle\,[p-]}{\langle +p\rangle\,p_\oplus^-}, \quad (\epsilon_+)^- = 0, \quad (\epsilon_-)^+ = 0, \quad (\epsilon_-)^- = \frac{\langle +p\rangle\,[-+]}{[p-]\,p_\ominus^+}. \tag{3.11}$$

For what is to come remember this: *The positive (negative) helicity polarization vector can be identified with the positive (negative) light-cone direction*. Following the logic of [1] we notice an inherent feature with the spinor helicity formalism: There exists a direction in space transverse to both positive and negative helicity gluons spanned by

$$N = |+\rangle\,[-|\,. \tag{3.12}$$

Since $A_\mu \propto \epsilon_\mu$ we may equivalently write

$$N \cdot A = \langle +|\,|A\rangle\,[A|\,|-] = a = 0 \tag{3.13}$$

Now this condition must hold no matter what gauge we choose for A_μ and especially it must hold if we choose this as our gauge. That this is a legitimate gauge choice we will not prove here but rather refer to [31] for details regarding axial gauge choices. The gauge is space-like since the components of N are in the $i = 1, 2$ direction. At the same time N is a null vector, since $\langle qq \rangle = [qq] = 0$. This is of course a result of constructing N from spinors, which are manifestly "massless". In real Minkowski space it is not possible to choose a direction in space which is also a null direction, you always need to include time. The price we pay for choosing the space-cone gauge is that we have complexified the gluon field. Many recent results in the research field of amplitudes comes directly from considering the amplitudes as functions of complex momenta. We should therefore not be concerned that this gauge will lead us into problems , on the contrary we should rather hope that we may obtain some recent results with ease since our theory is naturally complex.

The space-cone gauge, just like all other gauges, leaves some gauge redundancy since (3.13) is invariant under covariant gauge transformations involving the coordinates $x^\pm$ and x [31, 32]. This can be seen since $\partial\alpha(x^+, x^-, x) = 0$. We still need this gauge redundancy to pick reference vectors since even though we have constructed explicit reference vectors, we have yet to actually identify them with physical vectors.

Our next goal is to gauge-fix the Yang-Mills lagrangian using the space-cone gauge.

3.2 Yang-Mills Lagrangian in Space-Cone Gauge

We set out to rewrite (3.1) in terms of (2.9) using the gauge (3.13). That is, we set

$$a = 0.$$

To derive the space-cone gauge lagrangian requires a bit of tedious algebra. There are in principles four logical steps in the derivation. 1) We write out the different terms of $F^a_{\mu\nu}$ in terms of light-cone coordinates (2.9). 2) We gauge fix each of these terms (trivial). 3) We eliminate $\bar{a}$ by its equation of motion. 4) We write out the full lagrangian reorganizing it to our liking. When equalities are not purely algebraic, we have applied the chain rule for differentiation a number of times. We also use that total derivatives in the lagrangian do not contribute in the action, and hence we are allowed to remove them from the lagrangian. The derivation of the space-cone gauge lagrangian was not done in detail in [1], so it is also our goal to present the derivation in full detail.

In order to tidy up the notation we only write the group index when there is a reason for it, usually when a structure constant is in the vicinity. Otherwise it is to be understood implicitly. We start by expanding $F_{\mu\nu}F^{\mu\nu}$ in terms of the new coordinates

$$\begin{aligned} F_{uv}F^{uv} &= -F^t_v F^{\bar{t}v} - F^{\bar{t}}_v F^{tv} + F^+_v F^{-v} + F^-_v F^{+v} \\ &= F^{t\bar{t}}F^{\bar{t}t} - F^{t+}F^{\bar{t}-} - F^{t-}F^{\bar{t}+} \\ &\quad + F^{\bar{t}t}F^{t\bar{t}} - F^{\bar{t}+}F^{t-} - F^{\bar{t}-}F^{t+} \\ &\quad - F^{+t}F^{-\bar{t}} - F^{+\bar{t}}F^{-t} + F^{+-}F^{-+} \\ &\quad - F^{-t}F^{+\bar{t}} - F^{-t}F^{+\bar{t}} + F^{-+}F^{+-} \\ &= -2(F^{t\bar{t}})^2 - 2(F^{+-})^2 - 4F^t_l F^{\bar{t}l} \end{aligned} \tag{3.14}$$

Where we have used the antisymmetry of the field tensor and use the notation that a repeated index l means: "Sum over $\pm$". Using our gauge choice the four different components are given by

$$\begin{aligned}
F^{t\bar{t}a} &= \partial \bar{a}^a \\
F^{+-a} &= \partial^+ a^{-a} - \partial^- a^{+a} + g f^{abc} a^{+b} a^{-c} \\
F^{tla} &= \partial a^{la} \\
F^{\bar{t}la} &= \bar{\partial} a^{la} - \partial^l \bar{a}^a + g f^{abc} \bar{a}^b a^{lc}
\end{aligned} \tag{3.15}$$

Specifying the gauge removes one degree of freedom for the gauge field. We know that the massless gluon field only has two degrees of freedom and we want to get rid of the last one. The trick is to use the equations of motion for $\bar{a}$ to eliminate it from the lagrangian. Solving $\nabla_u F^{uta} = 0$ yields

$$\begin{aligned}
\nabla_u F^{uta} &= \nabla_u \partial A^{ua} \\
&= \partial_u \partial A^{ua} + g f_{abc} A_u^b \partial A^{ua} \\
&= \partial_l \partial A^{la} - \partial^2 \bar{a}^a + g f_{abc} A_l^b \partial A^{la} \\
&\Rightarrow \bar{a}^a = \frac{1}{\partial^2} \nabla_l \partial a^{la}
\end{aligned} \tag{3.16}$$

where we have used the gauge condition $a = 0$ in the third line. One should of course immediately stop and wonder whether the expression $\frac{1}{\partial^2}$ is well-defined. In the light-cone gauge these kinds of non-local terms are usually called *spurious poles*, as they introduce poles in loop-integrals that are not easily calculated using an ordinary ϵ-prescription and much work have been done in investigating them since the 70's. Leibbrandt [31] discusses this in great details and provides references for all original work as well as many strategies. In this thesis we shall not concern ourselves with these poles as we are only interested in providing a formalism at tree-level where they under any circumstances are no problem. However, it should be noted, that since ∂ is a complex derivative, no singularities along the real line are introduced due to the spurious poles in the Lagrangian. We refer to [1], where it is concluded that the poles are not a problem. Inserting (3.16) back into (3.15) yields the following

$$\begin{aligned}
F^{t\bar{t}a} &= \partial \bar{a}^a = \frac{1}{\partial} \nabla_l \partial a^{la} \\
&= \partial_l a^{la} + g f_{abc} \frac{1}{\partial} \left(a_l^b \partial a^{lc} \right).
\end{aligned} \tag{3.17}$$

The equation of motion for $\bar{a}$ is per this equation also given by $\bar{a} = \frac{1}{\partial} F^{t\bar{t}}$. We will use this to evaluate $F^{\bar{t}la}$ as it brings some simplification in the next couple of steps

$$\begin{aligned}
F^{\bar{t}la} &= \bar{\partial} a^{la} - \partial^l \bar{a}^a + g f_{abc} \bar{a}^b a^{lc} \\
&= \bar{\partial} a^{la} - \partial^l \frac{1}{\partial} F^{t\bar{t}a} + g f_{abc} \left(\frac{1}{\partial} F^{t\bar{t}b} \right) a^{lc}.
\end{aligned} \tag{3.18}$$

The last term with an $\bar{a}$ component is $4F^{\bar{t}la}F^{tla}$ which can also be expressed in terms of $F^{t\bar{t}}$

$$\begin{aligned}
F^{\bar{t}la}F^{tla} &= \left(\bar{\partial}a^{la} - \partial^{l}\frac{1}{\partial}F^{t\bar{t}a} + gf^{abc}\left(\frac{1}{\partial}F^{t\bar{t}b}\right)a^{lc}\right)\partial a^{la} \\
&= \bar{\partial}a^{la}\partial a^{la} - \partial^{l}\left(\frac{1}{\partial}F^{t\bar{t}a}\right)\partial a^{la} + gf^{abc}\left(\frac{1}{\partial}F^{t\bar{t}b}\right)a^{lc}\partial a^{la} \\
&= -a^{la}\bar{\partial}\partial a^{la} - F^{t\bar{t}a}\partial^{l}a^{la} - F^{t\bar{t}b}gf^{abc}\left(\frac{1}{\partial}a^{lc}\partial a^{la}\right) \\
&= -a^{la}\bar{\partial}\partial a^{la} - F^{t\bar{t}a}\left(\partial^{l}a^{la} + gf^{abc}\left(\frac{1}{\partial}a^{lb}\partial a^{lc}\right)\right) \\
&= -2a^{+a}\bar{\partial}\partial a^{-a} - \left(F^{t\bar{t}a}\right)^{2} \qquad (3.19)
\end{aligned}$$

Where we have used in line three that total derivatives do not contribute in the action. Adding everything together, we see that the lagrangian takes the form

$$\begin{aligned}
\mathcal{L}_{YM} &= -\frac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu a} \\
&= -2a^{+a}\bar{\partial}\partial a^{-a} + \frac{1}{2}(F^{+-a})^{2} - \frac{1}{2}(F^{t\bar{t}a})^{2} \\
&= -2a^{+a}\bar{\partial}\partial a^{-a} + \frac{1}{2}\left(F^{+-a} - F^{t\bar{t}a}\right)\left(F^{+-a} + F^{t\bar{t}a}\right) \\
&= -2a^{+a}\bar{\partial}\partial a^{-a} + \frac{1}{2}\left(-2\partial^{-}a^{+a} + gf^{abc}a^{+b}a^{-c} - gf^{abc}\frac{1}{\partial}\left(a^{b}_{l}\partial a^{lc}\right)\right) \\
&\qquad \cdot\left(2\partial^{+}a^{-a} + gf^{ab'c'}a^{+b'}a^{-c'} + gf^{ab'c'}\frac{1}{\partial}\left(a^{b'}_{l}\partial a^{lc'}\right)\right) \\
&= -2a^{+a}\bar{\partial}\partial a^{-a} + \frac{1}{2}\left(-2\partial^{-}a^{+a} - 2gf^{abc}\frac{1}{\partial}\left(a^{-b}\partial a^{+c}\right)\right) \\
&\qquad \cdot\left(2\partial^{+}a^{-a} + 2gf^{ab'c'}\frac{1}{\partial}\left(a^{+b'}\partial a^{-c'}\right)\right) \\
&= 2\Bigg[a^{+a}(\partial^{+}\partial^{-} - \bar{\partial}\partial)a^{-a} + gf^{abc}\left(\frac{\partial^{+}}{\partial}a^{-a}\right)a^{-b}\partial a^{+c} + gf^{abc}\left(\frac{\partial^{-}}{\partial}a^{+a}\right)a^{+b}\partial a^{-c} \\
&\qquad + g^{2}f^{abc}f^{ab'c'}\frac{1}{\partial}\left(a^{-b}\partial a^{+c}\right)\frac{1}{\partial}\left(\partial a^{-b'}a^{+c'}\right)\Bigg] \qquad (3.20)
\end{aligned}$$

First of all we notice that the lagrangian only has two scalar degrees of freedom, $a^{\pm}$. This is in accordance with a massless vector-boson only having two degrees of freedom. That everything is expressed in terms of scalars make the Feynman rules particular easy to apply when calculating diagrams. Especially no algebra concerning contractions over vector indices is needed. Second of all comparing with (3.11) we see that the light-cone components $\{+,-\}$ of the gluons are nothing more that specifications of the helicity, and hence that vertices in Feynman diagrams will only figure in some helicity configurations. For instance a^{+} means the $+$-light-cone component of the gluon field and hence the $+$-component of its polarization vector. The only polarization state which has non-zero $+$-component is the $+$-helicity state and likewise for a^{-}.

Feynman rules can easily be obtained from this lagrangian. To do so will be our next goal.

3.3 Feynman Rules for Yang-Mills

We will now write down colour-ordered momentum space Feynman rules obtained from the space-cone gauge-fixed Yang-Mills lagrangian (3.20). It is straightforward to read of the Feynman rules. A derivative goes into a momentum of the same kind and a field component $a^{\pm}$ simply denotes the helicity of the line attached to a vertex. Since we are giving colour-ordered rules, we write out all structure constants in terms of (2.42) to determine the rules. For instance

$$\begin{aligned} 2gf^{abc}\left(\frac{\partial^+}{\partial}a^{-a}\right)a^{-b}\partial a^{+c} &= -ig\left(\mathrm{Tr}[T^aT^bT^c]-\mathrm{Tr}[T^bT^aT^c]\right)\left(\frac{\partial^+}{\partial}a^{-a}\right)a^{-b}\partial a^{+c} \\ &= -ig\mathrm{Tr}[T^aT^bT^c]\left[\left(\frac{\partial^+}{\partial}a^{-a}\right)a^{-b}\partial a^{+c}-\left(\frac{\partial^+}{\partial}a^{-b}\right)a^{-a}\partial a^{+c}\right]. \end{aligned} \tag{3.21}$$

This expression goes into the following Feynman rule, if we take the three four-momenta as (P^-, K^-, Q^+)

$$2q(\frac{p^+}{p}-\frac{k^+}{k}). \tag{3.22}$$

In this way we can derive all Feynman rules for the space-cone Lagrangian. We take all momentum to be outgoing and assume overall momentum conservation as

$$\sum |p\rangle\,[p| = 0. \tag{3.23}$$

The same identity holds at each vertex. Note that small letters like p denotes light-cone components, whereas capital letters like P denotes full four-momentum.

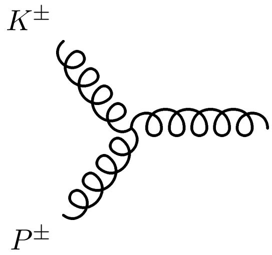
For external gluons the polarization vectors contribute with the $\pm$-light-cone component depending on the helicity of the gluons. This is again since the only polarization state which has non-zero $\pm$-component is the $\pm$-helicity state.

$P^{\pm}$  $= \quad (\epsilon_{\pm}(P))^{\pm}$ (3.24)

Although the propagator in (3.20) looks a little bit odd, we notice that it is just the ordinary propagator since $\partial^+\partial^- - \bar{\partial}\partial = \partial^2$.

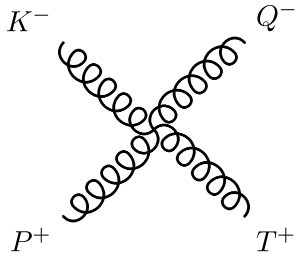
P  $= \quad \frac{1}{P^2}$ (3.25)

The propagator forces internal lines to have opposite helicity at each end. An ϵ-prescription is understood when needed. The three-point vertex comes in the two different helicity configurations $(++-)$ and $(--+)$ giving rise to the following vertices



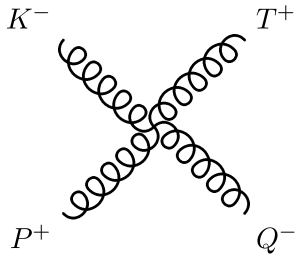
$$= \quad 2(k+p)\left(\frac{k^{\mp}}{k} - \frac{p^{\mp}}{p}\right). \tag{3.26}$$

The minus sign in front of $\frac{p^{\mp}}{p}$ comes from the antisymmetry of the structure constants as we showed above. The four-point vertex also comes in two helicity configurations given by $(++--)$ and $(+-+-)$ cyclically speaking. The former has a simpler structure than the latter, due to the fewer number of permutations of legs with that helicity structure. The vertices are given by



$$= \quad -2\frac{pq+tk}{(p+k)^2} \tag{3.27}$$

and



$$= \quad 2\frac{pk+tq}{(p+q)^2} + 2\frac{pq+tk}{(p+k)^2} \tag{3.28}$$

These are all the Feynman rules for lines and vertices without reference lines. It seems that a polarization vector that corresponds to a reference vector vanish, since by (3.11) it will be proportional to $[--]$ or $\langle ++ \rangle$ respectively. However this can be countered by $\frac{p^{\mp}}{p}$ in a three vertex which seem to diverge. We see that

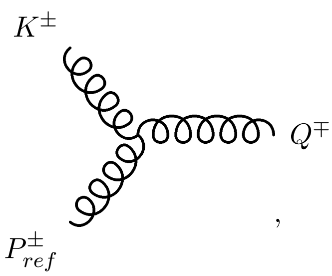
$$\frac{p^+}{p}(\epsilon_-)^- = \frac{\langle -p \rangle [-+]}{[-p]\sqrt{\langle -+ \rangle [+-]}} \overset{p \to +}{=} \frac{\langle -+ \rangle}{\sqrt{\langle -+ \rangle [+-]}} \tag{3.29}$$

$$\frac{p^-}{p}(\epsilon_+)^+ = \frac{[+p] \langle -+ \rangle}{\langle +p \rangle \sqrt{\langle -+ \rangle [+-]}} \overset{p \to -}{=} \frac{[-+]}{\sqrt{\langle -+ \rangle [+-]}} \tag{3.30}$$

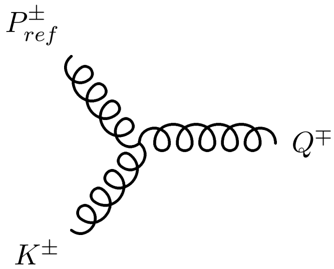
When we identify both reference vectors with that of two legs in a diagram, we always end up with the product of the two above expressions. We may therefore choose reference line factors as

P_{ref}  $= \quad i$ (3.31)

and three vertices as



$= \quad -2k$,



$= \quad 2k$ (3.32)

The formalism doesn't require of us to identify any of the reference vectors with gluon legs. If we choose only one, we may not apply the above rules directly, but have to make use of the full expressions of (3.29) or (3.30) for proper normalization. By the structure of the four vertex we see, that any diagram with a reference line attached to a four vertex *automatically* gives zero. This also holds for any diagram with two reference lines on a three vertex since there is only one factor in a three vertex to make a cancellation of the form of (3.29) or (3.30). For this reason one should mostly choose the reference legs to be adjacent. This simplifies calculations of Feynman diagrams greatly. It is also clear, that identifying less than two reference lines as gluon legs increases bot the number of diagrams and the number of terms in each diagram. Hence, for any calculations with the sole purpose of obtaining an amplitude, one should abstain from this. We will see later on how the number of diagrams change when we choose different numbers of external lines as reference lines. Another important thing to notice is that, since the polarization vectors have reduced to mere scalars, these will only contribute as an overall factor to all subamplitudes with the same helicity structure. Now the rules for computing an amplitude with a given ordering is

1. Draw all planar diagrams corresponding to the specific ordering of the external legs. On internal lines we assign a $+$ in one and a $-$ in the other end in all possible way. All vertices has to come in MHV- or googly-MHV-configuration.

2. Pick a positive helicity leg and call it $|-\rangle\,[-|$ and a negative helicity leg and call it $|+\rangle\,[+|$. An arbitrary set of reference legs can also be picked.

3. All diagrams with a reference leg attached to a four-point vertex or both reference legs attached to a three-vertex are automatically zero.

4. Evaluate the remaining diagrams according to the above vertices and line factors.

That the vertices only come in the MHV- and googly-MHV configuration is an indication that these Feynman rules are connected to the CSW formalism. This formalism has (googly)-MHV-amplitudes as vertices connected by scalar propagators which seems very similar to our Feynman rules. It has been shown in [33–35] that the CSW rules have a lagrangian origin. The transformations that are applied in these works are very similar to those we have applied here although no reference to the space-cone gauge is made. We will not go into their calculations but their results should be noted for completeness.

Before adding fermions to the Yang-Mills Lagrangian, we will give some examples of how to use the Feynman rules for Yang-Mills theory. First we will discuss in some detail two four- and five-point MHV amplitudes. Then we will show how to compute a six-point $NMHV$ amplitude which is the first amplitude where the four-point vertex enters explicitly. Lastly we will also compute the three-point MHV amplitude, as it requires a trick to compute and the result, although already well-known, is needed for our discussion of the BCFW recursion.

Some Useful Identities in Space-Cone Gauge

Before we do any calculations with our newly obtained Feynman rules, we would like to work out some useful expressions for the evaluation of these diagrams. In doing so, we anticipate the calculations that we will be doing. There is nothing deep in these expressions in the sense that we could just apply the above Feynman rules and do the necessary algebra. However, since one of the points of working in space-cone gauge is the reduced algebra we would like to as little work to do when actually evaluating diagrams.

First of all we very often have $\epsilon_\pm(p)^\pm$ multiplied with the space-cone component p of P. That always yields

$$\epsilon_-(P)^- p = \frac{[+-]\langle p+\rangle}{[-p]\, p^-_\oplus}\frac{\langle p+\rangle [p-]}{\sqrt{\langle -+\rangle [+-]}} = \frac{1}{p^-_\oplus}\sqrt{\frac{[-+]}{\langle +-\rangle}}\langle p+\rangle^2 \tag{3.33}$$

and similarly

$$\epsilon_+(P)^+ p = \frac{\langle -+\rangle [p-]}{\langle +p\rangle\, p^+_\ominus}\frac{\langle p+\rangle [p-]}{\sqrt{\langle -+\rangle [+-]}} = \frac{1}{p^+_\ominus}\sqrt{\frac{\langle +-\rangle}{[-+]}}[p-]^2 \tag{3.34}$$

Besides the funny looking normalization, these two expressions are very simple. In fact, in the original article by Chalmers and Siegel [1], the authors choose a normalization where $\langle -+\rangle [+-] = 1$ such that any normalization factor drops out. The price they pay is that their amplitudes have the wrong units and hence the have to restore their normalization in the end by counting the numbers of $\langle\cdot\rangle$ and $[\cdot]$. This is somewhat faster when you know the answer beforehand but not very instructive. We choose to keep the normalization explicit here as to make calculations more transparent for the reader. Every time we have a three-vertex without any attached reference lines we will need the following two expressions

$$\begin{aligned}
\frac{p^-}{p} - \frac{q^-}{q} &= \frac{\sqrt{\langle -+\rangle [+-]}}{p^-_\oplus}\left[\frac{\langle +p\rangle [p+]}{\langle p+\rangle [p-]} - \frac{\langle +q\rangle [q+]}{\langle q+\rangle [q-]}\right] \\
&= \sqrt{\frac{p^+_\ominus}{p^-_\oplus}}\left[\frac{[+p][q-] + [+q][-p]}{[p-][q-]}\right] \\
&= [-+]\sqrt{\frac{p^+_\ominus}{p^-_\oplus}}\frac{[pq]}{[p-][q-]} = p^+_\ominus\sqrt{\frac{[-+]}{\langle +-\rangle}}\frac{\langle +|\,p\,|q]}{\langle +|\,p\,|-]\,[q-]}
\end{aligned} \tag{3.35}$$

and doing the substitution $\langle\cdot\rangle \leftrightarrow [\cdot]$ and $+ \leftrightarrow -$ we also have

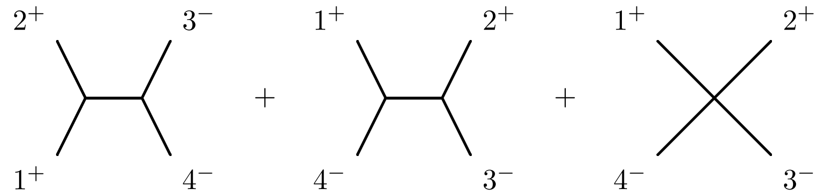
$$\frac{p^+}{p} - \frac{q^+}{q} = p_\oplus^- \sqrt{\frac{\langle +- \rangle}{[-+]}} \frac{\langle q|\, p\, |-]}{\langle +|\, p\, |-]\, \langle q+ \rangle}. \tag{3.36}$$

We have used the Schouten identity in the second line of (3.35).

A Four- and Five-point MHV Example

Computing the colour-ordered subamplitude $A_4(++--)$ requires us to write down all planar diagrams with the $(++--)$ helicity configurations. We will go through this example in very high detail, as many arguments used here will also have to be invoked for all higher order amplitudes. There are three different diagrams that contribute to the amplitude.

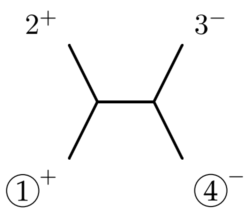
$A_4(++--) =$



(3.37)

To simplify the drawings all lines are straight instead of springy and no arrows are drawn since all momenta are taken to be outgoing as stated in section 3.3. Notice that the helicity on the internal lines are uniquely specified by the Feynman rules in the first diagram, whereas in the second both helicity configurations of the internal line has to be taken into consideration. At this point we have yet to choose reference lines. Now we start by picking P_1 as the reference line for negative helicity gluons[1]. That is, we choose $P_1 = |-\rangle\,[-|$. Doing so will instantly kill the third diagram, as it has no three vertex to absorb $\epsilon_+(1)^+$ which is zero. The two other diagrams survive. By choosing the positive helicity reference momentum as $P_4 = |+\rangle\,[+|$ the second diagram also vanishes, since it will have a three vertex with two reference momenta attached. All together we are left with

$A_4(\oplus + - \ominus) =$



(3.38)

where the circles indicates reference lines. To compute this diagram is almost trivial, as the

[1] Remember the conventions of section 3.1 that $|-\rangle\,[-|$ is a momentum of positive helicity and vice versa.

two vertices both have a reference line. Following the rules of section 3.3 we find

$$
\begin{aligned}
A_4(\oplus + - \ominus) &= i^2 \epsilon_+(2)^+ \epsilon_-(3)^-(-2p_2)(2p_3)\frac{1}{P_{12}^2} \\
&= 4\frac{\langle -+\rangle [2-]}{\langle +2\rangle p_\oplus^-}\frac{\langle +3\rangle [-+]}{[3-]\, p_\ominus^+}\frac{\langle 2+\rangle [2-]}{\sqrt{\langle -+\rangle [+-]}}\frac{\langle 3+\rangle [3-]}{\sqrt{\langle -+\rangle [+-]}}\frac{1}{\langle 12\rangle [21]} \\
&= 4\frac{[2-]^2 \langle 3+\rangle^2}{\langle +-\rangle [+-] \langle 12\rangle [21]} \\
&= 4\frac{[21] \langle 34\rangle^3}{[41] \langle 12\rangle \langle 34\rangle \langle 41\rangle}
\end{aligned}
\tag{3.39}
$$

Using overall momentum conservation and the fact that $\langle qq\rangle = [qq] = 0$ we have

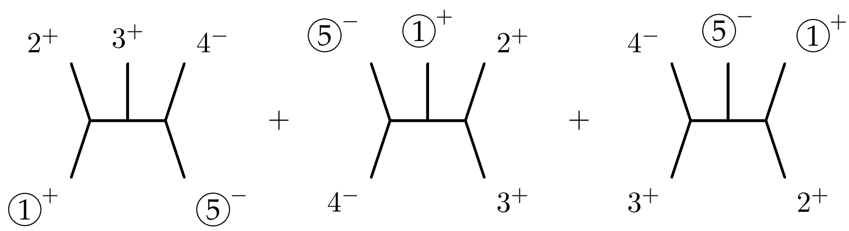
$$
\begin{aligned}
\sum |p\rangle [p| = 0 \quad &\Rightarrow \quad \langle 32\rangle [21] + \langle 34\rangle [41] = 0 \\
&\Leftrightarrow \quad \frac{[21]}{[41]} = \frac{\langle 34\rangle}{\langle 23\rangle}
\end{aligned}
\tag{3.40}
$$

Inserting this into (3.39) we arrive at the well known Parke-Taylor result

$$
A_4(\oplus + - \ominus) = 4\frac{\langle 34\rangle^4}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle}. \tag{3.41}
$$

We now proceed to evaluate the 5-point MHV amplitude $A_5(+++--)$. Since there are a lot of topological different diagrams that could contribute to this amplitude, we refrain from doing any drawing until we have argued what diagrams are non-zero. Taking into account all topologically different diagrams as well as all allowed helicity configurations of the internal lines, there are 13 diagrams all together. If we choose reference lines in the same manner as in the above example, that is $P_1 = |-\rangle [-|$ and $P_5 = |+\rangle [+|$, we end up with the following three diagrams

$A_5(\oplus + + - \ominus) =$



(3.42)

since all diagrams containing a 4-point vertex with attached reference lines are zero, as well as all diagrams with two reference lines attached to a single 3-point vertex. The evaluation of these three diagrams is now straightforward. Using the above Feynman rules we have

$$
\begin{aligned}
A_5(\oplus + + - \ominus) = i^2 \epsilon_+(2)^+ \epsilon_+(3)^+ \epsilon_-(4)^- \Bigg[&(-2p_2)(2p_4)\left(\frac{p_2^- + p_\oplus^-}{p_2} - \frac{p_3^-}{p_3}\right)(2p_4)\frac{1}{P_{12}^2 P_{45}^2} \\
&+ (2p_4)(2p_4)(2p_4)\left(\frac{p_2^-}{p_2} - \frac{p_3^-}{p_3}\right)\frac{1}{P_{45}^2 P_{23}^2} \\
&+ (2p_4)\left(\frac{p_2^- + p_\ominus^-}{p_2} - \frac{p_3^-}{p_3}\right)(2p_2)(2p_2)\frac{1}{P_{34}^2 P_{12}^2}\Bigg] \\
= 8\epsilon_+(2)^+ \epsilon_+(3)^+ \epsilon_-(4)^- p_4 \Bigg[&\left(\frac{p_2^- + p_\oplus^-}{p_2} - \frac{p_3^-}{p_3}\right)\frac{p_2 p_4}{P_{12}^2 P_{45}^2} \\
&- \left(\frac{p_2^-}{p_2} - \frac{p_3^-}{p_3}\right)\frac{p_4^2}{P_{45}^2 P_{23}^2} - \left(\frac{p_2^- + p_\oplus^-}{p_2} - \frac{p_3^-}{p_3}\right)\frac{p_2^2}{P_{34}^2 P_{12}^2}\Bigg]
\end{aligned}
\tag{3.43}
$$

The overall factor is given by

$$
\begin{aligned}
(i^2) 8\epsilon_+(2)^+ \epsilon_+(3)^+ \epsilon_-(4)^- p_4 &= -8\frac{\langle -+\rangle [2-]}{\langle +2\rangle p_\oplus^-}\frac{\langle -+\rangle [3-]}{\langle +3\rangle p_\oplus^-}\frac{1}{p_\ominus^+}\sqrt{\frac{[-+]}{\langle +-\rangle}}\langle 4+\rangle^2 \\
&= 8\frac{1}{p_\oplus^-}\sqrt{\frac{\langle -+\rangle}{[+-]}}\frac{[2-][3-]}{\langle +2\rangle\langle +3\rangle}\langle 4+\rangle^2 .
\end{aligned}
\tag{3.44}
$$

The three terms in the brackets are each given by

$$
\begin{aligned}
\left(\frac{p_2^- + p_\oplus^-}{p_2} - \frac{p_3^-}{p_3}\right)\frac{-p_2 p_4}{s_{12}s_{45}} &= -p_\oplus^-\sqrt{\frac{[-+]}{\langle +-\rangle}}\frac{\langle +|\,2\,|3] + \langle +|-|3]}{\langle +|\,2\,|-]\,[3-]}\frac{\langle 2+\rangle [2-]\langle 4+\rangle [4-]}{\langle -+\rangle [+-]\langle 12\rangle [21]\langle 45\rangle [54]} \\
&= \frac{1}{p_\ominus^+}\sqrt{\frac{[-+]}{\langle +-\rangle}}\frac{[2-][43]\langle +4\rangle^2 [-4]}{[2-][3-]\langle 12\rangle [21]\langle 54\rangle [54]}
\end{aligned}
\tag{3.45}
$$

$$
\begin{aligned}
\left(\frac{p_2^-}{p_2} - \frac{p_3^-}{p_3}\right)\frac{p_4^2}{s_{45}s_{23}} &= p_\oplus^-\sqrt{\frac{[-+]}{\langle +-\rangle}}\frac{\langle +|\,2\,|3]}{\langle +|\,2\,|-]\,[3-]}\frac{\langle 4+\rangle^2 [4-]^2}{\langle -+\rangle [+-]\langle 45\rangle [54]\langle 23\rangle [32]} \\
&= \frac{1}{p_\ominus^+}\sqrt{\frac{[-+]}{\langle +-\rangle}}\frac{\langle 4+\rangle^2 [4-]^2}{[2-][3-]\langle 45\rangle [54]\langle 32\rangle}
\end{aligned}
\tag{3.46}
$$

$$
\begin{aligned}
\left(\frac{p_2^- + p_\oplus^-}{p_2} - \frac{p_3^-}{p_3}\right)\frac{p_2^2}{s_{34}s_{12}} &= p_\oplus^-\sqrt{\frac{[-+]}{\langle +-\rangle}}\frac{\langle +|\,2\,|3] + \langle +|-|3]}{\langle +|\,2\,|-]\,[3-]}\frac{\langle 2+\rangle^2 [2-]^2}{\langle -+\rangle [+-]\langle 34\rangle [43]\langle 12\rangle [21]} \\
&= \frac{1}{p_\ominus^+}\sqrt{\frac{[-+]}{\langle +-\rangle}}\frac{\langle +4\rangle\langle 2+\rangle [2-]^2}{[2-][3-]\langle 34\rangle\langle 12\rangle [21]} .
\end{aligned}
\tag{3.47}
$$

In (3.45) and (3.47) we have applied momentum conservation in the form

$$
\langle +|\,2\,|3] + \langle +|-|3] = -\langle +|\,4\,|3] . \tag{3.48}
$$

Besides that no non-trivial manipulations have been done. Combining the overall factor of

$$\frac{1}{p_{\ominus}^{+}}\sqrt{\frac{[-+]}{\langle +- \rangle}}\frac{\langle +4 \rangle}{[2-]\,[3-]} \tag{3.49}$$

from the above three expressions with that of (3.44) and restoring $|+\rangle = |5\rangle$ and $|-\rangle = |1\rangle$ we may write the full amplitude as

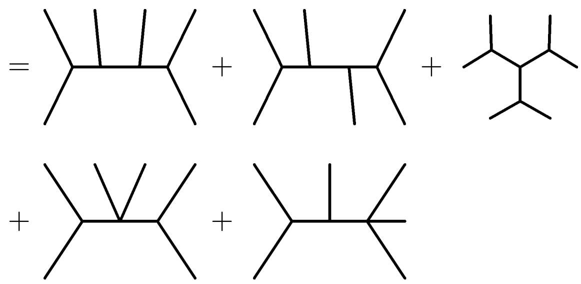
$$\begin{aligned} A_5(\oplus ++- \ominus) &= 8\frac{\langle 45\rangle^3}{\langle 15\rangle\,[51]\,\langle 25\rangle\,\langle 53\rangle}\left[\frac{[43]\,[14]}{\langle 12\rangle\,[54]} + \frac{[41]^2}{[54]\,\langle 23\rangle} + \frac{\langle 25\rangle\,[21]}{\langle 34\rangle\,\langle 12\rangle}\right] \\ &= 8\frac{\langle 45\rangle^3}{\langle 15\rangle\,[51]\,\langle 25\rangle\,\langle 53\rangle}\left[\frac{\langle 25\rangle\,[14]}{\langle 12\rangle\,\langle 23\rangle} + \frac{\langle 25\rangle\,[21]}{\langle 34\rangle\,\langle 12\rangle}\right] \\ &= 8\frac{\langle 45\rangle^3\,\langle 25\rangle}{\langle 15\rangle\,[51]\,\langle 25\rangle\,\langle 53\rangle}\left[\frac{\langle 35\rangle\,[51]}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle}\right] \\ &= 8\frac{\langle 45\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 45\rangle\,\langle 51\rangle} \end{aligned} \tag{3.50}$$

Where in the first line we have cancelled equal factors in the numerators and denominators, in the second and third line we have applied momentum conservation and in the last line we have rearranged to the familiar result. Even though this calculations is somewhat more difficult than the four-point case, it still shows significant improvement over conventional Feynman diagram methods.

A Six-point $NMHV$ Example

Common for the two above MHV examples is, that our choice of reference momenta cancelled all four-point vertices. Monteiro and O'Connell showed in [36] that all MHV amplitudes can be constructed from Feynman diagrams containing only three-point vertices, using arguments from the space-cone gauge. Because of this, it is instructive to look at an $NMHV$ example, to see that the four-point vertices derived above do really contribute. The simplest $NMHV$ amplitude is $A_6(+++---)$ which however is much more complicated than the five-point amplitude we calculated before. Altogether we find that 15 diagrams contribute to the amplitude. Schematically, we may write the amplitude as

$$\mathcal{A}_6(\oplus ++--\ominus) =$$



(3.51)

There are six diagrams of the first type, four diagrams of the third type, two diagrams of the third type, one diagram of the fourth type and two diagrams of the fifth type. All diagrams and their contributions are given in appendix A. The amount of work that needs to be done in order to reduce these 15 diagrams to a manageable object is huge. However we can say some non-trivial thing about the diagrams. First of all, the contribution from the three diagrams containing a four-point vertex is non-zero, since up to a common (non-zero) factor it is given by

$$\frac{p_2^2}{P_{12}^2 P_{126}^2} + \frac{p_5^2}{P_{56}^2 P_{156}^2} - \frac{p_2 p_5}{P_{12}^2 P_{56}^2}. \tag{3.52}$$

When rewriting the diagrams in terms of spinor helicity products, there are some symmetries one may use to facilitate calculations. From reflection symmetry of the entire amplitude we know that the following transformation leaves the amplitude invariant

$$1 \leftrightarrow 6, \quad 2 \leftrightarrow 5, \quad 3 \leftrightarrow 4, \quad - \leftrightarrow + \tag{3.53}$$

Now performing this transformation on the individual diagrams we find that some of them are manifestly invariant and some are not. Those that are not invariant turn into some of the other diagrams. For instance looking at (3.52) we see that the last term is invariant whereas the two other terms are dual to each other under this transformation. So, if we could find a simple expression in terms of spinor helicity products of the first term, we would immediately have an expression for the other. This symmetry can from inspection be seen to reduce the number of independent diagrams from 15 to 10. We have tested the sum of all terms against the expression for the amplitude one obtains from the BCFW recursion numerically. From [37] we have

$$\mathcal{A}_6(+++---) = \frac{\langle 6|\,1+2\,|3]^3}{\langle 12\rangle\langle 61\rangle [34][45]\langle 2|\,1+6\,|5]\, P_{126}^2} + \frac{\langle 4|\,5+6\,|1]^3}{\langle 23\rangle\langle 34\rangle [56][61]\langle 2|\,1+6\,|5]\, P_{156}^2}. \tag{3.54}$$

Such a test numerically verifies the four-point vertex rules derived above[2]. It should be stressed that even though this six-point calculations does not take up much space here it is by far the most important calculation as to our knowledge the four-point vertex has not been checked anywhere else in literature. We will return to this amplitude later on, as we inspect some of its factorization properties.

A Three-point MHV Amplitude

Having looked at a 4-, 5- and 6-point example it might seem weird to end by calculating the 3-point MHV amplitude. Isn't it trivial? Both yes and no. We of course already know the result from Parke and Taylor

$$A_3(+--) = \frac{\langle 23\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}. \tag{3.55}$$

However if we naively go forward as in the two above examples, we can quickly convince ourselves, that the amplitude should be zero. If we choose $P_1 = |-\rangle\,[-|$ and $P_3 = |+\rangle\,[+|$ we have attached two reference lines to the three-vertex, and hence it vanishes. To avoid this peculiarity we introduce another positive helicity momentum as reference $P_4 = |-\rangle\,[-|$ such that

$$\sum_{i=1}^{4} |p_i\rangle\,[p_i| = 0. \tag{3.56}$$

[2] The more complicated four-point vertex has also been tested in a similar manner

It is well-known that this amplitude vanishes for real momenta as can be seen from (2.57). However for complex values momenta the amplitude is non-zero. The amplitude is straightforward to calculate, since it is almost just equal to the three-vertex

$$\begin{aligned} A_3(+-\ominus) &= 2\epsilon_+(1)^+\epsilon_-(2)^- \frac{\langle -+\rangle}{\sqrt{\langle -+\rangle\,[+-]}} p_2 \\ &= 2\frac{\langle -+\rangle\,[1-]}{\langle +1\rangle\, p_\oplus^-}\frac{\langle +2\rangle\,[-+]}{[2-]\, p_\ominus^+}\frac{\langle -+\rangle}{\sqrt{\langle -+\rangle\,[+-]}}\frac{\langle 2+\rangle\,[2-]}{\sqrt{\langle -+\rangle\,[+-]}} \\ &= 2\frac{[14]\,\langle 23\rangle^2}{\langle 31\rangle\,[34]} \\ &= 2\frac{\langle 23\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,\langle 31\rangle} \end{aligned} \tag{3.57}$$

Where we in the last line have used momentum conservation in the form

$$\frac{[14]}{[34]} = \frac{\langle 23\rangle}{\langle 12\rangle} \tag{3.58}$$

There are several lessons to learn from this calculations that we will exploit when discussing the space-cone and on-shell recursion relations and factorization. First of all, we are not forced to choose two external lines as reference lines. The three-point amplitude was a little special since we could not choose both reference lines to be identified with external momenta. In general we have the choice and calculations should not depend on this choice. Second of all we saw that the three-point amplitude essentially was just the three-point vertex. Since the four-point example calculated above was nothing but the product of two three-point vertices we might suspect, that the amplitude could be expressed as a factorization of on-shell three-point amplitudes. This is indeed the case. We will derive this result and formulate it in a more concise way later on. The result is gauge-invariant but in the space-cone gauge the factorization is almost manifest as everything is expressed in terms of scalars. Before discussing this we will go on and add fermions to our lagrangian, thereby extending the theory to include all of QCD.

3.4 Extending to QCD

The apparent success of the space-cone gauge for Yang-Mills makes it obvious to try and extend the space-cone methods to include other theories. The natural extension is to include (fundamental) fermions thereby describing the theory of QCD. The Lagrangian describing fermions coupled to a gauge field is given by

$$\begin{aligned} \mathcal{L}_q &= \bar{\psi}(i\gamma_\mu \nabla^\mu)\psi - m\bar{\psi}\psi \\ &= \mathcal{L}_{q,0} + gA^{a\mu}\bar{\psi}\gamma_\mu T^a\psi\,, \end{aligned} \tag{3.59}$$

where $\mathcal{L}_{q,0}$ is the Lagrangian for the free Dirac field. The procedure for writing this Lagrangian in space-cone gauge is the same as it was for pure Yang-Mills theory. First we expand the interaction in terms of its light-cone components and use the gauge condition $a^a = 0$

$$gA^{au}\bar{\psi}\gamma_u T^a\psi = g\Big[a^{+a}\bar{\psi}\gamma^- T^a\psi + a^{-a}\bar{\psi}\gamma^+ T^a\psi - \underbrace{a^a\bar{\psi}\bar{\gamma}T^a\psi}_{=0} - \bar{a}^a\bar{\psi}\gamma T^a\psi\Big]. \tag{3.60}$$

Note that we have also written the γ-matrices in the light-cone notation now, *i.e.* $\gamma^+ = \frac{1}{\sqrt{2}}(\gamma^0 + \gamma^3)$, *etc.* The equation of motion for $\bar{a}^a$ is not given by (3.16) any more, since this was derived without fermions. Since the Euler-Lagrange equations (3.3) are linear, we just plug (3.60) into these and add the result to (3.4). The Euler-Lagrange equations for (3.60) read

$$-g(\bar{\psi}\gamma T^a \psi) = 0 \tag{3.61}$$

which together with (3.4) and what was derived in (3.16) gives the equations of motion for $\bar{a}^a$

$$\bar{a}^a = \frac{1}{\partial^2}\nabla_l \partial a^{la} - g\frac{1}{\partial^2}(\bar{\psi}\gamma T^a \psi) \tag{3.62}$$

Now the first part is just the old equations of motion. This means, that we recover the same rules for vertices with only gluons. This does not come as a surprise, but it is a good sanity check. The bad news however, is that the second part of the equations of motion will interfere with both (3.60) and parts of the Yang-Mills Lagrangian (3.15). The algebra is tedious but not very complicated and since it is very similar to the manipulations done in section 3.2 we will refer from doing it in details. Inserting (3.62) into (3.60) we obtain

$$\begin{aligned}
-g\bar{a}^a\bar{\psi}\gamma T^a\psi &= -g\left(\frac{1}{\partial^2}\nabla_l\partial a^{la} - g\frac{1}{\partial^2}(\bar{\psi}\gamma T^a\psi)\right)\bar{\psi}\gamma T^a\psi \\
&= -\left(\frac{\partial^-}{\partial}a^{+a}\right)\bar{\psi}\gamma T^a\psi - \left(\frac{\partial^+}{\partial}a^{-a}\right)\bar{\psi}\gamma T^a\psi - g^2\frac{1}{\partial}\left(\bar{\psi}\gamma T^a\psi\right)\frac{1}{\partial}\left(\bar{\psi}\gamma T^a\psi\right) \\
&\quad - g^2 f_{abc}\left[\frac{1}{\partial}\left((\partial a^{+b})a^{-c}\right)\frac{1}{\partial}\left(\bar{\psi}\gamma T^a\psi\right) + \frac{1}{\partial}\left((\partial a^{-b})a^{+c}\right)\frac{1}{\partial}\left(\bar{\psi}\gamma T^a\psi\right)\right]
\end{aligned} \tag{3.63}$$

In the Yang-Mills Lagrangian a lot of terms turns up, however most of them cancel and we are left with only one contribution

$$\Delta\mathcal{L}_{YM} = \frac{g^2}{2}\frac{1}{\partial}\left(\bar{\psi}\gamma T^a\psi\right)\frac{1}{\partial}\left(\bar{\psi}\gamma T^a\psi\right). \tag{3.64}$$

Substituting these two terms in the Lagrangian, we may write the full QCD Lagrangian in the following way

$$\mathcal{L}_{QCD} = \mathcal{L}_{YM} + \mathcal{L}_{q,0} + \mathcal{L}_{q,I}\,, \tag{3.65}$$

where $\mathcal{L}_{YM}$ is just (3.20), $\mathcal{L}_{q,0}$ is still the free Dirac Lagrangian and

$$\begin{aligned}
\mathcal{L}_{q,I} &= g\left[a^{+a}\bar{\psi}\gamma^- T^a\psi - \left(\frac{\partial^-}{\partial}a^{+a}\right)\bar{\psi}\gamma T^a\psi\right] + g\left[a^{-a}\bar{\psi}\gamma^+ T^a\psi - \left(\frac{\partial^+}{\partial}a^{-a}\right)\bar{\psi}\gamma T^a\psi\right] \\
&\quad + g^2 f_{abc}\left[\frac{1}{\partial}\left((\partial a^{+a})a^{-c}\right)\frac{1}{\partial}\left(\bar{\psi}\gamma T^b\psi\right) - \frac{1}{\partial}\left((\partial a^{-c})a^{+a}\right)\frac{1}{\partial}\left(\bar{\psi}\gamma T^b\psi\right)\right] \\
&\quad - \frac{g^2}{2}\frac{1}{\partial}\left(\bar{\psi}\gamma T^a\psi\right)\frac{1}{\partial}\left(\bar{\psi}\gamma T^a\psi\right)\,.
\end{aligned} \tag{3.66}$$

We immediately see that in this lagrangian there are effective gluon-gluon-quark-antiquark and double pair quark-antiquark vertices.

3.5 Feynman Rules Involving Quarks

Where the space-cone gauge immediately simplified the Yang-Mills Feynman rules, it has almost had the opposite effect on the fermion sector. Taking all momenta outgoing the propagator is just the usual Dirac propagator

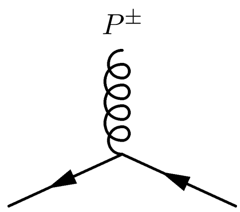
$$= \frac{\not{Q} + m}{Q^2 - m^2} \tag{3.67}$$

and the external states are

$$= \bar{u}^s(P),$$

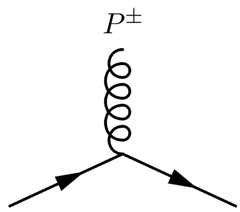
$$= v^s(P). \tag{3.68}$$

This was just the free part of the Lagrangian. We may read of the colour-ordered momentum space Feynman rules in the same way as we did for Yang-Mills. We first look at the three-point vertices which are given by



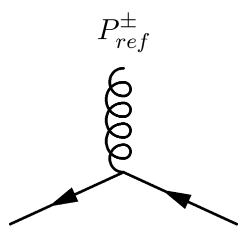
$$= \quad \gamma^{\mp} - \frac{p^{\mp}}{p}\gamma \tag{3.69}$$

and

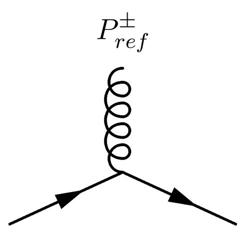


$$= \quad -\left(\gamma^{\mp} - \frac{p^{\mp}}{p}\gamma\right). \tag{3.70}$$

Since the vertices have a term with a $\frac{p^{\mp}}{p}$ we see that a reference leg may be attached to them as before. If we do so, we should of course follow the same prescription as before with the gluons concerning the external polarization states. If we pick two reference legs from the external legs and use the convention of (3.31) we have

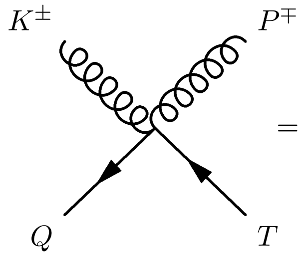


$$= \quad -\gamma$$



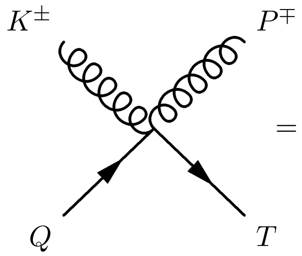
$$= \quad \gamma \tag{3.71}$$

If we only pick one external state as a reference we should use the expressions of (3.29) or (3.30). So far so good. However, when we eliminated the auxiliary component $\bar{a}^a$ from the Lagrangian we paid a price. In the Lagrangian we have introduced effective four-point interactions between gluons and quarks. In Lorentz-Feynman gauge these are absent and we therefore expect cancellations to occur when calculating amplitudes. The first vertex is an effective gluon-gluon-quark-antiquark interaction



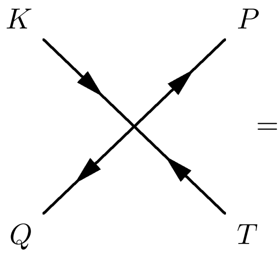
$$= \frac{(p-k)}{(k+p)^2}\gamma\,, \tag{3.72}$$

and



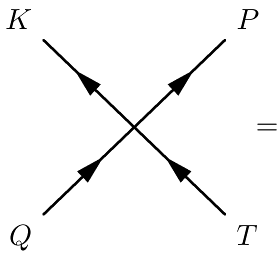
$$= -\frac{(p-k)}{(k+p)^2}\gamma\,. \tag{3.73}$$

The second one is a double pair quark-antiquark interaction



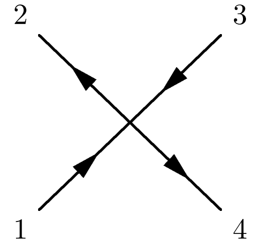
$$= \frac{\gamma}{(t+q)}\frac{\gamma}{(k+p)} + \frac{\gamma}{(q+k)}\frac{\gamma}{(p+t)}\,, \tag{3.74}$$

and



$$= -\frac{\gamma}{(q+k)}\frac{\gamma}{(p+t)}\,. \tag{3.75}$$

The most peculiar of the vertices are the two last one, since they indicate that quarks couple to each other. From eq. (3.62) we can understand the quarks that participate in this vertex as gluons under disguise. They are nothing more than part of the auxiliary field $\bar{a}^a$ and we expect all their contributions to be countered by other diagrams. Notice that the γ-matrices in eq. (3.74) and (3.75) are *not* multiplied together. In a real amplitude calculation they will appear between corresponding external spinors, for example like



$$= \frac{\bar{u}(P_2)\gamma v(P_1)}{p_1+p_2}\frac{\bar{u}(P_4)\gamma v(P_3)}{p_4+p_3} + (2 \leftrightarrow 4)\,. \tag{3.76}$$

This covers all the Feynman rules one obtains for QCD in the space-cone gauge. The rules are not immediately suited for doing analytical computations as we have to invoke some tricks. From the rules above we read of, that terms like

$$\bar{u}(p)\gamma v(q) = \langle p|\,\gamma\,|q] \tag{3.77}$$

has to be computed. When working in Lorentz-Feynman gauge, such terms will always appear contracted with a similar term like $\langle p|\,\gamma_\mu\,|q]\,\langle k|\,\gamma^\mu\,|l]$. Then one can usually invoke a Fierz identity to reduce expressions to simple spinor products. As there are no Lorentz indices left in our expressions we are not able to make such contractions and in principal the matrix products have to be evaluated. We will however use a trick to derive a simple expression for (3.77) from the three-point MHV amplitude in the next section along with some other identities which will allow us to evaluate amplitudes in a simpler way.

The Feynman rules may directly be implemented in numerical computations, since they do produce somewhat fewer terms than usual methods. All the above Feynman rules have been checked numerically and some of the rules have been checked analytical. We give one example below.

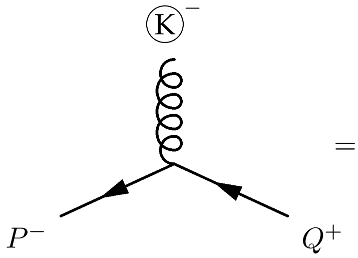
An Example

First we will derive some identities to facilitate calculations later on. We do not derive all identities that one may need in order to calculate any amplitude, but most identities can be derived in a manner similar to what is shown here. In the following we assume all fermions to be massless. First, let us clarify why (3.77) poses a problem in an amplitude calculation in the spinor helicity formalism as we have formulated it. To be able to compare our results with known expressions, we want to be able to represent everything in terms of spinor products $\langle\cdot\rangle$ and $[\cdot]$. If we write out $\langle p|$ and $|q]$ in terms of spinor components, we see that

$$\langle p|\,\gamma\,|q] \propto p_1 q_{\dot{2}}\,. \tag{3.78}$$

This is not something that we can interpret in terms of any simple combination of $\langle\cdot\rangle$ and $[\cdot]$, since it mixes the two types of spinors[3]. We may however derive an expression by a cunning use of the three-point MHV amplitude. From (3.71) we have

[3] James Ettle et al. propose in [34, 35] to introduce another set of objects called $(\cdot)$ and $\{\cdot\}$ that mixes spinor indices. Using their formalism it might be possible to directly relate (3.78) to such an object. This work was not discovered until very late in the process of writing this thesis and we have therefore not have the possibility to investigate it properly.



$$= \bar{u}(p)(-\gamma)v(Q)\frac{\langle -+\rangle}{\sqrt{\langle -+\rangle\,[+-]}} = \langle p|\,\gamma\,|q]\,\frac{\langle +-\rangle}{\sqrt{\langle -+\rangle\,[+-]}} \tag{3.79}$$

where we have chosen $|k\rangle = |+\rangle$ and some other momentum as $|-\rangle$. The factor $\frac{\langle +-\rangle}{\sqrt{\langle -+\rangle[+-]}}$ comes from the fact that only one reference line is attached to the diagram, just like the case of the three-point gluon amplitude. We are therefore also explicitly not working in real Minkowski space, as the amplitude should vanish here. The expression for this amplitude could of course be obtained from any other old gauge and by gauge invariance we expect such an expression to be equal to (3.79). We therefore read of the amplitude from (2.60) and thus we have

$$\begin{aligned}\langle p|\,\gamma\,|q]\,\frac{\langle +-\rangle}{\sqrt{\langle -+\rangle\,[+-]}} &= \frac{\langle pk\rangle^3\,\langle qk\rangle}{\langle pk\rangle\,\langle kq\rangle\,\langle qp\rangle} = \frac{\langle p+\rangle^2}{\langle pq\rangle} \\ \Rightarrow \quad \langle p|\,\gamma\,|q] &= \frac{\langle p+\rangle^2}{\langle pq\rangle}\frac{\sqrt{\langle -+\rangle\,[+-]}}{\langle +-\rangle}.\end{aligned} \tag{3.80}$$

Notice that this expression of course only makes sense for $p \neq q$. We may use this expression to evaluate terms like $\langle p|\,\gamma\not{K}\gamma\,|q]$

$$\begin{aligned}\langle p|\,\gamma\not{K}\gamma\,|q] &= \langle p|\,\gamma K_u\gamma^u\gamma\,|q] \\ &= \langle p|\,\gamma(k^+\gamma^- + k^-\gamma^+ - k\bar{\gamma} - \bar{k}\gamma)\gamma\,|q] \\ &= 2k\,\langle p|\,\gamma\,|q] \\ &= 2\frac{\langle k+\rangle\,[k-]}{\langle +-\rangle}\frac{\langle p+\rangle^2}{\langle pq\rangle}\end{aligned} \tag{3.81}$$

where we have used that

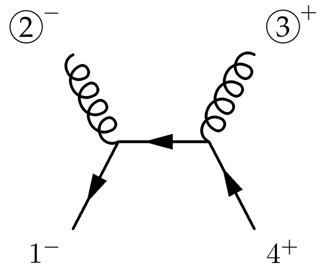
$$\gamma\gamma\gamma = 0 \qquad \gamma\bar{\gamma}\gamma = -2\gamma$$
$$\gamma\gamma^+\gamma = 0 \qquad \gamma\gamma^-\gamma = 0$$

which can easily be confirmed from (2.26). There are of course other such relations that can be worked out. For completeness we also give an expression for $\langle p|\,\gamma^\pm\,|q] - \frac{k^\pm}{k}\,\langle p|\,\gamma\,|q]$ although we wont use it in our computed example. The derivation follows that of $\langle p|\,\gamma\,|q]$ but now with no reference lines attached to the diagram.

$$\langle p|\,\gamma^+\,|q] - \frac{k^+}{k}\,\langle p|\,\gamma\,|q] = \frac{\langle pk\rangle^2\,[k-]\,p^+_\ominus}{\langle pq\rangle\,\langle +k\rangle\,[+-]} \tag{3.82}$$

$$\langle p|\,\gamma^-\,|q] - \frac{k^-}{k}\,\langle p|\,\gamma\,|q] = \frac{[qk]^2\,\langle k+\rangle\,p^-_\oplus}{[pq]\,[-k]\,\langle +-\rangle} \tag{3.83}$$

These expression can also be used to find $\langle p|\,\gamma^\pm\,|q]$ explicitly. It requires the use of momentum conservation to get rid of the k-dependence. With our new-found identities we can compute an amplitude right away. We take the simplest amplitude we can think of: $q\bar{q} \to gg$. We choose the two gluons with opposite helicity, so that we may choose them as reference lines. Doing so only one diagram survives



$$
\begin{aligned}
&= \bar{u}(p_1)(-\gamma)\frac{\not{p}_1 + \not{p}_2}{P_{12}^2}(-\gamma)v(p_4) \\
&= \Big[\langle 1|\, \gamma \not{p}_1 \gamma \,|4] + \langle 1|\, \gamma \not{p}_2 \gamma \,|4]\Big] \frac{1}{P_{12}^2} \\
&= \left[2\frac{\langle 12\rangle\,[13]}{\langle 23\rangle}\frac{\langle 12\rangle^2}{\langle 14\rangle}\right]\frac{1}{\langle 12\rangle\,[21]} \\
&= 2\frac{\langle 12\rangle^3}{\langle 12\rangle\,\langle 23\rangle\,\langle 41\rangle}\frac{[31]}{[21]} \\
&= 2\frac{\langle 12\rangle^3\,\langle 42\rangle}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 41\rangle}
\end{aligned} \tag{3.84}
$$

where we in the third line have used (3.81) to see that $\langle 1|\, \gamma \not{p}_2 \gamma \,|4] = 0$ and in the last line have used momentum conservation. This ends our discussion of direct amplitude calculations in the space-cone gauge. We will now turn to the BCFW recursion and how to factorize amplitudes into lower-point amplitudes through the space-cone gauge and the largest time equation.

— 4 —

BCFW recursion and the Largest Time Equation

Even with the simple Feynman rules derived in chapter 2 we were not able to obtain a simple expression for the six-point $NMHV$ amplitude. There exists a very powerful calculation method called the BCFW recursion relation, named after Britto, Cachazo, Feng and Witten, that enables calculations of practically all tree-level amplitudes. It delivers by far the most compact expressions for N^mMHV amplitudes in a very simple way. The purpose of this chapter is to introduce the BCFW recursion relation and derive its connection to the largest time equation. First we derive the BCFW recursion relation and give a simple example of how to use it. Following that we show how diagrams factorize in the space-cone gauge and we derive the largest time equation. Lastly we put the pieces together and show that there is a fundamental connection between the space-cone gauge, BCFW recursion and the largest time equation.

4.1 The Recursion

We will now derive the BCFW recursion relation in a manner similar to the one first introduced by Britto, Cachazo, Feng and Witten in [7]. The result itself was first conjectured by Britto, Cachazo and Feng [6] inspired by their their investigations of $\mathcal{N} = 4$ loop-amplitudes [10]. The content and simplicity of the result is astonishing. The recursion relation gives a prescription of how to express any tree-level amplitude in terms of lower point on-shell amplitudes. The main idea in the derivation is to deform two external legs and the consider the amplitude as a function of this deformation. The deformation is complex and such that the legs are still on-shell and momentum is conserved.

We start by looking at the n-point colour-ordered tree amplitude A_n. Here we will just look at gluonic amplitudes, but the results can be extended to include a much larger set of theories including massive particles, vector bosons and fermions [38–41]. Now we choose two external legs j and l and deform them in the following way

$$|j\rangle \quad \rightarrow \quad |\hat{j}\rangle = |j\rangle\,, \qquad\qquad |j] \quad \rightarrow \quad |\hat{j}] = |j] - z\,|l] \tag{4.1}$$

$$|l] \quad \rightarrow \quad |\hat{l}] = |l]\,, \qquad\qquad |l\rangle \quad \rightarrow \quad |\hat{l}\rangle = |l\rangle + z\,|j\rangle \tag{4.2}$$

Now it is clear that this deformation keeps the two legs on-shell since

$$(|\hat{j}\rangle\,[\hat{j}|)^2 = (|j\rangle\,[j| - z\,|j\rangle\,[l|)^2 = \langle jj\rangle\,[jj] - z\,\langle jj\rangle\,[lj] - z\,\langle jj\rangle\,[jl] + |z|^2\,\langle jj\rangle\,[ll] = 0 \tag{4.3}$$

and

$$(|\hat{l}\rangle\,[\hat{l}|)^2 = (|l\rangle\,[l| + z\,|j\rangle\,[l|)^2 = \langle ll\rangle\,[ll] + z\,\langle lj\rangle\,[ll] + z\,\langle jl\rangle\,[ll] + |z|^2\,\langle jj\rangle\,[ll] = 0. \tag{4.4}$$

Momentum is also conserved since

$$|\hat{j}\rangle [\hat{j}| + |\hat{l}\rangle [\hat{l}| = |j\rangle [j| - z\,|j\rangle [l| + |l\rangle [l| + z\,|j\rangle [l| = |j\rangle [j| + |l\rangle [l|\,. \tag{4.5}$$

At tree-level the amplitude is a function of products of spinor products. Hence the deformation turns the amplitude A_n into a rational function of the complex variable z. Let us now assume $A_n(z) \to 0$ as $z \to \infty$. At this point it is not obvious if and when this is the case. Suffice to say it depends on the helicity of the chosen deformed external legs. The helicity configurations $(h_j, h_l) = (-,-), (+,+)$ or $(-,+)$ will have the needed convergence whereas $(+,-)$ in general will not. If we apply Cauchy's Theorem with a contour enclosing all poles we have the following result

$$0 = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathrm{d}z \frac{A_n(z)}{z} = A_n(0) + \sum \frac{\mathrm{Res}(A_n(z), z_i)}{z_i} \tag{4.6}$$

where the sum is over all poles $z_i \neq 0$. The quantity $A_n(0)$ is of course nothing but the amplitude we started with and the one we want to derive. The remaining poles of $A_n(z)$ come from propagators going on-shell

$$P^2_{m,k}(z) \to 0 \quad \text{for} \quad z \to z_{m,k} \tag{4.7}$$

where

$$P_{m,k} = \sum_{i=m}^{k} p_i. \tag{4.8}$$

There can not be introduced poles from vertices, since these only add factors of momenta[1]. Now a pole in a propagator can only occur if this propagator explicitly depends on either leg j or l. If it depends on both we may infer from eq. (4.3) that $P_{h,k}$ does not depend on z any more. Let us without loss of generality assume $m < j < k < l$. Then

$$P^2_{m,k}(z) = (|m\rangle [m| + \cdots + |\hat{j}\rangle [\hat{j}| + \cdots + |k\rangle [k|)^2 = P^2_{m,k}(0) - z\,\langle j|\,P_{m,k}\,|l]\,. \tag{4.9}$$

Since by assumption $P^2_{m,k}(z_{m,k}) = 0$ it follows that

$$z_{m,k} = \frac{P^2_{m,k}(0)}{\langle j|\,P_{m,k}\,|l]} \tag{4.10}$$

Had we chosen a propagator with momentum l flowing through, we would have arrived at

$$z_{m,k} = -\frac{P^2_{m,k}(0)}{\langle j|\,P_{m,k}\,|l]} \tag{4.11}$$

[1] This statement is highly gauge dependent. In the space-cone gauge we have introduced spurious poles in the vertices, and hence one could fear that these could become dangerous when deforming external legs. In fact it turns out, that one can always choose the deformation such that all vertices are unaffected [8]. We will study this in section 4.2.

From (4.9) we see that the poles are actually simple poles. The residues are therefore straightforward to calculate

$$\begin{aligned}\frac{\text{Res}(A_n(z), z_{m,k})}{z_{m,k}} &= \frac{1}{z_{m,k}} \lim_{z \to z_{m,k}} \left[(z - z_{m,k}) A_n(z)\right] \\ &= \frac{\langle j|\, P_{m,k}\, |l]}{P^2_{m,k}} \lim_{z \to z_{m,k}} \left[-\frac{P^2_{m,k}(z)}{\langle j|\, P_{m,k}\, |l]} A_n(z) \right] \\ &= -\frac{1}{P^2_{m,k}} \lim_{P^2_{m,k} \to 0} \left[P^2_{m,k} A_n \right]. \end{aligned} \tag{4.12}$$

What happens with A_n as $P^2_{m,k} \to 0$ is simple to deduce. Since Yang-Mills theory is planar we know that the momenta that goes into the propagator have to be adjacent. Therefore the diagram splits up into a left-hand and a right-hand side separated by the propagator going on-shell. Since the helicity is not specified on the internal line we have to sum over the two possible helicity configurations $h = \pm$. We can therefore write

$$A_n \xrightarrow{P^2_{m,k} \to 0} \sum_h A_L(m, \ldots, k, -P^h_{m,k}) \frac{1}{P^2_{m,k}} A_R(P^{-h}_{m,k}, k+1, \ldots, m-1). \tag{4.13}$$

This is just the general factorization property of tree-level amplitudes [16]. We therefore have

$$\frac{\text{Res}(A_n(z), z_{m,k})}{z_{m,k}} = -\sum_h A_L(m, \ldots, k, -\hat{P}^h_{m,k}) \frac{1}{P^2_{m,k}} A_R(\hat{P}^{-h}_{m,k}, k+1, \ldots, m-1) \tag{4.14}$$

where the shifted propagator is given by

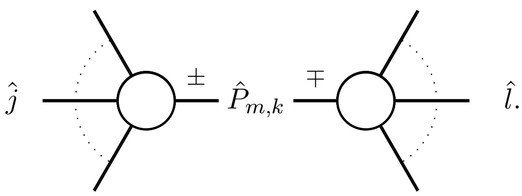
$$\hat{P}_{m,k} = P_{m,k} - \frac{P^2_{m,k}(0)}{\langle j|\, P_{m,k}\, |l]} |j\rangle\, [l|\,. \tag{4.15}$$

Combining this with (4.6) we arrive at the BCFW recursion relation

$$A_n = \sum_{m,k} \sum_h A_L(m, \ldots, k, -\hat{P}^h_{m,k}) \frac{1}{P^2_{m,k}} A_R(\hat{P}^{-h}_{m,k}, k+1, \ldots, m-1) \tag{4.16}$$

This extraordinary elegant result tells us that in order to calculate any n-point tree-level amplitude, all we have to do is pick two external legs and shift them so that they are still on-shell and conserve momentum. Then we cut all diagrams in the amplitude in all possible ways such that the shifted legs are on opposite sides of the shift and sum over helicity states. Diagrammatically we may write

$$A_n = \sum_{m,k}$$



(4.17)

In reality many of the terms in the sum are zero. One of the reasons for this is that the all plus and all minus amplitudes are zero along with all amplitudes with only one positive or one negative helicity gluon is zero. The only exception is the three-point amplitude evaluated with complex momenta.

A Simple Example

To illustrate the power of the BCFW recursion relation we compute the amplitude $A_5(1^+2^+3^-4^+5^-)$. We start by giving two useful relations. By acting with $\langle\cdot|$ and $|l]$ on (4.15) we get

$$\langle\cdot|\hat{P}_{m,k}\rangle = \frac{\langle\cdot|\, P_{m,k}\,|l]}{[\hat{P}_{m,k}|l]} \tag{4.18}$$

and likewise acting with $|\cdot]$ and $\langle j|$ we get

$$[\hat{P}_{m,k}|\cdot] = \frac{\langle j|\, P_{m,k}\,|\cdot]}{\langle j|\hat{P}_{m,k}\rangle}. \tag{4.19}$$

For the amplitude $A_5(1^+2^+3^-4^+5^-)$ we pick the following deformations

$$|5\rangle \quad\rightarrow\quad |\hat{5}\rangle = |5\rangle\,, \qquad\qquad |5] \quad\rightarrow\quad |\hat{5}] = |5] - z\,|1] \tag{4.20}$$

$$|1] \quad\rightarrow\quad |\hat{1}] = |1]\,, \qquad\qquad |1\rangle \quad\rightarrow\quad |\hat{1}\rangle = |1\rangle + z\,|5\rangle \tag{4.21}$$

The expansion of (4.16) then reads

$$\begin{aligned} A_5(\hat{1}^+2^+3^-4^+\hat{5}^-) &= A_L(\hat{1}^+,2^+,3^-,-\hat{P}_{45}^+)\frac{1}{P_{45}^2}A_R(\hat{P}_{45}^-,4^+,\hat{5}^-) \\ &+ A_L(\hat{1}^+,2^+,3^-,-\hat{P}_{45}^-)\frac{1}{P_{45}^2}A_R(\hat{P}_{45}^+,4^+,\hat{5}^-) \\ &+ A_L(\hat{1}^+,2^+,-\hat{P}_{12}^+)\frac{1}{P_{12}^2}A_R(\hat{P}_{12}^-,3^-,4^+,\hat{5}^-) \\ &+ A_L(\hat{1}^+,2^+,-\hat{P}_{12}^-)\frac{1}{P_{12}^2}A_R(\hat{P}_{12}^+,3^-,4^+,\hat{5}^-). \end{aligned} \tag{4.22}$$

The first and third terms are automatically zero since $A(++-+) = A(+++) = 0$. The second term

$$A_L(\hat{1}^+,2^+,3^-,-\hat{P}_{45}^-)\frac{1}{P_{45}^2}A_R(\hat{P}_{45}^+,4^+,\hat{5}^-) = \frac{\langle 3|-\hat{P}_{45}\rangle^3}{\langle\hat{1}2\rangle\,\langle 23\rangle\,\langle -\hat{P}_{45}|\hat{1}\rangle}\frac{1}{P_{45}^2}\frac{[\hat{P}_{45}|4]^3}{[4\hat{5}]\,[\hat{5}|\hat{P}_{45}]} \tag{4.23}$$

is also zero since $[\hat{P}_{45}|4] = \frac{\langle 5|P_{45}|4]}{\langle 5|\hat{P}_{45}\rangle} = 0$. Hence, the only term that contributes to the amplitude is the fourth given by

$$\begin{aligned} &A_L(\hat{1}^+,2^+,-\hat{P}_{12}^-)\frac{1}{P_{12}^2}A_R(\hat{P}_{12}^+,3^-,4^+,\hat{5}^-) = -8\frac{[\hat{1}2]^3}{[2|-\hat{P}_{12}]\,[-\hat{P}_{12}|\hat{1}]}\frac{1}{P_{12}^2}\frac{[4|\hat{P}_{12}]^4}{[\hat{P}_{12}|3]\,[34]\,[4\hat{5}]\,[\hat{5}|\hat{P}_{12}]} \\ &= -8\frac{[12]^3\,\langle 5|\,1+2\,|4]^4}{\langle 5|\,1+2\,|2]\,\langle 5|\,1+2\,|1]\,\langle 12\rangle\,[21]\,\langle 5|\,1+2\,|3]\,[34]\,[4|5-z_{12}1]\,\langle 5|\,1+2\,|5-z_{12}1]} \\ &= -8\frac{[12]^2\,\langle 5|\,3\,|4]^4}{\langle 5|\,1\,|2]\,\langle 5|\,2\,|1]\,\langle 12\rangle\,\langle 5|\,4\,|3]\,[34]\,([45]+\frac{\langle 12\rangle}{\langle 52\rangle}[41])(\langle 5|\,1+2\,|5]+\frac{\langle 12\rangle}{\langle 52\rangle}\langle 5|\,2\,|1])} \\ &= -8\frac{\langle 53\rangle^4\,[34]^2}{\langle 51\rangle\,\langle 12\rangle\,\langle 54\rangle\,\underbrace{(\langle 52\rangle\,[45]+\langle 12\rangle\,[41])}_{=-\langle 23\rangle[34]}\underbrace{(\langle 51\rangle\,[15]+\langle 52\rangle\,[25]+\langle 12\rangle\,[21])}_{=\langle 34\rangle[43]}} \\ &= 8\frac{\langle 53\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 45\rangle\,\langle 51\rangle}. \end{aligned} \tag{4.24}$$

In the last lines we have applied momentum conservation. In this way all tree amplitudes within Yang-Mills theory can be obtained. Compared to our calculation of a similar five-point amplitude in the space-cone gauge it is seen that the BCFW recursion improves somewhat. For $NMHV$ amplitudes and beyond the simplifications are much greater and in particular the six-point $NMHV$ amplitude that we calculated can be derived in only two pages of algebra.

4.2 BCFW and Space-Cone gauge

One should ask if the BCFW also works on a diagrammatic level and not just in terms of amplitudes. Say we wanted to calculate an n-point tree-level subamplitude A_n. We draw all contributing Feynman diagrams and then we identify two external momenta. We then shift these momenta according to (4.2) and cut propagators in all possible ways keeping the two shifted momenta separated by this cut. According to the BCFW recursion, the entire amplitude should be decomposed into lower-point amplitudes. We want to investigate, whether the diagrams that result from this procedure can be identified with the diagrams that one would write down to calculate the lower-point amplitudes. To do so we will find that the space-cone gauge is particular suited. We start by looking at an example. From that example it will clear for us, what general identities we need to establish in order to prove our result for all tree-level diagrams. This identification was first made by Vaman and Yao in [8] and we will follow their derivation most of the way. Another diagrammatic proof of the BCFW recursion was given almost simultaneously by Draggiotis et al. in [42] but we will only focus on the first proof.

Factorization of $A_4(++--)$

We start by looking at the simplest amplitude that has potential to be factorized into lower-point components. Assume for the moment, that the BCFW recursion does not exist. We ask ourselves if it is possible to factor the four-point amplitude into two on-shell three-point amplitudes. First we write down all diagrams for the amplitude. If we choose $P_1 = |-\rangle [-|$ and $P_4 = |+\rangle [+|$ as we did when we calculated the amplitude earlier, we know that only one diagram contribute, namely (3.38). Since we want to keep the amplitudes on-shell we want to shift two external momenta according to (4.2). We want the shifts to have as little impact as possible since we want to show that the amplitude does not change under such a shift. The most important thing to notice is, that no vertex in the space-cone formalism depends on the momentum component $\bar{p}$ and neither does any polarization vector. If we make the shift along this direction we therefore only shift the internal propagators and leave all vertices invariant. The $\bar{p}$ direction is according to (3.8) given by

$$|+\rangle [-| = |4\rangle [1| \,. \tag{4.25}$$

We therefore choose to shift the reference momenta along this direction without changing our spinor basis. That is after we shift $|1\rangle [1| \to |1\rangle [1| + z\,|4\rangle [1|$ and $|4\rangle [4| \to |1\rangle [1| - z\,|4\rangle [1|$ we still retain $|1\rangle [1| = |-\rangle [-|$ and $|4\rangle [4| = |+\rangle [+|$. The direction we choose to shift along is exactly along the space-cone gauge direction. The impact of the shift on the components of

P_1 and P_4 is trivial to work out

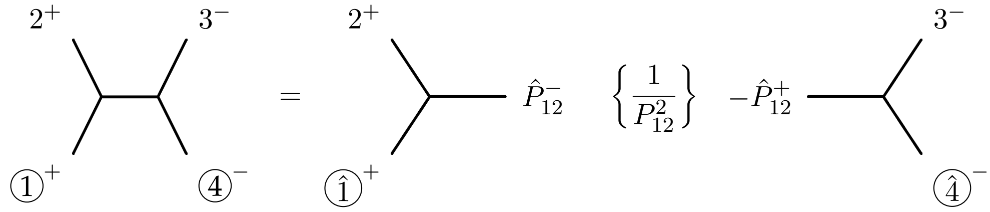
$$\hat{p}_1 = 0, \quad \hat{\bar{p}}_1 = \frac{z\,\langle 41\rangle\,[14]}{\sqrt{\langle 41\rangle\,[14]}}, \quad \hat{p}_1^+ = 0, \quad \hat{p}_1^- = p_\oplus^- \tag{4.26}$$

$$\hat{p}_4 = 0, \quad \hat{\bar{p}}_4 = \frac{-z\,\langle 41\rangle\,[14]}{\sqrt{\langle 41\rangle\,[14]}}, \quad \hat{p}_4^+ = p_\ominus^+, \quad \hat{p}_4^- = 0. \tag{4.27}$$

Just as expected the only components that change are $\hat{\bar{p}}_1$ and $\hat{\bar{p}}_4$. Now we cut the propagator such that the two resulting three-point diagrams are on-shell. From (4.15) we have

$$\hat{P}_{12} = P_{12} - \frac{P_{12}^2}{\langle 4|\,P_{12}\,|1]}\,|4\rangle\,|1]\,. \tag{4.28}$$

Inserting the unshifted propagator between the two three-point diagrams we arrive at the following diagrammatic factorization.



(4.29)

Now since the three-point vertex and the three-point function is identical up to two external polarization factors we can identify the two three-point diagrams with the full three-point amplitude. Since $\epsilon_+(k)^+\epsilon_-(k)^- = -1$ we are free to insert such factors. Algebraically the factorization takes the form

$$\begin{aligned} A_4 &= i^2\left[\epsilon_+(2)^+(-2p_2)\right]\frac{1}{P_{12}^2}\left[\epsilon_-(3)^-(2p_3)\right] \\ &= -i^2\left[\epsilon_+(2)^+\epsilon_-(\hat{P}_{12})^-\frac{\langle -+\rangle}{\sqrt{\langle -+\rangle\,[-+]}}(-2p_2)\right]\frac{1}{P_{12}^2}\left[\epsilon_-(3)^-\epsilon_+(-\hat{P}_{12})^+\frac{[-+]}{\sqrt{\langle -+\rangle\,[-+]}}(2p_3)\right] \end{aligned} \tag{4.30}$$

which can be verified to indeed be the product of the googly-MHV and MHV three-point amplitude as predicted by the BCFW recursion. Since $\epsilon_+(k)^+\epsilon_-(k)^- = -1$ we pick up a sign in the amplitude. In the original article by Vaman and Yao they work in the funny normalization of Chalmers and Siegel where $\langle -+\rangle\,[+-] = 1$ and hence $\epsilon_+(k)^+\epsilon_-(k)^- = 1$. This sign difference is of course trivial and we should not be concerned about it.

The factorization was almost trivial here, since after we made the cut, there were no more propagators left, on which the shift could leave an imprint. Notice, that in principle we first showed that the factorization on the right-hand side is equal to the diagram on the left-hand side and *then* we saw that the factorization was identical to the BCFW factorization. From this we learn that in general if we shift the two reference lines along the $|+\rangle\,|-]$ direction it leaves *no imprint on the vertices*. We now take a look at the $A_5(+++--)$ amplitude to see what happens with the shifted internal propagators.

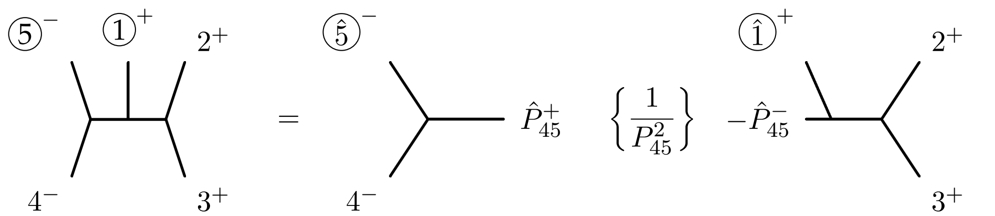
Factorization of $A_5(+++--)$

We proceed to evaluate the amplitude $A_5(+++--)$ in the same way as we did above. If we choose $P_1 = |-\rangle\,[-|$ and $P_5 = |+\rangle\,[+|$ there are three diagrams (3.42) that contribute to the amplitude. The shifts now take the form

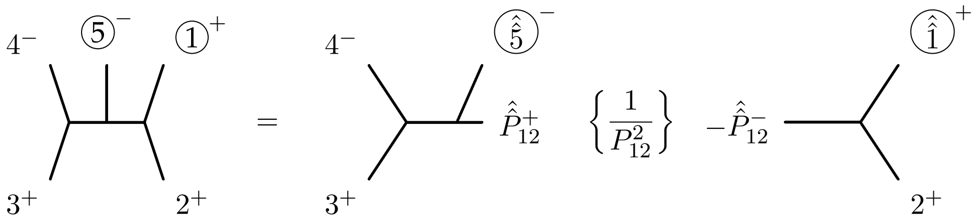
$$|1\rangle\,[1| \to |1\rangle\,[1| + z\,|5\rangle\,[1| \tag{4.31}$$

$$|5\rangle\,[5| \to |1\rangle\,[1| - z\,|5\rangle\,[1| \tag{4.32}$$

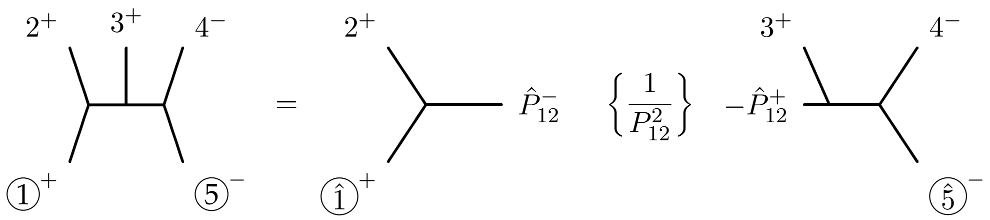
After cutting all propagators we have the following diagrammatic factorization



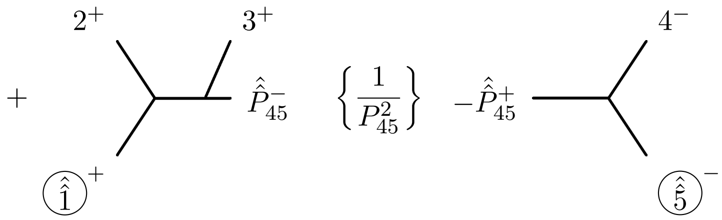
(4.33)



(4.34)



(4.35)



(4.36)

We have emphasized that the shifts are not the same in all diagrams by putting either one or two hats on the shifted variables. Let us now show that this factorization is indeed correct. That the right-hand side of (4.33) and (4.34) equals the left-hand side is obvious since no propagators are shifted. We may therefore freely insert a factor of $\epsilon_+(k)^+\epsilon_-(k)^- = -1$ just as before to obtain the factorization. As with the four-point amplitude above, we therefore factor out the vertices, which are the same on both sides of the equality and we then insert appropriate polarizations. To show that the sum of (4.35) and (4.36) equals the left-hand side diagram is not as trivial. We may still factor out vertices, as these are the same on both sides. What is left is then to show an identity between the propagators of the two sides

$$\frac{1}{P_{12}^2}\frac{1}{P_{45}^2} = \frac{1}{P_{12}^2}\frac{1}{P_{4\hat{5}}^2} + \frac{1}{P_{\hat{1}2}^2}\frac{1}{P_{45}^2}.$$

$$\Leftrightarrow 1 = \frac{P_{45}^2}{P_{4\hat{5}}^2} + \frac{P_{12}^2}{P_{\hat{1}2}^2} \tag{4.37}$$

From (4.10) and (4.11) we may write

$$P_{12}^2 = -\hat{z}_{12}\langle 5|P_{12}|1], \qquad P_{45}^2 = \hat{\hat{z}}_{45}\langle 5|P_{45}|1] \tag{4.38}$$

and

$$P_{\hat{1}2}^2 = (P_{\hat{1}} + P_2)^2 = (P_1 + \hat{\hat{z}}_{45}|5\rangle[1| + P_2)^2 = P_{12}^2 + \hat{\hat{z}}_{45}\langle 5|P_{12}|1] = (\hat{\hat{z}}_{45} - \hat{z}_{12})\langle 5|P_{12}|1] \tag{4.39}$$

$$P_{4\hat{5}}^2 = (P_4 + P_{\hat{5}})^2 = (P_4 + P_5 - \hat{z}_{12}|5\rangle[1|)^2 = P_{45}^2 - \hat{z}_{12}\langle 5|P_{45}|1] = (\hat{\hat{z}}_{45} - \hat{z}_{12})\langle 5|P_{45}|1] \tag{4.40}$$

Inserting this back into the right-hand side (4.37) we arrive at

$$1 = \frac{P_{45}^2}{P_{4\hat{5}}^2} + \frac{P_{12}^2}{P_{\hat{1}2}^2} = \frac{\hat{\hat{z}}_{45}\langle 5|P_{45}|1]}{(\hat{\hat{z}}_{45} - \hat{z}_{12})\langle 5|P_{45}|1]} + \frac{-\hat{z}_{12}\langle 5|P_{12}|1]}{(\hat{\hat{z}}_{45} - \hat{z}_{12})\langle 5|P_{12}|1]} = 1 \tag{4.41}$$

The factorization that we choose is indeed equal to the original diagrams. By inspecting (4.35)-(4.34) we see that (4.36) and (4.33) add up to

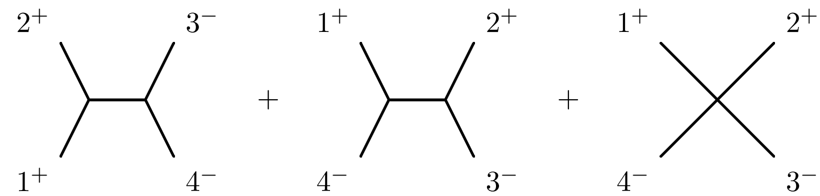
$$A_3(--+)\frac{1}{P_{45}^2}A_4(-+++) \tag{4.42}$$

while (4.35) and (4.34) add up to

$$A_3(++-)\frac{1}{P_{12}^2}A_4(++--). \tag{4.43}$$

This is exactly the BCFW factorization of (4.16). The $(-+++)$ amplitude is of course zero, so we may conclude that our factorization procedure will generate not only the non-zero terms of the BCFW recursion but also some terms that are zero. In general we will not produce all the zero terms of the BCFW recursion. When we calculated the four-point amplitude in section 3.3 we only had one diagram to compute. Why are there two diagrams that contribute now and why were we so fast at concluding that they really did sum up to the four-point amplitude? Remember that before we had chosen reference lines there were actually three diagrams

$A_4(++--) =$



(3.37)

If we pick one reference line the four-point vertex vanishes but the two diagrams with only three-point vertices survive. These are exactly the diagrams that contributed in our factorization. This is a general feature of the factorization since we always cut a propagator between the two shifted legs and we will therefore always have to identify the diagrams corresponding to the amplitude with only one reference line attached.

The identity (4.37) that we proved for the five-point case will be needed in a generalized version i order for us to be able to show, that this factorization always works out. Since it is clear that no matter what tree-level amplitude we look at, the vertices will always factorize, the only thing we need to show is that the propagators are shifted in the correct way. It is possible to show that any tree-level amplitude is invariant under our cut prescription directly but we would like to root this identity in a more fundamental relation in quantum field theory, the largest time equation. Hence, we will use some time on introducing this.

4.3 The Largest Time Equation

The largest time equation is at the core of quantum field theory. Unlike the space-cone gauge and the BCFW recursion which are both rather modern concepts the largest time equation was already introduced by Veltman in 1963 [9]. It is closely related to unitarity in quantum field theory and one of its main consequences is a set of cutting equations similar to those introduced by Cutkosky in 1960 [43]. In this section we will derive the largest time equation following Veltman and 't Hooft [44, 45] although it will differ in some places since they use the mostly plus metric. We will postpone the derivation of the cutting equations until we discuss loops in part III. Since the largest time equation was derived long before the MHV formalism it is formulated in the language of "real" Feynman diagrams. However, as we will see the derivation holds for all types of diagrams where propagators satisfy certain simple relations. We assume that any vertex part of a diagram is just a trivial coupling constant but more complicated vertices may enter as long as they do not introduce poles. For any local gauge theory, vertices will always be rational in the external momenta and for such theories we will expect the derivation to hold.[2]

We start by assuming that our theory may be described by Feynman rules and diagrams. In such diagrams we expect a propagator to turn up and for simplicity we only look at scalar propagators. With our metric the usual Feynman propagator in coordinate space takes the form

$$\Delta_F(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \mathrm{e}^{-ipx} \tag{4.44}$$

We can write the Feynman propagator in terms of positive and negative frequency parts also called cutting propagators. This name will become apparent later on. We write

$$\Delta_F(x) = \theta(x_0)\Delta^+(x) + \theta(-x_0)\Delta^-(x) \tag{4.45}$$

and define

$$\Delta^\pm(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^3} \mathrm{e}^{-ipx}\theta(\pm p_0)\delta(p^2 - m^2). \tag{4.46}$$

[2]The attentive reader should note that the space-cone formalism is actually non-local. Therefore we have also introduced the so-called spurious poles in our lagrangian. They will not be an issue for what is to come.

From this we have that Δ^+ and Δ^- are complex conjugated of each other and furthermore it holds that $\Delta^+(x) = \Delta^-(-x)$. Let us show that (4.45) is true by inserting (4.46) into (4.45).

$$\begin{aligned}\theta(\pm x_0)\Delta^\pm(x) &= \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \mathrm{d}k \frac{\mathrm{e}^{\pm ikx_0}}{k+i\epsilon}\int \frac{\mathrm{d}^4p}{(2\pi)^3}\mathrm{e}^{-ipx}\theta(\pm p_0)\delta(p^2-m^2) \\ &= \frac{i}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k\int\frac{\mathrm{d}^4p}{(2\pi)^3}\frac{\theta(\pm p_0)\delta(p^2-m^2)}{k+i\epsilon}\mathrm{e}^{\pm ikx_0-ipx} \\ &= \frac{i}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k\int\frac{\mathrm{d}^4p}{(2\pi)^3}\frac{\theta(\pm p_0)\delta(p^2-m^2)}{\mp k+i\epsilon}\mathrm{e}^{-ikx_0-ipx} \end{aligned} \tag{4.47}$$

Where we have made a smart change of k in the last line. Substituting $p_0 \to k + p_0$ we have

$$\theta(\pm x_0)\Delta^\pm(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k\int\frac{\mathrm{d}^4p}{(2\pi)^3}\frac{\theta(\pm k \pm p_0)\delta((k+p_0)^2-\mathbf{p}^2-m^2)}{\mp k+i\epsilon}\mathrm{e}^{-ipx} \tag{4.48}$$

Now performing the k-integral we single out the two roots of the δ-function of which only one survive due to the step function

$$k + p_0 = \pm\sqrt{\mathbf{p}^2+m^2} = \pm\omega_p \tag{4.49}$$

and we arrive at

$$\theta(\pm x_0)\Delta^\pm(x) = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{2\omega_p}\frac{i\mathrm{e}^{-ipx}}{\pm p_0-\omega_p+i\epsilon} \tag{4.50}$$

Inserting into (4.45) we have

$$\begin{aligned}\Delta_F(x) &= \int\frac{\mathrm{d}^4p}{(2\pi)^4}\left[\frac{1}{p_0-\omega_p+i\epsilon}+\frac{1}{-p_0-\omega_p+i\epsilon}\right]\frac{i\mathrm{e}^{-ipx}}{2\omega_p} \\ &= \int\frac{\mathrm{d}^4p}{(2\pi)^4}\left[\frac{-2\omega_p-2i\epsilon}{(p_0-\omega_p+i\epsilon)(-p_0-\omega_p+i\epsilon)}\right]\frac{i\mathrm{e}^{-ipx}}{2\omega_p} \\ &= \int\frac{\mathrm{d}^4p}{(2\pi)^4}\left[\frac{i\mathrm{e}^{-ipx}}{p^2-m^2+2i\epsilon\omega_p}\right]. \end{aligned} \tag{4.51}$$

By a innocent redefinition of ϵ we have recovered the Feynman propagator. We also have the complex conjugated of the Feynman propagator given by conjugating (4.45)

$$\Delta_F^*(x) = \theta(-x_0)\Delta^+(x) + \theta(x_0)\Delta^-(x). \tag{4.52}$$

Now the largest time equation is essentially a generalization of (4.45) which tells us that we may interchange Δ_F with $\Delta^\pm$ depending on the sign of x_0. We want to generalize this statement to diagrams containing more than one propagator. At the moment we do not include external states in our discussion. We are only interested in the structure of internal lines.

Assume that we have a diagram with n vertices located at space-time points $x_1 \ldots x_n$. Connecting two of these points x_i, x_j is a propagator $\Delta_F(x_j - x_i)$. Take one of the space-time points to have the largest time. That is $(x_m)^0 > (x_i)^0$ for all $i \neq m$. Then all propagators Δ_F containing the argument $x_m - x_i$ may be replaced by Δ^+ and all propagators containing $x_i - x_m$ may be replaced by Δ^-. This is the largest time equation in the case where only one vertex is singled out. We may define diagrammatic rules that allows us to invoke the largest time equation very simply. We define Feynman diagrams in the usual way but now we also include the possibility of circling a vertex. In the above example the vertex x_m was circled.

- Every circled vertex in a diagram we will give a minus sign
- A line connecting two uncircled vertices corresponds to a Feynman propagator Δ_F
- A line connecting two circled vertices corresponds to a conjugated Feynman propagator Δ_F^*
- A line connecting a circled to an uncircled vertex corresponds to Δ^-
- A line connecting an uncircled to a circled vertex corresponds to Δ^+

With these conventions we can now formulate the general form of the largest time equation.

The Largest Time Equation. *Given any diagram with any number of circled and uncircled vertices we may obtain minus the original diagram by either removing or adding a circle around the largest time x_m.*

Now this might neither seem as a very powerful nor useful equation. Are these circled and uncircled objects even related to something physical? And what are we to do with this equation when we have no idea which time is the largest? We can give an answer to the first question right away since the completely uncircled diagram is of course the "real" Feynman diagram and the fully circled diagram is just the complex conjugated Feynman diagram. All other diagrams are best understood when we discuss cutting equations. But what about diagrams where we do not know the ordering of the time. When we construct Feynman diagrams in x-space we are required to integrate over all space-time components. Doing so it becomes complicated to know the time-ordering of the vertices. This forces us to think of the largest time equation in another way. Let us define the function $F(x_1, \ldots, x_n)$. Each space-time point x_i can be underlined or not. The F with no underlines is just the diagram with no vertices circled and an F with one or more of the x_i's underlined is the diagram with the same vertices circled. Consider now a diagram and all its possible circlings. Even though we do not know which time is the largest we know that to every diagram exists the same diagram but with the largest time vertex circled or uncircled. From this we can conclude, that the sum over *all possible circlings* is zero. Formulated in the language of F we have

$$\sum_{\text{underlinings}} F(x_1, \ldots, x_n) = 0. \tag{4.53}$$

Inside this sum sits in particular the function with no underlinings and the function with all underlinings. Singling these two out we obtain a form of the above expression that will become important for us later on

$$F(x_1, \ldots, x_n) + F^*(x_1, \ldots, x_n) = - \sum_{\text{underlinings}_{\neq 0,n}} F(x_1, \ldots, x_n). \tag{4.54}$$

This states that the sum of the diagram and its complex conjugate is given by the sum over all possible circlings (underlinings) of diagrams. We will interpret this result later on. There is one form of (4.53) that will be important when discussing the BCFW recursion. Let us assume that we know that one of the vertices,x_k, does not carry the largest time. That means that there exists x_j such that $x_k^0 < x_j^0$ where x_j does not have to be globally the largest time. Then we can remove all diagrams with x_k underlined, since these by themselves add up to zero. We are therefore left with all diagrams where x_k is not underlined. We write this as

$$\sum_{\text{underlinings}\neq x_k} \theta(x_j^0 - x_k^0)F(x_1, \ldots, x_n) = 0 \tag{4.55}$$

Adding this equation to the same equation with x_k and x_j interchanged we obtain a special case of the largest time equation

$$F(x_i) = -F(x_k, x_j, x_i) + \theta(x_j^0 - x_k^0)F(x_k, \underline{x_j}, x_i) + \theta(x_k^0 - x_j^0)F(\underline{x_k}, x_j, x_i) \tag{4.56}$$

Here $F(x_i)$ has no underlinings, $F(x_k, \underline{x_j}, x_i)$ has x_j underlined but not x_k, $F(\underline{x_k}, x_j, x_i)$ has x_k underlined but not x_j and $F(x_k, x_j, \overline{x_i})$ has neither x_k nor x_j underlined. A sum over all possible underlinings of the remaining x_i's is to be understood. With this last identity we may show a connection between the largest time equation and the BCFW recursion.

4.4 Connecting the BCFW Recursion and the Largest Time Equation

By working in the space-cone gauge we explicitly saw that BCFW shifts do not impact vertices or polarization factors. The imprint the shifts leave on propagators was shown in the five-point amplitude to be equivalent to an algebraic identity amongst the propagators. We would now want to show that this identity is rooted in the largest time equation. The reason for doing this is that the largest time equation is a very fundamental quantum field theoretical result only relying on the structure of the Feynman propagator. We will not derive the general BCFW recursion from the largest time equation only show that the propagator identity for the five-point amplitude (4.37) can be recovered directly from the largest time equation. The reason for looking at a special case is that calculations become very non-transparent in the general case and as such do not bring any insight into the problem. That things work out in a special case is enough to prove that a connection between the BCFW recursion and the largest time equation exists.

There are certain problems we are faced with from the beginning. Our plan is to write out the propagators using (4.45). We will first rewrite the step function as

$$\theta(y) = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \mathrm{d}k \frac{\mathrm{e}^{-iky}}{k + i\epsilon} \to \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \mathrm{d}z \frac{\mathrm{e}^{-iz\eta\cdot x}}{z + i\epsilon} = \theta(\eta \cdot x) \tag{4.57}$$

where $\eta \cdot x$ needs only be proportional to y and positive. Now when we showed that

$$\Delta_F(x) = \theta(x^0)\Delta^+(x) + \theta(-x^0)\Delta^-(x) \tag{4.58}$$

we could have chosen other coordinates than x^0 to "time"-order the propagator. By manipulations similar to those above it can be shown [8] that the propagator may be decomposed into

$$\Delta_F(x) = \theta(\eta \cdot x)\Delta^+(x) + \theta(-\eta \cdot x)\Delta^-(x) \tag{4.59}$$

and

$$\Delta_F^*(x) = \theta(-\eta \cdot x)\Delta^+(x) + \theta(\eta \cdot x)\Delta^-(x) \tag{4.60}$$

if we redefine $\Delta^\pm$ as

$$\Delta^\pm = \int \frac{\mathrm{d}^4 p}{(2\pi)^3} \mathrm{e}^{-ipx} \theta(\pm\eta \cdot p) \delta(p^2 - m^2). \tag{4.61}$$

These equations only make sense if η is real since we can only order $\eta \cdot x$ if it is real. This is a problem since we in a moment would like to identify η with a complex null-direction. Following [8] what we will do is to assume η is real and just carry on with the calculations. Since it is possible to analytically complexify η at tree-level as it only enters in rational functions and that similar results can be obtained from axiomatic quantum field theory to extend the results into loop-level [46] we will assume that our result also holds for complex η. The largest time equation can equally well be formulated with this decomposition of the propagator. Now instead of the largest *time* equation we get the largest $\eta \cdot x$ equation. We will still just call this the largest time equation. We also introduce some shorthand notation

$$\Delta_{ij} = \Delta_F(x_i - x_j), \quad \theta_{ij} = \theta(\eta \cdot (x_i - x_j)), \quad \Delta^\pm_{ij} = \Delta^\pm(x_i - x_j) \tag{4.62}$$

We start by writing the left-hand side of the propagator identity (4.37) in terms of coordinate propagators. First we multiply with a δ-function of overall momentum conservation. We will do the calculation for the left-hand side in great detail. We work with massless propagators here, but the argument could also be given for propagators with mass

$$\begin{aligned}
\frac{\delta^{(4)}(P_1 + \cdots + P_5)}{P_{12}^2 P_{45}^2} &= \int \frac{\mathrm{d}^4 x_2}{(2\pi)^4} \mathrm{e}^{i(P_1 + \cdots + P_5)x_2} \frac{i}{(P_1+P_2)^2 + i\epsilon} \frac{i}{(P_4+P_5)^2 + i\epsilon} \\
&= \int \frac{\mathrm{d}^4 x_2}{(2\pi)^4} \frac{\mathrm{d}^4 Q_1}{(2\pi)^4} \frac{\mathrm{d}^4 Q_2}{(2\pi)^4} \mathrm{e}^{i(P_1 + \cdots + P_5)x_2} \frac{i\delta^{(4)}(P_1 + P_2 - Q_1)}{Q_1^2 + i\epsilon} \frac{i\delta^{(4)}(P_4 + P_5 - Q_2)}{Q_2^2 + i\epsilon} \\
&= \int \frac{\mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \mathrm{d}^4 x_3 \mathrm{d}^4 Q_1 \mathrm{d}^4 Q_2}{(2\pi)^4 (2\pi)^4 (2\pi)^4 (2\pi)^4 (2\pi)^4} \mathrm{e}^{i(P_3 + Q_1 + Q_2)x_2} \frac{i\mathrm{e}^{i(P_1 + P_2 - Q_1)x_1}}{Q_1^2 + i\epsilon} \frac{i\mathrm{e}^{i(P_4 + P_5 - Q_2)x_3}}{Q_2^2 + i\epsilon} \\
&= \int \frac{\mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \mathrm{d}^4 x_3}{(2\pi)^4 (2\pi)^4 (2\pi)^4} \mathrm{e}^{i(P_1 + P_2)x_1} \mathrm{e}^{iP_3 x_2} \mathrm{e}^{i(P_4 + P_5)x_3} \\
&\quad \times \int \frac{\mathrm{d}^4 Q_1}{(2\pi)^4} \frac{i\mathrm{e}^{iQ_1(x_2 - x_1)}}{Q_1^2 + i\epsilon} \int \frac{\mathrm{d}^4 Q_2}{(2\pi)^4} \frac{i\mathrm{e}^{iQ_2(x_2 - x_3)}}{Q_2^2 + i\epsilon} \\
&= \int \frac{\mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \mathrm{d}^4 x_3}{(2\pi)^4 (2\pi)^4 (2\pi)^4} \mathrm{e}^{i(P_1 + P_2)x_1} \mathrm{e}^{iP_3 x_2} \mathrm{e}^{i(P_4 + P_5)x_3} \Delta_F(x_1 - x_2) \Delta_F(x_2 - x_3)
\end{aligned} \tag{4.63}$$

This is of course just a rewriting of the momentum space propagator into space-time propagators saying that $P_1 + P_2$ flows out at space-time point x_1 and so forth. For the two shifted propagators on the right-hand side we use (4.59). As stated before we will assume that the shift made in (4.37) is real. It is easiest to follow the calculations backwards, so we work our way from the final result and back to the propagators

$$\begin{aligned}
&\int \frac{\mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3}{(2\pi)^4(2\pi)^4(2\pi)^4}\mathrm{e}^{i(P_1+P_2)x_1}\mathrm{e}^{iP_3x_2}\mathrm{e}^{i(P_4+P_5)x_3}\Delta_F(x_2-x_3)\\
&\times[\theta(\eta\cdot(x_1-x_3))\Delta^+(x_1-x_2)+\theta(-\eta\cdot(x_1-x_3))\Delta^-(x_1-x_2)]\\
&=\int \frac{\mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3}{(2\pi)^4(2\pi)^4(2\pi)^4}\mathrm{e}^{i(P_1+P_2)x_1}\mathrm{e}^{iP_3x_2}\mathrm{e}^{i(P_4+P_5)x_3}\int\frac{\mathrm{d}^4Q_2}{(2\pi)^4}\frac{i\mathrm{e}^{-iQ_2(x_2-x_3)}}{Q_2^2+i\epsilon}\\
&\times i\int \mathrm{d}z\mathrm{e}^{-iz\eta\cdot(x_1-x_3)}\int\frac{\mathrm{d}^4Q_1}{(2\pi)^4}\mathrm{e}^{-iQ_1(x_1-x_2)}\delta(Q_1^2)\left[\frac{\theta(-\eta\cdot Q_1)}{z+i\epsilon}-\frac{\theta(\eta\cdot Q_1)}{z-i\epsilon}\right]\\
&=\int\frac{\mathrm{d}z\mathrm{d}^4Q_1\mathrm{d}^4Q_2\mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3}{(2\pi)^4(2\pi)^4(2\pi)^4(2\pi)^4(2\pi)^4}\mathrm{e}^{i(P_1+P_2-Q_1-z\eta)x_1}\mathrm{e}^{i(P_3+Q_1-Q_2)x_2}\mathrm{e}^{i(P_4+P_5+Q_2+z\eta)x_3}\frac{i}{Q_2^2+i\epsilon}\\
&\times i\delta(Q_1^2)\left[\frac{\theta(-\eta\cdot Q_1)}{z+i\epsilon}-\frac{\theta(\eta\cdot Q_1)}{z-i\epsilon}\right]\\
&\overset{Q_1\to Q_1-z\eta}{=}\int\frac{\mathrm{d}z\mathrm{d}^4Q_1\mathrm{d}^4Q_2\mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3}{(2\pi)^4(2\pi)^4(2\pi)^4(2\pi)^4(2\pi)^4}\mathrm{e}^{i(P_1+P_2-Q_1)x_1}\mathrm{e}^{i(P_3+Q_1-Q_2-z\eta)x_2}\mathrm{e}^{i(P_4+P_5+Q_2+z\eta)x_3}\\
&\times\frac{i^2}{Q_2^2+i\epsilon}\delta((Q_1-z\eta)^2)\left[\frac{\theta(-\eta\cdot Q_1)}{z+i\epsilon}-\frac{\theta(\eta\cdot Q_1)}{z-i\epsilon}\right]\\
&=\int\mathrm{d}z\delta(P_1+\dots+P_5)\frac{i^2\delta((P_1+P_2-z\eta)^2)}{(P_4+P_5-z\eta)^2+i\epsilon}\left[\frac{\theta(-\eta\cdot(P_1+P_2))}{z+i\epsilon}-\frac{\theta(\eta\cdot(P_1+P_2))}{z-i\epsilon}\right] \qquad (4.64)
\end{aligned}$$

where we in the last line have integrated to first obtain three δ-functions and then used two of these to perform the momentum integrals. To perform the last integral we use the last δ-function to put z to the value $\hat{z}_{12}$ which was defined to put the propagator P_{12} on-shell.

$$\begin{aligned}
(4.64)&=\frac{\delta(P_1+\dots+P_5)}{P_{4\hat{5}}^2}\\
&\times\frac{i}{|2\eta\cdot(P_1+P_2)|}\left[\frac{-\theta(-\eta\cdot(P_1+P_2))2\eta\cdot(P_1+P_2)}{(P_1+P_2)^2-2i\epsilon\eta\cdot(P_1+P_2)}-\frac{-\theta(\eta\cdot(P_1+P_2))2\eta\cdot(P_1+P_2)}{(P_1+P_2)^2+2i\epsilon\eta\cdot(P_1+P_2)}\right]\\
&=\frac{\delta(P_1+\dots+P_5)}{P_{4\hat{5}}^2}i\left[\frac{\theta(-\eta\cdot(P_1+P_2))}{P_{12}^2+i\epsilon}-\frac{\theta(\eta\cdot(P_1+P_2))}{P_{12}^2+i\epsilon}\right]\\
&=\frac{\delta(P_1+\dots+P_5)}{P_{4\hat{5}}^2P_{12}^2} \qquad (4.65)
\end{aligned}$$

By similar manipulations we obtain the following expression for the second term on the right-hand side

$$\begin{aligned}
\frac{\delta(P_1+\dots+P_5)}{P_{45}^2P_{\hat{1}2}^2}&=\int\frac{\mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3}{(2\pi)^4(2\pi)^4(2\pi)^4}\mathrm{e}^{i(P_1+P_2)x_1}\mathrm{e}^{iP_3x_2}\mathrm{e}^{i(P_4+P_5)x_3}\Delta_F(x_1-x_2)\\
&\times[\theta(\eta\cdot(x_1-x_3))\Delta^+(x_2-x_3)+\theta(-\eta\cdot(x_1-x_3))\Delta^-(x_2-x_3)] \qquad (4.66)
\end{aligned}$$

Collecting the pieces the propagator identity now reads

$$\begin{aligned}
&\int\frac{\mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3}{(2\pi)^4(2\pi)^4(2\pi)^4}\mathrm{e}^{i(P_1+P_2)x_1}\mathrm{e}^{iP_3x_2}\mathrm{e}^{i(P_4+P_5)x_3}\\
&\times\left(\Delta_{12}\Delta_{23}-[\theta_{13}\Delta_{12}^++\theta_{31}\Delta_{12}^-]\Delta_{23}-[\theta_{13}\Delta_{23}^++\theta_{31}\Delta_{23}^-]\Delta_{12}\right)=0. \qquad (4.67)
\end{aligned}$$

By writing some of the propagators as $\Delta^\pm$ the integrand without exponentials is equal to

$$\begin{aligned}0 =&\Delta_{12}\Delta_{23} - \theta_{13}\Delta^+_{12}\Delta_{23} - \theta_{31}\Delta^-_{23}\Delta_{12} - \theta_{31}\theta_{23}\Delta^-_{12}\Delta^+_{23} \\ &- \theta_{31}\theta_{32}\Delta^-_{12}\Delta^-_{23} - \theta_{13}\theta_{12}\Delta^+_{23}\Delta^+_{12} - \theta_{13}\theta_{21}\Delta^+_{23}\Delta^-_{12}.\end{aligned} \tag{4.68}$$

Now comes the time to invoke the largest time equation in the form of (4.56) where we single out x_1 and x_3. Remember that circling (or underlining) a vertex will interchange propagators with their positive and negative frequency part or the conjugated propagator.

$$\Delta_{12}\Delta_{23} - \Delta^-_{12}\Delta^+_{23} - \theta_{13}\left[\Delta^+_{12}\Delta_{23} - \Delta^*_{12}\Delta^+_{23}\right] - \theta_{31}\left[\Delta_{12}\Delta^-_{23} - \Delta^-_{12}\Delta^*_{23}\right] = 0. \tag{4.69}$$

Writing out Δ^*_F as (4.60) we have

$$\begin{aligned}0 = &\Delta_{12}\Delta_{23} - \Delta^-_{12}\Delta^+_{23} - \theta_{13}\Delta^+_{12}\Delta_{23} - \theta_{31}\Delta^-_{23}\Delta_{12} \\ &+ \theta_{13}\theta_{12}\Delta^-_{12}\Delta^+_{23} + \theta_{13}\theta_{21}\Delta^+_{12}\Delta^+_{23} + \theta_{31}\theta_{23}\Delta^-_{12}\Delta^-_{23} + \theta_{31}\theta_{32}\Delta^-_{12}\Delta^+_{23}\end{aligned} \tag{4.70}$$

Now to show that this expression equals the integrand (not counting the exponentials) of (4.68) we subtract the two and end up with a remainder of

$$\theta_{13}\Delta^+_{12}\Delta^+_{23} + \theta_{31}\Delta^-_{12}\Delta^-_{23} = 0 \tag{4.71}$$

It is very easy to show that both terms give zero. By performing the integration that we have so far neglected and by writing the step function as an integral as we have done many times now, we can show that

$$\begin{aligned}&\int \frac{\mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3}{(2\pi)^4(2\pi)^4(2\pi)^4}\mathrm{e}^{i(P_1+P_2)x_1}\mathrm{e}^{iP_3x_2}\mathrm{e}^{i(P_4+P_5)x_3}\theta_{13}\Delta^+_{12}\Delta^+_{23} \\ &= \int \frac{2\pi i\mathrm{d}z}{z+i\epsilon}\theta(\eta\cdot(P_1+P_2))\delta((P_1+P_2-z\eta)^2)\theta(\eta\cdot(P_4+P_5))\delta((P_4+P_5+z\eta)^2)\end{aligned} \tag{4.72}$$

which is clearly zero.

We have now shown that the propagator identity needed to work out the BCFW recursion relation at a diagrammatic level is rooted in the largest time equation. The calculation was performed in the special case of a five-point amplitude and is not a proof that the connection exists to any point at tree-level. We will now prove the general propagator identity algebraically. As we will see, the identity is just the naïve generalization of (4.37) and we will therefore conclude that the largest time equation can also be invoked in the general case without proof.

4.5 The Propagator Identity for Arbitrary Tree-diagrams

As we have mentioned several times the factorization procedure relies on the special vertices in the space-cone gauge. If we look at an arbitrary tree-level diagram it is obviuos that we may always factor out all vertices and polarization vectors (scalars) and only look at the propagators. Now consider an N-point diagram and pick two external momenta p_+ and p_- that are not connected to the same vertex. At tree-level there is a unique path between these two external momenta and we label this path according to figure 4.1. That is, we call the vertex where p_+ is attached for x_1 and the vertex where p_- is attached x_n. There will be excatly $n-1$

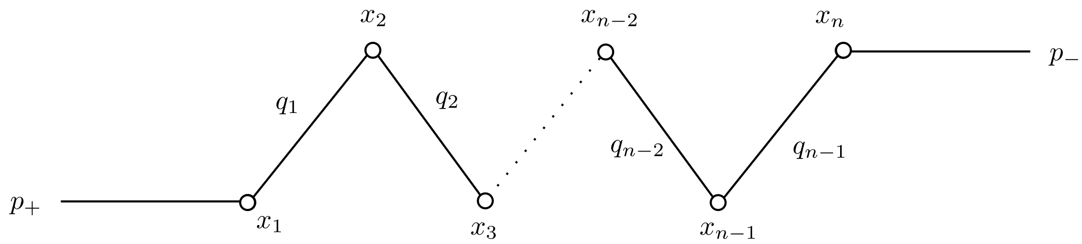


Figure 4.1: The labelling of the path connecting two external momenta p_+ and p_- at tree-level. There are n vertices and $n-1$ propagators connecting the vertices.

propagators in between which we label by their momenta $q_1 \dots q_{n-1}$. There could be more vertices and internal lines in the diagram under consideration, but we will only be interested in this segment as it is the only affected by shifts. The propagator connecting two vertices are assumed to carry different mass but could in principle be massless.

Now the factorization of this line segment is carried out by cutting each propagator in turn by shifting the two external momenta p_+ and p_-. By momentum conservation every propagator has a component p_+ or equivalently $-p_-$. We cut each line by shifting them by $z_i\eta$ where η is a light-like (null) vector

$$\hat{q}_i^2 - m_i^2 = 0, \quad \hat{q}_i = q_i - z_i\eta. \tag{4.73}$$

We want the left-hand side and the right-hand side of the factorization to each be on-shell, conserve momentum and glued together by the unshifted propagator $\frac{1}{q_i^2-m_i^2}$. To keep the two sides on-shell we have to require

$$(p_+ - z_i\eta)^2 - m_+^2 = 0, \quad (p_- - z_i\eta)^2 - m_-^2 = 0 \tag{4.74}$$

which implies that $p_+ \cdot \eta = p_- \cdot \eta = 0$. Since we have already assumed η to be light-like this must also hold for p_+ and p_-[3]. We therefore have $m_+ = m_- = 0$. This is in agreement with the specific choices of $p_+$and p_- as well as η that we made earlier on for the four- and five-point amplitudes. Now the propagator identity which we want to establish is simply (no factors of i or ϵ's)

$$\begin{aligned}
&\frac{1}{q_1^2 - m_1^2}\frac{1}{q_2^2 - m_2^2}\cdots\frac{1}{q_{n-1}^2 - m_{n-1}^2} = \frac{1}{q_1^2 - m_1^2}\frac{1}{(q_2 - z_1\eta)^2 - m_2^2}\cdots\frac{1}{(q_{n-1} - z_1\eta)^2 - m_{n-1}^2}\\
&+ \frac{1}{(q_1 - z_2\eta)^2 - m_1^2}\frac{1}{q_2^2 - m_2^2}\cdots\frac{1}{(q_{n-1} - z_2\eta)^2 - m_{n-1}^2}\\
&+ \cdots +\\
&+ \frac{1}{(q_1 - z_j\eta)^2 - m_1^2}\frac{1}{(q_2 - z_j\eta)^2 - m_2^2}\cdots\frac{1}{q_j^2 - m_j^2}\cdots\frac{1}{(q_{n-1} - z_j\eta)^2 - m_{n-1}^2}\\
&+ \cdots +\\
&+ \frac{1}{(q_1 - z_{n-1}\eta)^2 - m_1^2}\frac{1}{(q_2 - z_{n-1}\eta)^2 - m_2^2}\cdots\frac{1}{q_{n-1}^2 - m_{n-1}^2}
\end{aligned} \tag{4.75}$$

[3] p_+ and p_- could in principle be space-like but we will not consider this here.

Inspired by what we did for the five-point amplitude we turn the propagator identity into an identity in the shifts z_i. We can express the unshifted propagator $(q_i^2 - m_i^2)^{-1}$ in terms of the shift

$$\hat{q}_i^2 - m_i^2 = 0, \quad \Rightarrow \quad q_i^2 - m_i^2 = 2z_i\eta \cdot q_i. \tag{4.76}$$

and since the shifted propagators are given by

$$(q_i - z_j\eta)^2 - m_i^2 = q_i^2 - m_i^2 - 2z_j\eta \cdot q_i \tag{4.77}$$

we have

$$(q_i - z_j\eta)^2 - m_i^2 = -2(z_j - z_i)\eta \cdot q_i \tag{4.78}$$

Inserting this and dividing through with all factors of $-2\eta \cdot q_i$ we arrive at

$$\begin{aligned} \frac{(-1)^{n-2}}{z_1 z_2 \cdots z_{n-1}} &= \frac{1}{z_1(z_1 - z_2)\cdots(z_1 - z_{n-1})} + \frac{1}{(z_2 - z_1)z_2\cdots(z_2 - z_{n-1})} \\ &+ \cdots + \frac{1}{(z_j - z_1)(z_j - z_2)\cdots z_j \cdots (z_j - z_{n-1})} \\ &+ \cdots + \frac{1}{(z_{n-1} - z_1)(z_{n-1} - z_2)\cdots z_{n-1}} \end{aligned} \tag{4.79}$$

The easiest way to see that this identity holds is by expressing it as in integral

$$\oint \frac{\mathrm{d}z}{z(z - z_1)(z - z_2)\ldots(z - z_{n-1})} = 0 \tag{4.80}$$

By Cauchy's integral theorem the integral is zero if the contour encloses all poles and the identity follows directly. The integral is also equal to the sum over residues which each corresponds to a term in (4.79). This concludes our proof of the propagator identity. To conclude this section let us sum up what we have accomplished so far in this chapter. We started out by proving that tree-amplitudes can be expressed in a recursive way through products of lower-point tree amplitudes. By exploiting the structure of space-cone gauge vertices in Yang-Mills theory we could show, that the amplitudes also factorize on a diagrammatic level up to an algebraic identity involving only the internal propagators. We showed that this identity could be found in the largest time equation and then we showed that the identity holds for the factorization of any tree-level diagram. The only thing we have not shown is that the lower-point diagrams produced when factorizing exactly corresponds to the diagrams of the lower-point amplitudes. It could happen that the diagrams combined in such a way that cancellations would occur and the factorization would not be the same as the BCFW recursion relation predicts.

We have shown that the factorization works out for a four- and five-point amplitude example and end this chapter by showing that the six-point $NMHV$-amplitude calculated in section 3.3 correctly combines to reproduce the BCFW recursion. An interesting thing that was not addresed in [8] is the fact that the factorization of this amplitude (as well as higher-point amplitudes) introduces the four-point vertex in the MHV-amplitudes. This seems to be in contradiction with the fact, that is has been shown that the MHV-amplitudes are *independent* of the four-vertex [36]. This is just a consequence of only having one reference line attached to each diagram but here we see that the four-point vertex enters in the correct way.

4.6 Factorization of $A_6(+++---)$

We have already written down all diagrams that contribute to the $A_6(+++---)$ subamplitude in appendix A. Since we have proved the general propagator identity the only thing left for us to check, is whether the factorized diagrams correctly reproduces the expected amplitudes. From the BCFW recursion we expect the following two non-zero contributions

$$A_3(++-)A_5(++---), \qquad A_5(+++--)A_3(+--) \tag{4.81}$$

It is hard to illustrate how the factorization works out even with all the factorized diagrams. In principle there are nothing more to do than simply look at all the resulting diagrams and compare them with the diagrams that is needed to calculate the amplitudes. The work lies mainly in writing out all the diagrams and checking that everything works out. The 15 factorized diagrams can be found in appendix B written in the same way as the factorized four- and five-point amplitudes. We will be refering to the diagrams such that $(B.1)$ is 1, $(B.2)$ is 2 etc. In the same way as when we factorized the four- and five-point amplitudes we now receive many more terms than we did from calculating them in section 3.3. This is once again due to the number of external reference lines attached to a given diagram. If we draw all diagrams contributing to the $A_5(++---)$ amplitude with the third negative helicity line as the only external reference line, we see that these correspond exactly to the nine five-point diagrams found in factorization $1, 3, 6, 10, 14, 15, 16, 19, 22$. It holds that the factorizations $2, 4, 5, 8, 11, 12, 18, 20, 21$ corresponds to the nine diagrams of $A_5(+++--)$ with only the positive helicity reference line idenitfied with one of the external lines. Diagram $19, 20, 21$ and 22 all four have a four-point vertex, but as we have pointed out many times now, this is only due to having identified only one reference line on each diagram. The four remaining diagrams, $7, 9, 13$ and 17 sum up to $A_4(+++-)A_4(+---)$ which is zero. Hence we have shown that the factorization of a six-point $NMHV$-amplitude correctly reproduces the BCFW result.

Part III

Loops and Trees

— 5 —

From loops to trees

This chapter is somewhat different from the two previous although they are deeply connected. Where we up until now have been interested in tree-level diagrams from a quite modern viewpoint, we will now dive into the core of quantum field theory. We will return to the largest time equation and its relation to unitarity and loop-diagrams. What we will see is, that it is possible from the largest time equation to construct a set of *cutting equations* that will allow us to extract information regarding loop-diagrams from tree-diagrams. When we have made this connection we will look at a result introduced by Feynman in a series of lectures and articles [11, 12, 47] now known as *Feynman's Tree Theorem*. The tree theorem states that any one-loop diagram can be expressed as a sum over tree diagrams[1]. Both the largest time equation and the tree theorem, as we have seen and will see, relies on the structure of the Feynman propagator and especially its deconstruction into δ-functions. We will show that the decomposition of the propagator used to derive the largest time equation is the Fourier transform of the decomposition needed to derive Feynman's Tree Theorem. We will start by briefly discussing unitarity in quantum field theory. Then we revisit the largest time equation and derive its connection to the unitarity condition. We then derive Feynman's Tree Theorem and end by investigating the connection between the largest time equation and the tree theorem. Since we already know that there is a connection between the largest time equation and the BCFW recursion, we hope to indirectly show a connection between the tree theorem and the recursion relation.

5.1 Unitarity

Scattering amplitudes can be related to the so-called S-matrix as we briefly mentioned in the introduction. It was object to a lot of interest and research during the 60's and 70's and although today the S-matrix is out of fashion it is still a very instructive tool when discussing unitarity of quantum field theories. In general when describing scattering amplitudes we assume that the two states scattering are very far from each other prior to the interaction and thereby can be described as free particles. Now consider some initial state $|a\rangle_{in}$ in Hilbert space at time $t = -\infty$ and some final state $|b\rangle_{out}$ in the same Hilbert space at time $t = \infty$. The physical quantity we want to determine is the probability that the initial state transitions into the final state. Now we assume that all particles in the Hilbert space can be described both by a $|\cdot\rangle_{in}$ and a $|\cdot\rangle_{out}$ state, however the two states are not the same, as in general it is unlikely (however not impossible) for an interacting state to come out identical to its initial

[1]The theorem can be applied to higher order loop-diagrams by iteration. Here will only be interested in one-loop diagrams.

state. We therefore conclude that the Hilbert space must be spanned both by the initial states and the final states. Since the bases are expected to be orthonormal there exists a unitary matrix transforming from one basis to another

$$|a\rangle_{out} = S^\dagger |a\rangle_{in} . \tag{5.1}$$

The transition probability for the process we initially considered is then given by

$$|_{out}\langle b| |a\rangle_{in}|^2 = |_{in}\langle a| |b\rangle_{out}|^2 = \left|_{in}\langle a|S^\dagger |b\rangle_{in}\right|^2 = |_{in}\langle b|S |a\rangle_{in}|^2 \tag{5.2}$$

Unitarity of the S-matrix reads

$$SS^\dagger = 1. \tag{5.3}$$

If no interaction happens so that the initial and final states are the same, the S-matrix is just the identity. We may therefore write it in the form [48, 49]

$$S = 1 + iT \tag{5.4}$$

where the i is just a convention. The T-matrix now contains all relevant information regarding scattering of states in the Hilbert space. The unitarity condition can be recast into a statement about the T-matrix

$$TT^\dagger = -i\left(T - T^\dagger\right) = 2\text{Im}T. \tag{5.5}$$

Now inserting this equation in between two states and multiplying by 1 on the right-hand side yields

$$i\left(\langle a| T |b\rangle - \langle a| T^\dagger |b\rangle\right) = -\sum_c \langle a| T |c\rangle \langle c| T^\dagger |b\rangle \tag{5.6}$$

If we interpret this equation perturbatively in some coupling constant of our theory, it says that it is possible to extract information regarding the imaginary part of higher order perturbation terms from lower order perturbation terms. This can be seen from the fact that if $\langle a| T |b\rangle$ is to order g^2 then the right hand side matrix elements have to be of order g as they are multiplied together. Similar considerations apply for higher order terms. A special consequence of this equation is that we are able to say something about loop-diagrams from only considering tree-diagrams. We will make this statement more clear with help from the largest time equation.

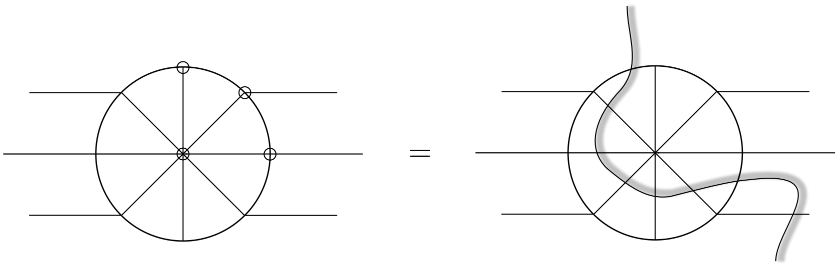
5.2 The Largest Time Equation Revisited

Recall the following form of the largest time equation

$$F(x_1, \ldots, x_n) + F^*(x_1, \ldots, x_n) = -\sum_{\text{underlinings}\neq 0,n} F(x_1, \ldots, x_n), \tag{5.7}$$

which tells us that the sum of a diagram and its complex conjugated is the sum of all possible underlinings (circled and uncircled vertices). Now what is the physical meaning of just circling one vertex in a diagram? From the rules we are to substitute all propagators connecting this vertex with $\Delta^\pm$. The $\Delta^\pm$ has a step function in it forcing the sign of the energy to be positive towards the vertex. We may conclude that energy must flow from an uncircled towards a circled vertex. Now this might pose a problem for energy conservation, since if the circled vertex is isolated, it will receive energy from all directions. The only way energy may be conserved is if an external outgoing line is connected to the vertex. Between two circled or uncircled vertices such restrictions do not apply, since the Feynman propagator or its complex conjugate allows both positive and negative energy flow. If a circled vertex has another circled vertex in its vicinity, energy may flow between these two. However, a set of circled vertices that form a connected region will always have energy flowing towards it. We conclude that a connected set of circled vertices has to have at least one outgoing line attached in order to not violate energy conservation. If they do not, the diagram must vanish.

Since the circled vertices has to form connected regions we will draw a line instead of circles, to indicate what regions have been circled and which have been not.



Now energy may only cross from the unshaded to the shaded region of a diagram. Notice that the external lines to the left are all ingoing whereas the external lines to the right are all outgoing. We only require that the shaded part of the diagram has at least on outgoing line and that the unshaded part of the diagrams has one ingoing line. An internal line which crosses the shaded line we will call a *cutted line*. The reason for this terminology should be clear. When substituting a propagator with Δ^+ we set the line on-shell due to the $\delta(p^2 - m^2)$ term. This also explains why we decided upon calling $\Delta^\pm$ for cut propagators. The largest time equation can now be cast into a set of cutting equations

$$F + F^* = - \sum_{\text{cuttings}} F. \tag{5.8}$$

These equations contain all possible cuts of a given diagram. Only diagrams in which energy conservation is respected contribute to the sum. A cutted loop diagram must necessarily be the product of lower order loop (or tree) diagrams. It is therefore easy to see what this has to do with our discussion about unitarity before. The left-hand side of (5.8) is the sum of the diagram and its complex conjugate. It should therefore yield $2\text{Re}F$. Now a diagram gives us information about the S-matrix and as we defined the T-matrix with an i in front the *real* part of S gives us information about the *imaginary* part of T. (5.8) is therefore equivalent to

$$i\left(\langle a|\, T\,|b\rangle - \langle a|\, T^\dagger\,|b\rangle\right) = -\sum_c \langle a|\, T\,|c\rangle\,\langle c|\, T^\dagger\,|b\rangle. \tag{5.9}$$

This is the connection between the cutting equations and thereby the largest time equation and unitarity. We should notice some things regarding the cuts the cutting equations allow. The number of cuts a given diagram can survive is always given by the number of independent loop integration performed. We can of course understand this if we remember that the δ-function only makes sense under an integration. We are also always required to cut at least two lines as a line entering the diagram also has to leave again. The only exception is when the cutted propagator is not part of a loop. When we establish a connection between the largest time equation and Feynman's Tree Theorem it will be through the cutting equations.

The Special Case of One-Loop Diagrams

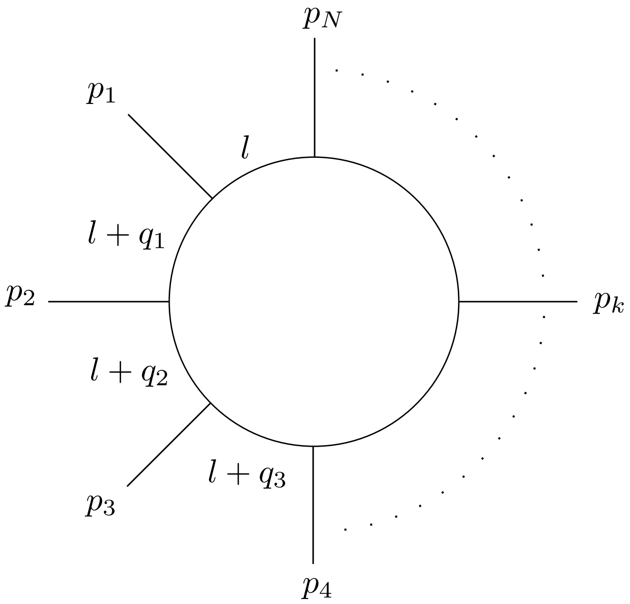


Figure 5.1: The momentum assignment for a generic one-loop N-point diagram. The loop momentum can be assigned both to go around clockwise and anticlockwise. In general the momenta $p_1, ..., p_N$ can be both in- and outgoing.

We will take special interest in the generic one-loop diagram as it is the simplest case. In our discussion of the largest time equation and cutting equations so far we have solely been working in x-space. All results and equations can be Fourier transformed to obtain the corresponding p-space results. Now in d dimensions the generic one-loop integral is given by

$$I^{(N)} = \int \frac{\mathrm{d}^d l}{(2\pi)^d} f(l, \{q^i\}) \prod_{i=1}^{N} \Delta_F(l + q^i) \tag{5.10}$$

whit N momenta going into the loop. Here l is the loop four-momentum, and q^i denotes a sum of external momenta such that starting at an arbitrary propagator i in the loop, q^i equals the sum of all momenta from that point and clockwise to the point 1. This is illustrated in figure 5.1. The product is simply the product over all propagators inside the loop. The function f comes from vertices, polarization factors etc., but the important thing is that it has no poles and that is doesn't explode at infinity. This is true if our underlying theory is unitary and local. We will assume that the integral is dimensionally regularized [50]. When we talk about four dimensions we actually mean $d = 4 - 2\epsilon$. This should not give rise to confusion.

Now if we investigate what cuts are possible in such diagram we know from the discussion above that no more that d cuts can be made in general. We are however able to say a little more than that for the one-loop diagram. Whenever we cut a line in the diagram the next line we cut will take us out of the diagram. Since the cut line has to start outside the diagram and end there as well, we can quickly convince ourselves that only an even number of cuts can be performed. For the special case of $d = 4$ dimension we may only perform 2- and 4-cuts. This of course also means that the physical information stored in the odd cuts, if any, has to be related to the real part of the diagram. We write

$$F_{1-loop} + F^*_{1-loop} = - \sum_{\text{even \# cuts}} F_{1-loop}. \tag{5.11}$$

This is essentially the result we want to retrieve from Feynman's Tree Theorem, to show its connection to the largest time equation. This is of course just a special case of the largest time equation but proving the connection here might give us insight into how to prove the connection for arbitrary loop-diagrams. First though, we introduce Feynman's Tree Theorem.

5.3 Feynman's Tree Theorem

In 1963 Feynman [11, 12, 47] showed a relation between loop diagrams and tree diagrams. Essential this result gives a prescription for cutting open loops in the right manner, as to reconstruct the loop-diagrams from sums of tree diagrams. This result is now called Feynman's Tree Theorem and the purpose of this section is to derive it. Since Feynman derived the result, it has almost been forgotten. However, recently a series of papers have been published exploiting the tree theorem to derive loop-dualities and to perform numerical calculations of loop-amplitudes [51–55]. It is therefore of interest to relate the tree theorem to other results like the largest time equation as we know this has connection with the modern BCFW recursion. We start by rewriting the Feynman propagator as well as the retarded and advanced propagators.

The Propagators

Feynman propagators along with retarded and advanced propagators take the form

$$\Delta_F(p) = \frac{i}{p^2 - m^2 + i\epsilon} = \frac{i}{p_0^2 - \omega_p^2 + i\epsilon} \tag{5.12}$$

$$\Delta^*_F(p) = \frac{-i}{p^2 - m^2 - i\epsilon} = \frac{-i}{p_0^2 - \omega_p^2 - i\epsilon} \tag{5.13}$$

$$\Delta_R(p) = \frac{i}{(p_0 + i\epsilon)^2 - \mathbf{p}^2 - m^2} = \frac{i}{(p_0 + i\epsilon)^2 - \omega_p^2} \tag{5.14}$$

$$\Delta_A(p) = \frac{i}{(p_0 - i\epsilon)^2 - \mathbf{p}^2 - m^2} = \frac{i}{(p_0 - i\epsilon)^2 - \omega_p^2} \tag{5.15}$$

where $\omega_p^2 = \mathbf{p}^2 + m^2$. These propagators can be cast into the form

$$\Delta_F(p) = \frac{i}{2\omega_p}\left[\frac{1}{p_0 - \omega_p + i\epsilon} - \frac{1}{p_0 + \omega_p - i\epsilon}\right] \tag{5.16}$$

$$\Delta^*_F(p) = \frac{i}{2\omega_p}\left[\frac{1}{p_0 + \omega_p + i\epsilon} - \frac{1}{p_0 - \omega_p - i\epsilon}\right] \tag{5.17}$$

$$\Delta_R(p) = \frac{i}{2\omega_p}\left[\frac{1}{p_0 - \omega_p + i\epsilon} - \frac{1}{p_0 + \omega_p + i\epsilon}\right] \tag{5.18}$$

$$\Delta_A(p) = \frac{i}{2\omega_p}\left[\frac{1}{p_0 - \omega_p - i\epsilon} - \frac{1}{p_0 + \omega_p - i\epsilon}\right]. \tag{5.19}$$

These decompositions can easily be verified by finding a common denominator and redefining ϵ. The equality here means, that the two expressions are equal when evaluated under an integral in the limit $\epsilon \to 0^+$. These decompositions are the main ingredient in deriving the tree theorem.

The Tree Theorem

In his paper [11], Feynman noticed a simple thing about the generic one-loop integral (5.10). If we replace all Feynman propagators with retarded (or advanced) propagators in (5.10) the entire integral vanishes

$$I_R^{(N)} = \int \frac{\mathrm{d}^d l}{(2\pi)^d} f(l, \{q^i\}) \prod_i \Delta_R(l + q^i) = 0. \tag{5.20}$$

This can be seen by looking at (5.14) or simply remembering, that the retarded propagator only has poles in the negative half-plane. By closing just one contour in the positive half-plane, the integral is seen to vanish. We want to recast this integral into another form, by first showing an identity between the Feynman propagator and the retarded propagator. Essentially, we want to show, that the Feynman propagator and the retarded (advanced) propagator differ only by a contribution from an on-shell δ-function. We will carry through with retarded propagators[2] but the argument holds for advanced propagators too. The difference between the Feynman propagator and the retarded propagator is given by

$$\Delta_F(p) - \Delta_R(p) = \frac{i}{2\omega_p}\left[\frac{1}{p_0 + \omega_p + i\epsilon} - \frac{1}{p_0 + \omega_p - i\epsilon}\right] \tag{5.21}$$

Now comes the real trick. Let us evaluate (5.21) inside the integral (5.20)

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} f(l, \{q^i\}) \prod_i \left(\Delta_F(l + q^i) - \frac{i}{2\omega^i}\left[\frac{1}{(l_0 + q_0^i) + \omega^i + i\epsilon} - \frac{1}{(l_0 + q_0^i) + \omega^i - i\epsilon}\right]\right) \tag{5.22}$$

where we have defined $\omega^i = \sqrt{(\mathbf{l} + \mathbf{q}^i)^2 + m_i^2}$. We are interested in the poles of this integral. First we notice, that all the factors that go into the product in general will have different poles. This holds as long as the q^i's differs from each other. Given one of the factors $\frac{i}{2\omega^i}\left[\frac{1}{(l_0+q_0^i)+\omega^i+i\epsilon} - \frac{1}{(l_0+q_0^i)+\omega^i-i\epsilon}\right]$ we see, that it always has two poles in the $l_0 + q_0^i$-plane. One above the real axis and one below. Denote these two poles by $l_i^\pm = -\omega^i \pm i\epsilon$, remembering that $\omega^i > 0$. The propagator under the integral will also have poles but we will not be interested

[2] This has been canonical since Feynman's paper

in these for the moment so we simply omit it. Singling out the k'th factor, we may write the integral (5.22) as

$$-\int \frac{\mathrm{d}^d l}{(2\pi)^d} f_k(l,\{q^i\}) \frac{i}{2\omega^k}\left[\frac{1}{(l_0+q_0^k)+\omega^k+i\epsilon} - \frac{1}{(l_0+q_0^k)+\omega^k-i\epsilon}\right] = -\int \frac{\mathrm{d}^d l}{(2\pi)^d} I_k \quad (5.23)$$

where f_k is a rational function with *no* poles. Evaluating the l_0 part of the integral, we may choose to close the contour in either the upper or lower half-plane, since the integrand goes to zero in both directions. Closing in the upper plane, we find the residue

$$\begin{aligned}\mathrm{Res}_{l_k^+}(I_k) &= \lim_{l_0+q_0^k \to l_k^+} \left(f_k(l,\{q^i\}) \frac{i}{2\omega^k}\left[\frac{(l_0+q_0^k)+\omega^k-i\epsilon}{(l_0+q_0^k)+\omega^k+i\epsilon} - 1\right]\right) \\ &\overset{e\to 0^+}{=} -f_k(l,\{q^i\})\frac{i}{2\omega^k}\delta(l_0+q_0^k+\omega^k). \end{aligned} \quad (5.24)$$

From this we conclude that

$$-\int \frac{\mathrm{d}^d l}{(2\pi)^d} I_k \overset{e\to 0^+}{=} -\int \frac{\mathrm{d}^d l}{(2\pi)^d} f_k(l,\{q^i\}) \frac{\pi}{\omega^k}\delta(l_0+q_0^k+\omega^k). \quad (5.25)$$

With this, we may go back to (5.21) and write

$$\begin{aligned}\Delta_R(p) &= \Delta_F(p) - \frac{i}{2\omega_p}\left[\frac{1}{p_0+\omega+i\epsilon} - \frac{1}{p_0+\omega-i\epsilon}\right] \\ &\overset{e\to 0^+}{=} \Delta_F(p) - \frac{\pi}{\omega}\delta(p_0+\omega_p) \end{aligned} \quad (5.26)$$

This is the result we are after. Notice that in principle what we have shown is

$$\delta(x) = \frac{i}{2\pi}\left[\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon}\right]. \quad (5.27)$$

It will prove useful to recast (5.26) using the delta function identity (2.63) proved earlier. By choosing $g(p_0) = p^2 - m^2$ we see that

$$\begin{aligned}2\pi\delta(p^2-m^2)\theta(-p_0) &= 2\pi\delta([p_0-\omega_p][p_0+\omega_p])\theta(-p_0) \\ &= 2\pi\left[\frac{\delta(p_0-\omega_p)}{2\omega_p} + \frac{\delta(p_0+\omega_p)}{2\omega_p}\right]\theta(-p_0) \\ &= \frac{\pi}{\omega_p}\delta(p_0+\omega_p) \end{aligned} \quad (5.28)$$

Thus,

$$\Delta_R(p) = \Delta_F(p) - 2\pi\delta(p^2-m^2)\theta(-p_0) = \Delta_F(p) - \Delta^-(p) \quad (5.29)$$

Plugging this identity into (5.20) and rearranging we arrive at Feynman's Tree Theorem

$$\begin{aligned}&\int \frac{\mathrm{d}^d l}{(2\pi)^d} f(l,\{q^i\}) \prod_i \Delta_F(l+q^i) = \\ &\qquad -\int \frac{\mathrm{d}^d l}{(2\pi)^d} f(l,\{q^i\}) \prod_i{}' \Big(\Delta_F(l+q^i) - \Delta^-(l+q^i)\Big) \end{aligned} \quad (5.30)$$

where the primed product omits the term found on the left hand side. For completeness, we state (5.26) for advanced propagators as well

$$\Delta_A(p) = \Delta_F(p) - \frac{\pi}{\omega}\delta(p_0 - \omega) = \Delta_F(p) - \Delta^+(p). \tag{5.31}$$

Some notes are in order here. First of all, we need to interpret (5.30) correctly. First one might ask what happens, if there are more than d external momenta. The formula indicates, that one should put in a δ-function for each of the loop-propagators. However, after d integrations, the δ-functions are meaningless. Hence, the terms with more than d δ-functions simply vanish. This can also be seen by looking at the first part of (5.26) in the case were there is no integration. In that case the expression that turned into a δ-function under an integration now vanishes in the limit $\epsilon \to 0^+$. The interpretation of each δ-function in the integrand is straightforward. Comparing (5.30) with (5.22) we see, that a δ-function substitutes a propagator with an on-shell particle. The Feynman Tree Theorem therefore tells us, that any one-loop integral can be expressed as a sum over cutted diagram. We can write this as

$$I^{(N)} = \sum_{i=1}^{m} I^{(N)}_{i-\text{cut}} \tag{5.32}$$

where $m \leq N$ and $m \leq d$. When we cut a propagator, we are left with a tree diagram and the tree theorem therefore tells us, that any one-loop diagram, may be expressed as a sum over tree-diagrams. In principle this prescription can be iterated, such that also higher order loop diagrams may be expressed in terms of trees. Our goal is to understand the tree theorem in relations to the largest time equation at the one-loop level.

Summary of Results

Before we move on, it is instructive to pause and identify a couple of quantities. If we Fourier transform the cutting propagators (4.46) that we defined as we introduced the largest time equation we arrive at

$$\Delta^{\pm}(p) = 2\pi\theta(\pm p_0)\delta(p^2 - m^2). \tag{5.33}$$

This can be rewritten using (5.28) to

$$\Delta^{\pm}(p) = \frac{\pi}{\omega_p}\delta(p_0 \mp \omega_p). \tag{5.34}$$

These are the same cut propagators that enter the tree theorem. The tree theorem comes in two forms, one if derived from retarded propagators

$$I^{(N)} = -\int \frac{\mathrm{d}^d l}{(2\pi)^d} f(l, \{q^i\}) \prod_i{}' \Big(\Delta_F(l + q^i) - \Delta^-(l + q^i)\Big) \tag{5.35}$$

or had we derived the result using advanced propagators

$$I^{(N)} = -\int \frac{\mathrm{d}^d l}{(2\pi)^d} f(l, \{q^i\}) \prod_i{}' \Big(\Delta_F(l + q^i) - \Delta^+(l + q^i)\Big). \tag{5.36}$$

There also exists complex conjugated versions of the tree theorem. They take the form

$$I^{*(N)} = -\int \frac{\mathrm{d}^d l}{(2\pi)^d} f^*(l,\{q^i\}) \prod_i{}' \Big(\Delta_F^*(l+q^i) - \Delta^-(l+q^i)\Big) \tag{5.37}$$

and

$$I^{*(N)} = -\int \frac{\mathrm{d}^d l}{(2\pi)^d} f^*(l,\{q^i\}) \prod_i{}' \Big(\Delta_F^*(l+q^i) - \Delta^+(l+q^i)\Big). \tag{5.38}$$

We will refer to these equations as the four tree theorems. From now on we will set $f(l,\{q^i\}) = 1$ as it is irrelevant for the rest of the discussion. In the following we will need the decomposition of the Feynman propagator and its complex conjugate as well as the δ-function. These have all been introduced before but we would like to remind the reader of their form at this point

$$\Delta_F(p) = \frac{i}{2\omega_p}\left[\frac{1}{p_0-\omega_p+i\epsilon} - \frac{1}{p_0+\omega_p-i\epsilon}\right] \tag{5.39}$$

$$\Delta_F^*(p) = \frac{i}{2\omega_p}\left[\frac{1}{p_0+\omega_p+i\epsilon} - \frac{1}{p_0-\omega_p-i\epsilon}\right] \tag{5.40}$$

$$\delta(x) = \frac{i}{2\pi}\left[\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon}\right]. \tag{5.41}$$

5.4 Connecting Feynman's Tree Theorem and The Largest Time Equation

As we have seen the tree theorem gives us a prescription for calculating a full one-loop diagram whereas the cutting equations only provide us with the real part of that same diagram. We will not be able to show that the tree theorem reproduces the Veltman's cutting equations for an arbitrary number of external legs but we will show how to proceed by inspecting two examples and show that the tree theorem and the cutting equations agree. To relate the two results it is necessary to identity the real part of the tree theorem[3]. The naive way to go about this is to simply complex conjugate the right-hand side of (5.35) and add it to the tree theorem to obtain $2\mathrm{Re}[I^{(N)}]$. What makes this difficult in general is that we only introduce Δ^- in the equations and therefore no terms of the tree theorem reassembles those of the cutting equations right away. A further problem is, that we would like to mix Δ_F and Δ_F^* since the cutting equations does this as propagators on the shaded side of a diagram has to be complex conjugated. So we might try and complex conjugate the other version of the tree theorem (5.36) which have a Δ^+. This still has separated terms with Δ_F and Δ_F^* but one might be better of going this way.

It is possible to derive another version of the tree theorem which is manifestly real. This enables us to directly relate some of the terms of the cutting equations with the tree theorem. When we derived the tree theorem we were focusing on relating the retarded (advanced) propagator to the Feynman propagator. It is however not difficult to see, that we could just as

[3]The opposite strategy, constructing the full diagram from the real part, doesn't seem very fruitful. This road was pursued during the 70's through dispersion relations but without real success.

well have expressed the complex conjugated propagator in terms of the retarded (advanced) propagator. Since the retarded and advanced propagators are related by complex conjugation we can immediately find the following from (5.29)

$$\Delta_A(p) = -\Delta_F^*(p) + \Delta^-(p). \tag{5.42}$$

Subtracting this from the result obtained with Feynman propagator we have

$$0 = \Delta_F(p) + \Delta_F^*(p) - \Delta^+(p) - \Delta^-(p). \tag{5.43}$$

The loop-integral over this quantity must also vanish. Extracting $(\Delta_F)^N$ and $(\Delta_F^*)^N$ in the same way as before we have the following modified tree theorem

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \left[\prod_i \Delta_F(l+q^i) + \prod_i \Delta_F^*(l+q^i) \right] = \\ - \int \frac{\mathrm{d}^d l}{(2\pi)^d} \prod_i{}' \Big[\Delta_F(l+q^i) + \Delta_F^*(l+q^i) - \Delta^+(l+q^i) - \Delta^-(l+q^i) \Big]. \tag{5.44}$$

where the prime is to be understood as neither the product of all Δ_F nor the product of all Δ_F^*. The right-hand side has the property that when expanded it will have products of all combinations of Δ_F, Δ_F^* and $\Delta^{\pm}$. There are of course many more terms than the cut equations include, but most importantly the terms that show up in the cutting equations are already present in this version of the tree theorem, however some of them with the wrong sign. It should be noted that this version is no longer a real tree theorem as it does not express the left-hand side in terms of only trees. These terms that have not cut propagators can be related to cut propagators as we will see shortly.

Now before we go on we should ask ourselves what we mean by connecting the tree theorem and the largest time equation (cutting equations). If we just show that the real part of Feynman's Tree Theorem agrees with the cutting equations all we have done is a consistency check. It should not come as a surprise that the two results agree as such an identity is purely mathematical. Such a consistency check is of course important, and we will perform it for the self-energy diagram. There are one of two things that we would really like to do to show a connection between the two results. If we could somehow show that the diagrams that the cut equations produce matches the diagrams of the real part of the tree theorem, we would have shown that the two different prescriptions are identical. This is the same as showing that the terms are identical one by one. This road has been pursued but it is hard to show in general that the terms match as the number of terms that do not agree, and hence cancel, seems to grow very rapidly and their cancellations has to be verified explicitly. We show for the self-energy diagram that the cut equations and the real part of the tree theorem produce the same terms and sketch how to generate the correct terms in general.

A more fundamental way to see a connection between the two theorems is to take a look at some of the principal identities that goes into the derivation of the two results. If we can show that the equations needed to derive the tree theorem also implies the equations that goes into the derivation of the largest time equation then we have a clear connection. Both results rely on the structure of the Feynman propagator and in particular on its decomposition into cut propagators. To remind the reader, the two space-time decompositions used to derive the largest time equation were

$$\Delta_F(x) = \theta(x_0)\Delta^+(x) + \theta(-x_0)\Delta^-(x) \tag{5.45}$$

and

$$\Delta_F^*(x) = \theta(-x_0)\Delta^+(x) + \theta(x_0)\Delta^-(x). \tag{5.46}$$

Adding these two together and Fourier transforming we obtain the simplest form of the largest time equation

$$\Delta_F(p) + \Delta_F^*(p) = \Delta^+(p) + \Delta^-(p). \tag{5.47}$$

This is just (5.43) which we obtained by adding the decompositions used to derive the tree theorem. We therefore see that the simplest realisation of the largest time equation may be obtained directly from the tree theorem. The question is whether or not we can show that (5.45) is the Fourier transform of

$$\Delta_F(p) = \Delta_R(p) + \Delta^-(p) \tag{5.48}$$

which was the main ingredient in deriving the tree theorem. Taking the Fourier transform we obtain

$$\Delta_F(x) = \Delta_R(x) + \Delta^-(x) \tag{5.49}$$

suggesting that

$$\Delta_R(x) = \theta(x_0)\Delta^+(x) - \theta(x_0)\Delta^-(x). \tag{5.50}$$

From a physical point of view this expression looks correct as it is zero unless $x_0 > 0$. The first term has already been evaluated when we derived the largest time equation. It is (4.50)

$$\theta(x_0)\Delta^+(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{2\omega_p}\frac{i\mathrm{e}^{-ipx}}{p_0 - \omega_p + i\epsilon}. \tag{5.51}$$

By manipulations similar to those performed when deriving this expression we may show that the second term is given by

$$\theta(x_0)\Delta^-(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{2\omega_p}\frac{i\mathrm{e}^{-ipx}}{p_0 + \omega_p + i\epsilon}. \tag{5.52}$$

The difference between these two terms is exactly the retarded propagator as stated earlier. We have therefore shown, that the decomposition of the Feynman propagator used to derive the largest time equation is the Fourier transform of the decomposition needed to derive Feynman's Tree Theorem. This short discussion shows that the starting point for the tree theorem and for the largest time equation is indeed the same, and therefore that the two results are deeply connected. It is important to understand that what we have shown is not just that $\Delta_F(x)$ is the Fourier transform of $\Delta_F(p)$ since this is obvious. The point is that the decomposition made in x-space turns into the correct decomposition in p-space.

We will now go on and calculate the self-energy diagram in great detail. We start by stating and deriving some identities that will be useful. Then we show that the tree theorem plus its complex conjugate really give the same terms as the cutting equations. After that we show that the modified tree theorem does also yield this exact same result. We end this chapter by briefly discussing the strategy for retrieving the cutting equations in general.

Some Useful Identities

From Feynman's Tree Theorem it is possible to derive identities between propagators and cut propagators that are not apparent to start with. The simplest one we have already stated twice and can be found in (5.43). The following identities are understood to hold under a loop integration. If we consider the tree theorem with two propagators $\Delta_F(l)$ and $\Delta_F(l+p)$ we get the following identities

$$\begin{aligned}\Delta_F(l)\Delta_F(l+p) &= \Delta_F(l)\Delta^+(l+p) + \Delta^+(l)\Delta_F(l+p) - \Delta^+(l)\Delta^+(l+p) \\ &= \Delta_F(l)\Delta^-(l+p) + \Delta^-(l)\Delta_F(l+p) - \Delta^-(l)\Delta^-(l+p) \end{aligned} \tag{5.53}$$

and the complex conjugated identities

$$\begin{aligned}\Delta_F^*(l)\Delta_F^*(l+p) &= \Delta_F^*(l)\Delta^+(l+p) + \Delta^+(l)\Delta_F^*(l+p) - \Delta^+(l)\Delta^+(l+p) \\ &= \Delta_F^*(l)\Delta^-(l+p) + \Delta^-(l)\Delta_F^*(l+p) - \Delta^-(l)\Delta^-(l+p). \end{aligned} \tag{5.54}$$

Obviously, similar identities can be derived for an arbitrary number of propagators. Another non-trivial identity comes not directly from the tree theorem but can be derived in a similar way. The identity reads

$$\Delta_F(p)\Delta_F^*(q) + \Delta_F^*(p)\Delta_F(q) = \Delta^-(p)\Delta^-(q) + \Delta^+(p)\Delta^+(q). \tag{5.55}$$

The identity is showed by using (5.39)-(5.41). We start by evaluating the first term on the left-hand side of the identity and then we proceed to add the complex conjugate.

$$\begin{aligned}&\int \frac{\mathrm{d}^d l}{(2\pi)^d}\Delta_F(l)\Delta_F^*(l+p) = \\ &\quad \int \frac{\mathrm{d}^d l}{(2\pi)^d}\frac{i(-i)}{4\omega_l\omega_{l+p}}\left[\frac{1}{l_0-\omega_l+i\epsilon}-\frac{1}{l_0+\omega_l-i\epsilon}\right]\left[\frac{1}{l_0+p_0-\omega_{l+p}-i\epsilon}-\frac{1}{l_0+p_0+\omega_{l+p}+i\epsilon}\right]. \end{aligned} \tag{5.56}$$

The integrand has four poles, two in the lower half-plane

$$l_1^- = \omega_l - i\epsilon, \qquad l_2^- = -p_0 - \omega_{l+p} - i\epsilon, \tag{5.57}$$

and two in the upper half-plane

$$l_1^+ = -\omega_l + i\epsilon, \qquad l_2^+ = -p_0 + \omega_{l+p} + i\epsilon. \tag{5.58}$$

We choose to close the contour in the lower half-plane thereby picking up a minus sign and a factor of $2\pi i$. We then arrive at

$$\int \frac{\mathrm{d}^{d-1} l}{(2\pi)^{d-1}}\frac{-i}{4\omega_l\omega_{l+p}}\left[\frac{1}{p_0+\omega_l-\omega_{l+p}-2i\epsilon}+\frac{1}{-p_0-\omega_{l+p}+\omega_l-2i\epsilon}\right]. \tag{5.59}$$

We wont need to perform any more integrations now. If we add the complex conjugate of this and redefine ϵ we have

$$\begin{aligned}&\int \frac{\mathrm{d}^d l}{(2\pi)^d}\left[\Delta_F(l)\Delta_F^*(l+p) + \Delta_F^*(l)\Delta_F(l+p)\right] = \\ &\quad \int \frac{\mathrm{d}^{d-1} l}{(2\pi)^{d-1}}\frac{-i}{4\omega_l\omega_{l+p}}\left[\frac{1}{p_0+\omega_l-\omega_{l+p}-i\epsilon}-\frac{1}{p_0+\omega_l-\omega_{l+p}+i\epsilon}\right. \\ &\qquad\qquad \left. + \frac{1}{p_0+\omega_{l+p}-\omega_l-i\epsilon}-\frac{1}{p_0+\omega_{l+p}-\omega_l+i\epsilon}\right]. \end{aligned} \tag{5.60}$$

This is nothing more that the sum of two δ-functions (5.41)

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \left[\Delta_F(l)\Delta_F^*(l+p) + \Delta_F^*(l)\Delta_F(l+p)\right] = \int \frac{\mathrm{d}^{d-1} l}{(2\pi)^{d-1}} \frac{\pi}{2\omega_l \omega_{l+p}} \left[\delta(p_0 + \omega_l - \omega_{l+p}) + \delta(p_0 + \omega_{l+p} - \omega_l)\right]. \tag{5.61}$$

We may restore one integration by a δ-function with which we have the result

$$\begin{aligned}\int \frac{\mathrm{d}^d l}{(2\pi)^d} &\left[\Delta_F(l)\Delta_F^*(l+p) + \Delta_F^*(l)\Delta_F(l+p)\right] = \\ &\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{\pi^2}{\omega_l \omega_{l+p}} \left[\delta(l_0 + p_0 - \omega_{l+p})\delta(l_0 - \omega_l) + \delta(l_0 + p_0 + \omega_{l+p})\delta(l_0 + \omega_l)\right] \\ &= \int \frac{\mathrm{d}^d l}{(2\pi)^d} \Big[\Delta^+(l)\Delta^+(l+p) + \Delta^-(l)\Delta^-(l+p)\Big]\end{aligned} \tag{5.62}$$

These are all the identities we need. More can be formed by cunningly combining the ones that we have already stated and by using the tree theorem on more propagators.

The Self-Energy Diagram

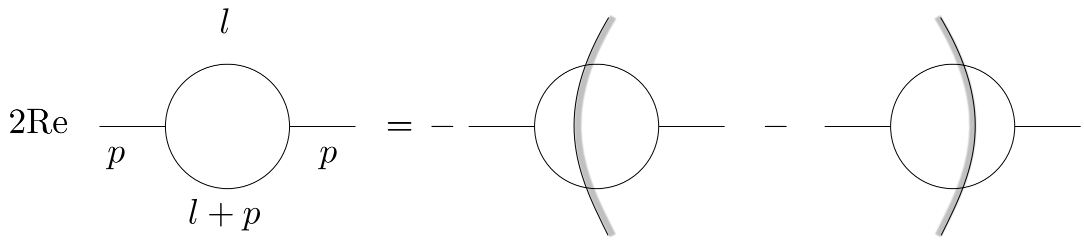


Figure 5.2: The two terms produced by the cutting equations for the self-energy diagram.

To begin the discussion of the connection between the largest time equation and the tree theorem we will look at the scalar self-energy diagram. The cutting equations gives the two terms seen in figure 5.2. The two terms are in principle mutually exclusive, since they restrict energy to either flow to the right or to the left. The right-hand side is given by

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \Big[\Delta^+(l)\Delta^-(l+p) + \Delta^-(l)\Delta^+(l+p)\Big] \tag{5.63}$$

where the vertex on the shaded side has received and extra minus according to the rules. These two terms are what we expect to find from the tree theorem. This simplest example already presents some complexity as (5.30) plus its complex conjugated produces six non-zero terms. These are given by

$$\begin{aligned}-\int \frac{\mathrm{d}^d l}{(2\pi)^d} \big[&- \Delta_F(l)\Delta^+(l+p) - \Delta^+(l)\Delta_F(l+p) \\ &- \Delta_F^*(l)\Delta^+(l+p) - \Delta^+(l)\Delta_F^*(l+p) + 2\Delta^+(l)\Delta^+(l+p))\big]\end{aligned} \tag{5.64}$$

These terms have to add up to the two terms from the cutting equations. The four first terms are each others complex conjugate, so we may start by just rewriting the first two

$$\begin{aligned}&\int \frac{\mathrm{d}^d l}{(2\pi)^d}\Delta_F(l)\Delta^+(l+p) = \\ &\quad \int \frac{\mathrm{d}^d l}{(2\pi)^d}\frac{i}{2\omega_l}\left[\frac{1}{l_0-\omega_l+i\epsilon}-\frac{1}{l_0+\omega_l-i\epsilon}\right]\Delta^+(l+p)\end{aligned} \tag{5.65}$$

and

$$\begin{aligned}&\int \frac{\mathrm{d}^d l}{(2\pi)^d}\Delta^+(l)\Delta_F(l+p) = \\ &\quad \int \frac{\mathrm{d}^d l}{(2\pi)^d}\frac{i}{2\omega_{l+p}}\Delta^+(l)\left[\frac{1}{l_0+p_0-\omega_{l+p}+i\epsilon}-\frac{1}{l_0+p_0+\omega_{l+p}-i\epsilon}\right].\end{aligned} \tag{5.66}$$

We integrate using the δ-function and obtain

$$\begin{aligned}&\int \frac{\mathrm{d}^d l}{(2\pi)^d}\Big[\Delta_F(l)\Delta^+(l+p)+\Delta^+(l)\Delta_F(l+p)\Big] = \\ &\quad \int \frac{\mathrm{d}^{d-1} l}{(2\pi)^{d-1}}\frac{i}{4\omega_l\omega_{l+p}}\left[\frac{1}{-p_0-\omega_{l+p}-\omega_l+i\epsilon}-\frac{1}{-p_0-\omega_{l+p}+\omega_l-i\epsilon}\right. \\ &\qquad\qquad \left.+\frac{1}{p_0-\omega_l-\omega_{l+p}+i\epsilon}-\frac{1}{p_0-\omega_l+\omega_{l+p}-i\epsilon}\right] \\ &= \int \frac{\mathrm{d}^{d-1} l}{(2\pi)^{d-1}}\frac{i}{4\omega_l\omega_{l+p}}\left[\frac{1}{-p_0-\omega_{l+p}-\omega_l+i\epsilon}-\frac{1}{-p_0+\omega_l+\omega_{l+p}-i\epsilon}\right. \\ &\qquad\qquad \left.-2\pi i\delta(p_0+\omega_l-\omega_{l+p})\right]\end{aligned} \tag{5.67}$$

where we have used (5.41) in the last line to identify a δ-function. We use this same identity when adding the complex conjugated to arrive at

$$\begin{aligned}&\int \frac{\mathrm{d}^d l}{(2\pi)^d}\Big[\Delta_F(l)\Delta^+(l+p)+\Delta^+(l)\Delta_F(l+p)+\Delta_F^*(l)\Delta^+(l+p)+\Delta^+(l)\Delta_F^*(l+p)\Big] = \\ &= \int \frac{\mathrm{d}^{d-1} l}{(2\pi)^{d-1}}\frac{\pi}{2\omega_l\omega_{l+p}}\Big[\delta(p_0+\omega_{l+p}+\omega_l)+\delta(p_0-\omega_l-\omega_{l+p}) \\ &\qquad\qquad +2\delta(p_0+\omega_l-\omega_{l+p})\Big].\end{aligned} \tag{5.68}$$

The last step is to reintroduce an integration by a δ-function at each term to end by our final result

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{\pi^2}{\omega_l \omega_{l+p}} \Big[\delta(l_0 - \omega_l)\delta(l_0 + p_0 + \omega_{l+p}) + \delta(p_0 + l_0 - \omega_{l+p})\delta(l_0 + \omega_l) + 2\delta(p_0 + l_0 - \omega_{l+p})\delta(l_0 - \omega_l) \Big]$$
$$= \int \frac{\mathrm{d}^d l}{(2\pi)^d} \left[\Delta^+(l)\Delta^-(l+p)) + \Delta^-(l)\Delta^+(l+p)) + 2\Delta^+(l)\Delta^+(l+p)) \right]. \tag{5.69}$$

Inserting this back into (5.64) we see that the tree theorem and the cutting equations are in perfect agreement. Although the calculation was related to the self-energy diagram it adds another identity to our arsenal, namely

$$\Delta_F(l)\Delta_F(l+p) + \Delta_F^*(l)\Delta_F^*(l+p) = \Delta^+(l)\Delta^-(l+p) + \Delta^-(l)\Delta^+(l+p). \tag{5.70}$$

Now that we have shown that the tree theorem and the cutting equations agree we want to show that the same result can be obtained from the modified tree theorem (5.44).

Connecting Through the Self-Energy Diagram

It is not very difficult to obtain the cutting equations from the modified tree theorem for the self-energy diagram. For clarity we will omit the loop integral and only look on the propagators. Also we will condense the notation such as $\Delta_F\Delta_F := \Delta_F(l)\Delta_F(l+p)$. This should not result in confusion. The modified tree theorem generates 14 terms on the right-hand side to be evaluated. The expansion takes the following form

$$\begin{aligned}\Delta_F\Delta_F + \Delta_F^*\Delta_F^* = &-\Delta_F\Delta_F^* - \Delta_F^*\Delta_F + \Delta_F\Delta^+ + \Delta^+\Delta_F + \Delta_F\Delta^- + \Delta^-\Delta_F \\ &+ \Delta_F^*\Delta^+ + \Delta^+\Delta_F^* + \Delta_F^*\Delta^- + \Delta^-\Delta_F^* - \Delta^+\Delta^+ - \Delta^-\Delta^- \\ &- \Delta^-\Delta^+ - \Delta^+\Delta^-. \end{aligned} \tag{5.71}$$

The two last terms are identical to the terms of the cutting equations except that they have the wrong sign. If we use the two identities (5.53) and (5.54) we may drastically reduce the number of terms to

$$\begin{aligned}\Delta_F\Delta_F + \Delta_F^*\Delta_F^* &= -\Delta_F\Delta_F^* - \Delta_F^*\Delta_F + 2\Delta_F\Delta_F + 2\Delta_F^*\Delta_F^* \\ &\quad + \Delta^+\Delta^+ + \Delta^-\Delta^- - \Delta^-\Delta^+ - \Delta^+\Delta^- \\ &= \Delta_F\Delta_F^* + \Delta_F^*\Delta_F - \Delta^+\Delta^+ - \Delta^-\Delta^- + \Delta^-\Delta^+ + \Delta^+\Delta^- \end{aligned} \tag{5.72}$$

where we in the last line have moved $2\Delta_F\Delta_F + 2\Delta_F^*\Delta_F^*$ to the left-hand sign and multiplied by -1. Notice also that the sign in front of $\Delta^+\Delta^+$ and $\Delta^-\Delta^-$ has changed as we have subtracted these *twice*. From (5.55) we have that $\Delta_F\Delta_F^* + \Delta_F^*\Delta_F - \Delta^+\Delta^+ - \Delta^-\Delta^- = 0$ and hence

$$\Delta_F\Delta_F + \Delta_F^*\Delta_F^* = \Delta^-\Delta^+ + \Delta^+\Delta^- \tag{5.73}$$

as wanted. In the first step of our calculation we did nothing more than use the original four tree theorems to eliminate a number of terms. It should be noted, that even though this calculations seemed simpler than the "brute force" method, it was actually just as lengthy since the last identity we used had to be derived in full detail, we only chose to do it higher up.

Generalizing the Connection

Now things become much more complicated already as we turn our attention to the generic one loop-diagram with three external legs. In general the modified tree theorem (5.44) produces $4^N - 2$ terms[4] which is 62 for a diagram with three external legs. If we try to mimic our above derivations the first thing we should do is to apply (5.53) and (5.54). Those four equations are nothing more than the four tree theorems. If we subtract each of these from the modified tree theorem we will have subtracted $2\Delta_F\Delta_F + 2\Delta_F^*\Delta_F^*$ from the left-hand side and on the right-hand side we will have subtracted $4(2^N - 2)$ terms as there are $2^N - 1$ terms in each of the four tree theorems and the terms of $\Delta^+ \ldots \Delta^+$ and $\Delta^- \ldots \Delta^-$ are subtracted twice according to our above derivations. This leaves $2^N(2^N - 4) + 6$ terms which is still 38 terms for a loop with three external legs. For comparison the cutting equations give 6 terms and only three of these are present in the modified tree theorem as the remaining 3 terms have the wrong sign. To show that the remaining 35 terms are equal to the 3 we are looking for is a somewhat lengthy derivation and we will not give it here. In general we expect that the cutting equations can be obtained from the tree theorem or the modified tree theorem in a manner similar to what we have shown here. There is however nothing new to be learned from doing this for more and more legs, as doing it for the self-energy diagram already show that there is a connection which is all we wanted.

[4]The -2 terms are on the left-hand side

— 6 —
Summary and Final Remarks

In this thesis we have discussed various properties of scattering amplitudes both at the tree- and loop-level. We started by introducing the spinor helicity formalism and colour-ordering of scattering amplitudes. These to notions greatly simplified the structure of the amplitudes as these could be decomposed in to simpler subamplitudes given by compact expressions in terms of spinor products.

We then constructed an explicit spinor basis to express all physical quantities in. From this basis we could read of a particular complex space-like direction which all polarization vectors are transverse to. This direction we turned into the space-cone gauge and we gauge-fixed the QCD lagrangian with it. From the lagrangian we read of colour-ordered Feynman rules and showed how these could be applied in the calculation of a four-, five- and six-point gluon amplitude. We also showed how to apply the Feynman rules involving quarks although this was not as straightforward.

We did not investigate the space-cone gauges direct relation to the CSW formalism. Some literature suggests that the CSW rules can be obtained directly from a lagrangian approach similar to the one presented here.

The space-cone gauge was then used to show a connection between the BCFW recursion and the largest time equation. The recursion gives a prescription for expressing tree-level amplitudes in terms of lower-order on-shell tree-level amplitudes. This factorization of the amplitude could be shown to also take place at a diagrammatical level. The main observation was that no of the vertices of the space-cone gauge fixed lagrangian depends on the momentum component $\bar{p}$ and therefore the propagators could be shifted along this direction to be set on-shell. From this it was clear that both polarization vectors and vertices factorizes in a given amplitude and we were left with showing that the shifted propagators recombined into the original propagators. This relation was showed to be rooted in the largest time equation.

The largest time equation results in a set of cutting equations which relates a diagram to its real part through diagrams of a lower order in the coupling constant. For the special case of a one-loop diagram this means that we extract the real part from tree-level diagrams. We introduced another result known as Feynman's Tree Theorem which is very similar to the cutting equations as it relates a one-loop diagram directly to tree-level diagrams. We proceeded to show that the cutting equations and the tree theorem are related as they derive from the same Feynman propagator identity. We also showed that the tree theorem can be used to obtain the cutting equations directly for the self-energy diagram.

As the largest time equation is related to the BCFW relation it is natural to think that a direct link between the tree theorem and the BCFW recursion can be obtained. When we showed the connection between the BCFW recursion and the largest time equation we had to rewrite everything from x-space to p-space. Since the tree theorem is already formulated in p-space the connection might even be simpler here. We hope that the work presented here can be a starting point for the derivation of such a connection.

Acknowledgements

I would like to thank my two thesis advisors Poul Henrik Damgaard and Emil Bjerrum-Bohr for many enlightening discussions without which this thesis had not been possible to write. I would also like to thank Diana Vaman for discussions regarding the space-cone gauge during here short visit in the summer 2011. Lastly I would like to thank Thomas Søndergaard for many helpful discussions.

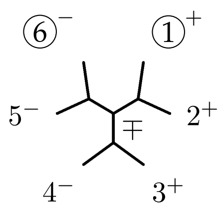
Appendices

— A —

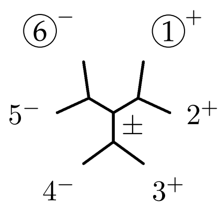
Six-point $NMHV$ amplitude

Here we present all diagrams and their expressions for the amplitude $A_6(+++---)$ discussed in section 3.3. This amplitude has been checked numerically against results obtained from the BCFW recursion. There is an overall factor multiplied on all diagrams given by

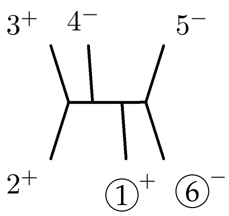
$$-\epsilon_+(2)^+\epsilon_+(3)^+\epsilon_-(4)^-\epsilon_-(5)^- \tag{A.1}$$



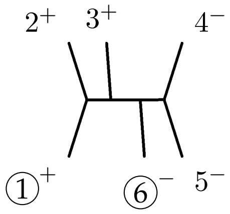
$$= -p_2^2 p_5 p_4 \left(\frac{p_5^- + p_2^- + p_1^-}{p_5 + p_2 + p_1} - \frac{p_3^-}{p_3} \right) \left(\frac{p_3^+ + p_4^+}{p_3 + p_4} - \frac{p_5^+ + p_6^+}{p_5 + p_6} \right) \frac{1}{P_{12}^2 P_{34}^2 P_{56}^2} \tag{A.2}$$



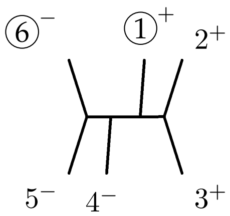
$$= -p_2 p_5^2 p_3 \left(\frac{p_4^+}{p_4} - \frac{p_5^+ + p_2^+ + p_6^+}{p_5 + p_2 + p_6} \right) \left(\frac{p_1^- + p_2^-}{p_1 + p_2} - \frac{p_3^- + p_4^-}{p_3 + p_4} \right) \frac{1}{P_{12}^2 P_{34}^2 P_{56}^2} \tag{A.3}$$



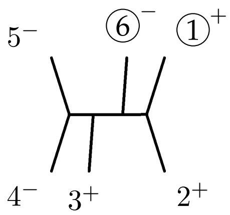
$$= p_5^2 (p_2 + p_3)(p_4 + p_5) \left(\frac{p_2^-}{p_2} - \frac{p_3^-}{p_3} \right) \left(\frac{p_4^+}{p_4} - \frac{p_5^+ + p_6^+}{p_5 + p_6} \right) \frac{1}{P_{23}^2 P_{56}^2 P_{156}^2} \tag{A.4}$$



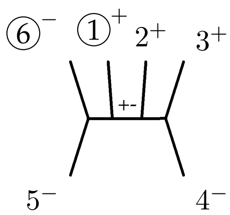
$$= -p_2(p_2+p_3)^2(p_4+p_5)\left(\frac{p_2^- + p_1^-}{p_2+p_1} - \frac{p_3^-}{p_3}\right)\left(\frac{p_5^+}{p_5} - \frac{p_4^+}{p_4}\right)\frac{1}{P_{12}^2 P_{45}^2 P_{123}^2} \tag{A.5}$$



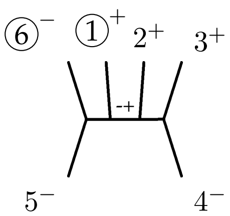
$$= -p_5(p_2+p_3)(p_4+p_5)^2\left(\frac{p_4^+}{p_4} - \frac{p_5^+ + p_6^+}{p_5+p_6}\right)\left(\frac{p_3^-}{p_3} - \frac{p_2^-}{p_2}\right)\frac{1}{P_{23}^2 P_{56}^2 P_{123}^2} \tag{A.6}$$



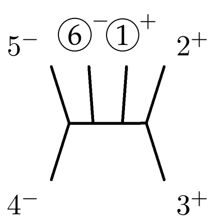
$$= p_2^2(p_4+p_5)^2\left(\frac{p_4^+}{p_4} - \frac{p_5^+}{p_5}\right)\left(\frac{p_3^-}{p_3} - \frac{p_2^- + p_1^-}{p_2+p_1}\right)\frac{1}{P_{126}^2 P_{45}^2 P_{12}^2} \tag{A.7}$$



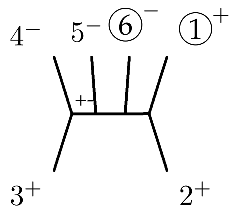
$$= p_5^3 p_3\left(\frac{p_2^-}{p_2} - \frac{p_3^- + p_4^-}{p_3+p_4}\right)\left(\frac{p_4^+}{p_4} - \frac{p_2^+ + p_5^+ + p_6^+}{p_2+p_5+p_6}\right)\frac{1}{P_{156}^2 P_{34}^2 P_{56}^2} \tag{A.8}$$



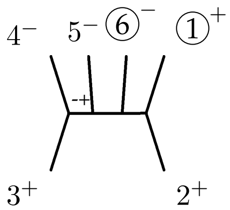
$$= p_5^2 p_2 p_4\left(\frac{p_3^+ + p_4^+}{p_3+p_4} - \frac{p_5^+ + p_6^+}{p_5+p_6}\right)\left(\frac{p_2^- + p_5^- + p_1^-}{p_2+p_5+p_1} - \frac{p_3^-}{p_3}\right)\frac{1}{P_{156}^2 P_{34}^2 P_{56}^2} \tag{A.9}$$



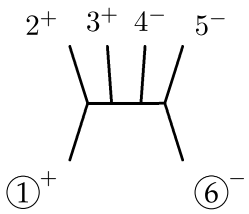
$$= -(p_4+p_5)^4\left(\frac{p_4^+}{p_4} - \frac{p_5^+}{p_5}\right)\left(\frac{p_2^-}{p_2} - \frac{p_3^-}{p_3}\right)\frac{1}{P_{123}^2 P_{23}^2 P_{45}^2} \tag{A.10}$$



$$= p_4 p_2^3 \left(\frac{p_5^+}{p_5} - \frac{p_3^+ + p_4^+}{p_3 + p_4} \right) \left(\frac{p_3^-}{p_3} - \frac{p_2^- + p_5^-}{p_2 + p_5} \right) \frac{1}{P_{126}^2 P_{12}^2 P_{34}^2} \tag{A.11}$$



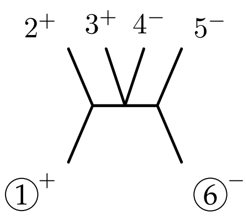
$$= p_5 p_3 p_2^2 \left(\frac{p_3^- + p_4^-}{p_3 + p_4} - \frac{p_1^- + p_2^-}{p_1 + p_2} \right) \left(\frac{p_2^+ + p_5^+ + P_6^+}{p_2 + p_5 + p_6} - \frac{p_4^+}{p_4} \right) \frac{1}{P_{126}^2 P_{12}^2 P_{34}^2} \tag{A.12}$$



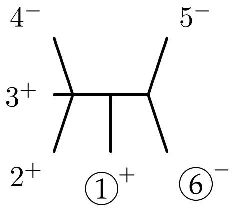
$$= p_5 p_2 (p_2 + p_3)^2 \left(\frac{p_4^+}{p_4} - \frac{p_5^+ + p_6^+}{p_5 + p_6} \right) \left(\frac{p_1^- + p_2^-}{p_1 + p_2} - \frac{p_3^-}{p_3} \right) \frac{1}{P_{123}^2 P_{12}^2 P_{56}^2} \tag{A.13}$$

The three four-point diagrams have the following common factor

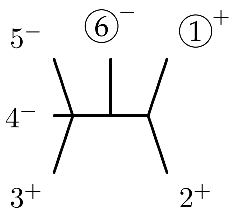
$$\frac{p_3 p_5 + p_2 p_4}{(p_5 + p_2)(p_3 + p_4)} \tag{A.14}$$



$$= -\frac{p_2 p_5}{P_{12}^2 P_{56}^2} \tag{A.15}$$



$$= \frac{p_5^2}{P_{56}^2 P_{156}^2} \tag{A.16}$$

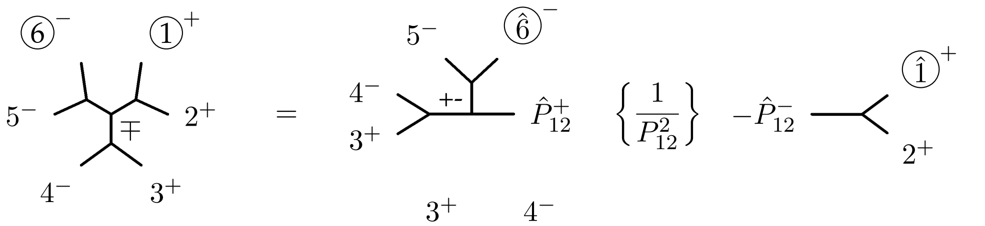


$$= \frac{p_2^2}{P_{12}^2 P_{126}^2} \tag{A.17}$$

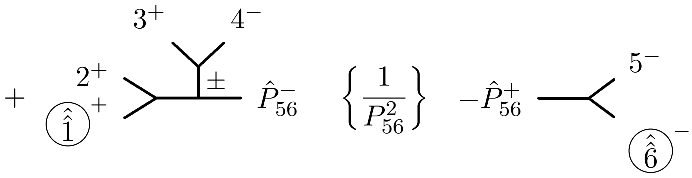
— B —

Factorization of Six-point $NMHV$ Amplitude

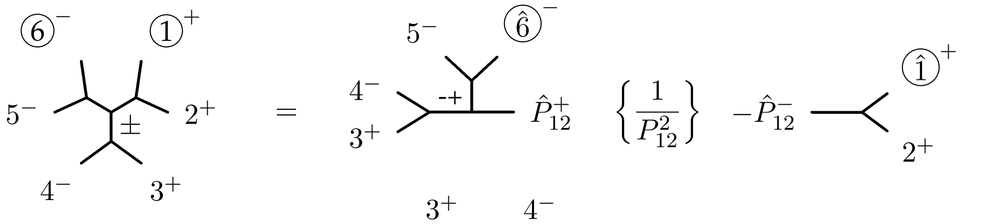
Here we write the full factorization of the six-point amplitude $A_6(+++---)$ following the prescription described in chapter 4. We factorize each diagram in the same order as in appendix A



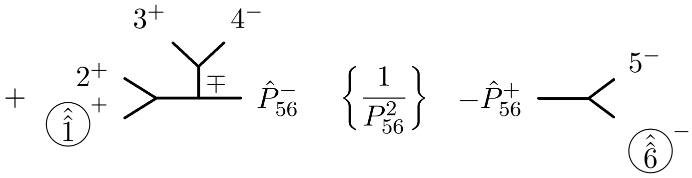
(B.1)



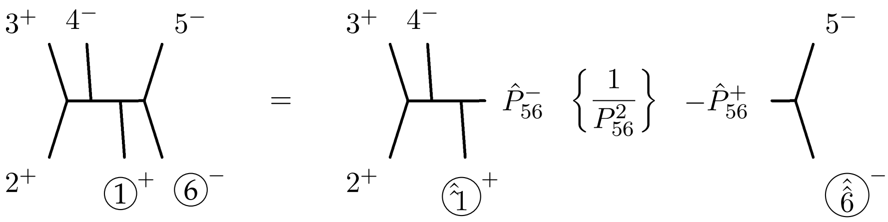
(B.2)



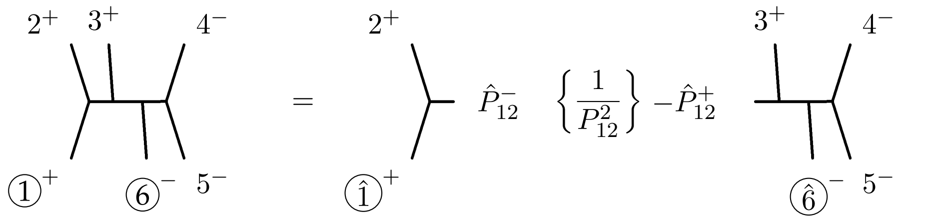
(B.3)



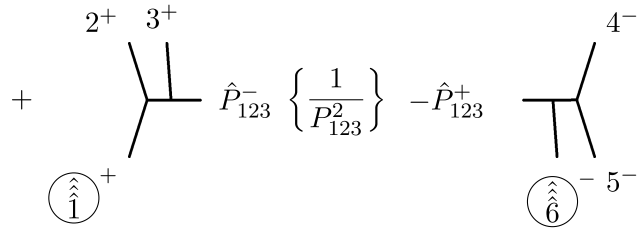
(B.4)



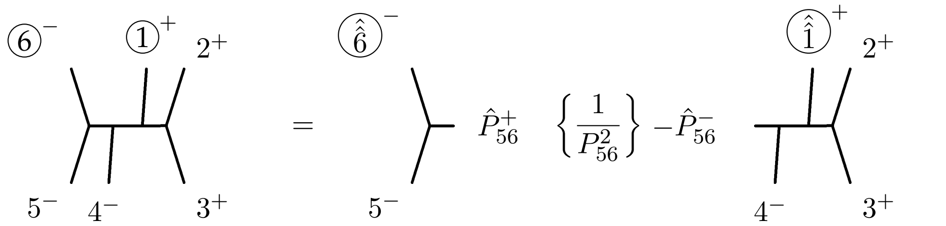
(B.5)



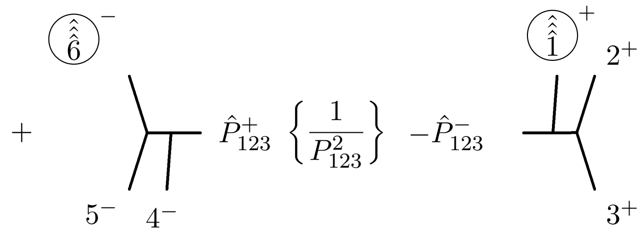
(B.6)



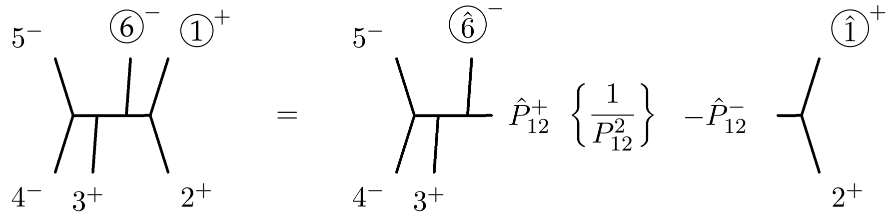
(B.7)



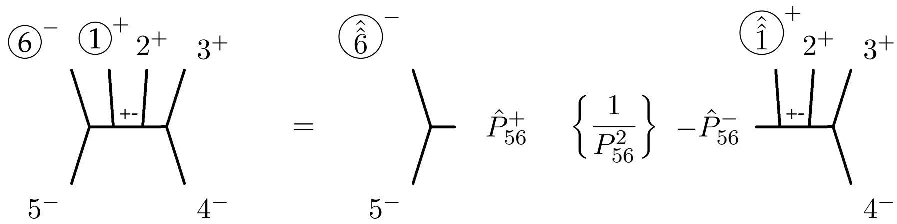
(B.8)



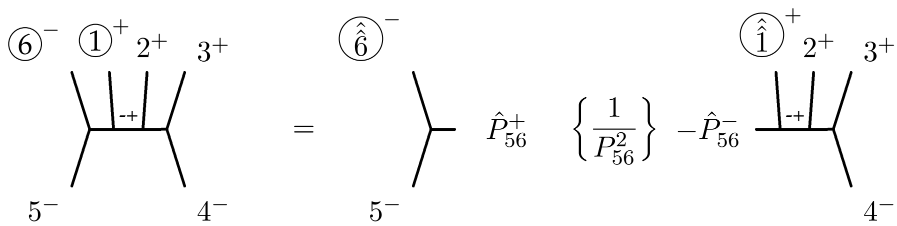
(B.9)



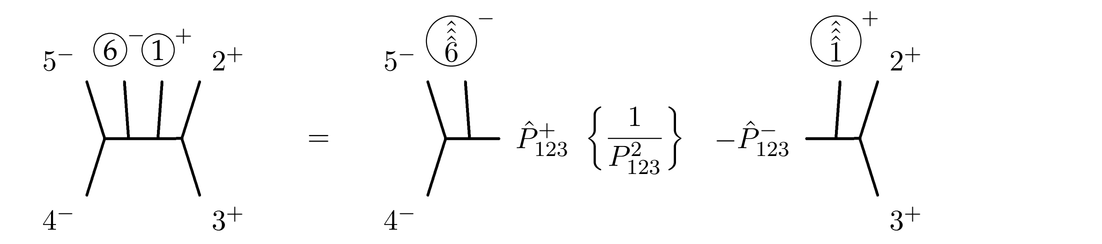
(B.10)



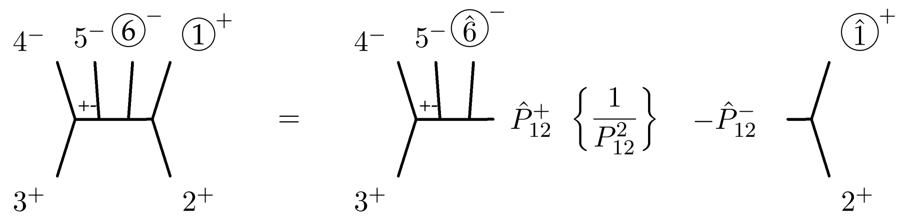
(B.11)



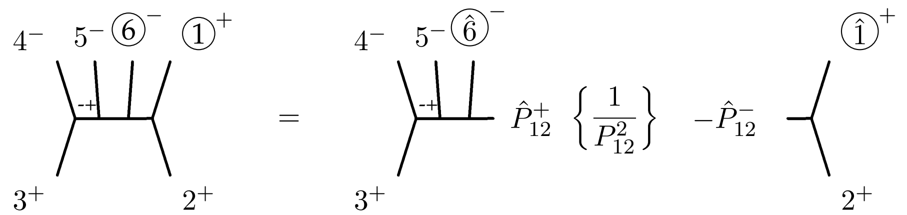
(B.12)



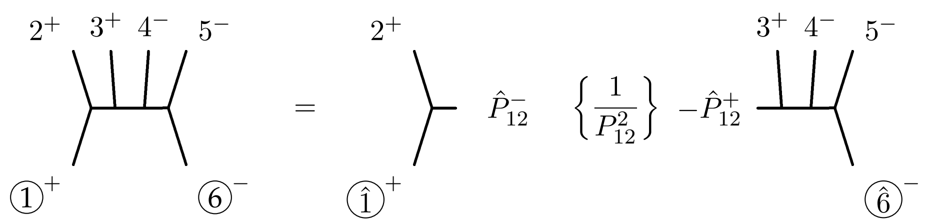
(B.13)



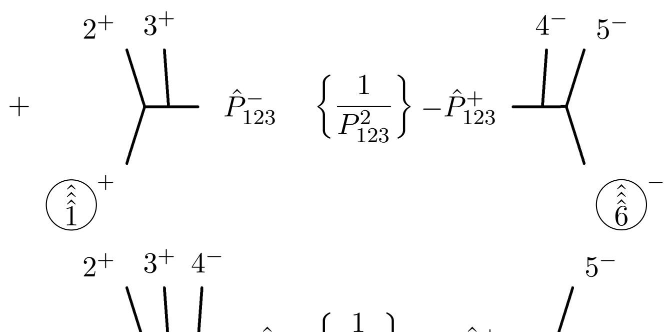
(B.14)



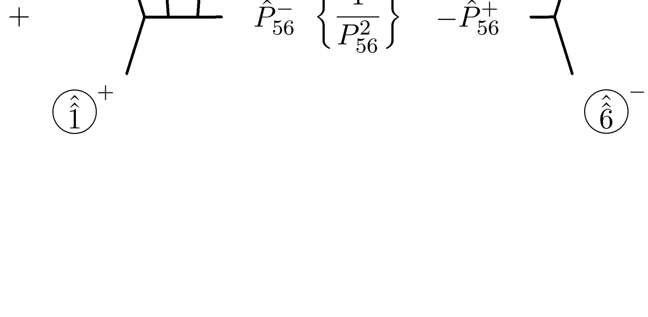
(B.15)



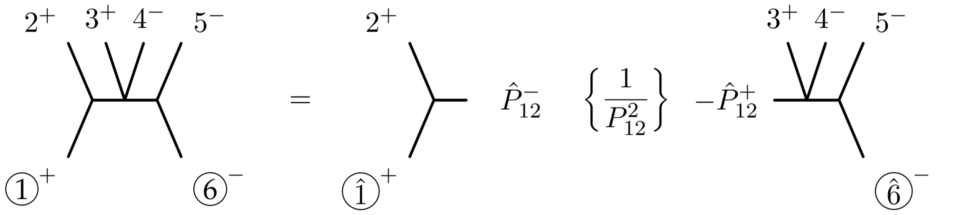
(B.16)



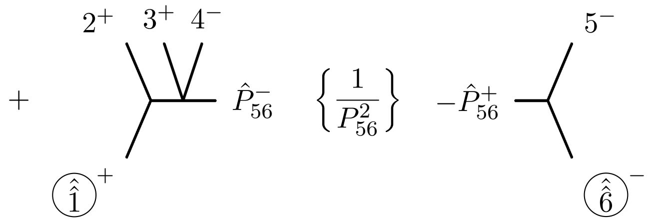
(B.17)



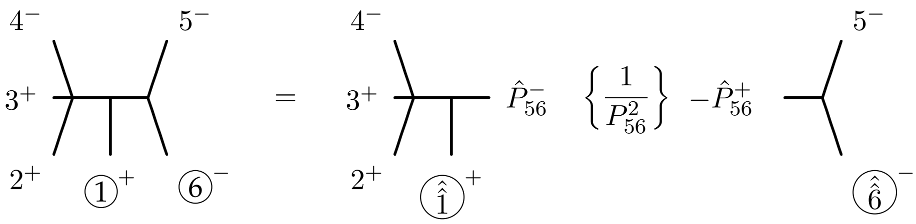
(B.18)



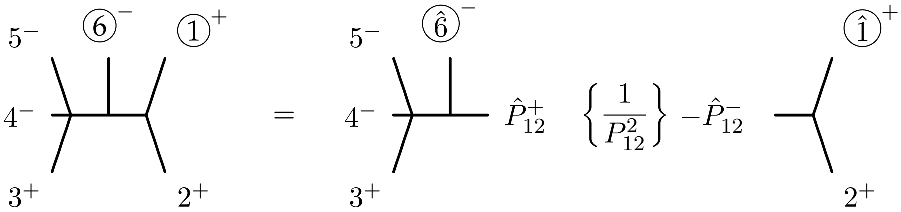
(B.19)



(B.20)



(B.21)



(B.22)

Bibliography

[1] G. Chalmers and W. Siegel, "Simplifying algebra in Feynman graphs. II. Spinor helicity from the spacecone," *Physical Review D*, vol. 59, no. 4, p. 045013, 1999.

[2] G. Chalmers and W. Siegel, "Simplifying algebra in Feynman graphs. I. Spinors," *Physical Review D*, vol. 59, no. 4, p. 045012, 1999.

[3] G. Chalmers and W. Siegel, "Simplifying algebra in Feynman graphs. III. Massive vectors," *Physical Review D*, vol. 63, no. 12, p. 125027, 2001.

[4] E. Witten, "Perturbative gauge theory as a string theory in twistor space," *Communications in mathematical physics*, vol. 252, no. 1, pp. 189–258, 2004.

[5] F. Cachazo, P. Svrcek, and E. Witten, "MHV vertices and tree amplitudes in gauge theory," *Journal of High Energy Physics*, vol. 09, p. 006, 2004.

[6] R. Britto, F. Cachazo, and B. Feng, "New recursion relations for tree amplitudes of gluons," *Nuclear Physics B*, vol. 715, no. 1-2, pp. 499–522, 2005.

[7] R. Britto, F. Cachazo, B. Feng, and E. Witten, "Direct proof of the tree-level scattering amplitude recursion relation in Yang-Mills theory," *Physical review letters*, vol. 94, no. 18, p. 181602, 2005.

[8] D. Vaman and Y. Yao, "QCD recursion relations from the largest time equation," *Journal of High Energy Physics*, vol. 04, p. 030, 2006.

[9] M. Veltman, "Unitarity and causality in a renormalizable field theory with unstable particles," *Physica*, vol. 29, no. 3, pp. 186–207, 1963.

[10] R. Britto, F. Cachazo, and B. Feng, "Generalized unitarity and one-loop amplitudes in $\mathcal{N} = 4$ super-Yang–Mills," *Nuclear Physics B*, vol. 725, no. 1, pp. 275–305, 2005.

[11] R. Feynman, "Closed loop and tree diagrams," *Magic Without Magic: John Archibald Wheeler*, vol. 1, p. 355, 1972.

[12] R. Feynman, "Problems in quantizing the gravitational field, and the massless Yang-Mills field," *Magic Without Magic: John Archibald Wheeler*, vol. 1, p. 377, 1972.

[13] F. Berends and W. Giele, "The six-gluon process as an example of Weyl-van der Waerden spinor calculus," *Nuclear Physics B*, vol. 294, pp. 700–732, 1987.

[14] Z. Bern and D. Kosower, "Color decomposition of one-loop amplitudes in gauge theories," *Nuclear Physics B*, vol. 362, no. 1, pp. 389–448, 1991.

[15] M. Mangano and S. Parke, "Multi-parton amplitudes in gauge theories," *Physics Reports*, vol. 200, no. 6, pp. 301–367, 1991.

[16] L. Dixon, "Calculating scattering amplitudes efficiently," *Arxiv preprint hep-ph/9601359*, 1996.

[17] M. Peskin, "Simplifying Multi-Jet QCD Computation," *Arxiv preprint arXiv:1101.2414*, 2011.

[18] F. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans, and T. Wu, "Single bremsstrahlung processes in gauge theories," *Physics Letters B*, vol. 103, no. 2, pp. 124–128, 1981.

[19] P. De Causmaecker, R. Gastmans, W. Troost, and T. Wu, "Multiple bremsstrahlung in gauge theories at high energies (I). General formalism for quantum electrodynamics," *Nuclear Physics B*, vol. 206, no. 1, pp. 53–60, 1982.

[20] R. Kleiss and W. Stirling, "Spinor techniques for calculating $pp \to W^{\pm}/Z^{0}$+ jets," *Nuclear Physics B*, vol. 262, pp. 235–262, 1985.

[21] J. Gunion and Z. Kunszt, "Improved analytic techniques for tree graph calculations and the $ggq\bar{q}ll$ subprocess," *Physics Letters B*, vol. 161, no. 4, pp. 333–340, 1985.

[22] Z. Xu, D. Zhang, and L. Chang, "Helicity amplitudes for multiple bremsstrahlung in massless non-abelian gauge theories," *Nuclear Physics B*, vol. 291, pp. 392–428, 1987.

[23] R. Gastmans and T. Wu, "The Ubiquitous Photon, Helicity Method for QED and QCD," *International Series of Monographs on Physics*, vol. 80, 1990.

[24] R. Kleiss and H. Kuijf, "Multigluon cross sections and 5-jet production at hadron colliders," *Nuclear Physics B*, vol. 312, no. 3, pp. 616–644, 1989.

[25] V. Del Duca, L. Dixon, and F. Maltoni, "New color decompositions for gauge amplitudes at tree and loop level," *Nuclear Physics B*, vol. 571, no. 1, pp. 51–70, 2000.

[26] Z. Bern, J. Carrasco, and H. Johansson, "New relations for gauge-theory amplitudes," *Physical Review D*, vol. 78, no. 8, p. 85011, 2008.

[27] T. Søndergaard, "New relations for gauge-theory amplitudes with matter," *Nuclear physics B*, vol. 821, no. 1, pp. 417–430, 2009.

[28] S. Parke and T. Taylor, "Amplitude for n-gluon scattering," *Physical Review Letters*, vol. 56, no. 23, pp. 2459–2460, 1986.

[29] F. Berends and W. Giele, "Recursive calculations for processes with n gluons," *Nuclear Physics B*, vol. 306, no. 4, pp. 759–808, 1988.

[30] K. Risager, "A direct proof of the CSW rules," *Journal of High Energy Physics*, vol. 2005, p. 003, 2005.

[31] G. Leibbrandt, "Introduction to noncovariant gauges," *Reviews of Modern Physics*, vol. 59, no. 4, p. 1067, 1987.

[32] S. Mandelstam, "Light-cone superspace and the ultraviolet finiteness of the $\mathcal{N} = 4$ model," *Nuclear Physics B*, vol. 213, no. 1, pp. 149–168, 1983.

[33] P. Mansfield, "The lagrangian origin of mhv rules," *Journal of High Energy Physics*, vol. 2006, p. 037, 2006.

[34] J. Ettle, *MHV Lagrangians for Yang-Mills and QCD*. PhD thesis, University of Southampton, 2008.

[35] J. Ettle, T. Morris, and Z. Xiao, "The MHV QCD Lagrangian," *Journal of High Energy Physics*, vol. 2008, p. 103, 2008.

[36] R. Monteiro and D. O Connell, "The kinematic algebra from the self-dual sector," *Journal of High Energy Physics*, vol. 2011, no. 7, pp. 1–24, 2011.

[37] T. Søndergaard, "Simplicity of Scattering Amplitudes and New Gauge-Theory Relations," Master's thesis, University of Copenhagen, 2009.

[38] S. Badger, E. Glover, V. Khoze, and P. Svrcek, "Recursion relations for gauge theory amplitudes with massive particles," *Journal of High Energy Physics*, vol. 2005, p. 025, 2005.

[39] S. Badger, E. Glover, and V. Khoze, "Recursion relations for gauge theory amplitudes with massive vector bosons and fermions," *Journal of High Energy Physics*, vol. 2006, p. 066, 2006.

[40] C. Schwinn and S. Weinzierl, "On-shell recursion relations for all Born QCD amplitudes," *Journal of High Energy Physics*, vol. 2007, p. 072, 2007.

[41] M. Luo and C. Wen, "Recursion relations for tree amplitudes in super gauge theories," *Journal of High Energy Physics*, vol. 2005, p. 004, 2005.

[42] P. Draggiotis, R. Kleiss, A. Lazopoulos, and C. Papadopoulos, "Diagrammatic proof of the BCFW recursion relation for gluon amplitudes in QCD," *The European Physical Journal C - Particles and Fields*, vol. 46, pp. 741–750, 2006. 10.1140/epjc/s2006-02484-y.

[43] R. Cutkosky, "Singularities and discontinuities of Feynman amplitudes," *Journal of Mathematical Physics*, vol. 1, p. 429, 1960.

[44] M. Veltman and G. 't Hooft, "Diagrammar," *CERN Yellow Report*, pp. 28–173, 1973.

[45] M. Veltman, *Diagrammatica: the path to Feynman rules*, vol. 4. Cambridge Univ Press, 1994.

[46] R. Streater and A. Wightman, *PCT, spin and statistics, and all that*. Princeton Univ Press, 2000.

[47] R. Feynman, "Quantum theory of gravitation," *Acta Phys. Polon.*, vol. 24, 1963.

[48] C. Møller, "General Properties of the Characteristic Matrix in the Theory of Elementary Particles I," *K. Danske. Vid. Selsk.*, vol. 23, no. 1, pp. 1–48, 1945.

[49] C. Møller, "General Properties of the Characteristic Matrix in the Theory of Elementary Particles II," *K. Danske. Vid. Selsk.*, vol. 22, no. 19, pp. 1–49, 1946.

[50] G. 't Hooft and M. Veltman, "Regularization and renormalization of gauge fields," *Nuclear Physics B*, vol. 44, no. 1, pp. 189–213, 1972.

[51] I. Bierenbaum, S. Catani, P. Draggiotis, and G. Rodrigo, "Feynman's Tree Theorem and Loop-Tree Dualities," *Arxiv preprint arXiv:1011.0585*, 2010.

[52] W. Kilian and T. Kleinschmidt, "Using Feynman's Tree Theorem to Evaluate Loop Integrals Numerically," *Arxiv preprint arXiv:1002.1289*, 2010.

[53] A. Brandhuber, B. Spence, and G. Travaglini, "From trees to loops and back," *Journal of High Energy Physics*, vol. 2006, p. 142, 2006.

[54] S. Catani, T. Gleisberg, F. Krauss, G. Rodrigo, and J. Winter, "From loops to trees bypassing Feynman's theorem," *Journal of High Energy Physics*, vol. 2008, p. 065, 2008.

[55] A. Brandhuber and G. Travaglini, "Quantum MHV diagrams," in *Continuous Advances in QCD 2006*, vol. 1, pp. 443–456, World Scientific Pub Co Inc, 2007.