



Neutrino masses and mixing

-a group and quantum field theoretical approach.

Christine Hartmann

Master's Thesis

October, 2011



Niels Bohr Institute



Neutrino masses and mixing, -a group and quantum field theoretical approach.

This report was prepared by

Christine Hartmann

Supervisors

Professor Poul Henrik Damgaard

Professor Anthony Zee

Release date: October, 2011

Comments: This report is part of the requirements to achieve the Master of Science in Physics (M.Sc.) at the University of Copenhagen.
This report represents 60 ECTS points.

Niels Bohr Institute
International Academy and Discovery Center
University of Copenhagen
Blegdamsvej 17
2100 Copenhagen
Denmark

www.nbi.ku.dk

Tel: (+45) 35 32 52 09

E-mail: eksp-blv@nbi.dk

Kavli Institute for Theoretical Physics
Kohn Hall
University of California, Santa Barbara
Santa Barbara
CA 93106-4030
USA

www.kitp.ucsb.edu

Tel: (+1) (805) 893 4111

E-mail: info@kitp.ucsb.edu

Abstract

Within the last decade after the discovery of neutrino oscillations, there has been an ongoing search to reveal the mysteries associated with massive neutrinos. Physicists have sought to explore this undiscovered area to find the physical masses and the mechanisms responsible for these.

Model building is a way to approach this, and especially group theory has been used to describe neutrino mixing. The Frobenius group T_{13} has turned out to be very compatible for this task. To achieve a more original approach, this group will be applied when investigating some interesting areas of neutrino physics.

Two popular mechanisms used to naturally describe the small neutrino masses consist of the see-saw mechanism and radiative loop corrections. These take into account that neutrinos don't have charge, can be their own antiparticles and therefore have Majorana abilities. These two mechanisms will be investigated and applied with the Frobenius group T_{13} .

The Frobenius group T_{13} forms an excellent fit with tribimaximal neutrino mixing. Implementing the see-saw mechanism with this group generates further constraints and leads to a predictive model, where neutrino mass eigenvalues and allowed regions for neutrinoless double beta decay are suggested. Since it has recently been insinuated that the $(e3)$ element of the neutrino mixing can not be exactly zero, it must deviate from tribimaximal mixing. The Zee model is introduced to create radiative loop corrections, thus proposing a way to achieve these small deviations. However, tribimaximal is a highly suitable first order mixing matrix that can be used to describe neutrino mixing. This has been exploited in this thesis.

Preface

In 2009 I started a year of exchange at the University of California in Santa Barbara. At the time I did not know the challenges I was going to face or the exciting experiences I was about to have during that year. Moreover, I did not know that the year would turn into almost two, as I would be starting my Master's Thesis working with a world-famous Professor at the Kavli Institute for Theoretical Physics.

I had only heard of the KITP and the Nobel laureates wondering around in the halls of the building unraveling the mysteries of the universe. I could only hope to learn and be inspired from such brilliant people, when I started the first school day at UCSB. The exchange year turned out to be tough, but I received a knowledge and experience I could not have found in my own country. Not only in academia, where I was taught by excellent professors, but also from the experience of being abroad, meeting physicist from a different part of the world and being surrounded by people, some of which, valued physics above all.

I found an excellent advisor, co-worker and friend in Tony Zee. He introduced me to the exciting area of neutrino physics and brought me to the level of research. Taking his job as my advisor seriously, I got the amazing opportunity to go to Taiwan for a couple of weeks, where I gave my first talk on the subject of neutrinos and got the chance to speak with other neutrino physics enthusiasts. I was stunned by the hospitality of the Academia Sinica and the friendly Taiwanese people. During the visit I was honored to attend the 1 month celebration of Max, Tony and Janice Zee's son.

I returned to Copenhagen in April 2011 to finish my Master's Thesis under the guidance of Poul Henrik Damgaard. I had speculations on whether he would be able to advise me on this subject outside his own fields. Without knowing the field, it seems, he somehow managed to ask the right questions, so that my own confidence within the field has grown and I have had the feeling of being led in the right direction. I have greatly appreciated his guidance during my application to CERN, which resulted in a 6 months grant carrying out research at CERN during my Ph.D.

My work on neutrino physics has been described in two papers:

C. Hartmann and A. Zee, Nucl. Phys. B **853**, 105 (2011) [arXiv:1106.0333[hep-ph]].
C. Hartmann (2011) [arXiv:1109.5143[hep-ph]]. Submitted to Nucl. Phys. B.

Christine Hartmann

October 3, 2011, Copenhagen

Acknowledgements

I want to thank Prof. A. Zee for his invaluable guidance throughout the process of this thesis and for very interesting discussions. Without his help, this thesis would not have existed in the first place. I appreciate the confidence he has shown in me. I also want to thank Prof. P. H. Damgaard for taking on the task of guiding a student on a subject outside his own field. His thoughtful questions have led to a better understanding of the field. Furthermore, he has been a tremendous help in practical areas associated with submitting papers and the application to CERN.

Supporting and loving as always, my parents are indispensable. I greatly appreciate their interest and encouragement. Last but not least, my incredible boyfriend Jon Fold von Bülow deserves a special thanks. Without his help the figures in the text would have been low resolution and some put together using keynote and LaTeXiT. Most importantly, however, he has been supportive during the whole period, showing only enthusiasm, interest and motivation. This thesis would - literally - not have been written on neutrinos with A. Zee, had it not been for him.

During the first part of the thesis writing process where I was abroad, I was funded by the Augustinus Foundation, the Loerup Foundation and the Reinholdt W. Jorck and Wife's Foundation.

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Introduction

According to the Standard Model, describing all known particles and how they interact, neutrinos do not have a mass. However, experiments have shown that these neutral particles oscillate, meaning they transform into each other. In order to do so, there must be a difference in their masses, so that at least two of them are massive. That neutrinos have masses, and more importantly, very small masses compared to the other fermions, is the first and only evidence of physics beyond the Standard Model. It is therefore a very exciting area to explore and many opportunities associated with extending the Standard Model have arisen. Since the discovery of neutrino masses, many physicists have plunged into this exciting area and sought to reveal the mysteries behind. This thesis is a step towards achieving this.

W. Pauli was the first to postulate the existence of the electron neutrino in 1930. Energy-momentum conservation could not be stored in β -decay, unless some new unknown and barely interacting particle was present. He saw this as a bad sign as can be analyzed from one of his famous quotes:

I have done a terrible thing. I have postulated a particle that can not be detected.

Fermi understood in 1933-34 that in the β -decay of the nuclei, an electron-proton pair is produced with an antineutrino,

$$n \rightarrow p + e^- + \bar{\nu}. \quad (1)$$

He even proposed how to measure this neutrino. It could be done, he stated, by measuring the end-point part of the electron spectrum, where the neutrino energy is comparable with the neutrino mass. This experiment was already carried out in 1949, where Hanna, Pontecorvo and Curran found the neutrino mass to be smaller than 500 eV [1, 2].

In 1957 parity violation was discovered in weak interactions. This lead Landau, Lee and Young to assume that the neutrino was massless, which became widely accepted [3-5].

Pontecorvo thought again about massive neutrinos in 1957-58, where he looked for an analogy

to the charge-parity (CP) violating K oscillations in the lepton sector. He proposed neutrino oscillations [6]:

If the two-component neutrino theory turns out to be incorrect (which at present seems to be rather improbable) and if the conservation law of neutrino charge would not apply, then in principle neutrino $\leftrightarrow$ antineutrino transitions could take place in vacuum.

He turned out to be right.

Evidence that neutrinos undergo oscillations and thereby have masses was first insinuated in the 1960's by Raymond Davies in Brookhaven.

Neutrinos are created in the sun from the nuclear reaction



and the β -decay



R. Davies measured the neutrinos in Brookhaven using 390.000 L of perchloroethylene,



whereafter the argon was collected and counted.

At this time, Pontecorvo had stated [7]:

... due to neutrino oscillations the observed flux of solar neutrinos could be two times smaller than the expected flux.

The experiment by R. Davies was the first to show a lack in the number of observed solar electron neutrinos compared to the expected amount. The observed amount was 2-3 times smaller than expected. The validity of the results were doubted by many stating that the counting of the few argon atoms in the vast amount of detergent was highly imprecise.

Several experiments have since then used solar and atmospheric experiments measuring neutrino events from cosmic ray showers created in the sun and the atmosphere. Artificial reactor and accelerator experiments have been set up to confirm data from natural sources using medium and long baselines to begin with and even short baseline experiments came into the picture. In 1998, the existence of neutrino masses were understood [8].

In 2002 R. Davies was awarded the Nobel prize.

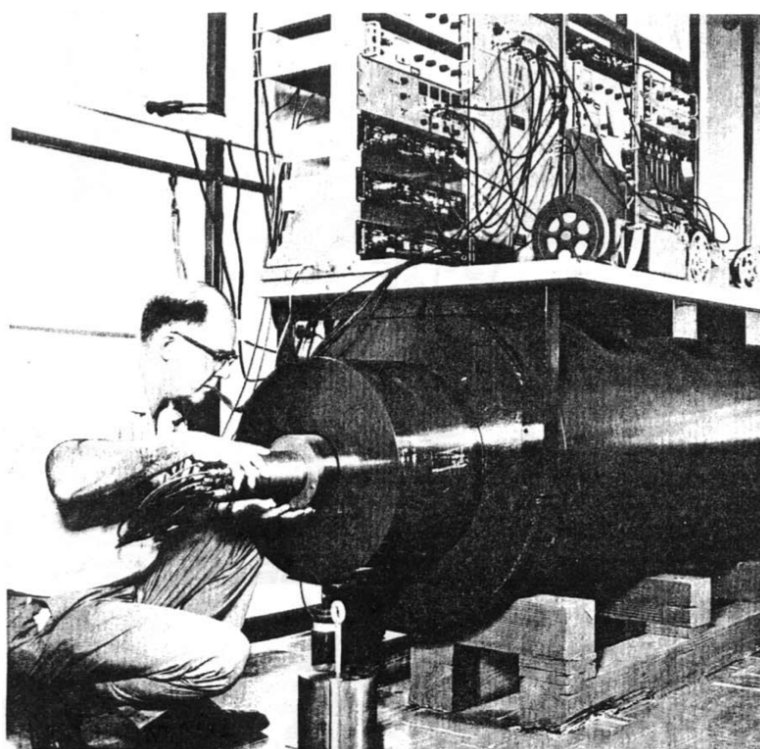


Figure 1: Dr. Ray Davis by the detector in the 1960's at Brookhaven Solar neutrino experiment [9]

Neutrino overview

1.1 Why neutrinos have mass

Neutrinos must have masses in order to undergo oscillations. This can be explained using a quantum field theory approach. If one neutrino is converted into the other, neither of the flavor eigenstates can be eigenstates of the Hamiltonian [10]. The flavor eigenstates must therefore be orthogonal linear combinations of the stationary eigenstates, that is the mass eigenstates. The neutrino flavor eigenstates are the partners of the charged lepton flavor eigenstates in the lefthanded weak SU(2) isospin group [11]. The idea can be illustrated in the case of two neutrinos, say the muon and electron neutrinos. The flavor eigenstates can then be expressed in terms of the mass eigenstates

$$\begin{aligned}\nu_e &= \cos \theta \nu_1 + \sin \theta \nu_2, \\ \nu_\mu &= -\sin \theta \nu_1 + \cos \theta \nu_2.\end{aligned}\tag{1.2}$$

The mixing matrix is therefore given by

$$U_{mix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix},\tag{1.3}$$

where

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = U_{mix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}.\tag{1.4}$$

In order to find the probability of an electron neutrino transforming into a muon neutrino, that is

$$\nu_e(0) = 1, \quad \nu_\mu(0) = 0, \quad \nu_1(0) = \cos \theta, \quad \nu_2(0) = \sin \theta,\tag{1.5}$$

the time evolution of the mass eigenstates can be used [10],

$$\nu_1(t) = \nu_1(0)e^{-iE_1t/\hbar},$$

$$\nu_2(t) = \nu_2(0)e^{-iE_2t/\hbar}, \quad (1.7)$$

so the mass eigenstates can be found at a later time,

$$\nu_1(t) = \cos\theta e^{-iE_1t/\hbar}, \quad \nu_2(t) = \sin\theta e^{-iE_2t/\hbar}. \quad (1.8)$$

The flavor eigenstate ν_μ at time t is therefore given by

$$\nu_\mu(t) = -\sin\theta\nu_1(t) + \cos\theta\nu_2(t) = \frac{1}{2}\sin(2\theta)(-e^{-iE_1t/\hbar} + e^{-iE_2t/\hbar}). \quad (1.9)$$

The probability to have a muon neutrino at time t after having an electron neutrino at time 0, is the squared norm of this result:

$$\begin{aligned} P_{\nu_e \rightarrow \nu_\mu} &= |\nu_\mu(t)|^2 = \frac{1}{4}\sin^2(2\theta)(e^{-iE_2t/\hbar} - e^{-iE_1t/\hbar})(e^{iE_2t/\hbar} - e^{iE_1t/\hbar}) \\ &= \frac{1}{4}\sin^2(2\theta)(2 - 2\cos\left(\frac{(E_2 - E_1)t}{\hbar}\right)) = \sin^2(2\theta)\sin^2\left(\frac{(E_2 - E_1)t}{2\hbar}\right). \end{aligned} \quad (1.11)$$

The energy of the relativistic neutrino is given by

$$\begin{aligned} E^2 &= |\vec{p}|^2 c^2 + m^2 c^4 = |\vec{p}|^2 c^2 \left(1 + \frac{m^2 c^2}{|\vec{p}|^2}\right) \Leftrightarrow \\ E &\simeq |\vec{p}|c \left(1 + \frac{1}{2} \frac{m^2 c^2}{|\vec{p}|^2}\right) = |\vec{p}|c + \frac{m^2 c^3}{2|\vec{p}|}. \end{aligned} \quad (1.13)$$

The difference in energy is then

$$E_2 - E_1 \simeq \frac{(m_2^2 - m_1^2)c^3}{2|\vec{p}|} \simeq \frac{m_2^2 - m_1^2}{2E}c^4, \quad (1.14)$$

giving the probability

$$P_{\nu_e \rightarrow \nu_\mu} = \sin^2(2\theta)\sin^2\left(\frac{\Delta m_{21}^2}{4E\hbar}c^4 t\right) = \sin^2(2\theta)\sin^2\left(\frac{\Delta m_{21}^2 L}{4E\hbar}c^3\right), \quad (1.15)$$

where L is the distance the neutrinos travel. In order for this probability to be nonzero, it is therefore necessary that Δm_{12}^2 is nonzero. That is, in the case of having two neutrinos, at least one of these must be massive. In the following we will set $\hbar = c = 1$.

The reality is that there exists three flavor neutrinos. In this case, the mixing matrix transforms the mass eigenstates into flavor eigenstates as follows:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = V \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \quad (1.16)$$

This mixing matrix is given by

$$V = \begin{pmatrix} e^{i\kappa_1} & 0 & 0 \\ 0 & e^{i\kappa_2} & 0 \\ 0 & 0 & e^{i\kappa_3} \end{pmatrix} U \begin{pmatrix} e^{i\Phi_1} & 0 & 0 \\ 0 & e^{i\Phi_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (1.17)$$

where

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \quad (1.18)$$

which is called the Pontecorvo-Maki-Nakagawa-Sakata mixing matrix, U_{PMNS} , can be found from combining three rotation matrices mixing the neutrinos pairwise. The three κ phases can be absorbed by rephasing the neutrino fields [12]. Therefore, the first matrix can be set equal to $\mathbf{1}$. Majorana neutrinos are their own antiparticles. This means, that they do not have invariant mass terms under rephasing of the neutrino fields, thus creating three Majorana phases whereof one can be rephased. Therefore, besides the Dirac phase δ , there exists two extra Majorana phases, Φ_1 and Φ_2 that can not be eliminated. These are relative phases among the Majorana masses,

$$m_{1,2} = |m_{1,2}|e^{i\Phi_{1,2}}. \quad (1.19)$$

The Majorana phases are only associated with the Majorana nature of neutrinos and can therefore only be observed in lepton-number violating processes. They are not observable in neutrino oscillations, so to carry out model building, only the mixing matrix U is to be considered.

The idea is the same when finding the probabilities for the transformations between the three neutrinos. The flavor eigenstates are again given by a linear combination of the mass eigenstates,

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle, \quad (1.20)$$

$$|\nu_i\rangle = \sum_\alpha U_{\alpha i}^* |\nu_\alpha\rangle, \quad (1.21)$$

where α is the flavor index and i the mass index. A neutrino ν_α at $t = 0$, becomes after a time evolution,

$$|\nu_\alpha(t)\rangle = \sum_i U_{\alpha i} |\nu_i\rangle e^{-iE_i t}. \quad (1.22)$$

The probability to have ν_α after a time t is given by [13]

$$|\langle \nu_\alpha | \nu_\alpha(t) \rangle|^2 = \left| \sum_i U_{\alpha i} U_{\alpha i}^* e^{-iE_i t} \right|^2. \quad (1.23)$$

The probability to have any neutrino ν_β at time t , rewriting the energy difference, is given by

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \left| \sum_i U_{\beta i} e^{-i\frac{\Delta m_{ij}^2 L}{2E}} U_{\alpha i}^* \right|^2 = \sum_{i,j} U_{\beta j}^* U_{\alpha j} U_{\beta i} U_{\alpha i}^* e^{-i\frac{\Delta m_{ij}^2 L}{2E}}, \quad (1.24)$$

where the approximation $|\mathbf{p}| \simeq E$ has been made. This can be written as

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \delta_{\alpha,\beta} - 4 \sum_{j < i=1} \text{Re}[U_{\beta j}^* U_{\alpha j} U_{\beta i} U_{\alpha i}^*] \sin^2 \left(\frac{\Delta m_{ij}^2 L}{4E} \right)$$

$$\pm 2 \sum_{j < i=1} \text{Im}[U_{\beta j}^* U_{\alpha j} U_{\beta i} U_{\alpha i}^*] \sin\left(\frac{\Delta m_{ij}^2 L}{2E}\right), \quad (1.26)$$

where the $\pm$ corresponds to neutrinos and antineutrinos respectively [14]. Charge and parity (CP) conservation is obtained when $\nu = \bar{\nu}$, so that

$$P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = P(\nu_\mu \rightarrow \nu_e). \quad (1.27)$$

In the case where the mixing matrix is real, the probability must then be given by

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \delta_{\alpha,\beta} - 4 \sum_{j < i=1} U_{\beta j} U_{\alpha j} U_{\beta i} U_{\alpha i} \sin^2\left(\frac{\Delta m_{ij}^2 L}{4E}\right). \quad (1.28)$$

This means that the term responsible for CP violation is given by $\text{Im}[U_{\beta j}^* U_{\alpha j} U_{\beta i} U_{\alpha i}^*]$. This is known as the Jarlskog factor. The Dirac phase δ is the phase that induces CP violation.

In order for the probability to be nonzero for all $\beta \neq \alpha$, at least two of the three neutrinos must have nonzero masses.

Explicitly, neutrino observations are observed if

$$|E_i - E_j|t \geq 1, (i \neq j). \quad (1.29)$$

This inequality is called the Mandelstam-Tamm time-energy uncertainty relation. It states, that the propagation time interval required for a change of the flavor neutrino state is given by

$$t \sim \frac{1}{|E_i - E_j|}. \quad (1.30)$$

In the ultrarelativistic case, we have $t \sim L$, where L is the propagation length. The length a neutrino needs to travel in order to oscillate is therefore inverse proportional to the mass difference squared, see (1.28), or the energy difference between this neutrino and the neutrino it oscillates into. We know from experiments that these are very small, and thus experiments are performed with oscillation over large distances, more precisely,

$$L_{osc}[km] = 2\pi \frac{E[GeV]}{1.27 \Delta m^2[eV^2]}. \quad (1.31)$$

1.2 Experiments

Since the experiments carried out by R. Davies, many have been set up around the world to establish the nature and values of neutrino masses. In the following I will mention a few important events.

The Kamiokande detector at the Kamioka-Cho experiment in Japan was situated 1000 m underground and consisted of 3000 tons of pure water. It detected neutrinos in 1985. The direction and energy of the neutrinos was measured for the first time [14].

The Super-Kamiokande and the Sudbury Neutrino Oscillation experiments followed up in 1998 and 2001, confirming the solar neutrino deficit. They showed that neutrinos came from

the sun as expected, by measuring the number of events as a function of the zenith angle. They also showed that the amount of muon and tauon neutrinos coming from the sun was twice the amount of electron neutrinos [11, 14]. The Super-Kamiokande proceeded in 2003 to measure the solar neutrino day/night and seasonal variations. In 2004 it turned to atmospheric oscillations $\nu_\mu \rightarrow \nu_\tau$. This was also done by the MINOS (Main Injector Neutrino Oscillation Search) experiment in 2005.

In 2002, the KamLAND disappearance experiment showed that the $\bar{\nu}_e$ coming from the reactors oscillated over a distance of mean value 175 km [14].

The solar neutrino measurements lead to a global fit for Δm_{12}^2 and $\sin^2 \theta_{12}$, which can be seen in the left figure in 1.1. The atmospheric global data from Super-Kamiokande and MINOS plotting Δm_{23}^2 and $\sin^2 \theta_{23}$ can be seen in the right figure in 1.1.

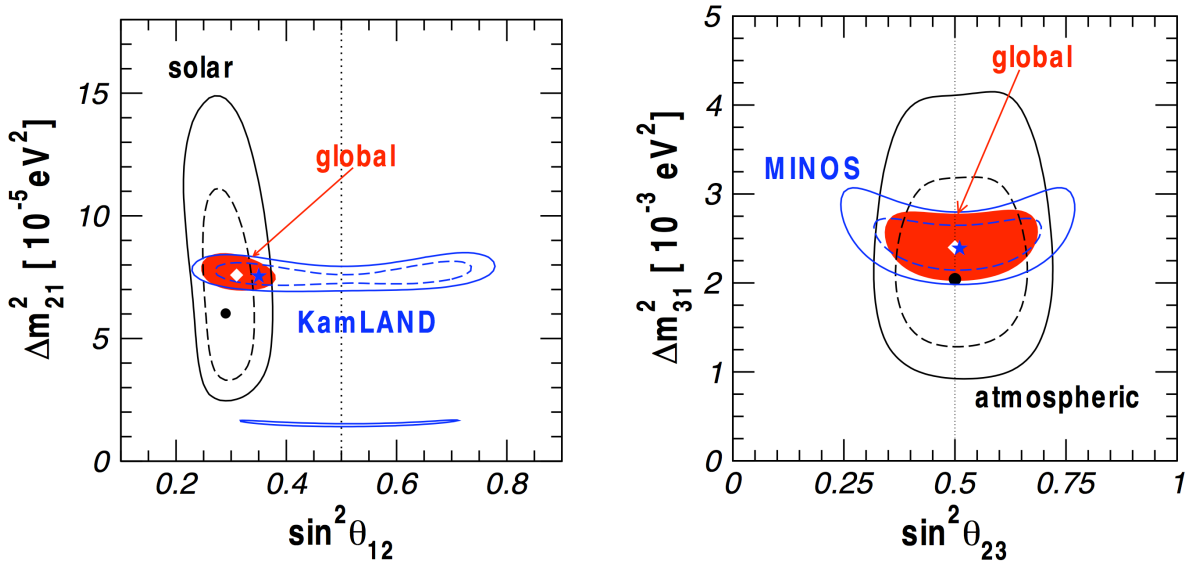


Figure 1.1: Global fit from solar neutrino oscillation experiments [15].

1.3 Experimental results

Even though many experiments have been carried out since the first experiments by R. Davies, only few experimental results are established. They include the differences between the mass values of the neutrinos squared and the mixing angles. A summary of the best fit values from combined data from solar, atmospheric, reactor and accelerator experiments described in section 1.2 can be seen in table 1.1, together with the 2σ and 3σ intervals [15].

From the table can be seen that $|\Delta m_{12}^2| \ll |\Delta m_{23}^2|$. Since the sign of the difference between m_2^2 and m_3^2 is unknown, there exists two possible mass hierarchies. The normal hierarchy, where $|m_3| > |m_2| \simeq |m_1|$ and the inverted hierarchy, where $|m_3| < |m_2| \simeq |m_1|$. These can be seen in figure 1.2, where an illustration of the mixture of the flavor states with respect to the mass eigenstates can also be seen. The solar mixing angle θ_{12} is the angle with the lowest uncertainty and deviations should therefore not be bigger than 0.05.

From the values in the table, the norm of the neutrino mixing matrix, see (1.18), is found with a global fit at 90 % Confidence Level (C.L.) [17]:

1.3. Experimental results

Parameter	Best fit	2 σ	3 σ
$\Delta m_{12}^2 [10^{-5} eV^2]$	$7.59^{+0.23}_{-0.18}$	7.22-8.03	7.03-8.27
$ \Delta m_{32}^2 [10^{-3} eV^2]$	$2.40^{+0.12}_{-0.11}$	2.18-2.64	2.07-2.75
$\sin^2 \theta_{12}$	$0.318^{+0.019}_{-0.016}$	0.29-0.36	0.27-0.38
$\sin^2 \theta_{23}$	$0.50^{+0.07}_{-0.06}$	0.39-0.63	0.36-0.67
$\sin^2 \theta_{13}$	$0.013^{+0.013}_{-0.009}$	≤ 0.039	≤ 0.053

Table 1.1: Experimental values for neutrino oscillations.

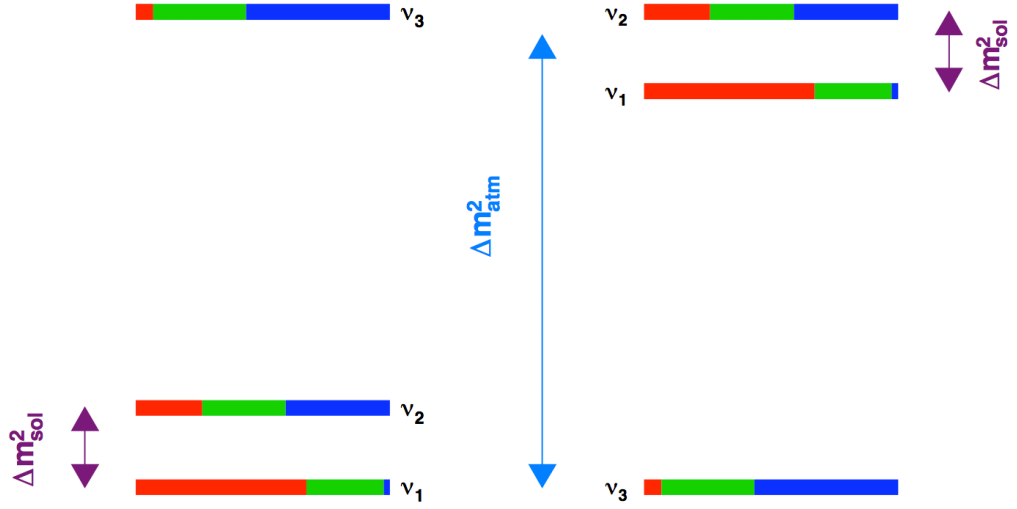


Figure 1.2: The hierarchies possible for the masses of the mass eigenstates [16].

$$|U_{exp}| = \begin{pmatrix} 0.79 - 0.86 & 0.50 - 0.61 & < 0.16 \\ 0.24 - 0.52 & 0.44 - 0.69 & 0.63 - 0.79 \\ 0.26 - 0.52 & 0.47 - 0.71 & 0.60 - 0.77 \end{pmatrix}. \quad (1.32)$$

The ratio between two masses squared is given by [15, 18, 19]

$$R \equiv \frac{\Delta m_{12}^2}{\Delta m_{23}^2} = 0.032^{+0.006}_{-0.005} \sim \frac{1}{30}. \quad (1.33)$$

This means that a hierarchical spectrum has $\frac{m_2}{m_3} \sim \sqrt{R} \sim 0.2$, comparable to the Cabibbo angle $\lambda_C \sim 0.22$ and to $\sqrt{\frac{m_\mu}{m_\tau}} \sim 0.24$. From cosmology (the large scale structure and the cosmic microwave background), the sum of the mass eigenstates has been given an upper limit [20]:

$$\sum m_i \leq 0.6 - 0.75 eV (90\% \text{C.L.}). \quad (1.34)$$

From the endpoints of tritium single β decay, the following upper limit on the neutrino mass associated with the β decay has been obtained [21, 22]:

$$m_\beta = (|U_{ei}|^2 m_i^2)^{1/2} \leq 2eV. \quad (1.35)$$

1.3.1 Neutrinoless double β decay

Since Majorana neutrinos are their own antiparticles, which will be justified later this requires total lepton number violation. Therefore, in order to test whether neutrinos are Dirac or Majorana fermions, a process is needed where the total lepton number is violated. This is the purpose of neutrinoless double β decay, $0\nu\beta\beta$. It consists of two subsequent β -decays. A neutrino is created in the first β -decay, which acts as an antineutrino and participates in a new β -decay, as seen in figure 1.3, given by

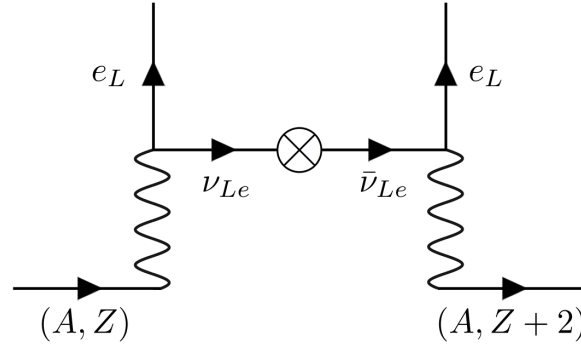


Figure 1.3: The double β -decay. A neutrino is introduced in the first β decay, which acts as an antineutrino in the second β decay.

$$(A, Z) \rightarrow (A, Z + 2) + e^- + e^-. \quad (1.36)$$

Since the neutrino mass eigenstates are mixtures of the flavor eigenstates, the neutrino propagator can be shown to be [23]

$$\sum_i U_{ei}^2 \left(\frac{1 - \gamma^5}{2} \right) \frac{\gamma \cdot p + m_i}{p^2 - m_i^2} \left(\frac{1 - \gamma^5}{2} \right) C \simeq m_{ee} \frac{1}{p^2} \left(\frac{1 - \gamma^5}{2} \right) C, \quad (1.37)$$

with $|m_{ee}|$, the effective neutrino mass:

$$|m_{ee}| = \left| \sum_i U_{ei}^2 m_i e^{i\Phi(i)} \right| = m_1 U_{e1}^2 + m_2 U_{e2}^2 e^{i2\Phi_1} + m_3 U_{e3}^2 e^{i2\Phi_2}. \quad (1.38)$$

The allowed regions for $|m_{ee}|$ from double β decay as a function of the lightest neutrino mass can be seen in figure 1.4 [17]. The upper limit on the effective neutrino mass from neutrinoless double β decay is given by [20, 24]

$$|m_{ee}| \leq 0.38eV \quad (1.39)$$

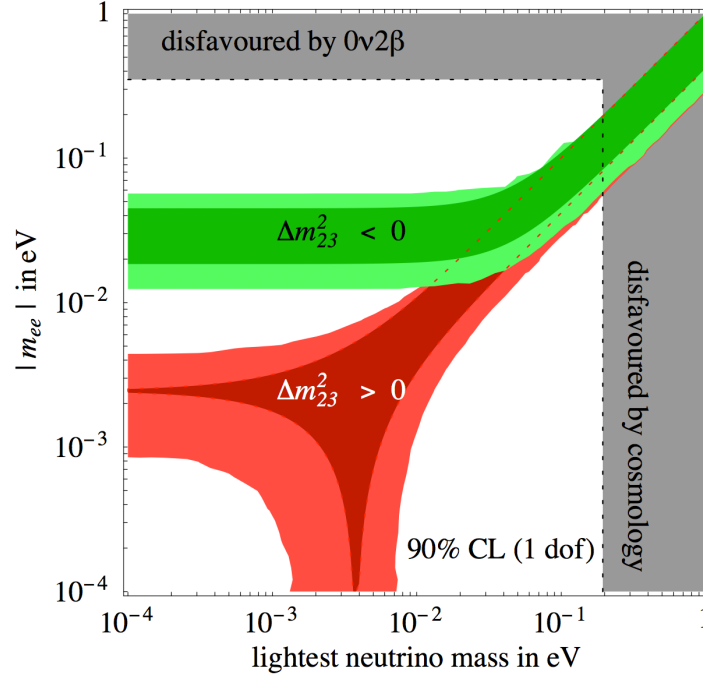


Figure 1.4: The allowed regions for double β decay in terms of the lightest neutrino [17].

1.3.2 Tritium β -decay

The tritium β decay is described by the process

$$\text{H}^3 \rightarrow {}^3\text{He} + e^- + \bar{\nu}_e. \quad (1.40)$$

The effect of the neutrino mass can be seen at the endpoint of the tritium single β decay energy spectrum, as seen in figure 1.5. The functional form of the curve is given by

$$K(E_e) \sim \sqrt{(E_0 - E_e)((E_0 - E_e)^2 - m_\nu^2)^{1/2}}. \quad (1.41)$$

The following upper limit has been obtained for the mass of the neutrino associated with the single β -decay event [21, 22, 25]:

$$m_\beta = (|U_{ei}|^2 m_i^2)^{1/2} \leq 2.2 \text{ eV}. \quad (1.42)$$

1.4 Mysteries of neutrino physics

Neutrinos are very mysterious fermions. The Standard Model can not explain the existence of neutrino masses, which have now been widely acknowledged and the experimental data differ a lot from the other fermions.

This can be seen as the mixing angles in the quark sector are all small, where neutrinos show two large and one small mixing angle. The masses of neutrinos can not be larger than of the order eV. For comparison can be seen the approximate values of the other fermions:

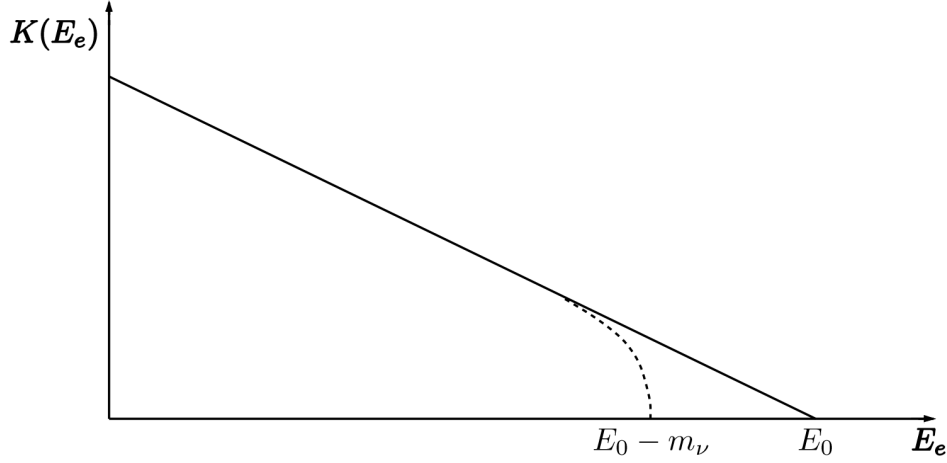


Figure 1.5: The endpoint from tritium β decay [14].

$$m_t \sim m_{EW} \sim 250 GeV, \quad m_b, m_c, m_\tau \sim 2 GeV, \quad (1.43)$$

$$m_s, m_\mu \sim 0.1 GeV, \quad m_u, m_d \sim 0.005 GeV, \quad m_e \sim 0.0005 GeV. \quad (1.44)$$

Neutrino masses are therefore at a much lower scale than the rest of the fermions in the Standard Model. This fact is pretty remarkable and great challenges are associated with it, when exploring the areas of neutrino physics.

To conserve charge, parity and time conservation (CPT), every charged lefthanded particle must have a right-handed antiparticle with opposite charge. If the fermions are neutral and the right-handed particle states do not exist, they could be Majorana particles [14]. This is the case for neutrinos. These can possibly be both left handed particles and right handed antiparticles:

$$\nu_R \rightarrow (\nu_L)^c = C\bar{\nu}_L^T = C\gamma_0\nu_L^*, \quad (1.45)$$

where C is the operator of charge conjugation in spinor space. Thus, according to Lorentz invariance, half of the degrees of freedom in the massive Dirac spinor disappear, when neutrinos are considered to be Majorana. This is only compatible with charge conservation of neutrinos [14]. Neutrinos can therefore be of Majorana type.

In order to explain the abnormal angles and masses, new mechanisms have been introduced to generate these. Some have introduced right-handed neutrinos, so neutrinos acquire masses in the same way as quarks and charged leptons, through Yukawa type couplings to vacuum expectation values of the scalar fields. These types of neutrinos are called Dirac neutrinos. Others have exploited the possibility of neutrinos being Majorana, and therefore obtain Majorana mass terms that break lepton number conservation. Both Majorana and Dirac type couplings to Higgs bosons can be introduced at the same time. This mechanism is called the see-saw mechanism.

However, it has not yet been determined whether neutrinos are in fact Majorana or Dirac particles. Furthermore, the value of the θ_{13} angle as well as the Dirac and Majorana phases are far from known.

In addition, the number of neutrinos has not yet been understood. The question is, if the number of neutrinos with definite masses in fact is equal to the number of flavor neutrinos, that is three, or larger. The LSND experiment, as will be described later, has questioned the limit of three neutrinos.

Many mysteries are yet to be revealed. This only contributes in making the area of neutrino physics more exciting.

1.5 Experimental anomalies

From 1993 to 1998 the Liquid Scintillator Neutrino Detector (LSND) showed controversial results, indicating a third distinct oscillation frequency. It was the first short baseline experiment established. It was built to carry out the search for neutrino oscillations looking for $\bar{\nu}_e$ appearance. Since Δm^2 is inverse proportional to the length, the short length of this baseline meant that this experiment was looking for evidence of oscillations leading to Δm^2 of order $\sim 1\text{eV}^2$. These events were discovered, which greatly contradicted other experiments. This can be the proof of either more than 3 neutrinos or CPT violation. [11]

The KARMEN experiment, which collected data in 1997-2001, was very similar to LSND, also searching for $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ oscillations. It was also looking for $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ oscillations. This experiment showed absence of these, and therefore increased the speculations around the LSND experiment.

In order to confirm or disprove the LSND experiment, the MiniBooNE (MB) experiment carried out measurements in 2007 studying $\nu_\mu \rightarrow \nu_e$ oscillations. The location of the detector satisfied $\frac{L[m]}{E[\text{MeV}]} \sim 1$, which was similar to LSND, and gave maximum oscillation at $\Delta m^2 \sim 1\text{eV}^2$. This experiment showed no evidence of excess of ν_e above 475 MeV, but it gave a 3.0σ excess of electron like events below 475 MeV, see figure 1.6. Thereby the MiniBooNE experiment seemed to disprove the LSND evidence from these first data [26]. The low excess energy was to begin with discarded as being due to a higher background noise in this area. However, in 2008, the MiniBooNE had improved its background analysis and still showed a 3σ excess. So it seemed like the LSND anomaly had just been traded for a MiniBooNE anomaly.

In 2009 the MiniBooNE experiment changed strategy to test the LSND more directly. A search in antineutrino mode $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ was carried out. It created a 2.8σ excess of $\bar{\nu}_e$ events at high energies, which was suddenly consistent with the LSND results. But it did not show an excess at low energies [27].

The search for $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ oscillations continued in 2010 with 1.7 times more protons on target (POT), that is $5.66 \cdot 10^{20}$ protons of 8 GeV. This created an excess of 20.9 ± 14 events in the range of $475 < E_\nu < 1250$ MeV, see figure 1.7. This was consistent with oscillations giving a Δm^2 in the range $0.1 - 1.0\text{eV}^2$ which is compatible with LSND [27]. The fit with the experiments KARMEN and BUGEY can be seen in figure 1.8.

The results of the LSND experiments have been questioned by "many". So far, it has not been possible to explain them and experiments are being carried out to seek answers to the anomalies arisen both from the LSND experiment and the MiniBooNE experiment. Since so much confusion is associated with the mass squared difference from these experiments, these results are mostly left out in the literature when analyzing neutrino data. This thesis will not

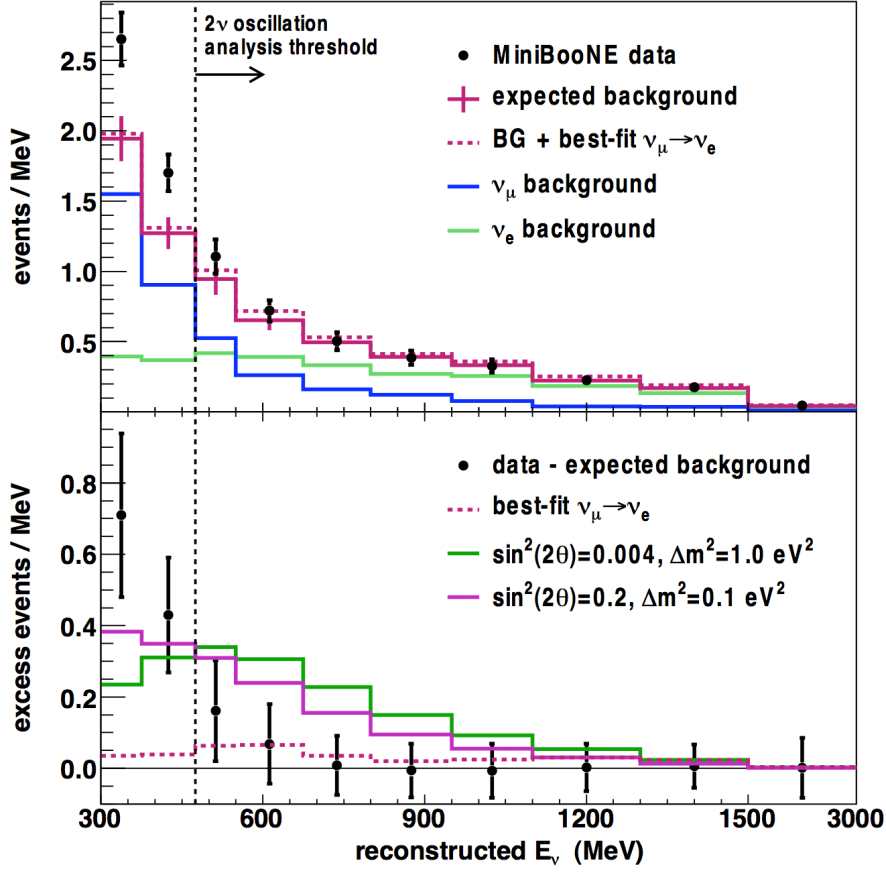


Figure 1.6: Top plot shows MiniBooNE excess events with background. The bottom plot shows excess events with the backgrounds subtracted. It also shows the two LSND solutions at high and low Δm^2 [26].

take these results into consideration, when carrying out the model building.

1.6 Setup of a short baseline experiment

As an example of an experimental setup measuring neutrino oscillations, the MiniBooNE experiment will in the following be used to describe this.

At 8 GeV kinetic energy, protons from Fermilab Booster are incident on a Beryllium target, where the protons interact with Beryllium and create the charged mesons, that is pions and kaons. This target lies inside a magnetic focusing horn, which produces a toroidal magnetic field, that focuses positive pions and kaons and defocuses negative to create pure antineutrino beams or opposite to create pure neutrino beams. These pass through a collimator and then a 50 m long tunnel, where the mesons decay. After having reached the beam stop/absorber, where the neutrinos are created, these pass through a 450 m dirt channel that stops the charged particles, in which they are allowed to oscillate. The center of the detector lies a total of 541m from the front of the Beryllium target, see figure 1.9. The detector consists of a 610 cm radius sphere. It is filled with 800 tons mineral oil. The neutrino interactions in the detector then creates final state electrons and muons, which are detected by the 1520 photomultiplier tubes

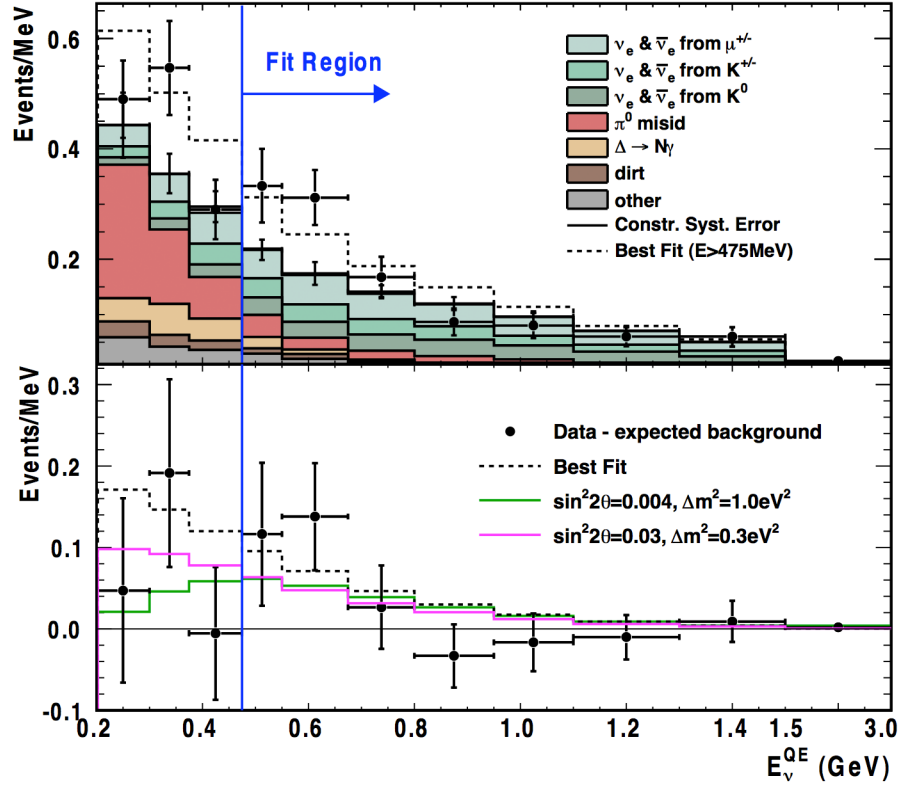


Figure 1.7: The top figure shows the MiniBooNE $\bar{\nu}_e$ event excess as a function of the energy distribution with the background. The figure at the bottom shows event excess as a function of the energy without the background. The expectations from the best fit for Δm^2 and $\sin^2(2\theta)$ above 475 MeV are also shown. This avoids the region of unexplained low-energy excess at neutrino mode [27].

[27].

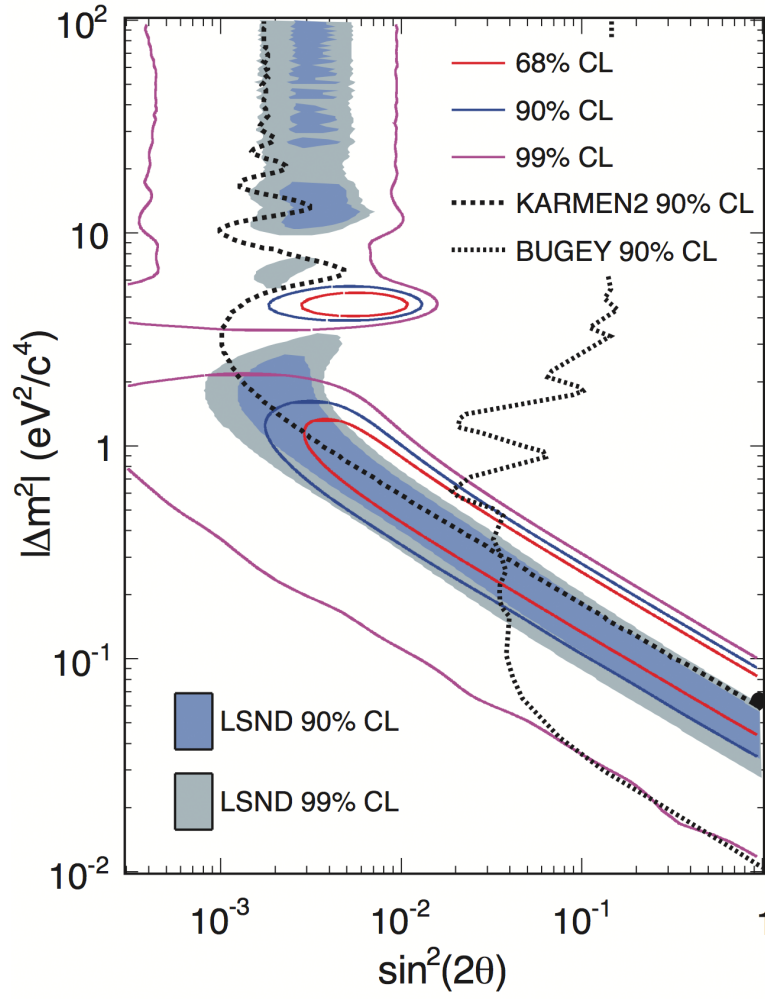


Figure 1.8: 68%, 90% and 99% CL closed contour for $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ oscillations for both LSND and MiniBooNE experiment. Agreement is shown in the allowed region of $0.1\text{--}1.0 \text{ eV}^2 \Delta m^2$ range [27].

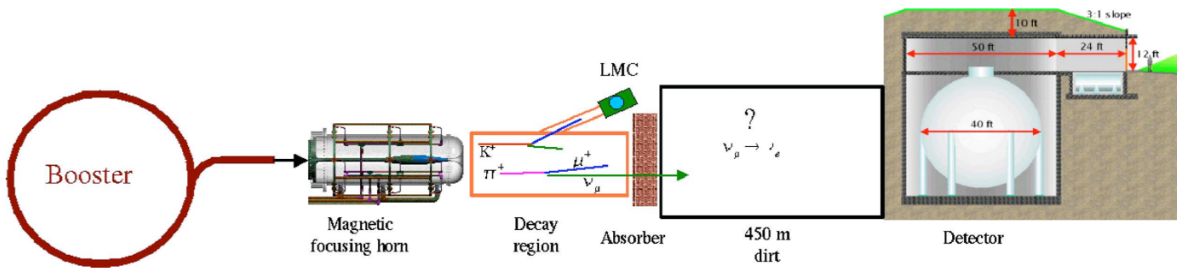


Figure 1.9: The MiniBooNE experiment [28].

2.1 Symmetries

Ever since Einstein identified the invariance group of space and time in 1905 [29], the understanding of the world has relied on symmetries and how these are broken to describe fields and their interactions. Particles have been found and their behavior explained on account of the use of symmetries. Symmetries will therefore be used extensively in this thesis. In order to understand physics beyond the Standard Model, it is of course necessary to understand the Standard Model and the particles that make up this theory. In order to do so, one must understand the way fields transform under a certain symmetry.

2.1.1 Nonabelian symmetries

A Lie group is represented by the unitary operators $U = e^{i\theta^a T^a}$ [30]. For infinitesimal transformations, these unitary operators are given by¹

$$U = 1 + i\theta^a T^a + \mathcal{O}(\theta^2), \quad (2.1)$$

where θ^a are arbitrary parameters and T^a are the generators of the group.

The rotation group in N dimensions, $SO(N)$, consists of orthogonal transformations of N -dimensional vectors, that preserve $\eta_a \xi_a$, the inner product. Under an infinitesimal transformation the orthogonal matrix is given by

$$R_{ij} = \delta_{ij} + \theta_{ij} + \mathcal{O}(\theta^2), \quad (2.2)$$

where θ_{ij} is some real and antisymmetric matrix. R_{ij} can therefore be expressed in terms of a linear combination of the hermitian matrices $(T^a)_{ij}$, so that $\theta_{ij} = -i\theta^a (T^a)_{ij}$. Since R_{ij} is orthogonal, θ_{ij} is real. As θ^a are real parameters, this means that $(T^a)_{ij}$ are complex hermitian matrices of which there are $\frac{1}{2}N(N-1)$. These are the generator matrices of $SO(N)$.

¹Einstein's sum convention is here understood.

The normalization condition requires

$$\text{Tr}(T^a T^b) = 2\delta^{ab}. \quad (2.3)$$

Since the product of two transformations in a group must generate another transformation in the same group, the commutation relation between two generators must be given by a linear combination of the generator matrices

$$[T^a, T^b] = if^{abc}T^c. \quad (2.4)$$

This is the Lie algebra commutation relation, where f^{abc} is the structure coefficient. The structure coefficient is found to be completely antisymmetric and real as can be seen by multiplying this relation with T^d , taking the trace and using the relation (2.3)

$$\begin{aligned} \text{Tr}([T^a, T^b]T^d) &= \text{Tr}(if^{abc}T^cT^d) = 2if^{abd} \Rightarrow \\ f^{abd} &= -\frac{1}{2}i\text{Tr}([T^a, T^b]T^d). \end{aligned} \quad (2.6)$$

That the structure coefficient is real can be seen from taking the complex conjugate and using that the generators are hermitian.

The group $U(N)$ contains the unitary transformation of N -dimensional vectors, meaning transformations that preserve $\eta_a^* \xi_a$. This group has $U(1)$ as a subgroup, that commutes with all the other unitary transformations. If this is removed, the group becomes $SU(N)$, that is the group of $N \times N$ unitary matrices with $\det U = 1$. Under $SU(N)$, an infinitesimal transformation is given by

$$U_{ij} = \delta_{ij} - i\theta^a (T^a)_{ij} + \mathcal{O}(\theta^2). \quad (2.7)$$

Since the group does not contain $U(1)$, there are $N^2 - 1$ traceless hermitian, but not necessarily antisymmetric generators. These consist of the $\frac{1}{2}N(N-1)$ $SO(N)$ generators and the $\frac{1}{2}N(N-1)$ off-diagonal symmetric matrices. These satisfy the Lie algebra:

$$\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \quad (2.8)$$

$$[T^a, T^b] = if^{abc}T^c. \quad (2.9)$$

2.1.2 Group properties

When a group is used to describe a system of fields, it is necessary to be familiar with its properties. The following properties also apply to discrete symmetry groups and will be of importance when investigating the Frobenius group T_{13} with neutrino mixing later.

A representation of a group R is reducible, if a unitary transformation of a representation $T_R^a \rightarrow U^{-1}T_R^a U$ can put all nonzero entries of the representation into the same diagonal block for each a . Otherwise, the representation is irreducible.

A representation can be decomposed if it is reducible

$$R = R_1 \oplus R_2. \quad (2.10)$$

which was also seen above for $SU(N) \otimes U(1) = U(N)$. The number of irreducible representations is equal to the number of different conjugacy classes. In a conjugacy class, the elements g_i can under a group transformation only transform into other elements of the same class. For m_n n -dimensional complex irreducible representations,

$$\sum_n m_n = \sum \text{Irreducible representations} = \sum_{C_i} 1, \quad (2.11)$$

with C_i being the conjugacy class of the elements g_i , and i the number of elements in the class. The sum of the dimensions of the complex irreducible representations squared is given by the number of elements N of the group:

$$N = \sum_i d_i^2. \quad (2.12)$$

This relation will become important when the irreducible representations of discrete family symmetries are found later.

2.1.3 Lorentz representation of fields

The Lorentz group makes up one of these Lie groups. It is an interesting group, as the fields within quantum field theory can be assigned to the four simplest representations of this group. It furthermore describes how a vector field can be decomposed into two Weyl spinors, a left-handed and a right-handed field. In this thesis, we are especially interested in these Weyl fields, as the properties of a Majorana field can be explained through the properties of these fields. The familiarity with Weyl fields will therefore give a better understanding of the nature of neutrinos.

Under a Lorentz transformation, a field transforms accordingly:

$$U(\Lambda)^{-1} \phi(x) U(\Lambda) = \phi(\Lambda^{-1}x), \quad (2.13)$$

where Λ is the transformation matrix. The transformation of the derivative is then given by

$$U(\Lambda)^{-1} \partial^\mu \phi(x) U(\Lambda) = \Lambda^\mu_\rho \bar{\partial}^\rho \phi(\Lambda^{-1}x), \quad (2.14)$$

where $\bar{\partial} = \frac{d}{d\bar{x}} = \frac{d}{d(\Lambda^{-1}x)}$. A vector field would then transform in the same way:

$$U(\Lambda)^{-1} A^\mu(x) U(\Lambda) = \Lambda^\mu_\rho A^\rho(\Lambda^{-1}x). \quad (2.15)$$

In order to find out which representations of the Lagrangian the fields in the Standard Model are assigned to, it is necessary to find the irreducible representations of the group. To all fields in the Lagrangian can be assigned a Lorentz index $\phi_A(x)$. Under a Lorentz transformation, the fields transform into fields which also have a Lorentz index, that might or might not be the same:

$$U(\Lambda)^{-1}\phi_A(x)U(\Lambda) = L_A^B(\Lambda)\phi_B(\Lambda^{-1}x), \quad (2.16)$$

where all $L_A^B(\Lambda)$'s form a representation of the Lorentz group. Under an infinitesimal transformation, $\Lambda_\nu^\mu = \delta_\nu^\mu + \delta\omega_\nu^\mu$, the representations are then

$$L_A^B(1 + \delta\omega) = \delta_A^B + \frac{i}{2}\delta\omega_{\mu\nu}(S^{\mu\nu})_A^B. \quad (2.17)$$

The unitary operator is given by

$$U(1 + \delta\omega) = I + \frac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu}, \quad (2.18)$$

where $M^{\mu\nu}$ are the hermitian generators of the Lorentz group, that obey the Lie algebra:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(g^{\mu\rho}M^{\nu\sigma} - (\mu \leftrightarrow \nu)) - (\rho \leftrightarrow \sigma). \quad (2.19)$$

The various sets of matrices $M^{\mu\nu}$, that obey (2.19) will give all the inequivalent irreducible representations. From the above commutation relation, the Lie algebra of the well-known angular momentum can be extracted: The angular momentum is given by

$$J^i = \frac{1}{2}\epsilon_{ijk}M^{jk}, \quad (2.20)$$

and the boost becomes

$$K^i \equiv M^{i0}, \quad (2.21)$$

so that

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad (2.22)$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k, \quad (2.23)$$

$$[K_i, K_j] = -i\epsilon_{ijk}J_k. \quad (2.24)$$

In $SO(3)$, three hermitian matrices of dimension $(2j+1) \times (2j+1)$, obeying (2.22) can be found with eigenvalues $-j, -j+1, \dots, +j$, where j is a positive half-integer and integer. For $j=1$, the irreducible representations of the rotation group in 3 dimensions are given by the three elements of rotation:

$$\mathcal{J}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & -\sin \theta_1 \\ 0 & \sin \theta_1 & \cos \theta_1 \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} \cos \theta_2 & 0 & \sin \theta_2 \\ 0 & 1 & 0 \\ -\sin \theta_2 & 0 & \cos \theta_2 \end{pmatrix}, \quad (2.25)$$

$$\mathcal{J}_3 = \begin{pmatrix} \cos \theta_3 & -\sin \theta_3 & 0 \\ \sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

When j is a half-integer, a 2π rotation leads to an overall minus sign, so these can not be representations of the group $SO(3)$. The half-integer j 's are only representations of $SU(2)$, which is the double cover group of $SO(3)$. Representations of $SO(3)$ are therefore specified by integers where representations of $SU(2)$ are specified by both integers and half-integers.

Since the Lie algebra of the Lorentz group is in 4 dimensions, it is given by the $SU(2) \times SU(2) = SO(3,1)$ group. This group must be specified by two integers or half-integers, n and n' , so that the number of components in $SU(2) \times SU(2)$ are $(2n+1, 2n'+1)$. In order for this group to encompass all of the relations (2.22)-(2.24), an operator and its hermitian conjugate can be created that are linear combinations of the angular momentum and boost operators, so that

$$N_i \equiv \frac{1}{2}(J_i - iK_i), \quad (2.27)$$

$$N_i^\dagger \equiv \frac{1}{2}(J_i + iK_i). \quad (2.28)$$

From this, the commutation relations become

$$[N_i, N_j] = i\epsilon_{ijk}N_k, \quad (2.29)$$

$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}N_k^\dagger, \quad (2.30)$$

$$[N_i, N_j^\dagger] = 0. \quad (2.31)$$

Within quantum field theory, the fields can now be assigned to the four simplest of the Lorentz representations $(2n+1)(2n'+1)$:

$$(1, 1) = \text{scalar/singlet}, j = 0, \quad (2.32)$$

$$(2, 1) = \text{left-handed spinor}, j = \frac{1}{2}, \quad (2.33)$$

$$(1, 2) = \text{right-handed spinor}, j = \frac{1}{2}, \quad (2.34)$$

$$(2, 2) = \text{vector}, j = 0, 1. \quad (2.35)$$

The (2,2) is the vector, since it has the right number of components and angular momentum values, namely 0 and 1 - a scalar component for time and a 3-vector for space. The Lorentz vector (2,2) can be split into the left-handed and right-handed spinor in the (2,1) and (1,2) representations, respectively. These are the so-called Weyl spinors.

2.1.4 Spinor equations

Now that we have seen the Lorentz vector decomposition into Weyl spinors, a suitable Lagrangian for these spinors can be found. This will determine which representations the fields can be assigned to in order to get invariant terms in the Lagrangian under a symmetry transformation. We can therefore also investigate what describes a Dirac and a Majorana field, which is highly relevant in neutrino physics.

The Lagrangian is required to be Lorentz invariant and hermitian. To get a linear equation of motion, and thereby a plane wave solution to describe free particles, another requirement is, that the Lagrangian is quadratic in both the left-handed and right-handed Weyl spinors.

In a theory with two left-handed spinors satisfying an $SO(2)$ symmetry, the Lagrangian is given by

$$\mathcal{L} = i\psi_i^\dagger \bar{\sigma}^\mu \partial_\mu \psi_i - \frac{1}{2}m\psi_i\psi_i - \frac{1}{2}m\psi_i^\dagger\psi_i^\dagger, \quad (2.36)$$

for $i = 1, 2$. Two complex conjugate fields can be written in terms of these two left-handed spinor fields:

$$\chi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2), \quad (2.37)$$

$$\xi = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2), \quad (2.38)$$

so that $\chi^\dagger = \xi$ and the Lagrangian becomes²

$$\mathcal{L} = i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi - m\chi\xi - m\chi^\dagger \xi^\dagger. \quad (2.39)$$

This Lagrangian is invariant under the U(1) transformation:

$$\chi \rightarrow e^{-i\alpha} \chi, \quad (2.40)$$

$$\xi \rightarrow e^{+i\alpha} \xi. \quad (2.41)$$

The Dirac equation can now be given in terms of the two complex fields:

$$\begin{pmatrix} m & -i\sigma^\mu \partial_\mu \\ -i\bar{\sigma}^\mu \partial_\mu & m \end{pmatrix} \begin{pmatrix} \chi \\ \xi^\dagger \end{pmatrix} = 0, \quad (2.42)$$

where the Dirac field is given by

$$\Psi = \begin{pmatrix} \chi \\ \xi^\dagger \end{pmatrix}. \quad (2.43)$$

This Dirac field combining the left-handed and right-handed Weyl fields obeys the Dirac equation:

$$(i\gamma^\mu \partial_\mu + m)\Psi = 0. \quad (2.44)$$

The conjugated Dirac field is given by

$$\Psi^\dagger = (\chi^\dagger, \xi). \quad (2.45)$$

In order to get an invariant term in the Lagrangian, this field needs to be multiplied by the γ_0 matrix. Defining

$$\bar{\Psi} \equiv \Psi^\dagger \gamma_0 = (\xi, \chi^\dagger), \quad (2.46)$$

the invariants of the Lagrangian are then given by

$$\bar{\Psi}\Psi = \xi\chi + \chi^\dagger \xi^\dagger, \quad (2.47)$$

and

$$\bar{\Psi}\gamma^\mu \partial_\mu \Psi = \xi\sigma^\mu \partial_\mu \xi^\dagger + \chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi \quad (2.48)$$

²In the following, a spinor notation is understood, so that $\xi\chi = \xi^a \chi_a$ and $\chi^\dagger \xi^\dagger = \chi_a^\dagger \xi^{\dagger a}$, where a and b are Lorentz spinor indices.

$$= \chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + \xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi + \partial_\mu (\xi \sigma^\mu \xi^\dagger), \quad (2.49)$$

where the last term is an irrelevant total divergence. This means, that the Lagrangian for the Dirac field becomes

$$\mathcal{L} = \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi. \quad (2.50)$$

Equation (2.36) is invariant under a discrete symmetry:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (2.51)$$

which is the charge conjugation symmetry associated with this U(1) symmetry. It therefore exchanges χ and ξ , so that

$$C^{-1} \chi(x) C = \xi(x), \quad (2.52)$$

$$C^{-1} \xi(x) C = \chi(x). \quad (2.53)$$

The charge conjugation of Ψ is then given by

$$C^{-1} \Psi(x) C = \Psi^c(x) = \begin{pmatrix} \xi \\ \chi^\dagger \end{pmatrix}, \quad (2.54)$$

which is the same as Ψ with χ and ξ interchanged. A Majorana field satisfies the condition $\Psi = \Psi^c$, where the left-handed and right-handed spinors are equal. It therefore contains a left-handed Weyl spinor, ψ_a and the hermitian conjugated right-handed Weyl spinor, $(\psi_a)^\dagger = \psi^{\dagger\dot{a}}$. We can express the Lagrangian in terms of the left-handed spinors:

$$\mathcal{L} = i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi - \frac{1}{2} m \psi \psi - \frac{1}{2} m \psi^\dagger \psi^\dagger, \quad (2.55)$$

where the mass is taken to be real and positive. The equation of motion can be used

$$0 = -\frac{\delta \mathcal{L}}{\delta \psi^\dagger} = -i\bar{\sigma}^\mu \partial_\mu \psi + m \psi^\dagger. \quad (2.56)$$

The same can be done in the hermitian conjugated case. Both of these contribute to the overall equation of motion:

$$\begin{pmatrix} m & -i\sigma^\mu \partial_\mu \\ -i\bar{\sigma}^\mu \partial_\mu & m \end{pmatrix} \begin{pmatrix} \psi \\ \psi^\dagger \end{pmatrix} = 0. \quad (2.57)$$

The Dirac equation of a Majorana field is then given by

$$(i\gamma^\mu \partial_\mu + m) \Psi = 0. \quad (2.58)$$

A Majorana particle has the property of being its own antiparticle. Speculations are circulating as to whether neutrinos are Majorana as outlined in the overview. The experiments carrying out neutrinoless double beta decay, as was seen in section 1.3.1 will hopefully reveal the truth about this in the near future.

2.2 Yang-Mills theory

Yang-Mills theory is an important part of the Standard Model. It describes the consequences of a global symmetry becoming local and how to account for this. This is certainly relevant when addressing particles, since their interactions can only be described locally.

Under a local $SU(N)$ symmetry transformation,

$$\phi(x) \rightarrow U(x)\phi(x), \quad (2.59)$$

the unitary matrix depends on position. In order to maintain the invariance of the Lagrangian upon local transformations, a gauge theory has to be introduced.

This can be illustrated for a $U(1)$ symmetry, where the Lagrangian should be invariant under a phase rotation of the Dirac field Ψ ,

$$\Psi(x) \rightarrow e^{i\alpha(x)}\Psi(x), \quad (2.60)$$

where α is now position dependent because we have made the global symmetry local. The mass term in the Lagrangian is the same as in the previous section, $m\bar{\Psi}\Psi$. The derivative of the field, however, changes accordingly:

$$\partial_\mu \Psi(x) \rightarrow \partial_\mu (e^{i\alpha(x)}\Psi(x)) = e^{i\alpha(x)}(\partial_\mu \Psi + (e^{-i\alpha(x)}\partial_\mu e^{i\alpha(x)})\Psi) = e^{i\alpha(x)}(\partial_\mu + i\alpha(x)\partial_\mu \alpha(x))\Psi. \quad (2.61)$$

Therefore, a gauge vector field, A_μ , is introduced, to compensate for the differences in the transformation at different locations. The derivative ∂_μ is therefore promoted to a covariant derivative $D_\mu = \partial_\mu - ieA_\mu$, so that

$$\bar{\Psi}(i\gamma^\mu D_\mu)\Psi, \quad (2.62)$$

is the invariant kinetic term. The gauge transformation of this covariant derivative is given by

$$D_\mu \rightarrow U(x)D_\mu U^\dagger(x) = U(x)\partial_\mu U^\dagger(x) - ieU(x)A_\mu(x)U^\dagger(x) \quad (2.63)$$

$$= e^{i\alpha(x)}\partial_\mu e^{-i\alpha(x)} - ie e^{i\alpha(x)}A_\mu(x)e^{-i\alpha(x)} = -i\partial_\mu \alpha(x) - ie e^{i\alpha(x)}A_\mu(x)e^{-i\alpha(x)}. \quad (2.64)$$

In order for this covariant derivative to remain invariant under the gauge transformation, the gauge field A_μ has to transform according to

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x) \Rightarrow \quad (2.65)$$

$$A_\mu(x) \rightarrow U(x)A_\mu(x)U^\dagger(x) + \frac{i}{e}U(x)\partial_\mu U^\dagger(x),$$

so the covariant derivative transforms as

$$D_\mu \Psi(x) \rightarrow [\partial_\mu + ie(A_\mu - \frac{1}{e}\partial_\mu \alpha)]e^{i\alpha(x)}\Psi(x)$$

$$\begin{aligned}
 &= e^{i\alpha(x)}[i\partial_\mu\alpha + \partial_\mu + ieA_\mu - i\partial_\mu\alpha]\Psi(x) \\
 &= e^{i\alpha(x)}D_\mu\Psi,
 \end{aligned} \tag{2.69}$$

which transforms the same way as Ψ as it should.

Under a local symmetry, the Lagrangian can thus be constructed just by replacing the derivatives with covariant derivatives. Regarding the new field, A_μ , it is necessary to find a corresponding kinetic term that is locally invariant and depends on this gauge field and its derivatives, but not on the Dirac field. Since $D_\mu\Psi(x)$ transforms the same way as $\Psi(x)$, so does $D_\mu D_\nu\Psi(x)$, so that

$$\begin{aligned}
 [D_\mu, D_\nu]\Psi(x) &\rightarrow e^{i\alpha(x)}[D_\mu, D_\nu]\Psi(x) \\
 &= [\partial_\mu, \partial_\nu]\Psi(x) + ie([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu])\Psi(x) - e^2[A_\mu, A_\nu]\Psi(x)
 \end{aligned} \tag{2.71}$$

$$= ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\Psi(x) \equiv ieF_{\mu\nu}\Psi(x), \tag{2.72}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor. The most general local invariant Lagrangian under $U(1)$ up to mass dimension 4, which is also parity and time invariant is then given by

$$\mathcal{L} = \bar{\Psi}i\not{D}\Psi - \frac{1}{4}(F_{\mu\nu})^2 - m\bar{\Psi}\Psi. \tag{2.73}$$

It is possible to generalize this local invariance to any continuous symmetry. In the case of a $SU(2)$ symmetry, the infinitesimal transformation is given by

$$U(x) = I - ig\theta^a(x)\frac{\sigma^a}{2} + \mathcal{O}(\theta^2), \tag{2.74}$$

where the generators are the Pauli spin matrices satisfying the Lie commutation rules:

$$\left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}\right] = i\epsilon^{abc}\frac{\sigma^c}{2}, \tag{2.75}$$

$$\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}. \tag{2.76}$$

The covariant derivative can now be defined in terms of three gauge fields corresponding to the three generators:

$$D_\mu = \partial_\mu - igA_\mu^a\frac{\sigma^a}{2} \equiv \partial_\mu - igA_\mu. \tag{2.77}$$

The Dirac field transforms as

$$\Psi(x) \rightarrow (1 + i\theta^a\frac{\sigma^a}{2})\Psi(x) + \mathcal{O}(\theta^2), \tag{2.78}$$

and the vector field

$$A_\mu^a\frac{\sigma^a}{2} \rightarrow A_\mu^a\frac{\sigma^a}{2} + \frac{1}{g}(\partial_\mu\theta^a)\frac{\sigma^a}{2} + i\left[\theta^a\frac{\sigma^a}{2}, A_\mu^b\frac{\sigma^b}{2}\right] + \mathcal{O}(\theta^2) \tag{2.79}$$

The covariant derivative transforms as a consequence like the Dirac field,

$$D_\mu \Psi(x) \rightarrow \left(\partial_\mu - ig A_\mu^a \frac{\sigma^a}{2} - i(\partial_\mu \theta^a) \frac{\sigma^a}{2} + g \left[\theta^a \frac{\sigma^a}{2}, A_\mu^b \frac{\sigma^b}{2} \right] \right) \left(1 + i\theta^c \frac{\sigma^c}{2} \right) \Psi(x) \quad (2.80)$$

$$= (1 + i\theta^a \frac{\sigma^a}{2}) D_\mu \Psi. \quad (2.81)$$

However, the analog of the electromagnetic field tensor turns out to be different, as the gauge fields are now non-commuting matrices:

$$F_{\mu\nu}(x) \equiv \frac{i}{g} [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu]. \quad (2.82)$$

Under a gauge transformation, the field strength transforms as

$$F_{\mu\nu}(x) \rightarrow U(x) F_{\mu\nu}(x) U^\dagger(x) = F_{\mu\nu} + [i\theta^a \frac{\sigma^a}{2}, F_{\mu\nu}], \quad (2.83)$$

to first order in θ . To get rid of these crossterms, the gauge invariant term we are looking for as a kinetic candidate is the following:

$$-\frac{1}{2} \text{Tr}(F^{\mu\nu} F_{\mu\nu}). \quad (2.84)$$

The kinetic term of the gauge field becomes

$$\mathcal{L}_{kin} = -\frac{1}{2} \text{Tr}(F^{\mu\nu} F_{\mu\nu}) = -\frac{1}{2} \text{Tr}(F^{a\mu\nu} T^a F_{\mu\nu}^b T^b) = -\frac{1}{2} F^{a\mu\nu} F_{\mu\nu}^b \text{Tr}(T^a T^b) = -\frac{1}{4} F^{c\mu\nu} F_{\mu\nu}^c, \quad (2.85)$$

where $F_{\mu\nu}^c = \partial_\mu A_\nu^c - \partial_\nu A_\mu^c + gf^{abc} A_\mu^a A_\nu^b$. The Yang-Mills Lagrangian can now be written up:

$$\mathcal{L} = \bar{\Psi}(i\not{D})\Psi - \frac{1}{4} F^{c\mu\nu} F_{\mu\nu}^c - m\bar{\Psi}\Psi, \quad (2.86)$$

which can be generalized to any continuous group that satisfies the Lie algebra. In general, the unitary matrix can be written in terms of the generators T^a :

$$U(x) = I + i\alpha^a(x) T^a + \mathcal{O}(\alpha^2). \quad (2.87)$$

These generators satisfy the Lie algebra as previously stated:

$$[T^a, T^b] = if^{abc} T^c. \quad (2.88)$$

The covariant derivative is given in terms of the gauge fields and the generators

$$D_\mu \rightarrow \partial_\mu - ig A_\mu^a T^a, \quad (2.89)$$

and the gauge field transforms as

$$A_\mu^a \rightarrow A_\mu^a + \frac{1}{g} \partial_\mu \alpha^a(x) + f^{abc} A_\mu^b \alpha^c(x). \quad (2.90)$$

The commutation of the covariant derivatives defines the field tensor:

$$[D_\mu, D_\nu] = -igF_{\mu\nu} = -ig(\partial_\mu A_\nu - \partial_\nu A_\mu) - ig^2 f^{abc} A_\mu A_\nu, \quad (2.91)$$

where the field tensor transforms as

$$F_{\mu\nu}^a \rightarrow F_{\mu\nu}^a - f^{abc} \alpha^b F_{\mu\nu}^c, \quad (2.92)$$

again to first order in α .

2.3 Renormalization

Funny things can happen in physics, when loop corrections are taken into consideration. Therefore, it is necessary to account for these loops, when studying the amplitude of an interaction. The loops might diverge, and more importantly, they might not be renormalizable. In a renormalizable quantum field theory, however, an amplitude involving divergent diagrams can get a finite result through renormalization. The renormalization is done using regulators to obtain the physical masses and coupling constants, that will not depend on these regulators. The resulting expression for the amplitude is then finite for the cut-off $\Lambda \rightarrow \infty$. As we will see in the following, the Lagrangian of the Standard Model is renormalizable.

In a ϕ^4 theory, the Lagrangian is defined as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2 \phi^2 - \frac{\lambda}{4!} \phi^4, \quad (2.93)$$

The divergent amplitudes come from the three 4-point loops, that can be seen in figure 2.1. The amplitude of the first diagram is given by

$$\mathcal{M} = \frac{1}{2}(-i\lambda)^2 \int \frac{d^4 l}{(2\pi)^4} \frac{i}{l^2 - m^2 + i\epsilon} \frac{i}{(k_1 + k_2 + l)^2 - m^2 + i\epsilon} \quad (2.94)$$

which diverges logarithmically. We can solve this integral using Feynman parameters:

$$\mathcal{M} = \frac{1}{2}(-i\lambda)^2 i^2 \int \frac{d^4 l}{(2\pi)^4} \int_0^1 dx \frac{1}{D^2} \quad (2.95)$$

where

$$D = x(k_1 + k_2 + l)^2 + (1-x)l^2 - m^2 + i\epsilon = (l + x(k_1 + k_2))^2 + x(1-x)(k_1 + k_2)^2 - m^2 + i\epsilon \quad (2.96)$$

Making a shift of variable $l \rightarrow l - x(k_1 + k_2)$, we have

$$D = l^2 + x(1-x)(k_1 + k_2)^2 - m^2 + i\epsilon \quad (2.97)$$

The integral over l can now be solved using the Pauli-Villars Regularization, introducing the cut-off Λ . At the same time, we can go to Euclidian space and polar coordinates

$$\int \frac{d^4 l}{(2\pi)^4} \int_0^1 dx \frac{1}{D^2} = i \frac{1}{(2\pi)^4} \int d\Omega \int dl_E \left(\frac{l_E^3}{(l_E^2 + m^2 - x(1-x)(k_1 + k_2)^2)^2} - \frac{l_E^3}{(l_E^2 + \Lambda^2)^2} \right)$$

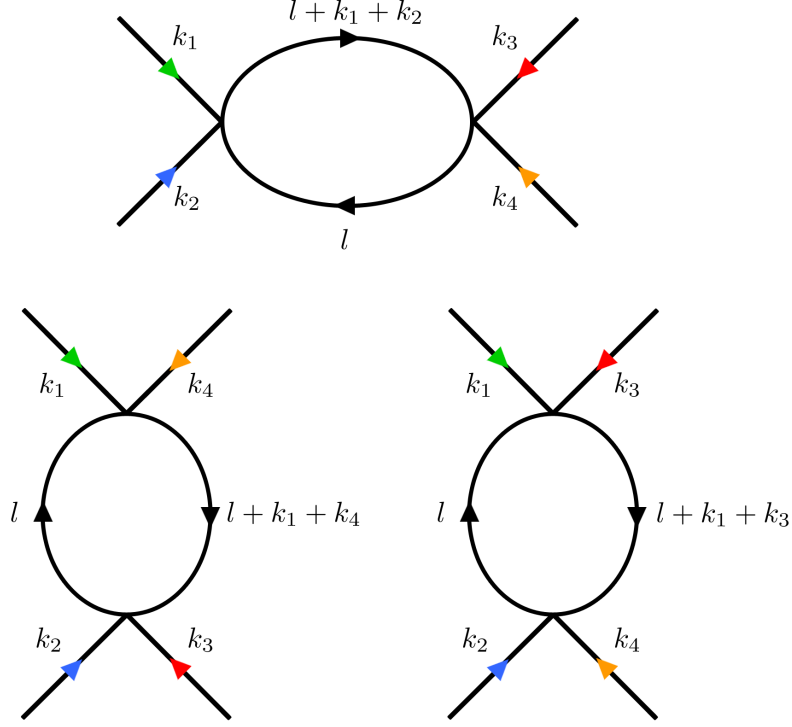


Figure 2.1: 4-point vertex corrections.

$$= \frac{i}{16\pi^2} \ln \left(\frac{\Lambda}{m^2 - x(1-x)(k_1 + k_2)^2} \right) \quad (2.99)$$

Since three loops are contributing with this logarithmically divergent term, the incoming and outgoing particles can conveniently be expressed in terms of the Mandelstam variables:

$$s = -(k_1 + k_2)^2, \quad t = -(k_1 + k_3)^2, \quad u = -(k_1 + k_4)^2 \quad (2.100)$$

so that the contribution to the amplitude including the tree-level amplitude is given by

$$\mathcal{M} = -i\lambda + iC\lambda^2 \left[\ln \frac{\Lambda^2}{s} + \ln \frac{\Lambda^2}{t} + \ln \frac{\Lambda^2}{u} \right] + \mathcal{O}(\lambda^3) \quad (2.101)$$

with C some constant. For $\Lambda \rightarrow \infty$, we want this term to be regularized. In order to do this, we define some physically measured amplitude:

$$-i\lambda_R = -i\lambda + iC\lambda^2 \left[\ln \frac{\Lambda^2}{s_0} + \ln \frac{\Lambda^2}{t_0} + \ln \frac{\Lambda^2}{u_0} \right] + \mathcal{O}(\lambda^3) \quad (2.102)$$

where s_0 , t_0 and u_0 are now associated with the physically measured amplitude. The amplitude can now be expressed in terms of these physical parameters:

$$\mathcal{M} = -i\lambda + iC\lambda^2 \left[\ln \frac{\Lambda^2}{s} + \ln \frac{\Lambda^2}{t} + \ln \frac{\Lambda^2}{u} \right] + \mathcal{O}(\lambda^3) \quad (2.103)$$

$$= -i\lambda_R - iC\lambda^2 \left[\ln \frac{\Lambda^2}{s_0} + \ln \frac{\Lambda^2}{t_0} + \ln \frac{\Lambda^2}{u_0} \right] + iC\lambda^2 \left[\ln \frac{\Lambda^2}{s} + \ln \frac{\Lambda^2}{t} + \ln \frac{\Lambda^2}{u} \right] + \mathcal{O}(\lambda^3) \quad (2.104)$$

$$= -i\lambda_R - iC\lambda^2 \left[\ln \frac{s_0}{s} + \ln \frac{t_0}{t} + \ln \frac{u_0}{u} \right] \quad (2.105)$$

which no longer depends on Λ . It has been regularized [31]. This means that a Lagrangian with up to dimension 4 operators is renormalizable. However, only a few quantum field theories are renormalizable. When operators of mass dimensions higher than four are introduced, the divergences blow up as Λ , Λ^2 , etc and we can no longer regularize as we just did. This has presented a constraint to the Standard Model, which does not allow non-renormalizable interactions [29]. However, when expanding the Standard Model, thus often being forced to introduce these non-renormalizable interactions, a new approach has to be used, namely an effective field theory approach, that will be described in the next chapter.

2.4 Effective field theory

An effective field theory approach is adopted when the field theory is non-renormalizable. When the Standard model is extended to describe new physics, higher dimensional operators are occasionally introduced and thus bring with them non-renormalizability. To generate neutrino masses, a dimension 5 operator is commonly used. However, the effective field theory approach takes care of this. It is a realistic theory used to describe physics at accessible energies, being a low energy approximation that includes an infinite number of non-renormalizable interactions.

In the following the Wilsonian approach will be used to illustrate how a theory becomes effective.

In this functional integral approach, the degrees of freedom of the quantum fields are variables of the integration. In a ϕ^4 theory, the generating functional, that is the partition function, is given by

$$Z(J) = \int \mathcal{D}\phi e^{i \int [\mathcal{L} + J\phi]} = \prod_k \int d\phi(k) e^{i \int [\mathcal{L} + J\phi]}. \quad (2.106)$$

A sharp ultraviolet cut-off Λ can then be imposed again by justifying $m^2 \ll \Lambda$. The number of integration variables k are then restricted to an integration over $\phi(k)$ with $|k| < \Lambda$ and $\phi(k) = 0$ for those outside. This means that quantum fluctuations at very short distances are neglected. The cut-off is imposed in Euclidian space to avoid the light-like very large momenta with very small $|k|^2$. After the cut-off has been applied, and $J = 0$, the partition function is given by

$$Z(J = 0) = \int [\mathcal{D}\phi]_{|k| < \Lambda} \exp \left[- \int d^d x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right] \right], \quad (2.107)$$

where now $(\partial_\mu \phi)^2 = (\frac{\partial \phi}{\partial \tau})^2 + (\Delta \phi)^2$ and no counterterms have been introduced to renormalize. In order to carry out this integration, $\phi(k)$ can be divided into two groups, where a new scalar can be introduced:

$$\hat{\phi}(k) = \begin{cases} \phi(k) & \text{if } b\Lambda \leq |k| < \Lambda; \\ 0 & \text{otherwise.} \end{cases} \quad (2.108)$$

$$\phi(k) = \begin{cases} \phi(k) & \text{if } |k| < b\Lambda; \\ 0 & \text{otherwise.} \end{cases} \quad (2.109)$$

This means that the Lagrangian consists of two fields, $\phi(k)$ and $\hat{\phi}(k)$, so that

$$Z = \int [\mathcal{D}\phi] \int [\mathcal{D}\hat{\phi}] \exp \left[- \int d^d x \left[\frac{1}{2} (\partial_\mu \phi + \partial_\mu \hat{\phi})^2 + \frac{1}{2} m^2 (\phi + \hat{\phi})^2 + \frac{\lambda}{4!} (\phi + \hat{\phi})^4 \right] \right] \quad (2.110)$$

$$= \int [\mathcal{D}\phi] e^{-\int \mathcal{L}(\phi)} \int [\mathcal{D}\hat{\phi}] \exp \left[- \int d^d x \left[\frac{1}{2} (\partial_\mu \hat{\phi})^2 + \frac{1}{2} m^2 \hat{\phi}^2 + \lambda \left(\frac{1}{6} \phi^3 \hat{\phi} + \frac{1}{4} \phi^2 \hat{\phi}^2 + \frac{1}{6} \phi \hat{\phi}^3 + \frac{1}{4!} \hat{\phi}^4 \right) \right] \right], \quad (2.111)$$

where the $\phi\hat{\phi}$ terms vanish from the orthogonality of Fourier components with different wavelength. After the integration over $\hat{\phi}$ is performed, we have

$$Z = \int [\mathcal{D}\phi]_{|k| < b\Lambda} \exp \left(- \int d^d x \mathcal{L}_{\text{eff}} \right). \quad (2.112)$$

This new effective Lagrangian, which contains only momenta below $b\Lambda$, can be found carrying out the integrals over $\hat{\phi}$. This is done perturbatively. Since we address $m^2 \ll \Lambda$, the $\hat{\phi}^2$ terms can also be considered as perturbations. The leading order term is then given by the derivative in Fourier space:

$$\int \mathcal{L}_0 = \frac{1}{2} \int_{b\Lambda \leq |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \hat{\phi}^*(k) k^2 \hat{\phi}(k). \quad (2.113)$$

The propagator for the $\hat{\phi}(k)\hat{\phi}(p)$ term is given by

$$\langle \hat{\phi}(k) \hat{\phi}(p) \rangle = \frac{\int \mathcal{D}\hat{\phi} e^{-\int \mathcal{L}_0} \hat{\phi}(k) \hat{\phi}(p)}{\int \mathcal{D}\hat{\phi} e^{-\int \mathcal{L}_0}} = \frac{1}{k^2} (2\pi)^d \delta^{(d)}(k+p) \Theta(k), \quad (2.114)$$

where $\Theta(k) = 1$ for $b\Lambda \leq |k| < \Lambda$ and 0 otherwise. This propagator can be written in Feynman diagram manner as a double line. A $\phi^2 \hat{\phi}^2$ term can be found from

$$- \int d^d x \frac{\lambda}{4} \phi^2 \hat{\phi} \hat{\phi} = - \frac{1}{2} \int \frac{d^d k_1}{(2\pi)^4} \Delta m^2 \phi(k_1) \phi(-k_1), \quad (2.115)$$

where

$$\Delta m^2 = \frac{\lambda}{2} \int_{b\Lambda \leq |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2}. \quad (2.116)$$

The integral from this term therefore has the form:

$$\exp \left(- \int d^d x \frac{1}{2} \Delta m^2 \phi^2 + \dots \right). \quad (2.117)$$

The Δm^2 gives a positive correction to the m^2 term of the Lagrangian. Writing a ϕ as a single line in Feynman diagrams, the amplitudes can thus be worked out. The $\frac{\lambda}{4} \phi^2 \hat{\phi} \hat{\phi}$ contributes with the first Feynman diagram in figure 2.2. At order λ^2 , the contributions are then given by the other two diagrams in the figure, where the first of these is part of the λ^2 term in the ...

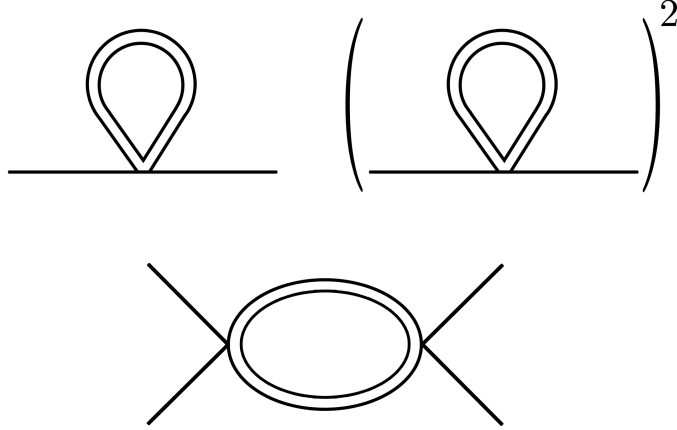


Figure 2.2: Order λ contribution to the effective Lagrangian [32].

of the exponential in (2.117). The last contribution is a direct correction to the ϕ^4 interaction of the Lagrangian (5.3):

$$-\frac{1}{4!} \int d^d x \Delta \lambda \phi^4, \quad (2.118)$$

where

$$\Delta \lambda = -4! \frac{2}{2!} \left(\frac{\lambda}{4} \right)^2 \int_{b\Lambda \leq |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 \quad (2.119)$$

$$\xrightarrow{d \rightarrow 4} -\frac{3\lambda^2}{16\pi^2} \log \frac{1}{b}. \quad (2.120)$$

The effective Lagrangian is then eventually given by

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 + (\text{sum of connected terms}). \quad (2.121)$$

Higher order contributions can be done similarly. Starting out with a partition function for all k lower than a very large cut-off Λ :

$$Z = \int [\mathcal{D}\phi]_{|k| < \Lambda} \exp \left[- \int d^d x \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right] \right], \quad (2.122)$$

a rescaling can be made, so that

$$k' = k/b, \quad x' = xb. \quad (2.123)$$

The integration of

$$\int d^d x \left[\frac{1}{2}(1 + \Delta Z)(\partial_\mu \phi)^2 + \frac{1}{2}(m^2 + \Delta m^2)\phi^2 + \frac{1}{4!}(\lambda + \Delta \lambda)\phi^4 + \Delta C(\partial_\mu \phi)^4 + \Delta D\phi^6 + \dots \right], \quad (2.124)$$

can then be done for $|k'| > \Lambda$, which is the same as $|k| > b\Lambda$, leaving the integration over $|k'| < \Lambda$, that is, the function

$$\int d^d x \mathcal{L}_{\text{eff}} = \int d^d x' \left[\frac{1}{2} (\partial'_\mu \phi')^2 + \frac{1}{2} m'^2 \phi'^2 + \frac{\lambda'}{4!} \phi'^4 + C' (\partial'_\mu \phi')^4 + D' \phi'^6 + \dots \right], \quad (2.125)$$

where

$$m'^2 = (m^2 + \Delta m^2)(1 + \Delta Z)^{-1} b^{-2}, \quad \lambda' = (\lambda + \Delta \lambda)(1 + \Delta Z)^{-2} b^{d-4}, \quad (2.126)$$

$$C' = (C + \Delta C)(1 + \Delta Z)^{-2} b^d, \quad D' = (D + \Delta D)(1 + \Delta Z)^{-3} b^{2d-6}. \quad (2.127)$$

If the perturbation condition is justified these corrections are very small and finite. This transformation can then be continued over another shell of momentum space, which transforms the Lagrangian once again, and so forth. This leads to an iteration of transformations, so that if $b \rightarrow 1$, the transformations become continuous. This is the Renormalization group.

The Wilsonian approach to effective field theory is therefore to define a quantum field theory through an action with a momentum cutoff. This cutoff can then be lowered by integrating out the higher momenta degrees of freedom and from this change the coefficients in the effective action. At an energy much lower than the initial cutoff, the results can then be approximated to the same as those for a renormalized perturbation theory up to a small correction proportional to the energy over the initial cutoff.

As a result, every Lagrangian, as long as the couplings are significantly weak, can be described at the energies of experiments by an effective Lagrangian. This is the effective field theory approach. It is used when higher mass-dimensional operators are introduced to describe physics beyond the Standard Model.

2.5 Spontaneous symmetry breaking

A renormalizable Lagrangian involving a complex field

$$\mathcal{L} = -\partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi - \frac{1}{4} \lambda (\phi^\dagger \phi)^2, \quad (2.128)$$

is invariant under the U(1) transformation $\phi(x) \rightarrow e^{-i\alpha} \phi(x)$. When m^2 is negative, the field can be written as

$$\phi(x) = \frac{1}{\sqrt{2}} (v + \rho(x)) e^{i\chi(x)/v}. \quad (2.129)$$

This means that there is a flat direction in the field space along which you can move without changing energy. The previous single global minimum of the potential has now been replaced by two minima, where $\phi = v$ and $\phi = -v$. Since we are dealing with fields, we are forced to choose one of these ground states, and this clearly breaks the Z_2 symmetry spontaneously. The Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \rho \partial_\mu \rho - \frac{1}{2} \left(1 + \frac{\rho}{v}\right)^2 \partial^\mu \chi \partial_\mu \chi - |m^2| \rho - \frac{1}{2} \lambda^{1/2} |m| \rho^3 - \frac{1}{16} \lambda \rho^4. \quad (2.130)$$

After the spontaneous symmetry breaking the χ remains massless. In the following section, we will see how this massless Goldstone boson disappears when introducing a gauge field, in the Higgs mechanism.

With N real fields $\phi_i(x)$, the renormalizable Lagrangian is given by [33]

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi_i\partial_\mu\phi_i - \frac{1}{2}m^2\phi_i^2 - \frac{1}{16}\lambda(\phi_i^2)^2. \quad (2.131)$$

This Lagrangian is invariant under a $SO(N)$ transformation, that is $\phi_i(x) \rightarrow R_{ij}\phi_j(x)$, where R_{ij} is orthogonal, as described in section 2.1.1. This symmetry is global since the orthogonal matrices do not depend on the position x . Under a spontaneous symmetry breaking, the potential becomes

$$V = \frac{1}{2}m^2\phi_i\phi_i + \frac{1}{16}\lambda(\phi_i\phi_i)^2 = -\frac{1}{2}|m^2|\phi_i\phi_i + \frac{1}{16}\lambda(\phi_i\phi_i)^2 = \frac{1}{16}\lambda[(\phi_i\phi_i - v^2)^2 - v^4], \quad (2.132)$$

where $v^2 \equiv \frac{4|m^2|}{\lambda}$. In an $SO(N)$ theory, v_i has N components. It is however possible to choose a basis, so that v_i only has a nonzero value in the last component, v_N , that is $v_i = v\delta_{iN}$. The Goldstone modes can be rotated away by a group transformation, in this case a $SO(N)$ transformation, that leaves the Lagrangian invariant [30].

The $SO(N)$ infinitesimal transformation of the vacuum expectation value components now becomes

$$v_i \rightarrow v_i - i\theta^a(T^a)_{ij}v_j = v\delta_{iN} - i\theta^a(T^a)_{iN}v. \quad (2.133)$$

All the fields transformed by the T^a 's that have zeroes in the last column do not change under this transformation - they are orthogonal to the Goldstone direction. $N-1$ T^a 's do not have zeroes in the last column. These generators are the ones that get spontaneously broken - a vacuum expectation value has been chosen for them. The condition $(T^a)_{iN}v \neq 0$ then applies and corresponds to a flat direction in field space. Each of the fields associated with a broken generator acquires a mass [33]. Symmetry is then only maintained by the unbroken generators. These will form a new group with $\frac{1}{2}N(N-1) - (N-1) = \frac{1}{2}(N-1)(N-2)$ unbroken generators. The group $SO(N)$ has been spontaneously broken down to the subgroup $SO(N-1)$. The fields corresponding to the unbroken generators are massless Goldstone bosons.

2.6 Higgs mechanism

The Higgs mechanism has the purpose of describing how gauge fields acquire their masses. In order to give masses to the fields, spontaneous symmetry breaking is implemented. In a $U(1)$ gauge theory, the Lagrangian contains a complex scalar field coupled to itself and an electromagnetic field as we have seen earlier:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_\mu\phi|^2 - V(\phi), \quad (2.134)$$

where $D_\mu = \partial_\mu - ieA_\mu$. This is invariant under the local transformation:

$$\phi(x) \rightarrow e^{i\alpha(x)}\phi(x), \quad A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x). \quad (2.135)$$

The invariance can be restated in polar coordinates, where the field is defined as

$$\phi(x) = \rho(x)e^{i\chi(x)}. \quad (2.136)$$

where ρ and χ are real. The covariant derivative is then given by

$$D_\mu\phi = [\partial_\mu\rho + i\rho(\partial_\mu\chi - eA_\mu)]e^{i\chi}, \quad (2.137)$$

and the Lagrangian by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \rho^2(\partial_\mu\chi - eA_\mu)^2 + (\partial\rho)^2 + \mu^2\rho^2 - \lambda\rho^4. \quad (2.138)$$

Under a gauge transformation:

$$\phi \rightarrow e^{i\alpha(x)} \Rightarrow \quad (2.139)$$

$$\chi \rightarrow \chi + \alpha(x), \quad A_\mu \rightarrow A_\mu - \frac{1}{e}\partial_\mu\alpha(x). \quad (2.140)$$

If a redefinition is made so that

$$B_\mu \equiv A_\mu + \frac{1}{e}\partial_\mu\chi, \quad (2.141)$$

this field is gauge invariant and the Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \rho^2e^2B_\mu^2 + (\partial\rho)^2 + \mu^2\rho^2 - \lambda\rho^4, \quad (2.142)$$

where now $F_{\mu\nu}F^{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. Under a spontaneous symmetry breaking, the field ρ acquires the vacuum expectation value:

$$\rho = \frac{1}{\sqrt{2}}(v + \theta), \quad (2.143)$$

where again $v = \sqrt{\frac{\mu^2}{\lambda}}$. The Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2B_\mu^2 + e^2v\theta B_\mu^2 + \frac{1}{2}e^2\theta^2 B_\mu^2 + \frac{1}{2}(\partial\theta)^2 - \mu^2\theta^2 - \sqrt{\lambda}\mu\theta^3 - \frac{\lambda}{4}\theta^4 + \frac{\mu^4}{4\lambda}, \quad (2.144)$$

where the mass of the newly defined gauge field B_μ is given by $M = ev$ and the new scalar field θ acquires the mass $\sqrt{2}\mu$. The scalar field χ has completely disappeared. It no longer appears as the massless Goldstone boson. Instead the gauge boson has acquired a mass and is said to have eaten the Goldstone boson [31]. Furthermore there has been a shift of degrees of freedom. The massless gauge boson had only two degrees of freedom and the massless Goldstone scalar had a single longitudinal degree of freedom. These have now gathered to form three degrees of freedom for the massive gauge boson. This is the Higgs mechanism.

2.6.1 Higgs mechanism and non-abelian symmetries

An SU(3) gauge theory has the covariant derivative:

$$D_\mu\phi_a = \partial_\mu\phi_a + gf_{abc}A_\mu^b\phi_c. \quad (2.145)$$

Making the definition:

$$\Phi \equiv \phi_c t^c, \quad (2.146)$$

where the generators t^c are the 8 3×3 traceless Hermitian Gell-Mann matrices, Φ must therefore be traceless and Hermitian. Using $[t^a, \phi_b t^b] = i f^{abc} \phi_c t^c$, the gauge field mass is in this case:

$$\Delta \mathcal{L} = \frac{g^2}{2} (f_{abc} A_\mu^b \phi_c)^2 = -g^2 \text{Tr}[[t^a, \Phi][t^b, \Phi]] A_\mu^a A^{b\mu}. \quad (2.147)$$

When Φ acquires a vacuum expectation value, it can have three eigenvalues, whose sum is zero (traceless). The vacuum expectation value can therefore have the orientation:

$$\Phi_0 = |\phi| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (2.148)$$

The only SU(3) generators that commute with this matrix are given by

$$t^a = \frac{1}{2} \begin{pmatrix} \sigma^a & 0 \\ 0 & 0 \end{pmatrix}, \quad t^8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (2.149)$$

These are the unbroken generators, so the four gauge fields corresponding to these generators will not acquire masses. The symmetry is broken down to $\text{SU}(2) \times \text{U}(1)$. The remaining four gauge fields acquire masses $m^2 = (3g|\phi|)^2$.

2.7 Standard Model

The Standard Model is the quantum field theory that has been describing our world in a unified way. In the Standard Model, the Lagrangian has mass dimension 4 and is invariant under the gauge symmetries $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ and Lorentz transformations. This gauge group is spontaneously broken down to the subgroup $\text{SU}(3)_{\text{color}} \times \text{U}(1)_{\text{em}}$, when the Higgs doublet acquires a vacuum expectation value as described in the previous section. The result is massive gauge bosons.

2.7.1 Electroweak theory

The electroweak theory consists of the group $\text{SU}(2) \times \text{U}(1)$. In order to study the interaction of the fields, they are assigned to representations within these two groups. Lepton numbers are separately conserved as a consequence of the $\text{SU}(2) \times \text{U}(1)$ gauge theory. This only leaves a few choices for the representations.

The most general gauge invariant and renormalizable Lagrangian involving the SU(2) gauge fields A_μ^a and the U(1) gauge field B_μ is

$$\mathcal{L} = \left(-\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} - \bar{\psi} \not{D} \psi\right)_{\text{SU}(2)} + \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi} \not{D} \psi\right)_{\text{U}(1)} \quad (2.150)$$

$$= -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gA_\mu^a \times A_\nu^a)^2 - \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 - \bar{\psi} \not{D} \psi, \quad (2.151)$$

where

$$D_\mu = \partial_\mu - i[g_2 A_\mu^a T^a + g_1 B_\mu Y], \quad (2.152)$$

and $T^a = -\frac{1}{2}\sigma^a$ and $Y = \frac{1}{2}I$ are the generators of the gauge groups SU(2) and U(1) respectively. The term in brackets in (2.152) can be written as

$$g_2 A_\mu^a T^a + g_1 B_\mu Y = \frac{1}{2} \begin{pmatrix} g_2 A_\mu^3 - g_1 B_\mu & g_2(A_\mu^1 - iA_\mu^2) \\ g_2(A_\mu^1 + iA_\mu^2) & -g_2 A_\mu^3 - g_1 B_\mu \end{pmatrix}. \quad (2.153)$$

The Higgs potential

$$-\frac{1}{2}\mu^2(\phi^\dagger \phi) + \frac{1}{4}\lambda(\phi^\dagger \phi)^2, \quad (2.154)$$

with $\mu^2 > 0$ is broken spontaneously, so the Higgs scalar ϕ gets a nonzero vacuum expectation value:

$$v = \sqrt{\frac{\mu^2}{\lambda}}, \quad (2.155)$$

and the scalar field is parameterized through a gauge transformation, so that

$$\phi = \begin{pmatrix} v + h(x) \\ 0 \end{pmatrix}. \quad (2.156)$$

In a unitary gauge, the potential term of the Lagrangian thus becomes

$$\frac{1}{2}\mu^2(v + h(x))^2 - \frac{1}{4}\lambda(v + h(x))^4 = \frac{\mu^4}{4\lambda} - \mu^2 h(x)^2 - \lambda v h(x)^3 - \frac{1}{4}\lambda h(x)^4, \quad (2.157)$$

where we can identify the mass of the Higgs boson as $m_h = \sqrt{2}\mu = \sqrt{2\lambda}v$

The kinetic term of the Higgs field becomes

$$-(D^\mu \phi)^\dagger D_\mu \phi = -\left(\partial^\mu \phi_i + i\phi_i^*[g_2 A_\mu^a T^a + g_1 B_\mu Y]_i^j\right)\left(\partial_\mu \phi_i - i[g_2 A_\mu^a T^a + g_1 B_\mu Y]_i^j \phi_j\right) \quad (2.158)$$

$$= -(\partial^\mu \phi_i \partial_\mu \phi_i + \phi_i^* \left([g_2 A_\mu^a T^a + g_1 B_\mu Y]_i^j\right)^2 \phi_j + i\phi_i^*[g_2 A_\mu^a T^a + g_1 B_\mu Y]_i^j \partial_\mu \phi_i \quad (2.159)$$

$$-i\partial^\mu \phi_i [g_2 A_\mu^a T^a + g_1 B_\mu Y]_i^j \phi_j), \quad (2.160)$$

where the second term is the mass term. If we let ϕ acquire the mentioned vacuum expectation value, we get the following mass term in the Lagrangian:

$$-\frac{1}{2}v^2 \begin{pmatrix} 1 & 0 \end{pmatrix} [g_2 A_\mu^a T^a + g_1 B_\mu Y]^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.161)$$

$$= -\frac{1}{8}v^2 \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} g_2 A_\mu^3 - g_1 B_\mu & g_2(A_\mu^1 - iA_\mu^2) \\ g_2(A_\mu^1 + iA_\mu^2) & -g_2 A_\mu^3 - g_1 B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (2.162)$$

In order to find the masses of the gauge fields, we want to diagonalize this mass matrix. We introduce the weak mixing angle and redefine the fields:

$$\tan \theta_w \equiv \frac{g_1}{g_2}, \quad (2.163)$$

$$W_\mu^\pm \equiv \frac{1}{\sqrt{2}}(A_\mu^1 \mp iA_\mu^2), \quad (2.164)$$

$$Z_\mu \equiv c_w A_\mu^3 - s_w B_\mu, \quad (2.165)$$

$$A_\mu \equiv s_w A_\mu^3 + c_w B_\mu, \quad (2.166)$$

with $s_w = \sin \theta_w$ and $c_w = \cos \theta_w$. Now the mass term becomes

$$-\frac{1}{8}v^2 g_2^2 \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{Z_\mu}{c_w} & \frac{\sqrt{2}W_\mu^+}{\sqrt{2}W_\mu^- - \frac{Z_\mu + 2s_w B_\mu}{c_w}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.167)$$

$$= -\frac{1}{8}v^2 g_2^2 \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{Z^\mu Z_\mu}{c_w^2} + 2W^{\mu+}W_\mu^- & 2\sqrt{2}\frac{Z^\mu W_\mu^+}{c_w} \\ 2\sqrt{2}\frac{Z^\mu W_\mu^-}{c_w} & 2W^{\mu+}W_\mu^- + \frac{Z_\mu + 2s_w B_\mu}{c_w^2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.168)$$

$$= -\frac{1}{8}v^2 g_2^2 \left(\frac{Z^\mu Z_\mu}{c_w^2} + 2W^{\mu+}W_\mu^- \right) = -\left(\frac{g_2 v}{2} \right)^2 W^{\mu+}W_\mu^- - \frac{1}{2} \left(\frac{g_2 v}{2c_w} \right)^2 Z^\mu Z_\mu \quad (2.169)$$

$$= -M_W^2 W^{\mu+}W_\mu^- - \frac{1}{2}M_Z^2 Z^\mu Z_\mu, \quad (2.170)$$

where the masses of the gauge fields obtained from this Higgs mechanism have been extracted. These have been measured together with $\cos \theta_w = 0.882$:

$$M_W = \frac{g_2 v}{2} = 80.4 \text{ GeV} \quad M_Z = \frac{M_W}{c_w} = 91.2 \text{ GeV}. \quad (2.171)$$

The A_μ gauge field corresponding to the unbroken generators T^3 and Y and does not acquire a mass. A_μ is the electromagnetic vector potential. The broken generators are T^1 and T^2 , leading to massive W and Z gauge fields.

The kinetic energy term for these massive gauge fields is given by, in terms of A_μ^a and B_μ :

$$-\frac{1}{4}F^{a\mu\nu}F_{\mu\nu}^a - \frac{1}{4}B^{\mu\nu}B_{\mu\nu}, \quad (2.172)$$

where

$$\frac{1}{\sqrt{2}}(F_{\mu\nu}^1 - iF_{\mu\nu}^2) = D_\mu W_\nu^+ - D_\nu W_\mu^+, \quad (2.173)$$

$$\frac{1}{\sqrt{2}}(F_{\mu\nu}^1 + iF_{\mu\nu}^2) = D_\mu^\dagger W_\nu^- - D_\nu^\dagger W_\mu^-, \quad (2.174)$$

In terms of A_μ^3 , the covariant derivative is given by

$$D_\mu = \partial_\mu - ig_2(s_w A_\mu + c_w Z_\mu). \quad (2.175)$$

Setting the charge of the W^+ to +1, the electromagnetic coupling constant is given by

$$e = g_2 s_w, \quad (2.176)$$

and the covariant derivative then becomes

$$D_\mu = \partial_\mu - ie(A_\mu + \cot \theta_w Z_\mu). \quad (2.177)$$

Using the following definitions:

$$c_w = \frac{g_2}{\sqrt{g_1^2 + g_2^2}}, \quad (2.178)$$

$$s_w = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}, \quad (2.179)$$

we have

$$D_\mu = \partial_\mu - i \frac{1}{\sqrt{g_1^2 + g_2^2}} Z_\mu (g_2^2 T^3 - g_1^2 Y) - i \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}} A_\mu (T^3 + Y). \quad (2.180)$$

From this we can easily extract the unbroken gauge generator that A_μ couples to. Associated with this unbroken symmetry, the charge Q is given by

$$Q = T^3 + Y. \quad (2.181)$$

2.7.2 Lepton sector

As we have seen, the covariant derivative determines the coupling between the gauge fields and the fermions. Experiments have shown that the W bosons only couple to left handed helicity states of quarks and leptons forming weak interactions. In order to ensure this, the left handed and right handed leptons are assigned to different representations of the gauge group. Left-handed fermion fields are assigned to a doublet representation of $SU(2)$ and right handed to a singlet representation. This means that $T^3 = 0$ and $Y = Q$ for right-handed fields and $T^3 = \pm \frac{1}{2}$ and $Y = Q \mp \frac{1}{2}$ for left-handed fields.

In the lepton sector we have the lepton doublet containing the neutrino:

$$\psi = \begin{pmatrix} \nu_l \\ l \end{pmatrix}_L, \quad (2.182)$$

with $l = e, \mu, \tau$ where $T^3(\nu_l) = \frac{1}{2}$ and $T^3(l) = -\frac{1}{2}$, so that $Y = -\frac{1}{2}$. The right-handed singlet l_R has $Y = Q = -1$. Since the left-handed and right-handed fermions are in different representations of the fundamental gauge group $SU(2) \times U(1)$, they can be treated as different particles. This will create two different covariant derivatives:

$$(D_\mu \psi)_i = \partial_\mu \psi_i - ig_2 A_\mu^a (T^a)_i^j \psi_j - ig_1 \left(-\frac{1}{2}\right) B_\mu \psi_i, \quad (2.183)$$

$$D_\mu l_R = \partial_\mu l_R - ig_1(+1)B_\mu l_R, \quad (2.184)$$

with the kinetic terms for these Weyl spinors:

$$\mathcal{L}_{kin} = i\psi^\dagger \bar{\sigma}^\mu (D_\mu \psi)_i + il_R^\dagger \bar{\sigma}^\mu D_\mu l_R. \quad (2.185)$$

The construction of the mass terms, that couple left-handed with right-handed leptons through a chiral symmetry, is forbidden by global gauge invariance;

$$\mathcal{L}_{mass} = -m_l(\bar{l}_L l_R + \bar{l}_R l_L), \quad (2.186)$$

since the components have different gauge quantum numbers as outlined above. We need to invoke the mechanism of spontaneous symmetry breaking once again.

If we couple the leptons to a Higgs boson thus forming a Yukawa coupling, the leptons can acquire masses through the term

$$\mathcal{L}_{Yuk} = -\lambda_e \bar{\psi} \cdot \phi l_R + h.c., \quad (2.187)$$

which requires the Higgs boson ϕ to be in a doublet SU(2) representation and have $Y = -\frac{1}{2}$ as the lepton doublet. The SU(2) and U(1) indices are contracted in the following way:

$$(2, -1/2) \otimes (2, -1/2) \otimes (1, +1) = (1, 0) \oplus (3, 0). \quad (2.188)$$

When the Higgs boson acquires a vacuum expectation value, $\langle \phi^0 \rangle = \frac{1}{\sqrt{2}}v$, this will give mass to the leptons:

$$\mathcal{L}_{Yuk} = -\frac{1}{\sqrt{2}}\lambda_l v \bar{l}_L l_R + h.c. \Rightarrow m_l = \frac{1}{\sqrt{2}}\lambda_l v. \quad (2.189)$$

No other invariant Yukawa coupling exists with mass dimension 4 or less, so in order to preserve renormalizability, no more terms can be added to the Lagrangian, thus constraining it.

If we have more than one lepton generation and the fields $\psi_{\alpha i}$ and l_{Ri} have the generation index $i = 1, 2, 3$, the Yukawa term in the Lagrangian becomes

$$\mathcal{L}_{Yuk} = -\phi_i \psi_i y_{ij} \bar{l}_{Rj} + h.c. \quad (2.190)$$

The fields can undergo a unitary transformation with independent unitary matrices in generation space in order to have a diagonal mass matrix, which is required for the charged leptons

$$\psi_i \rightarrow (U_L)_{ij} \psi_{Lj}, \quad l_{Ri} \rightarrow (U_R)_{ij} l_{Rj}, \quad (2.191)$$

where the kinetic terms are unchanged. The Yukawa matrix is then transformed into

$$y_{ij} \rightarrow (U_L)^T y U_R, \quad (2.192)$$

where the two unitary independent matrices are chosen, so that this term is diagonal with positive real entries, and the masses of the leptons become $m_{l_I} = y_I v / \sqrt{2}$. The crucial

thing to notice, is that neutrinos do not have right-handed partners in the Standard Model. This means that they can not acquire masses through dimension 4 or less operators. This is the main concern regarding neutrinos. Alternative mechanisms need to be introduced when dealing with neutrino masses.

2.8 Anomalies

When decomposing the vector field, we have seen that left-handed and right-handed Weyl spinors show up, which can be expressed as follows

$$\psi_L = \frac{1}{2}(1 - \gamma_5)\Psi, \quad \psi_R = \frac{1}{2}(1 + \gamma_5)\Psi. \quad (2.193)$$

A new symmetry arises corresponding to the chiral transformation $\Psi \rightarrow e^{i\gamma_5\alpha}\Psi$. This brings with it an axial vector current

$$j^{\mu 5} = \bar{\Psi}(x)\gamma^\mu\gamma^5\Psi(x), \quad (2.194)$$

which is not conserved for $m \neq 0$, since

$$\partial_\mu j^{\mu 5} = 2im\bar{\Psi}\gamma^5\Psi. \quad (2.195)$$

The currents associated with the left-handed and right-handed fields are then a combination of this axial vector current as well as the ordinary vector current

$$j_L^\mu = \bar{\Psi}\gamma^\mu\left(\frac{1 - \gamma^5}{2}\right)\Psi, \quad (2.196)$$

$$j_R^\mu = \bar{\Psi}\gamma^\mu\left(\frac{1 + \gamma^5}{2}\right)\Psi. \quad (2.197)$$

It is a general statement that the vector current should be conserved in order to conserve charge and thereby the number of fermions. The axial vector current, on the other hand is not necessarily a conserved quantity in quantum field theory. For a massless fermion, the axial vector is classically conserved. However classical symmetries might not be quantum symmetries. These require not only the Lagrangian, but the complete path integral including the Lagrangian to be invariant:

$$\int D\phi e^{iS(\phi)}. \quad (2.198)$$

In other words, the quantum fluctuations might result in the breaking of classical symmetries. This is called an anomaly [31]. These chiral symmetry anomalies do not impose any problems under global symmetries. It does not lead to severe damage, if the axial charge is not conserved. However, if the theory is gauged, the radiative corrections will involve gauge operators and since the axial current is gauge invariant, the anomaly must also be gauge invariant. Therefore, these radiative corrections could lead to violation of gauge invariance and are therefore required to cancel. The amplitude for the radiative corrections associated with the axial vector current can be found from the diagrams in figure 2.3. To get a conserved vector current, the Ward identity tells us that

$$k_{1\mu}\Delta^{\lambda\mu\nu} = k_{2\mu}\Delta^{\lambda\mu\nu} = 0, \quad (2.199)$$

In order to get a conserved axial vector, the same identity applies

$$q_\lambda\Delta^{\lambda\mu\nu} = 0. \quad (2.200)$$

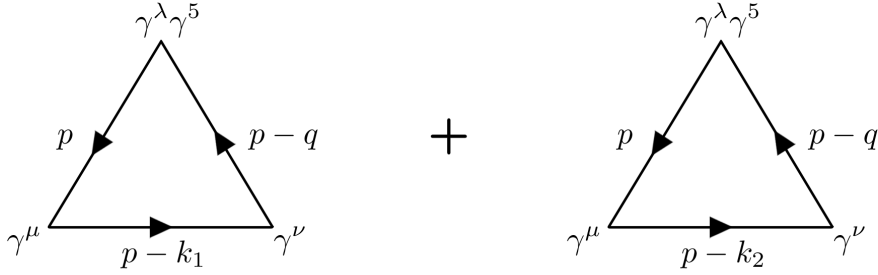


Figure 2.3: 3-vertex diagrams contributing to the U(1) anomalies [31].

The amplitude can be found for these diagrams. We have

$$\Delta^{\lambda\mu\nu}(k_1, k_2) = (-1)i^3 \int \frac{d^4p}{(2\pi)^3} \text{Tr} \left(\gamma^\lambda \gamma^5 \frac{1}{\not{p} - \not{q}} \gamma^\nu \frac{1}{\not{p} - \not{k}_1} \gamma^\mu \frac{1}{\not{p}} + \gamma^\lambda \gamma^5 \frac{1}{\not{p} - \not{q}} \gamma^\mu \frac{1}{\not{p} - \not{k}_2} \gamma^\nu \frac{1}{\not{p}} \right), \quad (2.201)$$

where $q = k_1 + k_2$ and the amplitude for the second diagram is the same as for the first with $\mu \leftrightarrow \nu$ and $k_1 \leftrightarrow k_2$. Letting $k_{1\mu}$ and $k_{2\mu}$ act on the first and second term respectively, we have,

$$(-1)i^3 \int \frac{d^4p}{(2\pi)^3} \text{Tr} \left(\gamma^\lambda \gamma^5 \frac{1}{\not{p} - \not{q}} \gamma^\nu \frac{\not{k}_1}{(\not{p} - \not{k}_1)(\not{p})} + \gamma^\lambda \gamma^5 \frac{\not{k}_2}{(\not{p} - \not{q})(\not{p} - \not{k}_2)} \gamma^\nu \frac{1}{\not{p}} \right). \quad (2.202)$$

Using $\not{k}_1 = \not{p} - (\not{p} - \not{k}_1)$ and $\not{k}_2 = (\not{p} - \not{k}_2) - (\not{p} - \not{q})$ we get

$$(-1)i^3 \int \frac{d^4p}{(2\pi)^3} \text{Tr} \left(\gamma^\lambda \gamma^5 \frac{1}{\not{p} - \not{q}} \gamma^\nu \frac{1}{\not{p} - \not{k}_1} - \gamma^\lambda \gamma^5 \frac{1}{\not{p} - \not{k}_2} \gamma^\nu \frac{1}{\not{p}} \right), \quad (2.203)$$

In order to have the axial vector current conserved, this should disappear. It can be seen, that if we can make the shift $p \rightarrow p - k_1$, this can be achieved. However, when shifting the integration variable of $\int_{-\infty}^{\infty} dp f(p)$ to $\int_{-\infty}^{\infty} dp f(p + a)$, the difference is actually equal to

$$\int_{-\infty}^{\infty} dp \left(a \frac{d}{dp} f(p) + \dots \right), \quad (2.204)$$

so that the two triangles do not cancel if this is nonzero. Moving to 4-dimensional Euclidian space, this term becomes

$$\int \frac{d^4 p}{(2\pi)^4} a^\mu \frac{\partial}{\partial p_\mu} f(p) = i \lim_{p \rightarrow \infty} a^\mu \int \frac{dS_\mu}{(2\pi)^4} f(p), \quad (2.205)$$

where the divergence theorem has been used and $dS_\mu = p^2 p_\mu d\Omega$ is the surface-area element. Defining

$$f(p) = \text{Tr} \left(\gamma^\lambda \gamma^5 \frac{1}{\not{p} - \not{k}_2} \gamma^\nu \frac{1}{\not{p}} \right) = \frac{\text{Tr}(\gamma^5 (\not{p} - \not{k}_2) \gamma^\nu \not{p} \gamma^\lambda)}{(p - k_2)^2 p^2} = \frac{4i\epsilon^{\tau\nu\sigma\lambda} k_{2\tau} p_\sigma}{(p - k_2)^2 p^2}. \quad (2.206)$$

Equation (2.205) reveals the real difference between the two terms in the trace

$$\begin{aligned} k_{1\mu} \Delta^{\lambda\mu\nu} &= \int \frac{d^4 p}{(2\pi)^4} (-k_1)^\mu \frac{\partial}{\partial p^\mu} f(p) = i \lim_{p \rightarrow \infty} \int \frac{d\Omega}{(2\pi)^4} (-k_1)^\mu \frac{4ip_\mu \epsilon^{\tau\nu\sigma\lambda} k_{2\tau} p_\sigma}{(p - k_2)^2} \\ &= \frac{i}{8\pi^2} \epsilon^{\tau\nu\sigma\lambda} k_{1\tau} k_{2\sigma}, \end{aligned} \quad (2.208)$$

where we have used that for the scalar function $f(p^2)$, the integrals $\int d^4 p f(p^2) \frac{p_\mu p_\sigma}{p^2}$ and $\int \frac{1}{4} d^4 p f(p^2) g_{\mu\sigma}$ are both convergent and it is allowed to replace one with the other in this case. An equivalent term can be found from

$$k_{2\nu} \Delta^{\lambda\mu\nu}, \quad (2.209)$$

interchanging μ and ν as well as k_1 and k_2 . The vector current is therefore not conserved choosing this shift of the integration variable. Since the original integral was linearly divergent, it is possible to retrieve conservation using a different shift. For an arbitrary a ,

$$\begin{aligned} \Delta^{\lambda\mu\nu}(a, k_1, k_2) &= \\ &(-1)i^3 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left(\gamma^\lambda \gamma^5 \frac{\not{p} + \not{a} - \not{q}}{(p + a - q)^2} \gamma^\nu \frac{\not{p} + \not{a} - \not{k}_1}{(p + a - k_1)^2} \gamma^\mu \frac{\not{p} + \not{a}}{(p + a)^2} \right) + \{\mu, k_1 \leftrightarrow \nu, k_2\}. \end{aligned} \quad (2.211)$$

The difference between this integral and the integral involving k_2 needs to be zero. Using

$$\Delta^{\lambda\mu\nu}(a, k_1, k_2) = (-1)i^3 I_{\alpha\beta\gamma}(a) \text{Tr}(\gamma^\lambda \gamma^5 \gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\mu \gamma^\gamma) + \{\mu, k_1 \leftrightarrow \nu, k_2\}, \quad (2.212)$$

so that

$$I_{\alpha\beta\gamma}(a) = \int \frac{d^4 p}{(2\pi)^4} \frac{(p + a - q)_\alpha (p + a - k_1)_\beta (p + a)_\gamma}{(p + a - q)^2 (p + a - k_1)^2 (p + a)^2}. \quad (2.213)$$

Subtracting the term with $a = 0$ and again using that the integrals $\int d^4 p f(p^2) \frac{p_\mu p_\sigma}{p^2}$ and $\int \frac{1}{4} d^4 p f(p^2) g_{\mu\sigma}$ are equal,

$$I_{\alpha\beta\gamma}(a) - I_{\alpha\beta\gamma}(0) = a^\delta \int \frac{d^4 p}{(2\pi)^4} \frac{\partial}{\partial p^\delta} \frac{(p - q)_\alpha (p - k_1)_\beta p_\gamma}{(p - q)^2 (p - k_1)^2 p^2} \quad (2.214)$$

$$= i \lim_{p \rightarrow \infty} a^\delta \int \frac{d\Omega}{(2\pi^4)} \frac{p^2 p_\delta (p - q)_\alpha (p - k_1)_\beta p_\gamma}{(p - q)^2 (p - k_1)^2 p^2} \quad (2.215)$$

$$= ia^\delta \frac{1}{8\pi^2} \frac{1}{24} (g_{\delta\alpha} g_{\beta\gamma} + g_{\delta\beta} g_{\gamma\alpha} + g_{\delta\gamma} g_{\alpha\beta}). \quad (2.216)$$

Using $g_{\alpha\beta} \gamma^\alpha \gamma^\mu \gamma^\beta = 2\gamma^\mu$ and $4i\epsilon^{\mu\nu\lambda\sigma} = \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^5$, the difference is given by

$$\Delta^{\lambda\mu\nu}(a, k_1, k_2) - \Delta^{\lambda\mu\nu}(k_1, k_2) = \frac{1}{96\pi^2} (a_\alpha \gamma^\alpha \gamma^\nu \gamma^\mu \gamma^\lambda \gamma^5 + a_\beta \gamma^\nu \gamma^\beta \gamma^\mu \gamma^\lambda \gamma^5 + a_\gamma \gamma^\nu \gamma^\mu \gamma^\gamma \gamma^\lambda \gamma^5) \quad (2.217)$$

$$= \frac{i}{8\pi^2} a_\sigma \epsilon^{\lambda\mu\nu\sigma} + \{\mu, k_1 \leftrightarrow \nu, k_2\}. \quad (2.218)$$

Since the Levi-Cevita symbol is antisymmetric in μ and ν , a_σ must also be antisymmetric in k_1 and k_2 . Thereby, it can be written as $a_\sigma = \beta(k_1 - k_2)_\sigma$. Substituting we have

$$\Delta^{\lambda\mu\nu}(a, k_1, k_2) - \Delta^{\lambda\mu\nu}(k_1, k_2) = \frac{i}{4\pi^2} \beta(k_1 - k_2)_\sigma \epsilon^{\lambda\mu\nu\sigma}. \quad (2.219)$$

From equation (2.208) we know that

$$k_{1\mu} \Delta^{\lambda\mu\nu}(k_1, k_2) = \frac{i}{8\pi^2} \epsilon^{\tau\nu\sigma\lambda} k_{1\tau} k_{2\sigma}. \quad (2.220)$$

This means that in order for (2.219) to vanish, we need $\beta = -\frac{1}{2}$.

It can now be seen whether the axial vector current is also constant using this condition. We have similarly, interchanging λ and μ and remembering that only one vertex contributes to this term,

$$q_\lambda \Delta^{\lambda\mu\nu}(a, k_1, k_2) = q_\lambda \Delta^{\lambda\mu\nu}(k_1, k_2) + \frac{i}{4\pi^2} \epsilon^{\mu\nu\tau\sigma} k_{1\tau} k_{2\sigma}. \quad (2.221)$$

Since we have from previously,

$$q_\lambda \Delta^{\lambda\mu\nu}(k_1, k_2) = \frac{i}{4\pi^2} \epsilon^{\mu\nu\tau\sigma} k_{1\tau} k_{2\sigma}, \quad (2.222)$$

we get

$$q_\lambda \Delta^{\lambda\mu\nu}(a, k_1, k_2) = \frac{i}{2\pi^2} \epsilon^{\mu\nu\tau\sigma} k_{1\tau} k_{2\sigma}, \quad (2.223)$$

which is not conserved. So far, this does not seem to be a problem. However, when the chiral symmetries are gauged, the non-vanishing derivative of the axial vector current can no longer be ignored, as argued previously.

When coupled to a gauge field, the derivative in the Dirac equation changes to a covariant derivative. In the case of non-abelian gauge theory, a left-handed Weyl spinor in a representation R of a gauge group has a chiral gauge current,

$$j^{a\mu} \equiv \bar{\Psi} T_R^a \gamma^\mu \frac{1}{2} (1 - \gamma^5) \Psi. \quad (2.224)$$

This leads to a covariant derivative of this axial current, which according to Noether should be zero in order to keep gauge invariance. However, this is not the case and the anomalies created are illustrated in the diagrams of figure 2.4, which are the same as in 2.3, with external lines representing nonabelian gauge bosons. The incoming gauge bosons are each associated with a gauge symmetry. In the case of adding the vector fields to the theory, we need to

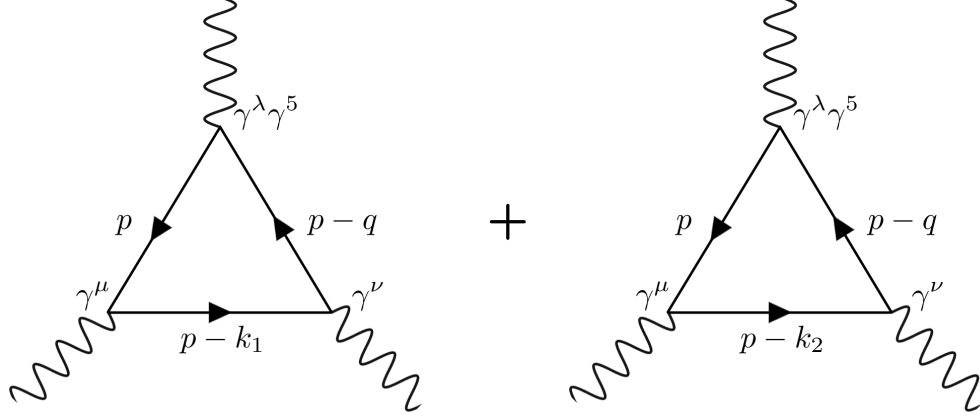


Figure 2.4: 3-vertex diagrams contributing to the gauge anomalies [31]

attach $\frac{1}{2}\text{Tr}(T_R^a T_R^b T_R^c)$ to one of the diagrams and $\frac{1}{2}\text{Tr}(T_R^a T_R^c T_R^b)$ to the other. The anomaly term becomes

$$\frac{i}{2\pi^2} \epsilon^{\mu\nu\tau\sigma} k_{1\tau} k_{2\sigma} \mathcal{A}^{abc}, \quad (2.225)$$

where $\mathcal{A}^{abc} = \frac{1}{2}\text{Tr}(\{T_R^a, T_R^b\}T_R^c)$ is the anomalous term that must cancel in order to maintain gauge invariance.

This is the fundamental consistency condition for chiral gauge theories. The three generators in $\mathcal{A}$ can be different or the same. The combinations of generators from the gauge groups in the Standard Model turn out to cancel perfectly.

For example, in $\text{SU}(2)$, we have that $T^a = \frac{\sigma^a}{2}$, so that $\{\sigma^a, \sigma^b\} = 2\delta^{ab}$. Since $\text{Tr}(\sigma^c) = 0$, $\mathcal{A}^{abc} = 0$. Combining one $\text{U}(1)$ boson with two $\text{SU}(2)$ bosons leads to the anomalous term,

$$\frac{1}{4}\text{Tr}[Q\{\sigma^b, \sigma^c\}] = \frac{1}{2}\text{Tr}[Q]\delta^{bc}. \quad (2.226)$$

The anomaly is thus the sum over the fermion electric charges. This means that there need to be equal numbers of quarks and lepton, since

$$\text{Tr}[Q] = 3\left(\frac{2}{3} - \frac{1}{3}\right) + (0 - 1) = 0. \quad (2.227)$$

The Standard Model is anomaly free. It is possible to introduce chiral fermions, as long as they are gauge invariant, as well as global symmetries. This will not result in anomalies. The right-handed neutrinos used to give masses to the neutrinos and the global discrete groups that constrain the interactions of the fields will therefore not involve any complications in this area.

Neutrino physics

3.1 Physics beyond the Standard Model

It is now widely accepted that neutrinos oscillate and therefore must be massive. What has not yet been understood is why neutrinos have such small masses compared to charged leptons and quarks. As a result, it has been suggested that neutrinos acquire masses in a different way than their cousins. The challenge then becomes to explain the mechanisms that can generate these small masses.

Neutrino masses can be understood in several ways. One way is to introduce right-handed neutrinos creating Dirac masses from Yukawa type coupling with the Higgs doublet thus extending the theory in a renormalizable way. Since these are gauge singlets, they must necessarily be anomaly free and can be added without breaking gauge invariance. However, including right-handed neutrinos alone will not explain the lightness of the neutrinos.

Another way is to generate neutrino masses through a non-renormalizable operator. It is interesting to note that all renormalizable interactions that conserve $SU(2) \times U(1)$ gauge symmetries also conserve baryon and lepton number. Introducing the non-renormalizable dimension 5 operator, however, breaks lepton number conservation, since it does not distinguish between neutrinos and antineutrinos. This should be possible, since no fundamental principle so far requires the conservation of lepton number. Baryon number is assumed to be violated, so it is even argued that there must exist some mechanism, whereby lepton number is also violated.

Neutrinos are in this case assumed to be Majorana. This is only possible for neutral particles, as postulated by Majorana, where the right-handed state is identified with the antiparticle of the left-handed state,

$$\nu_R \rightarrow (\nu_L)^c = C\bar{\nu}_L^T = C\gamma_0\nu_L^*. \quad (3.1)$$

The lepton number violation through non-renormalizable physics is assumed at low energy. This means that these violations can be described by an effective field theory. Beyond the Standard Model, the dimension 5 operator is the lowest dimension operator which can give masses to the neutrinos. The renormalizable theory of the Standard Model has been favored to get physics at low energy, not affected by physics at a much higher energy scale [34]. To

describe neutrino mixing, the opposite seems to be needed. At higher dimensions, more factors of the cut-off scale are introduced in the denominator, leading to naturally small masses.

A third option, is to have both right-handed neutrinos and lepton number violation. This happens in the see-saw mechanism, where the low masses arise from a relation between the Majorana and Dirac masses [11]. We will come back to this in chapter 5.

In the Majorana case, the mass terms for the leptons, with l being the charged leptons, are given by

$$\mathcal{L}_{mass} = -\bar{l}_{L\alpha}(M_l)_{\alpha\beta}l_{R\beta} - \frac{1}{2}\nu_{L\alpha}^T(M_\nu)_{\alpha\beta}C\nu_{L\beta} + h.c., \quad (3.2)$$

where α, β denote the flavour indices e, μ, τ .¹ The Fermi-Dirac statistics of the neutrino requires

$$\nu_{L\alpha}^T(M_\nu)_{\alpha\beta}C\nu_{L\beta} = -\nu_{L\alpha}^T(M_\nu)_{\alpha\beta}^TC^T\nu_{L\beta} = \nu_{L\alpha}^T(M_\nu)_{\alpha\beta}^TC\nu_{L\beta}, \quad (3.3)$$

since

$$C^T = -C. \quad (3.4)$$

This means that M_ν must be symmetric. The masses are then obtained by finding the eigenvalues of M_l and M_ν through diagonalization:

$$U_L^\dagger M_l U_R \equiv D_l = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \quad (3.5)$$

$$U_\nu^T M_\nu U_\nu \equiv D_\nu = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \quad (3.6)$$

If the neutrino mass matrix is real, the diagonalizing matrix must be orthogonal and CP conserved.²

The left handed charged lepton mass basis is given by

$$l_{L\alpha} \equiv (U_L)_{\alpha i} l_{Li}, \quad (3.7)$$

and the neutrino mass basis by

$$\nu_{L\alpha} \equiv (U_\nu)_{\alpha i} \nu_{Li}, \quad (3.8)$$

with $i = 1, 2, 3$ as in (1.16). As opposed to the Standard Model, the neutrinos and the charged leptons in the SU(2) doublet no longer transform in the same way. A lepton mixing matrix has come into the picture:

¹We have here used the two component spinor notation, so that each lepton and scalar carries the Lorentz spinor indices $a = 1, 2$. For example, $\psi\psi = \psi^a\psi_a = \epsilon_{ab}\psi^a\psi^b$ [35].

²The neutrinos, being CP eigenstates, $(CP)\nu_{La}(CP)^{-1} = \nu_{Ra}^c$, can have CP eigenvalues $\eta_n = \pm 1$ with opposite signs. This is corrected for by redefining the Majorana fields $\chi_n = \Sigma_a(U_{na}\nu_{La} + \eta_n U_{na}\nu_{Ra}^c)$, where U_{na} is the orthogonal matrix. Therefore, $UM_\nu U^T = \eta_n m_n$, and the masses m_n are positive [36]. However, when interactions are added the phase factors become physical, as in neutrinoless double beta decay.

$$\begin{pmatrix} \nu_{L\alpha} \\ l_{L\alpha} \end{pmatrix} = \begin{pmatrix} U_\nu \nu_{Li} \\ U_L l_{Li} \end{pmatrix} = U_L \begin{pmatrix} U_L^{-1} U_\nu \nu_{Li} \\ l_{Li} \end{pmatrix}, \quad (3.9)$$

from which the lepton mixing matrix can be found to be $U = U_L^\dagger U_\nu$. It mixes the leptons in the charged current interaction with the W boson:

$$\mathcal{L}_{cc} = -\frac{g}{\sqrt{2}} \bar{l}_L \gamma^\mu U \nu_L W_\mu^\dagger. \quad (3.10)$$

Neutrino oscillations open up for new possibilities in the form of new interactions at tree level through the dimension 5 operator.

3.2 Symmetries of the neutrino mixing matrix

A matrix can be found that fits data, is orthogonal and can be used for further model building.

U_{e3} , being very small, can be set to zero, so that $c_{13} \simeq 1$ and $U_{e2} \simeq \sin \theta_{12} \simeq 1/\sqrt{3}$. Atmospheric neutrino data have found that $U_{\mu 3} \simeq 1/\sqrt{2}$, so the mixing matrix can be found starting with three known values [38]:

$$V = \begin{pmatrix} X & \frac{1}{\sqrt{3}} & 0 \\ X & X & \frac{1}{\sqrt{2}} \\ X & X & X \end{pmatrix}. \quad (3.11)$$

The values of the rest of the components, based on the columns being orthogonal and orthonormal vectors are then found. The result is the following:

$$U_{TB} = \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad (3.12)$$

which reveals an interesting rotational symmetry:

$$\begin{aligned} \nu_1 &= -\frac{2}{\sqrt{6}}\nu_e + \frac{1}{\sqrt{6}}(\nu_\mu + \nu_\tau) = -\frac{2}{\sqrt{6}}\nu_e + \frac{1}{\sqrt{3}}\nu_x, \\ \nu_2 &= \frac{1}{\sqrt{3}}(\nu_e + \nu_\mu + \nu_\tau) = \frac{1}{\sqrt{3}}\nu_e + \sqrt{\frac{2}{3}}\nu_x, \\ \nu_3 &= \frac{1}{\sqrt{2}}(\nu_\mu - \nu_\tau) = \nu_y, \end{aligned} \quad (3.15)$$

as the basis $\{\nu_1, \nu_2\}$ is rotated from $\{\nu_e, \nu_x\}$ through $\arcsin \sqrt{\frac{1}{3}} \simeq 35^\circ$ [39]. The third neutrino is given by an even mixture of the two flavor neutrinos ν_μ and ν_τ . The second neutrino can be seen to mix equally in all three flavor neutrinos. These properties has given this mixing matrix the name "tribimaximal".

The symmetry of this matrix can also be seen when factorizing it so that $U_{TB} = U_{23}U_{12}$ [39] where

$$U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad (3.16)$$

and

$$U_{12} = \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.17)$$

That U_{TB} can be factorized like this is reminiscent of the Cabibbo-Kobayashi-Maskawa matrix for quarks.

When working in the weak basis, where the charged leptons are diagonalized, the neutrino mass matrix is in the tribimaximal case given by

$$M_\nu = \begin{pmatrix} \alpha & \beta & \beta \\ \beta & \gamma & \delta \\ \beta & \delta & \gamma \end{pmatrix},$$

where $\alpha = \frac{2}{3}m_1 + \frac{1}{3}m_2$, $\beta = \frac{1}{3}(m_2 - m_1)$, $\gamma = \frac{1}{6}m_1 + \frac{1}{3}m_2 + \frac{1}{2}m_3$ and $\delta = \alpha + \beta - \gamma$ is required. It is therefore common to look for a mass matrix of this form to agree with the tribimaximal mixing matrix and therefore experiments. In this thesis, the tribimaximal mixing will be used as a guideline carrying out the model building.

Speculations about whether the tribimaximal mixing matrix is exact are widespread. There are several motivations to question how precise this mixing matrix is. One of these considers the fact that $U_{e3} = 0$ leads to CP invariance. CP violation in the lepton sector is otherwise a very supported idea. It has contributed to explain why the universe is dominated by matter. New results from the T2K experiments have shown evidence that the U_{e3} value might indeed not be zero. At 90 % C.L., the T2K has defined a region for this element:

$$\text{Normal hierarchy: } 5^\circ < \theta_{13} < 16^\circ, \quad \text{Inverted hierarchy: } 5.8^\circ < \theta_{13} < 17.8^\circ \quad (3.19)$$

$$\text{Normal hierarchy: } 0.03 < \sin^2 \theta_{13} < 0.28, \quad \text{Inverted hierarchy: } 0.04 < \sin^2 \theta_{13} < 0.34 \quad (3.20)$$

with the best fit value at $\theta_{13} = 9.7^\circ(11^\circ)$ for normal (inverted) hierarchy. A nonzero θ_{13} leads to narrower allowed regions for global solar and atmospheric data. In figure 3.1 can be seen the allowed regions for Δm_{12}^2 as a function of $\tan \theta_{12}$ from low energy and high energy solar neutrino data for different values of θ_{13} . The allowed regions clearly decrease, as this value deviates from zero.

Furthermore, the best fit values from solar data and KamLAND analysis agree better with a nonzero θ_{13} as can be seen in figure 3.2.

This requires looking beyond tribimaximal mixing and ways to perturb or deviate from it. A way to do this will be shown in section 6.2, where the Frobenius group is applied with the Zee model to neutrino mixing.

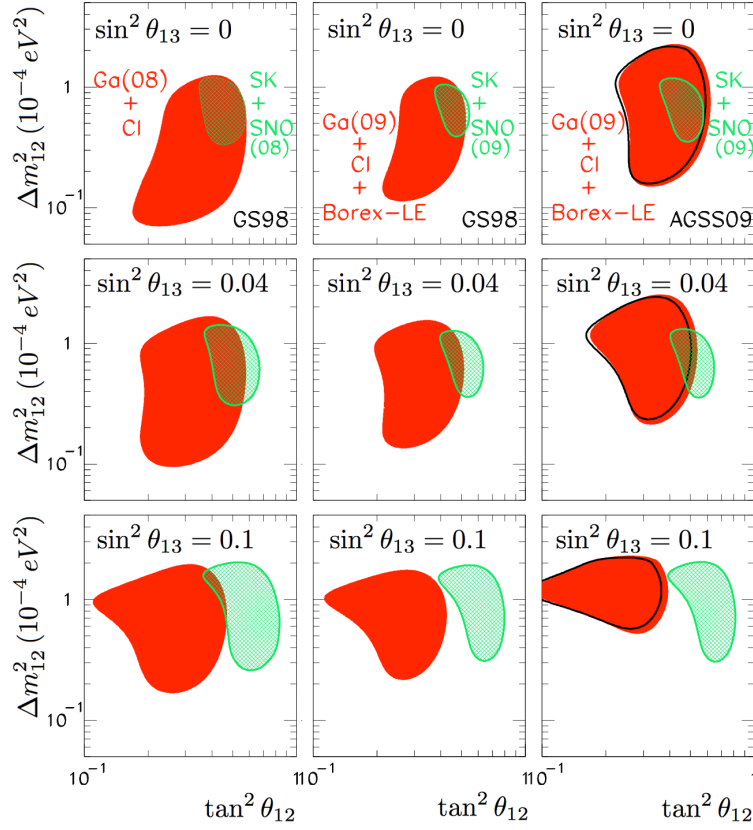


Figure 3.1: Allowed regions from solar neutrino data for different values of θ_{13} . Left column includes data before latest results from SAGE, Borexino and SNO. These are included in the central and right [40].

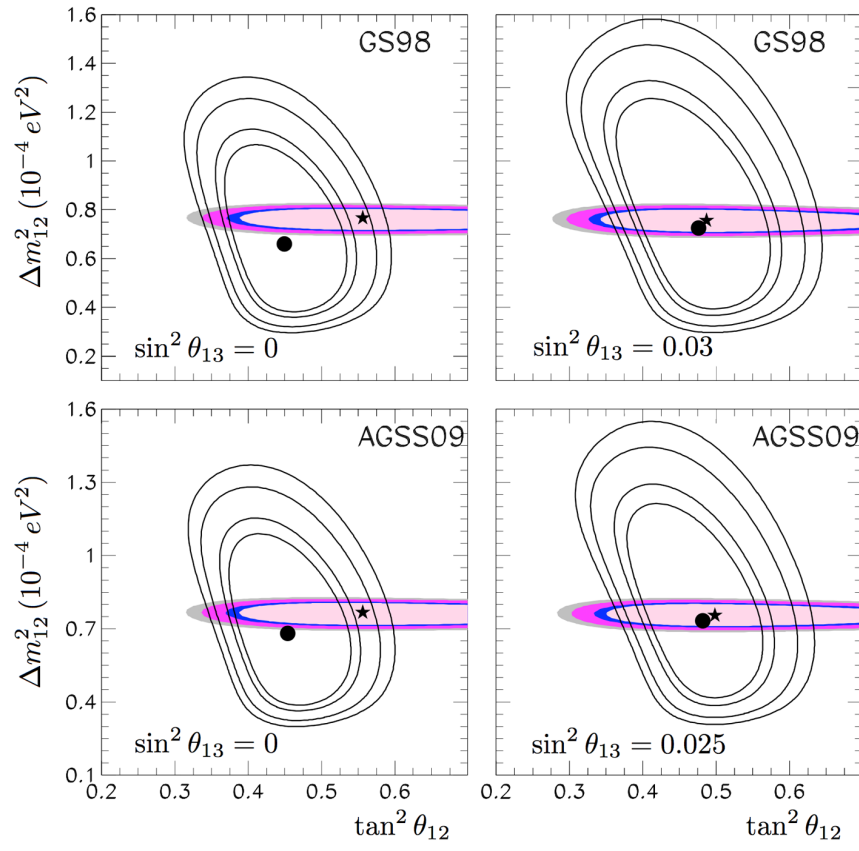


Figure 3.2: The allowed regions at 90 %, 95 %, 99 % and 99.73 % from KamLAND for two values of θ_{13} and two solar models [40].

Group theory and neutrino mixing

To find the mass and mixing matrices of neutrinos, group theory has been used as a common approach. The existence of a family symmetry was proposed long ago [41]. Under a family symmetry group F , the 3 quark and lepton generations transform into each other. A finite group is needed to avoid the appearance of Goldstone particles. The fields are placed in particular irreducible representations, whereafter the mass and mixing matrices for the charged leptons and neutrinos are found. The Higgs bosons, where there might be more than one, break the symmetry spontaneously. To match the triplication of chiral families in nature, it seems appropriate to choose groups with 2 or 3 dimensional irreducible representations. These are only found amongst subgroups of $SU(2)$, $SU(3)$ or $SO(3)$.

When a family symmetry group is added to the gauge groups of the Standard Model, this symmetry is assumed to be realized at a very high scale, Λ , the so-called cut-off scale. The symmetry breaking then occurs at a much lower scale than this cut-off and an effective field theory can therefore be assumed. In the following, a minimalist framework, assuming only $SU(2) \times U(1)$ as in [42] will be applied to carry out model building.

The tetrahedral group has been popular in the literature as an appropriate symmetry to describe neutrino mixing [11, 42–46]. So far, this group seems to present the best fit to describe neutrino oscillations. The group T_{13} has not been addressed much, but we will show here, that it produces very interesting results. The results obtained in [47] will be the main focus of this thesis. The group will then be applied with other models to carry out the search for neutrino masses.

The charged lepton masses are generated by the dimension 4 operator,

$$\mathcal{O}_4 = \bar{l}_R \phi \psi, \quad (4.1)$$

with ϕ the Higgs doublet, ψ the left-handed lepton doublet and l_R the right-handed lepton. In the following, the neutrino masses will be assumed to be generated by the mentioned dimension 5 operator, whose source to begin with, will be left unknown [48],

$$\mathcal{O}_5 = (\xi_1 \tau_2 \psi) C (\xi_2 \tau_2 \psi). \quad (4.2)$$

The Higgs doublets ξ_1 and ξ_2 may or may not be identical and there can be as many as needed to carry out the model building [49]. The τ_2 matrix is of course, up to a factor of i , the invariant antisymmetric tensor ϵ_{ab} in $SU(2)$.

4.1 The Frobenius group $T_{13} = Z_{13} \rtimes Z_3$

4.1.1 The family group

The Frobenius group T_{13} is a subgroup of $SU(3)$ generated by two elements a and b , such that $a^{13} = I$, $b^3 = I$ and $ba = a^3b$. All elements g of T_{13} can be written as $g = a^m b^n$ with $0 \leq m \leq 12$ and $0 \leq n \leq 2$. Thus, the group has $13 \cdot 3 = 39$ elements and 7 conjugacy classes given by

$$\begin{aligned}
 C_1 &: \{e\}, \\
 C_{13}^{(1)} &: \{b, ba, ba^2, \dots, ba^{11}, ba^{12}\}, \\
 C_{13}^{(2)} &: \{b^2, b^2a, b^2a^2, \dots, b^2a^{11}, b^2a^{12}\}, \\
 C_{3_1} &: \{a, a^3, a^9\}, \\
 C_{\bar{3}_1} &: \{a^4, a^{10}, a^{12}\}, \\
 C_{3_2} &: \{a^2, a^5, a^6\}, \\
 C_{\bar{3}_2} &: \{a^7, a^8, a^{11}\}.
 \end{aligned} \tag{4.9}$$

From $\sum_i d_i^2 = 39$, where d_i are the dimensions of the 7 irreducible representations and from $\sum_i m_i = 7$, where m_i is the number of IR with dimension d_i , we see that the group consists of three 1 dimensional irreducible representations $\mathbf{1}$, $\mathbf{1}'$ and $\bar{\mathbf{1}}'$ and four 3 dimensional irreducible representations, $\mathbf{3}_1$, $\bar{\mathbf{3}}_1$, $\mathbf{3}_2$ and $\bar{\mathbf{3}}_2$, since

$$m_1 + 4m_2 + 9m_3 + \dots = 39, \tag{4.10}$$

$$m_1 + m_2 + m_3 + \dots = 7, \tag{4.11}$$

$$\Rightarrow (m_1, m_2, m_3) = (3, 0, 4). \tag{4.12}$$

Explicitly, for $\mathbf{1}'$ and $\bar{\mathbf{1}}'$, b is represented respectively by ω and ω^2 and a by 1, where $\omega = e^{\frac{2\pi i}{3}}$ is the cube root of the identity.

Another way of defining the group $T_{13} = Z_{13} \rtimes Z_3$ is to demand that for any element b of Z_3 on any element a of Z_{13} [50], we have

$$bab^{-1} = a^r, \tag{4.13}$$

Letting the element b of Z_3 act on the element a of Z_{13} three times, we find that

$$b^3ab^{-3} = a^{r^3} = a \Rightarrow \tag{4.14}$$

$$r^3 = 1 + 13n \Rightarrow r = 3, \quad (4.15)$$

so that

$$bab^{-1} = a^3, \quad (4.16)$$

($r \neq 1$ is indeed what the symbol $\rtimes$ means). The Frobenius group T_{13} therefore has the presentation

$$\langle a, b \mid a^{13} = b^3 = I, bab^{-1} = a^3 \rangle. \quad (4.17)$$

Also

$$bab^{-1} = a^3 \Rightarrow a = b^{-1}a^3b \Rightarrow$$

$$a^9 = a^8b^{-1}ab = b^{-1}a^{27}b = b^{-1}ab, \quad (4.19)$$

$$ab^2 = b^3ab^2 = b^3ab^{-1} = b^2bab^{-1} = b^2a^3. \quad (4.20)$$

The representation of the Z_3 generator in 3 dimensions is given by

$$b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (4.21)$$

We can construct the representations of the Z_{13} generators by inducing them from the 1 dimensional representations of the subgroups Z_{13} and Z_3 [50]. In 1 dimension, the generator of Z_{13} has the representation $a = \rho^k = e^{2\pi ik/13}$ where $\rho^{13} = 1$, which acts on the 1 dimensional Hilbert space spanned by the complex numbers z . So, for every element of Z_{13} , these numbers transform as

$$a : z \rightarrow az = \rho^k z. \quad (4.22)$$

We can then define a set of elements (g, z) with $g \in T_{13}$ and $z \in$ Hilbert space of Z_{13} . The coset is made from the elements of T_{13} , which are not in Z_{13} , namely the elements of Z_3 , $(e, z), (b, z), (b^2, z)$.

Letting the generators of T_{13} act on this coset, the 3 dimensional representations of these generators can be found. Using the relations in (4.19) and (4.20), the representations of a follow from

$$a(e, z) = \rho^k(e, z), \quad (4.23)$$

$$a(b, z) = (ab, z) = (ba^9, z) = \rho^{9k}(b, z), \quad (4.24)$$

$$a(b^2, z) = (ab^2, z) = (b^2a^3, z) = \rho^{3k}b^2, z, \quad (4.25)$$

giving the 3 dimensional representations

$$a = \begin{pmatrix} \rho^k & 0 & 0 \\ 0 & \rho^{3k} & 0 \\ 0 & 0 & \rho^{9k} \end{pmatrix}. \quad (4.26)$$

The group T_{13} has two such generators with $k = 1, 2$ corresponding to $\mathbf{3}_1$ and $\mathbf{3}_2$,

$$a_1 = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho^3 & 0 \\ 0 & 0 & \rho^9 \end{pmatrix}, \quad (4.27)$$

and

$$a_2 = \begin{pmatrix} \rho^2 & 0 & 0 \\ 0 & \rho^6 & 0 \\ 0 & 0 & \rho^5 \end{pmatrix}, \quad (4.28)$$

with corresponding complex conjugates,

$$a_1^* = \begin{pmatrix} \rho^{12} & 0 & 0 \\ 0 & \rho^{10} & 0 \\ 0 & 0 & \rho^4 \end{pmatrix}, \quad (4.29)$$

and

$$a_2^* = \begin{pmatrix} \rho^{11} & 0 & 0 \\ 0 & \rho^7 & 0 \\ 0 & 0 & \rho^8 \end{pmatrix}. \quad (4.30)$$

To find the Kronecker products and the Clebsch-Gordan decompositions of the irreducible representations, it is convenient to use the fact, that the Frobenius group is a subgroup of $SU(3)$, and exploit the well-known properties of $SU(3)$. Denote the 3-component vector of the defining representation $\mathbf{3}$ by ψ^μ , with $\mu = 1, 2, 3$, and the vector of the conjugate representation $\bar{\mathbf{3}}$ by $\psi_\mu = (\psi^\mu)^*$. In $SU(3)$, we have

$$\mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6}, \quad (4.31)$$

as indicated by

$$\psi^\mu \psi^\nu \sim \psi^{\mu\nu} \sim \psi^{\mu\nu} \epsilon_{\mu\nu\lambda} + \psi^{\{\mu\nu\}} \equiv \psi^{[\mu\nu]} + \psi^{\{\mu\nu\}}. \quad (4.32)$$

Under the action of a generator c , $\psi^{\mu\nu}$ transforms as follows:

$$\psi'^{\mu\nu} = c^\mu_\sigma c^\nu_\rho \psi^{\sigma\rho}. \quad (4.33)$$

To find the decomposition of $\mathbf{3}_1 \otimes \mathbf{3}_1$, we use the generators a_1 and b to determine how $\psi^{\{\mu\nu\}}$ decomposes.

Consider the diagonal elements $\psi^{\{11\}}, \psi^{\{22\}}, \psi^{\{33\}}$. Under b , they transform as

$$\psi^{\{11\}} \rightarrow \psi^{\{22\}} \rightarrow \psi^{\{33\}} \rightarrow \psi^{\{11\}}, \quad (4.34)$$

while under a_1 , they transform as

$$\psi^{\{11\}} \rightarrow \rho^2 \psi^{\{11\}}, \quad \psi^{\{22\}} \rightarrow \rho^6 \psi^{\{22\}}, \quad \psi^{\{33\}} \rightarrow \rho^5 \psi^{\{33\}}. \quad (4.35)$$

Thus they form a $\mathbf{3}_2$ representation according to (4.28):

$$\begin{pmatrix} \psi^{\{11\}} \\ \psi^{\{22\}} \\ \psi^{\{33\}} \end{pmatrix}_{\mathbf{3}_2}. \quad (4.36)$$

Next, consider the off-diagonal elements of the symmetric representation $\psi^{\{\mu\nu\}}$. Under b they transform as

$$\psi^{\{12\}} \rightarrow \psi^{\{23\}} \rightarrow \psi^{\{31\}} \rightarrow \psi^{\{12\}}. \quad (4.37)$$

Under a_1 , they transform as

$$\psi^{\{12\}} \rightarrow \rho^4 \psi^{\{12\}}, \quad \psi^{\{23\}} \rightarrow \rho^{12} \psi^{\{23\}}, \quad \psi^{\{31\}} \rightarrow \rho^{10} \psi^{\{31\}}, \quad (4.38)$$

thus forming the $\bar{\mathbf{3}}_1$ representation according to (4.29):

$$\begin{pmatrix} \psi^{\{23\}} \\ \psi^{\{31\}} \\ \psi^{\{12\}} \end{pmatrix}_{\bar{\mathbf{3}}_1}. \quad (4.39)$$

Under a_1 and b , the off-diagonal elements of $\psi^{[\mu\nu]}$ and $\psi^{\{\mu\nu\}}$ transform the same way. The Kronecker product becomes

$$\mathbf{3}_1 \otimes \mathbf{3}_1 = \bar{\mathbf{3}}_1 \oplus \bar{\mathbf{3}}_1 \oplus \mathbf{3}_2. \quad (4.40)$$

The decompositions of $\mathbf{3}_2 \otimes \mathbf{3}_2$ and $\mathbf{3}_1 \otimes \mathbf{3}_2$ are found in the same way, using a_1 and a_2 appropriately. The result is displayed in Table 4.1.

In $SU(3)$ we have the product between a $\mathbf{3}$ and a $\bar{\mathbf{3}}$ representation:

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}, \quad (4.41)$$

where $\psi^\mu \psi_\nu \sim \psi_\nu^\mu$ is decomposed into a trace ψ_μ^μ that forms the singlet and a traceless ψ_ν^μ that forms the adjoint 8 representation.

For $\mathbf{3}_1 \otimes \bar{\mathbf{3}}_1$, the diagonal elements of ψ_ν^μ transform under b as $\psi_1^1 \rightarrow \psi_2^2 \rightarrow \psi_3^3 \rightarrow \psi_1^1$. We have the three possibilities

$$\mathbf{1} = \psi_1^1 + \psi_2^2 + \psi_3^3, \quad (4.42)$$

$$\bar{\phi} = \psi_1^1 + \omega \psi_2^2 + \omega^2 \psi_3^3, \quad (4.43)$$

$$\phi = \psi_1^1 + \omega^2 \psi_2^2 + \omega \psi_3^3. \quad (4.44)$$

Under b , these transform as

$$\bar{\phi} \rightarrow \psi_2^2 + \omega \psi_3^3 + \omega^2 \psi_1^1 = \omega^2 \bar{\phi}, \quad (4.45)$$

$$\phi \rightarrow \psi_2^2 + \omega^2 \psi_3^3 + \omega \psi_1^1 = \omega \phi. \quad (4.46)$$

Thus, we choose $\bar{\phi}$ to transform as a $\bar{\mathbf{1}}'$ and ϕ to transform as a $\mathbf{1}'$.

The off-diagonal terms split into two sets. They transform under b in the same obvious way as described in (4.37). Under a_1 and a_1^* they transform as

$$\psi_2^1 \rightarrow \rho^{11} \psi_2^1, \quad \psi_3^2 \rightarrow \rho^7 \psi_3^2, \quad \psi_1^3 \rightarrow \rho^8 \psi_1^3, \quad (4.47)$$

and

$$\psi_1^2 \rightarrow \rho^2 \psi_1^2, \quad \psi_2^3 \rightarrow \rho^6 \psi_2^3, \quad \psi_3^1 \rightarrow \rho^5 \psi_3^1, \quad (4.48)$$

giving a $\bar{\mathbf{3}}_2$ and a $\mathbf{3}_2$ representation:

$$\begin{pmatrix} \psi_2^1 \\ \psi_3^2 \\ \psi_1^3 \end{pmatrix}_{\bar{\mathbf{3}}_2}, \quad (4.49)$$

$$\begin{pmatrix} \psi_1^2 \\ \psi_2^3 \\ \psi_3^1 \end{pmatrix}_{\mathbf{3}_2}. \quad (4.50)$$

The rest of the decompositions can be found in the same way using the appropriate representations of the generators. The products between the 1 dimensional irreducible representations are found from seeing that under b , $\phi \rightarrow \omega\phi$ and $\bar{\phi} \rightarrow \omega^2\bar{\phi}$, so that for example $\phi\bar{\phi} \rightarrow \omega\phi\omega^2\bar{\phi} = \phi\bar{\phi}$ transforms as a $\mathbf{1}$. The Kronecker products are listed in table 4.1 and the Clebsch-Gordan decompositions in table 4.2, where we have dropped the $\{..\}$ on ψ .

$\mathbf{1}' \otimes \mathbf{1}' = \bar{\mathbf{1}}'$
$\bar{\mathbf{1}}' \otimes \bar{\mathbf{1}}' = \mathbf{1}'$
$\mathbf{1}' \otimes \bar{\mathbf{1}}' = \mathbf{1}$
$\mathbf{3}_1 \otimes \mathbf{3}_1 = \bar{\mathbf{3}}_1 \oplus \bar{\mathbf{3}}_1 \oplus \mathbf{3}_2$
$\mathbf{3}_2 \otimes \mathbf{3}_2 = \bar{\mathbf{3}}_2 \oplus \bar{\mathbf{3}}_1 \oplus \bar{\mathbf{3}}_2$
$\mathbf{3}_1 \otimes \bar{\mathbf{3}}_1 = \mathbf{1} \oplus \mathbf{1}' \oplus \bar{\mathbf{1}}' \oplus \mathbf{3}_2 \oplus \bar{\mathbf{3}}_2$
$\mathbf{3}_2 \otimes \bar{\mathbf{3}}_2 = \mathbf{1} \oplus \mathbf{1}' \oplus \bar{\mathbf{1}}' \oplus \mathbf{3}_1 \oplus \bar{\mathbf{3}}_1$
$\mathbf{3}_1 \otimes \mathbf{3}_2 = \bar{\mathbf{3}}_2 \oplus \mathbf{3}_1 \oplus \mathbf{3}_2$
$\mathbf{3}_1 \otimes \bar{\mathbf{3}}_2 = \bar{\mathbf{3}}_2 \oplus \mathbf{3}_1 \oplus \bar{\mathbf{3}}_1$
$\mathbf{3}_2 \otimes \bar{\mathbf{3}}_1 = \mathbf{3}_2 \oplus \mathbf{3}_1 \oplus \bar{\mathbf{3}}_1$

Table 4.1: $Z_{13} \rtimes Z_3$ Kronecker products

4.1.2 Model building

We are now ready to build a model for neutrino mixing based on the family symmetry group $F = Z_{13} \rtimes Z_3$. As in the standard procedure, we assign the lepton doublet fields ψ_a ($a = 1, 2, 3$), the right handed lepton singlet fields l_{Ra} , and the Higgs fields to various representations of T_{13} , and write down the various allowed forms of the operators $\mathcal{O}_4$ and $\mathcal{O}_5$. After the Higgs fields are allowed to acquire their vacuum expectation value, the resulting charged lepton mass matrix M_l and neutrino mass matrix M_ν are diagonalized, whereby the mixing matrix is obtained as outlined earlier.

In the charged lepton sector, since the fields can be placed either in one of the four 3 dimensional irreducible representations or in the three 1 dimensional irreducible representation, creating five possibilities, the operator $\mathcal{O}_4 = \phi \bar{l}_R \psi$ gives a priori $5^3 = 125$ ways of placing the three fields in different representations, but some of these are not allowed by the group theory. Also, two cases, where the representations are the conjugate of each other, will give the same result, so the number of different forms $\mathcal{O}_4$ can acquire, is down to 27, see Table 4.3. For each of these cases, we then consider the various possibilities for $\mathcal{O}_5$ assigning ξ_1 and ξ_2 to various representations of F . We then use tribimaximal mixing as a constraint when choosing the vacuum alignments of the Higgs fields. It turns out that the cases 1, 11, 22 and 25 will

$$\begin{aligned}
 \mathbf{3}_1 \otimes \mathbf{3}_1 &\rightarrow \begin{pmatrix} \psi^{11} \\ \psi^{22} \\ \psi^{33} \end{pmatrix}_{\mathbf{3}_2}, \quad \begin{pmatrix} \psi^{23} \\ \psi^{31} \\ \psi^{12} \end{pmatrix}_{\bar{\mathbf{3}}_1}, \quad \begin{pmatrix} \psi^{32} \\ \psi^{13} \\ \psi^{21} \end{pmatrix}_{\bar{\mathbf{3}}_1} \\
 \mathbf{3}_2 \otimes \mathbf{3}_2 &\rightarrow \begin{pmatrix} \psi^{22} \\ \psi^{33} \\ \psi^{11} \end{pmatrix}_{\bar{\mathbf{3}}_1}, \quad \begin{pmatrix} \psi^{23} \\ \psi^{31} \\ \psi^{12} \end{pmatrix}_{\bar{\mathbf{3}}_2}, \quad \begin{pmatrix} \psi^{32} \\ \psi^{13} \\ \psi^{21} \end{pmatrix}_{\bar{\mathbf{3}}_2} \\
 \mathbf{3}_1 \otimes \bar{\mathbf{3}}_1 &\rightarrow \begin{pmatrix} \psi_2^1 \\ \psi_3^2 \\ \psi_1^3 \end{pmatrix}_{\bar{\mathbf{3}}_2}, \quad \begin{pmatrix} \psi_1^2 \\ \psi_2^3 \\ \psi_3^1 \end{pmatrix}_{\mathbf{3}_2}, \quad (\psi_1^1 + \psi_2^2 + \psi_3^3)_{\mathbf{1}}, \\
 &\quad (\psi_1^1 + \omega\psi_2^2 + \omega^2\psi_3^3)_{\mathbf{1}'}, \quad (\psi_1^1 + \omega^2\psi_2^2 + \omega\psi_3^3)_{\bar{\mathbf{1}}'} \\
 \mathbf{3}_2 \otimes \bar{\mathbf{3}}_2 &\rightarrow \begin{pmatrix} \psi_2^3 \\ \psi_3^1 \\ \psi_1^2 \end{pmatrix}_{\bar{\mathbf{3}}_1}, \quad \begin{pmatrix} \psi_3^2 \\ \psi_1^3 \\ \psi_2^1 \end{pmatrix}_{\mathbf{3}_1}, \quad (\psi_1^1 + \psi_2^2 + \psi_3^3)_{\mathbf{1}}, \\
 &\quad (\psi_1^1 + \omega\psi_2^2 + \omega^2\psi_3^3)_{\mathbf{1}'}, \quad (\psi_1^1 + \omega^2\psi_2^2 + \omega\psi_3^3)_{\bar{\mathbf{1}}'} \\
 \mathbf{3}_1 \otimes \mathbf{3}_2 &\rightarrow \begin{pmatrix} \psi^{33} \\ \psi^{11} \\ \psi^{22} \end{pmatrix}_{\mathbf{3}_1}, \quad \begin{pmatrix} \psi^{31} \\ \psi^{12} \\ \psi^{23} \end{pmatrix}_{\bar{\mathbf{3}}_2}, \quad \begin{pmatrix} \psi^{32} \\ \psi^{13} \\ \psi^{21} \end{pmatrix}_{\mathbf{3}_2} \\
 \mathbf{3}_1 \otimes \bar{\mathbf{3}}_2 &\rightarrow \begin{pmatrix} \psi_1^1 \\ \psi_2^2 \\ \psi_3^3 \end{pmatrix}_{\bar{\mathbf{3}}_1}, \quad \begin{pmatrix} \psi_3^2 \\ \psi_1^3 \\ \psi_2^1 \end{pmatrix}_{\bar{\mathbf{3}}_2}, \quad \begin{pmatrix} \psi_1^2 \\ \psi_2^3 \\ \psi_3^1 \end{pmatrix}_{\mathbf{3}_1} \\
 \mathbf{3}_2 \otimes \bar{\mathbf{3}}_1 &\rightarrow \begin{pmatrix} \psi_1^1 \\ \psi_2^2 \\ \psi_3^3 \end{pmatrix}_{\mathbf{3}_1}, \quad \begin{pmatrix} \psi_2^1 \\ \psi_3^2 \\ \psi_1^3 \end{pmatrix}_{\bar{\mathbf{3}}_1}, \quad \begin{pmatrix} \psi_2^3 \\ \psi_3^1 \\ \psi_1^2 \end{pmatrix}_{\mathbf{3}_2}
 \end{aligned}$$

Table 4.2: $Z_{13} \rtimes Z_3$ Clebsch-Gordan decompositions.

lead to the same neutrino mass matrix. Another mass matrix can be found from the cases 23 and 24. Furthermore, the rest of the cases turn out to create degenerate charged lepton mass matrices, which are therefore not of phenomenological importance. Therefore, we can cover all possible results obtained from this group, by considering only four of the mentioned cases, where the lepton doublet is in two different representations. This leads to only two possible neutrino mass matrices. These cases will now be studied. In the following, we will not distinguish between upper and lower indices.

Case	ψ	ϕ	$\bar{l}_R$
1	$\mathbf{3}_1$	$\mathbf{3}_1$	$\mathbf{3}_1$
2	$\mathbf{3}_2$	$\bar{\mathbf{3}}_1$	$\bar{\mathbf{3}}_1$
3	$\mathbf{3}_1$	$\mathbf{3}_1$	$\bar{\mathbf{3}}_2$
4	$\mathbf{3}_1$	$\bar{\mathbf{3}}_2$	$\mathbf{3}_1$
5	$\mathbf{3}_2$	$\bar{\mathbf{3}}_1$	$\mathbf{3}_1$
6	$\mathbf{3}_1$	$\bar{\mathbf{3}}_1$	$\mathbf{3}_2$
7	$\mathbf{3}_2$	$\mathbf{3}_1$	$\bar{\mathbf{3}}_1$
8	$\mathbf{3}_1$	$\bar{\mathbf{3}}_1$	$\bar{\mathbf{3}}_2$
9	$\mathbf{3}_1$	$\mathbf{3}_2$	$\bar{\mathbf{3}}_1$
10	$\mathbf{3}_1$	$\bar{\mathbf{3}}_2$	$\bar{\mathbf{3}}_1$
11	$\mathbf{3}_2$	$\mathbf{3}_2$	$\mathbf{3}_2$
12	$\mathbf{3}_1$	$\mathbf{3}_2$	$\mathbf{3}_2$
13	$\mathbf{3}_2$	$\mathbf{3}_2$	$\mathbf{3}_1$
14	$\mathbf{3}_2$	$\mathbf{3}_1$	$\mathbf{3}_2$
15	$\mathbf{3}_1$	$\mathbf{3}_2$	$\bar{\mathbf{3}}_2$
16	$\mathbf{3}_2$	$\bar{\mathbf{3}}_2$	$\bar{\mathbf{3}}_1$
17	$\mathbf{3}_2$	$\bar{\mathbf{3}}_1$	$\bar{\mathbf{3}}_2$
18	$\mathbf{3}_2$	$\mathbf{3}_1$	$\bar{\mathbf{3}}_2$
19	$\mathbf{3}_2$	$\bar{\mathbf{3}}_2$	$\mathbf{3}_1$
20	$\mathbf{3}_1$	$\bar{\mathbf{3}}_2$	$\mathbf{3}_2$
21	$1, 1', \bar{1}'$	$\mathbf{3}_1$	$\bar{\mathbf{3}}_1$
22	$\mathbf{3}_1$	$\bar{\mathbf{3}}_1$	$1, 1', \bar{1}'$
23	$\mathbf{3}_1$	$1, 1', \bar{1}'$	$\bar{\mathbf{3}}_1$
24	$\mathbf{3}_2$	$1, 1', \bar{1}'$	$\bar{\mathbf{3}}_2$
25	$\mathbf{3}_2$	$\bar{\mathbf{3}}_2$	$1, 1', \bar{1}'$
26	$1, 1', \bar{1}'$	$\mathbf{3}_2$	$\bar{\mathbf{3}}_2$
27	$1, 1', \bar{1}'$	$1, 1', \bar{1}'$	$1, 1', \bar{1}'$

Table 4.3: Possible representations for the 3 fields.

4.1.2.1 Case 1:

The right-handed lepton, $\bar{l}_R$, and the lepton doublet, ψ , both in $\mathbf{3}_1$ can be combined in two ways,

$$\begin{aligned}
 \bar{\mathbf{3}}_1 : \quad & \{\bar{l}_{R2}\psi_3, \bar{l}_{R3}\psi_1, \bar{l}_{R1}\psi_2\}, \\
 \bar{\mathbf{3}}_1 : \quad & \{\bar{l}_{R3}\psi_2, \bar{l}_{R1}\psi_3, \bar{l}_{R2}\psi_1\}.
 \end{aligned} \tag{4.52}$$

With ϕ in $\mathbf{3}_1$, we get the invariant terms in the dimension 4 operator of the Lagrangian:

$$\begin{aligned}
 & h_1(\phi_1\bar{l}_{R2}\psi_3 + \phi_2\bar{l}_{R3}\psi_1 + \phi_3\bar{l}_{R1}\psi_2) + h_2(\phi_1\bar{l}_{R3}\psi_2 + \phi_2\bar{l}_{R1}\psi_3 + \phi_3\bar{l}_{R2}\psi_1) \\
 & = (\bar{l}_{R1}, \bar{l}_{R2}, \bar{l}_{R3}) \begin{pmatrix} 0 & h_1\phi_3 & h_2\phi_2 \\ h_2\phi_3 & 0 & h_1\phi_1 \\ h_1\phi_2 & h_2\phi_1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}.
 \end{aligned} \tag{4.53}$$

Upon spontaneous symmetry breaking, we let the Higgs field acquire a vacuum expectation value in the $(1, 1, 1)$ direction, and the mass matrix becomes

$$M_l = v \begin{pmatrix} 0 & h_1 & h_2 \\ h_2 & 0 & h_1 \\ h_1 & h_2 & 0 \end{pmatrix}. \quad (4.55)$$

To diagonalize M_l , we multiply by its hermitian conjugate,

$$M_l M_l^\dagger = M_l^\dagger M_l = \begin{pmatrix} |h_1|^2 + |h_2|^2 & h_1^* h_2 & h_1 h_2^* \\ h_1 h_2^* & |h_1|^2 + |h_2|^2 & h_1^* h_2 \\ h_1^* h_2 & h_1 h_2^* & |h_1|^2 + |h_2|^2 \end{pmatrix}. \quad (4.56)$$

This is a circulant matrix, that can be diagonalized by the following unitary matrix:

$$U_L^\dagger = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & 1 & \omega^2 \\ \omega^2 & 1 & \omega \end{pmatrix}. \quad (4.57)$$

We now look at the dimension 5 operator and ask what Higgs field vacuum expectation values will lead to tribimaximal mixing,

$$\mathcal{O}_5 = (\xi_1 \tau_2 \psi) C (\xi_2 \tau_2 \psi). \quad (4.58)$$

The Higgs fields ξ_1 and ξ_2 can each be any of the following five different Higgs in the representations:

$$\chi \sim \mathbf{1}, \quad \eta \sim \mathbf{3}_1, \quad \rho \sim \mathbf{3}_2, \quad \kappa \sim \bar{\mathbf{3}}_1, \quad \zeta \sim \bar{\mathbf{3}}_2. \quad (4.59)$$

We have a priori $5 + 4 + 3 + 2 + 1 = 15$ combinations of these in the dimension 5 operator. In Case 1, we have the ψ in the $\mathbf{3}_1$ representation. From Table 4.1, we can already exclude the combinations $(\chi\psi)^2$, $(\eta\psi)^2$, $(\chi\psi)(\kappa\psi)$ and $(\chi\psi)(\rho\psi)$. The possible invariants from combining two lepton doublets with two Higgs in the above mentioned representations and contracting the indices properly, can now be cataloged.¹

a) $(\rho\psi)^2$:

$$f_1(\rho_3 \psi_2 \rho_1 \psi_2 + \rho_1 \psi_3 \rho_2 \psi_3 + \rho_2 \psi_1 \rho_3 \psi_1). \quad (4.60)$$

b) $(\kappa\psi)^2$:

$$\begin{aligned} & f_2(\kappa_1 \psi_1 \kappa_1 \psi_1 + \kappa_2 \psi_2 \kappa_2 \psi_2 + \kappa_3 \psi_3 \kappa_3 \psi_3), \\ & + f_3(\kappa_1 \psi_1 \kappa_2 \psi_2 + \kappa_3 \psi_3 \kappa_1 \psi_1 + \kappa_2 \psi_2 \kappa_3 \psi_3), \\ & + f_4(\kappa_1 \psi_2 \kappa_2 \psi_1 + \kappa_2 \psi_3 \kappa_3 \psi_2 + \kappa_3 \psi_1 \kappa_1 \psi_3). \end{aligned} \quad (4.63)$$

c) $(\zeta\psi)^2$:

$$f_5(\zeta_1 \psi_1 \zeta_1 \psi_2 + \zeta_2 \psi_2 \zeta_2 \psi_3 + \zeta_3 \psi_3 \zeta_3 \psi_1). \quad (4.64)$$

¹In order to find the different invariant terms that enter the Lagrangian through this operator, it is necessary to keep in mind, that since the neutrinos are Majorana, we have here used the 2 component spinor notation, so that each lepton and scalar carries the Lorentz spinor indices $a = 1, 2$. For example, $\psi\psi = \psi^a \psi_a = \epsilon_{ab} \psi^a \psi^b$ [35]. We also have to take into account that the leptons are Grassman numbers, and thereby anticommute.

d) $(\chi\psi)(\eta\psi)$:

$$\begin{aligned} & f_6\chi(\psi_1\eta_2\psi_3 + \psi_2\eta_3\psi_1 + \psi_3\eta_1\psi_2), \\ & + f_7\chi(\psi_1\eta_3\psi_2 + \psi_2\eta_1\psi_3 + \psi_3\eta_2\psi_1). \end{aligned} \quad (4.66)$$

 e) $(\chi\psi)(\zeta\psi)$:

$$f_8\chi(\psi_1\zeta_1\psi_1 + \psi_2\zeta_2\psi_2 + \psi_3\zeta_3\psi_3). \quad (4.67)$$

 f) $(\eta\psi)(\rho\psi)$:

$$\begin{aligned} & f_9(\eta_2\psi_3\rho_3\psi_3 + \eta_3\psi_1\rho_1\psi_1 + \eta_1\psi_2\rho_2\psi_2), \\ & f_{10}(\eta_3\psi_2\rho_3\psi_3 + \eta_1\psi_3\rho_1\psi_1 + \eta_2\psi_1\rho_2\psi_2), \\ & + f_{11}(\eta_3\psi_3\rho_3\psi_2 + \eta_1\psi_1\rho_1\psi_3 + \eta_2\psi_2\rho_2\psi_1). \end{aligned} \quad (4.70)$$

 g) $(\eta\psi)(\kappa\psi)$:

$$f_{12}(\eta_1\psi_1\kappa_2\psi_1 + \eta_2\psi_2\kappa_3\psi_2 + \eta_3\psi_3\kappa_1\psi_3). \quad (4.71)$$

 h) $(\eta\psi)(\zeta\psi)$:

$$\begin{aligned} & f_{13}(\eta_3\psi_2\zeta_1\psi_2 + \eta_1\psi_3\zeta_2\psi_3 + \eta_2\psi_1\zeta_3\psi_1), \\ & f_{14}(\eta_2\psi_3\zeta_1\psi_2 + \eta_3\psi_1\zeta_2\psi_3 + \eta_1\psi_2\zeta_3\psi_1), \\ & + f_{15}(\eta_1\psi_1\zeta_3\psi_2 + \eta_2\psi_2\zeta_1\psi_3 + \eta_3\psi_3\zeta_2\psi_1). \end{aligned} \quad (4.74)$$

 i) $(\rho\psi)(\kappa\psi)$:

$$\begin{aligned} & f_{16}(\rho_2\psi_3\kappa_2\psi_1 + \rho_3\psi_1\kappa_3\psi_2 + \rho_1\psi_2\kappa_1\psi_3), \\ & + f_{17}(\rho_1\psi_3\kappa_1\psi_2 + \rho_2\psi_1\kappa_2\psi_3 + \rho_3\psi_2\kappa_3\psi_1). \end{aligned} \quad (4.76)$$

 j) $(\rho\psi)(\zeta\psi)$:

$$\begin{aligned} & f_{18}(\rho_2\psi_3\zeta_3\psi_2 + \rho_3\psi_1\zeta_1\psi_3 + \rho_1\psi_2\zeta_2\psi_1), \\ & + f_{19}(\rho_3\psi_3\zeta_1\psi_1 + \rho_1\psi_1\zeta_2\psi_2 + \rho_2\psi_2\zeta_3\psi_3). \end{aligned} \quad (4.78)$$

 k) $(\kappa\psi)(\zeta\psi)$:

$$f_{20}(\kappa_1\psi_2\zeta_3\psi_2 + \kappa_2\psi_3\zeta_1\psi_3 + \kappa_3\psi_1\zeta_2\psi_1). \quad (4.79)$$

Of course, in building a specific model, we are free to omit one or more of these Higgs fields, narrowing down the possibilities further. There should to begin with be five different terms in b), since $\mathbf{3}_1 \otimes \bar{\mathbf{3}}_1 = \mathbf{1} \oplus \mathbf{1}' \oplus \bar{\mathbf{1}}' \oplus \mathbf{3}_2 \oplus \bar{\mathbf{3}}_2$ multiplied by itself creates invariants from $\mathbf{1} \otimes \mathbf{1}$, $\mathbf{1}' \otimes \bar{\mathbf{1}}'$, $\mathbf{3}_2 \otimes \bar{\mathbf{3}}_2$ and two charge conjugates. However, when multiplying out the terms and seeing that $\omega + \omega^2 = -1$, we get the three listed terms.

In Case 1, we first consider including only the Higgs fields η and κ , to which we assign the vacuum expectation values

i)

$$\langle \eta \rangle = v_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \langle \kappa \rangle = u_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad (4.80)$$

so that we get contributions from invariants in b) and g). We could also add χ so we have the vacuum expectation values

ii)

$$\langle \chi \rangle = x, \quad \langle \eta \rangle = v_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \langle \kappa \rangle = u_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad (4.81)$$

(with η and κ having the same vacuum expectation values as in (4.80)). This would give another contribution from d).

Upon spontaneous symmetry breaking, the neutral components of the SU(2) Higgs doublets can acquire these vacuum expectation values and so will connect with the neutrinos giving masses to these. The charge conjugation operator, together with the neutrinos being Grassman, will then allow us to combine some of the coupling constants in b), d), f), h), i) and j), contributing to the off diagonal elements of the neutrino mass matrix, thus making this symmetric as it should be. In the following equations, we will, in some cases, combine f_a and f_{a+1} into a new coupling constant, which we call f'_a . The terms contributing to the neutrino mass matrix in i) are then given by

$$f_2 u_1^2 (\nu_1 \nu_1 + \nu_3 \nu_3) + f'_3 u_1^2 \nu_1 \nu_3 + f_{12} v_1 u_1 \nu_2 \nu_2, \quad (4.82)$$

and for $\langle \chi \rangle$ getting a vacuum expectation value in ii), we also get the contribution:

$$f'_6 x v_1 \nu_1 \nu_3. \quad (4.83)$$

This leads to the possible mass matrices:

i)

$$M_\nu = \begin{pmatrix} f_2 u_1^2 & 0 & f'_3 u_1^2 \\ 0 & f_{12} v_1 u_1 & 0 \\ f'_3 u_1^2 & 0 & f_2 u_1^2 \end{pmatrix}, \quad (4.84)$$

and

ii)

$$M_\nu = \begin{pmatrix} f_2 u_1^2 & 0 & f'_3 u_1^2 + f'_6 x v_1 \\ 0 & f_{12} v_1 u_1 & 0 \\ f'_3 u_1^2 + f'_6 x v_1 & 0 & f_2 u_1^2 \end{pmatrix}, \quad (4.85)$$

with and without the Higgs χ respectively, both having the form:

$$M_\nu = \begin{pmatrix} \alpha & 0 & \beta \\ 0 & \gamma & 0 \\ \beta & 0 & \alpha \end{pmatrix}, \quad (4.86)$$

which is diagonalized by the unitary matrix:

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}. \quad (4.87)$$

We can now find the lepton mixing matrix:

$$\begin{aligned}
 U = U_L^\dagger U_\nu &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & 1 & \omega^2 \\ \omega^2 & 1 & \omega \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \Phi = U_{TB} \Phi,
 \end{aligned} \tag{4.89}$$

where

$$\Phi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}, \tag{4.90}$$

is a phase factor that does not affect neutrino mixing. This is an interesting result. The neutrino mass matrix in (6.66), which leads to tribimaximal mixing in case 1, is exactly achieved, by choosing the right vacuum expectation values in the neutrino sector. This was not possible in [42] for the tetrahedral group. In this paper, when choosing vacuum expectation values for the Higgs fields in the neutrino sector, the exact form of the mass matrix in (6.66) could not be achieved. The mass matrix had to be perturbed in the following way:

$$M_\nu = \begin{pmatrix} \alpha + \epsilon & 0 & \beta \\ 0 & \gamma & 0 \\ \beta & 0 & \alpha - \epsilon \end{pmatrix}, \tag{4.91}$$

which did not allow for exact tribimaximal mixing. It was instead assumed, that ϵ being small, could be seen as a perturbation, and therefore only lead to a small deviation from tribimaximal. If tribimaximal mixing is the best fit for lepton mixing, we already here have evidence, that the Frobenius group can be used to describe neutrino mixing. In order to determine which of the two groups, the tetrahedral or the Frobenius, is the best candidate to describe neutrino mixing, it will be necessary to test the Frobenius group further, using more constraints.

4.1.2.2 Case 23:

The lepton doublets are in the $\mathbf{3}_1$ representation, the charged lepton singlets in the $\bar{\mathbf{3}}_1$ representation, and the Higgs bosons ϕ in the $\mathbf{1}$, $\mathbf{1}'$ and $\bar{\mathbf{1}}'$ representations. $\mathbf{3}_1$ and $\bar{\mathbf{3}}_1$ can be combined to form the three 1 dimensional representations:

$$\begin{aligned}
 \mathbf{1} &\simeq \psi_1 \bar{l}_{R1} + \psi_2 \bar{l}_{R2} + \psi_3 \bar{l}_{R3}, \\
 \mathbf{1}' &\simeq \psi_1 \bar{l}_{R1} + \omega \psi_2 \bar{l}_{R2} + \omega^2 \psi_3 \bar{l}_{R3}, \\
 \bar{\mathbf{1}}' &\simeq \psi_1 \bar{l}_{R1} + \omega^2 \psi_2 \bar{l}_{R2} + \omega \psi_3 \bar{l}_{R3}.
 \end{aligned} \tag{4.94}$$

Combined with the ϕ , we get the following contributions:

$$(\bar{l}_{R1}, \bar{l}_{R2}, \bar{l}_{R3}) \begin{pmatrix} h_1\phi_1 + h_2\omega\phi_2 + h_3\omega^2\phi_3 & 0 & 0 \\ 0 & h_1\phi_1 + h_2\phi_2 + h_3\phi_3 & 0 \\ 0 & 0 & h_1\phi_1 + h_2\omega^2\phi_2 + h_3\omega\phi_3 \end{pmatrix} \times \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \quad (4.96)$$

When the Higgs field ϕ again acquires a vacuum expectation value in the $(1, 1, 1)$ direction, the charged lepton mass matrix is already diagonal. To get tribimaximal lepton mixing, we then need the neutrino mixing matrix to be tribimaximal. We use the invariants as listed for case 1 above for the ψ in the $\mathbf{3}_1$ representation to achieve this by letting the Higgs bosons acquire the following vacuum expectation values:

i)

$$\begin{aligned} \langle \eta \rangle &= v_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \rho \rangle = v_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \langle \kappa \rangle &= u_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \end{aligned} \quad (4.98)$$

This will give contributions from b), f), g) and i). Letting χ acquire a vacuum expectation value in addition to these could also be a possibility:

ii)

$$\begin{aligned} \langle \chi \rangle &= x, \quad \langle \eta \rangle = v_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \rho \rangle = v_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \langle \kappa \rangle &= u_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \end{aligned} \quad (4.100)$$

thereby gaining a contribution from d). Writing up these terms in the neutrino mass matrix, we find:

i)

$$M_\nu = \begin{pmatrix} f_2 u_1^2 + f_9 v_1 v_2 & f'_3 u_1^2 & f'_3 u_1^2 \\ f'_3 u_1^2 & f_2 u_1^2 + f_{12} v_1 u_1 & f'_3 u_1^2 + f'_{16} v_2 u_1 \\ f'_3 u_1^2 & f'_3 u_1^2 + f'_{16} v_2 u_1 & f_2 u_1^2 + f_{12} v_1 u_1 \end{pmatrix}, \quad (4.101)$$

and

ii)

$$M_\nu = \begin{pmatrix} f_2 u_1^2 + f_9 v_1 v_2 & f'_3 u_1^2 + f'_6 x v_1 & f'_3 u_1^2 + f'_6 x v_1 \\ f'_3 u_1^2 + f'_6 x v_1 & f_2 u_1^2 + f_{12} v_1 u_1 & f'_3 u_1^2 + f'_{16} v_2 u_1 \\ f'_3 u_1^2 + f'_6 x v_1 & f'_3 u_1^2 + f'_{16} v_2 u_1 & f_2 u_1^2 + f_{12} v_1 u_1 \end{pmatrix}, \quad (4.102)$$

leading to a matrix with $\mu - \tau$ symmetry:

$$M_\nu = \begin{pmatrix} \gamma & \beta & \beta \\ \beta & \alpha & \delta \\ \beta & \delta & \alpha \end{pmatrix}, \quad (4.103)$$

which is diagonalized by the tribimaximal mixing matrix, if we impose without justification, $\delta = \gamma + \beta - \alpha$.

4.1.2.3 Case 25:

We now have the l_R in the three 1D representations, the lepton doublet in the $\mathbf{3}_2$ representation and the ϕ in the $\bar{\mathbf{3}}_2$ representation. The lepton doublet and the Higgs then join to form the three 1 dimensional representations:

$$\begin{aligned} \mathbf{1} &\simeq \phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3, \\ \mathbf{1}' &\simeq \phi_1\psi_1 + \omega\phi_2\psi_2 + \omega^2\phi_3\psi_3, \\ \bar{\mathbf{1}}' &\simeq \phi_1\psi_1 + \omega^2\phi_2\psi_2 + \omega\phi_3\psi_3, \end{aligned} \quad (4.106)$$

which results in invariants when connected with $\mathbf{1} \simeq \bar{l}_{R1}$, $\bar{\mathbf{1}}' \simeq \bar{l}_{R3}$ and $\mathbf{1}' \simeq \bar{l}_{R2}$ respectively. The mass terms of the charged leptons in the Lagrangian are then (using the freedom to incorporate phases into the coupling constants h_i):

$$\begin{aligned} &h_1\bar{l}_{R1}(\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3) + h_3\omega^2\bar{l}_{R3}(\phi_1\psi_1 + \omega\phi_2\psi_2 + \omega^2\phi_3\psi_3) \\ &+ h_2\omega\bar{l}_{R2}(\phi_1\psi_1 + \omega^2\phi_2\psi_2 + \omega\phi_3\psi_3) = h_1\bar{l}_{R1}(\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3) \\ &+ h_2\bar{l}_{R2}(\omega\phi_1\psi_1 + \phi_2\psi_2 + \omega^2\phi_3\psi_3) + h_3\bar{l}_{R3}(\omega^2\phi_1\psi_1 + \phi_2\psi_2 + \omega\phi_3\psi_3) \\ &= \begin{pmatrix} \bar{l}_{R1} & \bar{l}_{R2} & \bar{l}_{R3} \end{pmatrix} \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \omega & 1 & \omega^2 \\ \omega^2 & 1 & \omega \end{pmatrix} \\ &\quad \times \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \end{aligned} \quad (4.111)$$

Upon spontaneous symmetry breaking, again with the Higgs vacuum expectation value in the $(1, 1, 1)^T$ direction, the mass matrix is diagonalized by the same unitary matrix as in case 1:

$$U_L^\dagger = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & 1 & \omega^2 \\ \omega^2 & 1 & \omega \end{pmatrix}. \quad (4.112)$$

Proceeding to the neutrino sector, we now have ψ in the $\mathbf{3}_2$ representation, leading to different invariants in the dimension 5 operator than for case 1 and 3. The Higgs fields ξ_1 and ξ_2 in (4.58) can each still be any of the following five Higgs with the assigned representations:

$$\chi \sim \mathbf{1}, \quad \eta \sim \mathbf{3}_1, \quad \rho \sim \mathbf{3}_2, \quad \kappa \sim \bar{\mathbf{3}}_1, \quad \zeta \sim \bar{\mathbf{3}}_2. \quad (4.113)$$

Using table 4.1 once again, we can omit the combinations $(\chi\psi)^2$, $(\rho\psi)^2$, $(\chi\psi)(\zeta\psi)$ and $(\chi\psi)(\kappa\psi)$. This leaves the possibilities:

a) $(\eta\psi)^2$:

$$f_1(\eta_3\psi_1\eta_3\psi_2 + \eta_1\psi_2\eta_1\psi_3 + \eta_2\psi_3\eta_2\psi_1). \quad (4.114)$$

b) $(\kappa\psi)^2$:

$$f_2(\kappa_1\psi_1\kappa_2\psi_1 + \kappa_2\psi_2\kappa_3\psi_2 + \kappa_3\psi_3\kappa_1\psi_3). \quad (4.115)$$

c) $(\zeta\psi)^2$:

$$\begin{aligned} & f_3(\zeta_1\psi_1\zeta_1\psi_1 + \zeta_2\psi_2\zeta_2\psi_2 + \zeta_3\psi_3\zeta_3\psi_3) \\ & + f_4(\zeta_1\psi_1\zeta_2\psi_2 + \zeta_3\psi_3\zeta_1\psi_1 + \zeta_2\psi_2\zeta_3\psi_3) \\ & + f_5(\zeta_1\psi_2\zeta_2\psi_1 + \zeta_3\psi_1\zeta_1\psi_3 + \zeta_2\psi_3\zeta_3\psi_2). \end{aligned} \quad (4.118)$$

d) $(\chi\psi)(\eta\psi)$:

$$f_6\chi(\psi_1\eta_3\psi_1 + \psi_2\eta_1\psi_2 + \psi_3\eta_2\psi_3). \quad (4.119)$$

e) $(\chi\psi)(\rho\psi)$:

$$\begin{aligned} & f_7\chi(\psi_1\rho_2\psi_3 + \psi_2\rho_3\psi_1 + \psi_3\rho_1\psi_2) \\ & + f_8\chi(\psi_1\rho_3\psi_2 + \psi_2\rho_1\psi_3 + \psi_3\rho_2\psi_1). \end{aligned} \quad (4.121)$$

f) $(\eta\psi)(\rho\psi)$:

$$\begin{aligned} & f_9(\eta_3\psi_2\rho_3\psi_2 + \eta_1\psi_3\rho_1\psi_3 + \eta_2\psi_1\rho_2\psi_1) \\ & f_{10}(\eta_3\psi_2\rho_2\psi_3 + \eta_1\psi_3\rho_3\psi_1 + \eta_2\psi_1\rho_1\psi_2) \\ & + f_{11}(\eta_3\psi_3\rho_2\psi_2 + \eta_1\psi_1\rho_3\psi_3 + \eta_2\psi_2\rho_1\psi_1). \end{aligned} \quad (4.124)$$

g) $(\eta\psi)(\kappa\psi)$:

$$\begin{aligned} & f_{12}(\eta_3\psi_1\kappa_2\psi_3 + \eta_1\psi_2\kappa_3\psi_1 + \eta_2\psi_3\kappa_1\psi_2) \\ & + f_{13}(\eta_3\psi_3\kappa_2\psi_1 + \eta_1\psi_1\kappa_3\psi_2 + \eta_2\psi_2\kappa_1\psi_3). \end{aligned} \quad (4.126)$$

h) $(\eta\psi)(\zeta\psi)$:

$$f_{14}(\eta_3\psi_3\zeta_2\psi_3 + \eta_1\psi_1\zeta_3\psi_1 + \eta_2\psi_2\zeta_1\psi_2). \quad (4.127)$$

i) $(\rho\psi)(\kappa\psi)$:

$$\begin{aligned} & f_{15}(\rho_3\psi_2\kappa_2\psi_3 + \rho_1\psi_3\kappa_3\psi_1 + \rho_2\psi_1\kappa_1\psi_2) \\ & + f_{16}(\rho_2\psi_2\kappa_1\psi_1 + \rho_3\psi_3\kappa_2\psi_2 + \rho_1\psi_1\kappa_3\psi_3) \\ & + f_{17}(\rho_2\psi_3\kappa_2\psi_3 + \rho_3\psi_1\kappa_3\psi_1 + \rho_1\psi_2\kappa_1\psi_2). \end{aligned} \quad (4.130)$$

j) $(\rho\psi)(\zeta\psi)$:

$$f_{18}(\rho_2\psi_2\zeta_3\psi_2 + \rho_3\psi_3\zeta_1\psi_3 + \rho_1\psi_1\zeta_2\psi_1). \quad (4.131)$$

k) $(\kappa\psi)(\zeta\psi)$:

$$\begin{aligned} & f_{19}(\kappa_1\psi_1\zeta_2\psi_3 + \kappa_2\psi_2\zeta_3\psi_1 + \kappa_3\psi_3\zeta_1\psi_2) \\ & + f_{20}(\kappa_2\psi_1\zeta_3\psi_2 + \kappa_3\psi_2\zeta_1\psi_3 + \kappa_1\psi_3\zeta_2\psi_1). \end{aligned} \quad (4.133)$$

Evidently, the coefficients here are not the same as for ψ in the $\mathbf{3}_1$ representation. In this case, the possible Higgs field vacuum expectation values needed to eventually get tribimaximal mixing are numerous. We list them all in the following:

i)

$$\langle \rho \rangle = v_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \langle \zeta \rangle = u_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad (4.134)$$

leading to contributions from c) and j).

ii)

$$\langle \chi \rangle = x, \quad \langle \rho \rangle = v_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \langle \zeta \rangle = u_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad (4.135)$$

which gives the same contributions as in i) with one extra from e).

iii)

$$\langle \kappa \rangle = u_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \zeta \rangle = u_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad (4.136)$$

with b), c) and k) contributing to the mass matrix.

iv)

$$\langle \chi \rangle = x, \quad \langle \kappa \rangle = u_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \zeta \rangle = u_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad (4.137)$$

leading to the same result as in iii) with no extra contribution.

v)

$$\langle \rho \rangle = v_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \langle \kappa \rangle = u_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad (4.138)$$

giving terms to the mass matrix from b) and i).

vi)

$$\langle \eta \rangle = v_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \langle \zeta \rangle = u_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad (4.139)$$

contributing with terms from a), c) and h).

When the neutral component of the Higgs bosons acquire these vacuum expectation values we can then combine the coupling constants as we did for the invariants with $\psi \sim \mathbf{3}_1$. The possible mass matrices from these couplings and vacuum expectation values will then become:

i)

$$M_\nu = \begin{pmatrix} f_3 u_2^2 & 0 & f'_4 u_2^2 \\ 0 & f_{18} v_2 u_2 & 0 \\ f'_4 u_2^2 & 0 & f_3 u_2^2 \end{pmatrix}, \quad (4.140)$$

ii)

$$M_\nu = \begin{pmatrix} f_3 u_2^2 & 0 & f_4' u_2^2 + f_7' x v_2 \\ 0 & f_{18} v_2 u_2 & 0 \\ f_4' u_2^2 + f_7' x v_2 & 0 & f_3 u_2^2 \end{pmatrix}, \quad (4.141)$$

iii), iv)

$$M_\nu = \begin{pmatrix} f_2 u_1^2 & 0 & f_{19}' u_1 u_2 \\ 0 & f_2 u_1^2 + f_3 u_2^2 & 0 \\ f_{19}' u_1 u_2 & 0 & f_2 u_1^2 \end{pmatrix}, \quad (4.142)$$

v)

$$M_\nu = \begin{pmatrix} f_2 u_1^2 & 0 & f_{15}' v_2 u_1 \\ 0 & f_2 u_1^2 + f_{17} v_2 u_1 & 0 \\ f_{15}' v_2 u_1 & 0 & f_2 u_1^2 \end{pmatrix}, \quad (4.143)$$

vi)

$$M_\nu = \begin{pmatrix} f_3 u_2^2 & 0 & f_1 v_1^2 + f_4' u_2^2 \\ 0 & f_{14} v_1 u_2 & 0 \\ f_1 v_1^2 + f_4' u_2^2 & 0 & f_3 u_2^2 \end{pmatrix}, \quad (4.144)$$

all with the same form as in case 1:

$$M_\nu = \begin{pmatrix} \alpha & 0 & \beta \\ 0 & \gamma & 0 \\ \beta & 0 & \alpha \end{pmatrix}, \quad (4.145)$$

and as in case 1, tribimaximal mixing is possible from this mass matrix.

4.1.2.4 Case 24:

This case gives the same result as case 23 in the charged lepton sector. We therefore have the following contributions to the charged lepton mass matrix:

$$(\bar{l}_{R1}, \bar{l}_{R2}, \bar{l}_{R3}) \begin{pmatrix} h_1 \phi_1 + h_2 \omega \phi_2 + h_3 \omega^2 \phi_3 & 0 & 0 \\ 0 & h_1 \phi_1 + h_2 \phi_2 + h_3 \phi_3 & 0 \\ 0 & 0 & h_1 \phi_1 + h_2 \omega^2 \phi_2 + h_3 \omega \phi_3 \end{pmatrix} \times \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (4.147)$$

where the charged lepton mass matrix is already diagonal upon alignment of the ϕ vacuum expectation value in the $(1, 1, 1)$ direction, as in case 23. We therefore want a tribimaximal neutrino mixing matrix once again, now with the ψ in the $\mathbf{3}_2$ representation. The invariants in the dimension 5 operator are as in case 25. We can get a tribimaximal neutrino mixing matrix by choosing the following different options for the vacuum expectation values:

i)

$$\langle \kappa \rangle = u_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \langle \zeta \rangle = u_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

where b), c) and k) will contribute.

ii)

$$\langle \chi \rangle = x, \quad \langle \kappa \rangle = u_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \langle \zeta \rangle = u_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

leading to the same result as in i).

iii)

$$\langle \eta \rangle = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \langle \kappa \rangle = u_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \langle \zeta \rangle = u_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad (4.150)$$

with contributions as in i) plus extra terms from a) and h).

iv)

$$\langle \rho \rangle = v_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \langle \kappa \rangle = u_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \zeta \rangle = u_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad (4.151)$$

with contributions from b), c), i) and k).

v)

$$\langle \rho \rangle = v_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \langle \kappa \rangle = u_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \zeta \rangle = u_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad (4.152)$$

with terms contributing from b), c), i) and k).

The following corresponding neutrino mass matrices will be possible:

i), ii)

$$M_\nu = \begin{pmatrix} f_2 u_1^2 & f'_{19} u_1 u_2 & f'_{19} u_1 u_2 \\ f'_{19} u_1 u_2 & f_3 u_2^2 & f'_4 u_2^2 \\ f'_{19} u_1 u_2 & f'_4 u_2^2 & f_3 u_2^2 \end{pmatrix}, \quad (4.153)$$

iii)

$$M_\nu = \begin{pmatrix} f_2 u_1^2 + f_{14} v_1 u_2 & f'_{19} u_1 u_2 & f'_{19} u_1 u_2 \\ f'_{19} u_1 u_2 & f_3 u_2^2 & f_1 v_1^2 + f'_4 u_2^2 \\ f'_{19} u_1 u_2 & f_1 v_1^2 + f'_4 u_2^2 & f_3 u_2^2 \end{pmatrix}, \quad (4.154)$$

iv)

$$M_\nu = \begin{pmatrix} f_2 u_1^2 + f_{17} v_2 u_1 & f'_{19} u_1 u_2 & f'_{19} u_1 u_2 \\ f'_{19} u_1 u_2 & f_2 u_1^2 + f_3 u_2^2 & f'_4 u_2^2 + f'_{15} v_2 u_1 \\ f'_{19} u_1 u_2 & f'_4 u_2^2 + f'_{15} v_2 u_1 & f_2 u_1^2 + f_3 u_2^2 \end{pmatrix}, \quad (4.155)$$

v)

$$M_\nu = \begin{pmatrix} f_2 u_1^2 + f_3 u_2^2 & f'_{15} v_2 u_1 & f'_{15} v_2 u_1 \\ f'_{15} v_2 u_1 & f_2 u_1^2 + f_{17} v_2 u_1 & f'_{19} u_1 u_2 \\ f'_{15} v_2 u_1 & f'_{19} u_1 u_2 & f_2 u_1^2 + f_{17} v_2 u_1 \end{pmatrix}, \quad (4.156)$$

all with the same form as in case 23:

$$M_\nu = \begin{pmatrix} \gamma & \beta & \beta \\ \beta & \alpha & \delta \\ \beta & \delta & \alpha \end{pmatrix}, \quad (4.157)$$

This will again give tribimaximal mixing, if $\delta = \gamma + \beta - \alpha$ is fulfilled.

We have covered all the cases of the group, by just studying these two forms of the neutrino mass matrix, with the ψ in either a $\mathbf{3}_1$ or a $\mathbf{3}_2$ representation. This group can then be applied with other models to place further constraints on neutrino mixing and achieve a more predictive model.

4.2 General Frobenius groups

The Frobenius group T_{13} is one of many Frobenius groups. It is interesting to see that it is not the only available group within these semidirect products of cyclic groups, that can describe neutrino mixing, leading to tribimaximal mixing. It turns out that a whole series of Frobenius groups, $T_n = Z_n \rtimes Z_3$ contain three 1 dimensional representations and a number of 3 dimensional representations with their corresponding charge conjugated representations. These are all subgroups of $SU(3)$ and the group theory of all these can therefore be generalized. The Frobenius groups that constitute the possible candidates for neutrino mixing are

$$T_n = Z_n \rtimes Z_3, \quad (4.158)$$

The dimensions of the irreducible representations can be found from

$$\sum_i d_i^2 = 3n = 1 + 1 + 1 + 3(n-1). \quad (4.159)$$

The number of 3 dimensional irreducible representations within these groups is then given by

$$\frac{n-1}{3}, \quad (4.160)$$

including the charge conjugated representations. n must therefore fulfil the requirement

$$\frac{n-1}{6} = \text{integer}. \quad (4.161)$$

All of these groups have three 1 dimensional irreducible representation. They are all subgroups of $SU(3)$. We can again denote the generators by $a \in Z_n$ and $b \in Z_3$. The generator of b in 1 dimension is given by ω , where $\omega^3 = 1$. The generator of a in 1 dimension is given by $\rho = e^{2\pi i/n}$. The generator b in 3 dimensions is the same for all these Frobenius groups:

$$b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (4.162)$$

where again $b^3 = 1$.

The meaning of a semidirect product is given by

$$bab^{-1} = a^r, \quad (4.163)$$

where

$$r^3 = 1 + nk, \quad (4.164)$$

and r and k are integers. The Frobenius group T_n thereby has the presentation

$$\langle a, b \mid a^n = b^3 = I, bab^{-1} = a^r \rangle. \quad (4.165)$$

Also

$$bab^{-1} = a^r \Rightarrow b^2ab^{-2} = a^{r^2} = b^{-1}ab, \quad (4.166)$$

and

$$ab^2 = b^3ab^{-1} = b^2bab^{-1} = b^2a^r. \quad (4.167)$$

Using that $a^n = I$, the entries of the 3 dimensional representations have to be powers of ρ as before. We call the first entry ρ^i , and the other two are then fixed by bab^{-1} as previously,

$$a_i = \begin{pmatrix} \rho^i & 0 & 0 \\ 0 & \rho^{ir} & 0 \\ 0 & 0 & \rho^{ir^2} \end{pmatrix}, \quad (4.168)$$

$$a_i^* = \begin{pmatrix} \rho^{ir+ir^2} & 0 & 0 \\ 0 & \rho^{i+ir^2} & 0 \\ 0 & 0 & \rho^{i+ir} \end{pmatrix}, \quad (4.169)$$

with $\rho^{1+r+r^2} = 1$. There are as many of these generators as there are 3 dimensional representations. In order to find the applications of the group on neutrino mixing, it is necessary to know the Kronecker products and Clebsch-Gordon coefficients of the representations. These can be worked out in the same way as before. In $SU(3)$, we have

$$\mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6}. \quad (4.170)$$

To find the decomposition of $\mathbf{3}_i \otimes \mathbf{3}_i$, we use the generators a_i and b to determine how $\psi^{\{\mu\nu\}}$ decomposes.

Consider the diagonal elements $\psi^{\{11\}}, \psi^{\{22\}}, \psi^{\{33\}}$. Under b , they transform as

$$\psi^{\{11\}} \rightarrow \psi^{\{22\}} \rightarrow \psi^{\{33\}} \rightarrow \psi^{\{11\}}, \quad (4.171)$$

while under a_i , they transform as

$$\psi^{\{11\}} \rightarrow \rho^{2i}\psi^{\{11\}}, \quad \psi^{\{22\}} \rightarrow \rho^{2ir}\psi^{\{22\}}, \quad \psi^{\{33\}} \rightarrow \rho^{2ir^2}\psi^{\{33\}}. \quad (4.172)$$

Thus they form a $\mathbf{3}_{2i}$ representation:

$$\begin{pmatrix} \psi^{\{11\}} \\ \psi^{\{22\}} \\ \psi^{\{33\}} \end{pmatrix}_{\mathbf{3}_{2i}}. \quad (4.173)$$

Next, consider the off-diagonal elements of the symmetric representation $\psi^{\{\mu\nu\}}$. Under b these transform as

$$\psi^{\{12\}} \rightarrow \psi^{\{23\}} \rightarrow \psi^{\{31\}} \rightarrow \psi^{\{12\}}. \quad (4.174)$$

Under a_i , they transform as

$$\psi^{\{12\}} \rightarrow \rho^{i(1+r)}\psi^{\{12\}}, \quad \psi^{\{23\}} \rightarrow \rho^{i(r+r^2)}\psi^{\{23\}}, \quad \psi^{\{31\}} \rightarrow \rho^{i(r^2+1)}\psi^{\{31\}}, \quad (4.175)$$

thus forming the $\bar{\mathbf{3}}_i$ representation:

$$\begin{pmatrix} \psi^{\{23\}} \\ \psi^{\{31\}} \\ \psi^{\{12\}} \end{pmatrix}_{\bar{\mathbf{3}}_i}. \quad (4.176)$$

We thus obtain, seeing that the antisymmetric entries again transform the same way as the symmetric under a_i and b ,

$$\mathbf{3}_i \otimes \mathbf{3}_i = \bar{\mathbf{3}}_i \oplus \bar{\mathbf{3}}_i \oplus \mathbf{3}_{2i}. \quad (4.177)$$

The Kronecker product between two different 3 dimensional representations $\mathbf{3}_i \otimes \mathbf{3}_j$ with $i \neq j$, is then obtained from seeing that under a_i and a_j , the diagonal elements transform as

$$\psi^{\{11\}} \rightarrow \rho^{i+j}\psi^{\{11\}}, \quad \psi^{\{22\}} \rightarrow \rho^{(i+j)r}\psi^{\{22\}}, \quad \psi^{\{33\}} \rightarrow \rho^{(i+j)r^2}\psi^{\{33\}}, \quad (4.178)$$

forming the $\mathbf{3}_{i+j}$ representation:

$$\begin{pmatrix} \psi^{\{11\}} \\ \psi^{\{22\}} \\ \psi^{\{33\}} \end{pmatrix}_{\mathbf{3}_{i+j}}, \quad (4.179)$$

and the off-diagonal elements transform as

$$\psi^{\{12\}} \rightarrow \rho^{i+jr}\psi^{\{12\}}, \quad \psi^{\{23\}} \rightarrow \rho^{ir+jr^2}\psi^{\{23\}}, \quad \psi^{\{31\}} \rightarrow \rho^{ir^2+j}\psi^{\{31\}}, \quad (4.180)$$

$$\psi^{\{21\}} \rightarrow \rho^{j+ir}\psi^{\{21\}}, \quad \psi^{\{32\}} \rightarrow \rho^{jr+ir^2}\psi^{\{32\}}, \quad \psi^{\{13\}} \rightarrow \rho^{jr^2+i}\psi^{\{13\}}. \quad (4.181)$$

Thus forming the $\mathbf{3}_{i+jr}$ and $\mathbf{3}_{ir+j}$ representations:

$$\begin{pmatrix} \psi^{\{12\}} \\ \psi^{\{23\}} \\ \psi^{\{31\}} \end{pmatrix}_{\mathbf{3}_{i+jr}}, \quad (4.182)$$

$$\begin{pmatrix} \psi^{\{21\}} \\ \psi^{\{32\}} \\ \psi^{\{13\}} \end{pmatrix}_{\mathbf{3}_{\mathbf{i}+\mathbf{j}}} . \quad (4.183)$$

In summary, we have the Kronecker product:

$$\mathbf{3}_{\mathbf{i}} \otimes \mathbf{3}_{\mathbf{j}} = \mathbf{3}_{\mathbf{i}+\mathbf{j}} \oplus \mathbf{3}_{\mathbf{i}+\mathbf{j}} \oplus \mathbf{3}_{\mathbf{i}+\mathbf{j}\mathbf{r}} \quad (4.184)$$

In $\text{SU}(3)$ we have the product between a $\mathbf{3}$ and a $\bar{\mathbf{3}}$ representation:

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}, \quad (4.185)$$

For $\mathbf{3}_{\mathbf{i}} \otimes \bar{\mathbf{3}}_{\mathbf{j}}$ with $i = j$ the diagonal elements of ψ_{ν}^{μ} transform under b as $\psi_1^1 \rightarrow \psi_2^2 \rightarrow \psi_3^3 \rightarrow \psi_1^1$. We have the two possibilities

$$\bar{\phi} = \psi_1^1 + \omega\psi_2^2 + \omega^2\psi_3^3, \quad (4.186)$$

$$\phi = \psi_1^1 + \omega^2\psi_2^2 + \omega\psi_3^3. \quad (4.187)$$

Under b , these transform as

$$\bar{\phi} \rightarrow \psi_2^2 + \omega\psi_3^3 + \omega^2\psi_1^1 = \omega^2\bar{\phi}, \quad (4.188)$$

$$\phi \rightarrow \psi_2^2 + \omega^2\psi_3^3 + \omega\psi_1^1 = \omega\phi. \quad (4.189)$$

Thus, we choose $\bar{\phi}$ to transform as $\bar{\mathbf{1}}'$ and ϕ to transform as $\mathbf{1}'$.

The off-diagonal terms split into two sets. They transform under b in the same obvious way as described in (4.174). These transform under a_i and a_i^* as

$$\psi_2^1 \rightarrow \rho^{i(2+r^2)}\psi_2^1, \quad \psi_3^2 \rightarrow \rho^{i(2r+1)}\psi_3^2, \quad \psi_1^3 \rightarrow \rho^{i(2r^2+r)}\psi_1^3, \quad (4.190)$$

and

$$\psi_1^2 \rightarrow \rho^{i(2r+r^2)}\psi_1^2, \quad \psi_2^3 \rightarrow \rho^{2r^2+1}\psi_2^3, \quad \psi_3^1 \rightarrow \rho^{i(2+r)}\psi_3^1, \quad (4.191)$$

Seeing that $2 + r^2 + 2r + r^2 = 2n$, these give a $\bar{\mathbf{3}}_{\mathbf{i}(1-\mathbf{r})}$ and a $\mathbf{3}_{\mathbf{i}(1-\mathbf{r})}$ representation:

$$\begin{pmatrix} \psi_2^1 \\ \psi_3^2 \\ \psi_1^3 \end{pmatrix}_{\mathbf{3}_{\mathbf{i}(1-\mathbf{r})}}, \quad (4.192)$$

$$\begin{pmatrix} \psi_1^2 \\ \psi_2^3 \\ \psi_3^1 \end{pmatrix}_{\bar{\mathbf{3}}_{\mathbf{i}(1-\mathbf{r})}}. \quad (4.193)$$

For $i \neq j$, the diagonal terms transform under a_i and a_j^* as

$$\psi^{\{11\}} \rightarrow \rho^{i+j(r+r^2)}\psi^{\{11\}}, \quad \psi^{\{22\}} \rightarrow \rho^{ir+j(1+r^2)}\psi^{\{22\}}, \quad \psi^{\{33\}} \rightarrow \rho^{ir^2+j(1+r)}\psi^{\{33\}}. \quad (4.194)$$

The off-diagonal terms transform as

$$\psi_2^1 \rightarrow \rho^{i+j(1+r^2)} \psi_2^1, \quad \psi_3^2 \rightarrow \rho^{ir+j(r+1)} \psi_3^2, \quad \psi_1^3 \rightarrow \rho^{ir^2+j(r+r^2)} \psi_1^3, \quad (4.195)$$

and

$$\psi_1^2 \rightarrow \rho^{ir+j(r+r^2)} \psi_1^2, \quad \psi_2^3 \rightarrow \rho^{ir^2+j(1+r^2)} \psi_2^3, \quad \psi_3^1 \rightarrow \rho^{i+j(1+r)} \psi_3^1. \quad (4.196)$$

which leads to the 3 dimensional decompositions:

$$\begin{pmatrix} \psi^{\{11\}} \\ \psi^{\{22\}} \\ \psi^{\{33\}} \end{pmatrix}_{\mathbf{3}_{i-j}}, \quad (4.197)$$

$$\begin{pmatrix} \psi_2^1 \\ \psi_3^2 \\ \psi_1^3 \end{pmatrix}_{\mathbf{3}_{i-jr}}, \quad (4.198)$$

$$\begin{pmatrix} \psi_1^2 \\ \psi_2^3 \\ \psi_3^1 \end{pmatrix}_{\mathbf{3}_{ir-j}}. \quad (4.199)$$

The Kronecker products are listed in table 4.4 and the Clebsch-Gordan decompositions in table 4.5.

$\mathbf{1}' \otimes \mathbf{1}' = \bar{\mathbf{1}}'$
$\bar{\mathbf{1}}' \otimes \bar{\mathbf{1}}' = \mathbf{1}'$
$\mathbf{1}' \otimes \bar{\mathbf{1}}' = \mathbf{1}$
$\mathbf{3}_i \otimes \mathbf{3}_i = \bar{\mathbf{3}}_i \oplus \bar{\mathbf{3}}_i \oplus \mathbf{3}_{2i}$
$\mathbf{3}_i \otimes \bar{\mathbf{3}}_i = \mathbf{1} \oplus \mathbf{1}' \oplus \bar{\mathbf{1}}' \oplus \mathbf{3}_{i(1-r)} \oplus \bar{\mathbf{3}}_{i(1-r)}$
$\mathbf{3}_i \otimes \mathbf{3}_j = \mathbf{3}_{ir+j} \oplus \mathbf{3}_{i+j} \oplus \mathbf{3}_{i+jr}$
$\mathbf{3}_i \otimes \bar{\mathbf{3}}_j = \mathbf{3}_{i-j} \oplus \mathbf{3}_{i-jr} \oplus \mathbf{3}_{ir-j}$

Table 4.4: $Z_n \rtimes Z_3$ Kronecker products

It can be seen that only the product between a 3 dimensional representation and it's own charge conjugated representation will create an invariant. This is an important aspect of the Frobenius groups.

We can now assign the fields of the dimension 4 operator in the charged lepton sector to various representations within these general Frobenius groups, as done previously. The fields can be placed in the three 1 dimensional representations and in the $\frac{n-1}{3}$ 3 dimensional representations. The possible assignments can be seen in Table 4.6.

These three cases will cover all the possibilities for neutrino mixing using Frobenius groups. For a group with $\frac{n-1}{6}$ 3 dimensional representations, all possibilities can be described by $\frac{n-1}{2}$ cases. Furthermore, it turns out that all Frobenius groups will only generate the two mentioned neutrino mass matrices.

$$\begin{aligned}
 \mathbf{3}_i \otimes \mathbf{3}_i &\rightarrow \begin{pmatrix} \psi^{\{11\}} \\ \psi^{\{22\}} \\ \psi^{\{33\}} \end{pmatrix}_{\mathbf{3}_{2i}}, \quad \begin{pmatrix} \psi^{\{23\}} \\ \psi^{\{31\}} \\ \psi^{\{12\}} \end{pmatrix}_{\bar{\mathbf{3}}_i}, \quad \begin{pmatrix} \psi^{32} \\ \psi^{13} \\ \psi^{21} \end{pmatrix}_{\bar{\mathbf{3}}_i} \\
 \mathbf{3}_i \otimes \bar{\mathbf{3}}_i &\rightarrow \begin{pmatrix} \psi_2^1 \\ \psi_3^2 \\ \psi_1^3 \end{pmatrix}_{\mathbf{3}_{i(1-r)}}, \quad \begin{pmatrix} \psi_1^2 \\ \psi_2^3 \\ \psi_3^1 \end{pmatrix}_{\bar{\mathbf{3}}_{i(1-r)}}, \quad (\psi_1^1 + \psi_2^2 + \psi_3^3)_{\mathbf{1}}, \\
 &\quad (\psi_1^1 + \omega\psi_2^2 + \omega^2\psi_3^3)_{\mathbf{1}'}, \quad (\psi_1^1 + \omega^2\psi_2^2 + \omega\psi_3^3)_{\bar{\mathbf{1}}'}, \\
 \mathbf{3}_i \otimes \mathbf{3}_j &\rightarrow \begin{pmatrix} \psi^{\{11\}} \\ \psi^{\{22\}} \\ \psi^{\{33\}} \end{pmatrix}_{\mathbf{3}_{i+j}}, \quad \begin{pmatrix} \psi^{\{12\}} \\ \psi^{\{23\}} \\ \psi^{\{31\}} \end{pmatrix}_{\mathbf{3}_{i+jr}}, \quad \begin{pmatrix} \psi^{\{21\}} \\ \psi^{\{32\}} \\ \psi^{\{13\}} \end{pmatrix}_{\mathbf{3}_{ir+j}} \\
 \mathbf{3}_i \otimes \bar{\mathbf{3}}_j &\rightarrow \begin{pmatrix} \psi^{\{11\}} \\ \psi^{\{22\}} \\ \psi^{\{33\}} \end{pmatrix}_{\mathbf{3}_{i-j}}, \quad \begin{pmatrix} \psi_2^1 \\ \psi_3^2 \\ \psi_1^3 \end{pmatrix}_{\mathbf{3}_{i-jr}}, \quad \begin{pmatrix} \psi_1^2 \\ \psi_2^3 \\ \psi_3^1 \end{pmatrix}_{\mathbf{3}_{ir-j}}
 \end{aligned}$$

Table 4.5: $Z_n \rtimes Z_3$ Clebsch-Gordan decompositions.

Case	ψ	$\phi^\dagger$	l^c
1	$\mathbf{3}_i$	$\mathbf{3}_i$	$\mathbf{3}_i$
2	$\mathbf{3}_i$	$\bar{\mathbf{3}}_i$	$\mathbf{1}, \mathbf{1}', \bar{\mathbf{1}}'$
3	$\mathbf{3}_i$	$\mathbf{1}, \mathbf{1}', \bar{\mathbf{1}}'$	$\bar{\mathbf{3}}_i$

Table 4.6: Possible representations for the 3 fields.

The first two cases will upon the Higgs acquiring a vacuum expectation value in the $(1, 1, 1)$ direction, lead to the charged lepton mixing matrix

$$U_L^\dagger = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & 1 & \omega^2 \\ \omega^2 & 1 & \omega \end{pmatrix}. \quad (4.200)$$

Case 3 will lead to a diagonal charged lepton mass matrix. In order to get tribimaximal neutrino mixing, it is therefore necessary in case 1 and 2 to find a dimension 5 operator with a neutrino mass matrix with the form

$$M_\nu = \begin{pmatrix} \alpha & 0 & \beta \\ 0 & \gamma & 0 \\ \beta & 0 & \alpha \end{pmatrix}, \quad (4.201)$$

and in case 3 to find a mass matrix of the form

$$M_\nu = \begin{pmatrix} \gamma & \beta & \beta \\ \beta & \alpha & \delta \\ \beta & \delta & \alpha \end{pmatrix}. \quad (4.202)$$

The study of the group can then be continued by looking at the various possibilities for $\mathcal{O}_5$, assigning ξ_1 and ξ_2 to various representations of F once again. The doublet ψ is in the $\mathbf{3}_i$ representations and the aim is then to align the Higgs fields to get the shapes of two mass matrices above. The Higgs fields ξ_1 and ξ_2 in (4.58) can each be any of the following different Higgs in the representations:

$$\chi \sim \mathbf{1}, \quad \eta_j \sim \mathbf{3}_j, \quad \kappa_j \sim \bar{\mathbf{3}}_j. \quad (4.203)$$

The ψ is in the $\mathbf{3}_i$ representation. From Table 4.1, we can already exclude the combinations $(\chi\psi)^2$ and $(\eta_i\psi_i)^2$. The possible invariants from combining two lepton doublets with two Higgs in the above mentioned representations and contracting the indices properly, can now be cataloged. The list of invariants can be seen in Appendix D.

An example of finding these invariants will be worked out in the following, where $i \neq j$ applies.

$(\eta_j\psi_i)^2$: From $\mathbf{3}_{i+j} \otimes \mathbf{3}_{j+ir} = \mathbf{3}_{i+j} \otimes \bar{\mathbf{3}}_{-j-ir}$, we can have an invariant if $i + j = -j - ir$, that is if $j = \frac{1}{2}ir^2$. However, we can also have invariants if the representations are permuted, that is for $\mathbf{3}_{i+j} \otimes \bar{\mathbf{3}}_{-jr-ir^2}$ and $\mathbf{3}_{i+j} \otimes \bar{\mathbf{3}}_{-jr^2-i}$, leading to invariants if $i = -jr$ and $i = \frac{1}{2}jr$. The invariants are then given by

$$j = \frac{1}{2}ir^2 : \sim (\eta_1\psi_1\eta_1\psi_2 + \eta_2\psi_2\eta_2\psi_3 + \eta_3\psi_3\eta_3\psi_1), \quad (4.204)$$

$$i = -jr : \sim (\eta_1\psi_1\eta_2\psi_3 + \eta_2\psi_2\eta_3\psi_1 + \eta_3\psi_3\eta_1\psi_2), \quad (4.205)$$

$$i = \frac{1}{2}jr : \sim (\eta_1\psi_1\eta_3\psi_1 + \eta_2\psi_2\eta_1\psi_2 + \eta_3\psi_3\eta_2\psi_3). \quad (4.206)$$

From $\mathbf{3}_{i+j} \otimes \mathbf{3}_{i+jr} = \mathbf{3}_{i+j} \otimes \bar{\mathbf{3}}_{-i-jr}$, the invariants above have interchanged i and j and the two last indices,

$$i = \frac{1}{2}jr^2 : \sim (\eta_1\psi_1\eta_2\psi_1 + \eta_2\psi_2\eta_3\psi_2 + \eta_3\psi_3\eta_1\psi_3), \quad (4.207)$$

$$j = -ir : \sim (\eta_1\psi_1\eta_3\psi_2 + \eta_2\psi_2\eta_1\psi_3 + \eta_3\psi_3\eta_2\psi_1), \quad (4.208)$$

$$j = \frac{1}{2}ir : \sim (\eta_1\psi_1\eta_1\psi_3 + \eta_2\psi_2\eta_2\psi_1 + \eta_3\psi_3\eta_3\psi_2). \quad (4.209)$$

Finally, from $\mathbf{3}_{i+jr} \otimes \bar{\mathbf{3}}_{-j-ir}$, we have

$$i = -j : \sim (\eta_1\psi_2\eta_2\psi_1 + \eta_2\psi_3\eta_3\psi_2 + \eta_3\psi_1\eta_1\psi_3), \quad (4.210)$$

$$j = 2i : \sim (\eta_1\psi_2\eta_3\psi_2 + \eta_2\psi_3\eta_1\psi_3 + \eta_3\psi_1\eta_2\psi_1), \quad (4.211)$$

$$j = \frac{1}{2}ir : \sim (\eta_1\psi_2\eta_1\psi_3 + \eta_2\psi_3\eta_2\psi_1 + \eta_3\psi_1\eta_3\psi_2). \quad (4.212)$$

The vacuum alignments can then be found from the listed invariants aiming at getting one of the possible structures for the neutrino mass matrix. The case of T_{13} has been worked out in 4.1. With even more possibilities for $n > 3$, the Frobenius groups constitute an infinite number of possibilities to describe neutrino mixing, as n gets bigger.

The see-saw mechanism

The see-saw mechanism assumes that there is some unknown physics beyond the cutoff scale Λ generating higher dimensional operators at scales much below Λ . The operator needed in the see-saw mechanism, and the lowest dimensional operator beyond the Standard Model, is the dimension 5 operator,

$$\mathcal{O}_5 = \frac{1}{M} (\phi \tau_2 \psi)^T C (\phi \tau_2 \psi), \quad (5.1)$$

where M is the cutoff scale of the effective field theory.

In the minimal standard electroweak model there exists three types of see-saw models, leading to this effective operator at tree-level, using only renormalizable interactions. The standard model lepton doublet $\psi = (\nu_L, l_L)^T$ and the Higgs doublet $\phi = (\phi^+, \phi^0)^T$ are used in all three models, and interact in the following ways:

The type I see-saw mechanism, also known as the canonical see-saw mechanism, introduces two or more right-handed neutrinos, N_{Ra} . In this type, the lepton doublet and the Higgs doublet couple to form a fermion singlet as can be seen in figure 5.1[51].

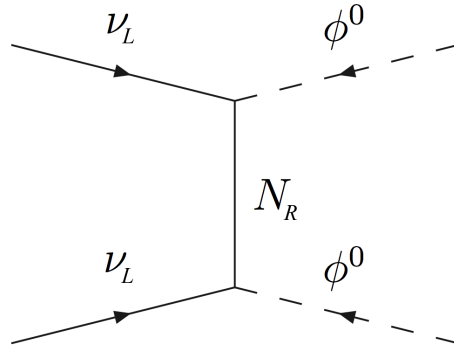


Figure 5.1: Type I see-saw mechanism [51].

In the type II see-saw mechanism, a heavy Higgs triplet is introduced, $\xi = (\xi^{++}, \xi^+, \xi^0)$. Two lepton doublets form in this case the scalar triplet as can be seen in figure 5.2.

Finally, in the type III see-saw mechanism, a heavy Majorana fermion triplet Σ is introduced.

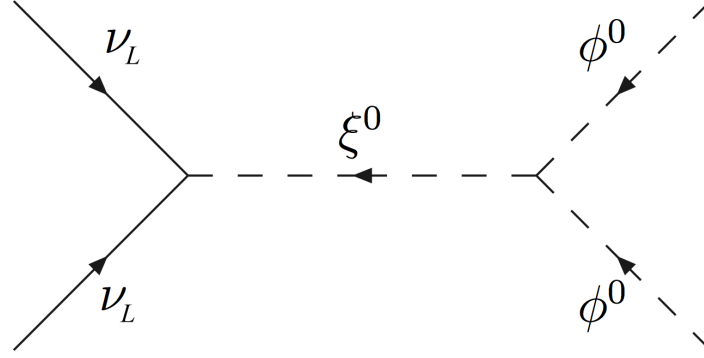


Figure 5.2: Type II see-saw mechanism [51].

This is formed by coupling the lepton doublet and the Higgs doublet, where the figure is the same as figure 5.1 with N replaced by Σ^0 .

These three models all explain the smallness of neutrinos in a natural way. However, the heavy fields introduced are of the order of the cutoff scale. This means that it will be difficult to detect them in future experiments. In the following, the models will be investigated in greater detail. The type I model will furthermore be used in the study of neutrino mixing, implementing the Frobenius group T_{13} .

5.1 Type I see-saw model

The canonical see-saw mechanism is currently the most investigated mechanism for understanding neutrino masses [52–56]. The effective dimension 5 operator is, as mentioned, introduced to give masses to the neutrinos,

$$\mathcal{O}_5 = \frac{1}{M}(\phi\tau_2\psi)^T C(\phi\tau_2\psi), \quad (5.2)$$

with M the cutoff scale. This operator is produced, for example, when integrating out the very heavy right-handed neutrinos, N_{Ra} , which are singlets under $SU(2) \times U(1)$. The Lagrangian from which this operator is extracted consists of a coupling between these, the left-handed neutrinos and the Standard Model Higgs doublet, as well as a Majorana mass term for the right-handed neutrinos,

$$\mathcal{L}^\nu = \bar{\psi}\lambda_\phi\phi N_R + \frac{1}{2}N_R^T C M_R N_R + \text{h.c.} \quad (5.3)$$

When the Higgs doublet ϕ acquires a vacuum expectation value, v , under spontaneous symmetry breaking, this leads to a Dirac neutrino mass, $M_D = \lambda_\phi v$. The mass matrix M_D could therefore be expected to be at the same order of magnitude as M_q and M_l , the masses of quarks and charged leptons.

A Yukawa coupling term is allowed between right-handed neutrinos and a singlet scalar field, χ . The mass matrix of the right-handed neutrinos, M_R , therefore consists of a bare mass term M and a mass term from the Yukawa coupling, $\lambda_\chi\langle\chi\rangle$ when the scalar singlet acquires a vacuum expectation value. The Majorana term is more explicitly given by

$$\frac{1}{2}N_R^T C M_R N_R = \frac{1}{2}N_R^T C M N_R + \frac{1}{2}\lambda_\chi N_R^T C N_R \chi. \quad (5.4)$$

One way of understanding the type I see-saw mechanism is to use the fact that the masses of the right-handed neutrinos are assumed to be very large. These Majorana masses are not related to the electroweak symmetry breaking and there is no symmetry protecting them, so they are naturally of the order of the cut-off scale [11]. Therefore, it is appropriate to integrate the massive right-handed neutrinos out at zero momentum,

$$\frac{\partial \mathcal{L}^\nu}{\partial N_R} = \lambda_\phi(\phi\psi) + M_R N_R = 0 \leftrightarrow N_R = -M_R^{-1}\lambda_\phi(\phi\psi). \quad (5.5)$$

The Lagrangian becomes

$$\begin{aligned} \mathcal{L}^\nu &= -(M_R^{-1}\lambda_\phi\phi\psi)^T \lambda_\phi(\phi\psi) + \frac{1}{2}(M_R^{-1}\lambda_\phi\phi\psi)^T M_R M_R^{-1}\lambda_\phi(\phi\psi) \\ &= -\frac{1}{2}(\phi\psi)^T \lambda_\phi^T M_R^{-1}\lambda_\phi(\phi\psi), \end{aligned} \quad (5.7)$$

and using $\frac{2\lambda}{M} \equiv -\lambda_\phi^T M_R^{-1}\lambda_\phi$,

$$\mathcal{L}^\nu = (\phi\psi)^T \frac{\lambda}{M}(\phi\psi) = \mathcal{O}_5. \quad (5.8)$$

Thus, the result is a dimension 5 operator, leading to Majorana masses for neutrinos. The M in the denominator of the dimension 5 operator is now the mass matrix of the very heavy right-handed neutrinos, N_{Ra} . At least two of these are needed to explain the mass squared differences found experimentally.

We see that the neutrino mass matrix is given by

$$\mathcal{M}_\nu = -v^2 \lambda_\phi^T M_R^{-1} \lambda_\phi = -M_D^T M_R^{-1} M_D. \quad (5.9)$$

This is the type I see-saw result for the light neutrino masses.

A different way of understanding the see-saw mechanism is by diagonalizing the mass matrix. The Lagrangian (5.3) can be written in the form,

$$\mathcal{L}^\nu \rightarrow \frac{1}{2}(\nu_L^T C \lambda_\phi v N_R + N_R^T C \lambda_\phi^T v \nu_L + N_R^T C M_R N_R) \quad (5.10)$$

$$= \frac{1}{2} \begin{pmatrix} \nu_L^T & N_R^T \end{pmatrix} C \begin{pmatrix} 0 & \lambda_\phi v \\ \lambda_\phi^T v & M_R \end{pmatrix} \begin{pmatrix} \nu_L \\ N_R \end{pmatrix}. \quad (5.11)$$

Diagonalizing the mass matrix

$$\mathcal{M}_\nu = \begin{pmatrix} 0 & \lambda_\phi v \\ \lambda_\phi^T v & M_R \end{pmatrix} = \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix}, \quad (5.12)$$

the eigenvalues can be found

$$\lambda_\pm = \frac{M_R \pm \sqrt{M_R^2 + 4M_D^2}}{2}. \quad (5.13)$$

5.2. Type II see-saw model

When $M_R \gg v$, these are to second order in v given by

$$\lambda_+ \simeq M_R, \quad \lambda_- \simeq -\frac{M_D^2}{M_R}, \quad (5.14)$$

corresponding to the masses of the heavy and light neutrinos respectively. If one of these eigenvalues increases, the other decreases and vice versa. This is what is understood by the name "see-saw".

Therefore, the term of the Lagrangian giving mass to the light neutrinos is given by

$$\mathcal{L}^\nu = -\psi^T \phi_1^T C \left(\lambda_\phi \frac{1}{M_R} \lambda_\phi^T \right) \phi_2 \psi. \quad (5.15)$$

Comparing this with the dimension 5 operator, we have that these are the same if

$$\frac{1}{M} \rightarrow -\lambda_\phi \frac{1}{M_R} \lambda_\phi^T, \quad (5.16)$$

as before.

5.2 Type II see-saw model

In the type II see-saw model, a triplet scalar field, ξ , is introduced,

$$\xi = \begin{pmatrix} \xi^+ & \sqrt{2}\xi^{++} \\ \sqrt{2}\xi^0 & -\xi^+ \end{pmatrix}, \quad (5.17)$$

which is (1,3,2) under $SU(3) \times SU(2) \times U(1)$. The light neutrinos gain masses when the neutral component of this triplet acquires a vacuum expectation value, $\langle \xi^0 \rangle = u$.

The Yukawa coupling between the leptons and the Higgs triplet looks like

$$\psi^T C \xi \psi. \quad (5.18)$$

This means that for u very small, there exists a very small neutrino mass term, $\sim \langle u \rangle$ from

$$u \nu_L^T C \nu_L. \quad (5.19)$$

5.3 Type III see-saw model

The heavy neutrinos are in this model contained in the leptonic triplet representation. A heavy triplet fermion, $\Sigma = (\Sigma^+, \Sigma^0, \Sigma^-)$, (1,3,0) participates in a Yukawa coupling with the left-handed neutrino ψ and the Higgs doublet ϕ . Majorana masses for the leptonic triplets are now possible. The heavy mass is in this model given by the mass of the heavy fermion triplet. The neutrino mass terms in the Lagrangian becomes

$$\bar{\psi} \phi l_R + \lambda_\phi \bar{\psi} \Sigma \tau_2 \phi - \frac{1}{2} M_\Sigma \Sigma \Sigma + \text{h.c.}, \quad (5.20)$$

where the lepton triplet is given by

$$\Sigma = \begin{pmatrix} \Sigma^0/\sqrt{2} & \Sigma^+ \\ \Sigma^- & -\Sigma^0/\sqrt{2} \end{pmatrix}. \quad (5.21)$$

The neutrino mass matrix therefore has the same form as in type I

$$M_\nu = \begin{pmatrix} 0 & M_D \\ M_D^T & M_\Sigma \end{pmatrix}, \quad (5.22)$$

in the $(\nu_L, \Sigma_R)^T$ basis. Diagonalizing this matrix, the eigenvalues can be found in a similar way as in type I, so that the neutrino mass matrix becomes

$$M_\nu = -M_D^T M_\Sigma^{-1} M_D. \quad (5.23)$$

The charged lepton matrix also gets extended in this model from the newly added lepton triplets, so that

$$M_E = \begin{pmatrix} M_l & M_D \\ 0 & M_\Sigma \end{pmatrix}, \quad (5.24)$$

in the $(e_R, \Sigma_R^-)^T$ basis.

Some other see-saw mechanisms have been addressed in the literature. The linear and inverse see-saw mechanisms have been introduced to lower the scale of the heavy fields. This is done by further extending the model with gauge singlets, leading to contributions in the numerator of the neutrino mass matrix. If these are small enough, the scale of the heavy fields can be lowered as well.

5.4 The mixing matrix

The neutrino mass matrix results from integrating out the heavy right-handed neutrinos as mentioned. The Lagrangian containing the Majorana mass term for the right-handed neutrinos and the Dirac mass term coupling the left- and right-handed neutrinos can be diagonalized in the same way as for charged leptons and left-handed neutrinos,

$$U_{DL}^\dagger M_D U_{DR} \equiv D_D = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}_D, \quad (5.25)$$

$$U_R^T M_R U_R \equiv D_R = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}_R. \quad (5.26)$$

This means that the neutrino mass matrix can be written up in terms of these unitary matrices and the diagonal Dirac and right-handed mass matrices

$$M_\nu = -M_D^T M_R^{-1} M_D = -U_{DR}^* D_D U_{DL}^T U_R D_R^{-1} U_R^T U_{DL} D_D U_{DR}^\dagger, \quad (5.27)$$

which is then diagonalized as previously, to obtain the physical neutrino masses.

5.5 The see-saw mechanism and the Frobenius group T_{13}

The canonical see-saw mechanism has previously been applied with a non-abelian symmetry group to neutrino mixing [57, 58].

The compatibility between the Frobenius group T_{13} and the canonical see-saw mechanism with neutrino mixing will be examined in the following. The Standard Model is therefore extended minimally, by introducing a family symmetry and three right-handed neutrinos. To fit experiments and place constraints on the possibilities, tribimaximal mixing is used as a guideline. The application of both a family symmetry group and the canonical see-saw mechanism naturally generates small neutrino masses and produces enough constraints to narrow down the parameters of the neutrino mass matrix to two. This is therefore a predictive model where the physical neutrino masses are found. Values are suggested for the modulus of the m_{ee} parameter of the neutrino mass matrix which is proportional to the neutrinoless double β decay rate. The validity of this model can therefore be tested in the near future.

The lepton doublet fields ψ_a ($a = 1, 2, 3$), the right-handed lepton singlet fields l_{Ra} , the right-handed neutrino singlet fields N_{Ra} and the scalar fields are assigned to various representations of T_{13} in the same way as we did previously. The invariants in the Lagrangian are found. After the scalar fields are allowed to acquire their vacuum expectation values, the resulting right-handed neutrino mass matrix M_R , the charged lepton mass matrix M_l , the Dirac mass matrix M_D and the neutrino mass matrix M_ν are diagonalized, whereby the mixing matrix is obtained as we did in section 4.1.

The Lagrangian terms of importance are

$$\mathcal{L}^\nu = \frac{1}{2} M N_R^T C N_R + \frac{1}{2} \lambda_\chi N_R^T C N_R \chi + \frac{1}{2} \lambda_\xi N_R^T C N_R \xi + \lambda_\phi \bar{\psi} \phi N_R + \bar{\psi} \phi l_R, \quad (5.28)$$

where an extra singlet, ξ , has been introduced.¹ The right-handed neutrinos are assigned to 3 dimensional representations.² Looking at the Clebsch-Gordon decompositions, group theory excludes the plain mass term $M N_R^T C N_R$ for this choice. In table 5.1 the possible representations of the right-handed neutrinos and the scalar singlets χ and ξ in the terms $\lambda_\chi N_R^T C N_R \chi$ and $\lambda_\xi N_R^T C N_R \xi$ are listed.

Case	N_R	χ	ξ
1	3₁	3₁	3₂
2	3₂	3₂	3₁

Table 5.1: Possible representations for the three fields in the right-handed neutrino mass term.

When the family symmetry is broken, the scalar singlets acquire the vacuum expectation values $\langle \chi \rangle$ and $\langle \xi \rangle$. The right-handed neutrino mass matrices are then given by

$$\text{Case 1 : } M_R = \begin{pmatrix} x \langle \xi_1 \rangle & y \langle \chi_3 \rangle & y \langle \chi_2 \rangle \\ y \langle \chi_3 \rangle & x \langle \xi_2 \rangle & y \langle \chi_1 \rangle \\ y \langle \chi_2 \rangle & y \langle \chi_1 \rangle & x \langle \xi_3 \rangle \end{pmatrix}, \quad (5.29)$$

¹A single scalar singlet, χ , is not sufficient to give solutions to this coupling between the Frobenius group and the canonical see-saw mechanism.

²An analysis has been carried out showing no solutions when placing the N_R 's in the 1 dimensional representations, when fitting with tribimaximal mixing.

$$\text{Case 2 : } M_R = \begin{pmatrix} x\langle\xi_3\rangle & y\langle\chi_3\rangle & y\langle\chi_2\rangle \\ y\langle\chi_3\rangle & x\langle\xi_1\rangle & y\langle\chi_1\rangle \\ y\langle\chi_2\rangle & y\langle\chi_1\rangle & x\langle\xi_2\rangle \end{pmatrix}, \quad (5.30)$$

where x and y are coupling constants. The inverse right-handed neutrino mass matrices are then given by

Case 1:

$$M_R^{-1} = \frac{1}{\text{Det}(M_R)} \begin{pmatrix} -y^2\langle\chi_1\rangle^2 + x^2\langle\xi_2\rangle\langle\xi_3\rangle & y^2\langle\chi_1\rangle\langle\chi_2\rangle - xy\langle\chi_3\rangle\langle\xi_3\rangle & y^2\langle\chi_1\rangle\langle\chi_3\rangle - xy\langle\chi_2\rangle\langle\xi_2\rangle \\ y^2\langle\chi_1\rangle\langle\chi_2\rangle - xy\langle\chi_3\rangle\langle\xi_3\rangle & -y^2\langle\chi_2\rangle^2 + x^2\langle\xi_1\rangle\langle\xi_3\rangle & y^2\langle\chi_2\rangle\langle\chi_3\rangle - xy\langle\chi_1\rangle\langle\xi_1\rangle \\ y^2\langle\chi_1\rangle\langle\chi_3\rangle - xy\langle\chi_2\rangle\langle\xi_2\rangle & y^2\langle\chi_2\rangle\langle\chi_3\rangle - xy\langle\chi_1\rangle\langle\xi_1\rangle & -y^2\langle\chi_3\rangle^2 + x^2\langle\xi_1\rangle\langle\xi_2\rangle \end{pmatrix}, \quad (5.31)$$

Case 2:

$$M_R^{-1} = \frac{1}{\text{Det}(M_R)} \begin{pmatrix} -y^2\langle\chi_1\rangle^2 + x^2\langle\xi_1\rangle\langle\xi_2\rangle & y^2\langle\chi_1\rangle\langle\chi_2\rangle - xy\langle\chi_3\rangle\langle\xi_2\rangle & y^2\langle\chi_1\rangle\langle\chi_3\rangle - xy\langle\chi_2\rangle\langle\xi_1\rangle \\ y^2\langle\chi_1\rangle\langle\chi_2\rangle - xy\langle\chi_3\rangle\langle\xi_2\rangle & -y^2\langle\chi_2\rangle^2 + x^2\langle\xi_3\rangle\langle\xi_2\rangle & y^2\langle\chi_2\rangle\langle\chi_3\rangle - xy\langle\chi_1\rangle\langle\xi_3\rangle \\ y^2\langle\chi_1\rangle\langle\chi_3\rangle - xy\langle\chi_2\rangle\langle\xi_1\rangle & y^2\langle\chi_2\rangle\langle\chi_3\rangle - xy\langle\chi_1\rangle\langle\xi_3\rangle & -y^2\langle\chi_3\rangle^2 + x^2\langle\xi_3\rangle\langle\xi_1\rangle \end{pmatrix}. \quad (5.32)$$

As seen from (5.29) and (5.30) the only difference lies in the vacuum alignment of the ξ .

In the dimension 4 operator, the assignments of the fields to the irreducible representations of T_{13} have already been investigated in section 4.1. A summary of the representations in the six cases that led to useful solutions and the charged lepton mixing matrices, $U_L^\dagger$, corresponding to these six cases can be seen in table 5.2.

Case	$\bar{\psi}$	ϕ	l_R	$U_L^\dagger$
A	$\mathbf{3}_1$ $\mathbf{3}_2$ $\mathbf{3}_1$ $\mathbf{3}_2$	$\mathbf{3}_1$ $\mathbf{3}_2$ $\bar{\mathbf{3}}_1$ $\bar{\mathbf{3}}_2$	$\mathbf{3}_1$ $\mathbf{3}_2$ $1, 1', \bar{1}'$ $1, 1', \bar{1}'$	$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & 1 & \omega^2 \\ \omega^2 & 1 & \omega \end{pmatrix}$
B	$\mathbf{3}_1$ $\mathbf{3}_2$	$1, 1', \bar{1}'$ $1, 1', \bar{1}'$	$\bar{\mathbf{3}}_1$ $\bar{\mathbf{3}}_2$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Table 5.2: Possible representations for the three fields in the dimension 4 operator, and the corresponding charged lepton mixing matrix.

When using tribimaximal mixing as a guideline, the neutrino mixing matrix U_ν can be found from the charged lepton mixing matrix $U_L^\dagger$ through the relation

$$U_{TB} = U_L^\dagger U_\nu. \quad (5.33)$$

The neutrino mixing matrices must therefore in cases A and B of table 5.2 be given by

$$U_{\nu A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad (5.34)$$

and

$$U_{\nu B} = U_{TB} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad (5.35)$$

respectively, corresponding to the two possible charged lepton mixing matrices, in order to get tribimaximal mixing. The corresponding mass matrices have the shapes

$$M_{\nu A} \sim \begin{pmatrix} \alpha & 0 & \beta \\ 0 & \gamma & 0 \\ \beta & 0 & \alpha \end{pmatrix}, \quad (5.36)$$

and

$$M_{\nu B} \sim \begin{pmatrix} \gamma & \beta & \beta \\ \beta & \alpha & \delta \\ \beta & \delta & \alpha \end{pmatrix}. \quad (5.37)$$

The last matrix needs to satisfy the condition $\delta = \gamma + \beta - \alpha$ to result in tribimaximal mixing.

The Dirac mass matrix can acquire three forms when aiming at tribimaximal mixing, namely

$$M_D \sim I, \quad M_D \sim \begin{pmatrix} 0 & X & 0 \\ 0 & 0 & X \\ X & 0 & 0 \end{pmatrix}, \quad M_D \sim \begin{pmatrix} 0 & 0 & X \\ X & 0 & 0 \\ 0 & X & 0 \end{pmatrix}. \quad (5.38)$$

The Higgs doublet ϕ is aligned in the (1,1,1) direction, when placed in a 3 dimensional representation, to acquire non-degenerate masses for the charged leptons in the dimension 4 operator. The Dirac mass matrices will lead to the same neutrino mass matrices, since the last two will only perturb the vacuum alignments in the case where the Dirac mass matrix is diagonal. The shapes can be obtained by choosing representations for $\bar{\psi}$ and ϕ in the term $\lambda_\phi \bar{\psi} N_R \phi$ in various ways. The six possibilities from the dimension 4 operator need to be taken into consideration when making these choices. It turns out that only cases where the Higgs doublet is placed in a 3 dimensional representation are possible. Therefore, case B of table 5.2 can be excluded. All the possible ways of placing the various fields in different representations are assembled in table 5.3.

M_D	N_R	χ	ξ	$\bar{\psi}$	ϕ
Diagonal	$\mathbf{3}_2$	$\mathbf{3}_2$	$\mathbf{3}_1$	$\mathbf{3}_1$	$\bar{\mathbf{3}}_1$
	$\mathbf{3}_2$	$\mathbf{3}_2$	$\mathbf{3}_1$	$\bar{\mathbf{3}}_1$	$\bar{\mathbf{3}}_1$
Off-diagonal	$\mathbf{3}_1$	$\mathbf{3}_1$	$\bar{\mathbf{3}}_2$	$\mathbf{3}_2$	$\bar{\mathbf{3}}_2$
	$\mathbf{3}_1$	$\mathbf{3}_1$	$\bar{\mathbf{3}}_2$	$\mathbf{3}_2$	$\mathbf{3}_2$
	$\mathbf{3}_1$	$\mathbf{3}_1$	$\bar{\mathbf{3}}_2$	$\bar{\mathbf{3}}_2$	$\mathbf{3}_2$
	$\mathbf{3}_2$	$\mathbf{3}_2$	$\mathbf{3}_1$	$\bar{\mathbf{3}}_1$	$\mathbf{3}_1$

Table 5.3: Possible representations for the five fields and the corresponding form of the Dirac mass matrix.

In the two possibilities where M_D is diagonal, the neutrino mass matrix will have the same structure as M_R^{-1} . In the four possibilities where M_D is off-diagonal, the neutrino mass matrix is from (5.9) given by

$$M_\nu \sim \langle \phi \rangle^2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} M_R^{-1} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (5.39)$$

The form of $M_{\nu A}$ can be obtained by choosing the following vacuum expectation values for the neutral component of the two scalar singlets and the Higgs doublet in the diagonal case,

$$\langle \phi \rangle = v \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \chi \rangle = u_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \langle \xi \rangle = u_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad (5.40)$$

and in the off-diagonal case,

$$\langle \phi \rangle = v \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \chi \rangle = u_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \langle \xi \rangle = u_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad (5.41)$$

After symmetry breaking with these vacuum alignments, the neutrino mass matrix will in both cases be given by

$$M_\nu = -M_D^T M_R^{-1} M_D = \frac{1}{x^3 u_2^3 - x y^2 u_2 u_1^2} \begin{pmatrix} x^2 u_2^2 & 0 & -x y u_1 u_2 \\ 0 & -y^2 u_1^2 + x^2 u_2^2 & 0 \\ -x y u_1 u_2 & 0 & x^2 u_2^2 \end{pmatrix}, \quad (5.42)$$

which has the desired shape of (5.36). Note, however, the constraint $\gamma = \alpha - \frac{\beta^2}{\alpha}$, so that the form of the mass matrix is now more constrained:

$$M_{\nu A} \sim \begin{pmatrix} \alpha & 0 & \beta \\ 0 & \alpha - \frac{\beta^2}{\alpha} & 0 \\ \beta & 0 & \alpha \end{pmatrix}. \quad (5.43)$$

The possible choices for assigning representations to the various fields have now been exhausted leading to only a single possible neutrino mass matrix. The result can now be explored.

5.5.1 Predictions

The result of this paper is that the neutrino mass matrix can have a single shape that is highly constrained, and depends on only two parameters:

$$M_{\nu A} \sim \begin{pmatrix} \alpha & 0 & \beta \\ 0 & \alpha - \frac{\beta^2}{\alpha} & 0 \\ \beta & 0 & \alpha \end{pmatrix}. \quad (5.44)$$

Taking into account the experimental knowledge about neutrino masses, the model allows some predictions. The eigenvalues of the mass matrix are given by

$$m_1 = \alpha + \beta, \quad m_2 = \alpha - \frac{\beta^2}{\alpha}, \quad m_3 = \alpha - \beta. \quad (5.45)$$

The mass squared differences can be found in terms of α and β :

$$\Delta m_{12}^2 = |m_2|^2 - |m_1|^2 = \left|\alpha - \frac{\beta^2}{\alpha}\right|^2 - |\alpha + \beta|^2, \quad (5.46)$$

$$\Delta m_{23}^2 = |m_3|^2 - |m_2|^2 = |\alpha - \beta|^2 - \left|\alpha - \frac{\beta^2}{\alpha}\right|^2. \quad (5.47)$$

At best fit, the experimental values are

$$\Delta m_{12}^2 = 7.59 \pm 0.23 \cdot 10^{-5} eV^2, \quad (5.48)$$

$$|\Delta m_{32}^2| = 2.40 \pm 0.12 \cdot 10^{-3} eV^2. \quad (5.49)$$

The two equations (5.46) and (5.47) can be solved for α and β . In the normal hierarchy, $m_3 \gg m_1 \simeq m_2$. In the inverted hierarchy, $m_3 \ll m_1 \sim m_2$.

Since this paper assumes no CP violation, using tribimaximal mixing as a first order approximation, α and β must also be real to first order. The solution for these as well as the neutrino mass eigenvalues can be seen in table 5.4.³

Hierarchy	Normal		Inverted
$\alpha [10^{-2} eV]$	-2.80 ± 0.07	-2.28 ± 0.06	1.70 ± 0.05
$\beta [10^{-2} eV]$	2.21 ± 0.06	2.72 ± 0.07	3.42 ± 0.08
$m_1 [10^{-2} eV]$	-0.59 ± 0.01	0.45 ± 0.01	5.12 ± 0.13
$m_2 [10^{-2} eV]$	-1.05 ± 0.02	0.98 ± 0.01	-5.19 ± 0.13
$m_3 [10^{-2} eV]$	-5.01 ± 0.13	-5.00 ± 0.13	-1.72 ± 0.05

Table 5.4: Predictions for neutrino mass eigenvalues

From these values, the possible regime for neutrinoless double β decay can be explored. The rate of this decay is proportional to the modulus of the first entry of the effective neutrino mass matrix, given by

$$|m_{ee}| = |m_1 U_{11}^2 + m_2 U_{12}^2 + m_3 U_{13}^2|. \quad (5.50)$$

The (ee) entry of the neutrino mass matrix with tribimaximal mixing and CP conservation is then given by

$$|m_{ee}| = \frac{1}{3} |2m_1 + m_2|. \quad (5.51)$$

The absolute value corresponding to the 3 possible values of the neutrino masses can be seen in table 5.5. The upper limit for this entry is today given by $0.38 eV$. These predictions are well below this limit. Furthermore, this model can be tested in the near future, since upcoming neutrinoless double β decay experiments will have sensitivities down to the order of $10^{-2} eV$ [59, 60]. Therefore, this model could soon be confronted with experiment.

³In order to acquire positive masses for these Majorana neutrinos, it is possible to redefine the Majorana fields $\chi_n = \Sigma_a (U_{na} \nu_{La} + \eta_n U_{na} \nu_{Ra}^c)$, where $\eta_n = \pm 1$ are CP eigenvalues, U_{na} is the orthogonal matrix that diagonalizes the real neutrino mass matrix and we have $(CP) \nu_{La} (CP)^{-1} = \nu_{Ra}^c$. Therefore, $UM_\nu U^T = \eta_n m_n$, and the masses m_n are positive [36]. The solutions that differ only by a sign can therefore be omitted.

Hierarchy	$ m_{ee} [10^{-2}eV]$
Normal	0.74 ± 0.01
	0.62 ± 0.01
Inverted	1.68 ± 0.05

Table 5.5: Predictions for the modulus of the (ee) entry of the neutrino mass matrix.

If these results survive future experiments, e.g. neutrinoless double β decay, the model of combining the Frobenius group T_{13} with the canonical see-saw mechanism can be seen as a good candidate to predict neutrino masses.

Radiative loop corrections

6.1 The Zee Model

The dimension 5 operator can also originate from radiative loop corrections. In the Zee model, the electroweak sector of the Standard Model is extended with an extra charged scalar field, which transforms as a singlet under $SU(2)$ and couples to the lepton and Higgs doublets [61]. To preserve gauge invariance, h^+ can not acquire a vacuum expectation value, only Higgs doublets ϕ break symmetry spontaneously. The Majorana neutrinos can thus couple to this newly introduced scalar field and generate the dimension 5 operator. The new charged scalar field has lepton number 2. Lepton number is therefore broken in the coupling with the Higgs doublets.

A radiative contribution as seen in figure 6.1, generates the neutrino mass matrix.

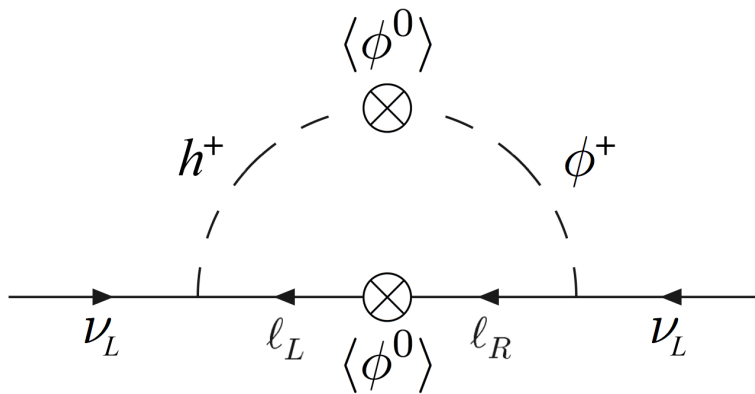


Figure 6.1: Zee model

The coupling between two left handed doublets and the charged Higgs singlet gives the following contribution to the Lagrangian:

$$f^{ac}(\psi_{aL}^i C \psi_{cL}^j) \epsilon_{ij} h + \text{h.c.}, \quad (6.1)$$

where f^{ac} is antisymmetric due to fermi statistics. The coupling between the right handed charged lepton and the left handed doublet leads to the term

$$f_\phi^{\beta db} \bar{l}_{dR} \phi_\beta^i \psi_{bL}^j \epsilon_{ij} + \text{h.c.} \quad (6.2)$$

This contributes to a mass term for the charged leptons:

$$f_\phi^{\beta cd} \bar{l}_{dR} \phi_\beta^0 l_{cL} + \text{h.c.} \quad (6.3)$$

When the Higgs doublet acquires a vacuum expectation value, the charged leptons acquire the masses

$$m_{cd} = \Sigma_\beta f_\phi^{\beta cd} v_\beta, \quad (6.4)$$

with $\langle \phi_\beta^0 \rangle = v_\beta$.

The same term contributes to a Yukawa coupling between the right handed lepton and the neutrino,

$$f_\phi^{\beta db} \bar{l}_{dR} \phi_\beta^+ \nu_{bL} + \text{h.c.} \quad (6.5)$$

The lepton number violating term is located in the scalar potential, where h^+ with lepton number 2 couples to two Higgs doublets,

$$M^{\alpha\beta} \phi_\alpha \phi_\beta h. \quad (6.6)$$

Since $M^{\alpha\beta}$ is antisymmetric, more than one Higgs doublet is needed. We can find the amplitude of the diagram in the rest frame of the incoming neutrino. All the propagators around the loop will then have the same momentum. We can also assume that the lepton mass is negligible, compared to the rest of the parameters.

The contribution to the neutrino mass from this loop is calculated in Appendix A. The result is given by

$$M_{ab} = \frac{1}{16\pi^2} f^{ac} m_{cd} f_\phi^{\beta db} v_\alpha M_{\alpha\beta} \frac{1}{m_h^2} \left[\ln \left[\frac{m_\phi^2}{m_h^2} \right] \right]. \quad (6.7)$$

Since the terms are off-diagonal, the neutrino mass matrix looks like

$$\begin{pmatrix} 0 & m_{e\mu} & m_{e\tau} \\ m_{e\mu} & 0 & m_{\mu\tau} \\ m_{e\tau} & m_{\mu\tau} & 0 \end{pmatrix}, \quad (6.8)$$

which is traceless. This requires

$$m_1 = -m_2 - m_3, \quad (6.9)$$

and therefore one of the neutrino masses must have opposite CP parity from the others. This also means, that no neutrinoless double beta decay will be observed, if this model holds, since the (ee) entry of the mass matrix is zero.

6.1.1 The General Zee Model (GZM)

In the general Zee model, both Higgs doublets couple to the charged leptons. The charged gauge singlet mixes with the charged components of the Higgs doublets ϕ_1, ϕ_2 , creating the Higgs eigenstates, $H_i^\pm, i = 1, 2$. Since both of the Higgs doublets can couple to leptons, both will acquire a vacuum expectation value, so that $\langle \phi_i \rangle = \frac{1}{\sqrt{2}} v_i$. The total magnitude of the two vacuum expectation values must then be $v = \sqrt{v_1^2 + v_2^2} \simeq 246 \text{ GeV}$, the known weak scale, if the coupling constant proportional to the Higgs mass is approximately one. The ratio of these two vacuum expectation values can then be parametrized as

$$\tan \beta = \frac{v_1}{v_2}, \quad (6.10)$$

since the Higgs potential is invariant under a global $\text{SO}(2)$ transformation of the Higgs doublets,

$$\begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} = \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad (6.11)$$

and the Yukawa couplings, so

$$\begin{pmatrix} f_\phi'^1 \\ f_\phi'^2 \end{pmatrix} = \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} f_\phi^1 \\ f_\phi^2 \end{pmatrix}, \quad (6.12)$$

where the couplings here are matrices [62]. We are now in a position to redefine the doublet scalars by making this arbitrary $\text{SO}(2)$ transformation. We choose the transformation that makes only one of the Higgs doublets acquire a vacuum expectation value, so that $\tan \beta = \frac{v'_1}{v'_2} = 0$. This is known as the Higgs basis [62],

$$\langle \phi_1'^0 \rangle = 0, \quad \langle \phi_2'^0 \rangle = \frac{1}{\sqrt{2}} v = \sqrt{\frac{v_1^2 + v_2^2}{2}}. \quad (6.13)$$

From this, the $\phi_2'^\pm$ is found to be the Nambu-Goldstone boson. The physically charged fields are therefore h^+ and $\phi_1'^\pm$. From the chosen vacuum expectation value, the lepton violating term becomes, omitting the primes,

$$M^{12} \phi_1^i \phi_2^j h \epsilon_{ij} \rightarrow M^{12} \frac{v}{\sqrt{2}} \phi_1^+ h^-. \quad (6.14)$$

Together with the mass term for the two fields respectively in the scalar potential, this constructs a mass matrix for these physically charged fields, which can be diagonalized to find their masses,

$$\mathcal{L}_{Higgs} = \begin{pmatrix} h^+ & \phi_1^+ \end{pmatrix} \begin{pmatrix} m_h^2 & \frac{M^{12}v}{\sqrt{2}} \\ \frac{M^{12}v}{\sqrt{2}} & (M_1^\pm)^2 \end{pmatrix} \begin{pmatrix} h^- & \phi_1^- \end{pmatrix} \quad (6.15)$$

$$= \begin{pmatrix} H_1^+ & H_2^+ \end{pmatrix} \begin{pmatrix} M_1^2 & 0 \\ 0 & M_2^2 \end{pmatrix} \begin{pmatrix} H_1^- \\ H_2^- \end{pmatrix}, \quad (6.16)$$

where H_i are the mass eigenstates of the physical fields and M_i are the physical masses. The mass eigenstates can be written as a linear combination of the physical states [63],

$$\begin{pmatrix} H_1^- \\ H_2^- \end{pmatrix} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} h^- \\ \phi_1^- \end{pmatrix} \quad (6.17)$$

With

$$U_{\phi h}^\dagger = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}, \quad (6.18)$$

where

$$\tan(2\alpha) = -\frac{\sqrt{2}M^{12}v}{(M_1^\pm)^2 - m_h^2}, \quad (6.19)$$

and

$$M_{1,2}^2 = \frac{m_h^2 + (M_1^\pm)^2 \pm \sqrt{[m_h^2 - (M_1^\pm)^2]^2 + 2(M^{12})^2 v^2}}{2}. \quad (6.20)$$

When the Higgs doublets acquire the mentioned vacuum expectation values, the terms where these couple to other leptons become

$$\mathcal{L}_{SM}^{Yukawa} = \bar{\psi}_a f_\phi^1 \phi_{1,a} l_R + \bar{\psi}_a f_\phi^2 \phi_{2,a} l_R + h.c. \rightarrow \bar{\nu}_L f_\phi^1 \phi_1^+ l_R + \bar{l}_L f_\phi^2 \frac{v}{\sqrt{2}} l_R + h.c., \quad (6.21)$$

where a denotes the SU(2) index. The unphysical Nambu-Goldstone Boson ϕ_2^+ has been absorbed by the gauge bosons W^+ and Z^0 . We now have

$$\mathcal{L} = M^{12} \frac{v}{\sqrt{2}} \phi_1^+ h^- + \bar{\nu}_L f_\phi^1 \phi_1^+ l_R + \bar{l}_L f_\phi^2 \frac{v}{\sqrt{2}} l_R. \quad (6.22)$$

These can now be written in terms of the physical mass eigenstates. Since

$$\begin{pmatrix} h^- \\ \phi_1^- \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} H_1^- \\ H_2^- \end{pmatrix}, \quad (6.23)$$

we have

$$\begin{aligned} & M^{12} \frac{v}{\sqrt{2}} \phi_1^+ h^- + \bar{\nu}_L f_\phi^1 \phi_1^+ l_R + \bar{l}_L f_\phi^2 \frac{v}{\sqrt{2}} l_R + f \psi_L C \psi_L h \\ &= M^{12} \frac{v}{\sqrt{2}} (-\sin \alpha H_1^+ + \cos \alpha H_2^+) (\cos \alpha H_1^- + \sin \alpha H_2^-) + \bar{\nu}_L f_\phi^1 (-\sin \alpha H_1^+ + \cos \alpha H_2^+) l_R \\ & \quad + \bar{l}_L f_\phi^2 \frac{v}{\sqrt{2}} l_R + 2\bar{\nu}_L f C l_L (\cos \alpha H_1^- + \sin \alpha H_2^-) + h.c. \end{aligned} \quad (6.26)$$

The term

$$\bar{l}_L f_\phi^2 \frac{v}{\sqrt{2}} l_R, \quad (6.27)$$

will give masses to the charged leptons, obtained from diagonalizing the mass matrix

$$U_L^\dagger f_\phi^2 \frac{v}{\sqrt{2}} U_R = U_L^\dagger M_l U_R = D_l = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}. \quad (6.28)$$

In this basis, when the Higgs doublets acquire the vacuum expectation values mentioned previously, the Majorana neutrino masses are calculated from the two one-loop diagrams seen in figure 6.2. We can now find the neutrino mass matrix in terms of these physical fields which will give us the physical masses of the neutrinos. In order to do this, we again need to solve an integral over the internal propagators of the diagram. This can be seen in Appendix B. The result is given by

$$M_\nu = \frac{i}{4^4\pi^2} \ln\left(\frac{M_2^2}{M_1^2}\right) \sin 2\alpha M_l f f_\phi^1 + \text{h.c.} \quad (6.29)$$

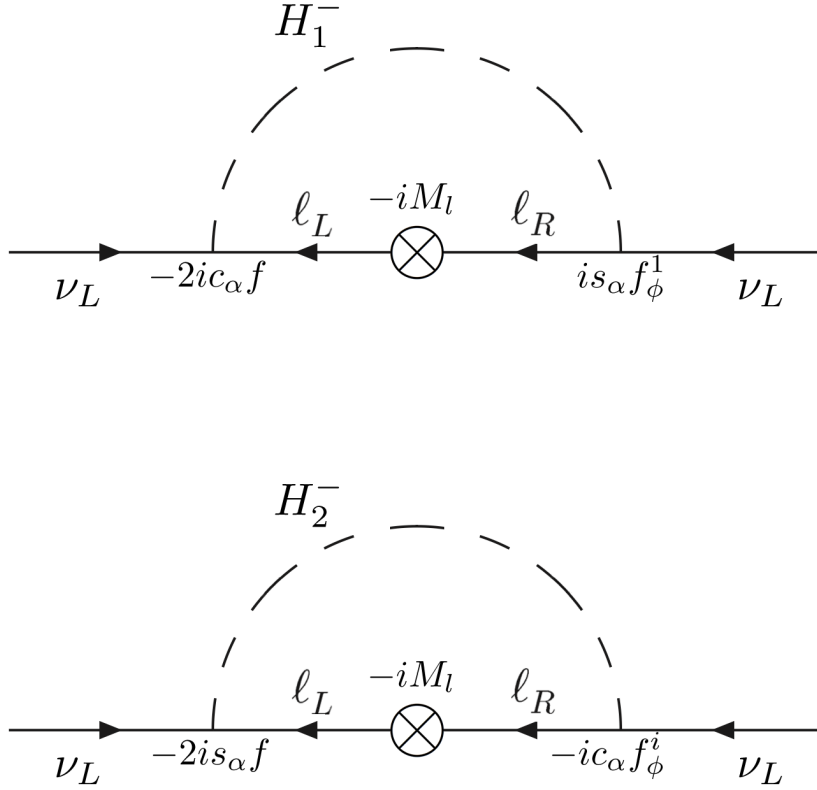


Figure 6.2: Two one-loop diagrams generating the Majorana neutrino masses.

The neutrino masses are then found by diagonalizing this mass matrix.

6.1.2 The Minimal Zee Model (MZM)

We can also assume that only one of the Higgs doublets, ϕ_β , couples to the leptons by imposing a discrete symmetry [64]. This is called the Minimal Zee Model (MZM). In this case, there is no need for the parameterization we just did.

6.2 Applying the Frobenius group T_{13} with the Zee model

The Frobenius group can be applied with the Zee model in two ways. One way is to let the dimension 5 operator be generated only from the radiative loop correction. Another way is to see the radiative loop correction as a perturbation to some already generated dimension 5 operator from a different mechanism, which could be the see-saw mechanism. Tribimaximal mixing can then be used to first order in the other unknown dimension 5 operator, and the perturbation from the radiative loop correction can create a deviation from tribimaximal mixing. Both will be done in the following, using the GZM with the first possibility and the MZM with the second.

6.2.1 The Frobenius group and the GZM

The general Zee model can create the dimension 5 operator to first order. A nonabelian symmetry group can be applied to the general Zee model, placing the fields in various representations as we have been doing to oblivion. The mass matrices are then diagonalized to obtain a basis where all the fields are diagonal. The neutrino mixing and mass matrix can then be found.

Introducing only one Higgs doublet ϕ , placed in a 3 dimensional representation, we can test whether results are possible to obtain TB mixing. We have the fields ψ , l_R , ϕ and h , that enter the terms $\psi l_R \phi$, $\psi \psi h^-$ and $\phi^0 \phi^+ h^-$. For $\psi \psi h^-$ to be antisymmetric, we can only place ψ in a 3 dimensional representation. Due to group theory, the fields ψ and h^- have to be in the same representation. Also, the Higgs doublet and the h^- have to be in the same representations in order for the term $\phi^0 \phi^+ h^-$ to be antisymmetric. Therefore, the only cases that will work for the GZM with only one Higgs doublet are cases 22 and 25 in section 4.1. In case 25, we have $\bar{\psi} \sim \mathbf{\bar{3}}_2$, $\phi \sim \mathbf{\bar{3}}_2$, $l_R \sim \mathbf{1}, \mathbf{1}', \mathbf{\bar{1}}'$ and therefore $h^+ \sim \mathbf{3}_2$ and $h^- \sim \mathbf{\bar{3}}_2$. In this case, the unitary matrix that diagonalizes the charged lepton mass matrix is given by

$$U_L^\dagger = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega^2 & 1 & \omega \\ \omega & 1 & \omega^2 \end{pmatrix}, \quad (6.30)$$

where the vacuum expectation value for ϕ has been chosen to be

$$\langle \phi^0 \rangle = \frac{1}{\sqrt{6}} v \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad (6.31)$$

so that the masses of the charged leptons are given by

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} = \frac{v}{\sqrt{6}} \begin{pmatrix} y_{\phi 1} & 0 & 0 \\ 0 & y_{\phi 2} & 0 \\ 0 & 0 & y_{\phi 3} \end{pmatrix} \quad (6.32)$$

Moving on to the term $M h^- \phi^0 \phi^+$, the invariant in the Lagrangian will be

$$\begin{pmatrix} h_1^- & h_2^- & h_3^- \end{pmatrix} \begin{pmatrix} 0 & \phi_3^0 & -\phi_2^0 \\ -\phi_3^0 & 0 & \phi_1^0 \\ \phi_2^0 & -\phi_1^0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1^+ \\ \phi_2^+ \\ \phi_3^+ \end{pmatrix}. \quad (6.33)$$

This will lead to the mass matrix

$$M_{h\phi} = \frac{v}{\sqrt{6}} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad (6.34)$$

which can be diagonalized by the two unitary matrices U_ϕ and U_h , so that [65]

$$U_h^\dagger M_{h\phi} U_\phi = D_{h\phi}, \quad (6.35)$$

with

$$U_h^\dagger = U_\phi^\dagger = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega^2 & 1 & \omega \\ \omega & 1 & \omega^2 \end{pmatrix}. \quad (6.36)$$

In the basis where this mass matrix is diagonal, we can look at the term $Y_{\phi i} \bar{\psi} l_R \phi_i$ where the neutrino couples to the right handed charged lepton. We can find the couplings in the basis where the charged lepton mass matrix is diagonal [65],

$$\begin{pmatrix} \bar{\psi}_1 & \bar{\psi}_2 & \bar{\psi}_3 \end{pmatrix} \begin{pmatrix} y_{\phi 1} \phi_1^+ & y_{\phi 2} \omega \phi_1^+ & y_{\phi 3} \omega^2 \phi_1^+ \\ y_{\phi 1} \phi_2^+ & y_{\phi 2} \phi_2^+ & y_{\phi 3} \phi_2^+ \\ y_{\phi 1} \phi_3^+ & y_{\phi 2} \omega^2 \phi_3^+ & y_{\phi 3} \omega \phi_3^+ \end{pmatrix} \begin{pmatrix} l_{R1} \\ l_{R2} \\ l_{R3} \end{pmatrix} \quad (6.37)$$

$$= \begin{pmatrix} \bar{\nu}_{L1} & \bar{\nu}_{L2} & \bar{\nu}_{L3} \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega^2 & 1 & \omega \\ \omega & 1 & \omega^2 \end{pmatrix} \begin{pmatrix} y_{\phi 1} \phi_1^+ & y_{\phi 2} \omega \phi_1^+ & y_{\phi 3} \omega^2 \phi_1^+ \\ y_{\phi 1} \phi_2^+ & y_{\phi 2} \phi_2^+ & y_{\phi 3} \phi_2^+ \\ y_{\phi 1} \phi_3^+ & y_{\phi 2} \omega^2 \phi_3^+ & y_{\phi 3} \omega \phi_3^+ \end{pmatrix} \begin{pmatrix} l_{R1} \\ l_{R2} \\ l_{R3} \end{pmatrix}. \quad (6.38)$$

In the basis where the Higgs mass matrix is diagonal, we have

$$U_\phi \begin{pmatrix} P_1^+ \\ P_2^+ \\ P_3^+ \end{pmatrix} = \begin{pmatrix} \phi_1^+ \\ \phi_2^+ \\ \phi_3^+ \end{pmatrix}, \quad (6.39)$$

where P_i^+ are the mass eigenstates. The equation can then be written in terms of these,

$$\begin{pmatrix} \bar{\nu}_{L1} & \bar{\nu}_{L2} & \bar{\nu}_{L3} \end{pmatrix} \begin{pmatrix} y_{\phi 1} P_1^+ & y_{\phi 2} P_3^+ & y_{\phi 3} P_2^+ \\ y_{\phi 1} P_2^+ & y_{\phi 2} P_1^+ & y_{\phi 3} P_3^+ \\ y_{\phi 1} P_3^+ & y_{\phi 2} P_2^+ & y_{\phi 3} P_1^+ \end{pmatrix} \begin{pmatrix} l_{R1} \\ l_{R2} \\ l_{R3} \end{pmatrix} \quad (6.40)$$

$$= \begin{pmatrix} \bar{\nu}_{L1} & \bar{\nu}_{L2} & \bar{\nu}_{L3} \end{pmatrix} \times \quad (6.41)$$

$$\left(P_1^+ \begin{pmatrix} y_{\phi 1} & 0 & 0 \\ 0 & y_{\phi 2} & 0 \\ 0 & 0 & y_{\phi 3} \end{pmatrix} + P_2^+ \begin{pmatrix} 0 & 0 & y_{\phi 3} \\ y_{\phi 1} & 0 & 0 \\ 0 & y_{\phi 2} & 0 \end{pmatrix} + P_3^+ \begin{pmatrix} 0 & y_{\phi 2} & 0 \\ 0 & 0 & y_{\phi 3} \\ y_{\phi 1} & 0 & 0 \end{pmatrix} \right) \quad (6.42)$$

$$\times \begin{pmatrix} l_{R1} \\ l_{R2} \\ l_{R3} \end{pmatrix}, \quad (6.43)$$

so that

$$Y_{\phi 1} = \frac{\sqrt{2}}{v} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}, \quad Y_{\phi 2} = \frac{\sqrt{2}}{v} \begin{pmatrix} 0 & 0 & m_\tau \\ m_e & 0 & 0 \\ 0 & m_\mu & 0 \end{pmatrix},$$

$$Y_{\phi 3} = \frac{\sqrt{2}}{v} \begin{pmatrix} 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \\ m_e & 0 & 0 \end{pmatrix}. \quad (6.45)$$

From the term $F_\alpha \psi C \psi h_\alpha^-$, we get

$$f \begin{pmatrix} \psi_1^c & \psi_2^c & \psi_3^c \end{pmatrix} \begin{pmatrix} 0 & h_3^- & -h_2^- \\ -h_3^- & 0 & h_1^- \\ h_2^- & -h_1^- & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \quad (6.46)$$

$$= 2f \begin{pmatrix} \nu_{L1}^c & \nu_{L2}^c & \nu_{L3}^c \end{pmatrix} U_L^T \begin{pmatrix} 0 & h_3^- & -h_2^- \\ -h_3^- & 0 & h_1^- \\ h_2^- & -h_1^- & 0 \end{pmatrix} U_L \begin{pmatrix} l_{L1} \\ l_{L2} \\ l_{L3} \end{pmatrix}. \quad (6.47)$$

In the basis where the Higgs mass matrix is diagonal, we have

$$U_h \begin{pmatrix} H_1^- \\ H_2^- \\ H_3^- \end{pmatrix} = \begin{pmatrix} h_1^- \\ h_2^- \\ h_3^- \end{pmatrix}, \quad (6.48)$$

so that the term becomes

$$2f \begin{pmatrix} \nu_{L1}^c & \nu_{L2}^c & \nu_{L3}^c \end{pmatrix} \left(H_1^- \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + H_2^- \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + H_3^- \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} l_{L1} \\ l_{L2} \\ l_{L3} \end{pmatrix}, \quad (6.49)$$

and we have

$$F_1 = 2f \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_2 = 2f \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$F_3 = 2f \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (6.51)$$

We are now in a position to find the neutrino mass matrix [65],

$$M_\nu = \sum_{i,\alpha,l''} \frac{1}{16\pi^2} \frac{1}{m_{H_\alpha}^2} \ln \frac{m_{P_i}^2}{m_{H_\alpha}^2} m_{l''} (D_{\phi h})_{\alpha i}^2 \{ (F_\alpha)_{ll''} (Y_{\phi i}^\dagger)_{l''l'} + (Y_{\phi i}^*)_{ll''} (F_\alpha^T)_{l''l'} \}. \quad (6.52)$$

The loops generating this mass can be seen in figure 6.3.

Since there is no constraint on the masses of the Higgs doublets or the coupling constants, this mass matrix contains too many parameters to be predictive at this point.

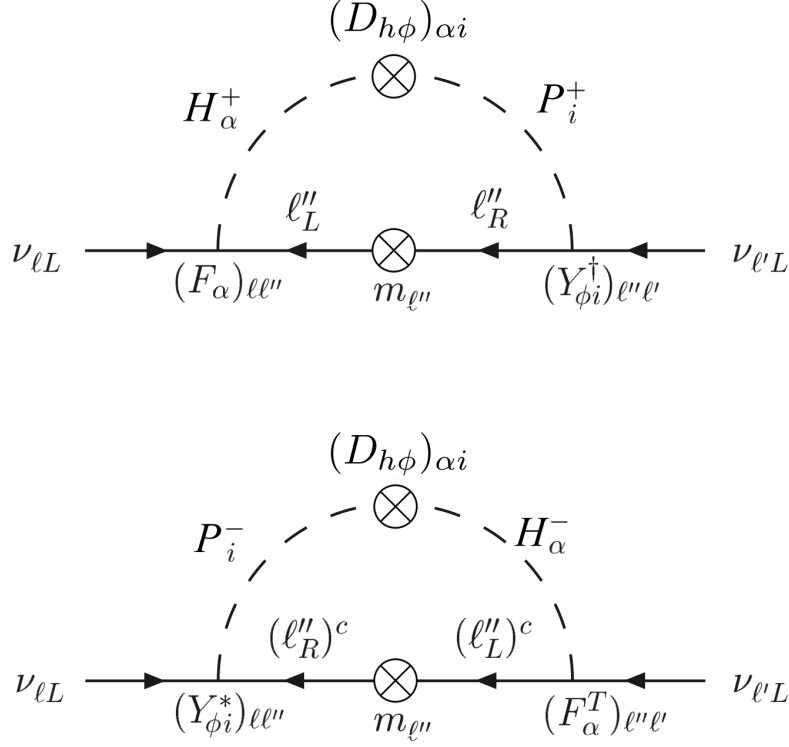


Figure 6.3: The loops in the basis of the mass eigenstates of the Higgs bosons [65].

6.2.2 The Frobenius group T_{13} and the MZM used as a perturbation

As we have seen in section 3.2, there are indications of deviations from tribimaximal mixing. It seems plausible to use tribimaximal to first order and perturb it, or look at minimal modifications to it. A dimension 5 operator generated from loop corrections could be a possibility to achieve such deviations.

The Zee model can be used for this purpose with the constraints from the Frobenius group. Explicitly, we will look at case 1 in section 4.1, leave the charged lepton mass matrix unchanged, and see what changes the Zee model will bring to the mixing matrix. We will furthermore assume, to begin with a minimalist approach, that the neutrino mass matrix is real in the following.

In case 1, we have the fields in the following representations:

$$\psi \sim \mathbf{3}_1, \quad \phi \sim \mathbf{3}_1, \quad \bar{l}_R \sim \mathbf{3}_1. \quad (6.53)$$

We have the neutrino mass matrix,

$$M_\nu = \begin{pmatrix} \alpha & 0 & \beta \\ 0 & \gamma & 0 \\ \beta & 0 & \alpha \end{pmatrix}, \quad (6.54)$$

which is diagonalized by the unitary matrix,

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}. \quad (6.55)$$

This mass matrix was seen to be obtained through two different combinations of vacuum alignments of the Higgs fields:

i)

$$\langle \eta \rangle = v_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \langle \kappa \rangle = u_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad (6.56)$$

and

ii)

$$\langle \chi \rangle = x, \quad \langle \eta \rangle = v_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \langle \kappa \rangle = u_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad (6.57)$$

with

$$\chi \sim \mathbf{1}, \quad \eta \sim \mathbf{3}_1, \quad \kappa \sim \bar{\mathbf{3}}_1. \quad (6.58)$$

Comparing the fields in this Frobenius group with those in the Zee model, we see that ϕ_β , which gives mass to the charged leptons in the Zee model, corresponds to ϕ in the Frobenius group model. ϕ_α , which acquires a vacuum expectation value in the coupling with the other Higgs doublet and the Higgs singlet, must then be either η , κ or both, using in this case $\phi_\alpha \neq \phi_\beta$.

In the coupling between two lepton doublets and the singlet scalar in equation (6.1), we can choose to place the singlet scalar h^- in a $\mathbf{3}_1$ representation of T_{13} . The coupling constant is antisymmetric, so we need a antisymmetric coupling between the fields

$$f(\nu_1 \bar{l}_{R2} h_3 + \nu_2 \bar{l}_{R3} h_1 + \nu_3 \bar{l}_{R1} h_2) - f_2(\nu_1 \bar{l}_{R3} h_2 - \nu_2 \bar{l}_{R1} h_3 - \nu_3 \bar{l}_{R2} h_1). \quad (6.59)$$

This contributes to all the off diagonal elements of the neutrino mass matrix. We can assume that the coupling constants are roughly equal [66], and seeing that $m_\tau \gg m_{e,\mu}$, we can neglect the terms proportional to m_e and m_μ [67]. Since the righthanded lepton couples to the antineutrino of the same flavor, the mass matrix has then obtained a contribution to the (23) and (12) entries,

$$M_\nu = \begin{pmatrix} \alpha & 0 & \beta - \epsilon \\ 0 & \gamma & \epsilon \\ \beta - \epsilon & \epsilon & \alpha \end{pmatrix}, \quad (6.60)$$

where

$$\epsilon = f m_\tau f_\phi v M \frac{1}{m_h^2} \log \left[\frac{m_{\phi^2}}{m_h^2} \right]. \quad (6.61)$$

Looking at the vertex with the coupling between the three Higgs, we have two possible contributions in i):

$$M_1 \epsilon_{ij} \eta^i \phi^j h, \quad (6.62)$$

and

$$M_2 \epsilon_{ij} \kappa^i \phi^j h. \quad (6.63)$$

With the singlet scalar in a $\mathbf{3}_1$ representation, the second coupling will not lead to any invariants, leaving only the first term, where again an antisymmetric combination is needed,

$$M_{1a}(\eta_1^T i\tau_2 \phi_2^\dagger h_3 + \eta_2^T i\tau_2 \phi_3^\dagger h_1 + \eta_3^T i\tau_2 \phi_1^\dagger h_2) - M_{1b}(\eta_1^T i\tau_2 \phi_3^\dagger h_2 + \eta_2^T i\tau_2 \phi_1^\dagger h_3 + \eta_3^T i\tau_2 \phi_2^\dagger h_1). \quad (6.64)$$

From (6.59), we see that only h_1 and h_2 enter the loop for $m_\tau \gg m_{e,\mu}$. Furthermore, the vacuum expectation value of η points in the 2-direction. These two conditions remove most of the terms, leaving only the following contribution:

$$M_{1a} \eta_2^T i\tau_2 \phi_3^\dagger h_1. \quad (6.65)$$

This coupling changes the mass matrix slightly

$$M_\nu = \begin{pmatrix} \alpha & 0 & \beta \\ 0 & \gamma & \epsilon \\ \beta & \epsilon & \alpha \end{pmatrix}. \quad (6.66)$$

Since ϵ in this case is just a radiative correction, we can treat the addition to the mass matrix

$$\epsilon M' = \epsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (6.67)$$

as a perturbation and then find the new neutrino mixing matrix. The perturbation calculations can be seen in Appendix C. The modified mixing matrix becomes

$$U_{mod} = U_\nu + \frac{1}{\sqrt{2}} \epsilon \begin{pmatrix} 0 & \frac{\sqrt{2}\beta}{(\gamma-\alpha)^2-\beta^2} & 0 \\ \frac{1}{\alpha+\beta-\gamma} & 0 & \frac{1}{\alpha-\beta-\gamma} \\ 0 & \frac{\sqrt{2}(\gamma-\alpha)}{(\gamma-\alpha)^2-\beta^2} & 0 \end{pmatrix}. \quad (6.68)$$

The overall lepton mixing matrix is given by

$$\begin{aligned} U = U_L^\dagger U_{mod} &= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & 1 & \omega^2 \\ \omega^2 & 1 & \omega \end{pmatrix} \left(\begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & \frac{\sqrt{2}\beta}{(\gamma-\alpha)^2-\beta^2} & 0 \\ \frac{1}{\alpha+\beta-\gamma} & 0 & \frac{1}{\alpha-\beta-\gamma} \\ 0 & \frac{\sqrt{2}(\gamma-\alpha)}{(\gamma-\alpha)^2-\beta^2} & 0 \end{pmatrix} \right) \\ &= U_{TB} + \frac{1}{\sqrt{6}} \epsilon \begin{pmatrix} \frac{1}{\alpha+\beta-\gamma} & \sqrt{2} \frac{\gamma-\alpha+\beta}{(\gamma-\alpha)^2-\beta^2} & \frac{1}{\alpha-\beta-\gamma} \\ \frac{1}{\alpha+\beta-\gamma} & \sqrt{2} \frac{\omega^2(\gamma-\alpha)+\omega\beta}{(\gamma-\alpha)^2-\beta^2} & \frac{1}{\alpha-\beta-\gamma} \\ \frac{1}{\alpha+\beta-\gamma} & \sqrt{2} \frac{\omega(\gamma-\alpha)+\omega^2\beta}{(\gamma-\alpha)^2-\beta^2} & \frac{1}{\alpha-\beta-\gamma} \end{pmatrix}. \end{aligned} \quad (6.70)$$

These terms will diverge for $\alpha + \beta = \gamma$, $|\gamma - \alpha| = |\beta|$ and $\alpha - \beta = \gamma$ which corresponds to $m_2 = m_1$ or $m_2 = m_3$. Luckily we know from experiments that $\Delta m_{12} \neq 0$ and $\Delta m_{23} \neq 0$,

so this will not create any problems. We have here a very small contribution to the mixing matrix. Therefore, we can neglect all elements except the contribution to the otherwise zero U_{e3} , which now has the value of $\frac{1}{\alpha-\beta-\gamma}$. This model can then be extended to include complex results. Therefore, CP violation is possible when using the Zee model with the Frobenius group. At this point, however, we do not have enough constraints to predict any values.

Grand Unification

A big goal in particle physics is to find some great symmetry that can describe the interactions of all fields in nature. A Grand Unified Theory contains such a symmetry. This is a gauge group that contains all gauge symmetries of the Standard Model [50]. With the extension of the Standard Model to include neutrino masses, it must be possible to implement some grand unified theory with a family symmetry that describes neutrino mixing and/or find a grand unified theory that can contain right-handed neutrinos. To do this, it seems appropriate to consider the grand unified theories that have already been applied for the Standard Model.

The gauge extensions must meet the criteria, that when broken at a very large energy scale, the Standard Model gauge groups are recovered. The representation of the fermions in the left-handed representations are given by

$$\begin{pmatrix} u \\ d \end{pmatrix} \sim (\mathbf{3}, \mathbf{2}, \frac{1}{6}), \quad d^c \sim (\bar{\mathbf{3}}, \mathbf{1}, \frac{1}{3}), \quad u^c \sim (\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3}) \quad (7.1)$$

$$\begin{pmatrix} \nu_l \\ l \end{pmatrix} \sim (\mathbf{1}, \mathbf{2}, -\frac{1}{2}), \quad l^c \sim (\mathbf{1}, \mathbf{1}, 1). \quad (7.2)$$

The representations for the gauge fields are given by

$$\text{Gluons} \sim (\mathbf{8}, \mathbf{1}, 0), \quad W \sim (\mathbf{1}, \mathbf{3}, 0), \quad B \sim (\mathbf{1}, \mathbf{1}, 0). \quad (7.3)$$

The Higgs doublet is then assigned to the representation,

$$\phi \sim (\mathbf{1}, \mathbf{2}, -\frac{1}{2}), \quad (7.4)$$

in order to allow Yukawa couplings of fermions.

The SU(5) Grand Unified Theory, the Georgi-Glashow model, is the simplest grand unified model that can be broken in one step to the Standard Model. However, the SU(5) GUT has turned out to not agree with experimental data [68]. Unless we want to extend the theory to a supersymmetric version, a different candidate for a grand unified model has to be implemented.

The group $\text{SO}(10)$ turns out to be that candidate. To illustrate the idea of a grand unified theory, let's start out with a look at the $\text{SU}(5)$ group.

The $\text{SU}(5)$ group has $5^2 - 1 = 24$ generators. The generators can be represented by traceless hermitian matrices acting on five objects $\psi^\mu, \mu = 1, 2, 3, 4, 5$. The 24 matrices contain eight of the form

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad (7.5)$$

from $\text{SU}(3)$. Three of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, \quad (7.6)$$

from $\text{SU}(2)$, and one

$$\frac{1}{2}Y = \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad (7.7)$$

from $\text{U}(1)$. With $\mu = \{\alpha, i\}$, $\alpha = 1, 2, 3$ and $i = 4, 5$, this means that $\text{SU}(3)$ acts on α and $\text{SU}(2)$ acts on i . We have $\psi^\alpha \sim (3, 1, -\frac{1}{3})$ and $\psi^i \sim (1, 2, \frac{1}{2})$. When a field Φ in a **24** representation of the group acquires a vacuum expectation value

$$\langle \Phi \rangle = \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} V, \quad (7.8)$$

the group is broken down to the subgroup $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ [33]. This means that the number of broken generators is given by $24 - 8 - 3 - 1 = 12$.

Writings up the embeddings of these groups within $\text{SU}(5)$ as $(\text{SU}(3), \text{SU}(2), \text{U}(1))$, we have

$$\mathbf{5} = (\mathbf{1}, \mathbf{2}, \frac{1}{2}) + (\mathbf{3}, \mathbf{1}, -\frac{1}{3}), \quad (7.9)$$

$$\bar{\mathbf{5}} = (\mathbf{1}, \mathbf{2}, -\frac{1}{2}) + (\bar{\mathbf{3}}, \mathbf{1}, \frac{1}{3}). \quad (7.10)$$

The representation containing the rest of the fermions can be found by looking at the Kronecker product

$$\begin{aligned} \mathbf{5} \otimes \mathbf{5} &= \mathbf{10}_A + \mathbf{15}_S = ((\mathbf{1}, \mathbf{2}, \frac{1}{2}) + (\mathbf{3}, \mathbf{1}, -\frac{1}{3}))((\mathbf{1}, \mathbf{2}, \frac{1}{2}) + (\mathbf{3}, \mathbf{1}, -\frac{1}{3})), \\ &= (\mathbf{1}, \mathbf{1}, 1)_A + (\mathbf{1}, \mathbf{3}, 1)_S + (\mathbf{3}, \mathbf{2}, \frac{1}{6})_S + (\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_A + (\mathbf{6}, \mathbf{1}, -\frac{2}{3})_S, \end{aligned} \quad (7.12)$$

where

$$\mathbf{10} = (\mathbf{3}, \mathbf{2}, \frac{1}{6}) + (\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3}) + (\mathbf{1}, \mathbf{1}, 1). \quad (7.13)$$

The left-handed up and down type quarks, the right-handed up type quarks and the right-handed leptons therefore fit into the $\mathbf{10}$ representation [31]. The fields

$$\psi_\mu = \begin{pmatrix} \psi^\alpha \\ \psi^i \end{pmatrix} = \begin{pmatrix} \bar{d}_\alpha \\ \nu \\ e \end{pmatrix}, \quad (7.14)$$

are in the $\bar{\mathbf{5}}$ representation and the antisymmetric tensor fields, $\chi^{\mu\nu}$, can be written as

$$\chi^{\mu\nu} = \{\chi^{\alpha\beta}, \chi^{\alpha i}, \chi^{ij}\} = \begin{pmatrix} 0 & \bar{u} & -\bar{u} & d & u \\ -\bar{u} & 0 & \bar{u} & d & u \\ \bar{u} & -\bar{u} & 0 & d & u \\ -d & -d & -d & 0 & \bar{e} \\ -u & -u & -u & -\bar{e} & 0 \end{pmatrix}, \quad (7.15)$$

with $\bar{u}$ the right-handed up type quark, u and d the left-handed up and down type quarks respectively and $\bar{e}$ the right-handed lepton [31]. The vector fields can be found from the Kronecker product

$$\mathbf{5} \otimes \bar{\mathbf{5}} = \mathbf{24} + \mathbf{1} = (\mathbf{1}, \mathbf{3}, 0) + (\mathbf{8}, \mathbf{1}, 0) + (\mathbf{1}, \mathbf{1}, 0) + (\mathbf{3}, \mathbf{2}, -\frac{5}{6}) + (\bar{\mathbf{3}}, \mathbf{2}, \frac{5}{6}) + (\mathbf{1}, \mathbf{1}, 0). \quad (7.16)$$

In the adjoint representation $\mathbf{24}$, we have, in order, the weak bosons, the 8 gluons, the hypercharge and the gauge fields corresponding to 12 unbroken generators.

We now proceed to look at the $\text{SO}(10)$ gauge group.

The two representations $\bar{\mathbf{5}}$ and $\mathbf{10}$ can instead be fitted into a single representation of a bigger group, which contains $\text{SU}(5)$. $\text{SO}(10)$ turns out to be a suitable candidate for this. The decomposition of the $\mathbf{16}$ representation is compatible:

$$\mathbf{16} \rightarrow \mathbf{1} \oplus \mathbf{10} \oplus \bar{\mathbf{5}}. \quad (7.17)$$

This means that the $\text{SO}(10)$ gauge group contains all the fermions in its $\mathbf{16}$ representation. In fact, it is the smallest simple Lie group that can contain all the Standard Model fermions in a single anomaly free irreducible representation. This means that there would be a perfect fit, if yet another field in a $\mathbf{1}$ representation could be found. This representation has so far been the problem in this grand unifications. However, with the revelation of neutrino masses, a very plausible candidate is now the right handed neutrino! If this is a singlet under $\text{SU}(5)$, it would be able to get the large Majorana mass without breaking $\text{SU}(5)$.

The adjoint representation is given by

$$\mathbf{45} = \mathbf{24} + \mathbf{1} + \mathbf{10} + \bar{\mathbf{10}}. \quad (7.18)$$

This means that there are more gauge bosons in $\text{SO}(10)$ than in $\text{SU}(5)$.

It would be very interesting to apply the family group T_{13} with this $\text{SO}(10)$ group or some other large gauge group, that contains the gauge symmetries of the Standard Model. A way to

do this could be to place the Higgs fields in different representations of this large gauge group and have the fermions and Higgs fields in specific representations of the T_{13} group. Model building could then be carried out to see if this unification is compatible with experiments.

Discussions and conclusion

In order to constrain the systematic search, some assumptions have been used during the model building. These can be summarized as follows:

- i) The vacuum expectation value of the Higgs doublet in the dimension 4 operator and the Dirac mass term was aligned in the $(1,1,1)$ direction.
- ii) No coupling was assumed between the scalar fields in the charged lepton sector and the neutrino sector, that is, the sequestering problem has not been addressed.
- iii) Since tribimaximal mixing gives a fairly good description of the experimental mixing matrix, at least as a first approximation, it was used as a phenomenological guideline.
- iv) It was assumed that the region of the parameter space of the scalar potential where the heavy scalar singlets in the see-saw mechanism develop vacuum expectation values lies in a different region than at the scale where the Higgs doublet acquires its vacuum expectation values.
- v) CP conservation has been assumed to leading order.

Since the first postulation of their existence in 1930 by Pauli, the field of neutrino physics has opened up many questions. However, experiments are extremely difficult to carry out, dealing with almost non-interacting particles. If neutrinos acquire their masses through some very high scale, as in the see-saw mechanism, this also imposes big challenges for neutrino experiments. Theory relies on very few experimental data and not many constraints are imposed when carrying out model building. However, the many unknown territories and the lack of data allow for greater liberty and creativity when working in this field.

In this thesis I have addressed the areas I find most fascinating in neutrino physics today. My approach has been to find a group that describes neutrino mixing well and then use this group throughout the thesis, implementing the interesting models.

The Frobenius group T_{13} , with its two 3 dimensional irreducible representations, seems to

be a suitable group to use, in order to study the family symmetry of neutrinos and charged leptons as was shown in chapter 4. It can even be generalized, as it turns out that a series of Frobenius groups could be applied to neutrino mixing.

There is a very good compatibility between the Frobenius group T_{13} and the canonical see-saw mechanism when applied to neutrino mixing, using tribimaximal mixing as a guideline. In chapter 5 it was also interesting to see that the Frobenius group does not allow a bare mass term for the right-handed neutrinos, but only a Yukawa coupling of these with two scalar singlets. Furthermore, only one Higgs doublet is needed. The assumptions allow only six possibilities for the representations of the fields. The constraints from applying both the family symmetry group T_{13} and the canonical see-saw mechanism narrow down the parameters in the neutrino mass matrix enough to make predictions. Using experimental data, I have predicted the physical neutrino masses. Moreover, I have predicted values for the $|m_{ee}|$ parameter which can be tested in future neutrinoless double beta decay experiments. In the inverted hierarchy case, the $|m_{ee}|$ parameter is almost a magnitude higher than in the normal case. If this is the case, the next neutrinoless double beta decay experiments may be able to reveal information about the mass hierarchy as well.

In chapter 6 it was shown that the Frobenius group could be implemented with the Zee model as well. However, in both approaches, it was not possible to constrain the model enough to make predictions. New implementations are needed, and it would be interesting to seek these in a further study.

In this thesis, the sequestering problem has not been addressed. When the Higgs of the dimension 4 operator acquires a vacuum expectation value in the (1,1,1) direction, the T_{13} symmetry is broken down to Z_3 . As the various scalar fields of the dimension 5 operators acquire vacuum expectation values, the symmetry is also broken down to Z_2 . The Z_2 subgroup does not commute with the Z_3 subgroup, and the vacuum alignments cannot be sequestered from each other unless the interaction terms in the Higgs potential which connect these fields vanishes [69]. Tribimaximal mixing is therefore a lowest order approach and we need to study allowed deviations within the T_{13} structure. Certain articles have already described modifications to tribimaximal mixing [70, 71]. These could be used with this theory. CP violation could also be considered.

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Radiative one-loop calculation

In the Zee model, the integral to be solved in order to find the neutrino mass contribution from the loop is given by:

$$\int \frac{d^4 k}{(2\pi^4)} \frac{1}{k^2 + i\epsilon} \frac{1}{k^2 - m_\phi^2 + i\epsilon} \frac{1}{k^2 - m_h^2 + i\epsilon}. \quad (\text{A.1})$$

We can write the integrand up in terms of Feynman parameters:

$$\frac{1}{k^2 + i\epsilon} \frac{1}{k^2 - m_\phi^2 + i\epsilon} \frac{1}{k^2 - m_h^2 + i\epsilon} = \int_0^1 dx dy dz \delta(x + y + z - 1) \frac{2}{D^3}, \quad (\text{A.2})$$

with

$$D = xk^2 + y(k^2 - m_\phi^2) + z(k^2 - m_h^2) + (x + y + z)i\epsilon. \quad (\text{A.3})$$

Integrating over x with $x = 1 - y - z$, we have

$$D_{x=1-y-z} = k^2 - ym_\phi^2 - zm_h^2 + i\epsilon = k^2 - \Delta \equiv D. \quad (\text{A.4})$$

We can now make a Wick rotation to solve the integral over k . Setting $k^0 = ik_E^0$, $\mathbf{k} = \mathbf{k}_E$ and having $k = k^0 - \mathbf{k}$, we get:

$$\begin{aligned} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - \Delta)^3} &= \frac{i}{(-1)^3} \frac{1}{(2\pi)^4} \int d^4 k_E \frac{1}{(k_E^2 + \Delta)^3} \\ &= -\frac{i}{(2\pi)^4} \int d\Omega_4 \int_0^\infty dk_E \frac{k_E^3}{(k_E^2 + \Delta)^3} = -\frac{i}{2^3 \pi^2} \frac{1}{4\Delta}, \end{aligned} \quad (\text{A.6})$$

for $\Delta > 0$, which it clearly is. We can now solve the integrals over x and y :

$$\begin{aligned} -\frac{i}{2^5 \pi^2} \int_0^1 dy dz \frac{2}{ym_\phi^2 + zm_h^2} &= -\frac{i}{16\pi^2} \int_0^1 dy \frac{1}{m_h^2} \ln\left(\frac{m_h^2 + ym_\phi^2}{ym_\phi^2}\right) \\ &= -\frac{i}{16\pi^2} \frac{1}{m_h^2} \left(\frac{m_h^2}{m_\phi^2} + 1\right) \left(\ln\left(\frac{m_h^2}{m_\phi^2} + 1\right)\right). \end{aligned} \quad (\text{A.8})$$

We now make the assumption that $m_h \gg m_\phi$. This leads to the approximation:

$$\frac{1}{m_h^2} \left(\frac{m_h^2}{m_\phi^2} + 1 \right) \left(\ln \left(\frac{m_h^2}{m_\phi^2} + 1 \right) \right) \simeq \frac{1}{m_h^2} \frac{m_h^2}{m_\phi^2} \ln \left(\frac{m_h^2}{m_\phi^2} \right) = \frac{1}{m_\phi^2} \ln \left(\frac{m_h^2}{m_\phi^2} \right). \quad (\text{A.9})$$

The resulting contribution to the neutrino mass matrix is then given by:

$$M_{ab} = \frac{1}{16\pi^2} f^{ac} m_{cd} f_\phi^{\beta db} v_\alpha M_{\alpha\beta} \frac{1}{m_h^2} [\log[\frac{m_\phi^2}{m_h^2}]]. \quad (\text{A.10})$$

Radiative two one-loop calculation

The integral of the H_1 diagram is given by:

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + i\epsilon} \frac{1}{k^2 - M_1^2 + i\epsilon}. \quad (\text{B.1})$$

Again we parametrize the fractions into Feynman parameters:

$$\frac{1}{k^2 + i\epsilon} \frac{1}{k^2 - M_1^2 + i\epsilon} = \int_0^1 dx dy \delta(x + y - 1) \frac{1}{D^2}, \quad (\text{B.2})$$

where $D = xk^2 + y(k^2 - M_1^2) + (x + y)i\epsilon$. Carrying out the integral over the delta function,

$$D_{x=1-y} = k^2 - yM_1^2 + i\epsilon \equiv D. \quad (\text{B.3})$$

The integral can then be performed over k :

$$\frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - yM_1^2)^2} = \frac{i}{(2\pi)^4} \int d^4 k_E \frac{1}{(k_E^2 + yM_1^2)^2} = \frac{i}{(2\pi)^4} \int d\Omega_4 \int_0^\infty dk_E \frac{k_E^3}{(k_E^2 + yM_1^2)^2}. \quad (\text{B.4})$$

This integral diverges! So we need to regulate it using the Pauli-Villars procedure, introducing the cut-off Λ ,

$$\frac{1}{k^2 - M_1^2 + i\epsilon} \rightarrow \frac{1}{k^2 - M_1^2 + i\epsilon} - \frac{1}{k^2 - \Lambda^2 + i\epsilon}. \quad (\text{B.5})$$

Thereby the integral becomes:

$$\begin{aligned} \frac{i}{2^3 \pi^2} \int_0^\infty dk_E \frac{k_E^3}{(k_E^2 + yM_1^2)^2} &\rightarrow \frac{i}{2^3 \pi^2} \left[\int_0^\infty dk_E \frac{k_E^3}{(k_E^2 + yM_1^2)^2} - \int_0^\infty dk_E \frac{k_E^3}{(k_E^2 + y\Lambda^2)^2} \right] \\ &= \frac{i}{4^4 \pi^2} \ln\left(\frac{\Lambda^2}{M_1^2}\right). \end{aligned} \quad (\text{B.7})$$

Since we have two loops with opposite signs, the cutoff parameter will disappear when adding these contributions together, and we get the propagator contribution from the two loops:

$$\frac{i}{4^4\pi^2} \ln\left(\frac{M_2^2}{M_1^2}\right). \quad (\text{B.8})$$

Perturbation calculation

Using $m_1^{(0)} = \alpha + \beta$, $m_2^{(0)} = \gamma$, $m_3^{(0)} = \alpha - \beta$ and

$$|1^{(0)}\rangle = \frac{1}{\sqrt{2}}(1, 0, 1), \quad (\text{C.1})$$

$$|2^{(0)}\rangle = (0, 1, 0), \quad (\text{C.2})$$

$$|3^{(0)}\rangle = \frac{1}{\sqrt{2}}(-1, 0, 1), \quad (\text{C.3})$$

corresponding to the columns of the mixing matrix

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}. \quad (\text{C.4})$$

Using the equation

$$|n^{(1)}\rangle = \frac{\sum_k \langle k^{(0)} | M' | n^{(0)} \rangle}{m_n^{(0)} - m_k^{(0)}} |k^{(0)}\rangle, \quad (\text{C.5})$$

we then find

$$\begin{aligned} |1^{(1)}\rangle &= \frac{1}{\sqrt{2}(\alpha + \beta - \gamma)} (0, 1, 0) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &+ \frac{1}{2\sqrt{2}(2\beta)} (-1, 0, 1) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \frac{1}{\alpha + \beta - \gamma} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \end{aligned} \quad (\text{C.7})$$

$$|2^{(1)}\rangle = \frac{1}{2(\gamma - \alpha - \beta)} (1, 0, 1) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$+\frac{1}{2(\gamma-\alpha+\beta)}(-1,0,1)\begin{pmatrix}0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 \\ 1 \\ 0\end{pmatrix}\begin{pmatrix}-1 \\ 0 \\ 1\end{pmatrix}=\frac{1}{(\gamma-\alpha)^2-\beta^2}\begin{pmatrix}\beta \\ 0 \\ \gamma-\alpha\end{pmatrix}, \quad (\text{C.9})$$

$$\begin{aligned} |3^{(1)}\rangle &= \frac{1}{2\sqrt{2}(-2\beta)}(1,0,1)\begin{pmatrix}0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}-1 \\ 0 \\ 1\end{pmatrix}\begin{pmatrix}1 \\ 0 \\ 1\end{pmatrix}+ \\ &\frac{1}{\sqrt{2}(\alpha-\beta-\gamma)}(0,1,0)\begin{pmatrix}0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}-1 \\ 0 \\ 1\end{pmatrix}\begin{pmatrix}0 \\ 1 \\ 0\end{pmatrix}=\frac{1}{\sqrt{2}}\frac{1}{\alpha-\beta-\gamma}\begin{pmatrix}0 \\ 1 \\ 0\end{pmatrix}, \quad (\text{C.11}) \end{aligned}$$

and we get the new mixing matrix

$$U_{mod}=U_\nu+\frac{1}{\sqrt{2}}\epsilon\begin{pmatrix}0 & \frac{\sqrt{2}\beta}{(\gamma-\alpha)^2-\beta^2} & 0 \\ \frac{1}{\alpha+\beta-\gamma} & 0 & \frac{1}{\alpha-\beta-\gamma} \\ 0 & \frac{\sqrt{2}(\gamma-\alpha)}{(\gamma-\alpha)^2-\beta^2} & 0\end{pmatrix}. \quad (\text{C.12})$$

Dimension 5 operator invariants for the general Frobenius groups

In the following can be seen the invariants possible. It is quite a long list. However, there are certain requirements for the invariants to exist. Therefore, the possibilities are narrowed down when a group is chosen. In building a specific model, we are also free to omit one or more of the Higgs fields, narrowing down the possibilities further. The following tables can therefore be seen as a look-up when finding the invariants of a certain Frobenius group.

	$(\eta_j \psi_i)^2$
$j = \frac{1}{2}ir^2$	$(\eta_1 \psi_1 \eta_1 \psi_2 + \eta_2 \psi_2 \eta_2 \psi_3 + \eta_3 \psi_3 \eta_3 \psi_1)$
$i = -jr$	$(\eta_1 \psi_1 \eta_2 \psi_3 + \eta_2 \psi_2 \eta_3 \psi_1 + \eta_3 \psi_3 \eta_1 \psi_2)$
$i = \frac{1}{2}jr$	$(\eta_1 \psi_1 \eta_3 \psi_1 + \eta_2 \psi_2 \eta_1 \psi_2 + \eta_3 \psi_3 \eta_2 \psi_3)$
$i = \frac{1}{2}jr^2$	$(\eta_1 \psi_1 \eta_2 \psi_1 + \eta_2 \psi_2 \eta_3 \psi_2 + \eta_3 \psi_3 \eta_1 \psi_3)$
$j = -ir$	$(\eta_1 \psi_1 \eta_3 \psi_2 + \eta_2 \psi_2 \eta_1 \psi_3 + \eta_3 \psi_3 \eta_2 \psi_1)$
$j = \frac{1}{2}ir$	$f_1(\eta_1 \psi_1 \eta_1 \psi_3 + \eta_2 \psi_2 \eta_2 \psi_1 + \eta_3 \psi_3 \eta_3 \psi_2)$
$i = -j$	$(\eta_1 \psi_2 \eta_2 \psi_1 + \eta_2 \psi_3 \eta_3 \psi_2 + \eta_3 \psi_1 \eta_1 \psi_3)$
$j = 2i$	$(\eta_1 \psi_2 \eta_3 \psi_2 + \eta_2 \psi_3 \eta_1 \psi_3 + \eta_3 \psi_1 \eta_2 \psi_1)$
$i = 2j$	$(\eta_1 \psi_2 \eta_1 \psi_3 + \eta_2 \psi_3 \eta_2 \psi_1 + \eta_3 \psi_1 \eta_3 \psi_2)$

Table D.1

	$(\kappa_i \psi_i)^2$
all	$(\kappa_1 \psi_1 \kappa_1 \psi_1 + \kappa_2 \psi_2 \kappa_2 \psi_2 + \kappa_3 \psi_3 \kappa_3 \psi_3)$
all	$(\kappa_1 \psi_1 \kappa_2 \psi_2 + \kappa_3 \psi_3 \kappa_1 \psi_1 + \kappa_2 \psi_2 \kappa_3 \psi_3)$
all	$(\kappa_1 \psi_2 \kappa_2 \psi_1 + \kappa_2 \psi_3 \kappa_3 \psi_2 + \kappa_3 \psi_1 \kappa_1 \psi_3)$

Table D.2

	$(\kappa_j \psi_i)^2$
$i = j$	$(\kappa_1 \psi_1 \kappa_1 \psi_1 + \kappa_2 \psi_2 \kappa_2 \psi_2 + \kappa_3 \psi_3 \kappa_3 \psi_3)$ $(\kappa_1 \psi_1 \kappa_2 \psi_2 + \kappa_2 \psi_2 \kappa_3 \psi_3 + \kappa_3 \psi_3 \kappa_1 \psi_1)$ $(\kappa_1 \psi_1 \kappa_3 \psi_3 + \kappa_2 \psi_2 \kappa_1 \psi_1 + \kappa_3 \psi_3 \kappa_2 \psi_2)$ $(\kappa_1 \psi_1 \kappa_3 \psi_1 + \kappa_2 \psi_2 \kappa_1 \psi_2 + \kappa_3 \psi_3 \kappa_2 \psi_3)$ $(\kappa_2 \psi_1 \kappa_1 \psi_2 + \kappa_3 \psi_2 \kappa_2 \psi_3 + \kappa_1 \psi_3 \kappa_3 \psi_1)$
$i = jr$	$(\kappa_2 \psi_1 \kappa_2 \psi_1 + \kappa_3 \psi_2 \kappa_3 \psi_2 + \kappa_1 \psi_3 \kappa_1 \psi_3)$ $(\kappa_2 \psi_1 \kappa_3 \psi_2 + \kappa_3 \psi_2 \kappa_1 \psi_3 + \kappa_1 \psi_3 \kappa_2 \psi_1)$ $(\kappa_2 \psi_1 \kappa_1 \psi_3 + \kappa_3 \psi_2 \kappa_2 \psi_1 + \kappa_1 \psi_3 \kappa_3 \psi_2)$ $(\kappa_1 \psi_1 \kappa_2 \psi_3 + \kappa_2 \psi_2 \kappa_3 \psi_1 + \kappa_3 \psi_3 \kappa_1 \psi_2)$ $(\kappa_2 \psi_1 \kappa_3 \psi_1 + \kappa_3 \psi_2 \kappa_1 \psi_2 + \kappa_1 \psi_3 \kappa_2 \psi_3)$
$j = ir$	$(\kappa_1 \psi_2 \kappa_1 \psi_2 + \kappa_2 \psi_3 \kappa_2 \psi_3 + \kappa_3 \psi_1 \kappa_3 \psi_1)$ $(\kappa_1 \psi_2 \kappa_2 \psi_3 + \kappa_2 \psi_3 \kappa_3 \psi_1 + \kappa_3 \psi_1 \kappa_1 \psi_2)$ $(\kappa_1 \psi_2 \kappa_3 \psi_1 + \kappa_2 \psi_3 \kappa_1 \psi_2 + \kappa_3 \psi_1 \kappa_2 \psi_3)$ $(\kappa_1 \psi_1 \kappa_3 \psi_2 + \kappa_2 \psi_2 \kappa_1 \psi_3 + \kappa_3 \psi_3 \kappa_2 \psi_1)$
$j = -2ir$	$(\kappa_1 \psi_1 \kappa_2 \psi_1 + \kappa_2 \psi_2 \kappa_3 \psi_2 + \kappa_3 \psi_3 \kappa_1 \psi_3)$
$i = -2jr$	$(\kappa_1 \psi_1 \kappa_1 \psi_2 + \kappa_2 \psi_2 \kappa_2 \psi_3 + \kappa_3 \psi_3 \kappa_3 \psi_1)$
$i = -2j$	$(\kappa_2 \psi_1 \kappa_2 \psi_3 + \kappa_3 \psi_2 \kappa_3 \psi_1 + \kappa_1 \psi_3 \kappa_1 \psi_2)$

Table D.3

	$(\chi \psi_i)(\eta_i \psi_i)$
all	$\chi(\psi_1 \eta_2 \psi_3 + \psi_2 \eta_3 \psi_1 + \psi_3 \eta_1 \psi_2)$
all	$\chi(\psi_1 \eta_3 \psi_2 + \psi_2 \eta_1 \psi_3 + \psi_3 \eta_2 \psi_1)$

Table D.4

	$(\chi \psi_i)(\eta_j \psi_i)$
$j = -2i$	$\chi(\psi_1 \eta_1 \psi_1 + \psi_2 \eta_2 \psi_2 + \psi_3 \eta_3 \psi_3)$
$j = -2ir^2$	$\chi(\psi_1 \eta_2 \psi_1 + \psi_2 \eta_3 \psi_2 + \psi_3 \eta_1 \psi_3)$
$j = -2ir$	$\chi(\psi_1 \eta_3 \psi_1 + \psi_2 \eta_1 \psi_2 + \psi_3 \eta_2 \psi_3)$
$j = ir$	$\chi(\psi_3 \eta_1 \psi_1 + \psi_1 \eta_2 \psi_2 + \psi_2 \eta_3 \psi_3)$
$j = i$	$\chi(\psi_2 \eta_3 \psi_1 + \psi_3 \eta_1 \psi_2 + \psi_1 \eta_2 \psi_3)$ $\chi(\psi_3 \eta_2 \psi_1 + \psi_1 \eta_3 \psi_2 + \psi_2 \eta_1 \psi_3)$
$j = ir$	$\chi(\psi_2 \eta_2 \psi_1 + \psi_3 \eta_3 \psi_2 + \psi_1 \eta_1 \psi_3)$
$j = ir^2$	$\chi(\psi_1 \eta_3 \psi_3 + \psi_2 \eta_1 \psi_1 + \psi_3 \eta_2 \psi_2)$ $\chi(\psi_1 \eta_1 \psi_2 + \psi_2 \eta_2 \psi_3 + \psi_3 \eta_3 \psi_1)$

Table D.5

	$(\chi \psi_i)(\kappa_i \psi_i)$
$r = 2$	$\chi(\psi_1 \kappa_2 \psi_1 + \psi_2 \kappa_3 \psi_2 + \psi_3 \kappa_1 \psi_3)$
$r^2 = 2$	$\chi(\psi_1 \kappa_3 \psi_1 + \psi_2 \kappa_1 \psi_2 + \psi_3 \kappa_2 \psi_3)$

Table D.6

	$(\chi\psi_i)(\kappa_j\psi_i)$
$j = 2i$	$\chi(\psi_1\kappa_1\psi_1 + \psi_2\kappa_2\psi_2 + \psi_3\kappa_3\psi_3)$
$j = -i$	$\chi(\psi_3\kappa_2\psi_1 + \psi_1\kappa_3\psi_2 + \psi_2\kappa_1\psi_3)$
	$\chi(\psi_2\kappa_3\psi_1 + \psi_3\kappa_1\psi_2 + \psi_1\kappa_2\psi_3)$
$j = -ir$	$\chi(\psi_3\kappa_1\psi_1 + \psi_1\kappa_2\psi_2 + \psi_2\kappa_3\psi_3)$
	$\chi(\psi_2\kappa_2\psi_1 + \psi_3\kappa_3\psi_2 + \psi_1\kappa_1\psi_3)$
$j = 2ir$	$\chi(\psi_1\kappa_3\psi_1 + \psi_2\kappa_1\psi_2 + \psi_3\kappa_2\psi_3)$
$j = 2ir^2$	$\chi(\psi_1\kappa_2\psi_1 + \psi_2\kappa_3\psi_2 + \psi_3\kappa_1\psi_3)$
$j = -ir^2$	$\chi(\psi_2\kappa_1\psi_1 + \psi_3\kappa_2\psi_2 + \psi_1\kappa_3\psi_3)$
	$\chi(\psi_3\kappa_3\psi_1 + \psi_1\kappa_1\psi_2 + \psi_2\kappa_2\psi_3)$

Table D.7

	$(\eta_i\psi_i)(\eta_j\psi_i)$
$j = -3i$	$(\eta_1\psi_1\eta_1\psi_1 + \eta_2\psi_2\eta_2\psi_2 + \eta_3\psi_3\eta_3\psi_3)$
$j = i(r - r^2)$	$(\eta_3\psi_3\eta_1\psi_1 + \eta_1\psi_1\eta_2\psi_2 + \eta_2\psi_2\eta_3\psi_3)$
	$(\eta_1\psi_2\eta_2\psi_1 + \eta_2\psi_3\eta_3\psi_2 + \eta_3\psi_1\eta_1\psi_3)$
	$(\eta_2\psi_1\eta_2\psi_1 + \eta_3\psi_2\eta_3\psi_2 + \eta_1\psi_3\eta_1\psi_3)$
$j = i(r^2 - r)$	$(\eta_2\psi_2\eta_1\psi_1 + \eta_3\psi_3\eta_2\psi_2 + \eta_1\psi_1\eta_3\psi_3)$
	$(\eta_1\psi_2\eta_1\psi_2 + \eta_2\psi_3\eta_2\psi_3 + \eta_3\psi_1\eta_3\psi_1)$
	$(\eta_2\psi_1\eta_1\psi_2 + \eta_3\psi_2\eta_2\psi_3 + \eta_1\psi_3\eta_3\psi_1)$
$j = i(r^2 - 1)$	$(\eta_1\psi_1\eta_1\psi_2 + \eta_2\psi_2\eta_2\psi_3 + \eta_3\psi_3\eta_3\psi_1)$
	$(\eta_1\psi_2\eta_1\psi_1 + \eta_2\psi_3\eta_2\psi_2 + \eta_3\psi_1\eta_3\psi_3)$
	$(\eta_2\psi_1\eta_1\psi_1 + \eta_3\psi_2\eta_2\psi_2 + \eta_1\psi_3\eta_3\psi_3)$
$j = i(1 - r^2)$	$(\eta_3\psi_3\eta_1\psi_2 + \eta_1\psi_1\eta_2\psi_3 + \eta_2\psi_2\eta_3\psi_1)$
	$(\eta_3\psi_1\eta_2\psi_1 + \eta_1\psi_2\eta_3\psi_2 + \eta_2\psi_3\eta_1\psi_3)$
	$(\eta_1\psi_3\eta_2\psi_1 + \eta_2\psi_1\eta_3\psi_2 + \eta_3\psi_2\eta_1\psi_3)$
$j = -3ir$	$(\eta_2\psi_2\eta_1\psi_2 + \eta_3\psi_3\eta_2\psi_3 + \eta_1\psi_1\eta_3\psi_1)$
$i = j(r^2 - 1)$	$(\eta_1\psi_1\eta_2\psi_1 + \eta_2\psi_2\eta_3\psi_2 + \eta_3\psi_3\eta_1\psi_3)$
$i = j(1 - r^2)$	$(\eta_3\psi_3\eta_2\psi_1 + \eta_1\psi_1\eta_3\psi_2 + \eta_2\psi_2\eta_1\psi_3)$
$i = -3jr$	$(\eta_2\psi_2\eta_2\psi_1 + \eta_3\psi_3\eta_3\psi_2 + \eta_1\psi_1\eta_1\psi_3)$
$j = i(r - 1)$	$(\eta_3\psi_1\eta_1\psi_1 + \eta_1\psi_2\eta_2\psi_2 + \eta_2\psi_3\eta_3\psi_3)$
	$(\eta_1\psi_3\eta_1\psi_1 + \eta_2\psi_1\eta_2\psi_2 + \eta_3\psi_2\eta_3\psi_3)$
$j = i(1 - r)$	$(\eta_2\psi_3\eta_1\psi_2 + \eta_3\psi_1\eta_2\psi_3 + \eta_1\psi_2\eta_3\psi_1)$
	$(\eta_3\psi_2\eta_1\psi_2 + \eta_1\psi_3\eta_2\psi_3 + \eta_2\psi_1\eta_3\psi_1)$

Table D.8

	$(\eta_i\psi_i)(\kappa_i\psi_i)$
$r = 3$	$(\eta_1\psi_1\kappa_2\psi_1 + \eta_2\psi_2\kappa_3\psi_2 + \eta_3\psi_3\kappa_1\psi_3)$
$r^2 = 3$	$(\eta_1\psi_1\kappa_3\psi_1 + \eta_2\psi_2\kappa_1\psi_2 + \eta_3\psi_3\kappa_2\psi_3)$
$r = 2$	$(\eta_2\psi_3\kappa_1\psi_2 + \eta_3\psi_1\kappa_2\psi_3 + \eta_1\psi_2\kappa_3\psi_1)$
	$(\eta_3\psi_2\kappa_1\psi_2 + \eta_1\psi_3\kappa_2\psi_3 + \eta_2\psi_1\kappa_3\psi_1)$
$r^2 = 2$	$(\eta_2\psi_3\kappa_1\psi_3 + \eta_3\psi_1\kappa_2\psi_1 + \eta_1\psi_2\kappa_3\psi_2)$
	$(\eta_3\psi_2\kappa_1\psi_2\psi_3 + \eta_1\psi_3\kappa_2\psi_1 + \eta_2\psi_1\kappa_3\psi_2)$

Table D.9

	$(\eta_i \psi_i)(\kappa_j \psi_i)$
$j = 3i$	$(\eta_1 \psi_1 \kappa_1 \psi_1 + \eta_2 \psi_2 \kappa_2 \psi_2 + \eta_3 \psi_3 \kappa_3 \psi_3)$
$j = 3ir$	$(\eta_2 \psi_2 \kappa_1 \psi_2 + \eta_3 \psi_3 \kappa_2 \psi_3 + \eta_1 \psi_1 \kappa_3 \psi_1)$
$j = 3ir^2$	$(\eta_1 \psi_1 \kappa_2 \psi_1 + \eta_2 \psi_2 \kappa_3 \psi_2 + \eta_3 \psi_3 \kappa_1 \psi_3)$
$j = i(r^2 - 1)$	$(\eta_3 \psi_3 \kappa_1 \psi_2 + \eta_1 \psi_1 \kappa_2 \psi_3 + \eta_2 \psi_2 \kappa_3 \psi_1)$
	$(\eta_3 \psi_1 \kappa_2 \psi_1 + \eta_1 \psi_2 \kappa_3 \psi_2 + \eta_2 \psi_3 \kappa_1 \psi_3)$
	$(\eta_1 \psi_3 \kappa_2 \psi_1 + \eta_2 \psi_1 \kappa_3 \psi_2 + \eta_3 \psi_2 \kappa_1 \psi_3)$
$j = i(r^2 - r)$	$(\eta_3 \psi_3 \kappa_1 \psi_1 + \eta_1 \psi_1 \kappa_2 \psi_2 + \eta_2 \psi_2 \kappa_3 \psi_3)$
	$(\eta_1 \psi_2 \kappa_2 \psi_1 + \eta_2 \psi_3 \kappa_3 \psi_2 + \eta_3 \psi_1 \kappa_1 \psi_3)$
	$(\eta_2 \psi_1 \kappa_2 \psi_1 + \eta_3 \psi_2 \kappa_3 \psi_2 + \eta_1 \psi_3 \kappa_1 \psi_3)$
$j = i(r - r^2)$	$(\eta_2 \psi_2 \kappa_1 \psi_1 + \eta_3 \psi_3 \kappa_2 \psi_2 + \eta_1 \psi_1 \kappa_3 \psi_3)$
	$(\eta_1 \psi_2 \kappa_1 \psi_2 + \eta_2 \psi_3 \kappa_2 \psi_3 + \eta_3 \psi_1 \kappa_3 \psi_1)$
	$(\eta_2 \psi_1 \kappa_1 \psi_2 + \eta_3 \psi_2 \kappa_2 \psi_3 + \eta_1 \psi_3 \kappa_3 \psi_1)$
$j = i(r - 1)$	$(\eta_3 \psi_3 \kappa_2 \psi_1 + \eta_1 \psi_1 \kappa_3 \psi_2 + \eta_2 \psi_2 \kappa_1 \psi_3)$
	$(\eta_2 \psi_3 \kappa_1 \psi_2 + \eta_3 \psi_1 \kappa_2 \psi_3 + \eta_1 \psi_2 \kappa_3 \psi_1)$
	$(\eta_3 \psi_2 \kappa_1 \psi_2 + \eta_1 \psi_3 \kappa_2 \psi_3 + \eta_2 \psi_1 \kappa_3 \psi_1)$
$j = i(1 - r)$	$(\eta_2 \psi_2 \kappa_2 \psi_1 + \eta_3 \psi_3 \kappa_3 \psi_2 + \eta_1 \psi_1 \kappa_1 \psi_3)$
	$(\eta_3 \psi_1 \kappa_1 \psi_1 + \eta_1 \psi_2 \kappa_2 \psi_2 + \eta_2 \psi_3 \kappa_3 \psi_3)$
	$(\eta_1 \psi_3 \kappa_1 \psi_1 + \eta_2 \psi_1 \kappa_2 \psi_2 + \eta_3 \psi_2 \kappa_3 \psi_3)$
$j = i(1 - r^2)$	$(\eta_1 \psi_1 \kappa_1 \psi_2 + \eta_2 \psi_2 \kappa_2 \psi_3 + \eta_3 \psi_3 \kappa_3 \psi_1)$
	$(\eta_1 \psi_2 \kappa_1 \psi_1 + \eta_2 \psi_3 \kappa_2 \psi_2 + \eta_3 \psi_1 \kappa_3 \psi_3)$
	$(\eta_2 \psi_1 \kappa_1 \psi_1 + \eta_3 \psi_2 \kappa_2 \psi_2 + \eta_1 \psi_3 \kappa_3 \psi_3)$

Table D.10

	$(\eta_j \psi_i)(\kappa_i \psi_i)$
$j = i(r - 2)$	$(\eta_1 \psi_1 \kappa_2 \psi_1 + \eta_2 \psi_2 \kappa_3 \psi_2 + \eta_3 \psi_3 \kappa_1 \psi_3)$
$j = i(r^2 - 2)$	$(\eta_2 \psi_2 \kappa_1 \psi_2 + \eta_3 \psi_3 \kappa_2 \psi_3 + \eta_1 \psi_1 \kappa_3 \psi_1)$
$j = i(1 - 2r^2)$	$(\eta_2 \psi_1 \kappa_2 \psi_1 + \eta_3 \psi_2 \kappa_3 \psi_2 + \eta_1 \psi_3 \kappa_1 \psi_3)$
$j = i(r^2 - 2r)$	$(\eta_3 \psi_1 \kappa_2 \psi_1 + \eta_1 \psi_2 \kappa_3 \psi_2 + \eta_2 \psi_3 \kappa_1 \psi_3)$
$j = i(1 - 2r)$	$(\eta_1 \psi_2 \kappa_1 \psi_2 + \eta_2 \psi_3 \kappa_2 \psi_3 + \eta_3 \psi_1 \kappa_3 \psi_1)$
$j = i(r - 2r^2)$	$(\eta_3 \psi_2 \kappa_1 \psi_2 + \eta_1 \psi_3 \kappa_2 \psi_3 + \eta_2 \psi_1 \kappa_3 \psi_1)$
$j = -ir^2$	$(\eta_2 \psi_2 \kappa_2 \psi_1 + \eta_3 \psi_3 \kappa_3 \psi_2 + \eta_1 \psi_1 \kappa_1 \psi_3)$
	$(\eta_3 \psi_1 \kappa_1 \psi_2 + \eta_1 \psi_2 \kappa_2 \psi_3 + \eta_2 \psi_3 \kappa_3 \psi_1)$
$j = -ir$	$(\eta_1 \psi_1 \kappa_1 \psi_2 + \eta_2 \psi_2 \kappa_2 \psi_3 + \eta_3 \psi_3 \kappa_3 \psi_1)$
	$(\eta_2 \psi_1 \kappa_1 \psi_3 + \eta_3 \psi_2 \kappa_2 \psi_1 + \eta_1 \psi_3 \kappa_3 \psi_2)$
$j = -i$	$(\eta_2 \psi_1 \kappa_1 \psi_2 + \eta_3 \psi_2 \kappa_2 \psi_3 + \eta_1 \psi_3 \kappa_3 \psi_1)$
	$(\eta_1 \psi_2 \kappa_2 \psi_1 + \eta_2 \psi_3 \kappa_3 \psi_2 + \eta_3 \psi_1 \kappa_1 \psi_3)$
$j = 2ir^2$	$(\eta_3 \psi_3 \kappa_2 \psi_1 + \eta_1 \psi_1 \kappa_3 \psi_2 + \eta_2 \psi_2 \kappa_1 \psi_3)$
	$(\eta_2 \psi_3 \kappa_1 \psi_2 + \eta_3 \psi_1 \kappa_2 \psi_3 + \eta_1 \psi_2 \kappa_3 \psi_1)$
$j = 2ir$	$(\eta_3 \psi_3 \kappa_1 \psi_2 + \eta_1 \psi_1 \kappa_2 \psi_3 + \eta_2 \psi_2 \kappa_3 \psi_1)$
	$(\eta_1 \psi_3 \kappa_2 \psi_1 + \eta_2 \psi_1 \kappa_3 \psi_2 + \eta_3 \psi_2 \kappa_1 \psi_3)$
$j = 2i$	$(\eta_2 \psi_3 \kappa_2 \psi_1 + \eta_3 \psi_1 \kappa_3 \psi_2 + \eta_1 \psi_2 \kappa_1 \psi_3)$
	$(\eta_1 \psi_3 \kappa_1 \psi_2 + \eta_2 \psi_1 \kappa_2 \psi_3 + \eta_3 \psi_2 \kappa_3 \psi_1)$

Table D.11

	$(\eta_j \psi_i)(\kappa_j \psi_i)$
$2i = j(r - 1)$	$(\eta_1 \psi_1 \kappa_2 \psi_1 + \eta_2 \psi_2 \kappa_3 \psi_2 + \eta_3 \psi_3 \kappa_1 \psi_3)$
$2i = j(1 - r^2)$	$(\eta_3 \psi_1 \kappa_1 \psi_1 + \eta_1 \psi_2 \kappa_2 \psi_2 + \eta_2 \psi_3 \kappa_3 \psi_3)$
$2i = j(1 - r)$	$(\eta_2 \psi_1 \kappa_1 \psi_1 + \eta_3 \psi_2 \kappa_2 \psi_2 + \eta_1 \psi_3 \kappa_3 \psi_3)$
$2i = j(r^2 - 1)$	$(\eta_2 \psi_2 \kappa_1 \psi_2 + \eta_3 \psi_3 \kappa_2 \psi_3 + \eta_1 \psi_1 \kappa_3 \psi_1)$
$2i = j(r - r^2)$	$(\eta_2 \psi_3 \kappa_1 \psi_3 + \eta_3 \psi_1 \kappa_2 \psi_1 + \eta_1 \psi_2 \kappa_3 \psi_2)$
$2i = j(r^2 - r)$	$(\eta_3 \psi_2 \kappa_1 \psi_2 + \eta_1 \psi_3 \kappa_2 \psi_3 + \eta_2 \psi_1 \kappa_3 \psi_1)$
$i = j(r - r^2)$	$(\eta_3 \psi_3 \kappa_1 \psi_1 + \eta_1 \psi_1 \kappa_2 \psi_2 + \eta_2 \psi_2 \kappa_3 \psi_3)$
	$(\eta_1 \psi_2 \kappa_2 \psi_1 + \eta_2 \psi_3 \kappa_3 \psi_2 + \eta_3 \psi_1 \kappa_1 \psi_3)$
$i = j(r^2 - r)$	$(\eta_2 \psi_2 \kappa_1 \psi_1 + \eta_3 \psi_3 \kappa_2 \psi_2 + \eta_1 \psi_1 \kappa_3 \psi_3)$
	$(\eta_2 \psi_1 \kappa_1 \psi_2 + \eta_3 \psi_2 \kappa_2 \psi_3 + \eta_1 \psi_3 \kappa_3 \psi_1)$
$i = j(r^2 - 1)$	$(\eta_3 \psi_3 \kappa_1 \psi_2 + \eta_1 \psi_1 \kappa_2 \psi_3 + \eta_2 \psi_2 \kappa_3 \psi_1)$
	$(\eta_1 \psi_3 \kappa_2 \psi_1 + \eta_2 \psi_1 \kappa_3 \psi_2 + \eta_3 \psi_2 \kappa_1 \psi_3)$
$i = j(1 - r^2)$	$(\eta_3 \psi_2 \kappa_2 \psi_1 + \eta_1 \psi_3 \kappa_3 \psi_2 + \eta_2 \psi_1 \kappa_1 \psi_3)$
	$(\eta_2 \psi_3 \kappa_1 \psi_1 + \eta_3 \psi_1 \kappa_2 \psi_2 + \eta_1 \psi_2 \kappa_3 \psi_3)$
$i = j(1 - r)$	$(\eta_3 \psi_2 \kappa_1 \psi_1 + \eta_1 \psi_3 \kappa_2 \psi_2 + \eta_2 \psi_1 \kappa_3 \psi_3)$
	$(\eta_3 \psi_1 \kappa_1 \psi_2 + \eta_1 \psi_2 \kappa_2 \psi_3 + \eta_2 \psi_3 \kappa_3 \psi_1)$
$i = j(r - 1)$	$(\eta_3 \psi_1 \kappa_2 \psi_3 + \eta_1 \psi_2 \kappa_3 \psi_1 + \eta_2 \psi_3 \kappa_1 \psi_2)$
	$(\eta_3 \psi_3 \kappa_2 \psi_1 + \eta_1 \psi_1 \kappa_3 \psi_2 + \eta_2 \psi_2 \kappa_1 \psi_3)$

Table D.12

	$(\kappa_i \psi_i)(\kappa_j \psi_i)$
$j = i(2 - r)$	$(\kappa_2 \psi_1 \kappa_1 \psi_1 + \kappa_3 \psi_2 \kappa_2 \psi_2 + \kappa_1 \psi_3 \kappa_3 \psi_3)$
$j = i(2r^2 - 1)$	$(\kappa_2 \psi_1 \kappa_2 \psi_1 + \kappa_3 \psi_2 \kappa_3 \psi_2 + \kappa_1 \psi_3 \kappa_1 \psi_3)$
$j = i(2r - 1)$	$(\kappa_1 \psi_2 \kappa_1 \psi_2 + \kappa_2 \psi_3 \kappa_2 \psi_3 + \kappa_3 \psi_1 \kappa_3 \psi_1)$
$j = i(2 - r^2)$	$(\kappa_1 \psi_2 \kappa_2 \psi_2 + \kappa_2 \psi_3 \kappa_3 \psi_3 + \kappa_3 \psi_1 \kappa_1 \psi_1)$
$j = i(2r - r^2)$	$(\kappa_3 \psi_2 \kappa_1 \psi_2 + \kappa_1 \psi_3 \kappa_2 \psi_3 + \kappa_2 \psi_1 \kappa_3 \psi_1)$
$j = i(2r^2 - r)$	$(\kappa_3 \psi_1 \kappa_2 \psi_1 + \kappa_1 \psi_2 \kappa_3 \psi_2 + \kappa_2 \psi_3 \kappa_1 \psi_3)$
$j = -2ir^2$	$(\kappa_3 \psi_1 \kappa_1 \psi_2 + \kappa_1 \psi_2 \kappa_2 \psi_3 + \kappa_2 \psi_3 \kappa_3 \psi_1)$
	$(\kappa_3 \psi_2 \kappa_1 \psi_1 + \kappa_1 \psi_3 \kappa_2 \psi_2 + \kappa_2 \psi_1 \kappa_3 \psi_3)$
$j = -2ir$	$(\kappa_3 \psi_2 \kappa_2 \psi_1 + \kappa_1 \psi_3 \kappa_3 \psi_2 + \kappa_2 \psi_1 \kappa_1 \psi_3)$
	$(\kappa_2 \psi_3 \kappa_1 \psi_1 + \kappa_3 \psi_1 \kappa_2 \psi_2 + \kappa_1 \psi_2 \kappa_3 \psi_3)$
$j = -2i$	$(\kappa_3 \psi_1 \kappa_3 \psi_2 + \kappa_1 \psi_2 \kappa_1 \psi_3 + \kappa_2 \psi_3 \kappa_2 \psi_1)$
	$(\kappa_3 \psi_2 \kappa_3 \psi_1 + \kappa_1 \psi_3 \kappa_1 \psi_2 + \kappa_2 \psi_1 \kappa_2 \psi_3)$
$j = ir$	$(\kappa_1 \psi_3 \kappa_2 \psi_1 + \kappa_2 \psi_1 \kappa_3 \psi_2 + \kappa_3 \psi_2 \kappa_1 \psi_3)$
$j = ir^2$	$(\kappa_2 \psi_3 \kappa_1 \psi_2 + \kappa_3 \psi_1 \kappa_2 \psi_3 + \kappa_1 \psi_2 \kappa_3 \psi_1)$
	$(\kappa_1 \psi_3 \kappa_1 \psi_1 + \kappa_2 \psi_1 \kappa_2 \psi_2 + \kappa_3 \psi_2 \kappa_3 \psi_3)$
$j = i$	$(\kappa_3 \psi_2 \kappa_2 \psi_3 + \kappa_1 \psi_3 \kappa_3 \psi_1 + \kappa_2 \psi_1 \kappa_1 \psi_2)$

Table D.13

	$(\eta_j \psi_i)(\eta_k \psi_i)$
$j = -2i - k$	$(\eta_1 \psi_1 \eta_1 \psi_1 + \eta_2 \psi_2 \eta_2 \psi_2 + \eta_3 \psi_3 \eta_3 \psi_3)$
$j = ir^2 - kr$	$(\eta_3 \psi_3 \eta_1 \psi_1 + \eta_1 \psi_1 \eta_2 \psi_2 + \eta_2 \psi_2 \eta_3 \psi_3)$
	$(\eta_3 \psi_1 \eta_1 \psi_3 + \eta_1 \psi_2 \eta_2 \psi_1 + \eta_2 \psi_3 \eta_3 \psi_2)$
$j = ir - kr^2$	$(\eta_2 \psi_2 \eta_1 \psi_1 + \eta_3 \psi_3 \eta_2 \psi_2 + \eta_1 \psi_1 \eta_3 \psi_3)$
$j = -k - 2ir$	$(\eta_1 \psi_2 \eta_1 \psi_2 + \eta_2 \psi_3 \eta_2 \psi_3 + \eta_3 \psi_1 \eta_3 \psi_1)$
$j = -kr + i$	$(\eta_1 \psi_2 \eta_2 \psi_3 + \eta_2 \psi_3 \eta_3 \psi_1 + \eta_3 \psi_1 \eta_1 \psi_2)$
$j = ir^2 - kr^2$	$(\eta_2 \psi_3 \eta_1 \psi_2 + \eta_3 \psi_1 \eta_2 \psi_3 + \eta_1 \psi_2 \eta_3 \psi_1)$
	$(\eta_3 \psi_3 \eta_2 \psi_1 + \eta_1 \psi_1 \eta_3 \psi_2 + \eta_2 \psi_2 \eta_1 \psi_3)$
$j = -k - 2ir^2$	$(\eta_2 \psi_1 \eta_2 \psi_1 + \eta_3 \psi_2 \eta_3 \psi_2 + \eta_1 \psi_3 \eta_1 \psi_3)$
$j = -kr + ir$	$(\eta_1 \psi_3 \eta_2 \psi_1 + \eta_2 \psi_1 \eta_3 \psi_2 + \eta_3 \psi_2 \eta_1 \psi_3)$
$j = -2ir^2 - kr^2$	$(\eta_2 \psi_1 \eta_1 \psi_3 + \eta_3 \psi_2 \eta_2 \psi_1 + \eta_1 \psi_3 \eta_3 \psi_2)$
$j = ir^2 - k$	$(\eta_1 \psi_1 \eta_1 \psi_2 + \eta_2 \psi_2 \eta_2 \psi_3 + \eta_3 \psi_3 \eta_3 \psi_1)$
$j = ir - k$	$(\eta_3 \psi_3 \eta_1 \psi_2 + \eta_1 \psi_1 \eta_2 \psi_3 + \eta_2 \psi_2 \eta_3 \psi_1)$
$j = -kr^2 - 2i$	$(\eta_2 \psi_2 \eta_1 \psi_2 + \eta_3 \psi_3 \eta_2 \psi_3 + \eta_1 \psi_1 \eta_3 \psi_1)$
$j = -kr - 2i$	$(\eta_1 \psi_1 \eta_2 \psi_1 + \eta_2 \psi_2 \eta_3 \psi_2 + \eta_3 \psi_3 \eta_1 \psi_3)$
$j = -ir - k$	$(\eta_2 \psi_2 \eta_2 \psi_1 + \eta_3 \psi_3 \eta_3 \psi_2 + \eta_1 \psi_1 \eta_1 \psi_3)$
$j = -kr^2 - 2ir$	$(\eta_1 \psi_2 \eta_3 \psi_2 + \eta_2 \psi_1 \eta_3 \psi_2 + \eta_3 \psi_1 \eta_1 \psi_2)$
$j = i - k$	$(\eta_3 \psi_1 \eta_3 \psi_2 + \eta_1 \psi_2 \eta_1 \psi_3 + \eta_2 \psi_3 \eta_2 \psi_1)$

Table D.14

	$(\eta_j \psi_i)(\kappa_k \psi_i)$
$j = kr - 2i$	$(\eta_1 \psi_1 \kappa_2 \psi_1 + \eta_2 \psi_2 \kappa_3 \psi_2 + \eta_3 \psi_3 \kappa_1 \psi_3)$
$j = k - 2i$	$(\eta_1 \psi_1 \kappa_1 \psi_1 + \eta_2 \psi_2 \kappa_2 \psi_2 + \eta_3 \psi_3 \kappa_3 \psi_3)$
$j = (k - 2i)r$	$(\eta_3 \psi_1 \kappa_1 \psi_1 + \eta_1 \psi_2 \kappa_2 \psi_2 + \eta_2 \psi_3 \kappa_3 \psi_3)$
$j = (k - 2i)r^2$	$(\eta_2 \psi_1 \kappa_1 \psi_1 + \eta_3 \psi_2 \kappa_2 \psi_2 + \eta_1 \psi_3 \kappa_3 \psi_3)$
$j = k - 2ir^2$	$(\eta_2 \psi_1 \kappa_2 \psi_1 + \eta_3 \psi_2 \kappa_3 \psi_2 + \eta_1 \psi_3 \kappa_1 \psi_3)$
$j = kr^2 - 2i$	$(\eta_2 \psi_2 \kappa_1 \psi_2 + \eta_3 \psi_3 \kappa_2 \psi_3 + \eta_1 \psi_1 \kappa_3 \psi_1)$
$j = kr^2 - 2ir$	$(\eta_2 \psi_3 \kappa_1 \psi_3 + \eta_3 \psi_1 \kappa_2 \psi_1 + \eta_1 \psi_2 \kappa_3 \psi_2)$
$j = k - 2ir$	$(\eta_2 \psi_3 \kappa_2 \psi_3 + \eta_3 \psi_1 \kappa_3 \psi_1 + \eta_1 \psi_2 \kappa_1 \psi_2)$
$j = kr - 2ir^2$	$(\eta_3 \psi_2 \kappa_1 \psi_2 + \eta_1 \psi_3 \kappa_2 \psi_3 + \eta_2 \psi_1 \kappa_3 \psi_1)$
$j = ir^2 + kr$	$(\eta_3 \psi_3 \kappa_1 \psi_1 + \eta_1 \psi_1 \kappa_2 \psi_2 + \eta_2 \psi_2 \kappa_3 \psi_3)$
	$(\eta_1 \psi_2 \kappa_2 \psi_1 + \eta_2 \psi_3 \kappa_3 \psi_2 + \eta_3 \psi_1 \kappa_1 \psi_3)$
$j = ir + kr^2$	$(\eta_2 \psi_2 \kappa_1 \psi_1 + \eta_3 \psi_3 \kappa_2 \psi_2 + \eta_1 \psi_1 \kappa_3 \psi_3)$
	$(\eta_2 \psi_1 \kappa_1 \psi_2 + \eta_3 \psi_2 \kappa_2 \psi_3 + \eta_1 \psi_3 \kappa_3 \psi_1)$
$j = (i + k)r$	$(\eta_3 \psi_3 \kappa_1 \psi_2 + \eta_1 \psi_1 \kappa_2 \psi_3 + \eta_2 \psi_2 \kappa_3 \psi_1)$
	$(\eta_1 \psi_3 \kappa_2 \psi_1 + \eta_2 \psi_1 \kappa_3 \psi_2 + \eta_3 \psi_2 \kappa_1 \psi_3)$
$j = i + k$	$(\eta_1 \psi_3 \kappa_1 \psi_2 + \eta_2 \psi_1 \kappa_2 \psi_3 + \eta_3 \psi_2 \kappa_3 \psi_1)$
	$(\eta_1 \psi_2 \kappa_1 \psi_3 + \eta_2 \psi_3 \kappa_2 \psi_1 + \eta_3 \psi_1 \kappa_3 \psi_2)$
$j = i + kr^2$	$(\eta_3 \psi_2 \kappa_2 \psi_1 + \eta_1 \psi_3 \kappa_3 \psi_2 + \eta_2 \psi_1 \kappa_1 \psi_3)$
	$(\eta_2 \psi_3 \kappa_1 \psi_1 + \eta_3 \psi_1 \kappa_2 \psi_2 + \eta_1 \psi_2 \kappa_3 \psi_3)$
$j = i + kr$	$(\eta_3 \psi_2 \kappa_1 \psi_1 + \eta_1 \psi_3 \kappa_2 \psi_2 + \eta_2 \psi_1 \kappa_3 \psi_3)$
	$(\eta_3 \psi_1 \kappa_1 \psi_2 + \eta_1 \psi_2 \kappa_2 \psi_3 + \eta_2 \psi_3 \kappa_3 \psi_1)$
$j = (i + k)r^2$	$(\eta_3 \psi_1 \kappa_2 \psi_3 + \eta_1 \psi_2 \kappa_3 \psi_1 + \eta_2 \psi_3 \kappa_1 \psi_2)$
	$(\eta_3 \psi_3 \kappa_2 \psi_1 + \eta_1 \psi_1 \kappa_3 \psi_2 + \eta_2 \psi_2 \kappa_1 \psi_3)$
$j = k + ir$	$(\eta_1 \psi_1 \kappa_1 \psi_3 + \eta_2 \psi_2 \kappa_2 \psi_1 + \eta_3 \psi_3 \kappa_3 \psi_2)$
	$(\eta_1 \psi_3 \kappa_1 \psi_1 + \eta_2 \psi_1 \kappa_2 \psi_2 + \eta_3 \psi_2 \kappa_3 \psi_3)$
$j = k + ir^2$	$(\eta_1 \psi_1 \kappa_1 \psi_2 + \eta_2 \psi_2 \kappa_2 \psi_3 + \eta_3 \psi_3 \kappa_3 \psi_1)$
	$(\eta_1 \psi_2 \kappa_1 \psi_1 + \eta_2 \psi_3 \kappa_2 \psi_2 + \eta_3 \psi_1 \kappa_3 \psi_3)$

Table D.15

	$(\kappa_j \psi_i)(\kappa_k \psi_i)$
$j = 2i - k$	$(\kappa_1 \psi_1 \kappa_1 \psi_1 + \kappa_2 \psi_2 \kappa_2 \psi_2 + \kappa_3 \psi_3 \kappa_3 \psi_3)$
$j = (2i - k)r^2$	$(\kappa_2 \psi_1 \kappa_1 \psi_1 + \kappa_3 \psi_2 \kappa_2 \psi_2 + \kappa_1 \psi_3 \kappa_3 \psi_3)$
$j = 2i - kr$	$(\kappa_1 \psi_1 \kappa_2 \psi_1 + \kappa_2 \psi_2 \kappa_3 \psi_2 + \kappa_3 \psi_3 \kappa_1 \psi_3)$
$j = 2ir^2 - k$	$(\kappa_2 \psi_1 \kappa_2 \psi_1 + \kappa_3 \psi_2 \kappa_3 \psi_2 + \kappa_1 \psi_3 \kappa_1 \psi_3)$
$j = 2ir - k$	$(\kappa_1 \psi_2 \kappa_1 \psi_2 + \kappa_2 \psi_3 \kappa_2 \psi_3 + \kappa_3 \psi_1 \kappa_3 \psi_1)$
$j = (2i - k)r$	$(\kappa_1 \psi_2 \kappa_2 \psi_2 + \kappa_2 \psi_3 \kappa_3 \psi_3 + \kappa_3 \psi_1 \kappa_1 \psi_1)$
$j = 2ir^2 - kr$	$(\kappa_3 \psi_2 \kappa_1 \psi_2 + \kappa_1 \psi_3 \kappa_2 \psi_3 + \kappa_2 \psi_1 \kappa_3 \psi_1)$
$j = 2ir - kr^2$	$(\kappa_3 \psi_1 \kappa_2 \psi_1 + \kappa_1 \psi_2 \kappa_3 \psi_2 + \kappa_2 \psi_3 \kappa_1 \psi_3)$
$j = 2i - kr^2$	$(\kappa_1 \psi_1 \kappa_3 \psi_1 + \kappa_2 \psi_2 \kappa_1 \psi_2 + \kappa_3 \psi_3 \kappa_2 \psi_3)$
$j = -ir^2 - kr$	$(\kappa_3 \psi_3 \kappa_1 \psi_1 + \kappa_1 \psi_1 \kappa_2 \psi_2 + \kappa_2 \psi_2 \kappa_3 \psi_3)$
	$(\kappa_1 \psi_2 \kappa_2 \psi_1 + \kappa_2 \psi_3 \kappa_3 \psi_2 + \kappa_3 \psi_1 \kappa_1 \psi_3)$
$j = -ir - kr^2$	$(\kappa_2 \psi_2 \kappa_1 \psi_1 + \kappa_3 \psi_3 \kappa_2 \psi_2 + \kappa_1 \psi_1 \kappa_3 \psi_3)$
	$(\kappa_3 \psi_2 \kappa_2 \psi_3 + \kappa_1 \psi_3 \kappa_3 \psi_1 + \kappa_2 \psi_1 \kappa_1 \psi_2)$
$j = -i - kr$	$(\kappa_3 \psi_1 \kappa_1 \psi_2 + \kappa_1 \psi_2 \kappa_2 \psi_3 + \kappa_2 \psi_3 \kappa_3 \psi_1)$
	$(\kappa_3 \psi_2 \kappa_1 \psi_1 + \kappa_1 \psi_3 \kappa_2 \psi_2 + \kappa_2 \psi_1 \kappa_3 \psi_3)$
$j = -i - kr^2$	$(\kappa_3 \psi_2 \kappa_2 \psi_1 + \kappa_1 \psi_3 \kappa_3 \psi_2 + \kappa_2 \psi_1 \kappa_1 \psi_3)$
	$(\kappa_2 \psi_3 \kappa_1 \psi_1 + \kappa_3 \psi_1 \kappa_2 \psi_2 + \kappa_1 \psi_2 \kappa_3 \psi_3)$
$j = -i - k$	$(\kappa_3 \psi_1 \kappa_3 \psi_2 + \kappa_1 \psi_2 \kappa_1 \psi_3 + \kappa_2 \psi_3 \kappa_2 \psi_1)$
	$(\kappa_3 \psi_2 \kappa_3 \psi_1 + \kappa_1 \psi_3 \kappa_1 \psi_2 + \kappa_2 \psi_1 \kappa_2 \psi_3)$
$j = -(i + k)r$	$(\kappa_1 \psi_3 \kappa_2 \psi_1 + \kappa_2 \psi_1 \kappa_3 \psi_2 + \kappa_3 \psi_2 \kappa_1 \psi_3)$
	$(\kappa_1 \psi_1 \kappa_2 \psi_3 + \kappa_2 \psi_2 \kappa_3 \psi_1 + \kappa_3 \psi_3 \kappa_1 \psi_2)$
$j = -(i + k)r^2$	$(\kappa_2 \psi_3 \kappa_1 \psi_2 + \kappa_3 \psi_1 \kappa_2 \psi_3 + \kappa_1 \psi_2 \kappa_3 \psi_1)$
	$(\kappa_1 \psi_1 \kappa_3 \psi_2 + \kappa_2 \psi_2 \kappa_1 \psi_3 + \kappa_3 \psi_3 \kappa_2 \psi_1)$
$j = -ir - k$	$(\kappa_1 \psi_3 \kappa_1 \psi_1 + \kappa_2 \psi_1 \kappa_2 \psi_2 + \kappa_3 \psi_2 \kappa_3 \psi_3)$
	$(\kappa_1 \psi_1 \kappa_1 \psi_3 + \kappa_2 \psi_2 \kappa_2 \psi_1 + \kappa_3 \psi_3 \kappa_3 \psi_2)$
$j = -ir^2 - k$	$(\kappa_1 \psi_1 \kappa_1 \psi_2 + \kappa_2 \psi_2 \kappa_2 \psi_3 + \kappa_3 \psi_3 \kappa_3 \psi_1)$
	$(\kappa_1 \psi_2 \kappa_1 \psi_1 + \kappa_2 \psi_3 \kappa_2 \psi_2 + \kappa_3 \psi_1 \kappa_3 \psi_3)$

Table D.16

www.nbi.ku.dk

Niels Bohr International Academy and Discovery Center
Niels Bohr Institute
Blegdamsvej 17
2100 Copenhagen
Denmark
Tel: (+45) 35 32 52 09
Email: info@nano.ku.dk