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M.Sc. Thesis

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Multi-loop Feynman integrals in maximally symmetric gauge theory

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Abstract

Scattering amplitudes are indispensable objects for studying the interaction of subatomic particles in different particle physics theories. They are a key ingredient for calculating the cross section, which is the main physical observable measured in collider experiments at the LHC. These scattering amplitudes also reflect the symmetries of the theory they are calculated in and therefore, provide a ground of study for yet undiscovered symmetries and mathematical structures. For example, [1] discovered that the recently found dual superconformal symmetry in planar $\mathcal{N} = 4$ sYM introduced in [2] combines with the long known ordinary (spacetime related) superconformal symmetry enjoyed by the same theory, to create an infinite dimensional symmetry algebra acting on scattering amplitudes called *Yangian symmetry*. The action of this algebra determines that planar $\mathcal{N} = 4$ sYM has an integrability structure; the integrability of the theory gives additional constraints on the structure of scattering amplitudes. The thesis mainly focuses on the integrals contributing to these scattering amplitudes, rather than on the amplitudes themselves. More precisely, in this thesis we compute the symbol of the 2-loop 10-point double-pentagon integral I_{dp} contributing to 2-loop MHV scattering amplitude in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills (sYM) theory in $d = 4$ spacetime dimensions, whose alphabet evaluates to both rational and algebraic letters. The relevant mathematical objects and structures for studying such integrals are reviewed beforehand. The calculation is performed using the duality of a certain class of Feynman-integrals with the vacuum expectation value (VEV) of null polygonal Wilson loops. However, some of these integrations over the edges require the rationalization of the square roots contained in the integrand. In total, we obtain a resulting symbol alphabet with 122 rational letters and 54 algebraic letters. This thesis is accompanied with a `Mathematica` notebook which contains the results presented in this thesis.

Acknowledgements

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Chapter 1

Introduction

Scattering amplitudes are fundamental objects in high energy theory, as they are an essential ingredient for linking theoretical models with experimental results obtained from modern particle accelerators like the Large Hadron Collider (LHC) at CERN. From them the scattering cross-section σ is calculated, which is also the primary physical parameter measured by the experimentalists in collider experiments. These amplitudes are also crucial for discovering new structures in quantum field theory (QFT) as well and their structures reflects the symmetries of the theory they are examined in. Amplitudes can be calculated perturbatively order-by-order, where usually the perturbative expansion is with respect to some suitable coupling parameter. Hence, the scattering amplitude A_n describing the interaction of n particles in some theory is calculated perturbatively as

$$A_n = A_n^{(L=0)} + A_n^{(L=1)} + A_n^{(L=2)} + \cdots \tag{1.1}$$

where in $A_n^{(L)}$, the L is the loop-order of the scattering amplitude, i.e. the number of independent internal loop momenta we have to integrate over. The $A_n^{(L=0)}$ is often also referred to as Leading Order (LO), $A_n^{(L=1)}$ as Next-to-Leading Order (NLO) and so forth. At each order in the perturbation series, one must write down all possible Feynman diagrams and calculate them using the associated Feynman rules that we have extracted from the Lagrangian of the theory. This is often a complex and tedious task that requires the summation over a large number of diagrams even at tree level, that is, for the calculation of $A_n^{(L=0)}$. The computational difficulty increases immensely with respect to the loop-order L. The thesis focuses on the calculation of a certain $A_n^{(L=2)}$ amplitude, specifically on the integral arising in the computation of the 2-loop (NNLO), $n = 10$-point (external leg) MHV scattering amplitude in planar $\mathcal{N} = 4$ super Yang-Mills theory in $d = 4$ spacetime dimensions.

Choosing this theory as the main laboratory is by no means random! Theories of Nature are often difficult to handle and stand out as some of the most difficult QFTs. Other models, equipped with natural and also not yet discovered symmetries offer major simplifications in the calculations of physical quantities. Such a model is planar $\mathcal{N} = 4$ super Yang-Mills (sYM) theory, in which exceptional computational advancements have been made in the last decades. For example, in [3] a recursive relation for determining all L-loop scattering amplitude integrands has been given, which manifests the Yangian symmetry of theory at all orders. Furthermore, high-loop scattering amplitudes have been computed analytically in papers such as [4],[5] and [6], even for large n numbers of external legs such as in [7] and [8]. These simplifications are due to multiple reasons. First, the theory enjoys both superconformal and dual superconformal symmetry. These symmetry groups lead to additional constraints on the amplitudes and allow for the use of powerful mathematical techniques, such as the use of momentum twistor space. *Physical constraints* that need to be externally enforced on momenta, become simply manifest at the level of scattering amplitudes with momentum-twistor variables. However, these connections are non-trivial and require the introduction of some mathematical concepts, such as differential and algebraic geometry. Moreover, due to planarity the diagrams are simpler; all Feynman-diagrams in planar $\mathcal{N} = 4$ can we drawn in the plane with no propagators crossing over. Large number of diagrams in $\mathcal{N} = 4$ SYM are related to each other by certain symmetry transformations, leading to a smaller number of distinct diagrams that are needed to be evaluated. Further interest in planar $\mathcal{N} = 4$ stems from the fact that it has an integrable structure, which in [9] gave insights into the strong coupling behavior of the theory in terms of the AdS/CFT duality. The existence of an infinite amount of conserved charges (one for each degree of freedom) is intimately related to the Yangian symmetry of this theory. Actually planar $\mathcal{N} = 4$ is conjectured to be exactly solvable at all loop orders! Least, but not last $\mathcal{N} = 4$ has no QCD like-confinement meaning that no hadronic states need to be accounted for, which also fuels the interest in this theory.

In $\mathcal{N} = 4$ there exists a continuous spacetime symmetry called superysmmetry (SUSY) that is realized by a set of transformations (or one such transformation) that map(s) bosons into fermions and fermions into bosons. These transformations are generated by fermionic spinor operators called spinor supercharges [10] that satisfy certain anti-commutator relations. These SUSY generators enlarge the Poincaré-algebra to create a so-called super-Poincaré-algebra; SUSY is a symmetry of spacetime. Different supersymmetric theories contain different numbers of these transformation generators; planar $\mathcal{N} = 4$ has 4 of them (hence the name). As will be argued in **Chapter 3.1**, in $4d$ this is the maximal possible number of SUSY generators for a maximum spin-1 theory, so often the theory is called the *maximally-symmetric* theory in $4d$. Although $\mathcal{N} = 4$ sYM has been long known to be a ”toy-model” of QFT and not a physical theory of Nature, supersymmetry itself has not been yet observed in collider experiments. Different SUSY theories put the scale of the SUSY symmetry breaking at different energies. For example, the energy scale of dynamical supersymmetry breaking in low energy supersymmetry models is estimated to be between (from [11])

$$100 \text{ GeV} \leq \text{SUSY breaking} \leq 30000 \text{ GeV}. \tag{1.2}$$

On the other hand, other high energy SUSY breaking models such as [12] predict this upper bound to be at the GUT scale $\Lambda_{GUT} \sim 10^{16}$ GeV, so during the inflationary epoch of our Universe. Henceforth, any direct experimental verification of supersymmetry may be very well ahead in the future, if there ever will be any. Current LHC data suggests SUSY may not exists in Nature at all, which would rule out $\mathcal{N} = 4$, or any other theory with SUSY completely as a natural theory.

Nevertheless, the theory has been the main ”toy model” of the amplitudeologists in the past decades as its many symmetries make it simpler than quantum chromodynamics (QCD) and scattering amplitudes in this theory uncovered a rich structure of planar $\mathcal{N} = 4$ sYM. It is also mainly interesting because it has similar aspects to QCD, as both theories enjoy a $SU(N_c)$ gauge symmetry, where for QCD in particular $N_c = 3$. The approach in recent years has been to develop new computational technologies by analyzing brute-force Feynman-diagram calculations and trying to find a set of variables for scattering amplitudes A_n that are more adapt to the problem. In works first pioneered by Parke and Taylor in [13], it turned out that the set of variables one uses for expressing the amplitudes can greatly simplify the resulting expressions. Using the so-called *spinor-helicity* formalism applicable to both QCD in the massless fermion limit and $\mathcal{N} = 4$, they found a succinct single expression for the tree-level n-gluon scattering amplitude. This is remarkable as the computational complexity greatly increases with increasing external legs as well, even when considering tree-level amplitudes $A_n^{(L=0)}$. To savour this difficulty, a small excerpt from [14] of the brute-force Feynman diagram calculation of the 5-point tree-level gluon scattering is shown in Figure 1.1. The idea is the same for loop-level, where other mathematical objects that are more suitable for the problem than the usual 4-momenta have been found. However, these are variables that are related to a new type of symmetry called dual conformal symmetry, so need to be introduced appropriately.

This thesis addresses the calculation of the integral arising in the calculation of the 2-loop, $n = 10$ point maximally helicity violating (MHV) amplitude in planar $\mathcal{N} = 4$ sYM in $d = 4$ spacetime dimensions. The $n = 6$ external gluon amplitude was first calculated in [15] and provided a non-trivial consistency check for the all loop Bern-Dixon-Smirnov (BDS) ansatz first presented in [16]. The general case for $n \geq 12$ external legs was first calculated in a novel way in [7] by using the duality of superamplitudes and the VEV of light-like edged Wilson loops with the insertion of fermionic fields at edges of the null polygon. The power of this method is therefore that it allows for the analytic computation of scattering amplitudes with large number of external legs n, high-loop numbers and different helicity configurations. The thesis uses this novel approach to determine the symbol of the integral I_{dp} arising in 10-point MHV scattering amplitude case. We remark that here the focus is on the integral and not on the actual MHV amplitude obtained by cyclically summing over the fermionic insertions in I_{dp}. The calculation is also of great importance because the MHV component is the simplest 2-loop case and also gives a large class of components for the NMHV amplitude.

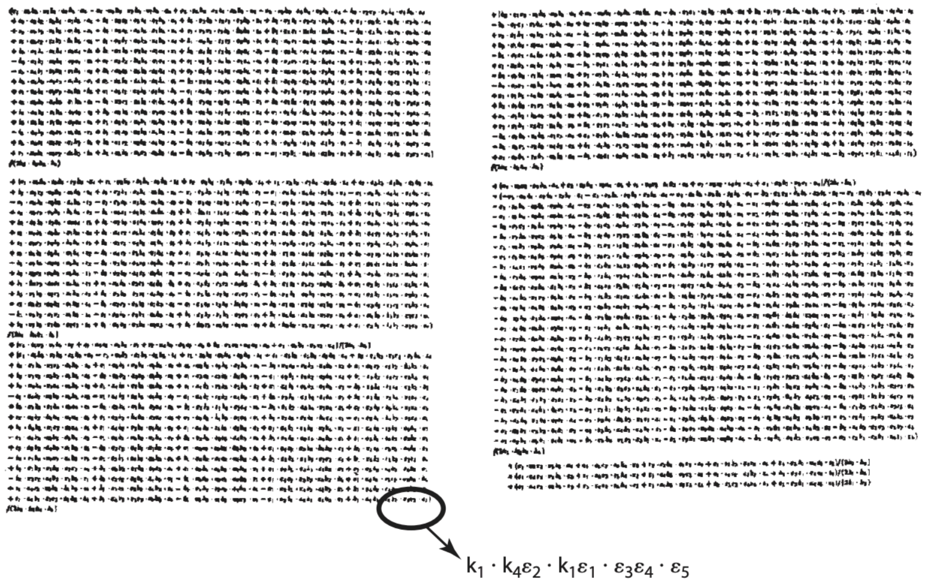


Figure 1.1: Small excerpt of 5-gluon scattering amplitude calculation using the Feynman diagram formalism

The thesis is structured as follows:

- **Chapter 2** contains an introduction of on-shell kinematics for scattering amplitudes, required for transitioning from the classical momentum picture to other more convenient variables that manifest the physical constraints. The concept of color-ordered partial amplitudes and spinor-spinor helicity variables are introduced.

- **Chapter 3** introduces and elaborates on the structure of 1-loop integrands and scattering amplitudes in $\mathcal{N} = 4$ sYM theory. The dual-coordinate space is introduced. Then, the embedding space and the momentum-twistor variables are discussed with the introduction of some projective and algebraic geometry. The box expansion is described for dealing with integrals having $m > 4$ internal propagators. Coalgebras are introduced and the *symbol* is defined.

- **Chapter 4** introduces the form and structure of integrands in 2-loop scattering amplitudes. A way is presented for reducing a 2-loop problem to a 1-loop problem with deformed external legs dependent on some auxiliary variables.

- **Chapter 5** describes and outlines the calculation of the I_{dp} integral and presents the result of the calculation at the symbol level.

- **Chapter 6** is the conclusion and outlook for future research directions.

Chapter 2

Kinematics for On-Shell Scattering Amplitudes in $4d$

In this chapter, we elaborate on methods for organizing the spin- and color information of external particle states in a way that facilitates the computation of scattering amplitudes. A set of variables is introduced for gauge-invariant on-shell amplitudes that makes the physical on-shell constraints manifest, which will also allow for a better classification of these amplitudes.

2.1 Ordering of external color information

The computation of scattering amplitudes in Yang-Mills theory can be a complex and challenging task, especially when considering scattering processes with large numbers of external states, or when requiring higher levels of precision. The amplitudes are defined by evaluating the scattering matrix (S-matrix) between the initial and final particle states of positive energy and they are functions of the external n particle momenta. Henceforth, a pure Yang-Mills (only gluons) scattering amplitude involving n external particles (states) with i-th particle momentum p_i and polarization ε_i is a $\mathbb{C}$ valued function

$$A_n(p_1, \cdots, p_n, \varepsilon_1, \cdots, \varepsilon_n). \tag{2.1}$$

The major difficulties in calculating Yang-Mills amplitudes partly originate from the invariance of the pure Yang-Mills action $S_{YM} = -\frac{1}{2}\int d^4x \, \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$ under $SU(N_c)$ gauge transformations of the gluon fields $A^a_\mu(x)T^a = A_\mu(x)$. Under gauge transformations, the scattering amplitudes remain invariant as the transformations of the polarization vectors are compensated by changes in other terms in the amplitude, ensuring the gauge invariance. In a typical brute force computation of a scattering amplitude in some gauge theory, each Feynman diagram is gauge-dependent (color dependent) and this dependence leads to gauge redundancies. On the other hand, the calculated amplitudes are gauge-independent, so this immediately raises the question: Is there a way of organizing the color (gauge group parameter) information in a way that reduces the complexity of the amplitude calculation? The answer is yes, and is called amplitude color ordering. We now introduce the so-called color-ordered partial/primitive amplitudes.

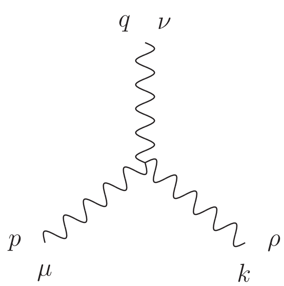
The main idea in reducing color-induced complexities in the calculation is to decompose the scattering amplitudes into a gauge-dependent and a gauge-independent part [17]. This is done by combining $[T^a, T^b] = i\sqrt{2} f^{abc} T^c$ (with $\mathfrak{su}(N_c) \ni T^a$) and $\mathrm{Tr}(T^a T^b) = \delta^{ab}$ in order to re-write the structure constant (products thereof) in terms of traces of the Lie-algebra generators. By following this path we can do a factorization in which the gauge-dependent part depends exclusively on the $SU(N_c)$ gauge group indices a_i with $i = 1, \cdots, N_c^2 - 1$ and the number of colors N_c. On the other hand, the gauge independent L-loop *partial amplitude* $\mathcal{P}_{n,k}$ depends only on the momenta and helicities of the n scattered particles. In general, for an L-loop amplitude we thus have that

$$A_n^{(L)}(\{p_i, h_i, a_i\}) = g^{n+2(L-1)} \sum_k C_{n,k}^{(L)}(\{a_i\}, N_c) \mathcal{P}_{n,k}^{(L)}(\{p_i, h_i\}), \tag{2.2}$$

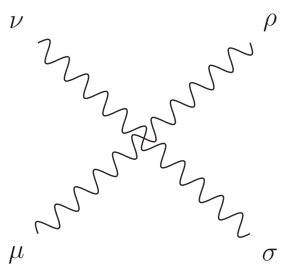
where $\mathcal{P}_{n,k}^{L-loop}$ denotes the partial amplitude and g denotes the gauge coupling. The limit $N_c \to \infty$ is called the *planar* limit of colors and was introduced by 't Hooft in [18]. In this special limit, the color decomposition described above takes the form

$$A_n^{(L)}(\{p_i, h_i, a_i\}) = g^{n+2(L-1)} N_c^L \left(\sum_{\sigma \in S_n / Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) \mathcal{A}_n^{(L)}(\sigma) + \mathcal{O}\left(\frac{1}{N_c}\right) \right), \tag{2.3}$$

where $\mathcal{A}_n^{L-loop}$ denotes the *L-loop color ordered partial amplitude*. The sum is over elements of the set of all possible permutations of n objects (called S_n) modulo the subset of cyclic rotations of n objects (called Z_n). This sum therefore only keeps the cyclically inequivalent orderings in the trace of the color algebra generator products. The arguments of the partial amplitudes hence only have a specific ordering of momenta with one term in the sum having $\sigma = (p_1, \cdots, p_n) \equiv (1, \cdots, n)$ and the other arguments having all the possible non-cyclic orderings of the n external momenta with respect to $(1, \cdots, n)$. The calculation of Yang-Mills amplitudes at this stage, therefore, reduces to the computation of the gauge-independent color-ordered partial amplitudes, which are determined by summing over the color-ordered Feynman diagrams with the use of the color-ordered Feynman rules shown in Figure 2.1 from [17]. Such diagrams can be only drawn in the plane (no crossing) given some specific external leg momentum ordering. This intrinsic ordering of the external legs fixed by the color factorization simplifies the singularity structure of the color-ordered partial amplitudes, as all singularities appear only in adjacent external legs of $\mathcal{A}_n(\sigma)$. So as [19] states, with Mandelstam variables $s_{ij} = -(p_i + p_j)^2$, only poles of the type: $s_{12}, s_{23}, \ldots$ appear, whereas $s_{13}, s_{25}, \ldots$ poles do not appear in the partial amplitudes. Furthermore, only cyclically adjacent momenta will have unitarity cuts at loop level. From this point on all amplitudes presented in this thesis are all color-ordered partial amplitudes $\mathcal{A}_n$ unless explicitly stated otherwise.



$$= (p-q)^\rho \eta^{\mu\nu} + \quad (q-k)^\mu \eta^{\nu\rho} + (k-p)^\nu \eta^{\mu\rho}$$



$$= 2\eta_{\mu\nu}\eta_{\rho\sigma} - \quad \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}$$

Figure 2.1: Color ordered Feynman rules for pure gluon vertices.

2.2 Spinor-helicity formalism

Having introduced color-ordering, we now focus on the definition domain of scattering amplitudes; more specifically, what are the most advantageous set of variables for representing loop amplitudes that make the physical conservation of momentum (all outgoing)

$$\sum_i^n p_i^\mu = 0 \tag{2.4}$$

and on-shell constraints on momenta

$$p_i^2 = 0 \quad , \forall\ i \in \{1, 2, ..., n\} \tag{2.5}$$

manifest. This will become evident in the next section here we lay the path for that by first introducing some kinematics in $d = 4$ dimensions.

The signature used is $(- + ++)$. The Lorentz-group $SO(1,3)$ upon complexification becomes isomorphic to $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$. Consequently, the finite-dimensional representations of $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ can be labelled by pairs (u, v) with $u, v \in \mathbb{Z}$ or $\mathbb{Z}/2$. So far, in order to express our amplitudes, we used quantities p_i^μ transforming under the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz-group, or equivalently $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$. As [20] discusses, we can equivalently represent these momenta as 2×2 Hermitian matrices, as the vector space of the 2×2 Hermitian matrices is isomorphic to M^4, i.e. 4-Minkowski spacetime

$$p_i^\mu \to p_i^{\alpha\dot{\alpha}} = p_i^\mu (\sigma_\mu)^{\alpha\dot{\alpha}} = \begin{pmatrix} -p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^0 - p^3 \end{pmatrix} \tag{2.6}$$

with $\det(p_i^{\alpha\dot{\alpha}}) = -p_{\mu,i}p_i^{\mu} = -p_i^2 = m_i^2$, which is a scalar; a Lorentz-invariant quantity. For a 2×2 matrix we can have at most 2 linearly independent columns, so the maximum rank is 2 and is less than 2 if and only if the determinant is 0. In high-energy processes fermions are ultra-relativistic and behave as massless particles so without further ado, we set $m = 0$ as in [5]. Since now $\det(p_i) = -p_i^2 = 0$, the rank is less than 2 and we can express the 2×2 Hermitian matrices as the product of a negative chirality (Left) spinor λ^{α} with $\alpha = 1, 2$ and a positive chirality (Right) spinor $\tilde{\lambda}^{\dot{\alpha}}$ with $\dot{\alpha} = 1, 2$

$$p_i^{\mu} \to p_i^{\alpha\dot{\alpha}} = p_i^{\mu}\sigma_{\mu}^{\alpha\dot{\alpha}} = -\lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}} \tag{2.7}$$

so that the on-shell momentum constraint $p_i^2 = 0$ is trivialized. The $p_i^{\alpha\dot{\alpha}}$ is often referred to as the bispinor. The representation $(\frac{1}{2}, 0)$ acts on the negative chirality spinor λ^{α} and the $(0, \frac{1}{2})$ representation acts on the positive chirality spinor $\tilde{\lambda}^{\dot{\alpha}}$. Altogether the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation and the $(\frac{1}{2}, \frac{1}{2})$ vector representation of $SO(1, 3)$ are isomorphic, allowing for 2 equivalent descriptions of scattering amplitudes. The indices of the negative chiral spinors can be risen using the $2d$ Levi-Civita totally anti-symmetric tensor $\varepsilon_{\alpha\beta}$ and on the other hand, the indices of the positive chiral spinors can be risen using $\varepsilon_{\dot{\alpha}\dot{\beta}}$. Then, we can define a Lorentz-invariant anti-symmetric form between 2 negative chirality spinors as $\langle\lambda_i\lambda_j\rangle = \varepsilon_{\alpha\beta}\lambda_i^{\alpha}\lambda_j^{\beta} = \lambda_{i,\beta}\lambda_j^{\beta}$. We define the Lorentz-invariant anti-symmetric product between 2 positive chirality spinors similarly with $[\tilde{\lambda}_i\tilde{\lambda}_j] = \varepsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_i^{\dot{\alpha}}\tilde{\lambda}_j^{\dot{\beta}} = \tilde{\lambda}_{\dot{\beta},i}\tilde{\lambda}_j^{\dot{\beta}}$. With the Leibniz formula for the determinant we can see that Mandelstam variables map to the product of spinors as: $s_{ij} = -(p_i + p_j)^2 = -2p_i \cdot p_j = -\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}(\lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}})(\mu_j^{\beta}\tilde{\mu}_j^{\dot{\beta}}) = -\langle\lambda_i\mu_j\rangle[\tilde{\lambda}_i\tilde{\mu}_j]$.
Given a negative chirality spinor λ^{α} and a positive chirality spinor $\tilde{\lambda}^{\dot{\alpha}}$ we can construct the light-like p_i momentum based on (2.7). However, let's examine the converse case when we have a given momentum p_i and we want to construct the spinors λ_i^{α} and $\tilde{\lambda}_i^{\dot{\alpha}}$ separately as functions of momentum. If $c \in \mathbb{C}\backslash\{0\}$, then the transformation on the separate spinors: $\lambda_i \to c\lambda_i$ and $\tilde{\lambda}_i \to c^{-1}\tilde{\lambda}_i$ leaves (2.7) invariant; thus, it not a unique representation as they can be determined only to modulo the scaling. So there is no general way of determining each spinor individually given the momentum p_i.

As a workaround in $d = 4$, we could label the external gauge boson with their momenta and projection of spin in the direction of motion, namely their helicity. Then, the amplitude would exclusively be a function of external momenta and helicities. However, as explicitly shown in (2.1), amplitudes for external states with spin are functions of momenta *and* polarizations ε_i^{μ}, with the constraint on longitudinal components $p \cdot \varepsilon = 0$. So to do this specification consistently we need to give: the momenta, polarizations and their helicities [21]. But given a massless gauge boson p_i and its projection of spin in the direction of motion there is still no consistent way to get the polarizations with good helicity. Although, for a gauge boson with momentum $p^{\alpha\dot{\alpha}}$ we can construct the correct negative helicity polarization vector if we are given λ^{α}. Then, by picking an arbitrary + chirality reference spinor $\tilde{\mu}^{\dot{\alpha}}$ we can write the $-$ helicity polarisation as $\varepsilon_-^{\alpha\dot{\alpha}} = \frac{\lambda^{\alpha}\tilde{\mu}^{\dot{\alpha}}}{[\tilde{\lambda}\tilde{\mu}]}$. The construction for the + helicity polarization requires picking an arbitrary $-$ chirality reference spinor μ^{α}: $\varepsilon_+^{\alpha\dot{\alpha}} = \frac{\mu^{\alpha}\tilde{\lambda}^{\dot{\alpha}}}{\langle\mu\lambda\rangle}$.

Summarizing, we cannot determine λ and $\tilde{\lambda}$ individually given p, but given $(\lambda, \tilde{\lambda})$ and the $h = \pm 1$ gauge boson helicities we can determine p and ε exactly. Henceforth, we may equivalently express our amplitudes with (p_i, ε_i), or $(\lambda_i, \tilde{\lambda}_i, h_i)$

$$\mathcal{A}_n(p_i, \varepsilon_i) \longleftrightarrow \mathcal{A}_n(\lambda_i, \tilde{\lambda}_i, h_i). \tag{2.8}$$

In the $(\lambda_i, \tilde{\lambda}_i, h_i)$ representation scattering amplitudes trivialize the physical constraint (2.5), but not (2.4). A set of variables trivializing (2.4) and (2.5) is discussed in **Chapter 3**.

Amplitudes can be classified according to the helicity configuration of their external legs. By introducing the following abbreviating notation $(\lambda_j, \tilde{\lambda}_j, h_j) \longleftrightarrow j^{h_j}$, we can express amplitudes as $\mathcal{A}_n = \mathcal{A}_n(1^{h_1}, \cdots, j^{h_j}, \cdots, n^{h_n})$ with j^{h_j} denoting the jth gluon with helicity $h_j = \pm 1$ [22]. It turns out that the helicity structure

$$[h_1, \cdots, h_j, \cdots, h_n] \tag{2.9}$$

for amplitude $\mathcal{A}_n$ with n external legs, greatly affects its structure and computational difficulty. At tree-level, it can be shown that the amplitude vanishes for all n helicities being positive, or all n helicities being negative. For $n = 3$, the 1 positive and 2 negative, or 1 negative and 2 positive helicities are non-0. For $n > 3$, the amplitude is 0 with 1 positive and $n-1$ negative, or 1 negative and $n-1$ positive helicities. This is not true however at the loop level in Yang-Mills theory and will be addressed in the next chapter.

The amplitude with 2 positive helicity and $n-2$ negative helicity gluons is classified as the *Maximally-Helicity-Violating*, or MHV amplitude, as this class realizes the maximal helicity conservation violation in the interactions. A formula was discovered by Parke-Taylor for the MHV tree level $gg \to gggg$ scattering in the planar limit. As an example, for the $n = 4$ MHV helicity configuration $[+1, -1, +1, -1]$, the amplitude is

$$\mathcal{A}^{(L=0)}_{4,MHV} = \frac{\langle \lambda_1 \lambda_3 \rangle^4}{\langle \lambda_1 \lambda_2 \rangle \langle \lambda_2 \lambda_3 \rangle \langle \lambda_3 \lambda_4 \rangle \langle \lambda_4 \lambda_1 \rangle}. \tag{2.10}$$

For $n = 4$ external legs we have 3 Feynman diagrams to evaluate. However, the number of diagrams increases rapidly, as for $n = 6$ we already have 220 diagrams to calculate. Therefore, its compactness is remarkable and illustrates the power of the helicity formalism. The result generalizes for arbitrary n outgoing gluons.

The next case is the one with 3 positive helicity and $n-3$ negative helicity gluons. This configuration is called the *Next-to-Maximally-Helicity-Violating*, or NMHV amplitude and is usually more complicated than the MHV case. Proceeding this way, we define the amplitude with $2+k$ positive and $n-k$ negative helicity gluons as the N^{k}MHV amplitude. The overall trend is that computational complexity increases with k.

Chapter 3

Planar 1-loop Integrals in $\mathcal{N} = 4$ sYM theory

Having introduced some $4d$ kinematics for scattering amplitudes, we can now advance the discussion to loop level. Traditional Yang-Mills theory becomes very difficult at loop level, although great simplifications are due to symmetry in its maximally supersymmetric planar extension, namely planar $\mathcal{N} = 4$ sYM theory.

3.1 Introduction to loop integrals in planar $\mathcal{N} = 4$ super Yang-Mills (sYM) theory

Supersymmetry is a spacetime symmetry that is realized by enlarging the Poincaré-algebra with so-called supersymmetry generators Q_α^A with $A = 1, ..., \mathcal{N}$ into a super-Poincaré-algebra. The maximum amount of SUSY generators we can have in $d = 4$ dimensions to constrain gauge bosons to having at most spin-1 is $\mathcal{N} = 4$; thus, $\mathcal{N} = 4$ super Yang-Mills theory is the maximally supersymmetric gauge theory in $d = 4$ spacetime dimensions with bosons having spin-1 at maximum. Its Lagrangian is unique and has the form

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}(&\frac{1}{2g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} - i\tilde{\lambda}^A \overline{\sigma}^\mu D_\mu \lambda_A - D_\mu \varphi^{\mathbf{I}} D^\mu \varphi^{\mathbf{I}} + \\ &+ gC_{\mathbf{I}}^{AB}\lambda_A[\varphi^{\mathbf{I}}, \lambda_B] + g\overline{C}_{\mathbf{I},AB}\tilde{\lambda}^A[\varphi^{\mathbf{I}}, \tilde{\lambda}^B] + \frac{g^2}{2}[\varphi^{\mathbf{I}}, \varphi^{\mathbf{J}}]^2).\end{aligned} \tag{3.1}$$

The particle content is given by the $\mathcal{N} = 4$ gauge (vector) multiplet $(A_\mu\ \lambda_A^\alpha\ \varphi^{\mathbf{I}})$ containing 1 A_μ spin-1 gauge field, 4 negative chiral Weyl-spinors λ^α labelled by $SU(4)$ indices $A = 1, 2, 3, 4$ (as we can "rotate" the A_μ into $\mathcal{N} = 4$ different spinors as we have $4 \times Q_\alpha$) and 6 real scalar fields $\varphi^{\mathbf{I}}$ with $\mathbf{I} = 1, 2, ..., 6$ labelling the $SO(6)_R \sim SU(4)_R$ global R-symmetry between the SUSY generators. In total this conformal theory has 30 bosonic and 32 fermionic symmetry generators; 4×4 fermionic SUSY generators $(Q_\alpha^A, \tilde{Q}_{A\dot{\alpha}})$, 4×4, fermionic conformal SUSY generators $(S_{A\alpha}, \tilde{S}_{\dot{\alpha}}^A)$, 4×1 bosonic translation generators $(P_{\alpha\dot{\alpha}})$,

3×2 Lorentz boost/rotation generators ($J_{\alpha\beta}, J_{\dot{\alpha}\dot{\beta}}$), 4×1 bosonic special conformal transformation (conformal boost) generators ($K_{\alpha\dot{\alpha}}$), 1×1 bosonic dilation generator (D), and $16 - 1 = 15$ $SU(4)_R$ bosonic R-symmetry generators (T^A). All combined, the fermionic and bosonic generators collectively form the $\mathfrak{su}(2,2|4)$ superconformal algebra [22]. It is the one of the most symmetric theories in $4d$ together with $\mathcal{N} = 8$ SUGRA, as [23] argues.

$\mathcal{N} = 4$ is UV finite at any loop order, eliminating the need of the renormalization of this theory. Since $\beta(g) = 0$ at any loop, the coupling is independent of the energy scale, which determines that the strength of the interaction is independent of the external particle states' energies. However, as we will see in detail in **Chapter 5**, IR divergences appear at the level of the box expansion due to the presence of massless legged boxes; by using some suitable regularization scheme such as dual conformal regularization, the divergence of each massless box can be regulated. Although certain boxes are divergent, when summing up all terms in the box expansion the divergent terms cancel between the boxes, leading to finite box expansion. On the other hand, IR divergences can appear again at the level of the actual scattering amplitudes $\mathcal{A}_n$; when appropriately summing over the integrals contributing to the scattering amplitude certain terms called boundary terms are divergent. These IR divergences arise due to the singular behavior of the soft (low momentum) gluon contributions to the scattering amplitude $\mathcal{A}_n$.

A general $4d$, n-external leg and L-loop amplitude $\mathcal{A}_n^{(L)}$ in planar $\mathcal{N} = 4$ sYM is some linear combination of Feynman-integrals of the form (in momentum representation)

$$I_n^{(L)} = \int \prod_{i=1}^{L} d^4 l_i \frac{\prod_{j=1}^{L} N(l_j)}{\prod_{k=1}^{L} P(l_k)} \frac{1}{R(l_1, \cdots, l_L)}, \tag{3.2}$$

where N, P, R are Lorentz-invariant functions made from Feynman-propagators depending on internal loop momenta l_i and/or external leg momenta p_m with $m = 1, \cdots, n$, or simply from some contraction of external kinematic variables [24]. As an example, the integral contributing to the $n = 4$ external leg box, $L = 1$ scattering amplitude is

$$I_4^{(1)} = \int d^4 l \frac{st}{l^2 (l - p_1)^2 (l - p_1 - p_2)^2 (l + p_4)^2}, \tag{3.3}$$

where $s = -(p_1 + p_2)^2$ and $t = -(p_2 + p_3)^2$ are the Mandelstam variables ensuring that the integral is properly normalized to have unit leading singularities. The study of leading singularities is crucial for understanding of scattering amplitudes. The idea is construct the amplitude from its leading singularities, as these encode important physical information about the scattering process of massless particles and can be used to reconstruct the full scattering amplitude. Unitarity based methods are readily available; the study of the branch cut structures of amplitudes is crucial for their complete understanding [25]. According to the important Generalized Residue Theorem (GRT), with good infinity conditions the sum over the residues of a given loop integrand is 0. If the non-0 residues are all equal up to a sign, then using a proper normalization factor in the numerator gives residues that are exclusively ± 1, or 0. Integrals with unit leading singularities are called *pure* integrals.

Pure integrals together with another class of integrals called *chiral* integrals play an important role in giving a set of basis integrals for loop amplitudes. These classifications are given in terms of their singularity structures. In the momentum-twistor representation of amplitudes, Schubert problems are intersection theory problems of finding the set of lines in projective space intersecting all other lines simultaneously as these elucidate the singularity structure of the integrands. If a loop integral has a Schubert problem in which the associated residues are not equal, then the integral is called chiral. As argued later in this chapter momentum twistors are projective objects that make conservation of momentum and on-shellness manifest at the level of scattering amplitudes $\mathcal{A}_n$ and can be regarded as the most natural set of variables for loop integrals as also in this language the pure and chiral aspects of these integrals are the most transparent.

The complete 1-loop MHV amplitude is given by the sum over the pure chiral pentagon as shown in Figure 3.1 from [24] where the wavy line between massless legs i and j represents

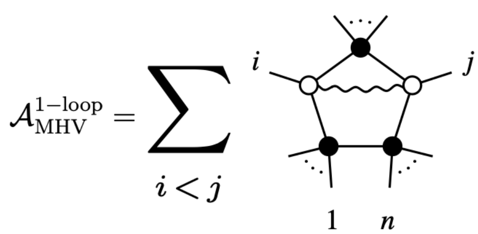


Figure 3.1: Color-ordered 1-loop MHV amplitude as the sum of pure chiral pentagons in planar $\mathcal{N} = 4$ sYM

the normalization numerator factor ensuring the unit leading singularity. The boundary terms $j = i + 1$ are box integrals.

It holds for all loop orders L in $\mathcal{N} = 4$ that

$$\mathcal{A}_n^{(L)}(1^+, 2^+, \cdots, j^+, \cdots, n^+) = \mathcal{A}_n^{(L)}(1^-, 2^-, \cdots, j^-, \cdots, n^-) = 0, \tag{3.4}$$

$$\mathcal{A}_n^{(L)}(1^+, 2^+, \cdots, j^-, \cdots, n^+) = \mathcal{A}_n^{(L)}(1^-, 2^-, \cdots, j^+, \cdots, n^-) = 0. \tag{3.5}$$

Therefore, the first non-vanishing amplitude at any order L is the MHV. If we were to construct the amplitude using the GRT formalism, then it would have been convenient to use the spinor-helicity variables introduced earlier to implement the unitarity cut conditions, as the $A_n(\lambda_i, \tilde{\lambda}_i, h_i)$ representation makes the on-shell condition (2.5) already manifest, so the propagators in the loop integrand are already on-shell as discussed in [26],[27] and [28]. Furthermore in the spinor-helicity representation of the $\mathfrak{su}(2,2|4)$ generators acting on an n external leg scattering amplitude, the only ones acting non-linearly are the generator of translations

$$P^{\alpha\dot{\alpha}} = \sum_i \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}} \tag{3.6}$$

and the conformal boost generator

$$K_{\alpha\dot{\alpha}} = \sum_i \frac{\partial}{\partial\lambda_i^{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{\alpha}}}. \tag{3.7}$$

Henceforth, our motivation is to find a set of variables for $\mathcal{A}_n$ that make the translational and conformal generator actions on the amplitudes linear. This will in return result in the emergence of a new type of hidden symmetry of planar $\mathcal{N}=4$ sYM called dual-conformal symmetry (DCI). Since in momentum representation translational invariance corresponds to constraint (2.4), the first step for linearizing the superconformal group action is to find a set of variables exhibiting constraint (2.4), then to find variables exhibiting the constraint linearly.

The physical conservation of momentum constraint can be made manifest for scattering amplitudes in planar $\mathcal{N}=4$ by introducing so-called dual-coordinates (zone variables) defined by

$$p_i^{\mu} = x_{i+1}^{\mu} - x_i^{\mu} \tag{3.8}$$

with periodicity condition for n external legs

$$x_{n+1}^{\mu} = x_1^{\mu} \tag{3.9}$$

so $p_n = x_1 - x_n$. The variable dual-space coordinate is denoted by $x_0^{\alpha\dot{\alpha}}$. According to (2.4), these newly introduced variables make conservation of momentum manifest (with Lorentz indices understood)

$$p_1 + p_2 + \cdots + p_n = (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_1 - x_n) = 0 \tag{3.10}$$

but the on-shell condition (2.5) has to be enforced manually, just oppositely to the case of spinor variables. The order of the amplitude external momenta is fixed by color-ordering (in [22]). From (3.8) and a notion of ordering, we have the distance between dual-coordinates in terms of 4-momenta

$$x_{ij} = x_i - x_j = p_i + p_{i+1} + \cdots + p_{j-1}. \tag{3.11}$$

This can be understood geometrically from the closed, dual-space polygon in Figure 3.2 exhibiting the conservation of momentum for the $n=4$ external momenta. For example, $x_{31} = p_1 + p_2$ as we can see from going along the edges of the polygon.

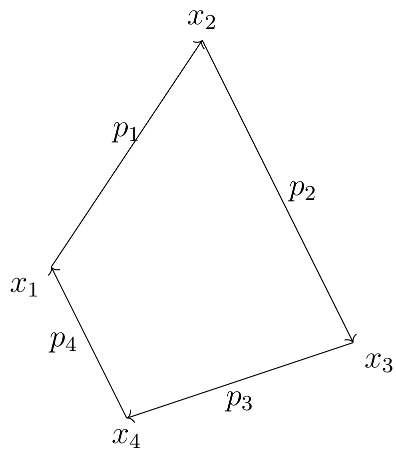


Figure 3.2: Dual-space closed polygon for $n=4$ external momenta.

For adjacent i,j, like in x_{12}, or x_{23} we have due to the on-shell condition (2.5) that $x_{12}^2 = x_{23}^2 = 0$. We can express the dual-coordinates equivalently as: $\left(x^\mu_{i+1} - x^\mu_i\right)\left(\sigma^{\alpha\dot\alpha}_\mu\right) = x^{\alpha\dot\alpha}_{i+1} - x^{\alpha\dot\alpha}_i$.
By using (2.7) and (3.8) we can use the manifest on-shellness of spinor variables to find a relation between dual-coordinates and spinor-variables. Since $x_i^{\alpha\dot\alpha}$ maps negative chiral spinors to positive chiral ones, we have

$$\left(x^{\alpha\dot\alpha}_{i+1} - x^{\alpha\dot\alpha}_i\right)\lambda_{\alpha,i} = 0 \implies x^{\alpha\dot\alpha}_i\lambda_{\alpha,i} = x^{\alpha\dot\alpha}_{i+1}\lambda_{\alpha,i} = \tilde\mu^{\dot\alpha}_i. \tag{3.12}$$

By shifting $i+1 \to i$ we obtain

$$x^{\alpha\dot\alpha}_i\lambda_{\alpha,i-1} = \tilde\mu^{\dot\alpha}_{i-1}. \tag{3.13}$$

Acting on $\tilde\mu^{\dot\alpha}_{i-1}$ with λ^α_i, on $\tilde\mu^{\dot\alpha}_i$ with λ^α_{i-1} and subtracting the two we obtain

$$\lambda^\alpha_{i-1}\tilde\mu^{\dot\alpha}_i - \lambda^\alpha_i\tilde\mu^{\dot\alpha}_{i-1} = x^{\alpha\dot\alpha}_i\left(\lambda^\alpha_{i-1}\lambda_{\alpha,i} - \lambda^\alpha_i\lambda_{\alpha,i-1}\right) = x^{\alpha\dot\alpha}_i\langle i-1\ i\rangle \tag{3.14}$$

with $\langle i-1\ i\rangle = \varepsilon_{\alpha\beta}\lambda^\alpha_{i-1}\lambda^\beta_i$. Therefore

$$x^{\alpha\dot\alpha}_i = \frac{\lambda^\alpha_{i-1}\tilde\mu^{\dot\alpha}_i - \lambda^\alpha_i\tilde\mu^{\dot\alpha}_{i-1}}{\langle i-1\ i\rangle}. \tag{3.15}$$

Let's now consider the dual-coordinate representation of our prototypical example $I_4^{(1)}$ as it will provide us with another important symmetry. Due to linearity we have a unit Jacobian, so the measure transformation is trivial

$$d^4l \to d^4x_0 \tag{3.16}$$

which yields the following dual-coordinate representation of $I_4^{(1)}$

$$I_4^{(1)}(p_1,p_2,p_3,p_4) \to I_4^{(1)} = \int d^4x_0 \frac{x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2}. \tag{3.17}$$

Now, by having a dual-coordinate representation of $I_4^{(1)}$, we can observe a remarkable fact. First, (3.17) is invariant under dual-coordinate dilation transformations $D\left(x_i^{\alpha\dot{\alpha}}\right) = tx_i^{\alpha\dot{\alpha}}$ with $t \in \mathbb{R}^+$

$$D(I_4^{(1)}) = \int \frac{(t^4 d^4 x_0)(tx_1 - tx_3)^2(tx_2 - tx_4)^2}{(tx_0 - tx_1)^2(tx_0 - tx_2)^2(tx_0 - tx_3)^2(tx_0 - tx_4)^2} = I_4^{(1)}. \tag{3.18}$$

Second, (3.17) is invariant under dual-coordinate inversion transformations (conformal boosts) $B(x_i^{\alpha\dot{\alpha}}) = \frac{x_i^{\alpha\dot{\alpha}}}{x_i^{\alpha\dot{\alpha}} x_{i,\alpha\dot{\alpha}}} = \frac{x_i^{\alpha\dot{\alpha}}}{x_i^2}$, so

$$B(I_4^{(1)}) = \int \frac{\frac{d^4 x_0}{x_0^8}\left(\frac{x_1}{x_1^2} - \frac{x_3}{x_3^2}\right)^2\left(\frac{x_2}{x_2^2} - \frac{x_4}{x_4^2}\right)^2}{\left(\frac{x_0}{x_0^2} - \frac{x_1}{x_1^2}\right)^2\left(\frac{x_0}{x_0^2} - \frac{x_2}{x_2^2}\right)^2\left(\frac{x_0}{x_0^2} - \frac{x_3}{x_3^2}\right)^2\left(\frac{x_0}{x_0^2} - \frac{x_4}{x_4^2}\right)^2} = \int \frac{d^4 x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} = I_4^{(1)}. \tag{3.19}$$

Together combined, we have that $I_4^{(1)}$ in the dual-coordinate representation is invariant under conformal transformations. This is called dual-conformal invariance, or DCI; we have a dual-conformal $\mathfrak{so}(2,4)$ symmetry algebra acting non-linearly on scattering amplitudes represented in the dual-coordinate picture. As argued by [2] and [29], dual-conformal symmetry is basically a conformal symmetry in dual-coordinate space and is independent of the usual $SO(2,4)$ conformal symmetry of $\mathcal{N} = 4$. Due to DCI, we have that the integral depends on cross-ratios that make the DCI manifest. With $I_4^{(1)} = f(u_1, u_2)$ we have that

$$u_1 = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad u_2 = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \tag{3.20}$$

where u_1 and u_2 contain all the kinematic data in the answer. Importantly, we are considering the integral $I_4^{(1)}$ in a Minkowskian-context in the off-shell regime, for which in general $x_{ii+1}^2 \neq 0$, as clarified in [2]. Hence, we found another representation for scattering amplitudes that elucidates this hidden DCI symmetry structure. Furthermore, in the strong coupling analysis it has been proven in [30] that the IR divergences of MHV (and NMHV) loop amplitudes are in one-to-one correspondence with the UV divergences of cusped Wilson loops. As [15] and [31] discusses, in $\mathcal{N} = 4$ sYM an object that manifests both the DCI and IR-UV matching properties is the VEV of null-polygonal Wilson-loops. For example, for $n = 4$ we define the null-polygonal Wilson-loop C_4 to be the dual-space polygon in Figure 3.2 with cusps corresponding to $x_1, \cdots, x_4$ and without momenta. Then we have the duality

$$M_4(p_1, p_2, p_3, p_4) = \frac{A_4^{(L=1)}}{A_4^{(tree)}} \iff$$
$$\iff W_4(C_4(x_1, x_2, x_3, x_4)) = \frac{1}{N_c} \langle 0|\, \mathrm{Tr}\left[\mathcal{P}\left(\exp(ig \oint_{C_4} dx^\mu A_\mu(x))\right)\right] |0\rangle,$$

where the $M_4(p_1, p_2, p_3, p_4) = \frac{A_4^{(L=1)}}{A_4^{(tree)}}$ ratio splits the amplitude into a universal IR divergent and a nontrivial finite part; this split is possible at any loop order L. This is a form of T-duality taking planar $\mathcal{N} = 4$ sYM to itself, as there is an equivalence between two quantities defined in different backgrounds within the same theory [32].

As we have seen so far, the dual-conformal group acts non-linearly on scattering amplitudes. Thus, carrying out conformal transformations and Lorentz-transformations in this representation is difficult and some aspects of this conformal symmetry are obfuscated. In the following, a formalism for linearizing the action of the dual-conformal group $SO(2,4)$ by *embedding* dual-coordinates into a special subset of a higher-dimensional space is presented.

This dual-conformal algebra expands into a dual-superconformal algebra $\mathfrak{psu}(2,2|4)$ when combined with the $\mathcal{N} = 4$ fermionic generators of the SUSY transformations. By combining together the generators of the superconformal algebra of $\mathcal{N} = 4$ sYM with that of the dual-superconformal algebra, we obtain an infinite dimensional algebra called *Yangian* algebra that is closely related to the integrability structure of the theory. This Lie superalgebra is therefore generated by the commutator and anti-commutator relations between the infinitesimal generators of the superconformal and dual-superconformal algebras. Generators of this Yangian algebra act non-locally on scattering amplitudes in the form of differential operators and are cyclically invariant due to the properties of its dual-conformal $\mathfrak{psu}(2,2|4)$ Lie subalgebra. These differential operators can be represented in terms of both the spinor-helicity variables introduced here and momentum-twistor variables. The role of the Yangian symmetry on the structure of scattering amplitudes at loop level is an area of active research; as [1] describes Yangian symmetry is well-established at tree-level in planar $\mathcal{N} = 4$ sYM theory, its precise form at loop-level is under investigation. Although, it has been consistently observed at loop level as well, such as in [3].

3.2 Embedding Formalism

Having discussed a set of variables for the manifestation of conservation of momentum, in this section we describe a new set of variables that makes the action of the dual-conformal group linear on the scattering amplitudes of planar $\mathcal{N}=4$ sYM, which are called embedding space variables. Embedding space provides a convenient way of doing dual conformal and Lorentz transformations by embedding dual coordinates from a non-compact Lorentz manifold M^4 into a special conformally compactified subset $\mathcal{M}^4$ of a $6d$ Lorentz spacetime that we denote with $\mathcal{M}^{2,4}$. After embedding the non-compact Lorentz manifold into a higher-dimensional compact manifold, it is easier to study and perform these transformations as the conformal group $SO(2,4)$ acts linearly on this special subset $\mathcal{M}^4 \subset \mathcal{M}^{2,4}$ of the $6d$ space. Importantly, $\dim(\mathcal{M}^4) = 4$ and $\dim(\mathcal{M}^{2,4}) = 6$; $\mathcal{M}^4$ is a $4d$ subspace of the $6d$ space $\mathcal{M}^{2,4}$. The embedding to embedding space $\mathcal{M}^4$ can be done in case of any relativistic theory. The conformal compactification is achieved by adding points at infinity, which are called ”null” points, and then introducing a special conformal transformation that maps the original space onto a compact subset of the higher-dimensional space. Then, as argued in [33] the constraints of conformal invariance are most transparent in this picture. The $4d$ fields are constructed as projections of fields on a projective hypercone in this $6d$ space, satisfying certain transversality conditions (detailed in [34]). This idea due to Dirac linearizes the action of the conformal group and Lorentz group. Given $x^\mu \in M^4$ we map it to a conformally compactified projective subset of a $6d$ Lorentz-space

$$x_i^\mu \to X_i^A = \begin{pmatrix} X^+ \\ X^- \\ x_i^\mu \end{pmatrix} \in \mathcal{M}^4 \subset \mathcal{M}^{2,4} \tag{3.21}$$

$A = 1, \cdots, 6$, with Lorentz invariant inner product on this embedding space given by

$$X_i \cdot X_j = \eta_{AB} X_i^A X_j^B = -\frac{1}{2} X_i^+ X_j^- - \frac{1}{2} X_i^- X_j^+ + \eta_{\mu\nu} X_i^\mu X_j^\nu \tag{3.22}$$

with $\eta_{+-} = \eta_{-+} = -\frac{1}{2}$, $\eta_{00} = -1$, $\eta_{ii} = +1$ and where $X^+ = X^0 + X^5$, $X^- = X^0 - X^5$. On this $\mathcal{M}^4 \ni X^A$ subset vectors are defined up to modulo the identification

$$X_i \sim tX_i \quad \text{with} \quad t \in \mathbb{R}\backslash\{0\}. \tag{3.23}$$

Let’s examine the structure of the embedding space further by considering a general $X_i^A \in \mathcal{M}^{2,4}$ and imposing the rescaling equivalence (3.23) on this $6d$ vector $X_i^A \in \mathcal{M}^{2,4}$. Then, the equivalence relation (3.23) implies that the vector X_i^A is in a projective space $\mathbb{RP}^5$ with $\dim(\mathbb{RP}^5) = 5$, as the function associating the equivalence class projection from general $6d$ Minkowski spacetime to this projective space is the canonical projection

$$\pi : \mathcal{M}^{2,4}\backslash\{0\} \to \mathbb{P}(\mathcal{M}^{2,4}) \tag{3.24}$$

where $\mathbb{P}(\mathcal{M}^{2,4}) = \mathbb{RP}^5$. Real projective space can be easily visualized for $3d$ Euclidean space $\mathbb{R}^3$ by taking all lines, that is, the $\mathbb{R}^1 \subset \mathbb{R}^3$ passing through the origin and looking at the intersection points of these lines with a fixed $2d$ plane not passing through the origin and located at $x^3 = 1$ let's say. Then this subset of $\mathbb{R}^3$ is a projective space $\mathbb{RP}^2$ as the points on this plane are independent of real rescalings of the lines through the origin. Also, the lines through the origin and in the x^1 and x^2 plane defining the points at infinity in $\mathbb{RP}^2$. We can therefore think about a projective space $\mathbb{P}(V^{n+1})$ over a vector space V^{n+1}, as the $1d$-subsets of V^{n+1} since the points in $\mathbb{P}(V^{n+1})$ correspond to line in V^{n+1}. So we have in general that if $\dim(V^{n+1}) = n+1 \implies \dim(\mathbb{P}(V^{n+1})) = n$. Projective space $\mathbb{P}(V^{n+1})$ therefore is like taking V^n together with its points at infinity, meaning that there is no concept of scale in the projective space. So by taking an arbitrary $(a_0 : ... :, a_{n+1}) \in \mathbb{RP}^n$ we have that $(a_0 : ... :, a_{n+1}) \equiv \lambda(a_0 : ... :, a_{n+1})$ with $\lambda \in \mathbb{R}$. Complexifying spacetime gives the equivalence relation $X_i \sim tX_i$ with $t \in \mathbb{C}\backslash\{0\}$ and this projective space becomes a complex $5d$ projective space that is described by homogeneous coordinates $\{X^1, X^2, X^3, X^4, X^5, X^6\}$ on $\mathbb{CP}^5$. If in $\mathbb{P}(\mathcal{M}^{2,4})$ the $\mathcal{M}^{2,4}$ is over $\mathbb{C}$, then $\mathbb{P}(\mathcal{M}^{2,4}) = \mathbb{CP}^5$.

In addition to this, because of the equivalence relation (3.23), a projective space is a space that is invariant under all general linear homogeneous transformations, let's say T, on its elements. If $\mathcal{M}^{2,4} \ni W, V$, $\mathbb{C} \ni t$ and $W = tV$, then $\mathbb{P}(W) \sim \mathbb{P}(V)$. That is, if V and W are equivalent in $\mathbb{CP}^5$, then $\mathbb{P}(T(v)) \sim \mathbb{P}(T(w))$ in $\mathbb{CP}^5$. Also, since T is homogeneous, it scales vectors equally, so T preserves lines and planes in $\mathbb{CP}^5$. Since lines and planes are defined as sets of equivalent vectors in $\mathbb{CP}^5$ and T preserves the structure of projective space, and any point in projective space remains in projective space after the application of T, which implies the $GL(1)$ invariance of the projective space. This equivalence is interpreted as a $GL(1)$ gauge freedom, which must be fixed in some way to get back the original dual-coordinate description in this formalism. Further details on projective spaces are found in [35].

Moreover, by taking any $\mathbb{CP}^5 \ni X$, we further have in the massless case that $X \cdot X = X^2 = 0$, which defines a Lorentz invariant subspace of $\mathbb{CP}^5$ that is $4d$, namely the projective null-cone, or projective light-cone. This projective null-cone naturally inherits the action of $SO(2,4)$ on the original M^4 spacetime on which $SO(2,4)$ acts linearly; on embedding space the conformal generators act linearly in contrast to the dual-coordinate picture. All in all, the *embedding space $\mathcal{M}^4$ can be considered as the subset of a $5d$ (complex) projective space $\mathbb{CP}^5$ on which vectors X^A are null-like*, i.e. $X \cdot X = 0$. This space is completely described by the Lorentz-invariant Klein-quadric

$$\eta_{AB}X^AX^B = X_AX^A = (X^1)^2 + (X^2)^2 - (X^3)^2 - (X^4)^2 - (X^5)^2 - (X^6)^2 = 0 \tag{3.25}$$

and rescaling equivalence $X_i \sim tX_i$ with $t \in \mathbb{C}\backslash\{0\}$. A general $\mathcal{M}^{2,4}$ vector has 6 degrees of freedom, but the rescaling invariance and null-condition combined yield $\mathcal{M}^4 \ni X_i^A$ vectors effectively described by 4 degrees of freedom. We can also consider the case where the i-th propagator is allowed to have mass m_i. In that case by gauge-fixing the rescaling freedom ($GL(1)$ gauge freedom) we can identify the original Minkowski spacetime M^4 with vectors that are not null. With massive propagators, the gauge-fixing condition $X^+ = 1$ called

Poincaré-section gives vectors outside of the lightcone of the form

$$\mathcal{X}_i = \begin{pmatrix} X^+ \\ X^- \\ x^\mu \end{pmatrix} = \begin{pmatrix} 1 \\ x_i^2 - m_i^2 \\ x_i^\mu \end{pmatrix}, \tag{3.26}$$

where $x^2 = x^\mu x_\mu$. In the limit $m_i \to 0$, we get the vectors on the lightcone that correspond to the i-th propagator being massless [36]. In this circumstance, the gauge-fixing condition $X^+ = 1$ gives vectors on the projective light-cone (projective subspace) of the form

$$X_i = \begin{pmatrix} 1 \\ x_i^2 \\ x_i^\mu \end{pmatrix} \tag{3.27}$$

that now satisfy the null-condition. From this point on we only consider vectors on the light-cone, so we exclusively consider massless propagators. In this gauge choice we can also see explicitly that for some $\mathcal{M}^4 \ni X, Y$ we have

$$X \cdot Y = -\frac{1}{2}x^2 - \frac{1}{2}y^2 + x_\mu y^\mu. \tag{3.28}$$

As evident in this gauge-choice, the embedding space vectors are $6d$ but have actually 4 degrees of freedom, same as the dual-coordinates x^μ manifesting momentum conservation. We indeed see the null-likeness of the embedding variables X^A by taking (3.28) with the same embedding space vectors

$$X \cdot X = X^2 = -\frac{1}{2}x^2 - \frac{1}{2}x^2 + x_\mu x^\mu = -x^2 + x^2 = 0. \tag{3.29}$$

Taking a point $X = (1, x^2, x^\mu)^T$ after gauge-fixing, we can act on it with a $D \in SO(2,4)$ conformal transformation: $D \to DX$. Then, rescaling the conformally transformed point as $DX \to \frac{DX}{(DX)^+}$ gives back the Poincaré-section (3.27). The composite non-linear tranformation $X \to \frac{DX}{(DX)^+}$ is then the non-linear action of $SO(2,4)$ on the original M^4.

Before the gauge-fixing we have that the dual-coordiantes propagators translate to the embedding space produce defined as

$$(x_i - x_j)^2 \to (X_i, X_j) = -2X_i \cdot X_j. \tag{3.30}$$

Importantly on the Poincaré-section, so after the gauge-fixig we have an equality with dual coordinates

$$(x_i - x_j)^2 = (X_i, X_j) = -2X_i \cdot X_j. \tag{3.31}$$

Since propagators in dual-coordinate representation have the form $\frac{1}{x_{ij}^2}$, it is useful to map the squared difference of the dual vectors x_i^μ, x_j^μ to the projective light-cone, which after fixing the gauge gives back exactly the original

$$(x_i - x_j)^2 \to (X_i - X_j)^2 = -2X_i \cdot X_j = x_i^2 + x_j^2 - 2x_{\mu,i}x_j^\mu = (x_i - x_j)^2. \tag{3.32}$$

We can conveniently express products of momenta as products of embedding space coordinates. Taking for example the 4-momentum p_2^μ

$$p_2^2 = 0 \mapsto x_{21}^2 = (x_2 - x_1)^2 = 0 \mapsto (X_2, X_1) = 0. \tag{3.33}$$

Due to on-shell constraint (2.5), we obtain the following projective space product relations for the adjacent embedding space vectors

$$(X_i, X_{i+1}) = 0, \quad \forall\, i \in \{1, 2, ..., n\}. \tag{3.34}$$

The conformal invariance of the embedding space $\mathcal{M}^4$ can be broken to Lorentz-invariance by picking a point at infinity in the embedding space. We can define this special point at infinity as the limit

$$I_\infty^A = \lim_{|x|\to\infty} \frac{1}{x^2}\begin{pmatrix} 1 \\ x^2 \\ x^\mu \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{3.35}$$

Similarly for the case of dual coordinates, for a variable loop momentum l we can assign a dual projective variable $\mathcal{M}^4 \ni L$ as $l \to L - X_n$ (to ensure momentum conservation). In this way we can express propagators as products of coordinates defined on the projective lightcone

$$(x_0 - x_i)^2 = x_{0i}^2 \mapsto (L, X_i). \tag{3.36}$$

Furthermore, in order to discuss loop amplitudes, conformal integrals must be discussed. The embedding space allows for the construction of conformal integrals, which again yield conformally invaraint quantities from the originals. For some scalar integrand $f(X)$ defined on the projective space $\mathcal{M}^4$ we have that $F(cX) = c^{-4}F(X)$ if the scalar function is of $d = 4$. We may obtain an $SO(2,4)$ invariant from the integral of the form $\mathcal{I} = \int d^6X\delta(X^2)f(X)$, but due to the scaling equivalence $X \simeq tX$ with $t \in \mathbb{C}\backslash\{0\}$, this is infinite. In order to regulate this we define the integral in embedding space by dividing with the gauge-group volume. As the scaling equivalnce is interpreted as a $GL(1)$ gauge invariance, we have that

$$\mathcal{I} = \int [d^6X] f(X) = \frac{2}{\mathrm{Vol}(GL(1))}\int_{X^+ + X^- \geq 0} d^6X\delta(X^2)f(X) \tag{3.37}$$

with $[d^6X] = 2\frac{d^6X\delta(X^2)}{\mathrm{Vol}(GL(1))}$. In the embedding space, the only scalar valued conformal integral depending on some arbitrary $\mathcal{M}^4 \ni Y$ on the conformally compactified subset is given by

$$\mathcal{I}(Y) = \int [d^6X]\frac{1}{(X,Y)^4} \quad \text{with} \quad Y^2 < 0 \tag{3.38}$$

corresponding to the integrand $f(X,Y) = \frac{1}{(X,Y)^4}$. Due to homogeneity in Y, we can pick $Y = Y_0 = (1,1,0)^T \implies Y_0^2 = -1$. Then, using $X^2 = -X^+X^- + X_\mu X^\mu = 0$ and

$(X, Y_0) = X^+ + X^-$ and $d^6X = dX^+dX^-d^4X$ together with $\Gamma(n) = (n-1)!$, we have

$$\begin{aligned}\mathcal{I}(Y_0) =& \frac{1}{\text{Vol}(GL(1))}\int_{X^++X^-\geq 0} dX^+dX^-d^4X\delta(-X^+X^- + X_\mu X^\mu)\frac{1}{(X^++X^-)^4} = \\ =& \frac{1}{\text{Vol}(GL(1))}\int d^4X\int_0^\infty \frac{dX^+}{X^+}\frac{1}{\left(X^+ + \frac{X_\mu X^\mu}{X^+}\right)^4} = \int d^4X\frac{1}{(1+X^\mu X_\mu)^4} = \frac{\pi^2}{6}.\end{aligned} \tag{3.39}$$

From the dimensional analysis of Y in the integrand in (3.38), it follows that for an arbitrary embedding space vector $Y \in \mathcal{M}^4$ that

$$\mathcal{I}(Y) = \int [d^6X]\frac{1}{(X,Y)^4} = \frac{\pi^2}{6}\frac{1}{(-Y^2)^2}. \tag{3.40}$$

Having $\mathcal{I}(Y)$, we can construct more complicated conformal integrals with non-trivial tensorial numerator structures [33].

$$\int [d^6X]\frac{X^{a_1}X^{a_2}\cdots X^{a_m}}{(X,Y)^{4+n}} = \frac{\Gamma(4)}{2^n\Gamma(4+n)}\left(\prod_{i=1}^m\frac{\partial}{\partial Y_{a_i}}\right)\mathcal{I}(Y) = \pi^2\frac{\Gamma(2+n)}{\Gamma(4+n)}\frac{Y^{a_1}Y^{a_2}\cdots Y^{a_m}}{(-Y^2)^{2+n}}. \tag{3.41}$$

Having discussed some aspects of conformal integrals, let's now consider some concrete loop amplitudes represented in the embedding space $\mathcal{M}^4$. As a first case, let's consider the zero-mass scalar box example discussed earlier in the context of dual-coordinates. By embedding (3.17) we obtain

$$\int \frac{d^4x_0x_{13}^2x_{24}^2}{x_{01}^2x_{02}^2x_{03}^2x_{04}^2} \to \mathcal{I}_4^{(1)} = \int [d^6L]\frac{(X_1,X_3)(X_2,X_4)}{(L,X_1)(L,X_2)(L,X_3)(L,X_4)}. \tag{3.42}$$

To demonstrate conformal invariance we must show that the integral is invaraint under dilations, as inversions are undefined here due to the null condition $X^2 = 0$. By doing the $L \to tL$ transformation on (3.42), the measure $[d^6L]$ gives a factor of t^4, and the integral remains the same

$$D\left(\mathcal{I}_4^{(1)}\right) = \int [d^6L]\frac{t^4(X_1,X_3)(X_2,X_4)}{t^4(L,X_1)(L,X_2)(L,X_3)(L,X_4)} = \mathcal{I}_4^{(1)}. \tag{3.43}$$

By embedding the integral on the projective light-cone, we have found a more natural representation that explicitly exhibits the conformal symmetry of the loop integral considered. The action of $SO(2,4)$ is linear on this integral, in opposition to the dual-coordinate representation. Henceforth, we have found a set of variables linearizing the $SO(2,4)$ action at the level of loop integrals in $\mathcal{N} = 4$. As an other example, let's consider the pentagon 1-loop integral. This is the first case where a tensor structured numerator appears, as the integrand's form is fixed to exhibit the $SO(2,4)$ conformal invariance

$$\mathcal{I}_5^{(1)} = \int [d^6L]\frac{L^a}{\prod_{i=1}^5(L,X_i)} = \int [d^6L]\frac{L^a}{(L,X_1)(L,X_2)(L,X_3)(L,X_4)(L,X_5)}. \tag{3.44}$$

By defining the scalar function with some general $\mathcal{M}^4 \ni W$ contracting the loop momentum

$$\mathcal{F}_5^{(1)}(L, W, X_i) = \frac{(W, L)}{(L, X_1)(L, X_2)(L, X_3)(L, X_4)L, X_5)}. \tag{3.45}$$

We can obtain the integrand by taking a derivative, which will result in a tensor integral

$$\frac{\partial}{\partial W_a}\mathcal{F}_5^{(1)} = \frac{L^a}{\prod_{i=1}^{5}(L, X_i)}. \tag{3.46}$$

This integrand construction method similar to (3.41) can be generalized to arbitrary m-gons as follows

$$\mathcal{F}_n^{(1)}(L, W_i, X_i) = \frac{(W_1, L)(W_2, L)\cdots(W_m, L)}{\prod_{i=1}^{n}(L, X_i)}. \tag{3.47}$$

Then, by taking the appropriate derivatives

$$\left(\prod_{i=1}^{m}\frac{\partial}{\partial W^i_{a_i}}\right)\mathcal{F}_n^{(1)}(L, W_i, X_i) = \frac{L^{a_1}L^{a_2}\cdots L^{a_{n-4}}}{\prod_{i=1}^{n}(L, X_i)} \tag{3.48}$$

meaning that we can define any tensor structured m-gon NLO correction with $\mathcal{F}_n^{(1)}$ as

$$\mathcal{I}_n^{(1)} = \int [d^6L]\left(\prod_{i=1}^{m}\frac{\partial}{\partial W_i^{a_i}}\right)\mathcal{F}_n^{(1)}(L, W_i, X_i). \tag{3.49}$$

3.3 Momentum-Twistors from Embedding Formalism

We have seen until this point that by embedding dual-coordinates to the embedding space, we can linearize the conformal group action at the level of amplitudes $\mathcal{A}_n$ in planar $\mathcal{N}=4$ sYM. We now discuss the natural variables for loop amplitudes that still exhibit this linear action of $SO(2,4)$ on them, but also manifest the physical constraints (2.4) and (2.5) altogether, without having to imposing them forcefully by hand. These variables are called momentum-twistor variables [37] and they follow naturally from the idea of embedding space defined earlier. The main idea is to use the anti-symmetric tensor representation of the conformal group $SO(2,4)$ to represent points of $\mathbb{CP}^5$ with skew-symmetric matrices made from $\mathbb{CP}^5$ homogeneous coordinates $\{X_1,...,X_6\}$. Then we have that $\mathbb{CP}^5 \ni X_{ab}$ and we can express the null-condition (3.29) as a conformal invariant product. Anti-symmetric 4×4 matrices have 6 free parameters, so this can be done by accounting for orthogonality in the case of the representation $SO(2,4)$, or unitarity if we do this by representing $SU(2,2)\simeq SO(2,4)$ instead. This in turn will imply that we can write X_{ab} as a wedge product of $\mathbb{CP}^3$ objects; the momentum-twistors, meaning that the decomposition to momentum-twistors follows naturally. This decomposition is also understandable from algebraic geometry, where it will follows from the embedding of the Grassmannian $\mathbb{G}(2,2)$ to $\mathbb{CP}^5$.

Let's first look at how we can map the previously described embedding space vectors to matrices anti-symmetrically representing $SU(2,2)$. In analogy to the (2.7) mapping of momenta to helicity-spinors through the use of the Pauli-matrices $(\sigma_\mu)^{\alpha\dot{\alpha}}$, we may map an embedding space vector $\mathcal{M}^{2,4}\supset\mathcal{M}^4 \ni X^A$ to the anti-symmetric tensor representation space of $SU(2,2)$, by using the $6d$ chiral Gamma-matrices Γ^A_{ab} where $a,b=1,2,3,4$ are $SU(2,2)$ indices and $A=1,2,\cdots,6$ is the usual $\mathcal{M}^{2,4}$ Lorentz index for $6d$ Minkowski space. These chiral $6d$ matrices are given by

$$\Gamma^+_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & 2i\epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \qquad \Gamma^-_{ab} = \begin{pmatrix} -2i\epsilon_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Gamma^\mu_{ab} = \begin{pmatrix} 0 & \sigma^\mu_{\dot{\alpha}\gamma}\epsilon^{\dot{\gamma}\dot{\beta}} \\ -\overline{\sigma}^{\mu,\dot{\alpha}\gamma}\epsilon_{\gamma\beta} & 0 \end{pmatrix}. \tag{3.50}$$

In addition to this, we also naturally have through $\tilde{\Gamma}^{A,ab} = \frac{1}{2}\varepsilon^{abcd}\Gamma^A_{cd}$ the dual Gamma-matrices given by

$$\tilde{\Gamma}^{+ab} = \begin{pmatrix} 2i\epsilon^{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix} \qquad \Gamma^{-ab} = \begin{pmatrix} 0 & 0 \\ 0 & -2i\epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix} \qquad \Gamma^\mu_{ab} = \begin{pmatrix} 0 & -\epsilon^{\alpha\gamma}\sigma^\mu_{\gamma\dot{\beta}} \\ \epsilon_{\dot{\alpha}\dot{\gamma}}\overline{\sigma}^{\mu,\dot{\gamma}\beta} & 0 \end{pmatrix} \tag{3.51}$$

These obey the following relations between the

$$\begin{aligned} \left(\Gamma^A\tilde{\Gamma}^B + \Gamma^B\tilde{\Gamma}^A\right)_a^{\ b} &= 2\eta^{AB}\delta_a^{\ b}, \\ \left(\tilde{\Gamma}^A\Gamma^B + \tilde{\Gamma}^B\Gamma^A\right)^a_{\ b} &= 2\eta^{AB}\delta^a_{\ b}, \\ \tilde{\Gamma}^{A,ab}\tilde{\Gamma}^{cd}_A &= 2\varepsilon^{abcd}, \\ \Gamma^A_{ab}\Gamma_{A,cd} &= 2\varepsilon_{abcd}, \\ \tilde{\Gamma}^{A,ab}\Gamma_{A,cd} &= 2\left(\delta^a_c\delta^b_d - \delta^b_c\delta^a_d\right). \end{aligned} \tag{3.52}$$

Then the mapping $\mathcal{M}^4 \to SU(2,2)$ from the embedding space to the space of anti-symmetric $SU(2,2)$ matrices is realized through the Gamma-matrices as

$$X_A \to X_{ab} = X_A \Gamma^A_{ab}, \tag{3.53}$$

$$X_A \to \overline{X}^{ab} = X_A \tilde{\Gamma}^{A,ab}. \tag{3.54}$$

By acting on (3.53) with $\tilde{\Gamma}^{B,ab}$ and using the identity $\operatorname{Tr}\left(\Gamma^A \tilde{\Gamma}^B\right) = 4\eta^{AB}$ we obtain that

$$X_{ab}\tilde{\Gamma}^{B,ab} = X_A \Gamma^A_{ab}\tilde{\Gamma}^{B,ab} = X_A \left(\Gamma^A \tilde{\Gamma}^B\right)_a^{\ a} = X_A \operatorname{Tr}\left(\Gamma^A \tilde{\Gamma}^B\right) = 4X_A \eta^{AB} = 4X^B. \tag{3.55}$$

From it follows that the inverse mapping from $SU(2,2) \to \mathcal{M}^4$ has the form

$$X^B = \frac{1}{4} X_{ab}\tilde{\Gamma}^{B,ab}. \tag{3.56}$$

Similarly using $\operatorname{Tr}\left(\tilde{\Gamma}^A \Gamma^B\right) = 4\eta^{AB}$ we obtain

$$X^B = \frac{1}{4} \overline{X}^{ab} \Gamma^B_{ab}. \tag{3.57}$$

Using the relation $\tilde{\Gamma}^{A,ab}\Gamma_{A,cd} = 2\left(\delta^a_c \delta^b_d - \delta^b_c \delta^a_d\right)$ for the other direction, we get back that

$$X^B \Gamma_{B,cd} = \frac{1}{4} X_{ab}\tilde{\Gamma}^{B,ab}\Gamma_{B,cd} = \frac{1}{2} X_{ab}\left(\delta^a_c \delta^b_d - \delta^b_c \delta^a_d\right) = \frac{1}{2}\left(X_{cd} - X_{dc}\right) = X_{cd}. \tag{3.58}$$

Henceforth, using the $6d$ chiral Γ matrices (3.50)-(3.51), we have established an isomorphism

$$\mathcal{M}^4 \longleftrightarrow SU(2,2). \tag{3.59}$$

So we can express $\mathbb{CP}^5$ coordinates both with embedding space vectors, whose products make the Lorentz invariance manifest, or we can express $\mathbb{CP}^5$ coordinates with anti-symmetric 4×4 tensors whose products make the $SU(2,2)$ conformal invariance manifest. Given this isomorphism we can map Lorentz invariants to $SU(2,2)$ invariants and vice-versa. By taking $\mathcal{M}^4 \ni X^A, Y^B$ and $SU(2,2) \ni X^{ab}, Y^{cd}$ we see using $\tilde{\Gamma}^{A,ab}\tilde{\Gamma}^{cd}_A = 2\varepsilon^{abcd}$ that in this representation the Loerentz invariant product becomes the conformal invariant

$$\eta_{AB}X^A Y^B = X_A Y^A = \tilde{\Gamma}^{ab}_A X_{ab} \tilde{\Gamma}^{A,cd} Y_{cd} = 2\varepsilon^{abcd} X_{ab} Y_{cd}. \tag{3.60}$$

This implies that the null-condition $X^2 = 0$ in this representation becomes a condition on the newly introduced $X_{ab} = -X_{ba} \in \mathbb{CP}^5$ homogeneous coordinates given by

$$\eta_{AB}X^A X^B = X_A X^A = \tilde{\Gamma}^{ab}_A \tilde{\Gamma}^{A,cd} X_{ab} X_{cd} = 2\varepsilon^{abcd} X_{ab} X_{cd} = 0 \implies \varepsilon^{abcd} X_{ab} X_{cd} = 0. \tag{3.61}$$

From this we see that conformally compactified Lorentz-spacetime can be described by the $SO(2,4) \simeq SU(2,2)$ invariant with Klein-quadric

$$\varepsilon^{abcd} X_{ab} X_{cd} = X_{ab} X^{ab} = X_1^2 + X_2^2 - X_3^2 - X_4^2 - X_5^2 - X_6^2 = 0 \tag{3.62}$$

with homogeneous coordinates $\{X^1, X^2, X^3, X^4, X^5, X^6\}$ for $\mathbb{CP}^5$, so we view the $X_{ab} = -X_{ba}$ as homogeneous coordinates for $\mathbb{CP}^5$. Thus, as $\mathbb{CP}^5 \ni X_{ab} : X_{ab} \sim tX_{ab}$ with $t \in \mathbb{C}\backslash\{0\}$. Since $\varepsilon^{abcd} = \varepsilon^{acdb} = \varepsilon^{adbc} = -\varepsilon^{abdc} = -\varepsilon^{adcb} = -\varepsilon^{acbd}$, we can write the null-condition $\varepsilon^{abcd} X_{ab} X_{cd} = 0$ as

$$\varepsilon^{abcd} X_{ab} X_{cd} + \varepsilon^{acdb} X_{ac} X_{db} + \varepsilon^{adbc} X_{ad} X_{bc} - \varepsilon^{abdc} X_{ab} X_{dc} - \varepsilon^{adcb} X_{ad} X_{cb} - \varepsilon^{acbd} X_{ac} X_{bd} = 0. \tag{3.63}$$

This implies

$$\frac{1}{3!}\left(X_{ab}X_{cd} + X_{ac}X_{db} + X_{ad}X_{bc} - X_{ab}X_{dc} - X_{ad}X_{cb} - X_{ac}X_{bd}\right) = X_{a[b}X_{cd]} = 0 \tag{3.64}$$

with $X_{a[b}X_{cd]}$ denoting the complete anti-symmetrization with respect to $SU(2,2)$ indices (bcd). The condition $\varepsilon^{abcd} X_{ab} X_{cd} = 0$ is satisfied if and only if

$$X_{ab} = V_{[a}W_{b]} = (V \wedge W)_{ab} = V_a W_b - W_a V_b \tag{3.65}$$

for some $\mathbb{T} \ni V, W$ with $\mathbb{T}$ denoting momentum-twistor space, as discussed in [38]. As we view the $X_{ab} = -X_{ba}$ as homogeneous coordinates for $\mathbb{CP}^5$, momentum-twistor space $\mathbb{T}$ is a copy of $\mathbb{CP}^3$: $\mathbb{T} \simeq \mathbb{CP}^3$. For any two arbitrary points in momentum-twistor space $V_a, W_b \in \mathbb{CP}^3$, we have a line $\mathbb{CP}^1 \subset \mathbb{CP}^3$ in momentum-twistor space $(V \wedge W)_{ab}$, which corresponds to a point in the conformally compactified and complexified spacetime: $X_{ab} = (V \wedge W)_{ab}$. That is

$$\textbf{\textit{Line in }} \mathbb{CP}^3 \iff \textbf{\textit{Point in }} \mathcal{M}^4 \iff \textbf{\textit{Point in dual-coordinate space}}. \tag{3.66}$$

On the other hand, for an arbitrary point $V_a \in \mathbb{CP}^3$, this point lies on the $\mathbb{CP}^1$ line $(K \wedge L)_{ab}$ corresponding to the point X_{ab} if

$$X_{[ab}V_{c]} = 0 \tag{3.67}$$

that is, if and only if $V_a = K_a + L_a$. Due to their projective nature, momentum-twistors are subject to the equivalence $V_a \sim tV_a$ with $t \in \mathbb{C}\backslash\{0\}$. Also, for an arbitrary $V_a \in \mathbb{CP}^3$, we have that $V_a V^a = 0$. Given two points $\mathcal{M}^4 \ni X^A, Y^B$ with corresponding bitwistors $X_{ab} = A_{[a}B_{b]}, Y_{cd} = C_{[c}D_{d]}$, the null-separation condition is given by condition (3.61)

$$\varepsilon^{abcd} X_{ab} Y_{cd} = 0 \tag{3.68}$$

which is the statement of intersection of lines $(A \wedge B), (C \wedge D) \in \mathbb{CP}^1 \subset \mathbb{CP}^3$ at some point in momentum-twistor space. So to null separated points X_{ab}, Y_{cd} in $\mathcal{M}^4$ corresponds a point in momentum-twistor space

$$\textbf{\textit{Point in }} \mathbb{CP}^3 \iff \textbf{\textit{Line in }} \mathcal{M}^4 \iff \textbf{\textit{Line in dual-coordinate space}}. \tag{3.69}$$

The algebraic geometry construction is based on the embedding of the Grassmanian $\mathbb{G}(2,2)$ to $\mathbb{CP}^5$. The Grassmanian $\mathbb{G}(m,n)$ is the set of m-dimensional subspaces of a vector space V^{n+m}. For example, $\mathbb{G}(m,0) = \mathbb{G}(0,m)$ are points in V^m. The $\mathbb{G}(1,m)$ are the $1d$ subsets of the vector space V^{m+1}, which are exactly points in m-dimensional projective space, so $\mathbb{G}(1,m) = \mathbb{P}^m$. In addition to this we have that $\mathbb{G}(m,n) \simeq \mathbb{G}(n,m)$, as the m-dimensional subspace of V^{n+m} is dual to the n-dimensional subspace of the dual vector space $(V^{m+n})^* \simeq V^{m+n}$. So we also have from this duality that $\mathbb{G}(m,1) = \mathbb{P}^m$. As argued by [39], the first non-trivial case that does not correspond to points in projective space is the Grassmannian

$$\mathbb{G}(2,2) = \text{set of 2d subspaces of } V^4. \tag{3.70}$$

By the canonical projection, we know that if the vector space V^4 is over the complex field $\mathbb{C}$, then $\pi : V^4 \to \mathbb{P}(V^4)$ with $\mathbb{P}(V^4) = \mathbb{CP}^3$.

We now embed $\mathbb{G}(2,2)$ to $\mathbb{CP}^5$. First, points in $\mathbb{G}(2,2)$ which correspond to $1d$ subsets in $\mathbb{CP}^3$, so lines in $\mathbb{CP}^3$. Now taking 2 non-equal points in $\mathbb{CP}^3$, we have

$$(a_0 : a_1 : a_2 : a_3), (b_0 : b_1 : b_2 : b_3) \in \mathbb{CP}^3. \tag{3.71}$$

We may produce from these a point in $\mathbb{CP}^5$, by putting them together as a 2×4 matrix

$$\begin{pmatrix} a_0 : & a_1 : & a_2 : & a_3 \\ b_0 : & b_1 : & b_2 : & b_3 \end{pmatrix} \tag{3.72}$$

and then taking minors $d_{ij} = \det(\ \textit{columns}\ i,j)$. For example, $d_{01} = a_0 b_1 - b_0 a_1$. We then group the possible non-0 minors and we obtain $(d_{01} : d_{02} : d_{03} : d_{12} : d_{13} : d_{23}) \in \mathbb{CP}^5$.
The point in $\mathbb{CP}^5$ obtained this way then only depends on the original $\mathbb{CP}^3$ line. This gives a well defined map between lines in $\mathbb{CP}^3$ and points in $\mathbb{CP}^5$. However, this map in not onto, as lines in $\mathbb{CP}^3$ are $\dim(\mathbb{G}(2,2)) = 4$ and on the other hand $\dim(\mathbb{CP}^5) = 5$. So we cannot cover all of $\mathbb{CP}^5$ with this map. Thus, there exists some internal relation between the minors in $(d_{01} : d_{02} : d_{03} : d_{12} : d_{13} : d_{23})$. This internal relation is called Plücker-relation and reads as $d_{01}d_{23} - d_{02}d_{13} + d_{03}d_{12} = 0$. There are no other relations satisfied by the minors d_{ij}, so the map from the lines in $\mathbb{CP}^3$ to the solution of the Plücker-relation is onto. Thus, the map $\mathbb{G}(2,2) \to$ *Solutions Plücker* is onto. This can be shown by noting that some of the $d_{ij} \neq 0$, so we may choose $d_{01} = 1$, then $d_{23} = d_{02}d_{13} - d_{03}d_{12} = 0$. Picking the 2 points in $\mathbb{G}(2,2)$, so points that satisfy the Plücker, we define the $\mathbb{G}(2,2)$ line as

$$\begin{pmatrix} 1 : & 0 : & d_{12} : & d_{13} \\ 0 : & 1 : & d_{02} : & d_{03} \end{pmatrix} \tag{3.73}$$

whose image is a point in $\mathbb{CP}^5$. So all points satisfying the Plücker, are images of points in $\mathbb{G}(2,2) \subset \mathbb{CP}^5$. This discussion elucidates that correspondence between lines in $\mathbb{CP}^3$ and points in $\mathbb{CP}^5$ through the Grassmannian.

Let's now discuss how we can obtain the orignal dual-coordinate description from the momentum-twistor formalism. As in the case of embedding coordinates we have due to the scaling equivalence $V_a \sim tV_a$ that there is no natural scale for any non-zero $X^{ab}Y_{ab}$. In order to introduce such a natural scale needed for a concept of Minkowski distance, the conformal group has to be broken to the Poincaré-group similarly to (3.35), so we introduce the conformal invariance breaking infinity momentum-twistor

$$I_{ab} = \begin{pmatrix} \varepsilon^{\dot{\alpha}\dot{\beta}} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.74}$$

Like before, using the totally anti-symmetric tensor we can define the dual infinity twistor

$$I^{ab} = \frac{1}{2}\varepsilon^{abcd} I_{cd} = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon^{\alpha\beta} \end{pmatrix}. \tag{3.75}$$

With I_{cd}, I^{jk} we can define the metric as

$$(x-y)^2 = \frac{X^{ab}Y_{ab}}{I_{cd}X^{cd}I_{jk}Y^{jk}} \tag{3.76}$$

which allows us to have a meaningful notion of scale. If now $(A \wedge B)$ corresponds to X_{ab} and $(C \wedge D)$ corresponds to Y_{ab}, their Minkowski distance given in the momentum-twistor representation is

$$(x-y)^2 = \frac{\langle ABCD\rangle}{\langle AB\rangle\langle CD\rangle} \tag{3.77}$$

with $\langle ABCD\rangle = \varepsilon^{abcd}A_aB_bC_cD_d = \det\{ABCD\}$ and $\langle AB\rangle = I^{ab}A_aB_b$. As evident from its structure, the infinity momentum-twistor plays the role of a projector acting on momentum-twistors $\mathbb{CP}^3 \ni V_a$. Acting on V with it, keeps its negative chiral component

$$I^{ab}V_b = \begin{pmatrix} 0 \\ \lambda^\alpha \end{pmatrix}. \tag{3.78}$$

The other component is a positive chiral $\mu^{\dot{\alpha}}$ spinor, so we have that momentum-twistors $V_a \in \mathbb{CP}^3$ have the form

$$V_a = \begin{pmatrix} \mu^{\dot{\alpha}} \\ \lambda_\alpha \end{pmatrix}. \tag{3.79}$$

In this representation, the skew-symmetric X_{ab} contains the zone-variables $x^{\alpha\dot{\alpha}}$ as

$$X_{ab} = \begin{pmatrix} -\frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}x^2 & -ix^{\alpha}{}_{\dot{\beta}} \\ ix_{\alpha}{}^{\dot{\beta}} & \varepsilon_{\alpha\beta} \end{pmatrix}, \qquad X^{ab} = \begin{pmatrix} \varepsilon_{\dot{\alpha}\dot{\beta}} & -ix_{\dot{\alpha}}{}^{\beta} \\ ix^{\alpha}{}_{\dot{\beta}} & -\frac{1}{2}\varepsilon^{\alpha\beta}x^2 \end{pmatrix}. \tag{3.80}$$

If X is a line passing through momentum-twistors $V=(\mu_V,\lambda_V), W=(\mu_W,\lambda_W)$, then the dual-coordinate takes the form

$$x^{\alpha\dot{\alpha}}=i\frac{(\mu_V\lambda_W-\mu_W\lambda_V)^{\alpha\dot{\alpha}}}{\langle VW\rangle} \tag{3.81}$$

which is the same as (3.15) up to a factor of i from complexification. Also, (3.67) becomes the incidence relation

$$\mu^{\dot{\alpha}}=-ix^{\alpha\dot{\alpha}}\lambda_\alpha \tag{3.82}$$

again same to (3.13) up to a complex factor.

In cases with more momentum-twistors involved, we may denote them simply with Z; for example, momentum twistor A is labelled as Z_A and so on. The lines defined by (Z_i,Z_{i+1}) $(\forall i)$ define a closed $\mathbb{CP}^1\subset\mathbb{CP}^3$ contour, determining momentum conservation. The twistor Z_a at the intersection of twistor lines $Z_b\wedge Z_c$ and $Z_d\wedge Z_e$ is given by

$$Z_a\langle b,c,d,e\rangle=-(Z_b\langle c,d,e,a\rangle+Z_c\langle d,e,a,b\rangle+Z_d\langle e,a,b,c\rangle+Z_e\langle a,b,c,d\rangle). \tag{3.83}$$

The point $\mathbb{CP}^3\ni Z_q$ corresponding to the intersection of $\mathbb{CP}^1\ni Z_i\wedge Z_j$ with the plane spanned by $\mathbb{CP}^2\ni(Z_k,Z_l,Z_m)$ is

$$Z_q=(Z_i,Z_j)\cap(Z_k,Z_l,Z_m)=Z_i\langle j,k,l,m\rangle+Z_j\langle k,l,m,i\rangle. \tag{3.84}$$

Furthermore, the line $\mathbb{CP}^1\ni Z_p\wedge Z_q$ formed by the intersection of $\mathbb{CP}^2$ planes spanned by (Z_i,Z_j,Z_k) and (Z_l,Z_m,Z_n) is

$$\begin{aligned}(Z_p,Z_q)&=(Z_i,Z_j,Z_k)\cap(Z_l,Z_m,Z_n)=\\ &=(Z_i,Z_j)\langle k,l,m,n\rangle+(Z_j,Z_k)\langle i,l,m,n\rangle+(Z_k,Z_i)\langle j,l,m,n\rangle.\end{aligned} \tag{3.85}$$

The cyclic ordering of dual variables (3.8) introduces some further interesting properties that are clearly visible in the momentum-twistor language. We have in correspondence to our null polygon (Wilson-loop) in the dual-coordinate space, that there are in total n momentum-twistors W_i located at the intersection of n lines X_i. So there is a momentum-twistor for each external particle, that is each edge of the null polygonal Wilson loop. This determines a polygon in $\mathbb{CP}^3$ momentum-twistor space whose edges are the lines X_i corresponding to the dual-coordinates. With the given cyclic ordering we have that $X_{i,ab}=W^{i-1}_{[a}W^i_{b]}$ and that $X_i\cap X_{i+1}=W^i$. This is illustrated below in Figure 3.2 from [38].
In order to get back the dual coordinate representation we must break the conformal invariance as the x^μ are Lorentz objects. The cyclic ordered region variable is obtained by the cyclic momentum-twistor description as

$$x_i^{\alpha\dot{\alpha}}=\frac{I^{ab}\left(W^{i-1}_bW^{i\dot{\alpha}}-W^{i-1,\dot{\alpha}}W^i_b\right)}{\langle i-1,i\rangle}. \tag{3.86}$$

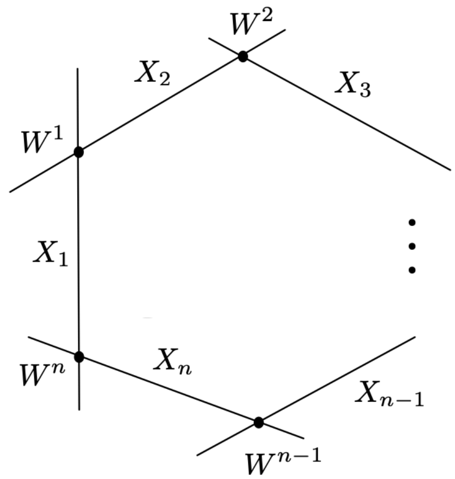


Figure 3.2: Intersection of $\mathbb{CP}^1$ lines with convention $X_i \cap X_{i+1} = W^i$

From this, the squared distance on the dual-space polygon (Wilson loop) between dual points x_j and x_k becomes the 4-bracket, which in turn allows us to express propagators

$$x_{jk}^2 = (x_j - x_k)^2 = \frac{\langle j-1, j, k-1, k\rangle}{\langle j-1, j\rangle\langle k-1, k\rangle} \tag{3.87}$$

with x_j corresponding to twistor line $Z_j \wedge Z_{j+1}$ and x_k corresponding to twistor line $Z_k \wedge Z_{k+1}$. In addition to this, to the variable dual-coordinate x_0 corresponds the $\mathbb{CP}^1 \ni Z_A \wedge Z_B$ line passing through momentum-twistor variables Z_A, Z_B, so

$$x_{0k}^2 = (x_0 - x_k)^2 = \frac{\langle A, B, k-1, k\rangle}{\langle AB\rangle\langle k-1, k\rangle}. \tag{3.88}$$

In order to evaluate loop integrals expressed in terms of momentum-twistors, we need to define the integration measure d^4x_0 first at the momentum-twistor level. As [24] argues, integrating over all x_0 in dual-coordinate space is equivalent to integrating over all $\mathbb{CP}^1 \subset \mathbb{CP}^3$. The integral over Z_A and Z_B can be decomposed, into the integral over all $\mathbb{CP}^1$ lines $Z_A \wedge Z_B$, and the integral over the separate momentum-twistors Z_A and Z_B moving along a particular line $Z_A \wedge Z_B$ (in [22]). By acting with a $GL(2)$ transformation on (Z_A, Z_B), the new resultant momentum-twistor pair $(Z_{A'}, Z_{B'})$ defines the same line in $\mathbb{CP}^1$ as the original pair (Z_A, Z_B), since the $GL(2, \mathbb{C})$ transformation on the momentum-twistors Z_A and Z_B leaves the line $Z_A \wedge Z_B$ invariant, due to the fact that $\mathbb{CP}^1$ is invariant under the group of all general linear homogeneous transformations (as it's a projective space). We can however parametrize the movement of Z_A and Z_B along $Z_A \wedge Z_B$ with the $GL(2)$ transformation. Separating the $GL(2)$ part out we obtain

$$\int d^4Z_A d^4Z_B = \int \frac{d^4Z_A d^4Z_B}{Vol(GL(2))} \int_{GL(2)}. \tag{3.89}$$

The integral on the momentum-twistor pair (Z_A, Z_B) modulo $GL(2)$ is just the integral over the usual and simplest non-trivial Grassmannian $\mathbb{G}(2,2)$ that can be parametrized by the 2×4 matrix in the following way

$$\begin{pmatrix} Z_A^1 & Z_A^2 & Z_A^3 & Z_A^4 \\ Z_B^1 & Z_B^2 & Z_B^3 & Z_B^4 \end{pmatrix} = \begin{pmatrix} \lambda_A^1 & \lambda_A^2 & \mu_A^{\dot{1}} & \mu_A^{\dot{2}} \\ \lambda_B^1 & \lambda_B^2 & \mu_B^{\dot{1}} & \mu_B^{\dot{2}} \end{pmatrix}. \tag{3.90}$$

The $GL(2)$ invariant measure is obtained by taking the integral over all Z_A, Z_B with the product of minors of the 2×4 matrix, which breaks the conformal invariance to Poincaré-invariance in order to relate the integral of the dual-coordinate measure d^4x_0 with the integral over all momentum-twistor lines $Z_A \wedge Z_B$ in $\mathbb{CP}^3$

$$\int d^4x_0 \longleftrightarrow \int \frac{d^4Z_Ad^4Z_B}{\mathrm{Vol}(GL(2))\langle Z_AZ_BI_\infty\rangle^{4-(n-m)}}. \tag{3.91}$$

As found by [40], for $\mathcal{N}=4$ sYM it is true that $n-m=4$, where m-denotes the tensor order of the integrand. However, it's important to note that we can have numerators structures that deviate from $n-m=4$, but then we must include the infinity twistor I_∞ to make the integral DCI. From what has been described, we can give a momentum-twistor representation of loop integrals. For example, the usual $n=4$ external leg and 1-loop integral $I_4^{(1)}$ in the momentum-twistor representation is

$$\int d^4x_0 \frac{x_{13}^2x_{24}^2}{x_{01}^2x_{02}^2x_{03}^2x_{04}^2} \longleftrightarrow \int \frac{d^4Z_Ad^4Z_B}{\mathrm{Vol}(GL(2))} \frac{\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle}. \tag{3.92}$$

For brevity, we introduce the notation

$$\int_{AB} \longleftrightarrow \int \frac{d^4Z_Ad^4Z_B}{\mathrm{Vol}(GL(2))}. \tag{3.93}$$

Some aspects of loop amplitudes are clearer in embedding space, whereas others are more explicit in the momentum-twistor representation described here. We can always translate between the 2 pictures and use the language most appropriate to the problem.

3.4 Van-Neerven-Vermaseren reduction

Now with being equipped with the powerful momentum-twistor and embedding space formalism, we proceed to the detailed study of 1-loop cases with more internal propagators and with $n > 4$ external legs. The most famous box reduction technique in Quantum Field Theory was first introduced by Passarino-Veltman in [41]. Here we sketch this general reduction method and then we present the reduction used in the thesis. The discussion follows [42].

The discussion generalizes to d-dimensions, but here we take $d = 4$ as it's the relevant special case. Suppose we are given a complicated vector structured loop integral in some general, even massive theory

$$I_n(l^\mu) = -i(4\pi)^2 \int \frac{d^4l}{(2\pi)^4} \frac{l^\mu}{(l^2 - m_0^2)((l+q_1)^2 - m_1^2) \cdots ((l+q_n)^2 - m_n^2)}. \tag{3.94}$$

We use the fact that the LHS is constructed of momenta $p_1, p_2, ..., p_n$ with $q_i = p_1 + p_2 + \cdots + p_i$. Due to conservation of momentum, we have that

$$p_1 + p_2 + \cdots + p_{n-1} = -p_n \tag{3.95}$$

so the number of linearly independent vectors is only $n-1$. This means that we can express the vector quantity on the LHS with a basis of $n-1$ momenta with some coefficients C_i; thus

$$i(4\pi)^2 \int \frac{d^4l}{(2\pi)^4} \frac{l^\mu}{(l^2 - m_0^2)((l+q_1)^2 - m_1^2) \cdots ((l+q_n)^2 - m_n^2)} = \sum_{i=1}^{n-1} C_i p_i^\mu. \tag{3.96}$$

Contracting both sides with p_j^μ gives

$$i(4\pi)^2 \int \frac{d^4l}{(2\pi)^4} \frac{l \cdot p_j}{(l^2 - m_0^2)((l+q_1)^2 - m_1^2) \cdots ((l+q_n)^2 - m_n^2)} = \sum_{i=1}^{n-1} C_i (p_i \cdot p_j) = \sum_{i=1}^{n-1} C_i \Delta^{ij}, \tag{3.97}$$

where Δ^{ij} is called the Grammian-matrix. We have that $p_j = q_j - q_{j-1}$ and we can re-express the term in the numerator by

$$l \cdot p_j = \frac{1}{2} \left((l+q_j)^2 - m_j^2\right) - \left((l+q_{j-1})^2 - m_{j-1}^2\right) + m_j^2 - m_{j-1}^2 - q_j^2 + q_{j-1}^2. \tag{3.98}$$

This is called the Passarino-Veltman-formula. Substituting back into the numerator, see that the that for example the term $\left((l+q_j)^2 - m_j^2\right)$ cancels the j-th propagator and the term $\left((l+q_{j-1})^2 - m_{j-1}^2\right)$ cancels the $(j-1)$-th propagator, so that we obtain $n-1$ equations for the coefficients C_i of the expansion. So by inverting the Grammian-matrix (assuming that

$\det(\Delta)\neq 0$), we get the solution for the expansion coefficients in terms of *scalar* n-point and $n-1$-point integrals

$$C_i=\frac{1}{2}\sum_j \Delta^{-1}_{ij}\left(I(j)_{n-1}[1]-I(j-1)_{n-1}[1]+(m_j^2-m_{j-1}^2-q_j^2+q_{j-1}^2)I^n[1]\right). \tag{3.99}$$

We described how to decompose these tensor structured integrals to a linear combination of scalar integrals. For a general theory with no rational terms, the integral reduction yields scalar integrals with bubble, triangle, or box topology. According to [23], in $\mathcal{N}=4$ the bubble and triangle terms are absent, there is no rational term and we have only a linear combination of scalar boxes. However, as [24] discusses, at the integrand level the expansion yields scalar boxes and chiral pentagons that are both pure and DCI. As we have seen, the main idea of the reduction is to contract the tensor structured integral with some tensor to yield a linear combination of scalar terms which in the usual momentum representation of loop integrals look like

$$\int d^4l\frac{l^{\mu_1}\cdots l^{\mu_m}}{\prod_{i=1}^n(l-p_i)^2}\to T_{\mu_1\cdots\mu_m}\int d^4l\frac{l^{\mu_1}\cdots l^{\mu_m}}{\prod_{i=1}^n(l-p_i)^2}. \tag{3.100}$$

In the embedding space representation, the same tensor structured $m=n-4$-gon loop integrand with n external legs has the form

$$\int[d^6L]\frac{L^{a_1}L^{a_2}\cdots L^{a_{n-4}}}{\prod_{i=1}^n(L,X_i)}. \tag{3.101}$$

In order to obtain a scalar integral we may directly work at the level of the function $\mathcal{F}_n^{(1)}$, where the embedding space momenta are already contracted with the $\mathcal{M}^4\ni W_i$ embedding space vectors, such that our loop integral becomes

$$\mathcal{I}_n^{(1)}=\int[d^6L]\frac{(W_1,L)(W_2,L)\cdots(W_{n-4},L)}{\prod_{i=1}^n(X_i,L)}. \tag{3.102}$$

The reduction consists of iteratively expanding the $\mathcal{M}^{2,4}\supset\mathcal{M}^4\ni W_i$ in the numerator on a $6d$ basis $\{X_1,X_2,X_3,X_4,X_5,X_6\}$. In the first iteration the W_1 is expanded in the m-gon numerator, which after simplification yields a linear combination of $m-1$-gons with some coefficients α_i. In the second iteration, the W_2 are expanded in a similar fashion and after simplifying, a linear combination of $m-2$-gons with coefficients $\alpha_i\beta_j$ is obtained. This is repeated until the chiral pentagons are expanded with different vectors, one of such is

$$W_m=c_1X_1+c_2X_2+c_3X_3+c_4X_4+c_5X_5+r_6R_6=c_iX_i+r_6R_6 \tag{3.103}$$

with R_6 denoting the special term satisfying $(X_i,R_6)=0$ and $(R_6,R_6)=1$. In each case the special term is chosen in a specific way to integrate out the remaining pentagon that cannot be simplified further to a scalar box. Suppose we are considering the reduction of a pentagon ($m=5$) with denominator terms $(L,X_1)\cdots(L,X_5)$. Then, the term is chosen to be R_6 as that term is not present in the numerator, so it cannot be simplified further. So in this pentagon by expanding the W_{n-4} in the numerator with the appropriate special term we get only a linear combination of scalar boxes. If we start directly with a pentagon, the reduction to scalar boxes is the following

$$\mathcal{I}_5^{(1)} = \int [d^6L] \frac{c_i(X_i,L) + r_6(R_6,L)}{(X_1,L)(X_2,L)(X_3,L)(X_4,L)(X_5,L)}. \tag{3.104}$$

Remarkably, the term (R_6, L) does not contribute to the integral. This can be seen using the method of Feynman/Schwinger parametrization of the loop integral

$$\begin{gathered}\mathcal{I}_5^{(1)} = \int [d^6L] \frac{r_6(R_6,L)}{(X_1,L)(X_2,L)(X_3,L)(X_4,L)(X_5,L)} = \\ = \prod_{i=2}^{5}\left(\int_0^\infty d\alpha_i\right)\int \frac{[d^6L]r_6(R_6,L)}{((X_1,L)+\alpha_2(X_2,L)+\alpha_3(X_3,L)+\alpha_4(X_4,L)+\alpha_5(X_5,L))^5}.\end{gathered} \tag{3.105}$$

By defining $\mathcal{W} = X_1+\alpha_2X_2+\alpha_3X_3+\alpha_4X_4+\alpha_5X_5$ in the denominator of the rational function, we obtain from the previous discussion on conformal integrals presented in **Chapter 3.2** that

$$\begin{gathered}\prod_{i=2}^{5}\left(\int_0^\infty d\alpha_i\right)\int[d^6L]\frac{(R_6,L)}{(\mathcal{W},L)^5} \sim \prod_{i=2}^{5}\left(\int_0^\infty d\alpha_i\right)(r_6R_6)\int[d^6L]\partial_{\mathcal{W}}\frac{1}{(\mathcal{W},L)^4} \sim \\ \sim \prod_{i=2}^{5}\left(\int_0^\infty d\alpha_i\right)(r_6R_6)\partial_{\mathcal{W}}\frac{1}{(\mathcal{W},\mathcal{W})^2} \sim \prod_{i=2}^{5}\left(\int_0^\infty d\alpha_i\right)r_6\frac{(R_6,\mathcal{W})}{(\mathcal{W},\mathcal{W})^3}.\end{gathered} \tag{3.106}$$

Since we have in general that $(R, X_i) = 0 \implies (R, \mathcal{W}) = 0$, we obtain that the term does not contribute to the integral

$$\prod_{i=2}^{5}\left(\int_0^\infty d\alpha_i\right)\frac{r_6(R_6,\mathcal{W})}{(\mathcal{W},\mathcal{W})^3} = 0 \tag{3.107}$$

Thus, we have that the original integral is effectively reduced to the form

$$\mathcal{I}_5^{(1)} = \int[d^6L]\frac{c_1(X_1,L)+c_2(X_2,L)+c_3(X_3,L)+c_4(X_4,L)+c_5(X_5,L)}{(X_1,L)(X_2,L)(X_3,L)(X_4,L)(X_5,L)}. \tag{3.108}$$

from which the decomposition to scalar box integrals follows simply by simplifying terms and expanding

$$\begin{gathered}\mathcal{I}_5^{(1)} = \int[d^6L]\frac{c_1}{(X_2,L)(X_3,L)(X_4,L)(X_5,L)} + \int[d^6L]\frac{c_2}{(X_1,L)(X_3,L)(X_4,L)(X_5,L)} + \\ + \int[d^6L]\frac{c_3}{(X_1,L)(X_2,L)(X_4,L)(X_5,L)} + \int[d^6L]\frac{c_4}{(X_1,L)(X_2,L)(X_3,L)(X_5,L)} + \\ + \int[d^6L]\frac{c_5}{(X_1,L)(X_2,L)(X_3,L)(X_4,L)}.\end{gathered} \tag{3.109}$$

Furthermore, the coefficients of the box expansion sketched above can be obtained by contracting the $\mathcal{M}^{2,4} \ni W$ vector using the anti-symmetric $6d$ Levi-Civita tensor defined as

$$\langle X_i X_j X_k X_l X_m X_p \rangle = \varepsilon^{ABCDFG} X_{iA} X_{jB} X_{kC} X_{lD} X_{mF} X_{pG}. \tag{3.110}$$

For example, we can get the c_1 coefficient of the box expansion by first contracting

$$W = c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 + c_5 X_5 + r_6 R_6. \tag{3.111}$$

Due to the null-condition on the embedding space vectors, we can write

$$\langle W X_2 X_3 X_4 X_5 R_6 \rangle = \langle (c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 + c_5 X_5 + r_6 R_6) X_2 X_3 X_4 X_5 \rangle \tag{3.112}$$

$$\langle W X_2 X_3 X_4 X_5 R_6 \rangle = \langle c_1 X_1 X_2 X_3 X_4 X_5 R_6 \rangle \implies c_1 = \frac{\langle W X_2 X_3 X_4 X_5 R_6 \rangle}{\langle X_1 X_2 X_3 X_4 X_5 R_6 \rangle}. \tag{3.113}$$

We can express the coefficient as a determinant by using the identity for the product of anti-symmetric Levi-Civita tensor, also called Cramer's Rule

$$\varepsilon^{ABCDFG} \varepsilon_{PQRSTV} = \delta^{ABCDFG}_{PQRSTV} = \begin{vmatrix} \delta^A_P & \delta^A_Q & \delta^A_R & \delta^A_S & \delta^A_T & \delta^A_V \\ \delta^B_P & \delta^B_Q & \delta^B_R & \delta^B_S & \delta^B_T & \delta^B_V \\ \delta^C_P & \delta^C_Q & \delta^C_R & \delta^C_S & \delta^C_T & \delta^C_V \\ \delta^D_P & \delta^D_Q & \delta^D_R & \delta^D_S & \delta^D_T & \delta^D_V \\ \delta^F_P & \delta^F_Q & \delta^F_R & \delta^F_S & \delta^F_T & \delta^F_V \\ \delta^G_P & \delta^G_Q & \delta^G_R & \delta^G_S & \delta^G_T & \delta^G_V \end{vmatrix}, \tag{3.114}$$

$$c_1 = \frac{\langle W X_2 X_3 X_4 X_5 R_6 \rangle}{\langle X_1 X_2 X_3 X_4 X_5 R_6 \rangle} = \frac{\langle W X_2 X_3 X_4 X_5 R_6 \rangle \langle X_1 X_2 X_3 X_4 X_5 R_6 \rangle}{\langle X_1 X_2 X_3 X_4 X_5 R_6 \rangle \langle X_1 X_2 X_3 X_4 X_5 R_6 \rangle} = \tag{3.115}$$

$$= \frac{\varepsilon^{ABCDFG} \varepsilon_{PQRSTV} W_A X_{2B} X_{3C} X_{4D} X_{5F} R_G X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_{6V}}{\varepsilon^{HJKLMN} \varepsilon_{PQRSTV} X_{1H} X_{2J} X_{3K} X_{4L} X_{5M} R_N X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_{6V}} \tag{3.116}$$

we obtain

$$
\begin{aligned}
c_1 &= \frac{\begin{vmatrix} \delta^A_P & \delta^A_Q & \delta^A_R & \delta^A_S & \delta^A_T & \delta^A_V \\ \delta^B_P & \delta^B_Q & \delta^B_R & \delta^B_S & \delta^B_T & \delta^B_V \\ \delta^C_P & \delta^C_Q & \delta^C_R & \delta^C_S & \delta^C_T & \delta^C_V \\ \delta^D_P & \delta^D_Q & \delta^D_R & \delta^D_S & \delta^D_T & \delta^D_V \\ \delta^F_P & \delta^F_Q & \delta^F_R & \delta^F_S & \delta^F_T & \delta^F_V \\ \delta^G_P & \delta^G_Q & \delta^G_R & \delta^G_S & \delta^G_T & \delta^G_V \end{vmatrix} W_A X_{2B} X_{3C} X_{4D} X_{5F} R_{6G} X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_{6V}}{\begin{vmatrix} \delta^H_P & \delta^H_Q & \delta^H_R & \delta^H_S & \delta^H_T & \delta^H_V \\ \delta^J_P & \delta^J_Q & \delta^J_R & \delta^J_S & \delta^J_T & \delta^J_V \\ \delta^K_P & \delta^K_Q & \delta^K_R & \delta^K_S & \delta^K_T & \delta^K_V \\ \delta^L_P & \delta^L_Q & \delta^L_R & \delta^L_S & \delta^L_T & \delta^L_V \\ \delta^M_P & \delta^M_Q & \delta^M_R & \delta^M_S & \delta^M_T & \delta^M_V \\ \delta^N_P & \delta^N_Q & \delta^N_R & \delta^N_S & \delta^N_T & \delta^N_V \end{vmatrix} X_{1H} X_{2J} X_{3K} X_{4L} X_{5M} R_N X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_V} = \\
&= \frac{\begin{vmatrix} W\cdot X_1 & W\cdot X_2 & W\cdot X_3 & W\cdot X_4 & W\cdot X_5 & W\cdot R_6 \\ X_2\cdot X_1 & X_2\cdot X_2 & X_2\cdot X_3 & X_2\cdot X_4 & X_2\cdot X_5 & X_2\cdot R_6 \\ X_3\cdot X_1 & X_3\cdot X_2 & X_3\cdot X_3 & X_3\cdot X_4 & X_3\cdot X_5 & X_3\cdot R_6 \\ X_4\cdot X_1 & X_4\cdot X_2 & X_4\cdot X_3 & X_4\cdot X_4 & X_4\cdot X_5 & X_4\cdot R_6 \\ X_5\cdot X_1 & X_5\cdot X_2 & X_5\cdot X_3 & X_5\cdot X_4 & X_5\cdot X_5 & X_5\cdot R_6 \\ R\cdot X_1 & R_6\cdot X_2 & R_6\cdot X_3 & R_6\cdot X_4 & R_6\cdot X_5 & R_6\cdot R_6 \end{vmatrix}}{\begin{vmatrix} X_1\cdot X_1 & X_1\cdot X_2 & X_1\cdot X_3 & X_1\cdot X_4 & X_1\cdot X_5 & X_1\cdot R_6 \\ X_2\cdot X_1 & X_2\cdot X_2 & X_2\cdot X_3 & X_2\cdot X_4 & X_2\cdot X_5 & X_2\cdot R_6 \\ X_3\cdot X_1 & X_3\cdot X_2 & X_3\cdot X_3 & X_3\cdot X_4 & X_3\cdot X_1 & X_3\cdot R_6 \\ X_4\cdot X_1 & X_4\cdot X_2 & X_4\cdot X_3 & X_4\cdot X_4 & X_4\cdot X_5 & X_4\cdot R_6 \\ X_5\cdot X_1 & X_5\cdot X_2 & X_5\cdot X_3 & X_5\cdot X_4 & X_5\cdot X_5 & X_5\cdot R_6 \\ R_6\cdot X_1 & R_6\cdot X_2 & R_6\cdot X_3 & R_6\cdot X_4 & R_6\cdot X_5 & R_6\cdot R_6 \end{vmatrix}} = \\
&= \frac{\begin{vmatrix} W\cdot X_1 & W\cdot X_2 & W\cdot X_3 & W\cdot X_4 & W\cdot X_5 & 1 \\ 0 & 0 & 0 & X_2\cdot X_4 & X_2\cdot X_5 & 0 \\ X_3\cdot X_1 & 0 & 0 & 0 & X_3\cdot X_5 & 0 \\ X_4\cdot X_1 & X_4\cdot X_2 & 0 & 0 & 0 & 0 \\ X_5\cdot X_1 & X_5\cdot X_2 & X_5\cdot X_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & X_1\cdot X_3 & X_1\cdot X_4 & X_1\cdot X_5 & 0 \\ 0 & 0 & 0 & X_2\cdot X_4 & X_2\cdot X_5 & 0 \\ X_3\cdot X_1 & 0 & 0 & 0 & X_3\cdot X_5 & 0 \\ X_4\cdot X_1 & X_4\cdot X_2 & 0 & 0 & 0 & 0 \\ X_5\cdot X_1 & X_5\cdot X_2 & X_5\cdot X_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}}.
\end{aligned}
\tag{3.117}
$$

It is important to consider the structure of the numerator also in the $\mathbb{CP}^3$ picture, as some features of the integrands examined here are more transparent in this framework. In the momentum-twistor representation, the general m-gon (3.102) loop integral has the form

$$I_n^{(1)} = \int_{AB} \frac{\langle ABW_1\rangle\langle ABW_2\rangle \cdots \langle ABW_{n-4}\rangle}{\langle AB12\rangle\langle AB23\rangle \cdot\cdot \langle ABn1\rangle}, \tag{3.118}$$

where here W_i^{ab} are the generic bitwistors corresponding to the embedding space vectors W_A according to the $\mathcal{M}^4 \longleftrightarrow SU(2,2)$ isomorphism. The fact that bitwistors appear reflects that it is the momentum-twistor line $Z_A \wedge Z_B$ we are integrating over, not the single twistors Z_A and Z_B. As argued previously, these bitwistors have six degrees of freedom, so we could have equivalently expanded them on a basis $Z_1Z_2, Z_2Z_3, ..., Z_6Z_7$ and contract to obtain six independent equations for the coefficients of the expansion. Moreover, in order for these loop integrals to be DCI, pure and chiral, a suitable normalization is chosen for the numerator and these bitwistors exactly serve that purpose. As found in [24], chiral and pure integrals are manifestly IR finite. The bitwistors W_i^{ab} are chosen in a way that the integrand is properly normalized; the momentum-twistors chosen are the ones solving the Schubert-problem of the integral, as this will give pure integrals. This normalization can be essentially done in two ways, as we may have momentum twistor lines solving the Schubert problems, or dual momentum twistor lines defined by the intersection of planes $\mathbb{CP}^2$ solving the Schubert-problem. Restricting ourselves to chiral pentagons for the moment, the possible normalizations are in the momentum-twistor line case represented by a dashed line, whereas for the momentum-twistor line formed by the intersection of $\mathbb{CP}^2$ planes (which are parity conjugate to the dashed) and are represented by wavy line. This point is illustrated in Figure 3.3 and Figure 3.4.

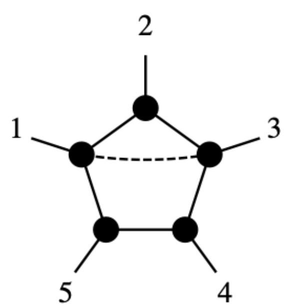


Figure 3.3: $W = (Z_1, Z_3)$ used for normalization.

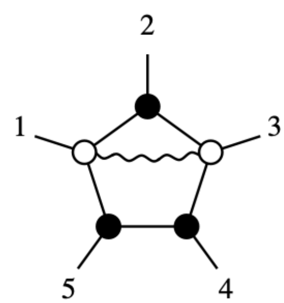


Figure 3.4: $W = (Z_5Z_1Z_2) \cap (Z_2Z_3Z_4)$ for normalization.

The dual twistor represented by the wavy line in Figure 3.4 is the parity conjugate of the momentum-twistor line used for normalizatioon in Figure 3.3.

As the form of the momentum-bitwistors is already specified by the definition of the loop integral, we just need to insert them accordingly to (3.117). For example, in the case of the dashed line normalization $W = (13)$ represented in Figure 3.3, the determinant for c_1 becomes in the momentum-twistor representation

$$c_1 = \frac{\begin{vmatrix} 0 & 0 & 0 & 0 & \langle 1345\rangle & 1 \\ 0 & 0 & 0 & \langle 1234\rangle & \langle 1245\rangle & 0 \\ \langle 5123\rangle & 0 & 0 & 0 & \langle 2345\rangle & 0 \\ \langle 5134\rangle & \langle 1234\rangle & 0 & 0 & 0 & 0 \\ 0 & \langle 1245\rangle & \langle 2345\rangle & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & \langle 5123\rangle & \langle 5134\rangle & 0 & 0 \\ 0 & 0 & 0 & \langle 1234\rangle & \langle 1245\rangle & 0 \\ \langle 5123\rangle & 0 & 0 & 0 & \langle 2345\rangle & 0 \\ \langle 5134\rangle & \langle 1234\rangle & 0 & 0 & 0 & 0 \\ 0 & \langle 1245\rangle & \langle 2345\rangle & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}}. \tag{3.119}$$

In this way we can obtain any coefficient of the expansion by evaluating the determinants $\langle ABCD\rangle = \langle Z_A Z_B Z_C Z_D\rangle$. So given a $n-4$-gon with n-external legs, we have a prescription for reducing to a linear combination of scalar boxes, whose coefficients we can determine in this way.

3.5 Polylogarithms and the Symbol

This section closely follows [43], [44] and [45]. At 1-loop and higher loop orders, we encounter and need to evaluate multiple integrals in the calculation of the scattering amplitudes. In $\mathcal{N} = 4$ sYM these integrals often evaluate to a class of special functions called *polylogarithms*, which have interesting analytic properties. Moreover, as [44] argues, unitarity determines that these special functions have complicated branch cut structures. The difficulty in evaluating these integrals increases exponentially with the loop order. At 1-loop, there is an integration over one loop momentum l and over Feynman-parameters, which in the embedding space representation of the m-gon, n-external leg integral has the form

$$\begin{aligned}&\int [d^6L]\frac{(W_1,L)(W_2,L)\cdots(W_{n-4},L)}{\prod_{i=1}^{n}(X_i,L)} = \\ &= \prod_{i=1}^{n}\left(\int_0^\infty d\alpha_i\right)\int [d^6L]\delta(1-\sum_{k=1}^{n}\alpha_k)\frac{(W_1,L)(W_2,L)\cdots(W_{n-4},L)}{\left(\sum_{j=1}^{n}\alpha_j(X_j,L)\right)^n}.\end{aligned} \tag{3.120}$$

In addition to this, at multi-loop order we have an integration over each loop momentum corresponding to every loop in Feynman-diagram(s). Thus, we have repeated integrations which must be evaluated to obtain the final result. It is hence useful to consider such iterated integrals in more detail. The general definition of the iterated integral in arbitrary dimensions requires the use of some differential geometry. If $\mathbb{V}$ is a linear space, then a 1-form on $\mathbb{V}$ is a linear functional on $\mathbb{V}$, where a linear functional α on $\mathbb{V}$ is a linear transformation

$$\alpha : \mathbb{V} \to \mathbb{C} \implies \alpha(a\mathbf{v} + b\mathbf{w}) = a\alpha(\mathbf{v}) + b\alpha(\mathbf{w}), \tag{3.121}$$

where therefore $\alpha(\mathbf{v}), \alpha(\mathbf{w}) \in \mathbb{C}$. On a manifold X, the 1-form is defined as a mapping from the tangent bundle of X ($TM = \sqcup_{x\in M} T_x M$, where T_x is the tangent space to X at x) to $\mathbb{R}$ such that by constraining the 1-form on each $T_x M$ we have a linear functional $T_x M \to \mathbb{R}$. So on a manifold X, a 1-form is

$$\alpha : TM \to \mathbb{R}, \qquad \alpha_x = \alpha|_{T_x M} : T_x M \to \mathbb{R}. \tag{3.122}$$

Given a smooth (differentiable) manifold X over the field $\mathbb{C}$ (or equivalently $\mathbb{R}$), a smooth curve $\gamma : [0,1] \to X$, n smooth 1-forms $\alpha_1, \alpha_2, ..., \alpha_n \in \Lambda^1(X)$ on X with $\alpha_k = \sum_i f_{k,i} dx^i$ and n pullbacks of the 1 form α on the curve γ expressed as $\gamma^*(\alpha_1), ..., \gamma^*(\alpha_n) \in \Lambda^1(\mathbb{C})$ on X with $\gamma^*(\alpha_k) = f_k(t)dt = \Sigma_i f_k \frac{dx^i}{dt} dt$, we define the iterated integral on the smooth curve γ as

$$\begin{aligned}\int_\gamma \alpha_1 \circ \alpha_2 \circ ... \circ \alpha_n &= \int_0^1 \gamma^*(\alpha_1) \circ \gamma^*(\alpha_2) \circ ... \circ \gamma^*(\alpha_n) = \\ &= \int_{0\leq t_1 \leq \cdots \leq t_n \leq n} f_1(t)dt \cdots f_n(t)dt,\end{aligned} \tag{3.123}$$

where the pullbacks are basically 1-forms on the interval $[0,1]$. The iterated integral does not depend on the parametrization of the smooth curve γ on which we integrate on. Moreover, if $\gamma^{-1}(t)=\gamma(1-t)$ is the reversed path on X with parametrization from 1 to 0, then

$$\int_{\gamma^{-1}} \alpha_n \circ \alpha_{n-1} \circ \ldots \circ \alpha_1 = (-1)^n \int_\gamma \alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_n. \tag{3.124}$$

In addition, if $\beta, \gamma : [0,1] \to X$ are smooth paths, such that the end point on X of path β is the initial point of path γ on X, then we have that

$$\int_{\alpha\beta} \alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_n = \sum_{i=1}^{n} \int_\beta \alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_i \int_\gamma \alpha_{i+1} \circ \alpha_{i+2} \circ \ldots \circ \alpha_n. \tag{3.125}$$

Furthermore, given a smooth curve γ on the smooth manifold X, we define the functional F of on the path γ as

$$F[\gamma] = \int_\gamma \alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_n. \tag{3.126}$$

If the function F is independent of the path γ and we consider the functional as a function of the endpoint $\gamma(1)$, we have that

$$dF[\gamma] = \alpha_n\left(\gamma(1)\right) \int_\gamma \alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_{n-1}. \tag{3.127}$$

According to the discussion above, in the special case of $1d$ we define the n-fold iterated integral recursively as

$$\int_a^b f_1(t)dt \circ f_2(t)dt \circ \ldots \circ f_n(t)dt = \int_a^b \left(\int_a^t f_1(u)du \circ f_2(u)du \circ \ldots \circ f_{n-1}(u)du \right) f_n(t)dt, \tag{3.128}$$

where the first integral of the recursion is the integral over f_1 and so on [43]. For example, the iterated integral special case with $n=2$ is

$$\int_a^b f_1(t)dt \circ f_2(t)dt = \int_a^b \left(\int_a^t f_1(u)du \right) f_2(t)dt. \tag{3.129}$$

By choosing the following 1-forms for (3.129)

$$\alpha_1 = \frac{dt}{1-t}, \qquad \alpha_2 = \frac{dt}{t} \tag{3.130}$$

we obtain the definition of the dilogarithm $\mathrm{Li}_2(z)$

$$\begin{aligned}\mathrm{Li}_2(z) = \int_0^z -\frac{dt}{1-t} \circ \frac{dt}{t} = \int_0^z \left(\int_0^{t_2} -\frac{dt_1}{1-t_1} \right) \frac{dt_2}{t_2} = \int_0^z \left(\int_0^{t_2} \mathrm{dlog}(1-t_1) \right) \mathrm{dlog}(t_2) = \\ = \int_0^z \frac{dt_2}{t_2} \log(1-t_2) = \int_0^z \frac{dt_2}{t_2} \mathrm{Li}_1(t_2),\end{aligned} \tag{3.131}$$

where $\mathrm{Li}_1(t) = \log(1-t) = -\int_0^z \frac{dt}{1-t}$. In this fashion, we can define any n-weight polylogarithm through iterated integrations by choosing the smooth path $\gamma : [0,1] \to \mathbb{C}\backslash\{0,1\}$, with endpoints $\gamma(0) = 0$ and $\gamma(1) = z$ and pullbacks on $X = \mathbb{C}$ given by $\gamma^*(\alpha_1) = \mathrm{dlog}(1-t_1)$ and $\gamma^*(\alpha_i) = \mathrm{dlog}(t_i)$ with $i = 2, ..., n$

$$\begin{aligned}\mathrm{Li}_n(z) = \int_0^z -\frac{dt}{1-t} \circ \underbrace{\frac{dt}{t} \circ ... \circ \frac{dt}{t}}_{n-1} = \int_0^z \int_0^{t_n} \cdots \int_0^{t_2} -\frac{dt_1}{1-t_1}\frac{dt_2}{t_2} \cdots \frac{dt_n}{t_n} = \\ = \int_0^z \int_0^{t_n} \cdots \int_0^{t_2} (-\,\mathrm{dlog}(1-t_1))\,\mathrm{dlog}(t_2) \cdots \mathrm{dlog}(t_n) = \int_0^z \frac{dt_n}{t_n}\,\mathrm{Li}_{n-1}(t_n).\end{aligned} \tag{3.132}$$

However, there exists an infinite amount of paths from 0 to z in $\mathbb{C}$, which introduces some serious ambiguity in the definition of the iterated integration. This was resolved by [46], who found that if the different paths with same initial points and endpoints can be continuously deformed into each other without encountering poles, branch cuts, etc., and the 1-form α on X being integrated is closed, i.e. $d\alpha = 0$, then the integral $\int_\gamma \alpha$ is invariant under choices of the path γ, so is invariant under small path variations. Paths with this property are called *homotopically equivalent* paths. Thus, if $\gamma_1 \simeq \gamma_2$ homotopically and $d\alpha = 0$ we have that

$$\int_{\gamma_1} \alpha = \int_{\gamma_2} \alpha. \tag{3.133}$$

Returning to the example of the polylogarithm, we see that if the 1-form α is exact then the 1-form α is closed, i.e. $d\alpha = 0$, then it can be written as the exterior derivative of some other 1-form β: $\alpha = d\beta \implies d\alpha = d^2(\beta) = 0$ by identity. Using the Stokes-theorem we see that

$$\int_\gamma \alpha = \int_\gamma d\beta = \int_{\partial\gamma} \beta = \beta(\gamma(1)) - \beta(\gamma(0)). \tag{3.134}$$

So according to te above in the special case of the 1-form given by $\alpha = \frac{dt}{1-t}$, the value of the integral

$$\int_0^z \frac{dt}{1-t} = -\log(1-z) \tag{3.135}$$

is invariant along a path not encountering the pole $z = 1$. Furthermore, advancing this discussion for an iterated integral

$$I = \int_\gamma \alpha_1 \circ \alpha_2 \tag{3.136}$$

with $F(z) = \int_0^z \alpha_1$, then with small variations of the path and the prescription $d(F\alpha_2) = d(F)\alpha_2 + Fd(\alpha_2) = 0$, we have that for

$$I = \int_\gamma \alpha_1 \circ \alpha_2 = \int_\gamma F\alpha_2 \tag{3.137}$$

the integral I is the same. If, $d(\alpha_2)=0$ and $dF=\alpha_1$, then $\alpha_1\alpha_2=0 \implies \alpha_1\wedge\alpha_2=0$, which is an *integrability condition*. According to this discussion, there is no more ambiguity in the defintion of polylogarithms, or other general multiple polylogarithms. In general for the general iterated integral case

$$I=\int_\gamma \sum_{i_1,\dots,i_n} c_{i_1,\dots,i_n}\alpha_{i_1}\circ\cdots\circ\alpha_{i_n} \tag{3.138}$$

the integrability condition reads as $\alpha_{i_j}\wedge\alpha_{i_{j+1}}=0$,$\forall j=1,...,n-1$.

Let's now focus on other properties of polylogarithms. The differential form $\frac{dt}{t}$ appearing in the definition of the polylogarithm is scale invariant; with a scaling transformation $u=\lambda t$, with $\lambda\in\mathbb{C}$, then

$$\frac{du}{u}=\frac{d(\lambda t)}{\lambda t}=\frac{dt}{t} \tag{3.139}$$

as argued by [45]. From the Taylor-series expansion of the logarithm about $z=0$, we can determine the series expansion of the dilogarithm about the same point

$$-\log(1-t)=\sum_{k=1}^{\infty}\frac{t^k}{k} \implies -\int_0^z dt\frac{\log(1-t)}{t}=\sum_{k=1}^{\infty}\int_0^z\frac{t^{k-1}}{k}=\sum_{k=1}^{\infty}\frac{z^k}{k^2}=\mathrm{Li}_2(z). \tag{3.140}$$

By proceeding this way iteratively, we can see that the Taylor-series expansion of the n-weight polylogarithm about $z=0$ is given by

$$\mathrm{Li}_n(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^n}. \tag{3.141}$$

Due to the integral representation, we see that we can produce lower weight polylogarithms from higher weight polylogarithms with derivatives with respect to z

$$d\,\mathrm{Li}_2(z)=-\frac{\log(1-z)}{z}dz=\frac{\mathrm{Li}_1(z)}{z}dz \implies z\frac{d}{dz}\mathrm{Li}_2(z)=\mathrm{Li}_1(z). \tag{3.142}$$

From the series expansion of the n-weight polylogarithm this is evident as

$$\frac{d}{dz}\mathrm{Li}_n(z)=\sum_{k=1}^{\infty}\frac{z^{k-1}}{k^{n-1}} \implies z\frac{d}{dz}\mathrm{Li}_n(z)\sum_{k=1}^{\infty}\frac{z^k}{k^{n-1}}=\mathrm{Li}_{n-1}(z). \tag{3.143}$$

Polylogarithms have multiple interesting identities. For example, it is true $\forall n\in\mathbb{N}$ that

$$\mathrm{Li}_n(z)+\mathrm{Li}_n(-z)=2^{1-n}\,\mathrm{Li}_n(z^2). \tag{3.144}$$

Dilogarithms $\mathrm{Li}_2(z)$ satisfy the so called inversion identity

$$\mathrm{Li}_2(z)+\mathrm{Li}_2\left(\frac{1}{z}\right)=-\log(-z)^2-\mathrm{Li}_2(1) \tag{3.145}$$

which gives a link between polylogarithms and the Riemann-zeta function defined by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \text{ with } \Re(s) > 1. \tag{3.146}$$

From it follows that we have the equality

$$\mathrm{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6} \implies \mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) = -\log(-z)^2 - \frac{\pi^2}{6}. \tag{3.147}$$

We can define more general iterated integrals dependent on arbitrary complex parameters a_i, which are known as *Goncharov*-polylogarithms, or multiple-polylogarithms [44]. These generalize the concept of polylogarithms and naturally lead to the notion of the *shuffle*-algebra. They are obtained by choosing different 1-forms in (3.123). The Goncharov-polylogarithm is defined recursively starting with the 1-form $\alpha_1 = \frac{dt}{t-a_1}$ with complex parameter a_1 and $\mathrm{G}(\cdot, z) = 1$

$$\mathrm{G}(a_1, z) = \int_0^z \frac{dt}{t - a_1} G(, z) = \log\left(\frac{a_1 - z}{a_1}\right). \tag{3.148}$$

Following this, we define the next Goncharov with complex parameters a_1, a_2 with

$$\mathrm{G}(a_1, a_2, z) = \int_0^z \frac{dt}{t - a_1} \int_0^t \frac{du}{u - a_2} = \int_0^z \frac{dt}{t - a_1} \log\left(\frac{a_2 - t}{a_2}\right) = \int_0^z \frac{dt}{t - a_1} \mathrm{G}(a_2, t). \tag{3.149}$$

In general, the n-weight Goncharov-polylogarithm is traditionally defined as

$$\begin{aligned}
\mathrm{G}(a_1, a_2, ..., a_n, z) &= \int_0^z \frac{dt}{t - a_1} \circ \frac{dt}{t - a_2} \circ ... \circ \frac{dt}{t - a_n} = \\
&= \int_0^z \int_0^{t_1} \cdots \int_0^{t_{n-1}} \frac{dt_1}{t_1 - a_1} \frac{dt_2}{t_2 - a_2} \cdots \frac{dt_n}{t_n - a_n} = \\
&= \int_0^z \int_0^{t_1} \cdots \int_0^{t_{n-1}} \mathrm{dlog}(t_1 - a_1)\, \mathrm{dlog}(t_2 - a_2) \cdots \mathrm{dlog}(t_n - a_n) = \\
&= \int_0^z \mathrm{dlog}(t_1 - a_1)\, \mathrm{G}(a_2, a_3, ..., a_n, t_1).
\end{aligned} \tag{3.150}$$

An alternative, more general definition for multiple-polylogarithms is given by the definition

$$\mathrm{I}(a_0, a_1, ..., a_n, z) = \int_{a_0}^z \frac{dt}{t - a_n} \mathrm{I}(a_0, a_1, ..., a_{n-1}, t) = \int_{a_0}^z \mathrm{dlog}(t - a_n)\, \mathrm{I}(a_0, a_1, ..., a_{n-1}, t), \tag{3.151}$$

which is related to the Goncharovs as

$$\mathrm{G}(a_n, ..., a_1, z) = \mathrm{I}(0, a_1, ..., a_n, z). \tag{3.152}$$

An interesting special case for the n-weight Goncharov is

$$\mathrm{G}(\underbrace{0,0,\cdots,0}_{n},z)=\frac{1}{n!}\log^n(z). \tag{3.153}$$

Moreover, we have the similar identity

$$\mathrm{G}(\underbrace{a,a,\cdots,a}_{n},z)=\frac{1}{n!}\log^n\left(1-\frac{z}{a}\right). \tag{3.154}$$

The definition of $\mathrm{Li}_1(z)$ can be obtained from $\mathrm{G}(a_1,z)$

$$-\mathrm{G}(1,z)=-\mathrm{G}\left(\frac{1}{z},1\right)=-\log\left(\frac{\frac{1}{z}-1}{\frac{1}{z}}\right)=-\log(1-z)=\int_0^z\frac{dt}{1-t}=\mathrm{Li}_1(z). \tag{3.155}$$

Similarly, we can obtain the relationship between the weight-2 Goncharov and the dilogarithm $\mathrm{Li}_2(z)$ as

$$-\mathrm{G}(0,1,z)=-\mathrm{G}\left(0,\frac{1}{z},1\right)=-\int_0^z\frac{dt}{t}\underbrace{\mathrm{G}\left(\frac{1}{t},1\right)}_{\mathrm{G}(1,t)}=-\int_0^z\frac{dt}{t}\log(1-t)=\int_0^z\frac{dt}{t}\mathrm{Li}_1(t)=\mathrm{Li}_2(z). \tag{3.156}$$

We can obtain polylogarithms of arbitrary weight n following this procedure; for the n-weight polylogarithm $\mathrm{Li}_n(z)$ we have from $\mathrm{G}(a_1,a_2,...,a_n,z)$ that

$$\mathrm{Li}_n(z)=-\mathrm{G}\left(\underbrace{0,0,...,0}_{n-1},\frac{1}{z},1\right)=-\int_0^z\frac{dt}{t}\mathrm{G}\left(\underbrace{0,0,...,0}_{n-2},\frac{1}{t},1\right)=\int_0^z\frac{1}{t}\mathrm{Li}_{n-1}(t). \tag{3.157}$$

Furthermore, the product of two distinct weight-n_1 and weight-n_2 Goncharov-polylogarithms defines the *shuffle product*

$$\mathrm{G}(a_1,a_2,...,a_{n_1},z)\,\mathrm{G}(a_{n_1+1},a_{n_1+2},...,a_{n_1+n_2},z)=\mathrm{G}(a_1,a_2,...,a_{n_1}\sqcup\!\sqcup\, a_{n_1+1},a_{n_1+2},...,a_{n_1+n_2},z). \tag{3.158}$$

Consider the special case with Goncharovs having both weight $n=1$, we see that

$$\mathrm{G}(a_1,z)\,\mathrm{G}(a_2,z)=\int_0^z\frac{dt_1}{t_1-a_1}\int_0^z\frac{dt_2}{t_2-a_2}=\int\int_{\Box}\frac{dt_1dt_2}{(t_1-a_1)(t_2-a_2)}, \tag{3.159}$$

where the integrals are combined using Fubini's theorem and the resulting integration is over the box $\Box$ whose corners are $(0,0),(z,0),(0,z),(z,z)$. By splitting the integration over the $\Box$, into two integrations over triangles $\triangle$ with corners $(0,0),(z,0),(z,z)$ and $(0,0),(0,z),(z,z)$ we obtain

$$\begin{aligned}\mathrm{G}(a_1,z)\,\mathrm{G}(a_2,z) &= \int\int_{0\le t_2\le t_1\le z}\frac{dt_1dt_2}{(t_1-a_1)(t_2-a_2)}+\int\int_{0\le t_1\le t_2\le z}\frac{dt_1dt_2}{(t_1-a_1)(t_2-a_2)}=\\ &=\int_0^z\frac{dt_1}{t_1-a_1}\int_0^{t_1}\frac{dt_2}{t_2-a_2}+\int_0^z\frac{dt_2}{t_2-a_2}\int_0^{t_2}\frac{dt_1}{t_1-a_1}=\mathrm{G}(a_1,a_2,z)+\mathrm{G}(a_2,a_1,z).\end{aligned} \tag{3.160}$$

The important take-home message from the discussion is that by multiplying two Goncharov-polylogarithms, one with weight n_1 and the other with weight n_2, we obtain a sum of weight n_1+n_2 Goncharov-polylogarithms of the form

$$\mathrm{G}(a_1,a_2,...,a_n,z)\,\mathrm{G}(a_{n_1+1},a_{n_1+2},...,a_{n_1+n_2},z)=\sum_{\sigma\in\Sigma(n_1,n_2)}\mathrm{G}(a_{\sigma(1)},a_{\sigma(2)},...,a_{\sigma(n_1+n_2)},z) \tag{3.161}$$

with $\Sigma(n_1,n_2)$ being the set of shuffles of n_1+n_2 number of elements, which is the subset of the symmetric group $S_{n_1+n_2}$ that is given by the set of (n_1,n_2) shuffles

$$\Sigma(n_1,n_2)=\{\sigma\in S_{n_1+n_2}|\sigma^{-1}(1)<\cdots<\sigma^{-1}(n_1)\ \mathit{and}\ \sigma^{-1}(n_1+1)<\cdots<\sigma^{-1}(n_1+n_2)\} \tag{3.162}$$

so the $S_{n_1+n_2}$ subset that preserves the ordering of $(a_1,a_2,...,a_{n_1})$ and the ordering of $(a_{n_1+1},a_{n_1+2},...,a_{n_1+n_2})$ simultaneously. Actually, this is a general attribute of iterated integrals as for the product of a weight n_1 iterated integral and a weight n_2 iterated integral over the same smooth curve γ, we have that

$$\begin{aligned}\left(\int_\gamma\alpha_1\circ\alpha_2\circ\cdots\circ\alpha_{n_1}\right)\left(\int_\gamma\alpha_{n_1+1}\circ\alpha_{n_1+2}\circ\cdots\circ\alpha_{n_1+n_2}\right)&=\\ =\sum_{\sigma\in\Sigma(n_1,n_2)}\int_\gamma\left(\alpha_{\sigma(1)}\circ\alpha_{\sigma(2)}\circ\cdots\circ\alpha_{\sigma(n_1+n_2)}\right).&\end{aligned} \tag{3.163}$$

Hence, the set of Goncharov-polylogarithm $\mathcal{G}$ equipped with a multiplication $\sqcup\!\sqcup$, so $(\mathcal{G},\sqcup\!\sqcup)$, is an algebra (vector space with closed product $\sqcup\!\sqcup$ on the vector space that is associative and and a has unit element) called the *shuffle-algebra*. Since, the product $\sqcup\!\sqcup$ preserves the weight of the Goncharov-polylogarithms $\mathcal{G}\ni\mathrm{G}$, we say that this a *graded*-algebra. As a further example, if $\mathrm{G}_1,\mathrm{G}_2\in\mathcal{G}$, G_1 has weight $n_1=2$ and G_2 has weight $n_2=1$, then as $(\mathcal{G},\sqcup\!\sqcup)$ is graded, we have a sum of n_1+n_2-weight Goncharov-polylogarithms G according to (3.161)

$$\mathrm{G}_1(a_1,a_2,z)\,\mathrm{G}_2(b_1,z)=\mathrm{G}(a_1,a_2,b_1,z)+\mathrm{G}(a_1,b_1,a_2,z)+\mathrm{G}(b_1,a_1,a_2,z). \tag{3.164}$$

Let's advance the discussion on algebraic concepts further as this will provide us with further tools for evaluating loop integrals efficiently. An algebra is a 2 $(\mathcal{A},\otimes)$ where $\mathcal{A}$ is a vector space with product $\otimes$ on it, that is associative and there exists a unit element. This means that there exists a closed map $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ associated to the product on the algebra such that for $a_1 \otimes a_2 \in \mathcal{A} \otimes \mathcal{A}$ $\mu(a_1, a_2) \to a_3$ with $a_3 \in \mathcal{A}$ where if $1 \in \mathcal{A}$ is the unit element, then $\mu(1, a_1) = \mu(a_1, 1) = 1 \otimes a_1 = a_1 \otimes 1$. For an algebra $\mathcal{A}$ the product space $\mathcal{A} \otimes \mathcal{A}$ is an algebra as well. Also, $\forall a_1, a_2, a_3 \in \mathcal{A}$ and $\forall k \in \mathcal{A}$, domain elements satisfy the conditions

$$\begin{aligned} (a_1 + a_2) \otimes a_3 &= a_1 \otimes a_3 + a_2 \otimes a_3, \\ a_1 \otimes (a_2 + a_3) &= a_1 \otimes a_2 + a_1 \otimes a_3, \\ (ka_1) \otimes a_2 = a_1 \otimes (ka_2) &= k(a_1 \otimes a_2). \end{aligned} \tag{3.165}$$

The product on the algebra $\mathcal{A} \otimes \mathcal{A}$ is the mapping $\rho : (\mathcal{A} \otimes \mathcal{A}) \bigotimes (\mathcal{A} \otimes \mathcal{A}) \to (\mathcal{A} \otimes \mathcal{A})$ where $\forall a_1, a_2, a_3, a_4 \in \mathcal{A}$ the product on $\mathcal{A} \otimes \mathcal{A}$ is defined as

$$(a_1 \otimes a_2)(a_3 \otimes a_4) = (a_1 a_3) \otimes (a_2 a_4). \tag{3.166}$$

We can also think of of the converse situation, where we want to assign to $\mathcal{A}$ elements an element of the product algebra $\mathcal{A} \otimes \mathcal{A}$. This idea leads to a further algebraic structure called the *coalgebra*, on which we have a *coproduct* realized by the linear mapping

$$\triangle : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \tag{3.167}$$

$$\text{such that } a \mapsto \triangle(a) \in \mathcal{A} \otimes \mathcal{A}. \tag{3.168}$$

The coproduct $\triangle$ is coassociative, so $(\triangle \otimes \mathrm{id}_\triangle)\triangle = (\mathrm{id}_\triangle \otimes \triangle)\triangle$.
Due to coassociativity, the coproduct is independent on the order of the iteration of the coproduct, i.e. $a \in \mathcal{A} \mapsto \triangle(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)} \mapsto \sum_i \triangle(a_i^{(1)}) \otimes a_i^{(2)}$ is equivalent to $a \in \mathcal{A} \mapsto \triangle(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)} \mapsto \sum_i a_i^{(1)} \otimes \triangle(a_i^{(2)})$.

The iteration of the coproduct is a mapping $(\triangle \otimes \mathrm{id}_\triangle) : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$.

All in all, if $\mathcal{A}$ is an algebra equipped with product μ *and* coproduct $\triangle$, then $\mathcal{A}$ is a bialgebra. Similarly to before, if the bialgebra is graded, then the coproduct conserves the weight and $\triangle(a_1 a_2) = \triangle(a_1)\triangle(a_2)$. A bialgebra equipped with antipode (analogous to inversion map $g \to g^{-1}$ for group elements in group theory) is called a *Hopf-algebra.*

Having described some mathematical preliminaries, let's now construct a concrete bialgebra structure. Considering a set of *letters* $\{a_1, a_2, a_3\}$ we consider the linear space $\mathcal{A}$ spanned by all possible linear combination of *words* with coefficients in $\mathbb{Q}$. Since the algebra is graded, the weight is given by the words' length. We have the product μ on $\mathcal{A}$ such that $(a_1 a_2) \otimes a_3 \mapsto a_1 a_2 a_3 \in \mathcal{A}$ and we have a coproduct $\triangle$ on $\mathcal{A}$ defined on single letters (length 1 words) of $\mathcal{A}$ as

$$\triangle(a) = 1 \otimes a + a \otimes 1 \tag{3.169}$$

with identity element $1 \in \mathcal{A}$ and satisfying $\triangle(1) = 1 \otimes 1$. For words of length ≥ 2 we use $\triangle(a_1 a_2) = \triangle(a_1)\triangle(a_2)$ together with (3.169). So $\forall a_1, a_2 \in \mathcal{A}$

$$\begin{aligned} \triangle(a_1 a_2) = \triangle(a_1)\triangle(a_2) = (1 \otimes a_1 + a_1 \otimes 1)(1 \otimes a_2 + a_2 \otimes 1) = \\ = 1 \otimes (a_1 a_2) + (a_1 a_2) \otimes 1 + a_1 \otimes a_2 + a_2 \otimes a_1. \end{aligned} \tag{3.170}$$

We similarly have for length 3 words that $\forall a_1, a_2, a_3 \in \mathcal{A}$

$$\begin{aligned} \triangle(a_1 a_2 a_3) &= \triangle(a_1 a_2)\triangle(a_3) = \triangle(a_1)\triangle(a_2 a_3) = \\ &= (1 \otimes a_1 + a_1 \otimes 1)(1 \otimes (a_2 a_3) + (a_2 a_3) \otimes 1 + a_2 \otimes a_3 + a_3 \otimes a_2) = \\ &= 1 \otimes (a_1 a_2 a_3) + (a_1 a_2 a_3) \otimes 1 + (a_1 a_2) \otimes a_3 + \\ &+ a_2 \otimes (a_1 a_3) + a_3 \otimes (a_1 a_2) + (a_1 a_3) \otimes a_2 + a_1 \otimes (a_2 a_3) + (a_2 a_3) \otimes a_1. \end{aligned} \tag{3.171}$$

Due to the coassociativity of the coproduct, the way we iterate does not matter. For example, $(\triangle \otimes \mathrm{id}_\triangle)\triangle(a_1 a_2) = (\mathrm{id}_\triangle \otimes \triangle)\triangle(a_1 a_2)$. If for $a \in \mathcal{A}$ we have that $\triangle(a) = 1 \otimes a + a \otimes 1$, then a is *primitive*. If $a \in \mathcal{A}$ is primitive, then a cannot be decomposed non-trivially.

Furthermore, if the Hopf-algebra is graded, then we can introduce the map $\triangle_{\{i_1 i_2 \cdots i_k\}}$ which maps elements of $\mathcal{A}$ to the coproduct sector where factors have the weight $(i_1 i_2 \cdots i_k)$. For example, in $\triangle(a_1 a_2)$ we have that

$$\triangle_{1,1}(a_1 a_2) = a_1 \otimes a_2 + a_2 \otimes a_1. \tag{3.172}$$

For length-3 words we have the relevant iterated coproduct component

$$\begin{aligned} \triangle_{1,1,1}(a_1 a_2 a_3) = a_1 \otimes a_2 \otimes a_3 + a_2 \otimes a_3 \otimes a_1 + a_3 \otimes a_1 \otimes a_2 + \\ + a_3 \otimes a_2 \otimes a_1 + a_2 \otimes a_1 \otimes a_3 + a_1 \otimes a_3 \otimes a_2. \end{aligned} \tag{3.173}$$

Other unevenly weighted coproduct components are given by

$$\triangle_{1,2}(a_1 a_2 a_3) = a_1 \otimes (a_2 a_3) + a_2 \otimes (a_1 a_3) + a_3 \otimes (a_1 a_2), \tag{3.174}$$

$$\triangle_{2,1}(a_1 a_2 a_3) = (a_1 a_2) \otimes a_3 + (a_2 a_3) \otimes a_1 + a_1 a_3 \otimes a_2 \tag{3.175}$$

in [44].

As studied by [47], multiple-polylogarithms form a Hopf-algebra $\mathcal{H}$ themselves with coproduct

$$\triangle(\mathrm{I}(a_0, a_1, ..., a_n, z)) = \sum_{0 = i_1 \leq \cdots \leq i_{k+1} = n} \mathrm{I}(a_0, a_{i_1}, ..., a_{i_k}, z) \otimes \left(\prod_{j=0}^{k} \mathrm{I}(a_{i_j}, a_{i_j + 1}, ..., a_{i_{j+1} - 1}, a_{i_j + 1}) \right). \tag{3.176}$$

For example, for weight $n = 2$ we have the coproduct expansion

$$\begin{aligned} \triangle(I(a_0, a_1, a_2, z)) = \mathrm{I}(a_0, a_1, a_2, z) \otimes 1 + 1 \otimes \mathrm{I}(a_0, a_1, a_2, z) + \\ + \mathrm{I}(a_0, a_1, z) \otimes \mathrm{I}(a_0, a_2, z) + \mathrm{I}(a_0, a_2, z) \otimes \mathrm{I}(a_0, a_1, a_2). \end{aligned} \tag{3.177}$$

The terms of the coproduct have a diagrammatic description for describing the various terms [43]. Some terms are divergent, so the functions have to be regularized according to some suitable scheme. One such scheme is shuffle-regularization. The original definition of the coproduct is contradictory for even ζ-values and $\log(-1) = i\pi$. So to allow for transcendental coefficients $i\pi$ for elements of $\mathcal{H}$ we redefine $\mathcal{A} = \mathbb{Q}[i\pi] \otimes \mathcal{H}$, so the coproduct is redefined to the map $\triangle : \mathbb{Q}[i\pi] \otimes \mathcal{H} \to (\mathbb{Q}[i\pi] \otimes \mathcal{H}) \otimes \mathcal{H}$.
Therefore, in this definition of the coproduct, we have a mapping to an asymmetric space $(\underbrace{\mathbb{Q}[i\pi] \otimes \mathcal{H}}_{\mathcal{A}}) \otimes \mathcal{H}$. The rightmost factor $\mathcal{H}$ describes the behaviour of $F \in \mathcal{A}$ under action of the derivative

$$\triangle\left(\frac{\partial}{\partial z}F\right) = \left(\mathrm{id}_{\triangle} \otimes \frac{\partial}{\partial z}\right)\triangle(F). \tag{3.178}$$

On the other hand, the leftmost factor $\mathcal{A} = \mathbb{Q}[i\pi] \otimes \mathcal{H}$ describes the discontinuity of $F \in \mathcal{H}$. So if Disc gives the discontinuity of the function across a branch cut, then

$$\triangle(\mathrm{Disc}(F)) = (\mathrm{Disc} \otimes \mathrm{id}_{\triangle})\triangle(F). \tag{3.179}$$

We can reduce n-weight polylogarithm functions to the n-fold tensor product of weight-1 polylogarithms, so ordinary logarithms using the maximal iteration of the coproduct $\triangle$, showed previously in (3.172) and (3.173). With function $F \in \mathcal{H}$, the maximal iteration of the coproduct modulo $i\pi$ is of special importance and is called the *symbol* and is defined as

$$\mathcal{S}(F) = \triangle_{1,\ldots,1}(F) \ \mathrm{Mod} \ i\pi. \tag{3.180}$$

The union of all distinct letters (entries) of the symbol $\mathcal{S}(F)$ is called the *alphabet* of F. The set of functions applicable is not defined, but usually concrete applications are with polylogarithms, or some generalization thereof. Since all entries are weight-1 polylogs (logs) it is customary to express the entries of $\mathcal{S}(F)$ not with $\log a_1 \otimes \cdots \otimes \log a_n$ but with $a_1 \otimes \cdots \otimes a_n$. Importantly, for $\mathcal{A} \ni F, G$ we have that the symbol of the product of functions F, G maps to the shuffle multiplication of the symbols of F and G

$$\mathcal{S}(FG) = \mathcal{S}(F) \sqcup\!\sqcup\, \mathcal{S}(G). \tag{3.181}$$

The logarithm identity $\log(a_1 a_2) = \log(a_1) + \log(a_2)$ manifests itself at the symbol level as

$$\cdots \otimes (ab) \otimes \cdots = \cdots \otimes a \otimes \cdots + \cdots \otimes b \otimes \cdots \tag{3.182}$$

The logarithm identity $\log(a^n) = n\log(a)$ manifests itself at the symbol level as

$$\cdots \otimes a^n \otimes \cdots = n(\cdots \otimes a \otimes \cdots). \tag{3.183}$$

Since the definition eliminates terms with $i\pi$, if $\rho^n = 1$, for some n, then

$$\cdots \otimes \rho \otimes \cdots = 0. \tag{3.184}$$

We can write a general element in the symbol space as

$$S = \sum_{i_1,...,i_k} c_{i_1,...,i_k} a_{i_1} \otimes \cdots \otimes a_{i_k}. \tag{3.185}$$

To the symbol S there exists a corresponding function $F \in \mathcal{A}$ such that $\mathcal{S}(F) = S$, if and only if the symbol S is integrable, so if

$$\sum_{i_1,...,i_k} c_{i_1,...,i_k} \operatorname{dlog} a_{i_j} \wedge \operatorname{dlog} a_{i_{j+1}} a_{i_1} \otimes \cdots \otimes a_{i_{j-1}} \otimes a_{i_{j+2}} \otimes a_{i_k} = 0 \quad, \forall i \leq j \leq k-1. \tag{3.186}$$

Henceforth, by unifying with the discussion on iterated integrals in the beginning of this section it follows that the integrable words determine iterated integrals that are path-independent up to homotopically equivalent paths. These integrals can be viewed as functionals of the paths endpoint as illustrated in (3.134), and conversely, any iterated integral that is homotopically invariant is determined by the integrable word corresponding to it, in accordance with (3.186). In this circumstance of calculation of loop integrals, this is advantageous as polylogarithms appearing in integral calculations can lead to numerous identities that make it difficult to work with these iterated integrals and determine their uniqueness. Goncharov, Spradlin, Vergu and Volovich firstly introduced the symbol by using the differential equation for multiple polylogarithms recursively to arrive its definition [47]. By using integrable words instead, linear algebra can be used to verify equality and automatically account for functional relations where the symbol provides a mapping between iterated integrals and integrable words as presented below. Let now $f(z) = \int_0^z \alpha_{i_1} \circ \cdots \circ \alpha_{i_n}$ with $\alpha_i \in \Omega = \{\alpha_{i_j} = \operatorname{dlog} a_{i_j} |\ a_{i_j} \in \mathbb{C}\ rational\ functions\ \forall j\}$ and require $\int_0^z \alpha_{i_1} \circ \cdots \circ \alpha_{i_n}$ to satisfy the integrability condition $\alpha_{i_j} \wedge \alpha_{i_{j+1}} = 0$, $\forall j = 1, ..., n-1$, then we get the connection between the iterated integral and the symbol as

$$\mathcal{S}(f(z)) = \mathcal{S}\left(\int_0^z \alpha_{i_1} \circ \cdots \alpha_{i_n}\right) = \alpha_{i_1} \otimes \cdots \otimes \alpha_{i_n} \tag{3.187}$$

(which is actually equal to $\log \alpha_{i_1} \otimes \cdots \otimes \log \alpha_{i_n}$, but we adhere to the convention). In [47], the definition is extended to k-variable n-weight transcendental functions $F \in \mathcal{A}$, where if $F_n(x_1, ..., x_k)$ can be expressed as

$$dF_n = \sum_i F_{i,n-1} \operatorname{dlog}(R_i), \tag{3.188}$$

where R_i is a complex valued rational function of $(x_1, ..., x_k)$, then the symbol of $F \in \mathcal{A}$ is defined recursively as

$$\mathcal{S}(F_n) = \sum_i \mathcal{S}(F_{i,n-1}) \otimes R_i. \tag{3.189}$$

As a first example, let's examine $\mathrm{Li}_n(z)$. Since $\mathrm{Li}_n(z) = \int_0^z \mathrm{Li}_{n-1}(t) \operatorname{dlog}(t)$ and $\mathrm{Li}_1(z) = \int_0^z -\operatorname{dlog}(1-t)$ we have that the symbol is

$$\mathcal{S}(\mathrm{Li}_n) = -(1 - z \otimes z \cdots \otimes z). \tag{3.190}$$

Another very important example is the symbol of the Goncharov-polylogarithm, which is also determined using the recursion (3.189). In order to obtain an important relationship between the total derivative of an n-weight Goncharov-polylogarithm and $n-1$-weight Goncharov polylogarithms, let's take the total derivative of $G(a_1, a_2, z)$

$$dG(a_1, a_2, z) = \frac{\partial G}{\partial z}dz + \frac{\partial G}{\partial a_1}da_1 + \frac{\partial G}{\partial a_2}da_2 \tag{3.191}$$

$$\frac{\partial G}{\partial z} = \frac{\partial}{\partial z}\int_0^z \frac{dt}{t-a_1}G(a_2, t) = \frac{1}{z-a_1}G(a_2, z) \tag{3.192}$$

$$\begin{aligned}
\frac{\partial G}{\partial a_1} &= \int_0^z \frac{dt}{(t-a_1)^2}G(a_2, t) = -\int_0^z d\left(\frac{1}{t-a_1}\right)G(a_2, t) = \\
&= -\frac{G(a_2, t)}{t-a_1}\Big|_0^z + \int_0^z \frac{1}{t-a_1}dG(a_2, t) = -\frac{G(a_2, z)}{z-a_1} + \int_0^z \frac{1}{t-a_1}\frac{dt}{t-a_2} = \\
&= -\frac{G(a_2, z)}{z-a_1} + \frac{1}{a_1-a_2}\int_0^z dt\left(\frac{1}{t-a_1} - \frac{1}{t-a_2}\right) = -\frac{G(a_2, z)}{z-a_1} + \frac{1}{a_1-a_2}\log\left(\frac{t-a_1}{t-a_2}\right)\Big|_0^z = \\
&= -\frac{G(a_2, z)}{z-a_1} + \frac{1}{a_1-a_2}\left(\log\left(\frac{z-a_1}{z-a_2}\right) - \log\left(\frac{-a_1}{-a_2}\right)\right) = \\
&= -\frac{G(a_2, z)}{z-a_1} + \frac{1}{a_1-a_2}(G(a_1, z) - G(a_2, z))
\end{aligned}$$

$$\frac{\partial G}{\partial a_2} = \int_0^z \frac{dt}{t-a_1}\frac{\partial G(a_2, t)}{\partial a_2} = \int_0^z \frac{dt}{t-a_1}\left(\int_0^t \frac{du}{(u-a_2)^2}\right) = \int_0^z \frac{dt}{t-a_1}\left(-\frac{1}{a_2} - \frac{1}{t-a_2}\right) = \tag{3.193}$$

$$= -\frac{G(a_1, z)}{a_2} - \frac{1}{a_1-a_2}(G(a_1, z) - G(a_2, z)). \tag{3.194}$$

So

$$dG(a_1, a_2, z) = dz\frac{G(a_2, z)}{z-a_1} - da_1\frac{G(a_2, z)}{z-a_1} + da_1\frac{G(a_1, z)}{a_1-a_2} - da_1\frac{G(a_2, z)}{a_1-a_2} - \tag{3.195}$$

$$-da_2\frac{G(a_1, z)}{a_2} - da_2\frac{G(a_1, z)}{a_1-a_2} + da_2\frac{G(a_2, z)}{a_1-a_2} = \tag{3.196}$$

$$= \left(\frac{dz}{z-a_1} - \frac{da_1}{z-a_1} - \frac{da_1}{a_1-a_2} + \frac{da_2}{a_1-a_2}\right)G(a_2, z) + \left(\frac{da_1}{a_1-a_2} - \frac{da_2}{a_1-a_2} - \frac{da_2}{a_2}\right)G(a_1, z). \tag{3.197}$$

The structure of the result suggests that we can write the integration measures more compactly. By taking $a_0 = z$ and $a_3 = 0$ we can write

$$d\log\left(\frac{a_i - a_{i-1}}{a_i - a_{i+1}}\right) = \frac{a_i - a_{i+1}}{a_i - a_{i-1}} d\left(\frac{a_i - a_{i-1}}{a_i - a_{i+1}}\right) = \tag{3.198}$$

$$= \frac{a_i - a_{i+1}}{a_i - a_{i-1}} \frac{d(a_i - a_{i-1})(a_i - a_{i+1}) - d(a_i - a_{i+1})(a_i - a_{i-1})}{(a_i - a_{i+1})^2} = \frac{da_i - da_{i-1}}{a_i - a_{i-1}} - \frac{da_i - da_{i+1}}{a_i - a_{i+1}}. \tag{3.199}$$

For $i = 1$

$$d\log\left(\frac{a_1 - z}{a_1 - a_2}\right) = \frac{dz}{z - a_1} - \frac{da_1}{z - a_1} - \frac{da_2}{a_1 - a_2} + \frac{da_1}{a_1 - a_2}, \tag{3.200}$$

whereas for $i = 2$

$$\text{dlog}\left(\frac{a_2 - a_1}{a_2}\right) = \frac{da_1}{a_1 - a_2} - \frac{da_2}{a_1 - a_2} - \frac{da_2}{a_2}. \tag{3.201}$$

Thus, we see that the total derivative can be expressed as

$$dG^{(n=2)}(a_1, a_2, z) = \sum_{i=1}^{n=2} G^{(n=1)}(\mathbf{a}_i)\,\text{dlog}\left(\frac{a_i - a_{i-1}}{a_i - a_{i+1}}\right), \tag{3.202}$$

where the components of $\mathbf{a}_i$ are $\mathbf{a}_0 = z, \mathbf{a}_1 = a_1, \mathbf{a}_2 = a_2, \mathbf{a}_3 = 0$. The result also derived in [6], holds for arbitrary n weight Goncharov-polylogarithms

$$dG^{(n)}(\mathbf{a}) = \sum_{i=1}^{n} G^{(n-1)}(\mathbf{a}_i)\,\text{dlog}\left(\frac{a_i - a_{i-1}}{a_i - a_{i+1}}\right). \tag{3.203}$$

Then, the symbol of the Goncharov-polylogarithms is given according to (3.189) by

$$\mathcal{S}(G^{(n)}(\mathbf{a}) = \sum_{i=1}^{n} \mathcal{S}(G^{(n-1)}(\mathbf{a}_i)) \otimes \frac{a_i - a_{i-1}}{a_i - a_{i+1}}. \tag{3.204}$$

For example, for $n = 1$ weight Goncharov-polylogarithm we have that its symbol is

$$\mathcal{S}\left(G^{(1)}(a_1, z)\right) = \frac{a_1 - z}{a_1}. \tag{3.205}$$

Then, for $n = 2$ weight Goncharov-polylogarithm we have that its symbol is

$$\mathcal{S}\left(G^{(2)}(a_1, a_2, z)\right) = \sum_{i=1}^{2} \mathcal{S}\left(G^{(1)}(\mathbf{a}_i)\right) \otimes \frac{a_i - a_{i-1}}{a_i - a_{i+1}} = \frac{a_1 - z}{a_1} \otimes \frac{a_1 - z}{a_1 - a_2} + \frac{a_2 - z}{a_2} \otimes \frac{a_2 - a_1}{a_2}. \tag{3.206}$$

Since the algebra $\mathcal{A}$ is graded, the weight of the words is equal to their length.

It is often useful to perform integrations at the symbol level. Let $S = F(t) \otimes G(t)$ be symbol that is linearly reducible in the variable t, i.e. a symbol having entries that are products of some linear functions of t, with last entry being $G(t)$. We define the symbol integral as

$$\int_a^b \mathrm{dlog}(t+c) S = \int_a^b \mathrm{dlog}(t+c) \left(F(t) \otimes G(t)\right). \tag{3.207}$$

Given that the last entry has the structure $G(t) = t + d$, the total derivative of the symbol integral (3.207) receives contributions from the boundaries of integration

$$\mathrm{dlog}(t+c) \left(F(t) \otimes G(t)\right) \Big|_{t=a}^{t=b} \implies \left(F(t) \otimes G(t) \otimes (t+c)\right) \Big|_{t=a}^{t=b} \tag{3.208}$$

and receives contributions from the last entry

$$\left(\int_a^b \mathrm{dlog} \left(\frac{t+c}{t+d}\right) F(t) \right) \mathrm{dlog}(c-d) \implies \left(\int_a^b \mathrm{dlog} \left(\frac{t+c}{t+d}\right) F(t) \right) \otimes (c-d), \tag{3.209}$$

where we see that the integral receives contribution from the branch cut $(c-d)$. Thus we have that

$$d\left(\int_a^b \mathrm{dlog}(t+c) \left(F(t) \otimes G(t)\right) \right) = \left(F(t) \otimes G(t) \otimes (t+c)\right) \Big|_{t=a}^{t=b} + \left(\int_a^b \mathrm{dlog} \left(\frac{t+c}{t+d}\right) F(t) \right) \otimes (c-d). \tag{3.210}$$

The symbol integration method is presented and the statements are proved in [48].

Chapter 4

Planar 2-loop Integrals in $\mathcal{N} = 4$ sYM theory

We now advance the discussion from 1-loop to 2-loop (multi-loop) integrals in planar $\mathcal{N} = 4$ sYM. We will moslty sketch the main differences compared to the 1-loop case and go into further details where it is needed. The mathematical formalisms described in **Chapter 2** generalize to 2-loop and are used extensively throughout. A reduction technique from 2-loop to 1-loop is presented, which will be crucial in the I_{dp} calculation briefly described in the **Introduction**.

4.1 Structure of 2-loop integrals in planar $\mathcal{N} = 4$ sYM theory

In general, 2-loop integrals are considerably more difficult than their 1-loop counterparts. They are required to produce higher-precision predictions for the theory as they correspond to $NNLO$ corrections in the perturbative expansion. Their computation involves integrations over 2-loop momenta l_1, l_2 and Feynman-parameters. However, as [23] and [49] describe, the many symmetries of planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory have facilitated the analytic multi-loop computation program tremendously in the past decades.

Since we used the 1-loop zero-mass box integral $I_4^{(L=1)}$ as our prototypical example for the 1-loop review, here we will describe the main characteristics of 2-loop integrals using the 2-loop zero-mass box integral $I_4^{(L=2)}$. The integral has the following representation in embedding space $\mathcal{M}^4$

$$\begin{aligned}\mathcal{I}_4^{(2)} = &\int \frac{1}{2}[d^6L_1][d^6L_2]\left(\frac{(X_1,X_3)^2(X_2,X_4)}{(X_1,L_1)(X_3,L_1)(X_4,L_1)(X_1,L_2)(X_2,L_2)(X_3,L_2)(L_1,L_2)}\right)+\\ &+\int \frac{1}{2}[d^6L_1][d^6L_2]\left(\frac{(X_1,X_3)^2(X_2,X_4)}{(X_1,L_2)(X_3,L_2)(X_4,L_2)(X_1,L_1)(X_2,L_1)(X_3,L_1)(L_1,L_2)}\right),\end{aligned} \tag{4.1}$$

where the factor $\frac{1}{2}$ reflects the complete symmetrization of the integral required to fix the ambiguity in the labelling of internal points. Given an L-loop integral, we always symmetrize it with a factor of $\frac{1}{L!}$ to eliminate such ambiguities. The normalization ensures the integral purity and the DCI. We can map this loop integral to momentum-twistor space $\mathbb{CP}^3$ by mapping the measures to the twistor variable lines as $L_1 \longleftrightarrow (A_1B_1)$ and $L_2 \longleftrightarrow (A_2B_2)$ and

$$I_4^{(L=2)} = \int_{(A_1B_1,A_2,B_2)} \frac{\langle 1234\rangle^3}{\langle A_1B_141\rangle\langle A_1B_112\rangle\langle A_1B_123\rangle\langle A_2B_223\rangle\langle A_2B_234\rangle\langle A_2B_241\rangle\langle A_1B_1A_2B_2\rangle}, \tag{4.2}$$

where we include the symmetrization factor in $\int_{AB} \longleftrightarrow \int \frac{1}{L!}\frac{d^4Z_Ad^4Z_B}{\mathrm{Vol}(GL(2))}$. The factor of (L_1, L_2) can be understood by going back briefly to 1-loop for the moment, more precisely to the chiral and pure pentagon integral represented in Figure 3.1, and given by

$$\mathcal{A}_{MHV}^{(L=1)} = \sum_{i<j<i}\left\{\int_{AB} \frac{\langle AB(i-1ii+1)\cap(j-1jj+1)\rangle\langle Xij\rangle}{\langle ABi-1i\rangle\langle ABii+1\rangle\langle ABj-1j\rangle\langle ABjj+1\rangle\langle ABX\rangle}\right\}. \tag{4.3}$$

The amplitude $\mathcal{A}_{MHV}^{(L=1)}$ has a numerator given by $\langle AB(i-1ii+1)\cap(j-1jj+1)\rangle \in \mathbb{CP}^2 \subset \mathbb{CP}^3$ to eliminate the non-chiral leading singularity originating from one of its Schubert-problems, whereas the new factor proportional to the bitwistor X ensures proper Wigner little group weights. The legs next to the bitwistor line X do not affect the result and the sum over all terms including boundary terms yields the leading singularity of the colored graphs, i.e. reproduces the complete 1-loop MHV amplitude. Now, "gluing" another pentagon to the pentagon described yields a 2-loop problem and the bitwistor line is the denominator is replaced with another momentum loop momentum, which in momentum-twistor place is a momentum-twistor line variable. In the numerator X is replaced by the "good" Schubert-solution of the other pentagon, namely (kl). All in all, this procedure gives back exactly an amplitude whose *leading singularities match that of the colored diagrams of the 2-loop MHV amplitude*. The general double-pentagon integral $I_n^{(L=2)}$, used as the basis of the 2-loop MHV amplitude is given by the Feynman diagram in Figure 4.1 from [24].

$$\mathcal{A}_{\mathrm{MHV}}^{2-\mathrm{loop}} = \frac{1}{2}\sum_{i<j<k<l<i}$$

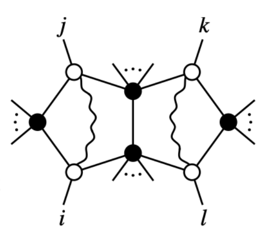
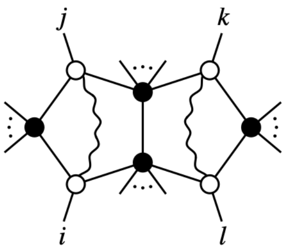


Figure 4.1: 2-loop MHV amplitude.

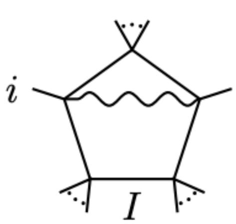


$$= \int_{(A_1B_1,A_2B_2)} \frac{\langle ijkl\rangle\langle A_1B_1(i-1ii+1)\cap(j-1jj+1)\rangle\langle A_2B_2(k-1kk+1)\cap(l-1ll+1)\rangle}{\langle\langle A_1B_1i\rangle\rangle\langle\langle A_1B_1j\rangle\rangle\langle\langle A_2B_2k\rangle\rangle\langle\langle A_2B_2l\rangle\rangle\langle A_1B_1A_2B_2\rangle}.$$

We use the notation $\langle\langle A_1B_1i\rangle\rangle = \langle A_1B_1i-1i\rangle\langle A_1B_1ii+1\rangle$ and we remark that the value of the integral depends exclusively on the external legs $(ijkl)$, which correspond to fermionic insertions to the dual space Wilson-loop. These are exploited in the next section for reducing the double integral to a single loop problem, with some external legs depending on continuous variables that need to be integrated over. The calculation of integral for $n=10$ is presented in **Chapter 5**.

4.2 The Star-Triangle Identity

The proof presented here follows [6]. In circumstances involving 2-loop integrals with one, or both loops being chiral pentagons, we can use an identity to reduce the 2-loop integral to a 1-loop problem. The reduction relies on massless fermionic insertions to the dual Wilson-loop. This reduction procedure allows also for a transparent computation of the integral, not relying on the GRT formalism. Let's start with the example chiral pentagon briefly described earlier, that depend exclusively on the fermionic insertions i,j and the general bitwistor line I. We define the following line integrals in the momentum-twistor space over the auxiliary



$$= \int_{AB} \frac{\langle AB(i-1ii+1)\cap(j-1jj+1)\rangle\langle Iij\rangle}{\langle ABi-1i\rangle\langle ABii+1\rangle\langle ABj-1j\rangle\langle ABjj+1\rangle\langle ABI\rangle}$$

variables defined as $X_1 = Z_{i-1}+\tau_1 Z_{i+1}$ and $X_2 = Z_{j-1}+\tau_2 Z_{j+1}$, which interpolate between the massles fermionic legs i and j. Using this, we can express propagator terms such as $1/\langle\langle ABi\rangle\rangle$ as

$$\frac{1}{\langle\langle ABi\rangle\rangle} = \int_0^\infty \frac{d\tau_1}{\langle ABiX_1\rangle^2}. \tag{4.4}$$

Thus, the chiral pentagon can be re-expressed using the auxiliary twistor line variables X_1, X_2 as

$$\int_{AB} \frac{\langle AB(i-1ii+1)\cap(j-1jj+1)\rangle\langle Iij\rangle}{\langle ABi-1i\rangle\langle ABii+1\rangle\langle ABj-1j\rangle\langle ABjj+1\rangle\langle ABI\rangle} = \int_{AB}\int_0^\infty d\tau_1 d\tau_2 \frac{\langle AB(i-1ii+1)\cap(j-1jj+1)\rangle\langle Iij\rangle}{\langle ABiX_1\rangle^2\langle ABjX_2\rangle^2\langle ABI\rangle}. \tag{4.5}$$

Translating the integral over the loop momentum to the embedding space representation yields the integral

$$\int_{AB} \frac{\langle AB(i-1ii+1)\cap(j-1jj+1)\rangle\langle Iij\rangle}{\langle ABiX_1\rangle^2\langle ABjX_2\rangle^2\langle ABI\rangle} \to \int [d^6L]\frac{(L,\overline{Z})(Z,W)}{(L,\mathcal{X}_1)^2(L,\mathcal{X}_2)^2(L,W)}, \tag{4.6}$$

where $(I)\sim W$ and the solutions to the Schubert problems $Z\sim(ij)$ and $\overline{Z}\sim(i-1ii+1)\cap(j-1jj+1)$ are null-separated from $\mathcal{X}_1,\mathcal{X}_2\in\mathcal{M}^4$, therefore $(\overline{Z},\mathcal{X}_1)=(\overline{Z},\mathcal{X}_2)=0$.

By Feynman/Schwinger parametrizing the integral over the loop momentum we get the expression

$$\int [d^6L]\frac{(\overline{Z},L)(Z,W)}{(\mathcal{X}_1,L)^2(\mathcal{X}_2,L)^2(W,L)} = \int [d^6L]\int_0^\infty d\alpha_1 d\alpha_2 \frac{3\alpha_1\alpha_2(\overline{Z},L)(Z,W)}{[L(\alpha_1\mathcal{X}_1+\alpha_1\mathcal{X}_2+W)]^5}, \tag{4.7}$$

where we define the embedding space quantities dual to the momentum-twistor space quantities as $\mathcal{W}=\alpha_1\mathcal{X}_1+\alpha_1\mathcal{X}_2+W$, such that $(\overline{Z},\mathcal{W})=(\overline{Z},W)$ by the null-separation condition $(\overline{Z},\mathcal{X}_1)=(\overline{Z},\mathcal{X}_2)=0$. Thus

$$\begin{aligned}\int_0^\infty d\alpha_1 d\alpha_2 \int [d^6L]\frac{\alpha_1\alpha_2(\overline{Z},L)(Z,W)}{(L,\mathcal{W})^5} &= \int_0^\infty d\alpha_1 d\alpha_2\alpha_1\alpha_2\int [d^6L]\partial_{\mathcal{W}}\frac{\overline{Z}(Z,W)}{(L,\mathcal{W})^4} = \\ = \int_0^\infty \frac{1}{2}d\alpha_1 d\alpha_2\alpha_1\alpha_2\partial_{\mathcal{W}}\frac{\overline{Z}(Z,W)}{(\mathcal{W},\mathcal{W})^2} &= \frac{(Z,W)(W,\overline{Z})}{(\mathcal{X}_1,\mathcal{X}_2)(\mathcal{X}_1,W)(\mathcal{X}_2,W)}.\end{aligned} \tag{4.8}$$

In Figure 4.2 from [7], the star-triangle identity is represnetend diagrammatically.
This beautiful identity is expressed diagrammatically as follows. The first diagram is the

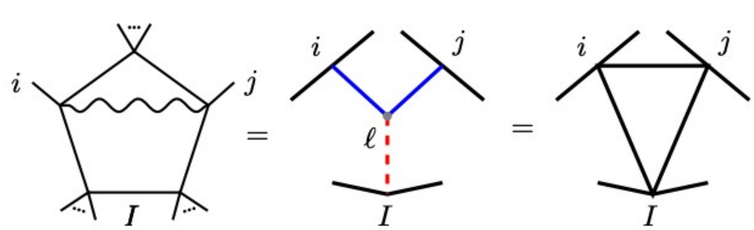


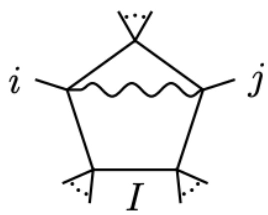
Figure 4.2: Diagramatic representation of the Star-Triangle identity

colorless Feynman-diagram, the second is the (bosonic) Wilson-loop representation and the last is the integrated Wilson-loop again written as a Feynman diagram. The identity can be used to integrate out the chiral pentagon in cases where one of the loops is a chiral pentagon, or both loops are chiral pentagons.

Converting back to the momentum-twistor representation, the integration over the variable twistor line $Z_A \wedge Z_B$ gives

$$\int_{AB} \frac{\langle AB(i-1ii+1)\cap(j-1jj+1)\rangle\langle Iij\rangle}{\langle ABiX_1\rangle^2\langle ABjX_2\rangle^2\langle ABI\rangle} = \frac{\langle I(i-1ii+1)\cap(j-1jj+1)\rangle\langle Iij\rangle}{\langle iX_1jX_2\rangle\langle iX_1I\rangle\langle jX_2I\rangle}. \tag{4.9}$$

So that the complete chiral pentagon integral can be written as the integral over the massless insertion point i and j as



$$= \int_0^\infty d\tau_1 d\tau_2 \frac{\langle I(i-1ii+1)\cap(j-1jj+1)\rangle\langle Iij\rangle}{\langle iX_1jX_2\rangle\langle iX_1I\rangle\langle jX_2I\rangle}$$

with the auxiliary variables $X_1(\tau_1) = Z_{i-1} + \tau_1 Z_{i+1}$ and $X_2(\tau_2) = Z_{j-1} + \tau_2 Z_{j+1}$. The partial fractioning with respect to τ_1, then τ_2 together with the use of $\mathrm{dlog}(a + \tau b) = \frac{1}{b}\frac{d\tau}{a+\tau b}$ gives

$$\begin{aligned} &\int_0^\infty d\tau_1 d\tau_2 \frac{\langle I(i-1ii+1)\cap(j-1jj+1)\rangle\langle Iij\rangle}{\langle iX_1jX_2\rangle\langle iX_1I\rangle\langle jX_2I\rangle} = \\ &= \int_{\mathbb{R}\geq 0} \mathrm{dlog}\, \frac{\langle jX_2I\rangle}{\langle jX_2iI\cap(i-1ii+1)\rangle}\, \mathrm{dlog}\, \frac{\langle iX_1jX_2\rangle}{\langle iX_1I\rangle}. \end{aligned} \tag{4.10}$$

By integrating first with respect to τ_1, then with respect to τ_2 the integral evaluates to logs and dilogs of DCI cross ratios, as presented in [6]

$$\begin{aligned} \int_0^\infty d\tau_1 d\tau_2 \frac{\langle I(i-1ii+1)\cap(j-1jj+1)\rangle\langle Iij\rangle}{\langle iX_1jX_2\rangle\langle iX_1I\rangle\langle jX_2I\rangle} &= \log(u_1)\log(u_2) + \mathrm{Li}_2(1-u_1) + \\ &+ \mathrm{Li}_2(1-u_2) + \mathrm{Li}_2(1-u_3) - \mathrm{Li}_2(1-u_1u_3) - \mathrm{Li}_2(1-u_2u_3) \end{aligned} \tag{4.11}$$

with cross-ratios given by

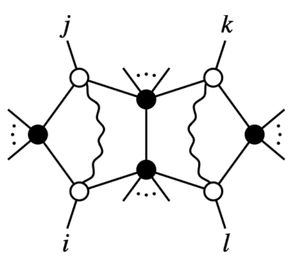
$$u_1 = \frac{\langle i-1iI\rangle\langle jj+1ii+1\rangle}{\langle i-1ijj+1\rangle\langle Iii+1\rangle},\ u_2 = \frac{\langle jj+1I\rangle\langle ii-1j-1j\rangle}{\langle jj+1ii-1\rangle\langle Ij-1j\rangle},\ u_3 = \frac{\langle i-1ijj+1\rangle\langle ij-1jii+1\rangle}{\langle i-1ij-1j\rangle\langle jj+1ii+1\rangle}. \tag{4.12}$$

Chapter 5

The computation of I_{dp}

5.1 Integration of loop momenta in the I_{dp} integral

We now turn to the computation of the 2-loop 10-point integral referred to as $I_{10}^{(L=2)}$ by using the techniques outlined in the previous chapters. The 10-point case corresponds to choosing $(ijkl) = (1469)$ and corresponds to the calculation of NNLO contribution of 10 particles scattering in planar $\mathcal{N} = 4$ sYM. The general case is $n \geq 12$ and was first calculated in [7]. We denote the $\mathbb{CP}^2$ planes as $\bar{i} = (i-1ii+1)$ and the auxilliary variables have the form $X_1(\tau_1) = Z_{10} + \tau_1 Z_2$ and $X_2(\tau_2) = Z_3 + \tau_2 Z_5$. So to begin with we have the integral



$$= I_{dp} = \int_{(A_1B_1,A_2B_2)} \frac{\langle 1469\rangle\langle A_1B_1\bar{1}\cap\bar{4}\rangle\langle A_2B_2\bar{6}\cap\bar{9}\rangle}{\langle\langle A_1B_11\rangle\rangle\langle\langle A_1B_14\rangle\rangle\langle\langle A_2B_26\rangle\rangle\langle\langle A_2B_29\rangle\rangle\langle A_1B_1A_2B_2\rangle}.$$

By using $\langle\langle ABi\rangle\rangle = \langle ABi-1i\rangle\langle ABii+1\rangle$, the auxiliary variables X_1 and X_2 together with $\frac{1}{\langle\langle ABi\rangle\rangle} = \int_0^\infty \frac{d\tau}{\langle ABiX(\tau)\rangle^2}$ we can rewrite the expression to

$$\begin{aligned} I_{dp} &= \int_{(A_1B_1,A_2B_2)} \frac{\langle 1469\rangle\langle A_1B_1\bar{1}\cap\bar{4}\rangle\langle A_2B_2\bar{6}\cap\bar{9}\rangle}{\langle\langle A_1B_11\rangle\rangle\langle\langle A_1B_14\rangle\rangle\langle\langle A_2B_26\rangle\rangle\langle\langle A_2B_29\rangle\rangle\langle A_1B_1A_2B_2\rangle} \\ &= \int_0^\infty \int_{(A_1B_1,A_2B_2)} \frac{\langle 1469\rangle\langle A_1B_1\bar{1}\cap\bar{4}\rangle\langle A_2B_2\bar{6}\cap\bar{9}\rangle}{\langle A_1B_11X_1\rangle^2\langle A_1B_14X_2\rangle^2\langle\langle A_2B_26\rangle\rangle\langle\langle A_2B_29\rangle\rangle\langle A_1B_1A_2B_2\rangle}. \end{aligned} \tag{5.1}$$

By using the Star-Triangle identity (4.9), we can perform one of the integrations on the loop momentum, which gives

$$\begin{aligned} I_{dp} &= \int_0^\infty \int_{(A_1B_1,A_2B_2)} \frac{\langle 1469\rangle\langle A_1B_1\bar{1}\cap\bar{4}\rangle\langle A_2B_2\bar{6}\cap\bar{9}\rangle}{\langle A_1B_11X_1\rangle^2\langle A_1B_14X_2\rangle^2\langle\langle A_2B_26\rangle\rangle\langle\langle A_2B_29\rangle\rangle\langle A_1B_1A_2B_2\rangle} \\ &= \int_0^\infty \frac{\langle 1469\rangle}{\langle 1X_14X_2\rangle}\int_{(A_2B_2)} \frac{\langle A_2B_2\bar{1}\cap\bar{4}\rangle\langle A_2B_2\bar{6}\cap\bar{9}\rangle}{\langle A_2B_21X_1\rangle\langle A_2B_24X_2\rangle\langle\langle A_2B_26\rangle\rangle\langle\langle A_2B_29\rangle\rangle}. \end{aligned} \tag{5.2}$$

The remaining integral over the momentum-twistor line (A_2B_2) is the 1-loop hexagon with deformed legs $X_1(\tau_1)$ and $X_2(\tau_2)$ instead of leg 2 and leg 3 respectively

$$I_{hex} = \frac{\langle A_2B_2\bar{1}\cap\bar{4}\rangle\langle A_2B_2\bar{6}\cap\bar{9}\rangle}{\langle A_2B_21X_1\rangle\langle A_2B_24X_2\rangle\langle\langle A_2B_26\rangle\rangle\langle\langle A_2B_29\rangle\rangle}. \tag{5.3}$$

At this point, the Van-Neerven-Vermaseren reduction of the hexagon is performed by evaluating the appropriate determinants for the box expansion coefficients. This yields box expansion coefficients of the form $c_i(\tau_1,\tau_2)$. The box expansion yields a linear combination of 15 boxes. These boxes appearing in the expansion are:

Four boxes of the type *4-mass* box

$$\begin{aligned} I_{4m} &= \int_{(AB)} \frac{-\langle a-1,a,c-1,c\rangle\langle b-1,b,d-1,d\rangle\Delta}{\langle ABa-1a\rangle\langle ABb-1b\rangle\langle ABc-1c\rangle\langle ABd-1d\rangle} = \\ &= -\operatorname{Li}_2(z) + \operatorname{Li}_2(\bar{z}) - \frac{1}{2}\log(z\bar{z})\log\left(\frac{1-z}{1-\bar{z}}\right), \end{aligned} \tag{5.4}$$

$$u = \frac{\langle a-1,a,b-1,b\rangle\langle c-1,c,d-1,d\rangle}{\langle a-1,a,c-1,c\rangle\langle b-1,b,d-1,d\rangle}, \qquad v = \frac{\langle b-1,b,c-1,c\rangle\langle a-1,a,d-1,d\rangle}{\langle a-1,a,c-1,c\rangle\langle b-1,b,d-1,d\rangle}, \tag{5.5}$$

$$z = \frac{1}{2}(1+u-v+\Delta), \tag{5.6}$$

$$\bar{z} = \frac{1}{2}(1+u-v-\Delta), \tag{5.7}$$

$$\Delta = \sqrt{(1-u-v)^2-4uv}. \tag{5.8}$$

Ten divergent boxes of the type *3-mass* box, with divergence regularized in ε. Here we use the usual DCI regularazation to regulate the IR divergence appearing in the calculation. This regulator method is described in detail in [50].

$$-I_{3m}^{\varepsilon} = \operatorname{Li}_2(1-v) + \frac{1}{2}\log(u')\log(v) + \frac{1}{2}\log(\epsilon)\log(v) + \mathcal{O}(\epsilon) \tag{5.9}$$

with

$$u' = \frac{\langle a-2, a-1, b-1, b\rangle\langle a-1, a, b, b+1\rangle\langle a-1, a, d-1, d\rangle}{\langle a-2, a-1, b, b+1\rangle\langle a-1, a, c-1, c\rangle\langle b-1, b, d-1, d\rangle}, \tag{5.10}$$

$$v = \frac{\langle b-1, b, c-1, c\rangle\langle a-1, a, d-1, d\rangle}{\langle a-1, a, c-1, c\rangle\langle b-1, b, d-1, d\rangle}. \tag{5.11}$$

One divergent box of the type *2-mass-easy* box, with divergence again regularized using the DCI regulator in ε

$$-I^{\varepsilon}_{2me} = \mathrm{Li}_2(1-v) + \log(u')\log(v) + \log(\varepsilon)\log(v) + \mathcal{O}(\varepsilon) \tag{5.12}$$

with

$$u' = \frac{\langle a-2, a-1, b-1, b\rangle\langle a-1, a, b, b+1\rangle\langle c-2, c-1, d-1, d\rangle\langle c-1, c, d, d+1\rangle}{\langle a-2, a-1, b, b+1\rangle\langle c-2, c-1, d, d+1\rangle\langle a-1, a, c-1, c\rangle\langle b-1, b, d-1, d\rangle}, \tag{5.13}$$

$$v = \frac{\langle b-1, b, c-1, c\rangle\langle a-1, a, d-1, d\rangle}{\langle a-1, a, c-1, c\rangle\langle b-1, b, d-1, d\rangle}. \tag{5.14}$$

The reference for the box functions introduced here is [50]. The divergent boxes can be obtained from the *4-mass* box functions as collinear limits of the massive corners. However, massless legs make the integral (5.4) divergent, making the need for a regularization scheme necessary to regulate the logarithmically divergent part. Although 11 of the 15 boxes are divergent, the *divergences cancel* in the linear combination corresponding to the Van-Neerven-Versamseren box expansion of the hexagon described in **Chapter 3.4**. For numeric evaluation of momentum-twistor 4-brackets we use the momentum-twistor support package in [51]. Thus, until this stage we have the finite linear combination in the form

$$\begin{aligned}
I_{dp} &= \int_0^\infty \frac{\langle 1469\rangle}{\langle 1X_1 4X_2\rangle} \int_{(A_2B_2)} \frac{\langle A_2B_2\bar{1}\cap\bar{4}\rangle\langle A_2B_2\bar{6}\cap\bar{9}\rangle}{\langle A_2B_2 1X_1\rangle\langle A_2B_2 4X_2\rangle\langle\langle A_2B_2 6\rangle\rangle\langle\langle A_2B_2 9\rangle\rangle} \\
&= \int_0^\infty \frac{\langle 1469\rangle}{\langle 1X_1 4X_2\rangle}(c_1(\tau_1,\tau_2)I_{4679} + c_2(\tau_1,\tau_2)I_{2679} + c_3(\tau_1,\tau_2)I_{67910} + c_4(\tau_1,\tau_2)I_{2479} + \\
&+ c_5(\tau_1,\tau_2)I_{47910} + c_6(\tau_1,\tau_2)I_{2790} + c_7(\tau_1,\tau_2)I_{2469} + c_8(\tau_1,\tau_2)I_{46910} + c_9(\tau_1,\tau_2)I_{26910} + \\
&+ c_{10}(\tau_1,\tau_2)I_{24910} + c_{11}(\tau_1,\tau_2)I_{2467}c_{12}(\tau_1,\tau_2)I_{46710} + c_{13}(\tau_1,\tau_2)I_{26710} + c_{14}(\tau_1,\tau_2)I_{24710} + \\
&+ c_{15}(\tau_1,\tau_2)I_{24610}),
\end{aligned} \tag{5.15}$$

where the external leg structure determines I_{abcd} as implied from Figure 5.1 from [52]. Following the box expansion, the τ_1 and τ_2 dependent coefficients need to be integrated. This is done at the symbol level, as presented in **Chapter 3.5**.

5.2 Integration of I_{dp} at the symbol level

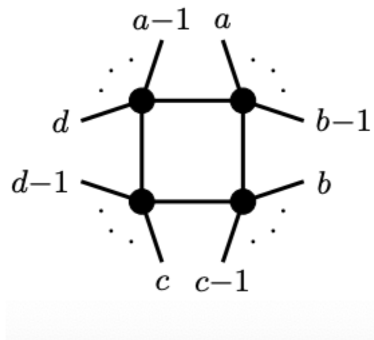


Figure 5.1: The leg structure of the box integral defines its label; the box I_{abcd} has the structure depicted in the figure.

The integrations over the auxilliary variables are performed at the symbol level. The weight of the symbol reflects the number of iterated integrations; for the 4-mass function the weight is 2 as two integrations have been performed over $(A_1 B_1)$ and $(A_2 B_2)$ respectively. For the 4-mass box $I_{4m} = I_{abcd}$ in (5.4), the symbol is given by

$$\mathcal{S}(I_{abcd}) = \frac{1}{2}\left(v \otimes \frac{z}{\bar{z}} + u \otimes \frac{1-\bar{z}}{1-z}\right). \tag{5.16}$$

In order to perform the integrations over the parameters τ_1, τ_2, the coefficients $c_i(\tau_1, \tau_2)$ have to be expressed as 2-fold dlog forms, or a linear combination thereof, of the parameters τ_1, τ_2. This can be done by solving the corresponding Schubert-problems and using the GRT. As a consistency check, the obtained dlogs have to match the expansion coefficients for concrete values of (τ_1, τ_2). Having the correct dlog forms then allows for direct integration of the *3-mass* and *2-mass-easy* coefficients with boxes using the functions' package provided in [53].

For the *4-mass* box the expressions for the coefficients also involve square-roots of some letters in the integrand, which makes the rationalization procedure necessary, as in (3.187) the entries need to be complex valued *rational* functions. The coefficients requiring rationalization are c_4, c_7, c_{14}, c_{15} corresponding to the 4-mass boxes I_{2479}, $I_{2469}, I_{24710}, I_{24610}$. In the following part of the chapter we label $\tau_1 = \tau$, until not noted otherwise. We get rid of the square root by doing a variable change of the integration variable $\tau \to t$, such that the integrand becomes square root free in terms of the new variable t and square roots are only present in the integration domain as the integration measure becomes $\int_0^\infty d\tau = \int_z^\infty dt \frac{d\tau}{dt}$. So we look for a rational function $\tau(t)$ for which Δ^2 is a perfect square in t, i.e. for which Δ is square root free in t. So we look for $\tau(t)$ according to

$$\begin{gathered} u(\tau), v(\tau) \text{ rational in } \tau \to \Delta(\tau) \text{ algebraic in } \tau \to z(\tau), \bar{z}(\tau) \text{ algebraic in } \tau, \\ \tau(t) \text{ rational in } t \to \Delta(\tau(t)) = \Delta(t) \text{ rational in } t \to z(t), \bar{z}(t) \text{ rational in } t. \end{gathered} \tag{5.17}$$

As an example, we consider the rational parametrization of the quadratic curve described by

$$y^2 = x^2 + 2ax + b. \tag{5.18}$$

First we need a rational point on this quadratic, so a point (x_0, y_0) for which $x_0, y_0 \in \mathbb{Q}(a, b)$. Then, using this rational point we parametrize points on the quadratic curve through a set of lines passing through the rational point (x_0, y_0). The point at (x_0, y_0) is kept fixed, whereas the other point on the locus moves according to the parameter t. Different rational points will yield different expressions for $x(t), y(t)$. With the rational point choice $(x_0, y_0) = (-\frac{b}{2a}, \frac{b}{2a})$, we then get by using $y - \frac{b}{2a} = t(x + \frac{b}{2a})$ that $x(t)$ is given by $(a \neq 0)$

$$x(t) = \frac{4a^2 - b - 2bt - bt^2}{2a(t^2 - 1)}. \tag{5.19}$$

Another suitable parametrization of the same quadratic curve is

$$x(t) = \frac{2(a + bt)}{bt^2 - 1}. \tag{5.20}$$

Further details on rationalization are found in the paper [54]. Let's now turn to the case of the rational parametrization of the 4-mass box functions appearing in the box expansion of the deformed leg hexagon (5.3). The square-root is in the term Δ, so as a first step let's re-express it in a form that will facilitate the rationalization procedure. By factoring out a perfect squared denominator $d(\tau)^2$ from Δ, we obtain the following expression

$$\Delta^2 = (1 - u(\tau) - v(\tau))^2 - 4u(\tau)v(\tau) = \frac{1}{d(\tau)^2}\left[(d(\tau) - b(\tau) - c(\tau))^2 - 4b(\tau)c(\tau)\right]. \tag{5.21}$$

In this way, we get a quadratic polynomial in τ, so the rationalization is similar to that of the quadratic curve in (5.18). The part of Δ we need to rationalize is therefore

$$\Gamma^2 = (d(\tau) - b(\tau) - c(\tau))^2 - 4b(\tau)c(\tau). \tag{5.22}$$

The rational point τ_0 rationalizing Δ also rationalizes Γ, as both become perfect squares with either $u(\tau_0) = \frac{b(\tau_0)}{d(\tau_0)} = 0$, or $v(\tau_0) = \frac{c(\tau_0)}{d(\tau_0)} = 0$ for u, v given in (5.5). Then, as $\Gamma(\tau_0), \Delta(\tau_0) \in \mathbb{Q}$ we therefore have that $z(\tau_0), \overline{z}(\tau_0) \in \mathbb{Q}$ for $z, \overline{z}$ in (5.6) and (5.7) respectively. With the fixed rational point (τ_0, Γ_0) on the locus we can then write the parametrization of Γ in terms of t as: $\Gamma = t(\tau - \tau_0) + \Gamma_0$. This will provide a parametrization $\tau(t)$ that is absent of square-roots. The rational point used for rationalization is kept fixed and the other moves on the locus of points of the quadratic according to t. Let's consider the case with $u(\tau_0) = 0 \implies \Delta(\tau_0) = 1 - v(\tau_0) = \Delta_0$. In this way we obtain a quadratic equation of τ in terms of t

$$(t(\tau - \tau_0) + \Gamma_0)^2 = (d(\tau) - b(\tau) - c(\tau))^2 - 4b(\tau)c(\tau). \tag{5.23}$$

We choose the solution dependent on t, as this will allow us to parametrize the moving point in terms of the new variable t. We obtain a parametrizing function of the form

$$\tau(t) = \frac{(t + a_1)(ta_2 + a_3) + a_4}{a_5 - t^2 a_6}, \tag{5.24}$$

where $a_1, a_2, a_3, a_4, a_5, a_6$ are constant composed of momentum-twistor 4-brackets. By having this parametrization, we can express Δ and z rationally in terms of t as

$$\Delta(\tau(t)) = \Delta(t) = \frac{1}{d(t)}\Gamma(t) \implies z(t) = \frac{(1 + u(t) - v(t) + \Delta(t))}{2}. \tag{5.25}$$

Let's consider a specific example, namely the rationalization of the coefficient c_7 corresponding to the 4-mass box I_{2469}. We label the rational points giving $u(\tau_u) = 0$, or $v(\tau_v) = 0$ as τ_u and τ_v. In the case of I_{2469} these rational points are

$$\tau_u = -\frac{\langle 11034\rangle}{\langle 1234\rangle}, \quad \tau_v = -\frac{\langle 11089\rangle}{\langle 1289\rangle}. \tag{5.26}$$

The Γ_0 rational point corresponding to the τ_u point is

$$\Gamma_0 = \Gamma(\tau_0) = \frac{-\langle 1234\rangle(\langle 11089\rangle\langle 3456\rangle + \langle 11056\rangle\langle 3489\rangle) + \langle 11034\rangle(\langle 1289\rangle\langle 3456\rangle - \langle 1256\rangle\langle 3489\rangle)}{\langle 1234\rangle}. \tag{5.27}$$

Converting the coefficients to the appropriate dlog forms in the new variable t is required as in the case of the rational part of the symbol. Furthermore, we use the rationalizing function (5.24) to account for integration measure in $\int_0^\infty d\tau = \int_z^\infty dt \frac{d\tau}{dt}$. The variable z enters as in the expression in the integration boundary; the algebraic part of the symbol contains $z, \overline{z}$ explicitly and they appear in the resulting symbol alphabet. Then, we can use the usual symbol integration method to obtain the algebraic part of the result. This procedure is repeated for coefficients c_3, c_{14} and c_{15} corresponding to 4-mass boxes I_{2479}, I_{24710} and I_{24610} respectively.

The $\tau_1(t)$ integration gives *algebraic* weight-3 symbols of the form

$$\mathcal{S}(I_{abcd}) \otimes \left(\frac{\frac{\langle x_a x_b\rangle\langle x_d 46\rangle}{\langle x_d x_b\rangle\langle x_a 46\rangle} - z_{abcd}}{\frac{\langle x_a x_b\rangle\langle x_d 46\rangle}{\langle x_d x_b\rangle\langle x_a 46\rangle} - \overline{z}_{abcd}}\right) = \frac{1}{2}\left(v \otimes \frac{\overline{z}}{z} - u \otimes \frac{1-z}{1-\overline{z}}\right) \otimes \left(\frac{\frac{\langle x_a x_b\rangle\langle x_d 46\rangle}{\langle x_d x_b\rangle\langle x_a 46\rangle} - z_{abcd}}{\frac{\langle x_a x_b\rangle\langle x_d 46\rangle}{\langle x_d x_b\rangle\langle x_a 46\rangle} - \overline{z}_{abcd}}\right). \tag{5.28}$$

The square-root drops out already at the τ_2 integrand level; integration over the other auxiliary variable τ_2 involves no square root, so it doesn't need rationalization. Therefore, the 4-th entry of the symbol is rational even in all algebraic words. The final integration produces weight-4 symbol in the resulting alphabet.

5.3 Resulting rational letters

Using the shorthand notation $\bar{i} = (i-1ii+1)$ and recalling that $(ijkl) = (1469)$, the rational result is described in terms of the rational letters appearing in the final integrated symbol alphabet. The rational alphabet obtained contains 122 *rational letters*, which are of the following types:

$$\begin{gathered}\text{49 rational lettersof the form:}\\ \langle abcd\rangle.\end{gathered} \tag{5.29}$$

Terms such as $\langle a-1ajk\rangle, \langle a-1akl\rangle, \langle a-1ajl\rangle$ and cyclic combinations thereof appear in the last entries (fourth entries) of the resulting symbol, whereas terms with $a = i-1, i$, $b = j-1, j$, $c = k-1, k$, $d = l-1, l$, so like $\langle a-1ab-1b\rangle$, $\langle a-1ac-1c\rangle$, etc. corresponding to physical discontinuities appear only in the first entry of the symbol result. Moreover, terms of the form $\langle i\bar{j}\rangle$ and $\langle k\bar{l}\rangle$ and terms being cyclic combinations appear in the second entry.

$$\begin{gathered}\text{43 rational letters of the form:}\\ \langle a(bc)(de)(fg)\rangle = \langle abde\rangle\langle acfg\rangle - \langle acde\rangle\langle acfg\rangle.\end{gathered} \tag{5.30}$$

Terms with such structure appear in the second entry and have the form $\langle i(i-ii+1)(b-1b)(c-1c)\rangle$, $\langle i(i-ii+1)(b-1b)(d-1d)\rangle$ and $\langle i(i-ii+1)(c-1c)(d-1d)\rangle$ together with cyclic combinations of the insertion indices i, j, k, l. Furthermore, terms such $\langle i(b-1b)(c-1c)(d-1d)\rangle$ appear in the third entry of the result.

$$\begin{gathered}\text{26 rational letters of the form:}\\ \langle ab(cde)\cap(fgh)\rangle = \langle abde\rangle\langle cfgh\rangle + \langle abec\rangle\langle dfgh\rangle + \langle abcd\rangle\langle efgh\rangle.\end{gathered} \tag{5.31}$$

Such terms appear in the second entry of the symbol result as terms $\langle id(i-1i)(c-1c)(d-1d)$ and cyclic combinations thereof with indices j, k, l.

$$\begin{gathered}\text{4 rational letters of the form:}\\ \langle(a_1b_1c_1)\cap(a_1b_2c_2)\cap(a_3b_3c_3)\cap(a_4b_4c_4)\rangle = \langle(a_1b_1c_1)\cap(a_1b_2c_2),(a_3b_3c_3)\cap(a_4b_4c_4)\rangle.\end{gathered} \tag{5.32}$$

Terms with such structure appear in the third entry exclusively and have the form such as $\langle\bar{i}\cap(ib-1b)\cap\bar{k}\cap(kd-1d)\rangle$, together with cyclic combinations of the insertion indices i, j, k, l.

5.4 Resulting algebraic letters

For the $n = 10$ external leg hexagon with $(ijkl) = (1469)$, the rationalization procedure introduces 54 *independent algebraic letters* in the symbol alphabet. The evaluation of the integration endpoints yields 9 square-roots, instead of the 16 square roots arising in the $n = 12$ external case. These 9 square roots in the $n = 10$ case are:

$\Delta(1,4,6,9)$, $\Delta(2,4,6,9)$, $\Delta(1,4,7,9)$, $\Delta(2,4,7,9)$, $\Delta(2,4,6,10)$, $\Delta(1,5,7,9)$, $\Delta(2,5,7,9)$, $\Delta(2,4,7,10)$ and $\Delta(2,5,7,10)$.

The first two integrations over the loop momenta give that the first two entries are exactly those of the 4-mass box functions, so have the structure as (5.16). The non-trivial entry is the third entry, as the rationalization has to be performed at the third integration over the auxiliary variable τ_1 under the square root. The last entry of the algebraic words is therefore a usual rational function as τ_2 enters rationally. With the 4-mass box function denoted by $I_{4m} = I_{abcd}$, the algebraic part of the symbol of I_{dp} is composed of by a sum of terms such as

$$\mathcal{S}(I_{abcd}) \otimes W^{ijkl}_{a-i,b-j,c-k,d-l} \tag{5.33}$$

where the algebraic letters are contained in the term newly introduced term given by

$$W^{ijkl}_{a-i,b-j,c-k,d-l} = \chi^{jk}_{abcd} \otimes \frac{\langle x_a jk\rangle\langle x_b il\rangle}{\langle x_a jl\rangle\langle x_b jk\rangle} + \text{cyclic} + \frac{1}{2}\left(\frac{z(1-\bar{z})}{\bar{z}(1-z)}\prod\chi\right) \otimes \frac{\langle x_a jl\rangle\langle x_b ik\rangle\langle x_c jl\rangle\langle x_d ik\rangle}{\langle x_a kl\rangle\langle x_b il\rangle\langle x_c ij\rangle\langle x_d jk\rangle} \tag{5.34}$$

The novel type of algebraic letters appear in the third algebraic symbol entry and are introduced by the rationalization. Such letters have the form

$$\chi^{j,k}_{abcd} = \left(\frac{\frac{\langle x_a x_b\rangle\langle x_d jk\rangle}{\langle x_d x_b\rangle\langle x_a jk\rangle} - z_{abcd}}{\frac{\langle x_a x_b\rangle\langle x_d jk\rangle}{\langle x_d x_b\rangle\langle x_a jk\rangle} - \bar{z}_{abcd}}\right) \tag{5.35}$$

and the others are cyclic combination thereof, so terms like $\chi^{k,l}_{bcda}, \chi^{l,i}_{cdba}, \chi^{i,j}_{dbac}$.

Let's consider one of the algebraic letter appearing from the square-root $\Delta(1,4,6,9)$. This radical introduces an algebraic letter of the type

$$\chi_{1469} = \left(\frac{\frac{\langle 3589\rangle 10134\rangle}{\langle 3489\rangle\langle 10135\rangle} - z_{1469}}{\frac{\langle 3589\rangle 10134\rangle}{\langle 3489\rangle\langle 10135\rangle} - \bar{z}_{1469}}\right). \tag{5.36}$$

In the case where some 4-mass corners have the only 2 particles (legs), some algebraic letters become $\frac{z}{\bar{z}}\frac{1-z}{1-\bar{z}}$. All in all, all algebraic letters appearing in the third entry of the resulting symbol are of the types

$$\left\{\frac{z}{\bar{z}}\frac{1-\bar{z}}{1-z}, \chi^{*}_{abcd}, \prod\chi\right\}. \tag{5.37}$$

5.5 Consistency checks

Various consistency check are available to verify the consistency of the result obtained. First, the result must be **invariant under the simultaneous exchange** of $i \leftrightarrow j$ and $k \leftrightarrow l$, as the value of the integral I_{dp} is invariant under such transformation of the indices. Under the exchange $i-1 \leftrightarrow i+1$ we must have that the result is anti-symmetric under the exchange. Furthermore, the **dual conformal symmetry must be manifest** at the level of the resulting symbol, as planar $\mathcal{N}=4$ is a dual conformal invariant theory. Since the theory is DCI, scattering amplitudes must reflect this symmetry of the theory. Also, for scattering amplitudes the **first entries of the symbol must only contain physical poles**, such as $\langle i-1ij-1j\rangle$. These attributes have been observed in the resulting symbol data.

Another good consistency check would be to see if we obtain the same rational and algebraic alphabet by taking the **collinear limits** of the general case $n \geq 12$ presented in [7]. This check can also be completed by checking if the symbol data is **integrable**.

Chapter 6

Conclusion & outlook

In this thesis, the symbol of the finite double-pentagon integral I_{dp} has been calculated, where the total resulting symbol can be written as the sum of the rational and algebraic parts: $S_{tot} = S_{alg} + S_{rat}$. This computation corresponds to the Feynman integral computation of the IR finite part of 2-loop MHV amplitudes $\mathcal{A}_n^{MHV}$ and some components of 2-loop NMHV amplitudes $\mathcal{A}_n^{NMHV}$. The alphabet obtained from the symbol level integration consists of 54 algebraic letters entering the symbol in the third entry exclusively and 122 rational letters. The desired physical conditions on the first two entries have been observed on the alphabet entries. It would instructive to upgrade the resulting symbol to functions of weight-4. The thesis also presented some mathematical objects that are indispensable in the analytic studies of general loop amplitudes.

Furthermore, the super-Wilson-loop duality computation method has been successfully applied to the integral I_{dp} which was extensively used in [6] for calculation involving pentagons and ladder-pentagons. However, for other generic cases with large number of external particles, similar rationalization procedures are required as presented in this thesis. It is straightforward to compute other essential two-loop, or higher multi-loop integrals. Having calculated the I_{dp} integral, a step further and improvement of this thesis would be the calculation of the other diagram contributing to the NMHV amplitude depicted in Figure 6.1 in [24].

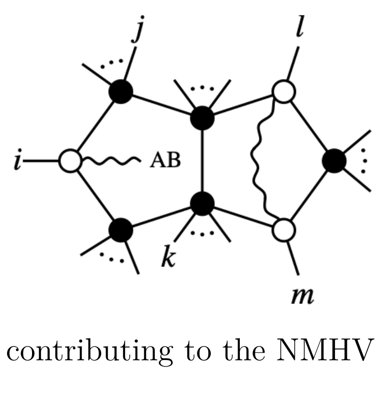


Figure 6.1: Integral contributing to the NMHV amplitude in $\mathcal{N} = 4$

The Wilson-loop duality method presented in this thesis has been also used extensively to compute other types of multi-loop integrals. The general cases still require the rationalization of square roots appearing in the calculation, but by employing this method it is possible to compute various important other cases that are more difficult than the one presented here. Examples include the double-pentagon integrals required for NMHV amplitudes depicted in Figure 6.1 and the penta-box integrals explored in [55]. They are of great importance as these calculations contribute to obtaining complete 2-loop NMHV amplitudes and components of N^2MHV amplitudes. As it is clear at this point, computational difficulty increases significantly also with higher powers of k in N^kMHV configurations. Moreover, also the 3-loop integrals contributing to the MHV amplitudes have been computed using the super-WL method in [48]. These require more effort as the computational difficulty increases exponentially with the loop order, but they are within reach of this powerful method as the $L > 3$ cases have not been computed yet in this way, which opens interesting directions for future research in multi-loop integrals. Furthermore, the $\bar{Q}$ formalism used with the WL method provides a good framework for tackling such difficult problems, as it has been readily used in [8] and [52] for the computation of their respective symbols.

All in all, the WL method used here can be used for the calculation of more difficult cases in terms of loop order and external leg numbers, together with higher helicity multiplicities situations. This thesis explores the integral arising in the calculation of 2-loop MHV scattering amplitude, but the approach to the computation is similar in higher loop and/or external leg cases. These novel methods provide an efficient way of tackling these problems and hopefully many new such powerful tools will see the light in the upcoming years.

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