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Master's thesis

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An Elliptic Double-Box

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Abstract

In [1], the authors found that a particular 'double-box' Feynman diagram in planar N=4 super Yang-Mills theory gives rise to a type of function known as an elliptic polylogarithm. Other authors have since developed a new formalism for elliptic polylogarithms in [2, 3], which gives control over relations between these functions. The goal of this thesis is to write a toy model of the elliptic double-box integral in the new formalism of elliptic multiple polylogarithms, and to investigate whether the new formalism uncovers all of the relations this integral satisfies.

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Contents

1 Introduction

In perturbative quantum field theory, the observables are expanded into a series in the coupling constants of the theory. The n-th order in perturbation theory involves a sum of n-loop Feynman diagrams with a fixed set of external legs that are integrated over the momentum flowing in each of the loops. Thus, the evaluation of higher orders is directly related with the computation of multi-loop Feynman integrals.

Feynman integrals have branch cuts that encode the physics of the theory. Some of the functions that reproduce the branch cut structure are the multiple polylogarithms (MPLs), that appear in multi-loop computations [4, 5, 6, 7]. The study of these functions has helped in the development of new techniques for the evaluation of Feynman integrals. For example, some use the concept of pure functions and pure integrals [8], i.e., integrals such that all the non-vanishing residues of the integrand are equal up to a sign.

Certain QFTs show that some observables can be expressed in terms of pure combinations of MPLs. An important concept is an 'invariant of the observable' called its weight, corresponding to the number of iterated integrations in its definition. It is conjectured that for certain QFTs like the N = 4 Super Yang-Mills (SYM) theory quantum corrections evaluate to pure combinations of MPLs of uniform weight. Therefore, the property of uniform weight might not be coincidental, instead it might be an intrinsic part of the mathematical structure of multi-loop integrals and QFT.

Not all Feynman integrals can be expressed in terms of MPLs. Non-polylogarithmic functions appear in different higher-order computations and these functions have shown a relation to elliptic curves in some cases. Since then, it has been crucial to understand the mathematical properties of these elliptic functions that appear in multi-loop computations.

From a mathematical point of view, the family of functions relevant to elliptic Feynman integrals are the so-called elliptic multiple polylogarithms (eMPLs) [9]. The eMPLs are defined as iterated integrals on an elliptic curve defined as a complex torus and they can also be described as iterated integrals on an elliptic curve defined by a polynomial equation. This description is convenient when working with elliptic Feynman integrals, since the explicit algebraic description of the elliptic curve is directly related to the kinematics of the process and Feynman parameter integrals.

A lot progress have been made in computing Feynman integrals that do not evaluate to ordinary MPLs. It has been particularly relevant to extend the notion of pure functions to the elliptic case and the study of Feynman integrals that rely on the concept of pure functions [10]. Therefore, a further analysis of purity in the elliptic case and the conjectured property of uniform weight is necessary. In this sense, the purpose of this master's thesis is to write a toy model of the elliptic double-box integral in a basis of elliptic multiple polylogarithms and see if the uniform weight property holds.

This paper is organized as follows. In Section 2, we give a short review of ordinary MPLs. In Section 3, we go through a review of the elliptic curves and elliptic multiple polylogarithms. In Section 4, we see the background of pure functions, we motivate the introduction of the elliptic multiple polylogarithms functions and define them and some of their most relevant properties. In Section 5, we review briefly the elliptic double-box integral and give some original results related to the eMPLs. Firstly, we introduce a toy model and, then, we present the expressions in terms of the eMPLs for this integral and the methodology developed to get them. Finally, in Section 6, we draw some conclusions and give the outlook.

2 Multiple polylogarithms

In this section we review the multiple polylogarithms, following closely ref. [11].

The Laurent coefficients of the expansion of Feynman integrals close to $\epsilon = 0$, working in dimensional regularisation $D = D_0 - 2\epsilon$,

$$I = \sum_{k \geq k_0} I_k \epsilon^k = I_{k_0}\epsilon^{k_0} + I_{k_0+1}\epsilon^{k_0+1} + I_{k_0+2}\epsilon^{k_0+2} + \dots, \quad k_0 \in \mathbb{Z} \tag{2.1}$$

form a restricted set numbers called periods, if all scalar products of momenta are negative or zero, internal masses are positive and all ratios of invariants are algebraic. These periods are defined as complex numbers whose real and imaginary parts can be written as integrals of an algebraic function with algebraic coefficients over a domain defined by polynomial inequalities with algebraic coefficients. This is the case of scalar Feynman integrals defined as

$$I = \int (\prod_{j=1}^{L} e^{\gamma_E \epsilon} \frac{d^D k_j}{i\pi^{D/2}}) \frac{\mathcal{N}(\{p_i, k_j\})}{(q_1^2 - m_1^2 + i0)^{\nu_1} \dots (q_N^2 - m_N^2 + i0)^{\nu_N}} \tag{2.2}$$

where $\nu_i \in \mathbb{Z}$ are integers and $m_i \geq 0$, $1 \leq i \leq N$ are the masses of the propagators. Some of the periods that appear in the computation of this kind of integrals are zeta values and polylogarithms. In the case of Feynman integrals with multiple loops and multiple legs that depend of many scales, a new kind of functions appear called multiple polylogarithms that generalise the logarithm function.

2.1 Definition

In an analogous manner of classical polylogarithms, defined by

$$\mathrm{Li}_n(z) = \int_0^z \frac{dt}{t} \mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \tag{2.3}$$

where the starting point is the ordinary logarithm $\mathrm{Li}_1(z) = -\log(1-z)$, multiple polylogarithms (MPLs) can be defined recursively as

$$G(a_1, \dots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_n; t) \tag{2.4}$$

with $n \geq 0$, $G(z) = G(;z) = 1$ and where $a_i \in \mathbb{C}$ and z is a complex number. In the case where all the a_i's are zero

$$G(\underbrace{0, \dots, 0}_{n \text{ times}}; z) = \frac{1}{n!} \log^n z, \quad \log z = \int_1^z \frac{dt}{t}, \tag{2.5}$$

The vector $\vec{a} = (a_1, \dots, a_n)$ is called the vector of singularities and the number of elements n is called weight of the MPL. From these definitions, it is clear that MPLs are periods.

It is obvious from the equation above that MPLs contain the logarithm but, more explicitly, we can see that

$$G(\vec{a}_n; z) = \frac{1}{n!} \log^n (1 - \frac{z}{a}) \quad \text{and} \quad G(\vec{0}_{n-1}, 1; z) = -\operatorname{Li}_n(z), \tag{2.6}$$

where $\vec{a}_n = (\underbrace{a, \dots, a}_{n})$.

2.2 Basic properties

In this section we include a general review of some of the properties that characterise the multiple polylogarithms.

Divergences Firstly, we can see from (2.4) that $G(a_1, \dots, a_n; z)$ is divergent at $z = a_1$ and is analytic at $z = 0$ whenever $a_n \neq 0$. When we consider that a_i's are constant, the MPLs will have branch cuts in z at most extending from $z = a_i$ to $z = \infty$ due to the singularities present at $z = a_i$.

Rescaling Secondly, if the rightmost index a_n is non-zero, the function $G(\vec{a}; x)$ is invariant under a rescaling of all the arguments:

$$G(k\vec{a}; kz) = G(\vec{a}; z), \quad a_n \neq 0, \tag{2.7}$$

where $k \in \mathbb{C}^*$.

Hölder convolution Thirdly, multiple polylogarithms satisfy the Hölder convolution, i.e, if $a_1 \neq 1$ and $a_n \neq 0$ we have

$$G(a_1, \dots, a_n; 1) = \sum_{k=0}^{n} (-1)^k G(1 - a_k, \dots, 1 - a_1; 1 - \frac{1}{p}) G(a_{k+1}, \dots, a_n; \frac{1}{p}), \tag{2.8}$$

with $\forall p \in \mathbb{C}^*$.

In the case that $p \to \infty$, this expression becomes

$$G(a_1, \dots, a_n; 1) = (-1)^n G(1 - a_n, \dots, 1 - a_1; 1). \tag{2.9}$$

Shuffle algebra The last property that we present explicitly in this section is the shuffle algebra. This property will be crucial in the computation of Feynman integrals expressed in terms of multiple polylogarithms and their regularisation. When we make the product of MPLs with weights n_1 and n_2, we obtain a sum of MPLs with weight $n_1 + n_2$,

$$G(a_1, \ldots, a_{n_1}; z)G(a_{n_1+1}, \ldots, a_{n_1+n_2}; z) = \sum_{\sigma \in \Sigma(n_1,n_2)} G(a_{\sigma(1)}, \ldots, a_{\sigma(n_1+n_2)}; z) \tag{2.10}$$

where $\Sigma(n_1, n_2)$ represents the set of all shuffles of $n_1 + n_2$ elements, i.e, the subset of the symmetric group $S_{n_1+n_2}$ defined by

$$\begin{aligned} \Sigma(n_1, n_2) = \{\sigma \in S_{n_1+n_2} \mid \sigma^{-1}(1) < \ldots < \sigma^{-1}(n_1) \\ \text{and } \sigma^{-1}(n_1 + 1) < \ldots < \sigma^{-1}(n_1 + n_2)\}\,, \end{aligned} \tag{2.11}$$

where it is clear that the subset $S_{n_1+n_2}$ preserves the ordering inside the vectors $(a_1, \ldots, a_{n_1})$ and $(a_{n_1+1}, \ldots, a_{n_1+n_2})$. Thus, the set of all MPLs form a shuffle algebra, i.e, a vector space equipped with the shuffle multiplication. It is important to note that the shuffle product preserves the weight of the multiple polylogarithms.

3 Elliptic curves and elliptic polylogarithms

This section includes a review of elliptic curves and the generalisation of multiple polylogarithms to elliptic curves, functions called elliptic multiple polylogarithms. The study of these functions and some of their properties will provide the basis to obtain one of the main results of this thesis, express a toy model double-box integral in terms of the so-called pure elliptic multiple polylogarithms. For the first part, we follow carefully the development made in ref. [2]. The definitions in the second part are mostly taken from ref. [10].

3.1 Elliptic curves

In this section we will consider a cubic polynomial $P_3(x)$ that defines an elliptic curve as the solution set of the following equation

$$y^2 = P_3(x) = (x - a_1)(x - a_2)(x - a_3). \tag{3.1}$$

It is of special interest to study how rational functions of this polynomial behave in an integral, since this structure is what we will encounter in the toy model double-box.

Following ref. [2], we work in projective space $\mathbb{CP}^2$ and interpret the polynomial equation in terms of homogeneous coordinates $[x, y, 1]$, where the elliptic curve also contains the point at infinity $[0, 1, 0]$. This point, together with the points $[a_i, 0, 1]$, will be referred to as branch points.

The elliptic curve defines a compact Riemann surface of genus one. A rational function on the elliptic curve is defined to be a rational function in the variables (x, y), that must fulfill $y^2 = P_3(x)$. Thus, a rational function on the elliptic curve can be written as

$$R(x, y) = \frac{p_1(x) + p_2(x)y}{q_1(x) + q_2(x)y} = \frac{p_1(x) + p_2(x)\sqrt{P_3(x)}}{q_1(x) + q_2(x)\sqrt{P_3(x)}}, \tag{3.2}$$

where p_i and q_i are polynomials in x. Multiplying both the numerator and denominator by the conjugate of the denominator, this can be rewritten as

$$R(x, y) = R_1(x) + \frac{1}{y}R_2(x) = R_1(x) + \frac{1}{\sqrt{P_3(x)}}R_2(x) \tag{3.3}$$

for some rational functions R_i.

Using the equation above, we see that $R_1(x)$ contributes with an ordinary integral of a rational function and can be computed in terms of rational functions and logarithms. When using partial fractioning in the contribution of $R_2(x)$, the only integrals that need to be considered are

$$\int \frac{dx}{y} x^k \quad \text{and} \quad \int \frac{dx}{y(x - c)^k}, \tag{3.4}$$

where k is an integer and c is a constant. After integration by parts, these integrals can be reduced to a linear combination of

$$\int \frac{dx}{y}, \quad \int \frac{xdx}{y}, \quad \int \frac{dx}{y(x-c)}. \tag{3.5}$$

These integrals cannot be reduced more and, therefore, can be considered as the analogues of the master integrals, i.e., the set of Feynman integrals that span all the Feynman integrals in a given family . They can be evaluated in terms of the incomplete elliptic integrals of the first, second and third kind

$$\begin{aligned} \mathrm{F}(x \mid m^2) &= \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-m^2t^2)}} \\ \mathrm{E}(x \mid m^2) &= \int_0^x dt \frac{1-m^2t^2}{\sqrt{(1-t^2)(1-m^2t^2)}} \\ \Pi(n^2, x \mid m^2) &= \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-m^2t^2)}} \frac{1}{1-n^2t^2}. \end{aligned} \tag{3.6}$$

The computation of these integrals on the elliptic curve can present obstructions to finding a rational primitive when integrating on an elliptic curve and, in this sense, the incomplete integral of the third kind Π can be considered as a generalisation of the logarithm in the case of the Riemann sphere.

There are some invariants that are attached to an elliptic curve. For the following definitions we will consider a quartic

$$y^2 = P_4(x) = (x-a_1)(x-a_2)(x-a_3)(x-a_4), \tag{3.7}$$

since this is the polynomial equation that will appear in the study of the double-box integral. Nevertheless, the definitions for the cubic polynomial are analogous. The periods of an elliptic curve are defined by integrating dx/y between two branch points:

$$\begin{aligned} \omega_1 &= 2c_4 \int_{a_2}^{a_3} \frac{dx}{y} = 2\,\mathrm{K}(\lambda) \\ \omega_2 &= 2c_4 \int_{a_1}^{a_2} \frac{dx}{y} = 2i\,\mathrm{K}(1-\lambda), \end{aligned} \tag{3.8}$$

where K corresponds to the complete elliptic integral of the first kind, $K(\lambda) = F(1 \mid \lambda)$, and

$$\begin{aligned} \lambda &= \mathrm{cr}(a_1, a_4, a_3, a_2) = \frac{a_{14}a_{23}}{a_{13}a_{24}} \\ c_4 &= \frac{1}{2}\sqrt{a_{13}a_{24}}, \end{aligned} \tag{3.9}$$

with $a_{ij} = a_i - a_j$.

The quasi-periods of the elliptic curve are defined by

$$\begin{aligned} \eta_1 &= -\frac{1}{2}\int_{a_2}^{a_3} dx\widetilde{\Phi}_4(x) = \mathrm{E}(\lambda) - \frac{2-\lambda}{3}\,\mathrm{K}(\lambda) \\ \eta_2 &= -\frac{1}{2}\int_{a_1}^{a_2} dx\widetilde{\Phi}_4(x) = -i\mathrm{E}(1-\lambda) + i\frac{1+\lambda}{3}\,\mathrm{K}(1-\lambda), \end{aligned} \tag{3.10}$$

with $E(\lambda) = E(1 \mid \lambda)$ and

$$\widetilde{\Phi}_4(x,\vec{a}) \equiv \frac{1}{c_4 y}(x^2 - \frac{s_1}{2}x + \frac{s_2}{6}), \tag{3.11}$$

where $s_n(\vec{a}) \equiv s_n(a_1,a_2,a_3)$ is the elementary symmetric polynomial of degree n in three variables. This function has the property that the differential one-form $dx\widetilde{\Phi}_4(x,\vec{a})$ has a double-pole with vanishing residue at $x=\infty$.

The periods and quasi-periods are not independent and are related through the Legendre relation

$$\omega_1\eta_2 - \omega_2\eta_1 = -i\pi. \tag{3.12}$$

The periods and quasi-periods are strictly speaking not invariants of the elliptic curve, since there can be different values of ω_i and η_i that correspond to the same elliptic curve. This redundancy is due to the fact that they only depend on the cross-ratio λ of the branch points. On the other hand, the j-invariant uniquely characterises an elliptic curve,

$$j = 256\frac{(1-\lambda(1-\lambda))^3}{\lambda^2(1-\lambda)^2}. \tag{3.13}$$

Every elliptic curve defined over the complex numbers is isomorphic to a complex torus, i.e, the quotient of the complex plane $\mathbb{C}$ by a two-dimensional lattice Λ. The relevant lattice that we will use in our results is $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ spanned by the two periods. A rescaling of the lattice can be done without changing the geometry, defining the lattice $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, where $\tau = \omega_2/\omega_1$, with Im $\tau > 0$. Thus, every τ in upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}\,\tau > 0\}$ defines a two-dimensional lattice and, therefore, an elliptic curve. Different values of τ, $\tau, \tau' \in \mathbb{H}$ define the same elliptic curve if and only if they are related by a modular transformation, i.e, a Möbius transformation for $SL(2,\mathbb{Z})$. Then, the space of geometrically-distinct tori (moduli space) can be identified with the quotient of the upper half-plane $\mathbb{H}$ by the modular group $SL(2,\mathbb{Z})$.

The map from the torus $\mathbb{C}/\Lambda$ to the curve defined by the polynomial equation $y^2 = P_4(x)$ can be realised by a function κ, $\kappa(.,\vec{a}) : \mathbb{C}/\Lambda \to \mathbb{C}$. This function is a meromorphic function of z and it is doubly-periodic, i.e, $\kappa(z+1,\vec{a}) = \kappa(z+\tau,\vec{a}) = \kappa(z,\vec{a})$. The functions that satisfy this property are called elliptic functions.

Taking $[X, Y, 1] \in \mathbb{CP}^2$ as a point satisfying $Y^2 = P_4(X)$, we can define its image on the torus, the inverse map to κ called Abel's map, as

$$z_X = \frac{c_4}{\omega_1}\int_{a_1}^{X}\frac{dx}{y} = \frac{\sqrt{a_{13}a_{24}}}{4\,\mathrm{K}(\lambda)}\int_{a_1}^{X}\frac{dx}{y}. \tag{3.14}$$

The image on the torus of the point $x = -\infty$ will be especially relevant in the development of the results,

$$z_* = \frac{c_4}{\omega_1}\int_{a_1}^{-\infty}\frac{dx}{y} = \frac{\sqrt{a_{13}a_{24}}}{4\,\mathrm{K}(\lambda)}\int_{a_1}^{-\infty}\frac{dx}{y}. \tag{3.15}$$

To conclude this section, we will consider the branch points as conjugate pairs with the following order from now on,

$$\begin{gathered} a_1 = a_2^*, \quad a_3 = a_4^* \\ \mathrm{Re}(a_1) < \mathrm{Re}(a_3), \quad \mathrm{Im}(a_2), \mathrm{Im}(a_3) > 0, \quad \mathrm{Im}(a_1), \mathrm{Im}(a_4) < 0, \end{gathered} \tag{3.16}$$

Therefore, the choice of the branch cut is

$$y = \sqrt{P_4(x)}. \tag{3.17}$$

3.2 Elliptic multiple polylogarithms

In this section we review the generalisation of polylogarithms to elliptic curves, the elliptic multiple polylogarithms (eMPLs). Firstly, we define these functions as iterated integrals on the complex torus and, secondly, what these integrals become in terms of the variables (x, y).

3.2.1 Elliptic multiple polylogarithms on a complex torus

We define the elliptic multiple polylogarithms by the iterated integral

$$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 \dots n_k \\ z_1 \dots z_k \end{smallmatrix}; z, \tau\right) = \int_0^z dz' g^{(n_1)}(z' - z_1, \tau)\widetilde{\Gamma}\left(\begin{smallmatrix} n_2 \dots n_k \\ z_2 \dots z_k \end{smallmatrix}; z', \tau\right), \tag{3.18}$$

where z_i are complex numbers and $n_i \in \mathbb{N}$ are positive integers. The integers k and $\sum_i n_i$ are called the length and the weight of the eMPL respectively. In the case where $(n_k, z_k) = (1, 0)$, the integral is divergent and requires regularisation. This matter will be further discussed in the results.

The integration kernels that appear in (3.18) are defined through a generating series known as the Eisenstein-Kronecker series,

$$F(z, \alpha, \tau) = \frac{1}{\alpha}\sum_{n\geq 0} g^{(n)}(z, \tau)\alpha^n = \frac{\theta_1'(0, \tau)\theta_1(z + \alpha, \tau)}{\theta_1(z, \tau)\theta_1(\alpha, \tau)}, \tag{3.19}$$

where θ_1 is the odd Jacobi theta function, and θ_1' is its derivative with respect to its first argument. The function $g^{(1)}(z,\tau)$ has a simple pole with residue 1 at every point of the lattice Λ. In the case that $n>1$, $g^{(n)}(z,\tau)$ has a simple pole only at the lattice points that do not lie on the real axis. Thus, the integrals (3.18) have at most logarithmic singularities. Adding to this, the functions $g^{(n)}$ have definite parity,

$$g^{(n)}(-z,\tau)=(-1)^n g^{(n)}(z,\tau), \tag{3.20}$$

and are invariant under translations by 1, but not by τ,

$$g^{(n)}(z+1,\tau)=g^{(n)}(z,\tau) \quad \text{and} \quad g^{(n)}(z+\tau,\tau)=\sum_{k=0}^{n}\frac{(-2\pi i)^k}{k!}g^{(n-k)}(z,\tau). \tag{3.21}$$

Between all the properties that define the eMPLs, it is relevant to mention that eMPLs form a shuffle algebra, in the same way that MPLs did in (2.10),

$$\widetilde{\Gamma}(A_1\cdots A_k;z,\tau)\widetilde{\Gamma}(A_{k+1}\cdots A_{k+l};z,\tau)=\sum_{\sigma\in\Sigma(k,l)}\widetilde{\Gamma}(A_{\sigma(1)}\cdots A_{\sigma(k+l)};z,\tau), \tag{3.22}$$

where $A_i=(\begin{smallmatrix} n_i \\ z_i \end{smallmatrix})$. Again, the shuffle product conserves both the length and the weight of the eMPLs.

3.2.2 Elliptic multiple polylogarithms in terms of (x,y)

In this section, we consider the map from the torus to the elliptic curve defined by the polynomial equation $y^2=P_4(x)$ to redefine the elliptic multiple polylogarithms in the variables (x,y). The iterated integrals now are

$$\mathrm{E}_4(\begin{smallmatrix} n_1 \dots n_k \\ c_1 \cdots c_k \end{smallmatrix};x,\vec{a})=\int_0^x dt\psi_{n_1}(c_1,t,\vec{a})\mathrm{E}_4(\begin{smallmatrix} n_2 \dots n_k \\ c_2 \cdots c_k \end{smallmatrix};t,\vec{a}), \tag{3.23}$$

with $n_i\in\mathbb{Z}$ and $c_i\in\widehat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$, and the recursion starts with $E_4(;x,\vec{a})=1$.

The integration kernels ψ_n are chosen to satisfy the following conditions.

1 The functions ψ_n are non-trivial, i.e, they cannot be expressed as total derivatives of a rational function on the elliptic curve. Thus, they are related to the irreducible integrands that we introduced in (3.5).

2 The integration kernels are linearly independent, we cannot find a linear combination that evaluates to a total derivative.

3 Each kernel ψ_n can have at most simple poles since elliptic polylogarithms should have at most logarithmic singularities (and no poles).

The integration kernels with $n = 0, 1, -1$ will appear extensively in the results and, therefore, we present a brief discussion of their construction based on [2]. For the case $n = 0$, we can define

$$\psi_0(c, x) = \frac{c_4}{y} = \frac{c_4}{\sqrt{P_4(x)}}, \tag{3.24}$$

where the right-hand side is independent of c and, consequently, we will set $c = 0$ in this kernel. The integral of ψ_0 is related to the incomplete elliptic integral of the first kind F, (3.6). This kernel defines a rational function that is free of poles, setting $x = 1/u^2$ we see that there is no pole at infinity,

$$\int dx\psi_0(0, x) = -2c_4 \int du(1 + \mathcal{O}(u)). \tag{3.25}$$

On the contrary, the functions $\psi_{\pm 1}(c, x)$ have a simple pole at $x = c$,

$$\begin{aligned} \psi_1(c, x) &= \frac{1}{x - c} \\ \psi_{-1}(c, x) &= \frac{y_c}{y(x - c)}, \end{aligned} \tag{3.26}$$

where we use $y_c = \sqrt{P_4(c)}$. Firstly, we see from these expressions that the integral involving ψ_{-1} with $c = a_i$ can be reduced to simpler integrals by using integration-by-parts identities. Therefore, the functions $\psi_{-1}(a_i, x)$ cannot be part of the basis of integration kernels. Secondly, the function ψ_1 is independent of the branch points a_i and it is identical to the integration kernel in the multiple polylogarithms, (2.4). Hence, MPLs are a subset of the eMPLs,

$$\mathrm{E}_4(\begin{smallmatrix} 1 \dots 1 \\ c_1 \dots c_k \end{smallmatrix}; x) = G(c_1, \dots, c_k; x), \quad c_i \neq \infty. \tag{3.27}$$

Here the differential xdx/y, which provided the differential of the second kind in the cubic case in (3.5), is no longer an option. Setting $u = 1/x$,

$$\int \frac{xdx}{y} = \int du \left[-\frac{1}{u} + \mathcal{O}(u^0) \right]. \tag{3.28}$$

We see that the differential has a simple pole at infinity and, therefore, defines a differential of the third kind. In the quartic case, we define the valid differential as

$$\frac{x^2 dx}{y} - \frac{s_1}{2} \frac{xdx}{y}. \tag{3.29}$$

Here it is important to notice that this choice is motivated by the fact that the first term cannot be broken into smaller parts by integrations-by-parts identities. Also, the second term is mandatory since the differential x^2dx/y has a non-vanishing residue at infinity,

$$\int \frac{x^2dx}{y} = \int du \left[-\frac{1}{u^2} - \frac{s_1}{2u} + \mathcal{O}(u^0) \right]. \tag{3.30}$$

We can add any multiple of dx/y to (3.29) and, thus, we will use $\widetilde{\Phi}_4(x, \vec{a})$ defined in (3.11).

Finally, we can summarize the previous results to write the relevant kernels [2, 3, 10],

$$\begin{aligned} \psi_0(0, x) &= \frac{c_4}{y}, \\ \psi_1(c, x) &= \frac{1}{x - c} \\ \psi_{-1}(c, x) &= \frac{y_c}{y(x - c)}, \\ \psi_1(\infty, x) &= \frac{c_4}{y} Z_4(x) \\ \psi_{-1}(\infty, x) &= \frac{x}{y}, \end{aligned} \tag{3.31}$$

where

$$Z_4(x, \vec{a}) \equiv \int_{a_1}^{x} dx' \Phi_4(x', \vec{a}) \tag{3.32}$$

and

$$\Phi_4(x) \equiv \widetilde{\Phi}_4(x) + 4c_4 \frac{\eta_1}{\omega_1} \frac{1}{y}. \tag{3.33}$$

The differential $dx\widetilde{\Phi}_4$ has a double pole without residue at infinity and, consequently, the function Z_4 has a simple pole there. In a similar manner, the kernels $\psi_n(c, x, \vec{a})$ with $|n| \geq 1$ have at most a simple pole at $x = c$.

3.2.3 Relations between $\widetilde{\Gamma}$ and E_4

We have defined the elliptic multiple polylogarithms in two different ways, in terms of $\widetilde{\Gamma}$ and E_4. These two functions are two different bases for the same space of functions. Thus, we can write the kernels ψ_n as linear combinations of the coefficients $g^{(n)}$ of the Eisenstein-Kronecker series in (3.19). Specifically, the kernels in (3.31) [10],

$$
\begin{aligned}
dx\psi_0(0,x,\vec{a}) &= dz \\
dx\psi_1(c,x,\vec{a}) &= dz\left[g^{(1)}(z-z_c,\tau)+g^{(1)}(z+z_c,\tau)-g^{(1)}(z-z_*,\tau)-g^{(1)}(z+z_*,\tau)\right] \\
dx\psi_{-1}(c,x,\vec{a}) &= dz\left[g^{(1)}(z-z_c,\tau)-g^{(1)}(z+z_c,\tau)+g^{(1)}(z_c-z_*,\tau)+g^{(1)}(z_c+z_*,\tau)\right] \\
dx\psi_1(\infty,x,\vec{a}) &= dz\left[-g^{(1)}(z-z_*,\tau)-g^{(1)}(z+z_*,\tau)\right] \\
dx\psi_{-1}(\infty,x,\vec{a}) &= \frac{a_1\omega_1 dz}{c_4}+dz\left[g^{(1)}(z-z_*,\tau)-g^{(1)}(z+z_*,\tau)+2g^{(1)}(z_*,\tau)\right]
\end{aligned}
\tag{3.34}
$$

where z_c is defined in (3.14) and z_* in (3.15). Using these relations, it is clear that there exists a one-to-one map between the functions $\widetilde{\Gamma}$ and E_4, and we can write a function from one class as a linear combination of the functions from the other class. Finally, we can also write the function Z_4 in terms of the coefficients of the Eisenstein-Kronecker series,

$$
Z_4(x,\vec{a}) = -\frac{1}{\omega_1}\left[g^{(1)}(z_x-z_*,\tau)+g^{(1)}(z_x+z_*,\tau)\right]. \tag{3.35}
$$

4 Pure functions and pure elliptic multiple polylogarithms

In this section, we review the concept of pure elliptic multiple polylogarithms based on ref. [10], including the form of the integration kernels that appear in the iterated integrals and some of their basic properties. Before defining these functions, we introduce the pure functions and we discuss why these functions are relevant.

4.1 Pure functions

The main goal of this section is to generalise the concept of pure functions to the elliptic case. Hence, we must define first what characterises a pure function. A pure integral is defined as an integral such that all non-vanishing residues of its integrand are the same up to a sign (it is possible then to normalise the integrals such that the non-vanishing residues are ± 1). Another definition uses the concept of weight, which we have introduced in previous sections. A pure function of weight n is a function whose total differential can be written in terms of pure functions of weight $n-1$ (multiplied by algebraic functions with at most single poles). This definition is therefore recursive and starts with assigning weight zero to algebraic functions. Sums and products of pure functions are still pure and the weight of a product of two pure functions is the sum of their weights. The concept of weight can be extended from functions to numbers, e.g, the weight of $i\pi$ is one and the weight of $\zeta_n = -G(\vec{0}_{n-1}, 1; 1)$ is n.

Now we can see clearly that MPLs are pure functions. They satisfy the following differential equation

$$dG(a_1, \ldots, a_n; z) = \sum_{i=1}^{n} G(a_1, \ldots, \hat{a}_i, \ldots, a_n; z) d \log \frac{a_{i-1} - a_i}{a_{i+1} - a_i}, \tag{4.1}$$

where $\hat{a}_i$ indicates that a_i is absent and it follows that $G(a_1, \ldots, a_n; z)$ is a pure function of weight n. If an integral can be evaluated in terms of algebraic functions and MPLs, this integral is pure if and only if can be expressed as a linear combination of MPLs whose coefficients are rational numbers.

In [12], it is argued that Feynman integrals usually do not evaluate to pure functions but we can make a choice of basis such that all members of a given of Feynman integrals can be written as a set of pure integrals (master integrals) with algebraic basis coefficients.

In the case of MPLs the basis of pure master integrals can be reached in an algorithmic way and the change of basis only involves algebraic functions. This can be illustrated on a one-loop example, specifically the family of bubble integrals with two massive propagators in $D = 2 - 2\epsilon$ dimensions,

$$B_{n_1 n_2}(p^2, m_1^2, m_2^2) = e^{\gamma_E \epsilon} \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2 - m_1^2)^{n_1}((k+p)^2 - m_2^2)^{n_2}}, \tag{4.2}$$

where $\gamma_E = -\Gamma'(1)$ is the Euler-Mascheroni constant. Using integration-by-parts identities [13, 14], every integral in this family can be written as a linear combination of three master integrals,

$$\begin{aligned}
B_{10}(p^2,m^2,0) = B_{01}(p^2,0,m^2) &= e^{\gamma_E\epsilon}\int\frac{d^Dk}{i\pi^{D/2}}\frac{1}{k^2-m^2}\\
&= -\frac{1}{\epsilon}+\log m^2+\mathcal{O}(\epsilon),\\
B_{11}(p^2,m_1^2,m_2^2) &= e^{\gamma_E\epsilon}\int\frac{d^Dk}{i\pi^{D/2}}\frac{1}{(k^2-m_1^2)((k+p)^2-m_2^2)}\\
&= \frac{1}{p^2(w-\bar{w})}\log(\frac{\bar{w}(1-w)}{w(1-\bar{w})})+\mathcal{O}(\epsilon),
\end{aligned} \tag{4.3}$$

with $w\bar{w}=m_1^2/p^2$ and $(1-w)(1-\bar{w})=m_2^2/p^2$. These master integrals are not pure since the logarithms are multiplied by algebraic prefactors, but we can perform a change of basis

$$\begin{pmatrix}B_{10}\\B_{01}\\B_{11}\end{pmatrix} = \begin{pmatrix}-1/\epsilon & 0 & 0\\ 0 & -1/\epsilon & 0\\ 0 & 0 & -2/(\epsilon p^2(w-\bar{w}))\end{pmatrix}\begin{pmatrix}\widetilde{B}_{10}\\\widetilde{B}_{01}\\\widetilde{B}_{11}\end{pmatrix}. \tag{4.4}$$

Now the functions $\widetilde{B}_{ij}$ are pure, i.e., the coefficient of ϵ^k is a linear combination of terms of uniform weight k. The algebraic factor in the one-loop bubble integral corresponds to the maximal cut of the integral,

$$\text{Cut}\left[B_{11|D=2}\right] = -\frac{2}{p^2(w-\bar{w})} \tag{4.5}$$

and, therefore,

$$B_{11} = \text{Cut}\left[B_{11|D=2}\right]\times[-\frac{1}{2}\log(\frac{\bar{w}(1-w)}{w(1-\bar{w})})+\mathcal{O}(\epsilon)]. \tag{4.6}$$

From this example, we can see that having a basis of pure master integrals facilitate their computation. Furthermore, the concept of purity and uniform weight has led to breakthroughs in the computation of master integrals.

4.2 Motivation

The previous section showed how the concepts of purity and uniform weight can be relevant in the study of Feynman integrals. In this section, we will see what conditions are needed for many of the properties of pure Feynman integrals to carry over to the elliptic case. In this case, we will demonstrate it with the example of the sunrise integral [3],

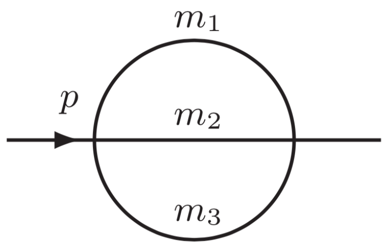

Figure 1: Sunrise integral diagram, where p is the momentum and m_1, m_2, m_3 are the masses of the propagators.

The two-loop sunrise integral in $D = 2 - 2\epsilon$ with three equal masses can be evaluated in terms of E_4. We consider the family of integrals

$$S_{n_1 n_2 n_3}(p^2, m^2) = -\frac{e^{2\gamma_E \epsilon}}{\pi^D} \int \frac{d^D k d^D l}{(k^2 - m^2)^{n_1} (l^2 - m^2)^{n_2} ((k + l + p)^2 - m^2)^{n_3}}, \quad (4.7)$$

with $n_i \in \mathbb{N}$. Using integration-by-parts identities, we obtain that every integral in this family can be written as a linear combination of the following master integrals

$$\begin{aligned} S_0(p^2, m^2) &= S_{110}(p^2, m^2), \\ S_1(p^2, m^2) &= S_{111}(p^2, m^2), \\ S_2(p^2, m^2) &= S_{112}(p^2, m^2). \end{aligned} \quad (4.8)$$

For the purpose of this example, it is sufficient to consider only the master integral S_1,

$$\begin{aligned} S_1(p^2, m^2) = \frac{1}{(m^2 + p^2) c_4} [&\frac{1}{c_4} \mathrm{E}_4(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}; 1, \vec{a}) - 2\mathrm{E}_4(\begin{smallmatrix} 0 & -1 \\ 0 & \infty \end{smallmatrix}; 1, \vec{a}) - \mathrm{E}_4(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}; 1, \vec{a}) \\ &- \mathrm{E}_4(\begin{smallmatrix} 0 & -1 \\ 0 & 1 \end{smallmatrix}; 1, \vec{a}) - \mathrm{E}_4(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}; 1, \vec{a})] + \mathcal{O}(\epsilon), \end{aligned} \quad (4.9)$$

where the branch points are

$$\vec{a} = (\frac{1}{2}(1 + \sqrt{1 + \rho}), \frac{1}{2}(1 + \sqrt{1 + \bar{\rho}}), \frac{1}{2}(1 - \sqrt{1 + \rho}), \frac{1}{2}(1 - \sqrt{1 + \bar{\rho}})) \quad (4.10)$$

with

$$\rho = -\frac{4m^2}{(m + \sqrt{-p^2})^2} \quad \text{and} \quad \bar{\rho} = -\frac{4m^2}{(m - \sqrt{-p^2})^2}. \quad (4.11)$$

The result for S_1 in (4.9) is not pure because not all the E_4 functions are multiplied by rational numbers. Nevertheless, some facts indicate that the two-loop sunrise integral should define a pure function. In [3] we see that, firstly, the result for S_1 obtained from dispersion relations can be written as a $\mathbb{Q}$-linear combination of E_4 functions, and no extra algebraic factors are needed. Secondly, in the case where one propagator is massless, the integral can be computed in terms of pure linear combination of MPLs. Thirdly, the equal-mass sunrise integral $S_1(p^2, m^2)$ can also be expressed in terms of iterated integrals of Eisenstein series and, again, no extra algebraic factors are needed.

Therefore, we conclude the basis of E_4 functions does not have the needed properties for elliptic purity. Nevertheless, we can consider the basis of $\widetilde{\Gamma}$ functions on the complex torus. With these functions, the equal-mass sunrise integral is a $\mathbb{Q}$-linear combination of $\widetilde{\Gamma}$ functions, i.e., it can be described as pure. The $\widetilde{\Gamma}$ functions, like the ordinary MPLs, have at most logarithmic singularities in all variables and no poles. On the contrary, the E_4 functions have logarithmic singularities and poles. Thus, considering the function $E_4(\begin{smallmatrix}-1\\c\end{smallmatrix}; x, \vec{a})$ with $c \neq \infty$, we obtain [10]

$$\begin{aligned}E_4(\begin{smallmatrix}-1\\c\end{smallmatrix}; x, \vec{a}) &= \widetilde{\Gamma}(\begin{smallmatrix}1\\z_c\end{smallmatrix}; z_x, \tau) - \widetilde{\Gamma}(\begin{smallmatrix}1\\-z_c\end{smallmatrix}; z_x, \tau) + [g^{(1)}(z_c - z_*, \tau) \\ &+ g^{(1)}(z_c + z_*, \tau)]\widetilde{\Gamma}(\begin{smallmatrix}0\\0\end{smallmatrix}; z_x, \tau) = \widetilde{\Gamma}(\begin{smallmatrix}1\\z_c\end{smallmatrix}; z_x, \tau) - \widetilde{\Gamma}(\begin{smallmatrix}1\\-z_c\end{smallmatrix}; z_x, \tau) \\ &- \omega_1 Z_4(c, \vec{a})\widetilde{\Gamma}(\begin{smallmatrix}0\\0\end{smallmatrix}; z_x, \tau).\end{aligned} \tag{4.12}$$

We see clearly that, while $\widetilde{\Gamma}$ functions have at most logarithmic singularities, the function Z_4 has a pole at $c = \infty$ and, consequently, E_4 functions have poles (but in the variable x they only have logarithmic singularities). Based on this, a pure function can be defined as a function that it is unipotent, i.e., a function that satisfy a differential equation without homogeneous term, and its total differential involves only pure functions and one-forms with at most logarithmic singularities. The sums and products of pure functions are also pure.

From the previous results, we have seen that the $\widetilde{\Gamma}$ functions provide a basis of pure eMPLs but they are not the most convenient when working with Feynman integrals due to the following reasons.

1 Feynman integrals have an intrinsic notion of parity, the final analytic result is independent of the choice of the branch of the root. Hence, the pure function must have definite parity with respect of changing the sign of the root. In the case of eMPLs, this corresponds to $(x, y) \leftrightarrow (x, -y)$ and, on the torus, this operation changes the sign of z. Therefore, $\widetilde{\Gamma}$ functions does not have definite parity.

2 Feynman integrals are more naturally expressed in the coordinates (x, y) because these variables are related to the kinematics of the physical process.

4.3 Pure elliptic multiple polylogarithms

The new class of iterated integrals can be introduced once we know what characteristics they should have.

1 They form a basis for the space of all eMPLs.

2 They are pure.

3 They have definite parity.

4 They manifestly contain ordinary MPLs.

4.3.1 Definition

The pure elliptic multiple polylogarithms can be defined as

$$\mathcal{E}_4(\begin{smallmatrix} n_1 \dots n_k \\ c_1 \dots c_k \end{smallmatrix}; x, \vec{a}) = \int_0^x dt \Psi_{n_1}(c_1, t, \vec{a}) \mathcal{E}_4(\begin{smallmatrix} n_2 \dots n_k \\ c_2 \cdots c_k \end{smallmatrix}; t, \vec{a}), \tag{4.13}$$

with $n_i \in \mathbb{Z}$ and $c_i \in \hat{\mathbb{C}}$.

The length and weight are specified in the same way as in the E_4 functions. The integration kernels can be expressed in terms of the coefficients of the Eisenstein-Kronecker series in the following way (for $n \geq 0$)

$$\begin{aligned} &dx\Psi_{\pm n}(c, x, \vec{a}) \\ &\quad = dz_x[g^{(n)}(z_x - z_c, \tau) \pm g^{(n)}(z_x + z_c, \tau) - \delta_{\pm n,1}(g^{(1)}(z_x - z_*, \tau) + g^{(1)}(z_x + z_*, \tau))]. \end{aligned} \tag{4.14}$$

Now we can check that the pure elliptic multiple polylogarithms fulfill all the necessary properties.

1 There is a one-to-one map between the $\mathcal{E}_4$ and $\widetilde{\Gamma}$ functions and, consequently, they define a basis of all eMPLs.

2 The coefficientes in (4.14) are all ± 1 and the $\mathcal{E}_4$ functions can be written as a $\mathbb{Q}$-linear combination of $\widetilde{\Gamma}$, therefore they are pure.

3 From (4.14), it is clear that they have definite parity under changing the sign of z_x. Then, the $\mathcal{E}_4$ functions define a pure basis of eMPLs with definite parity.

$$dx\Psi_{\pm n}(c,x,\vec{a}) = \frac{dx}{x-c}, \quad c \neq \infty, \tag{4.15}$$

we see that the ordinary MPLs are a subset of eMPLs

$$\mathcal{E}_4(\begin{smallmatrix} 1 \dots 1 \\ c_1 \dots c_k \end{smallmatrix}; x, \vec{a}) = G(c_1, \dots, c_k; x), \quad c_i \neq \infty. \tag{4.16}$$

4.3.2 Integration kernels

We have written the integration kernels of the pure eMPLs in terms of their relations to the coefficients of the Eisenstein-Kronecker series in (4.14). Now we can present the explicit form of the relevant kernels, up to $n = 1$. For $n = 0$, we have

$$\Psi_0(0,x,\vec{a}) = \frac{1}{\omega_1}\psi_0(0,x,\vec{a}) = \frac{c_4}{\omega_1 y}. \tag{4.17}$$

For $n = 1$ (and $c \neq \infty$), we obtain

$$\begin{aligned}
\Psi_1(c,x,\vec{a}) &= \psi_1(c,x,\vec{a}) = \frac{1}{x-c} \\
\Psi_{-1}(c,x,\vec{a}) &= \psi_{-1}(c,x,\vec{a}) + Z_4(c,\vec{a})\psi_0(0,x,\vec{a}) = \frac{y_c}{y(x-c)} + Z_4(c,\vec{a})\frac{c_4}{y} \\
\Psi_1(\infty,x,\vec{a}) &= -\psi_1(\infty,x,\vec{a}) = -Z_4(x,\vec{a})\frac{c_4}{y}, \\
\Psi_{-1}(\infty,x,\vec{a}) &= \psi_{-1}(\infty,x,\vec{a}) - [\frac{a_1}{c_4} + 2G_*(\vec{a})]\psi_0(0,x,\vec{a}) = \frac{x}{y} - \frac{1}{y}[a_1 + 2c_4 G_*(\vec{a})].
\end{aligned} \tag{4.18}$$

The quantity $G_*(\vec{a})$ corresponds to the image of z_* under the function $g^{(1)}$,

$$G_*(\vec{a}) \equiv \frac{1}{\omega_1} g^{(1)}(z_*, \tau). \tag{4.19}$$

We see that the integration kernels that define the pure elliptic multiple polylogarithms involve the functions Z_4 and $G_*(\vec{a})$. While, in general, these functions are transcendental, they can often be expressed in terms of algebraic quantities and will simplify the analytic structure of the integration kernels. Specifically, in the result of the double-box integral, we will see that all terms containing the function Z_4 will cancel.

4.3.3 Basic properties

Here we summarize some of the basic properties of the pure elliptic multiple polylogarithms. We can see that most of them are analogous to the properties of the E_4 and $\widetilde{\Gamma}$ functions.

Shuffle algebra In the same manner that we saw with the ordinary MPLs and the eMPLs, the pure eMPLs form a shuffle algebra,

$$\mathcal{E}_4(A_1\cdots A_k;x,\vec{a})\mathcal{E}_4(A_{k+1}\cdots A_{k+l};x,\vec{a}) = \sum_{\sigma\in\Sigma(k,l)} \mathcal{E}_4(A_{\sigma(1)}\cdots A_{\sigma(k+l)};x,\vec{a}) \tag{4.20}$$

with $A_i(\begin{smallmatrix} n_i \\ c_i \end{smallmatrix})$.

Rescaling of arguments Again, like ordinary MPLs, the $\mathcal{E}_4$ functions are invariant under a rescaling of the arguments,

$$\mathcal{E}_4(\begin{smallmatrix} n_1 & \dots & n_k \\ pc_1 & \cdots & pc_k \end{smallmatrix};px,p\vec{a}) = \mathcal{E}_4(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix};x,\vec{a}), \quad p,c_k\neq 0. \tag{4.21}$$

Value at infinite cusp For $\tau\to i\infty$, the pure eMPL always reduce to a pure combination of ordinary MPLs of weight $\sum_i n_i$. This follows from the coefficients of the Eisenstein-Kronecker series when studying the limit $\tau\to i\infty$. Specifically, they admit the Fourier expansions [15, 16, 17]

$$\begin{aligned} g^{(1)}(z,\tau) &= \pi\cot(\pi z) + 4\pi\sum_{m=1}^{\infty}\sin(2\pi m z)\sum_{n=1}^{\infty}q^{mn} \\ g^{(k)}(z,\tau)\big|_{k=2,4,\dots} &= -2\zeta_k - 2\frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\cos(2\pi m z)\sum_{n=1}^{\infty}n^{k-1}q^{mn} \\ g^{(k)}(z,\tau)\big|_{k=3,5,\dots} &= -2i\frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sin(2\pi m z)\sum_{n=1}^{\infty}n^{k-1}q^{mn} \end{aligned} \tag{4.22}$$

with $q=\exp(2\pi i\tau)$.

Regularisation In the case where $(n_k,c_k)=(\pm1,0)$, the integral in (4.13) is divergent and must be regularised. This is analogous to the case of ordinary MPLs where the iterated integral representation $G(0,\dots,0;x)$ diverges. In the case of eMPLs the divergence exists for $(n_k,c_k)=(\pm1,0)$ and, therefore, we need a special definition for the cases $\mathcal{E}_4(\begin{smallmatrix} n_1 & \dots & n_k \\ 0 & \dots & 0 \end{smallmatrix};x;\vec{a})$ with $n_i=\pm1$. For $A_i(\begin{smallmatrix} \pm1 \\ 0 \end{smallmatrix})$, we have

$$\begin{aligned} \mathcal{E}_4(A_1\dots A_k;x;\vec{a}) = \frac{1}{k!}\log^k x + &\sum_{l=0}^{k}\sum_{m=1}^{l}\sum_{\sigma}\frac{(-1)^{l+m}}{(k-l)!}\log^{k-l}x \\ &\times \mathcal{E}_4^{\mathrm{R}}(A^{(m)}_{\sigma(1)}\dots A^{(m)}_{\sigma(m-1)}A^{(m)}_{\sigma(m+1)}\dots A^{(m)}_{\sigma(l)} \mid A_m;x;\vec{a}), \end{aligned} \tag{4.23}$$

where the third sum runs over all shuffles and $A_i^{(m)}=A_i$ if $i<m$ and $A_i^{(m)}=(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix})$. The $\mathcal{E}_4^R$ are iterated integrals with certain subtractions to make the integrations finite,

$$\mathcal{E}_4^{\mathrm{R}}(\begin{smallmatrix} n_1 & \dots & n_k \\ 0 & \dots & 0 \end{smallmatrix} | \begin{smallmatrix} n_a \\ 0 \end{smallmatrix}; x; \vec{a}) = \int_0^x dt_1 \Psi_{n_1}(0, t_1) \int_0^{t_1} \cdots \int_0^{t_{k-1}} dt_k (\Psi_{n_a}(0, t_k) - \Psi_1(0, t_k)). \tag{4.24}$$

For instance,

$$\begin{aligned}
\mathcal{E}_4(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}; x; \vec{a}) &= \log x + \mathcal{E}_4^{\mathrm{R}}(| \begin{smallmatrix} -1 \\ 0 \end{smallmatrix}; x; \vec{a}) \\
\mathcal{E}_4(\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}; x; \vec{a}) &= \frac{1}{2} \log^2 x + \mathcal{E}_4^{\mathrm{R}}(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} | \begin{smallmatrix} -1 \\ 0 \end{smallmatrix}; x; \vec{a}) \\
\mathcal{E}_4(\begin{smallmatrix} -1 & 1 \\ 0 & 0 \end{smallmatrix}; x; \vec{a}) &= \frac{1}{2} \log^2 x + \log x \, \mathcal{E}_4^{\mathrm{R}}(| \begin{smallmatrix} -1 \\ 0 \end{smallmatrix}; x; \vec{a}) - \mathcal{E}_4^{\mathrm{R}}(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} | \begin{smallmatrix} -1 \\ 0 \end{smallmatrix}; x; \vec{a}) \\
\mathcal{E}_4(\begin{smallmatrix} -1 & -1 \\ 0 & 0 \end{smallmatrix}; x; \vec{a}) &= \frac{1}{2} \log^2 x + \log x \, \mathcal{E}_4^{\mathrm{R}}(| \begin{smallmatrix} -1 \\ 0 \end{smallmatrix}; x; \vec{a}) - \mathcal{E}_4^{\mathrm{R}}(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix} | \begin{smallmatrix} -1 \\ 0 \end{smallmatrix}; x; \vec{a}) + \mathcal{E}_4^{\mathrm{R}}(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} | \begin{smallmatrix} -1 \\ 0 \end{smallmatrix}; x; \vec{a})
\end{aligned} \tag{4.25}$$

with

$$\begin{aligned}
\mathcal{E}_4^{\mathrm{R}}(| \begin{smallmatrix} -1 \\ 0 \end{smallmatrix}; x; \vec{a}) &= \int_0^x dt (\Psi_{-1}(0, t) - \Psi_1(0, t)) \\
\mathcal{E}_4^{\mathrm{R}}(\begin{smallmatrix} \pm 1 \\ 0 \end{smallmatrix} | \begin{smallmatrix} -1 \\ 0 \end{smallmatrix}; x; \vec{a}) &= \int_0^x dt_1 \Psi_{\pm 1}(0, t_1) \int_0^{t_1} dt_2 (\Psi_{-1}(0, t_2) - \Psi_1(0, t_2)).
\end{aligned} \tag{4.26}$$

The regularisation of eMPLs fulfill the following properties.

1 The regularisation of eMPLs is consistent with the regularisation of ordinary MPLs.

2 The regularisation preserves the shuffle algebra structure.

3 The regularisation preserves the derivative with respect to x.

4 The regulated value for $\mathcal{E}_4(\begin{smallmatrix} n_1 & \dots & n_k \\ 0 & \dots & 0 \end{smallmatrix}; x; \vec{a})$ with $n_i \pm 1$ has a logarithmic singularity for $x = 0$,

$$\mathcal{E}_4(\begin{smallmatrix} n_1 & \dots & n_k \\ 0 & \dots & 0 \end{smallmatrix}; x, \vec{a}) \sim \mathcal{E}_4(\begin{smallmatrix} 1 & \dots & 1 \\ 0 & \dots & 0 \end{smallmatrix}; x, \vec{a}) = \frac{1}{k!} \log^k x, \quad \text{if } x \to 0. \tag{4.27}$$

All these requirements fix the form of the regulated pure eMPLs.

5 Elliptic double-box integral

In this section, we will review the double-box integral and we will introduce an elliptic toy model where a result of the integral in terms of pure elliptic multiple polylogarithms will be presented. The first subsection refers to [1] and, after that, we explain our original work.

The elliptic double-box integral is the simplest non-polylogarithmic contribution to scattering amplitudes of massless particles in four dimensions. This process can be represented as

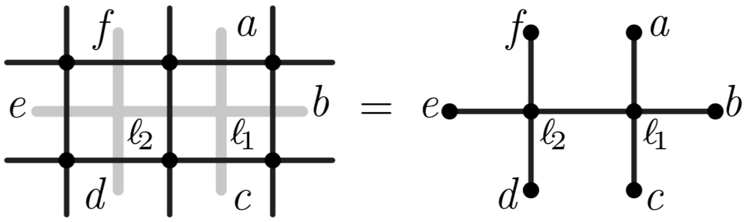

Figure 2: On the left, Feynman diagram representation of the elliptic double-box integral; on the right, its dual graph.

This integral can be understood as a contribution to the ten-point amplitude in massless φ^4 theory but also contributes to the (pure or supersymmetric) Yang-Mills and integrable fishnet theories. Furthermore, in the context of planar maximally supersymmetric Yang-Mills, it is the only diagram that contributes to a particular helicity configuration, making it the entire amplitude in this case. Following [18, 19], we will obtain a representation explicitly by direct integration of Feynman parameters.

5.1 Toy model

Here we present a toy model that is restricted to a particular three-dimensional subspace of ten-particle kinematics. Hence, it depends symmetrically on only three cross-ratios.

We consider dual-momentum x-coordinates, where the momentum of the a^{th} external particle is defined as $p_a = (x_{a-1} - x_a)$ (with cyclic labeling). Using these coordinates, we define

$$(a, b) = (b, a) \equiv (x_a - x_b)^2 = (p_a + \ldots + p_{b-1})^2. \tag{5.1}$$

where each loop momentum l_i can be represented by a dual point x_l, and inverse propagators as $(l_i, a) = (x_{l_i} - x_a)^2$.

In the case of the toy model, we take the dual coordinates to describe the momenta of six massless particles by assigning $x_a, \ldots, x_f$ in the diagram to $x_1, x_3, x_5, x_4, x_6, x_2$, i.e., we have the following restrictions

$$(a, f) = (f, b) = (b, d) = (d, c) = (c, e) = (e, a) = 0. \tag{5.2}$$

This choice of coordinates does not correspond to a physical process but can be evaluated on a well-defined three-dimensional subspace of ten-particle kinematics. We can write the integral as

$$I_{\text{toy}}^{\text{ell}} \equiv \int \frac{d^4\ell_1 d^4\ell_2 \quad \mathfrak{N}(1,4)(2,5)(3,6)}{(\ell_1,1)(\ell_1,3)(\ell_1,5)(\ell_1,\ell_2)(\ell_2,4)(\ell_2,6)(\ell_2,2)}, \tag{5.3}$$

where $\mathfrak{N}$ corresponds to a normalization factor. Now we can transform the integral into a manifestly dual-conformally invariant parametric integral. To achieve this, we must integrate one loop at a time. We assign associate Feynman parameters to the l_1 propagators according to

$$Y_1 \equiv (1) + \beta_1(3) + \beta_2(5) + \gamma_1(\ell_2) \equiv (R_1) + \gamma_1(\ell_2), \tag{5.4}$$

where (a) denotes the dual coordinate (x_a). The l_1 integration gives

$$\begin{aligned} \int d^4\ell_1 \mathcal{I}_{\text{toy}}^{\text{ell}} &= \int_0^\infty d^2\vec{\beta} \int_0^\infty d\gamma_1 \frac{\mathfrak{N}(1,4)(2,5)(3,6)}{(Y_1,Y_1)^2(\ell_2,2)(\ell_2,4)(\ell_2,6)} \\ &= \int_0^\infty d^2\vec{\beta} \frac{\mathfrak{N}(1,4)(2,5)(3,6)}{(R_1,R_1)(\ell_2,R_1)(\ell_2,2)(\ell_2,4)(\ell_2,6)} \end{aligned} \tag{5.5}$$

where we have considered that the γ_1 integral is a total derivative in the second step.

In a similar manner, we introduce the following Feynman parameters for l_2

$$Y_2 \equiv (R_1) + \alpha(6) + \beta_3(2) + \gamma_2(4) \equiv (R_2) + \gamma_2(4), \tag{5.6}$$

and repeat the same method as above. The four-fold representation is

$$I_{\text{toy}}^{\text{ell}} = \int_0^\infty d\alpha \int_0^\infty d^3\vec{\beta} \frac{\mathfrak{N}(1,4)(2,5)(3,6)}{(R_1,R_1)(R_2,4)(R_2,R_2)}. \tag{5.7}$$

We must rescale the Feynman parameters to render this manifestly dual-conformally invariant,

$$\alpha \mapsto \alpha\frac{(1,3)}{(3,6)}, \beta_1 \mapsto \beta_1\frac{(1,5)}{(3,5)}, \beta_2 \mapsto \beta_2\frac{(1,3)}{(3,5)}, \beta_3 \mapsto \beta_3\frac{(1,5)}{(2,5)} \tag{5.8}$$

and, therefore, $I_{\text{toy}}^{\text{ell}}$ becomes

$$I_{\text{toy}}^{\text{ell}} \equiv \int_0^\infty d\alpha \int_0^\infty d^3\vec{\beta} \frac{\mathfrak{N}}{f_1 f_2 f_3}, \left\{ \begin{array}{l} f_1 \equiv \beta_1 + \beta_2 + \beta_1\beta_2 \\ f_2 \equiv 1 + \alpha u_1 + u_2\beta_3 \\ f_3 \equiv f_1 + \alpha(\beta_1 + u_3\beta_3) + \beta_2\beta_3 \end{array} \right\}. \tag{5.9}$$

This expression depends on the usual six-particle cross-ratios $u_1 = (13;46)$, $u_2 = (24;51)$, $u_3 = (35;62)$, with

$$(ab;cd) \equiv \frac{(a,b)(c,d)}{(a,c)(b,d)}. \tag{5.10}$$

To conclude that the integral is elliptic, it is sufficient to see that

$$\mathrm{Res}_{f_i=0}(\frac{d^3\vec{\beta}}{f_1 f_2 f_3}) = \frac{1}{\sqrt{Q(\alpha)}} \tag{5.11}$$

where we introduce the irreducible quartic $Q(\alpha)$,

$$Q(\alpha) \equiv (1+\alpha(u_1+u_2+u_3+\alpha u_1 u_3))^2 - 4\alpha(1+\alpha u_1)^2 u_3. \tag{5.12}$$

After evaluating the β_i integrals, we obtain

$$I^{\mathrm{ell}}_{\mathrm{toy}} = \int_0^\infty d\alpha \frac{\mathfrak{N}}{\sqrt{Q(\alpha)}} H_{\mathrm{toy}}(\alpha) \tag{5.13}$$

where $H_{\mathrm{toy}}(\alpha)$ is sum of pure weight-three hyperlogarithms that depend on the final integration variable. Specifically, we get $H_{\mathrm{toy}}(\alpha) = F_1(\alpha) - F_2(\alpha)$, with

$$\begin{aligned}
F_i(\alpha) \equiv\ & G(\bar{w}_i, 0, 0; \alpha) + G(\bar{w}_i, \bar{0}, \bar{0}; \alpha) - G(\bar{w}_i, 0, \bar{0}; \alpha) - G(\bar{w}_i, \bar{0}, 0; \alpha) \\
& - G(\bar{w}_i, -\bar{w}_1\bar{w}_2, 0; \alpha) - G(\bar{w}_i, \frac{\bar{w}_1\bar{w}_2}{w_1+w_2}, \bar{0}; \alpha) + G(\bar{w}_i, \frac{\bar{w}_1\bar{w}_2}{w_1+w_2}; \alpha) \log(w_1 w_2 \bar{w}_1 \bar{w}_2) \\
& - G(\bar{w}_i, -\bar{w}_1\bar{w}_2; \alpha) \log(\frac{-1}{\bar{w}_1\bar{w}_2}) + (G(\bar{w}_i, 0; \alpha) - G(\bar{w}_i, \bar{0}; \alpha)) \log(-w_1 w_2) \\
& + G(\bar{w}_i; \alpha)\{\frac{1}{2}\log^2(\frac{1}{\overline{w_1+w_2}}) + \log(w_1 w_2)\log(\frac{-1}{\bar{w}_1\bar{w}_2}) \\
& - \log(\frac{1}{\overline{w_1+w_2}})\log(\frac{1}{\bar{w}_1\bar{w}_2}) + \mathrm{Li}_2(-\frac{\overline{w_1+w_2}}{\bar{w}_1\bar{w}_2})\},
\end{aligned} \tag{5.14}$$

where we have introduced the notation $\bar{x} \equiv -1/(1+x)$ (such that $\bar{0} = -1$). $G(\bar{w}_i; \alpha)$ is the ordinary multiple polylogarithm defined in (2.4), and

$$w_{1,2} \equiv \left[\alpha((\alpha u_3 - 1)u_1 - u_2 + u_3) - 1 \pm \sqrt{Q(\alpha)}\right] / (2\alpha u_2). \tag{5.15}$$

The goal of the following sections is to rewrite every term in (5.14) and, ultimately, $I^{\mathrm{ell}}_{\mathrm{toy}}$ in terms of $\widetilde{\Gamma}$ functions and pure elliptic multiple polylogarithms.

5.2 Methodology

In this section, we explain schematically the methodology used to obtain the results and illustrate the process in further detail with one example. Furthermore, we make some remarks about the numerical evaluation of eMPLs.

The process that we have followed to rewrite the logarithms and multiple polylogarithms in (5.14) in terms of pure elliptic multiple polylogarithms is the following.

1 Take the derivative of the logarithm or ordinary MPL with respect to α.

2 Apply a series of simplifications such as partial fractioning to the derivative and collect terms that are equal or proportional to the differentials that describe the kernels in (4.17) and (4.18).

3 Once we have a linear combination of kernels, we integrate back to obtain the correspondent eMPLs.

4 Evaluate numerically to check that the result matches with its original form.

These steps can be seen in detail in the following example, where all the considerations taken are fully explained.

5.2.1 Example

To illustrate the methodology that has been used in all the functions, we will rewrite $G(\bar{w}_1;\alpha)$ in terms of pure eMPLs.

The first step is taking the derivative respect to α of the function. To perform this, we use the package PolyLogTools [20] in Mathematica, that includes a function that takes the derivative of MPLs directly. Before simplifying the result, we must take two considerations.

First, we perform the following change of variables

$$x \to \frac{\alpha}{1+\alpha}, \tag{5.16}$$

such that the integrations limits in the integral (5.13) become $(0,\infty) \to (0,1)$. In this step, it is important to take into account a factor $1/(x-1)^2$ in the derivative of $G(\bar{w}_1;\alpha)$ that corresponds to the integration variable in $I^{\text{ell}}_{\text{toy}}$.

Second, the variable y that appears in the integration kernels in (4.17) and (4.18) does not correspond to $y^2 = Q(\alpha)$ (before changing the variable), since the coefficient of α^4 is not equal to 1. Thus, y^2 corresponds to the quartic polynomial after normalisation.

Now that we have these considerations, we can start simplifying the derivative, that now depends on x. After partial fractioning we obtain three terms. The first one is simply

$$-\frac{1}{x-1}. \tag{5.17}$$

We can instantly relate this expression with the kernel of the form $-\Psi_1(1,x)$.

The second term is

$$\frac{2u_1u_3x - u_1 - u_2 - 2u_3x + u_3 + 1}{2(u_1u_3x^2 - u_1x - u_2x + u_2 - u_3x^2 + u_3x + x - 1)}. \tag{5.18}$$

Here we can prove that the cross-ratios cannot factor rationally the denominator and, thus, we cannot perform partial fractioning. To proceed with the partial fractioning, we must introduce variables for the cross-ratios,

$$\begin{aligned} u_1 &= \frac{y_1(1-y_2)(1-y_3)}{(1-y_1y_2)(1-y_3y_1)} \\ u_2 &= \frac{y_2(1-y_3)(1-y_1)}{(1-y_2y_3)(1-y_1y_2)} \\ u_3 &= \frac{y_3(1-y_1)(1-y_2)}{(1-y_3y_1)(1-y_2y_3)}, \end{aligned} \tag{5.19}$$

where y_1, y_2, y_3 are now the cross-ratios that we consider in all the results. These variables correspond to familiar parametrizations of functions with hexagon kinematics [21]. We obtain

$$\frac{(y_1-1)((2x-1)(y_1-1)y_3 - y_1y_3^2 + 1)}{2(x^2(y_1-1)^2y_3 - x(y_1-1)(y_3+1)(y_1y_3-1) + (y_1y_3-1)^2)}, \tag{5.20}$$

and now the partial fractioning is direct, where we obtain two simple terms that can be related to the integration kernels of the pure eMPLs.

In an identical manner, we proceed with the third term of the derivative, that we omit here due to its length.

Finally, we identify the different terms with the integration kernels and, integrating them, we obtain the pure elliptic multiple polylogarithms. In this example, the result is

$$\begin{aligned} G(\bar{w}_1;\alpha) = &\frac{1}{2}\mathcal{E}_4(\begin{smallmatrix} -1 \\ \frac{1-y_1y_3}{y_3-y_1y_3} \end{smallmatrix};x) + \frac{1}{2}\mathcal{E}_4(\begin{smallmatrix} -1 \\ \frac{y_1y_3-1}{y_1-1} \end{smallmatrix};x) + \frac{1}{2}\mathcal{E}_4(\begin{smallmatrix} 1 \\ \frac{1-y_1y_3}{y_3-y_1y_3} \end{smallmatrix};x) \\ &+ \frac{1}{2}\mathcal{E}_4(\begin{smallmatrix} 1 \\ \frac{y_1y_3-1}{y_1-1} \end{smallmatrix};x) - \mathcal{E}_4(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix};x) - \mathcal{E}_4(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix};x). \end{aligned} \tag{5.21}$$

Clearly, the result we obtain is pure. It is important to remark that terms proportional to the function Z_4 appear before obtaining the final result but all of them cancel after applying the following identity,

$$\begin{aligned} Z_4(1) &= \frac{1}{2}\left(\frac{B}{c_4C} + Z_4\left(\frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)}\right) + Z_4(0)\right) \\ &= \frac{1}{2}\left(-\frac{A}{c_4C} + Z_4\left(\frac{1-y_1y_3}{y_3-y_1y_3}\right) + Z_4\left(\frac{y_1y_3-1}{y_1-1}\right)\right), \end{aligned} \tag{5.22}$$

where we have introduced

$$
\begin{aligned}
A = &-\frac{2(y_1^2y_2^2-2y_1^2y_2+y_1^2-2y_1y_2^2+4y_1y_2-2y_1+y_2^2-2y_2+1)}{(y_1-y_2)(y_1y_2-1)(y_1y_3-1)^2} \\
&+\frac{-3y_1^3y_2^2+6y_1^3y_2-3y_1^3+y_1^2y_2^3+4y_1^2y_2^2-11y_1^2y_2+6y_1^2-2y_1y_2^3+y_1y_2^2+4y_1y_2-3y_1+y_2^3-2y_2^2+y_2}{(y_1-y_2)^2(y_1y_2-1)(y_1y_3-1)} \\
&+\frac{y_1^3y_2^3-3y_1^3y_2^2+3y_1^3y_2-y_1^3+y_1^2y_2^4-3y_1^2y_2^3+3y_1^2y_2^2-y_1^2y_2-y_1y_2^4+3y_1y_2^3-3y_1y_2^2+y_1y_2-y_2^5+3y_2^4-3y_2^3+y_2^2}{y_2(y_1-y_2)^2(y_1y_2-1)(y_2y_3-1)} \\
&-\frac{-3y_1y_2+y_1+y_2^2+y_2}{y_2(y_1y_2-1)}+\frac{y_1(y_2-1)y_3}{y_1y_2-1}+\frac{y_2-1}{y_3(y_1y_2-1)}, \\
B = &\frac{(y_2-1)(y_3-1)(y_1^4y_2^2y_3^3+y_1^3y_2y_3^2(y_2-y_3-2)+y_1^2y_3(-2y_2^2y_3+3y_2(y_3-1)+2)+y_1(2y_2y_3+y_2-y_3)-1)}{(y_1y_2-1)(y_1y_3-1)^2(y_2y_3-1)(y_1y_2y_3-1)},
\end{aligned}
\tag{5.23}
$$

and C is the square root of the coefficient of α^4 in the quartic polynomial Q.

5.3 Result in terms of pure elliptic multiple polylogarithms

5.3.1 Result in terms of $\mathcal{E}_4$

Following the methodology explained in the previous section, we can rewrite all the terms in (5.14) in terms of pure elliptic multiple polylogarithms. Then, we simply substitute H_{toy} and $1/\sqrt{Q(x)} = c_4/(\omega_1 C)\Psi_0(0,x)$ in (5.13). Integrating in the variable x from 0 to 1, we get the final expression. $I_{\text{toy}}^{\text{ell}}$ in terms of pure eMPLs is

$$
\frac{\omega_1}{c_4 C} \times T, \tag{5.24}
$$

with

$$
\begin{aligned}
T = \; & 3\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{smallmatrix};1\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{smallmatrix};1\right) + 3\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{smallmatrix};1\right) \\
& - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} \end{smallmatrix};1\right) - 3\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 0 & \frac{(y_1y_3-1)(y_2y_3-1)}{(y_1-1)(y_2-1)y_3} & 0 \end{smallmatrix};1\right) \\
& - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 0 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & 1 \end{smallmatrix};1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 0 & \frac{(y_1y_3-1)(y_1y_2y_3-1)}{(y_1-1)(y_1y_2y_3^2-1)} & 1 \end{smallmatrix};1\right) \\
& - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 0 & \frac{(y_1y_3-1)(y_1y_2y_3-1)}{(y_1-1)(y_1y_2y_3^2-1)} & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} \end{smallmatrix};1\right) \\
& + 3\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 0 & \frac{(y_1y_3-1)(y_2y_3-1)}{-2y_2y_3+y_1(y_3y_2+y_2-2)y_3+y_3+1} & 0 \end{smallmatrix};1\right) \\
& - 3\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 0 & \frac{(y_1y_3-1)(y_2y_3-1)}{-2y_2y_3+y_1(y_3y_2+y_2-2)y_3+y_3+1} & 1 \end{smallmatrix};1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{smallmatrix};1\right) \\
& - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} \end{smallmatrix};1\right) + 3\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{smallmatrix};1\right) \\
& - 3\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{smallmatrix};1\right) + 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} \end{smallmatrix};1\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & \frac{(y_1y_3-1)(y_2y_3-1)}{(y_1-1)(y_2-1)y_3} & 0 \end{smallmatrix};1\right)-2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & 0 \end{smallmatrix};1\right)\\
&+3\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & 1 \end{smallmatrix};1\right)-2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} \end{smallmatrix};1\right)\\
&-\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & \frac{(y_1y_3-1)(y_1y_2y_3-1)}{(y_1-1)(y_1y_2y_3^2-1)} & 1 \end{smallmatrix};1\right)+\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & \frac{(y_1y_3-1)(y_1y_2y_3-1)}{(y_1-1)(y_1y_2y_3^2-1)} & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} \end{smallmatrix};1\right)\\
&-3\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & \frac{(y_1y_3-1)(y_2y_3-1)}{-2y_2y_3+y_1(y_3y_2+y_2-2)y_3+y_3+1} & 0 \end{smallmatrix};1\right)+3\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & \frac{(y_1y_3-1)(y_2y_3-1)}{-2y_2y_3+y_1(y_3y_2+y_2-2)y_3+y_3+1} & 1 \end{smallmatrix};1\right)\\
&-\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{1-y_1y_3}{y_3-y_1y_3} & 0 & 1 \end{smallmatrix};1\right)+\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{1-y_1y_3}{y_3-y_1y_3} & 0 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} \end{smallmatrix};1\right)\\
&-\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{1-y_1y_3}{y_3-y_1y_3} & 1 & 0 \end{smallmatrix};1\right)+\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{1-y_1y_3}{y_3-y_1y_3} & \frac{(y_1y_3-1)(y_2y_3-1)}{(y_1-1)(y_2-1)y_3} & 0 \end{smallmatrix};1\right)\\
&+\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{1-y_1y_3}{y_3-y_1y_3} & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & 0 \end{smallmatrix};1\right)-\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{1-y_1y_3}{y_3-y_1y_3} & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & 1 \end{smallmatrix};1\right)\\
&+\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{1-y_1y_3}{y_3-y_1y_3} & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} \end{smallmatrix};1\right)+\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{1-y_1y_3}{y_3-y_1y_3} & \frac{(y_1y_3-1)(y_1y_2y_3-1)}{(y_1-1)(y_1y_2y_3^2-1)} & 1 \end{smallmatrix};1\right)\\
&-\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{1-y_1y_3}{y_3-y_1y_3} & \frac{(y_1y_3-1)(y_1y_2y_3-1)}{(y_1-1)(y_1y_2y_3^2-1)} & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} \end{smallmatrix};1\right)-\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{y_1y_3-1}{y_1-1} & 0 & 1 \end{smallmatrix};1\right)\\
&+\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{y_1y_3-1}{y_1-1} & 0 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} \end{smallmatrix};1\right)-\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{y_1y_3-1}{y_1-1} & 1 & 0 \end{smallmatrix};1\right)\\
&+\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{y_1y_3-1}{y_1-1} & \frac{(y_1y_3-1)(y_2y_3-1)}{(y_1-1)(y_2-1)y_3} & 0 \end{smallmatrix};1\right)+\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{y_1y_3-1}{y_1-1} & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & 0 \end{smallmatrix};1\right)\\
&-\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{y_1y_3-1}{y_1-1} & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & 1 \end{smallmatrix};1\right)+\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{y_1y_3-1}{y_1-1} & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} \end{smallmatrix};1\right)\\
&+\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{y_1y_3-1}{y_1-1} & \frac{(y_1y_3-1)(y_1y_2y_3-1)}{(y_1-1)(y_1y_2y_3^2-1)} & 1 \end{smallmatrix};1\right)-\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{y_1y_3-1}{y_1-1} & \frac{(y_1y_3-1)(y_1y_2y_3-1)}{(y_1-1)(y_1y_2y_3^2-1)} & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} \end{smallmatrix};1\right)\\
&+\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & 1 & 0 \end{smallmatrix};1\right)-\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & 1 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} \end{smallmatrix};1\right)\\
&+\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & 1 \end{smallmatrix};1\right)-2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & \frac{(y_1y_3-1)(y_1y_2y_3-1)}{(y_1-1)(y_1y_2y_3^2-1)} & 1 \end{smallmatrix};1\right)\\
&+2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & \frac{(y_1y_3-1)(y_1y_2y_3-1)}{(y_1-1)(y_1y_2y_3^2-1)} & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} \end{smallmatrix};1\right)\\
&-2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 1 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} \end{smallmatrix};1\right)\log\left(\frac{(y_2-1)(y_1y_2-1)y_3}{y_2(y_3-1)(y_1y_3-1)}\right)\\
&+\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \frac{1-y_1y_3}{y_3-y_1y_3} & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} \end{smallmatrix};1\right)\log\left(\frac{(y_2-1)(y_1y_2-1)y_3}{y_2(y_3-1)(y_1y_3-1)}\right)\\
&+\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \frac{y_1y_3-1}{y_1-1} & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} \end{smallmatrix};1\right)\log\left(\frac{(y_2-1)(y_1y_2-1)y_3}{y_2(y_3-1)(y_1y_3-1)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & 1\end{matrix};1\right)\log\left(\frac{(y_1-1)^2(y_2-1)y_2(y_3-1)y_3}{(y_1y_2-1)(y_1y_3-1)(y_2y_3-1)^2}\right)\\
&+\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & 0 & \frac{(y_1y_3-1)(y_1y_2y_3-1)}{(y_1-1)(y_1y_2y_3^2-1)}\end{matrix};1\right)\log\left(\frac{(y_1-1)y_2(y_3-1)}{(y_1y_2-1)(y_2y_3-1)}\right)\\
&+2\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & 1 & 0\end{matrix};1\right)\log\left(\frac{(y_1-1)y_2(y_3-1)}{(y_1y_2-1)(y_2y_3-1)}\right)\\
&+\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & \frac{1-y_1y_3}{y_3-y_1y_3} & \frac{(y_1y_3-1)(y_1y_2y_3-1)}{(y_1-1)(y_1y_2y_3^2-1)}\end{matrix};1\right)\log\left(\frac{(y_1-1)y_2(y_3-1)}{(y_1y_2-1)(y_2y_3-1)}\right)\\
&+\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & \frac{y_1y_3-1}{y_1-1} & \frac{(y_1y_3-1)(y_1y_2y_3-1)}{(y_1-1)(y_1y_2y_3^2-1)}\end{matrix};1\right)\log\left(\frac{(y_1-1)y_2(y_3-1)}{(y_1y_2-1)(y_2y_3-1)}\right)\\
&-2\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)} & \frac{(y_1y_3-1)(y_1y_2y_3-1)}{(y_1-1)(y_1y_2y_3^2-1)}\end{matrix};1\right)\log\left(\frac{(y_1-1)y_2(y_3-1)}{(y_1y_2-1)(y_2y_3-1)}\right)\\
&-3\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & 0 & \frac{(y_1y_3-1)(y_2y_3-1)}{(y_1-1)(y_2-1)y_3}\end{matrix};1\right)\log\left(\frac{(y_1-1)(y_2-1)y_3}{(y_1y_3-1)(y_2y_3-1)}\right)\\
&+3\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & 0 & \frac{(y_1y_3-1)(y_2y_3-1)}{-2y_2y_3+y_1(y_3y_2+y_2-2)y_3+y_3+1}\end{matrix};1\right)\log\left(\frac{(y_1-1)(y_2-1)y_3}{(y_1y_3-1)(y_2y_3-1)}\right)\\
&+\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & 1 & \frac{(y_1y_3-1)(y_2y_3-1)}{(y_1-1)(y_2-1)y_3}\end{matrix};1\right)\log\left(\frac{(y_1-1)(y_2-1)y_3}{(y_1y_3-1)(y_2y_3-1)}\right)\\
&-3\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & 1 & \frac{(y_1y_3-1)(y_2y_3-1)}{-2y_2y_3+y_1(y_3y_2+y_2-2)y_3+y_3+1}\end{matrix};1\right)\log\left(\frac{(y_1-1)(y_2-1)y_3}{(y_1y_3-1)(y_2y_3-1)}\right)\\
&+\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & \frac{1-y_1y_3}{y_3-y_1y_3} & \frac{(y_1y_3-1)(y_2y_3-1)}{(y_1-1)(y_2-1)y_3}\end{matrix};1\right)\log\left(\frac{(y_1-1)(y_2-1)y_3}{(y_1y_3-1)(y_2y_3-1)}\right)\\
&+\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & \frac{y_1y_3-1}{y_1-1} & \frac{(y_1y_3-1)(y_2y_3-1)}{(y_1-1)(y_2-1)y_3}\end{matrix};1\right)\log\left(\frac{(y_1-1)(y_2-1)y_3}{(y_1y_3-1)(y_2y_3-1)}\right)\\
&+\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & 1 & \frac{(y_1y_3-1)(y_1y_2y_3-1)}{(y_1-1)(y_1y_2y_3^2-1)}\end{matrix};1\right)\log\left(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)}\right)\\
&+\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & \frac{1-y_1y_3}{y_3-y_1y_3} & 0\end{matrix};1\right)\log\left(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)}\right)\\
&+\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & \frac{y_1y_3-1}{y_1-1} & 0\end{matrix};1\right)\log\left(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)}\right)\\
&+\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & 1 & 1\end{matrix};1\right)\left(2\log\left(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)}\right)-3\log\left(\frac{(y_1y_3-1)(y_2y_3-1)}{(y_1-1)(y_2-1)y_3}\right)\right)\\
&+\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & \frac{1-y_1y_3}{y_3-y_1y_3} & 1\end{matrix};1\right)\log\left(\frac{(y_1y_3-1)(y_2y_3-1)}{(y_1-1)(y_2-1)y_3}\right)\\
&+\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & \frac{y_1y_3-1}{y_1-1} & 1\end{matrix};1\right)\log\left(\frac{(y_1y_3-1)(y_2y_3-1)}{(y_1-1)(y_2-1)y_3}\right)\\
&+\mathcal{E}_4\left(\begin{matrix}0 & -1 & 1\\ 0 & 0 & 1\end{matrix};1\right)\left(\log\left(\frac{(y_1-1)y_2(y_3-1)}{(y_1y_2-1)(y_2y_3-1)}\right)+2\log\left(\frac{(y_1y_3-1)(y_2y_3-1)}{(y_1-1)(y_2-1)y_3}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{E}_4(\begin{smallmatrix}0 & -1\\ 0 & 0\end{smallmatrix};1)(-\log(\frac{(y_1-1)y_2(y_3-1)}{(y_2-1)(y_1y_2y_3-1)})\log(\frac{(y_1-1)y_2(y_3-1)}{(y_1y_2-1)(y_2y_3-1)})\\
&+i\pi\log(\frac{(y_1-1)y_2(y_3-1)}{(y_1y_2-1)(y_2y_3-1)})-\mathrm{Li}_2(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)}))\\
&+\mathcal{E}_4(\begin{smallmatrix}0 & -1\\ 0 & \frac{1-y_1y_3}{y_3-y_1y_3}\end{smallmatrix};1)(-\frac{1}{2}\log^2(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)})\\
&+\log(\frac{(y_1-1)(y_1y_2-1)y_3}{(y_1y_3-1)(y_1y_2y_3-1)})\log(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)})-i\pi\log(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)})\\
&-\mathrm{Li}_2(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)})+\frac{\pi^2}{6})\\
&+\mathcal{E}_4(\begin{smallmatrix}0 & -1\\ 0 & \frac{y_1y_3-1}{y_1-1}\end{smallmatrix};1)(-\frac{1}{2}\log^2(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)})\\
&+(\log(\frac{(y_1-1)(y_1y_2-1)y_3}{(y_1y_3-1)(y_1y_2y_3-1)})-i\pi)\log(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)})-\mathrm{Li}_2(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)})+\frac{\pi^2}{6})\\
&+2\mathcal{E}_4(\begin{smallmatrix}0 & -1\\ 0 & \frac{(y_1y_2-1)(y_1y_3-1)}{(y_1-1)(y_1y_2y_3-1)}\end{smallmatrix};1)(\log(\frac{(y_1-1)y_2(y_3-1)}{(y_2-1)(y_1y_2y_3-1)})\log(\frac{(y_1-1)y_2(y_3-1)}{(y_1y_2-1)(y_2y_3-1)})\\
&-i\pi\log(\frac{(y_1-1)y_2(y_3-1)}{(y_1y_2-1)(y_2y_3-1)})+\mathrm{Li}_2(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)}))\\
&+\mathcal{E}_4(\begin{smallmatrix}0 & -1\\ 0 & 1\end{smallmatrix};1)(\log^2(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)})-2\log(\frac{(y_2-1)(y_1y_2-1)y_3}{y_2(y_3-1)(y_1y_3-1)})\log(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)})\\
&+\frac{1}{2}\log(\frac{(y_2-1)(y_1y_2y_3-1)}{(y_1-1)y_2(y_3-1)})\log(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)})+\frac{1}{2}i\pi\log(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)})\\
&+\frac{1}{2}i\pi\log(\frac{(y_2-1)(y_1y_2-1)y_3}{y_2(y_3-1)(y_1y_3-1)})+\frac{1}{2}i\pi\log(\frac{(y_1y_3-1)(y_2y_3-1)}{(y_1-1)(y_2-1)y_3})\\
&+\frac{1}{2}\log(\frac{(y_2-1)(y_1y_2-1)y_3}{y_2(y_3-1)(y_1y_3-1)})\log(\frac{(y_2-1)(y_1y_2y_3-1)}{(y_1-1)y_2(y_3-1)})\\
&+\frac{1}{2}\log(\frac{(y_1y_3-1)(y_2y_3-1)}{(y_1-1)(y_2-1)y_3})\log(\frac{(y_2-1)(y_1y_2y_3-1)}{(y_1-1)y_2(y_3-1)})+\mathrm{Li}_2(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)})-\frac{\pi^2}{3})\\
&+\mathcal{E}_4(\begin{smallmatrix}0\\ 0\end{smallmatrix};1)(-\frac{1}{6}\log^3(\frac{(y_1-1)y_2(y_3-1)}{(y_2-1)(y_1y_2y_3-1)})-\frac{1}{6}\pi^2\log(\frac{(y_1-1)y_2(y_3-1)}{(y_2-1)(y_1y_2y_3-1)})\\
&+\log^2(\frac{(y_1-1)y_2(y_3-1)}{(y_1y_2-1)(y_2y_3-1)})(\frac{i\pi}{2}-\frac{1}{2}\log(\frac{(y_1-1)y_2(y_3-1)}{(y_2-1)(y_1y_2y_3-1)}))\\
&+\log(\frac{(y_1-1)y_2(y_3-1)}{(y_1y_2-1)(y_2y_3-1)})(\log(\frac{(y_2-1)y_3}{y_2(y_3-1)})(\log(\frac{(y_2-1)(y_1y_2y_3-1)}{(y_1-1)y_2(y_3-1)})+i\pi)\\
&+\log(\frac{y_1y_2-1}{y_1y_3-1})(\log(\frac{(y_2-1)(y_1y_2y_3-1)}{(y_1-1)y_2(y_3-1)})+i\pi))\\
&-\log(\frac{(y_2-1)(y_1y_2-1)y_3}{y_2(y_3-1)(y_1y_3-1)})\mathrm{Li}_2(\frac{(1-y_1y_2)(1-y_2y_3)}{(1-y_1)y_2(1-y_3)})+\mathrm{Li}_3(\frac{(y_1y_2-1)(y_2y_3-1)}{(y_1-1)y_2(y_3-1)})\\
&-\mathrm{Li}_3(-\frac{(y_1-1)y_2(y_3-1)}{(y_2-1)(y_1y_2y_3-1)})+\zeta(3)).
\end{aligned}
\tag{5.25}
$$

5.3.2 Result in terms of $\widetilde{\Gamma}$

To find an expression for $I_{\text{toy}}^{\text{ell}}$ in terms of the $\widetilde{\Gamma}$ functions, we can simply take the previous result in (5.25) and use the relations in (3.34) (substituting the integration kernels ψ for Ψ according to (4.17) and (4.18)). We omit the result due to its length but we can state that it preserves the same properties as (5.25) and its pure form.

5.3.3 Numerical evaluation and properties

There are some checks that we can perform to provide evidence that the results obtained are right.

Numerical evaluation Firstly, it is straightforward to compute numerically the functions $\mathcal{E}_4$ from their definition as iterated integrals (4.13), for some given cross-ratios y_1, y_2, y_3. Clearly, this numerical implementation must include the proper regularisations (4.24). In a similar manner, we can obtain the regularised version by taking advantage of the shuffle algebra. Particularly, we can express any eMPL function in terms of a linear combination of products of other eMPLs where the indices $(n_k, c_k) \neq (\pm 1, 0)$ in the last place. We use this property to make sure that the $\widetilde{\Gamma}$ functions are not divergent and, then, take the q-expansion of the Eisenstein-Kronecker coefficients in (4.22) in the iterated integrals.

Finally, we can evaluate numerically the ordinary MPLs in (5.14) using GiNaC and check our results.

Properties There are two main conditions that our results must fulfil. In the first place, all terms in our results must have uniform weight and it must be four. To check this, we can look at the following table, where it shows the weight of the different functions [10]

Name	Unipotent	Length	Weight
Rational Functions	No	0	0
Algebraic Functions	No	0	0
$i\pi$	No	0	1
ζ_{2n}	No	0	$2n$
ζ_{2n+1}	Yes	0	$2n+1$
$\log x$	Yes	1	1
$\mathrm{Li}_n(x)$	Yes	n	n
$G(c_1,\ldots,c_k;x)$	Yes	k	k
ω_1	No	0	1
η_1	No	0	1
τ	Yes	1	0
$g^{(n)}(z,\tau)$	No	0	n
$h^{(n)}_{N,r,s}(\tau)$	No	0	n
$Z_4(c,\vec{a})$	No	0	0
$G_*(\vec{a})$	No	0	0
$\mathcal{E}_4(\begin{smallmatrix} n_1 & \ldots & n_k \\ c_1 & \ldots & c_k \end{smallmatrix};x,\vec{a})$	Yes	k	$\sum_i \|n_i\|$
$\widetilde{\Gamma}(\begin{smallmatrix} n_1 & \ldots & n_k \\ z_1 & \ldots & z_k \end{smallmatrix};z,\tau)$	Yes	k	$\sum_i n_i$
$I\left(\begin{smallmatrix} n_1 & N_1 \\ r_1 & s_1 \end{smallmatrix}\middle\|\ldots\middle\|\begin{smallmatrix} n_k & N_k \\ r_k & s_k \end{smallmatrix};\tau\right)$	Yes	k	$\sum_i n_i$

Figure 3: Weight and length of different building blocks encountered in elliptic Feynman integrals.

Indeed, we see that this is the case.

In the second place, our results must give the same numerical result under permutations of cross-ratios. This can be directly demonstrated by a numerical evaluation of the results with different values for y_1, y_2, y_3, using the methodology explained above.

6 Discussion

In this section, we give some remarks on the results obtained and discuss briefly what future research can be explored in the line of the work done in this thesis.

6.1 Summary

In this thesis, we have written a toy model of the elliptic double-box integral in terms of pure elliptic multiple polylogarithms.

To achieve this, we have reviewed the background needed to understand the motivation and the building blocks of the results. Firstly, we have reviewed the multiple polylogarithms, including some of their basic properties. Secondly, we have reviewed the elliptic curves and generalised the MPLs to the elliptic case, defining the elliptic MPLs both on a complex torus and in terms of the variables (x, y). After that, we have introduced the concept of pure functions and pure eMPLs, giving some motivation about why the properties of this class of functions are interesting in the study of elliptic Feynman integrals. Finally, after a brief review of the double-box integral, we have presented our results.

In summary, we can draw some conclusions about the results.

1 It is possible to write a toy model of the elliptic double-box integral expressed entirely in terms of the $\mathcal{E}_4$ and the $\widetilde{\Gamma}$ functions, and there exists a direct relation between both.

2 The results preserve the properties of the toy model written in terms of ordinary MPLs, i.e., they have uniform weight (equal to 4 in this case) and are symmetric with respect to permutations of the cross-ratios.

3 We found identities that relate the Z_4 functions and, as a consequence, we obtained pure results by cancelling some terms.

6.2 Outlook

A natural extension of this thesis would be to study the symbol of our results, similarly to [22]. The symbol is defined as a map which associates to an MPL a tensor whose entries are rational (or algebraic) functions and is a particularly powerful tool for computing with MPLs. The main advantage of the symbol is that it trivializes complicated functional relations among MPLs. The research would consist in extending this to eMPLs.

Another direction worth exploring is to seek identities between the eMPLs in the results that manifest the symmetry under permutation of the cross-ratios in the toy model. As we have stated on previous occasions, this property has been demonstrated numerically but it might be worth studying it analytically.

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