

CORRELATIONS IN 1D BOSON AND FERMION
SYSTEMS: EXACT RESULTS

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List of papers

Most of the results presented in this Thesis were reported in the following papers:

Vadim V. Cheianov and M.B. Zvonarev, Nonunitary Spin-Charge Separation in a One-Dimensional Fermion Gas, *Phys. Rev. Lett.* **92** (2004), 176401.

Vadim V. Cheianov and M.B. Zvonarev, Zero temperature correlation functions for the impenetrable fermion gas, *J. Phys. A: Math. Gen.* **37** (2004), 2261–2297.

Vadim V. Cheianov, H. Smith and M.B. Zvonarev, Low-temperature crossover in the momentum distribution of cold atomic gases in one dimension, *Phys. Rev. A* **71** (2005), 033610.

Vadim V. Cheianov, H. Smith and M.B. Zvonarev, Exact Results for Three-Body Correlations in a Degenerate One-Dimensional Bose Gas, cond-mat/0506609.

Abstract

In this Thesis I report the studies of the correlations in one-dimensional fermion and boson systems. For this, two complementary approaches are combined. The first one is an effective field theory approach, called the Luttinger Liquid theory; the second one is the Bethe Ansatz approach.

The main results presented in the Thesis are: (i) Motivated by recent experiments we derive an exact expression for the correlation function entering the three-body recombination rate for a one-dimensional gas of interacting bosons. The answer, given in terms of two thermodynamic parameters of the Lieb-Liniger model, is valid for all values of the dimensionless coupling γ and contains the previously known results for the Bogoliubov and Tonks-Girardeau regimes as limiting cases. (ii) For the two-component strongly repulsive quantum gases an emergence of a new “spin-disordered” state is predicted. A crossover to this state occurs at a temperature much lower than the Fermi temperature. This new temperature scale, which equals the Fermi temperature divided by the dimensionless coupling strength, is a feature of the two-component model and does not exist in the one-component case. The correlation functions in the ‘spin-disordered’ regime shows signs of spin-charge separation with scaling behavior in the charge part and exponential decay as a function of the space coordinate in the spin part. This behavior is strikingly different from that of in the Luttinger regime.

Introduction

The effects of reduced dimensionality are very dramatic for both boson and fermion systems. Particularly fascinating things happen in the quantum world of one spacial dimension [22, 20]. Such characteristic features of one dimension as statistical transmutation, spin-charge separation, absence of the long-range order, etc. gave a new insight into physics of strongly correlated systems. The field is developing actively, and many (if not most) subfields were opened in the last 25 years. A contribution from the experimentalists is increasing permanently. Systems such as carbon nanotubes, quantum wires, cold atoms in highly elongated traps, stripes in high T_c compounds are perfectly one-dimensional and can be studied in great details.

Theoretical tools for working with 1D systems are rather specific. On the one hand, the usual Fermi liquid theory, based on a quasi-particle picture, breaks down in one dimension. More generally, perturbative techniques have rather limited applications in 1D, as compared to higher dimensions. On the other hand, there are several powerful non-perturbative methods working in 1D, whose higher-dimensional analogues are not discovered yet.

I concentrate in this Thesis on applications of two non-perturbative techniques, Luttinger liquid theory and theory of the exactly solvable models, to some problems in condensed matter physics. These techniques are complementary to each other, and

most of the original results presented in this Thesis are obtained by combining them.

The Thesis consists of five chapters. Chapter 1 reviews the effective field theory approach to the description of correlated 1D fermion and boson systems at low energies. The Luttinger liquid concept is discussed and the Luttinger parameters are calculated for the microscopic models studied in the Thesis: Lieb-Liniger model and two-component impenetrable fermion gas. The long-distance asymptotics of correlation functions in these microscopic models are calculated using the Luttinger liquid approach. The content of this chapter is well presented in the literature.

In chapter 2 the coordinate Bethe ansatz solution of the Lieb-Liniger model is discussed. The exact eigenfunctions and the spectrum of the model are obtained for arbitrary number of particles. The ground state properties and the thermodynamic limit are studied in details. The Haldane's microscopic calculations [23] of the Luttinger parameters of the model are given. The Bethe ansatz solution of the Lieb-Liniger model works for arbitrary coupling strength. A significant part of the chapter is devoted to studies of the strong coupling (Bogoliubov) and weak coupling (Tonks-Girardeau) limits of the model. Like for chapter 1, the content of the chapter 2 is well presented in the literature, except, probably, the weak coupling limit studied in section 2.3.4: some of the results presented in that section have been obtained rather recently [47].

Chapter 3 is devoted to studies of the correlation functions in the Lieb-Liniger model. It consists of two non-connected parts. The first part, section 3.1, deals with the exact expression for the one-particle density matrix in the infinite repulsion

(Tonks-Girardeau) limit of the Lieb-Liniger model. The density matrix in the thermodynamic limit is expressed via the Fredholm determinant of a linear integral operator (Lenard's formula). The Fredholm determinant representation is a familiar object in the theory of exactly solvable models. The studies of section 3.1 are intended primarily to get an intuition about this object, more complicated Fredholm determinants are discussed in chapter 4. Note that the long-distance asymptotics of the density matrix obtained from the Lenard's formula coincides with that one obtained by using of the Luttinger liquid approach, thus giving the microscopic proof of a validity of the Luttinger liquid hypothesis. The content of section 3.1 is not original; the purpose of writing this section was mostly to introduce some objects used in chapter 4.

The second part of the chapter 3, section 3.2, contains my original studies on the local three-body correlation function g_3 in the Lieb-Liniger model. These studies were motivated by the experimental findings of Ref. [44]. Namely, by measuring a decay rate in a quasi-one-dimensional atomic cloud of cold ^{87}Rb atoms, the authors of Ref. [44] have obtained the value of g_3 for a particular value of the dimensionless coupling constant γ . I give the expression for g_3 valid for all γ , Eq. (3.2.13), and compare this expression with the experimental data in Fig. 3.5. These results are original and were reported in Ref. [9].

In chapter 4 I am studying the gas of strongly repulsive spin 1/2 fermions at infinitesimal temperature. It is show that the presence of the spin degrees of freedom results in the emergence of a new low-temperature spin-disordered state. The presence of this state is marked by a pronounced change of the momentum distribution with increasing temperature. This reconstruction is shown in figure 4.1 and, I expect it will be studied experimentally in the nearest future. These predictions are original and

were reported in Ref. [10]. Another original result presented in chapter 4 is long time and distance asymptotics of dynamical correlation functions in the spin-disordered regime, Eq. (4.2.32). These asymptotics shows signs of spin-charge separation with scaling behavior in the charge part and exponential decay as a function of the space coordinate in the spin part and are strikingly different from those calculated in the Luttinger regime. This result was reported in Refs. [11, 12].

Chapter 5 is technical: it is devoted to the calculation of Eq. (3.2.13). The outline of the calculations is given in section 3.2.3, so one can omit chapter 5 if not interested in technical details. The content of this chapter is not published yet and, unfortunately, the notations are not optimized and sometimes differ from those used in chapters 1–4 (I strongly suspect that the method of calculations could be optimized as well).

Chapter 1

Luttinger Liquid theory

As far as their low energy properties are concerned, most physically important one-dimensional systems belong to a certain universality class usually referred to as the Luttinger liquid [22, 20]. By bosonization the Luttinger liquid theory is mapped onto a theory of free massless boson fields, whose correlation functions are calculated trivially. This free boson theory is reviewed briefly in section 1.1. In section 1.2 we construct, following Ref. [25], the exact mapping from a fermion theory (non-interacting Luttinger model) onto the free boson theory. In section 1.3 we extend this mapping to interacting spinless fermion theories. A similar extension to the interacting boson theories is performed in section 1.4. We obtain critical exponents for a particular microscopic model (Lieb-Liniger model) from its thermodynamic properties. The bosonization procedure for the interacting fermion systems with spin is discussed in section 1.5, with particular emphasis on the theory of δ -interacting impenetrable fermion gas, whose critical exponents are obtained from the thermodynamic properties in much the same spirit as for the Lieb-Liniger model in section 1.4.

1.1 Free massless real scalar boson field

This section is devoted to the apparently trivial theory of the free massless real scalar boson field in one space and one time dimension. The purpose of this section is merely to fix the notations and the presentation will be rather brief.

1.1.1 Lagrangian and Hamiltonian

It is often more convenient to work with the Euclidean (imaginary) time τ rather than with the Minkovsky time t :

$$\tau = it. \quad (1.1.1)$$

The Lagrangian density in the Minkovsky space is

$$\mathcal{L}_M = -\frac{1}{2\pi K} \left[-\frac{1}{v}(\partial_t\phi)^2 + v(\partial_x\phi)^2 \right]. \quad (1.1.2)$$

and the corresponding Lagrangian density in the Euclidean space is

$$\mathcal{L}_E = \frac{1}{2\pi K} \left[\frac{1}{v}(\partial_\tau\phi)^2 + v(\partial_x\phi)^2 \right]. \quad (1.1.3)$$

The corresponding imaginary-time action

$$S_E = \frac{1}{2\pi K} \int dx d\tau \left[\frac{1}{v}(\partial_\tau\phi)^2 + v(\partial_x\phi)^2 \right] \quad (1.1.4)$$

is the action of the free massless scalar field. Note the relation $e^{iS_M} = e^{-S_E}$ which helps to control signs in Eqs. (1.1.2) and (1.1.3); S_M is the Minkovsky action. The canonical momentum is

$$\Pi = \frac{\partial\mathcal{L}_M}{\partial\partial_t\phi} = \frac{1}{\pi K v} \partial_t\phi = \frac{i}{\pi K v} \partial_\tau\phi. \quad (1.1.5)$$

Being quantized, the fields ϕ and Π obey canonical equal-time commutation relations

$$[\Pi(x), \phi(y)] = -i\delta(x - y). \quad (1.1.6)$$

The Hamiltonian of the system has the form

$$H = \frac{v_S}{2\pi} \int dx : K(\pi\Pi)^2 + \frac{1}{K}(\partial_x\phi)^2 : . \quad (1.1.7)$$

The symbol $::$ stands for the normal ordering; it is discussed below Eq. (1.1.18). The parameter v in Eq. (1.1.2) has the dimension of velocity. In condensed matter physics this velocity is associated with the velocity of sound waves, v_S . We will use the symbol v_S instead of v throughout the thesis. The dimensionless parameter K is often called Luttinger parameter.

The Hamiltonian (1.1.7) is a component of the energy-momentum tensor. Another component of this tensor gives the momentum operator (the corresponding calculations can be found in any textbook on Quantum Field theory, for example, in Ref. [17]):

$$P = - \int dx : \Pi(x) \left[\partial_x\phi(x) - \frac{\pi N_0}{L} \right] : . \quad (1.1.8)$$

1.1.2 Momentum space representation

Let us put the theory on a ring of circumference L and impose periodic boundary conditions on $\Pi(x)$ and $\partial_x\phi(x)$ fields. Write

$$\phi(x) = \phi_0 - \pi x \frac{N - N_0}{L} - \frac{i}{2} \sum_{q \neq 0} \left| \frac{2\pi K}{qL} \right|^{1/2} e^{-\alpha|q|/2} \text{sgn}(q) e^{-iqx} (b_q^\dagger + b_{-q}) \quad (1.1.9)$$

and define a field $\theta(x)$,

$$\theta(x) = \theta_0 + \pi x \frac{J}{L} + \frac{i}{2} \sum_{q \neq 0} \left| \frac{2\pi}{qLK} \right|^{1/2} e^{-\alpha|q|/2} e^{-iqx} (b_q^\dagger - b_{-q}). \quad (1.1.10)$$

whose derivative with respect to x gives the field $\Pi(x)$ canonically conjugated to $\phi(x)$, Eq. (1.1.6),

$$\Pi(x) = \frac{1}{\pi} \partial_x \theta(x), \quad (1.1.11)$$

therefore

$$\Pi(x) = \frac{J}{L} + \frac{1}{2\pi} \sum_{q \neq 0} \left| \frac{2\pi q}{LK} \right|^{1/2} e^{-\alpha|q|/2} \text{sgn}(q) e^{-iqx} (b_q^\dagger - b_{-q}). \quad (1.1.12)$$

Note a useful expression

$$\frac{1}{\pi} \partial_x \phi(x) = -\frac{N - N_0}{L} - \frac{1}{2\pi} \sum_{q \neq 0} \left| \frac{2\pi K q}{L} \right|^{1/2} e^{-\alpha|q|/2} e^{-iqx} (b_q^\dagger + b_{-q}). \quad (1.1.13)$$

After some algebra one gets the momentum space representation of the Hamiltonian (1.1.7):

$$H = \frac{\pi}{2L} [v_N (N - N_0)^2 + v_J J^2] + v_S \sum_{q \neq 0} |q| b_q^\dagger b_q. \quad (1.1.14)$$

Here

$$v_J = v_S K, \quad v_N = \frac{v_S}{K}. \quad (1.1.15)$$

The commutation relations between the operators N , J , ϕ_0 , and θ_0 are given in section 1.1.3. The summation index q in Eqs. (1.1.9)–(1.1.14) runs through the following set of values:

$$q = \frac{2\pi}{L} j, \quad j = \pm 1, \pm 2, \dots \quad (1.1.16)$$

The operators b_q^\dagger (b_q) are boson creation (annihilation) operators obeying canonical commutation relations

$$[b_q, b_{q'}^\dagger] = \delta_{qq'}. \quad (1.1.17)$$

Thus, the last term in the right hand side of Eq. (1.1.14) represents a set of decoupled harmonic oscillators with the frequencies $v_S |q|$; the ground state of the system should obey

$$b_k |0\rangle = 0 \quad \text{for} \quad k = \pm 1, \pm 2, \dots \quad (1.1.18)$$

The symbol $::$ stands in Eq (1.1.7) for the normal ordering: in any given monomial it puts creation operators b_q^\dagger to the left from the annihilation operators b_q . Note that

it is customary not to write the symbol $::$ explicitly when writing the Hamiltonian (1.1.7), though in most cases the normal ordering is assumed implicitly.

Concluding this section, write the mode expansion for the momentum operator (1.1.8). Substituting Eqs. (1.1.12) and (1.1.13) into Eq. (1.1.8) one gets

$$P = JN \frac{\pi}{L} + \sum_{q \neq 0} q b_q^\dagger b_q. \quad (1.1.19)$$

1.1.3 Boson and zero-mode excitations

One can see that the sum over q in Eq. (1.1.14) does not contain the term $b_q^\dagger b_q$ with $q = 0$; instead, the operator U_0 stands:

$$U_0 = v_N (N - N_0)^2 + v_J J^2. \quad (1.1.20)$$

The term (1.1.20) is sometimes called zero-frequency, or zero-mode term. The operators N_R and N_L obey the following commutation relations:

$$[N, J] = 0, \quad [N, b_q] = [N, b_q^\dagger] = 0, \quad [J, b_q] = [J, b_q^\dagger] = 0. \quad (1.1.21)$$

There are, thus, three different type of excitations in the theory: boson excitations coming from the term containing $\sum_{q \neq 0} b_q^\dagger b_q$ in the Hamiltonian (1.1.14), excitations coming from the term containing $(N - N_0)^2$, and excitations coming from the term containing J^2 . One should bear in mind, however, that the parameters v_J and v_N are constrained by the condition

$$v_S = \sqrt{v_N v_J}. \quad (1.1.22)$$

In Bethe-ansatz solvable models [33] the parameters v_S , v_N , and v_J can be defined by analyzing the thermodynamic Bethe ansatz equations. Since these parameters can be defined independently of each other, Eq. (1.1.22) can be used to test whether the

model under consideration belongs to the Luttinger liquid universality class. It is, of course, only a necessary condition: if Eq. (1.1.22) is violated in a given model, this model does not belong to the Luttinger liquid universality class.

Discuss now the properties of the operator ϕ_0 entering Eq. (1.1.9). This operator commutes with boson fields:

$$[\phi_0, b_q] = [\phi_0, b_q^\dagger] = 0. \quad (1.1.23)$$

Checking the consistency of the equal-time commutation relation (1.1.6) with the representations (1.1.9) and (1.1.12) (Eqs. (1.1.17) and (1.1.21) should be taken into account) one gets the following condition

$$[J, \phi_0] = -i. \quad (1.1.24)$$

Therefore N and ϕ_0 are conjugated action-angle variables. They obey

$$[J, e^{i\phi_0}] = e^{i\phi_0}. \quad (1.1.25)$$

The commutation relation between J and ϕ_0 can be obtained by considering the time evolution of the field $\phi(x)$:

$$\pi v_J \Pi(x, t) = \partial_t \phi(x, t) = i[H, \phi(x, t)]. \quad (1.1.26)$$

Substituting Eqs. (1.1.9) and (1.1.14) into Eq. (1.1.26), and comparing the result with Eq. (1.1.12) one gets

$$[N, \phi_0] = 0. \quad (1.1.27)$$

Concluding this section, consider the operator θ_0 entering Eq. (1.1.10). We have started from the Lagrangian approach to the free boson model, section 1.1.1. The field $\phi(x)$ and its derivatives were used in this approach, but not the field $\theta(x)$. The

only physical condition which fixes the field $\theta(x)$ is Eq. (1.1.11), so the operator θ_0 is not uniquely defined. We set

$$[\phi_0, \theta_0] = 0, \quad (1.1.28)$$

and, like for ϕ_0 ,

$$[\theta_0, b_q] = [\theta_0, b_q^\dagger] = 0. \quad (1.1.29)$$

Finally, take Eqs. (1.1.10) and (1.1.13) and calculate the commutator

$$[\theta(x), \frac{1}{\pi} \partial_y \phi(y)] = \frac{1}{L} ([N, \theta_0] - i) + i\delta(x - y). \quad (1.1.30)$$

We set

$$[N, \theta_0] = i, \quad (1.1.31)$$

which implies the following duality between $\phi(x)$ and $\theta(x)$: Eq. (1.1.30) transforms to Eq. (1.1.6) when $\phi(x)$ is replaced by $\theta(x)$ and vice versa. One gets from Eq. (1.1.31)

$$[N, e^{-i\theta_0}] = e^{-i\theta_0}. \quad (1.1.32)$$

For the commutator of J and θ_0 we set

$$[J, \theta_0] = 0. \quad (1.1.33)$$

We stress that it is ϕ_0 and θ_0 fields that are responsible for the transitions between the sectors with different N and J . This is clearly seen from Eqs. (1.1.25) and (1.1.32).

1.1.4 Gauge invariance

By using formulas from sections 1.1.2 and 1.1.3 one gets the following relations between the $\phi(x)$ and $\theta(x)$ fields:

$$\partial_t \theta(x) = \frac{v_S}{K} \partial_x \phi(x), \quad \partial_x \theta(x) = \frac{1}{K v_S} \partial_t \phi(x). \quad (1.1.34)$$

The Lagrangian (1.1.2) written via the θ -field is

$$\mathcal{L}_M = \frac{K}{2\pi} \left[-\frac{1}{v_S} (\partial_t \theta)^2 + v_S (\partial_x \theta)^2 \right]. \quad (1.1.35)$$

We denote by $\eta_{\mu\nu}$ the following diagonal metric tensor:

$$\eta_{\mu\nu} = \begin{pmatrix} -v_S & 0 \\ 0 & \frac{1}{v_S} \end{pmatrix}, \quad \eta^{\mu\nu} = \begin{pmatrix} -\frac{1}{v_S} & 0 \\ 0 & v_S \end{pmatrix}. \quad (1.1.36)$$

The conversion rules between covariant and contravariant indices are as usual:

$$a_\mu = \eta_{\mu\nu} a^\nu, \quad a^\nu = \eta^{\nu\mu} a_\mu, \quad \eta_{\mu\nu} \eta^{\nu\sigma} = \delta_\mu^\sigma. \quad (1.1.37)$$

The covariant form of the Lagrangian (1.1.35) is

$$\mathcal{L}_M = \frac{K}{2\pi} \partial_\mu \theta \partial^\mu \theta. \quad (1.1.38)$$

We want to make the free boson theory invariant under the following *local* transformation of the θ field:

$$\theta(x) \rightarrow \theta(x) + a(x). \quad (1.1.39)$$

Here x is a two-dimensional vector (x_0, x_1) . To ensure this invariance, we couple $\theta(x)$ with the gauge field A_μ (called the vector potential)

$$\mathcal{L}_M^{\text{gauge}} = \frac{K}{2\pi} (\partial_\mu \theta \partial^\mu \theta - 2A_\mu \partial^\mu \theta + A_\mu A^\mu), \quad S_M^{\text{gauge}} = \int dx dt \mathcal{L}_M^{\text{gauge}}, \quad (1.1.40)$$

where A_μ changes under the transformation (1.1.39) as follows:

$$A_\mu \rightarrow A_\mu + \partial_\mu a. \quad (1.1.41)$$

Therefore, the conserved gauge current $\mathcal{J}^\mu(A)$

$$\partial_\mu \mathcal{J}^\mu(A) = 0 \quad (1.1.42)$$

can be obtained by solving the equation of motion :

$$\frac{\delta S_M^{\text{gauge}}}{\delta A_\mu} = 0. \quad (1.1.43)$$

One gets

$$\mathcal{J}^\mu(A) = -\frac{K}{\pi}(A^\mu - \partial^\mu\theta). \quad (1.1.44)$$

The overall factor in the right hand side of Eq. (1.1.44) is fixed by the conditions (1.1.47). The current calculated at zero vector potential is

$$\mathcal{J}^\mu = \frac{K}{\pi}\eta^{\mu\nu}\partial_\nu\theta. \quad (1.1.45)$$

Written in components, this equation gives

$$\mathcal{J}^0 = -\frac{1}{\pi}\frac{K}{v_S}\partial_t\theta(x), \quad \mathcal{J}^1 = \frac{1}{\pi}Kv_S\partial_x\theta(x). \quad (1.1.46)$$

Using Eqs. (1.1.34), (1.1.11), and (1.1.15) one transforms Eq. (1.1.46) to

$$\mathcal{J}^0 = -\frac{1}{\pi}\partial_x\phi, \quad \mathcal{J}^1 = v_J\Pi(x). \quad (1.1.47)$$

The theory (1.1.40) in the Hamiltonian formalism is

$$\mathcal{H}^{\text{gauge}} = \mathcal{H} + \mathcal{J}^\mu A_\mu, \quad (1.1.48)$$

where \mathcal{H} is the Hamiltonian density corresponding to the Hamiltonian (1.1.7). We have dropped in Eq. (1.1.48) the terms quadratic in A_μ . Integrating $\mathcal{H}^{\text{gauge}}$ over x one gets

$$H^{\text{gauge}} = H + A_0(N - N_0) + A_1v_JJ. \quad (1.1.49)$$

1.1.5 Thermodynamics

The free boson model defined in the section 1.1.1 contains two free parameters: K and v_S . We define in this section these parameters from the thermodynamics of the model. The parameter N_0 in Eq. (1.1.14) is the number of particles in the absolute ground state of the system. In order to work within the grand canonical ensemble one should add the term $-\mu(N - N_0)$ to the Hamiltonian (1.1.14). The number of particles in the system at a given chemical potential μ is defined by the equation

$$\left(\frac{\partial \langle H - \mu N \rangle}{\partial N} \right)_\mu = 0. \quad (1.1.50)$$

Assuming that $\langle J \rangle = 0$ when $\langle N \rangle$ is changed, one gets from Eqs. (1.1.50) and (1.1.14):

$$\frac{\pi v_N}{L} (N - N_0) - \mu = 0. \quad (1.1.51)$$

Therefore

$$\left(\frac{\partial \mu}{\partial N} \right)_L = \frac{\pi v_N}{L}, \quad \text{that is} \quad \left(\frac{\partial \mu}{\partial \rho} \right)_L = \pi v_N. \quad (1.1.52)$$

One of the definitions of the compressibility is

$$\kappa_{\text{true}} = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial \mu} \right)_L. \quad (1.1.53)$$

In view of the equation (1.1.52) it is convenient to redefine the compressibility as follows:

$$\kappa = \left(\frac{\partial \rho}{\partial \mu} \right)_L, \quad (1.1.54)$$

therefore

$$\kappa = \frac{1}{\pi v_N}. \quad (1.1.55)$$

We will use the definition (1.1.54) hereafter. Note also the following useful representation for v_N :

$$v_N = \frac{L}{\pi} \left(\frac{\partial^2 \langle H \rangle}{\partial N^2} \right) \Big|_{N=N_0}. \quad (1.1.56)$$

One can easily recognize the chemical potential in the expression (1.1.49):

$$\mu = -A_0. \quad (1.1.57)$$

Varying A_0 one changes the number of particles in the absolute ground state of the system. The ground state average of the Hamiltonian (1.1.49) depend on A_0 as follows:

$$\langle H^{\text{gauge}} \rangle = -\frac{L}{2\pi} \frac{A_0^2}{v_N}, \quad (1.1.58)$$

therefore

$$\frac{1}{v_N} = -\frac{\pi}{L} \left(\frac{\partial^2 \langle H^{\text{gauge}} \rangle}{\partial A_0^2} \right)_L. \quad (1.1.59)$$

Consider now the opposite case: $A_0 = 0$ and A_1 is varying. The absolute ground state of the Hamiltonian H^{gauge} is achieved when the condition

$$A_1 = -\frac{\pi}{L} \langle J \rangle \quad (1.1.60)$$

is fulfilled. One has

$$\langle H^{\text{gauge}} \rangle = -\frac{\pi}{2L} v_J \langle J \rangle^2 = -\frac{L}{2\pi} v_J A_1^2, \quad (1.1.61)$$

therefore

$$v_J = -\frac{\pi}{L} \left(\frac{\partial^2 \langle H^{\text{gauge}} \rangle}{\partial A_1^2} \right)_L. \quad (1.1.62)$$

1.1.6 Chiral fields

Introduce the following fields:

$$\phi_r(x) = \frac{r\pi x}{L} N_r - i \sum_{q \neq 0} Y(rq) \left| \frac{2\pi}{Lq} \right|^{1/2} e^{-\alpha|q|/2} e^{iqx} b_q, \quad r = \pm 1, \quad (1.1.63)$$

where the function $Y(p)$ is the Heaviside step function:

$$Y(p) = \begin{cases} 1 & p > 0 \\ 0 & p < 0 \end{cases}. \quad (1.1.64)$$

The symbol r in Eq. (1.1.63) takes two values, $+1$ and -1 . The fields carrying the subindex r are called right(R), or right-moving, chiral fields when $r = +1$, and left(L), or left-moving, chiral fields when $r = -1$:

$$r = +1, -1 \quad \text{is equivalent to} \quad r = R, L. \quad (1.1.65)$$

The symbol r will be used as defined by Eq. (1.1.65) throughout the Thesis.

The commutation relations for the chiral fields (1.1.63) are

$$[\phi_r(x), \phi_{r'}(x')] = 0, \quad [\phi_r^\dagger(x), \phi_{r'}^\dagger(x')] = 0, \quad (1.1.66)$$

and

$$[\phi_r(x), \phi_{r'}^\dagger(x')] = -\delta_{rr'} \lim_{\alpha \rightarrow +0} \ln(1 - z^{2r} e^{-2\pi\alpha/L}), \quad z = \exp \frac{i\pi(x - x')}{L}. \quad (1.1.67)$$

In calculations of various correlation functions it is necessary to normal order the exponents of chiral fields. Recall that if the commutator of two operators, A and B , is a c -number, than

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}, \quad e^A e^B = e^B e^A e^{[A,B]}. \quad (1.1.68)$$

Using Eq. (1.1.67), one gets

$$\exp\{s[\phi_r(x), \phi_r^\dagger(x')]\} = (1 - z^{2r} e^{-2\pi\alpha/L})^{-s}, \quad (1.1.69)$$

where s is an arbitrary positive number. At $x = x'$ this expression diverges when $\alpha \rightarrow 0$:

$$\exp\{s[\phi_r(x), \phi_r^\dagger(x)]\} \simeq \left(\frac{L}{2\pi\alpha}\right)^s, \quad \alpha \rightarrow 0. \quad (1.1.70)$$

The parameter α plays a role of the ultraviolet cutoff in the momentum space and thus mimics a finite bandwidth in the theory. In the regime

$$|x - x'| \gg \alpha \quad \text{and} \quad x - x' \ll L \quad (1.1.71)$$

one gets for Eq. (1.1.70)

$$\exp\{s[\phi_r(x), \phi_r^\dagger(x')]\} \simeq \text{const} \frac{L^s}{|x - x'|^s}. \quad (1.1.72)$$

Equation (1.1.72) is crucial for the calculation of the correlation functions.

Write the fields $\phi(x)$ and $\theta(x)$ via the chiral fields:

$$\phi(x) - \phi_0 = -\frac{\sqrt{K}}{2} \sum_r r[\phi_r(x) + \phi_r^\dagger(x)], \quad (1.1.73)$$

$$\theta(x) - \theta_0 = \frac{1}{2\sqrt{K}} \sum_r [\phi_r(x) + \phi_r^\dagger(x)]. \quad (1.1.74)$$

Comparing Eqs. (1.1.73) and (1.1.74) with Eqs. (1.1.9) and (1.1.10) one gets

$$N_R = \frac{1}{2} \left(\frac{N - N_0}{\sqrt{K}} + J\sqrt{K} \right), \quad N_L = \frac{1}{2} \left(\frac{N - N_0}{\sqrt{K}} - J\sqrt{K} \right). \quad (1.1.75)$$

Introduce the fields

$$\bar{\phi}_R = \sqrt{K}\theta_0 - \frac{\phi_0}{\sqrt{K}}, \quad \bar{\phi}_L = \sqrt{K}\theta_0 + \frac{\phi_0}{\sqrt{K}}. \quad (1.1.76)$$

The commutation relations between $\bar{\phi}_r$ and N_r are

$$[N_r, N_{r'}] = 0, \quad [\bar{\phi}_r, \bar{\phi}_{r'}] = 0, \quad [N_r, \bar{\phi}_{r'}] = i\delta_{rr'}. \quad (1.1.77)$$

Unitary operators

$$\bar{U}_r = e^{i\bar{\phi}_r}, \quad U_r = (-1)^{rN_{-r}/2} \bar{U}_r. \quad (1.1.78)$$

play an important role in the theory. The commutation relations for \bar{U}_r are

$$[N_r, \bar{U}_{r'}] = -\bar{U}_r \delta_{rr'}, \quad [\bar{U}_r, \bar{U}_{r'}] = 0, \quad [\bar{U}_r, \bar{U}_{r'}^\dagger] = 0. \quad (1.1.79)$$

The operators U_r and U_{-r} anticommute:

$$[N_r, U_{r'}] = -U_r \delta_{rr'}, \quad [U_r, U_{-r}]_+ = 0, \quad [U_r, U_{-r}^\dagger]_+ = 0. \quad (1.1.80)$$

We will need the commutation relations between U_r and $\phi_r(x)$:

$$[\phi_r(x), U_{r'}] = [\phi_r^\dagger(x), U_{r'}] = \frac{r\pi x}{L} [N_r, U_{r'}] = -\frac{r\pi x}{L} U_r \delta_{rr'}, \quad (1.1.81)$$

therefore

$$\exp[i\phi_r(x)] U_r \exp[-i\phi_r(x)] = \exp[i\phi_r^\dagger(x)] U_r \exp[-i\phi_r^\dagger(x)] = \exp[-i\frac{r\pi x}{L}] U_r \quad (1.1.82)$$

and

$$\exp[-i\phi_r(x)] U_r \exp[i\phi_r(x)] = \exp[-i\phi_r^\dagger(x)] U_r \exp[i\phi_r^\dagger(x)] = \exp[i\frac{r\pi x}{L}] U_r. \quad (1.1.83)$$

When $L \rightarrow \infty$, the terms $\exp[\pm ir\pi x/L]$ in Eqs. (1.1.82) and (1.1.83) become negligible, and

$$[U_r, e^{\pm i\phi_r(x)}] = [U_r, e^{\pm i\phi_r^\dagger(x)}] = 0, \quad L \rightarrow \infty. \quad (1.1.84)$$

We will use this property extensively.

1.2 Non-interacting Luttinger model

As a first example of the model whose low-energy physics is described by the free boson theory Eq. (1.1.7) we consider the non-interacting Luttinger model. There is, an *exact* mapping between the fermion operators in this model and the boson fields $\phi(x)$ and $\theta(x)$. This mapping provides us with a lot of intuition how the fields $\phi(x)$ and $\theta(x)$ are related to the fermion (boson) operators in interacting fermion (boson) theories. A detailed discussion of the non-interacting (as well as of the interacting) Luttinger model is given in Ref. [25]; we will outline main results from this paper without deriving them.

1.2.1 Fermion representation

The non-interacting Luttinger model contains two type of fermions, right and left ones, both obeying strictly linear dispersion. The corresponding Hamiltonian is

$$H = v_F \sum_{kr} (rk - k_F)(n_{kr} - \langle n_{kr} \rangle), \quad \langle n_{kr} \rangle = Y(k_F - kr). \quad (1.2.1)$$

Here

$$n_{kr} = c_{kr}^\dagger c_{kr}, \quad (1.2.2)$$

the function $Y(p)$ is defined by Eq. (1.1.64), and the symbol r by Eq. (1.1.65). The right-moving c_{kR} and left-moving c_{kL} fermions obey canonical equal-time anti-commutation relations:

$$[c_{kr}, c_{k'r'}^\dagger]_+ = \delta_{kk'} \delta_{rr'}. \quad (1.2.3)$$

The model is defined on a ring of circumference L and the periodic boundary conditions are imposed. The possible values of k are quantized:

$$k = \frac{2\pi}{L} j, \quad j = 0, \pm 1, \pm 2, \dots \quad (1.2.4)$$

The parameter k_F is the Fermi momentum of the spinless fermion gas:

$$k_F = \pi \frac{N_0}{L}. \quad (1.2.5)$$

The structure of the ground state is as follows: the orbitals are occupied when $k < k_F$ and are empty when $k > k_F$ for the right fermions; the orbitals are occupied when $k > -k_F$ and are empty when $k < -k_F$ for the left fermions. To ensure the non-degeneracy of the ground state we assume N_0 to be odd. The number of right (left) fermions N_R (N_L) is

$$N_r = \sum_k (n_{kr} - \langle n_{kr} \rangle). \quad (1.2.6)$$

The chiral fermion operators in the real space are

$$\Psi_r(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} e^{-\alpha|k|/2} c_{kr}, \quad \Psi_r^\dagger(x) = \frac{1}{\sqrt{L}} \sum_k e^{-ikx} e^{-\alpha|k|/2} c_{kr}^\dagger. \quad (1.2.7)$$

The anti-commutation relations for Ψ_r and Ψ_r^\dagger are canonical:

$$[\Psi_r(x), \Psi_{r'}^\dagger(x')]_+ = \delta_{rr'} \delta(x - x'). \quad (1.2.8)$$

Written in terms of the chiral fermion fields (1.2.7), the Hamiltonian (1.2.1) has the form

$$H = v_F \int_0^L dx \sum_r : \Psi_r^\dagger(-ir\partial_x - k_F) \Psi_r(x) :, \quad (1.2.9)$$

where the symbol $::$ stands for the *fermion* normal ordering.

There is an infinite number of fermions in the ground state of the model. We include the (fermion) normal ordering prescription in the definition of the chiral density operator $\rho_r(x)$ to subtract this infinity:

$$\rho_r(x) = : \Psi_r^\dagger(x) \Psi_r(x) :. \quad (1.2.10)$$

The Fourier transform of the chiral density operator

$$\rho_{qr} = \int_0^L dx e^{-iqx} : \Psi_r^\dagger(x) \Psi_r(x) := \sum_k : c_{k-qr}^\dagger c_{kr} :, \quad \rho_{qr}^\dagger = \rho_{-qr}. \quad (1.2.11)$$

obeys *anomalous* commutation relations:

$$[\rho_{qr}, \rho_{-q'r'}] = \delta_{rr'} \delta_{qq'} \frac{Lr q}{2\pi}. \quad (1.2.12)$$

This anomaly appears due to a relativistic character of the model: an infinite depths of the Fermi sea. One can recognize in Eq. (1.2.12) the commutation relations for the $U(1)$ Kac-Moody algebra [22]. This algebraic structure provides one with a very efficient tool for classification of the operator content of the theory. Its discussion, however, lies outside the main line of the Thesis.

1.2.2 Boson representation

The fermion theory (1.2.1) is mapped onto the boson theory (1.1.7) taken at the point

$$v_J = v_S, \quad K = 1. \quad (1.2.13)$$

A relation between the boson operator b_q and the density operator (1.2.11) is simple and elegant:

$$b_q = \left| \frac{2\pi}{Lq} \right|^{1/2} \sum_r Y(qr) \rho_{qr}, \quad q \neq 0. \quad (1.2.14)$$

The inverse to Eq. (1.2.14) is

$$\rho_{qr} = N_r \delta_{q0} + \left| \frac{Lq}{2\pi} \right|^{1/2} [Y(qr)b_q + Y(-qr)b_{-q}^\dagger]. \quad (1.2.15)$$

The boson representation for the chiral fermion operator $\Psi_r(x)$ is a bit more complicated:

$$\Psi_r^\dagger(x) = L^{-1/2} e^{-irk_{\text{F}}x} e^{-i\phi_r^\dagger(x)} U_r^\dagger e^{-i\phi_r(x)}. \quad (1.2.16)$$

Using the property (1.1.84) one can write in the large L limit

$$\begin{aligned} \Psi_r^\dagger(x) &= L^{-1/2} e^{-irk_{\text{F}}x} U_r^\dagger e^{-i\phi_r^\dagger(x)} e^{-i\phi_r(x)} \\ &= L^{-1/2} e^{-irk_{\text{F}}x} e^{-i\phi_r^\dagger(x)} e^{-i\phi_r(x)} U_r^\dagger, \quad L \rightarrow \infty. \end{aligned} \quad (1.2.17)$$

Eqs. (1.1.68) and (1.1.70) allow us to rewrite Eq. (1.2.17) in the following form:

$$\Psi_r^\dagger(x) \simeq \frac{U_r^\dagger}{\sqrt{2\pi\alpha}} e^{-irk_{\text{F}}x} \exp\{-i[\phi_r^\dagger(x) + \phi_r(x)]\}, \quad \alpha \rightarrow 0 \quad (1.2.18)$$

This equation, in turn, can be transformed with the help of Eqs. (1.1.73) and (1.1.74):

$$\Psi_r^\dagger(x) \simeq \frac{U_r^\dagger}{\sqrt{2\pi\alpha}} e^{-irk_{\text{F}}x} \exp\{-i[\tilde{\theta}(x) - r\tilde{\phi}(x)]\}, \quad \alpha \rightarrow 0, \quad (1.2.19)$$

where

$$\tilde{\phi}(x) = \phi(x) - \phi_0, \quad \tilde{\theta}(x) = \theta(x) - \theta_0. \quad (1.2.20)$$

Eqs. (1.2.17) and (1.2.19) provide us with two alternative representations for the chiral fermion operator. The representation (1.2.17) looks much more attractive because Eq. (1.2.19) contains singular prefactor $(2\pi\alpha)^{-1/2}$ and non-normal-ordered operator $\exp\{-i[\theta(x) - r\phi(x)]\}$. The reason for writing the chiral fermion operator in the form given by Eq. (1.2.19) is that this form is well adopted for the generalization to the theories where the conditions (1.2.13) do not hold.

1.2.3 Correlation functions

We calculate the following two-point correlation function of the chiral fermion fields:

$$\tilde{\mathcal{C}}_r(x) = \langle \Psi_r^\dagger(x) \Psi_{r'}(0) \rangle. \quad (1.2.21)$$

The average $\langle \dots \rangle$ in Eq. (1.2.21) is taken over the absolute ground state of the system, that is $\langle N_r \rangle = 0$. Since the operator U_r changes the number of the chiral fermions, one has $\langle \Psi_r^\dagger(x) \Psi_{-r}(x') \rangle = 0$. For the fields of the same chirality one gets, using Eq. (1.2.17):

$$\tilde{\mathcal{C}}_r(x) = L^{-1} e^{-irk_{\text{F}}x} \langle e^{-i\phi_r(x)} e^{i\phi_r^\dagger(0)} \rangle. \quad (1.2.22)$$

Applying the formulas (1.1.68) and (1.1.72) to Eq. (1.2.22) one gets

$$\tilde{\mathcal{C}}_r(x) \simeq \text{const} e^{-irk_{\text{F}}x} |x|^{-1}. \quad (1.2.23)$$

It is evident from the representation (1.2.17) that the generalization of the correlation function (1.2.21) to the time-dependent case

$$\tilde{\mathcal{C}}_r(x, t) = \langle \Psi_r^\dagger(x, t) \Psi_{r'}(0, 0) \rangle \quad (1.2.24)$$

can be written as

$$\tilde{\mathcal{C}}_r(x) = L^{-1} e^{-irk_{\text{F}}x} \langle e^{-i\phi_r(x_r)} e^{i\phi_r^\dagger(0)} \rangle \simeq \text{const} e^{-irk_{\text{F}}x} |x_r|^{-1}, \quad (1.2.25)$$

where

$$x_r = x - rvst. \quad (1.2.26)$$

1.3 Interacting spinless fermions

We have considered in section 1.2 an example of the fermion theory whose operators can be expressed via the operators of the free boson theory. A great breakthrough in the theory of 1D strongly correlated systems was made possible by extending of this mapping to a wide class of interacting fermion (and boson) systems. Unlike Eq. (1.2.17), the mapping is not exact any more, and its usage should be limited to the description of the *low-energy* properties of interacting systems.

1.3.1 Boson form of the fermion operator

Both the microscopic analysis of the fermion system with the density-density interaction [25] and “first-principle” considerations [24] suggest the following low-energy approximation to the fermion operator of the spinless interacting fermion system

$$\Psi(x) \simeq C_L(x)e^{-ik_F x} + C_R(x)e^{ik_F x}, \quad (1.3.1)$$

where

$$C_r(x) = U_r \exp\{i[\tilde{\theta}(x) - r\tilde{\phi}(x)]\}. \quad (1.3.2)$$

Eqs. (1.3.1) and (1.3.2) deserve some comments: The right hand side of Eq. (1.3.1) contains an overall x -independent factor, not written explicitly. We know this factor for the non-interacting Luttinger model, considered in section 1.2: it is equal to $(2\pi\alpha)^{-1/2}$, as can be seen from Eq. (1.2.19). But, this factor depends on the details

of the interaction potential of a microscopic model for which the low-energy approximation (1.3.1) is applied. Given two microscopic models with the same Luttinger parameters K and v_S , this factor will, in principle, be different. Thus, the low-energy approximation (1.3.1) (which is often called the ‘‘Luttinger Liquid hypothesis’’) can be used to extract the critical exponents in the infrared asymptotics of correlation functions, but not their amplitudes.

We rewrite now Eq. (1.3.2) via the chiral fields $\phi_p(x)$. It follows from Eqs. (1.1.73), (1.1.74), and (1.2.20) that

$$\tilde{\theta}(x) - r\tilde{\phi}(x) = -\frac{1}{2} \sum_{r'} r' \left(\sqrt{K} + r \frac{r'}{\sqrt{K}} \right) [\phi_{r'}(x) + \phi_{r'}^\dagger(x)]. \quad (1.3.3)$$

The normal ordering procedure for the operators $C_r(x)$ is trivial: it follows from Eq. (1.1.70) that $C_r(x)$ differs from $:C_r(x):$ by a multiplicative factor. Since we do not care about an overall prefactor in the low-energy approximation (1.3.1), we will assume hereafter that the operators $C_r(x)$ are normal ordered. Substituting the representation (1.3.3) into Eq. (1.3.2), one gets

$$C_r(x) = U_r \exp \left[-\frac{i}{2} \sum_{r'} r' \left(\sqrt{K} + \frac{rr'}{\sqrt{K}} \right) \phi_{r'}^\dagger(x) \right] \\ \times \exp \left[-\frac{i}{2} \sum_{r'} r' \left(\sqrt{K} + \frac{rr'}{\sqrt{K}} \right) \phi_{r'}(x) \right]. \quad (1.3.4)$$

Note that the normal-ordered operators $:C_r(x):$, Eq. (1.3.4), are called *vertex operators* in the conformal field theory [17]. From now on, the normal ordering symbol will be implicitly assumed when writing C_r .

1.3.2 Correlation functions

One can see from the representation (1.3.1) for the fermion operator that any correlation function of a microscopic theory consists of the “elementary blocks”: correlation functions of the vertex operators. It will be sufficient for our purposes to calculate the two-point correlation function of vertex operators

$$\mathcal{C}_r(x) = \langle C_r^\dagger(x) C_r(0) \rangle. \quad (1.3.5)$$

Note that the correlator of vertex operators with different chiralities is equal to zero. We have removed the oscillating factor $e^{-irk_F x}$ from the definition of $\mathcal{C}_r(x)$, as compared to the definition of $\tilde{\mathcal{C}}_r(x)$, Eq. (1.2.21). Substituting Eq. (1.3.4) into Eq. (1.3.5) and using the commutation relations (1.1.68) and (1.1.72) one gets

$$\mathcal{C}_r(x) \simeq \text{const} |x|^{-\frac{1}{2}(K+\frac{1}{K})}, \quad x \rightarrow \infty. \quad (1.3.6)$$

A generalization to the time-dependent case is obvious:

$$\mathcal{C}_r(x, t) \simeq \text{const} |x_R|^{-\frac{1}{4}(\sqrt{K}+\frac{r}{\sqrt{K}})^2} |x_L|^{-\frac{1}{4}(\sqrt{K}-\frac{r}{\sqrt{K}})^2}, \quad x, t \rightarrow \infty. \quad (1.3.7)$$

The variables x_R and x_L are defined by Eq. (1.2.26).

Consider now a correlation function of the microscopic fermion fields

$$G_h(x, t) = \langle \Psi^\dagger(x, t) \Psi(0, 0) \rangle. \quad (1.3.8)$$

Here the subscript h indicates that this function is of “hole” type: the fermion annihilation operator stands on the right hand side from the fermion creation operator. Using the representation (1.3.1) for the fermion operator, and the expression (1.3.7)

for the correlation function of the vertex operators, one gets

$$\begin{aligned} G_h(x, t) &\simeq \sum_r e^{-ir k_F x} \mathcal{C}_r(x, t) \\ &= \text{const} \sum_r e^{-ir k_F x} |x_R|^{-\frac{1}{4}(\sqrt{K} + \frac{r}{\sqrt{K}})^2} |x_L|^{-\frac{1}{4}(\sqrt{K} - \frac{r}{\sqrt{K}})^2}, \quad x, t \rightarrow \infty. \end{aligned} \quad (1.3.9)$$

In case $t = 0$ the correlation function (1.3.8) is called the one-particle reduced density matrix $\rho(x)$:

$$\rho(x) = \langle \Psi^\dagger(x) \Psi(0) \rangle. \quad (1.3.10)$$

Its long distance asymptotics is

$$\rho(x) \simeq \text{const} \times x^{-\frac{1}{2}(K + \frac{1}{K})} \cos(k_F x), \quad x \rightarrow \infty. \quad (1.3.11)$$

The momentum distribution function $n(k)$ is given by

$$n(k) = \int dx e^{-ikx} \rho(x), \quad (1.3.12)$$

where $\rho(x)$ is the one-particle density matrix (1.3.10). The structure of the long-distance asymptotics (1.3.11) implies the power-law singularity of $n(k)$ near k_F :

$$n(k) \propto |k - k_F|^{\frac{1}{2}(K + \frac{1}{K}) - 1}, \quad k \rightarrow k_F. \quad (1.3.13)$$

To be more precise,

$$n(k) \simeq n(k_F) - \text{const} \times \text{sgn}(k - k_F) |k - k_F|^{\frac{1}{2}(K + \frac{1}{K}) - 1}, \quad k \rightarrow k_F. \quad (1.3.14)$$

A power-law singularity of the momentum distribution function near k_F is a characteristic property of the Luttinger liquid universality class. An example of such a singularity can be seen in Fig. 4.1.

1.4 Interacting bosons

We have presented in section 1.3.1, Eq. (1.3.1), the relation between the fermion operator of a microscopic theory and the vertex operators of the free boson theory. A similar expression will be given in section 1.4.1 for a *boson* operator of a microscopic boson theory. The long-distance asymptotics of the correlation functions of the microscopic theory then follow readily. In section 1.4.2 we discuss the correlation functions of the particular model: non-relativistic bosons interacting via the δ -function potential (Lieb-Liniger model).

1.4.1 Bosonization of boson theories

The analysis carried out in the paper [24] suggests the following representation for the boson operator in the U(1)-invariant theories:

$$\Psi(x) \simeq C_R(x) + C_L(x), \quad (1.4.1)$$

where

$$C_r(x) = \bar{U}_r \exp\{ir\tilde{\theta}(x)\}. \quad (1.4.2)$$

The operators U_r commute with each other

$$[\bar{U}_r, \bar{U}_{r'}] = 0, \quad [\bar{U}_r, \bar{U}_{r'}^\dagger] = 0, \quad (1.4.3)$$

which is a consequence of the boson statistics of the operator (1.4.1). We are interested in the long-distance asymptotics of the correlation function (1.3.5). Its calculation is very similar to that one given in section for fermion problem. Using the chiral fields representation (1.1.74) for the vertex operators (1.4.2) one gets

$$\mathcal{C}_r(x) \simeq \text{const } |x|^{-\frac{1}{2K}}, \quad x \rightarrow \infty. \quad (1.4.4)$$

The long-distance asymptotic of the one-particle density matrix (1.3.10) for boson theories follows from Eqs. (1.4.1) and (1.4.4):

$$\rho(x) \simeq \sum_r \mathcal{C}_r(x) = \text{const} \times |x|^{-\frac{1}{2K}}, \quad x \rightarrow \infty. \quad (1.4.5)$$

The momentum distribution function (1.3.12) has a singularity at $k = 0$:

$$n(k) \propto |k|^{\frac{1}{2K}-1}, \quad k \rightarrow 0. \quad (1.4.6)$$

1.4.2 Lieb-Liniger model

A detailed study of the Lieb-Liniger model is performed in chapters 2 and 3. In the present section we discuss exclusively the long-distance asymptotics of the density matrix, Eq. (1.4.5). It is shown in section 2.3.3 that the velocity does not depend on coupling strength and is given by Eq. (2.3.48). Therefore, the Luttinger parameter K depend on coupling only via the sound velocity:

$$K^{-1} = \frac{v_S}{2\pi\rho}. \quad (1.4.7)$$

It can be seen from Fig. 2.1 that K changes from infinity to one when the coupling changes from zero to infinity. It is a general feature of the boson theories with repulsive interactions to have the Luttinger parameter $K \geq 1$, while in fermion theories with repulsive interaction $K \leq 1$.

The limit of infinite repulsion in the Lieb-Liniger model, when $K = 1$, is known as the Tonks-Girardeau limit. Section (3.1) is devoted to microscopic calculations of the density matrix in this limit. These calculations confirm the Luttinger liquid theory predictions, as can be seen by comparing Eqs. (1.4.5) and (3.1.27).

1.5 Interacting fermions with spin

Concluding chapter 1, we discuss the Luttinger liquid approach to interacting fermions with spin. The primary purpose of this section is to give the asymptotic expressions for correlation functions of the impenetrable fermion gas. The presentation will be rather brief: we just summarize the necessary modifications of the procedure carried out for the interacting spinless fermions in section 1.3, and write the asymptotics used in chapter 4, leaving the technical details aside.

1.5.1 Bosonization procedure

In the theories with spin the low energy spectrum is described by the effective Hamiltonian [20, 22]

$$H_{\text{eff}} = H_c + H_\sigma \quad (1.5.1)$$

where H_c describes charge fluctuations, H_σ describes spin fluctuations and

$$[H_c, H_\sigma] = 0. \quad (1.5.2)$$

Independent dynamics of spin and charge is called spin-charge separation [22, 20]. Hamiltonians H_c and H_σ are the Hamiltonians of the two independent free boson theories, Eq. (1.1.7). There are thus four parameters in the model (1.5.1): K_c, v_c, K_σ , and v_σ , where v_σ and v_c are the propagation velocities of spin and charge excitations respectively. If the Hamiltonian of a microscopic theory possess a global $SU(2)$ symmetry associated with the spin-rotational invariance, then the following constraint on K_σ is imposed:

$$K_\sigma = 1. \quad (1.5.3)$$

Like for spinless case, Eq. (1.3.1), the fermion operator $\Psi_\sigma(x)$ is represented within the theory (1.5.1) as a sum of anti-commuting left and right fermion fields

$$\Psi_\sigma(x) = \Psi_{L,\sigma}(x)e^{-ik_F x} + \Psi_{R,\sigma}(x)e^{ik_F x}, \quad \sigma = \uparrow, \downarrow, \quad (1.5.4)$$

where k_F is the Fermi momentum,

$$k_F = \frac{\pi N_0}{2L}. \quad (1.5.5)$$

It should be stressed that the definitions of k_F for fermions with and without spin are different, compare Eqs. (1.5.5) and (1.2.5). The left and right fermion fields are products of spin and charge vertex operators

$$\Psi_{L,\sigma} = C_L S_{L,\sigma}, \quad \Psi_{R,\sigma} = C_R S_{R,\sigma}, \quad (1.5.6)$$

where the spin operators satisfy $S_{L(R),\downarrow} = S_{L(R),\uparrow}^\dagger$. Operators S and C commute with each other and satisfy

$$[S, H_c] = [C, H_s] = 0. \quad (1.5.7)$$

The correlation function

$$G_h(x, t) = \langle \Psi_\uparrow^\dagger(x, t) \Psi_\uparrow(0, 0) \rangle \quad (1.5.8)$$

has the following form within the Luttinger approximation

$$G_h(x, t) \simeq \sum_r e^{-irk_F x} \mathcal{C}_r(x, t) \mathcal{S}_r(x, t), \quad (1.5.9)$$

where

$$\mathcal{S}_r(x, t) = \langle S_r^\dagger(x, t) S_r(0, 0) \rangle = \frac{\text{const}}{(x - rv_\sigma t)^{1/2}}. \quad (1.5.10)$$

and

$$\mathcal{C}_r(x, t) = \langle C_r^\dagger(x, t) C_r(0, 0) \rangle = \frac{\text{const}}{(x - rv_c t)^{2\Delta_c} (x + rv_c t)^{2\bar{\Delta}_c}}. \quad (1.5.11)$$

The expressions for the vertex operators via the chiral fields, needed to calculate the correlation functions (1.5.10) and (1.5.11) can be found in much the same way as for the spinless case.

The anomalous dimensions Δ and $\bar{\Delta}$ are determined by the Luttinger parameter K_c which can be found from the charge compressibility $\kappa_c = \partial n_c / \partial \mu$ of the system as follows (compare with Eq. (1.1.55) for spinless case)

$$K_c = \frac{\pi v_c}{2} \kappa_c. \quad (1.5.12)$$

One has [22]

$$\Delta_c = \frac{1}{16} \left(\sqrt{K_c} + \frac{1}{\sqrt{K_c}} \right)^2, \quad \bar{\Delta}_c = \frac{1}{16} \left(\sqrt{K_c} - \frac{1}{\sqrt{K_c}} \right)^2. \quad (1.5.13)$$

The density matrix for the fermions with spin is defined by Eq. (4.2.24). When the interaction is invariant with respect to spin rotation, one has $\rho(x) = G_h(x, 0)$, where $G_h(x, 0)$ is defined by Eq. (1.5.8). Therefore,

$$\rho(x) \simeq \text{const} \times x^{-\frac{1}{2} - \frac{1}{4}(K_c + \frac{1}{K_c})}, \quad x \rightarrow \infty. \quad (1.5.14)$$

1.5.2 Impenetrable fermion gas

We are interested in the correlation functions of the two-component fermion gas (4.1.2) in the limit $c \rightarrow \infty$. For any finite c parameters v_c and κ_c can be calculated from the thermodynamic Bethe ansatz. For infinite c fermions become impenetrable, which from the point of view of thermodynamics means that they satisfy the Pauli principle independently of their spin orientation. For such a system parameters v_c and κ are calculated in the same way as for a system of spinless non-interacting fermions [42]. Calculating the right hand side of (1.5.12) for spinless fermions one finds

$$K_c = \frac{1}{2}. \quad (1.5.15)$$

For the anomalous dimensions in the charge sector this gives

$$\Delta_c = \frac{9}{32}, \quad \bar{\Delta}_c = \frac{1}{32}. \quad (1.5.16)$$

A correlation function for impenetrable fermions in the form (1.5.9) with anomalous exponents in the charge sector given by (1.5.16) was suggested in Ref. [16]. It was also discussed in [41, 40], where the $c \rightarrow \infty$ limit was analyzed microscopically.

For the density matrix one gets from Eqs. (1.5.14) and (1.5.15):

$$\rho(x) \simeq \text{const} \times x^{-9/8} \sin(k_F x), \quad x \rightarrow \infty. \quad (1.5.17)$$

This result implies the power-law singularity of the momentum distribution function, Eq. (1.3.12), at $k = k_F$:

$$n(k) \propto |k - k_F|^{1/8}, \quad k \rightarrow k_F. \quad (1.5.18)$$

This singularity can be seen in Fig. 4.1.

Chapter 2

Lieb-Liniger model

In this chapter we discuss the one-dimensional gas of bosons interacting via the δ -function potential. This model, often called the Lieb-Liniger model, is exactly solvable: its eigenfunctions and spectrum were found by Lieb and Liniger [35]. These authors used the model to check the validity of the Bogoliubov's theory of the weakly interacting ($\gamma \ll 1$) Bose gas (here γ is the dimensionless coupling strength, Eq. (2.1.2)). In the opposite limit, $\gamma = \infty$, known as the Tonks-Girardeau limit [45, 21], the spectrum becomes the same as that of a free-fermion gas. In lowering γ from $\gamma = \infty$ to arbitrarily small, the model exhibits no phase transition thus indicating that the difference between the boson- and fermion-type behavior in 1D is much less pronounced as compared to higher dimensions. The model has attracted further interest in 80-s, when the "Luttinger liquid" concept, discussed in chapter 1, was developed. In particular, the velocity parameters v_N , v_J , and v_S , obtained from the exact solution of the model were compared with those obtained within the Luttinger liquid approximation [23]. This was an important step towards the validation of the Luttinger liquid hypothesis. Another important property of the model, the complete integrability, was discovered at about the same time; its detailed description together with

an extensive list of references can be found in Ref. [33]. In the last decade, the Lieb-Liniger model was realized experimentally using cold atomic gases confined in highly elongated harmonic traps, see Refs. [44, 39, 29] and references therein.

In the present chapter we overview the (coordinate) Bethe ansatz solution and zero-temperature thermodynamics of the Lieb-Liniger model. In section 2.1 we discuss the Hamiltonian and the structure of the Fock space of the model. In the section 2.2 we give a detailed treatment of the model in the Tonks-Girardeau limit with the purpose to get an intuition about the Bethe ansatz solution for arbitrary coupling strength. We construct the complete set of eigenfunctions, find the eigenvalues, and discuss the zero-temperature thermodynamics of the model. The same objects are calculated in section 2.3 at arbitrary coupling strength.

The results of this chapter are not new and well presented in the literature. The only subject which is, probably, less known is discussed in section 2.3.4: it is the $\gamma \rightarrow 0$ limit of the model. The Bethe ansatz solution is very singular in this limit and a method for finding the small γ expansion of the thermodynamic quantities (such as the ground state energy or the sound velocity) beyond the leading order was developed only recently in Ref. [47].

2.1 The Hamiltonian

The Hamiltonian of the one-dimensional Bose gas with zero-range interaction potential is

$$H = \int dx [-\Psi^\dagger(x) \partial_x^2 \Psi(x) + c \Psi(x)^\dagger \Psi(x)^\dagger \Psi(x) \Psi(x)], \quad (2.1.1)$$

where the units have been chosen such that $\hbar = 2m = 1$. The interaction constant $c \geq 0$ has the dimension of inverse length. The dimensionless coupling strength γ is given by

$$\gamma = \frac{c}{\rho}. \quad (2.1.2)$$

The boson fields Ψ and Ψ^\dagger satisfy canonical equal-time commutation relations:

$$[\Psi(x), \Psi^\dagger(y)] = \delta(x - y), \quad [\Psi^\dagger(x), \Psi^\dagger(y)] = [\Psi(x), \Psi(y)] = 0. \quad (2.1.3)$$

The number of particles operator N is

$$N = \int dx \Psi^\dagger(x) \Psi(x), \quad (2.1.4)$$

and the momentum operator P is

$$P = -i \int dx \Psi^\dagger(x) \partial_x \Psi(x). \quad (2.1.5)$$

The number of particles and momentum are integrals of motion:

$$[H, N] = [H, P] = 0. \quad (2.1.6)$$

The equation of motion for the Hamiltonian (2.1.1)

$$i\partial_t \Psi = -\partial_x^2 \Psi + 2c\Psi^\dagger \Psi \Psi \quad (2.1.7)$$

is called the quantum nonlinear Schrödinger equation.

Since the Hamiltonian commutes with N , Eq. (2.1.6), the number of particles in the model is conserved and one can look for the eigenfunctions of the Hamiltonian (2.1.1) in a sector with the fixed number of particles. Write

$$|\Psi_N(k_1, \dots, k_N)\rangle = \frac{1}{\sqrt{N!}} \int d^N x \chi_N(x_1, \dots, x_N | k_1, \dots, k_N) \Psi^\dagger(x_1) \dots \Psi^\dagger(x_N) |0\rangle. \quad (2.1.8)$$

Here $|\Psi_N(k_1, \dots, k_N)\rangle$ is an eigenfunction of the second-quantized Hamiltonian (2.1.1):

$$H|\Psi_N(k_1, \dots, k_N)\rangle = E_N(k_1, \dots, k_N)|\Psi_N(k_1, \dots, k_N)\rangle, \quad (2.1.9)$$

the symbol $|0\rangle$ denotes the Fock vacuum state

$$\Psi(x)|0\rangle = 0, \quad \langle 0|\Psi^\dagger(x) = 0, \quad \langle 0|0\rangle = 1, \quad (2.1.10)$$

the symbol $d^N x \equiv dx_1 \dots dx_N$, and the parameters k_1, \dots, k_N are called quasi-momenta.

The coordinate wave function χ_N is an eigenfunction of the Hamiltonian (2.1.1) written in the first-quantized form:

$$\mathcal{H}_N = -\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + 2c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j), \quad (2.1.11)$$

$$\mathcal{H}_N \chi_N = E_N(k_1, \dots, k_N) \chi_N. \quad (2.1.12)$$

The Bose statistics of the model is encoded in the symmetry properties of χ_N : it should be symmetric with respect to the arbitrary permutations of the coordinates x_1, \dots, x_N . Finally, write the momentum operator (2.1.5) in the first-quantized form:

$$\mathcal{P}_N = -i \sum_{j=1}^N \frac{\partial}{\partial x_j}. \quad (2.1.13)$$

No boundary conditions were imposed so far. Since it is our purpose to get the gas at a finite density in the limit $N \rightarrow \infty$, we should compactify the system in the space direction when working at finite N . This can be done in several different ways: one can put the system on a ring of circumference L and impose the periodic boundary conditions, or put it in the hard-walls box of length L , or, more generally, impose twisted boundary conditions. The influence of the boundary conditions should disappear in the thermodynamic limit, but for finite N they are, of course, affect the

results. The exact eigenfunctions of the model are known for the general case of the twisted boundary conditions. The particular case of the periodic boundary conditions is, however, enough for our purposes and will be assumed henceforth.

2.2 The model in the $c \rightarrow \infty$ limit

The limit $c \rightarrow \infty$ is often called the Tonks-Girardeau limit [45, 21]. We will consider this limit in full details since it will give us a lot of intuition about the behavior of the model at arbitrary c .

2.2.1 Eigenfunctions and spectrum

The eigenfunctions of the Hamiltonian (2.1.11) should go to zero when $x_i \rightarrow x_j$ for arbitrary $i \neq j$:

$$\chi_N(x_1, \dots, x_i, \dots, x_j, \dots, x_N) \rightarrow 0, \quad x_i \rightarrow x_j, \quad i \neq j, \quad (2.2.1)$$

and they should solve the free Schrödinger equation when all x_j are different. A possible candidate is the determinant of an $N \times N$ matrix with the entries $\exp\{ik_j z_l\}$:

$$\det[\exp\{ik_j x_l\}]. \quad (2.2.2)$$

This function should be symmetrized in coordinates:

$$\chi_N(x_1, \dots, x_N | k_1, \dots, k_N) = \frac{C}{\sqrt{N!}} \det[\exp\{ik_j x_l\}] \prod_{1 \leq l < j \leq N} \text{sgn}(x_j - x_l), \quad (2.2.3)$$

where C is a normalization constant. It is easy to check that (2.2.3) is a common eigenfunction of the operators \mathcal{H} , \mathcal{P} and N with the corresponding eigenvalues being

equal to

$$E_N = \sum_{j=1}^N k_j^2, \quad P_N = \sum_{j=1}^N k_j. \quad (2.2.4)$$

In getting (2.2.4) we did not specify the boundary conditions. Let us now put a system on a ring of circumference L and impose the periodic boundary conditions. The wave function (2.2.3) should thus satisfy

$$\chi_N(x_1, \dots, x_j = 0, \dots, x_N | k_1, \dots, k_N) = \chi_N(x_1, \dots, x_j = L, \dots, x_N | k_1, \dots, k_N) \quad (2.2.5)$$

for all $j = 1, \dots, N$. This implies the following conditions on the quasimomenta k_j :

$$\exp\{ik_j L\} = (-1)^{N-1}, \quad j = 1, \dots, N. \quad (2.2.6)$$

These equations are called Bethe equations; in this particular case they can be solved easily:

$$k_j = \frac{2\pi}{L} n_j, \quad j = 1, \dots, N. \quad (2.2.7)$$

Here n_j are arbitrary odd numbers for N even and arbitrary even for N odd. Apart from this selection rule, the quantization condition for quasimomenta are the same as for free fermions: the allowed values of k_j are equidistant in the momentum space and multiple occupation of the same position in the momentum space is prohibited: the determinant in Eq. (2.2.6) becomes equal to zero. The fermion-like behavior of this boson system is very typical for the quantum systems in one spacial dimension; this phenomenon is called statistical transmutation. It should be stressed, however, that the Tonks-Girardeau gas is not a system of the non-interacting fermions: the wave function (2.2.6) is symmetric in coordinates, while the wave functions of the fermion system should be antisymmetric. Thus, correlation properties of the Tonks-Girardeau gas are very different in many aspects from those of the free fermion system. An

example illustrating this difference, the one-particle density matrix, will be studied in section 3.1.

The norm of the wave function (2.2.6) with the periodic boundary conditions can be calculated easily:

$$\int_0^L d^N x |\chi(x_1, \dots, x_N | k_1, \dots, k_N)|^2 = C^2 L^N, \quad (2.2.8)$$

therefore

$$C = \frac{1}{\sqrt{L^N}}. \quad (2.2.9)$$

It is not difficult to prove that the wave functions (2.2.3) with the quasi-momenta quantized by the conditions (2.2.6) form a complete set of eigenfunctions of the Hamiltonian (2.1.11) in the limit $c \rightarrow \infty$.

2.2.2 The ground state and thermodynamics at zero temperature

The ground state of the Tonks-Girardeau gas can be found easily. To minimize E_N , Eq. (2.2.4), one should take the following subset of k_j from the set (2.2.7):

$$k_j = \frac{2\pi}{L} \left(j - \frac{N+1}{2} \right), \quad j = 1, \dots, N. \quad (2.2.10)$$

The momentum P_N of the ground state is thus equal to zero.

The thermodynamic limit is the limit of infinite number of particles at finite density ρ :

$$\rho = \frac{N}{L} = \text{const}, \quad N, L \rightarrow \infty. \quad (2.2.11)$$

One has in this limit

$$\sum_{j=1}^N \rightarrow \frac{L}{2\pi} \int_{-\pi\rho}^{\pi\rho} dk. \quad (2.2.12)$$

The ground state energy in the canonical ensemble is, therefore,

$$E_0 = \frac{L}{2\pi} \int_{-\pi\rho}^{\pi\rho} dk k^2 = \frac{L}{3} \pi^2 \rho^3. \quad (2.2.13)$$

In the grand canonical ensemble the number of particles is not fixed and depends on the chemical potential; if the average number of particles in the system is N then

$$E_N = \sum_{j=1}^N k_j^2 - \mu. \quad (2.2.14)$$

In the thermodynamic limit the chemical potential is related to the density of particles as follows:

$$\mu = (\pi\rho)^2. \quad (2.2.15)$$

One can see that the thermodynamics of the Tonks-Gigardeau gas at zero temperature is the same as that of the spinless free-fermion gas, with $\pi\rho$ playing a role of the Fermi momentum k_F :

$$k_F = \pi\rho. \quad (2.2.16)$$

The pressure of the 1D gas is given by the formula

$$P = - \left(\frac{\partial E_0}{\partial L} \right)_{\sigma, N} = - \left(\frac{\partial F}{\partial L} \right)_{T, N}. \quad (2.2.17)$$

Here σ is the entropy of the gas, and F is the Helmholtz free energy, $F = E_0 - T\sigma$. The partial derivatives taken at a fixed T coincide with the partial derivatives taken at a fixed σ for the zero temperature case. Because of this fact we will omit the corresponding subscripts in the zero temperature case. For example, we will write $P = -(\partial E_0/\partial L)_N$. The thermodynamic definition of the sound velocity is

$$v_S = \sqrt{-\frac{L}{m\rho} \left(\frac{\partial P}{\partial L} \right)_N}. \quad (2.2.18)$$

We set $m = 1/2$ in the definition of the Lieb-Liniger model (2.1.1). One thus gets for the Tonks-Girardeau gas

$$P = \frac{2}{3}\pi^2\rho^3 \quad (2.2.19)$$

and

$$v_S = 2\pi\rho. \quad (2.2.20)$$

One can see from the expression for k_F that the Fermi velocity v_F is

$$v_F \equiv \frac{k_F}{m} = 2\pi\rho \quad (2.2.21)$$

(recall that we set $m = 1/2$). Comparing Eqs. (2.2.20) and (2.2.21) one concludes that the Fermi velocity coincides with the sound velocity in the Tonks-Girardeau gas. This is the expected result since the spectrum of the Tonks-Girardeau gas is the same as that of the free-fermion gas. In our studies of the Lieb-Liniger model with finite coupling c we will use the notion of the Fermi velocity as given by Eq. (2.2.21); the sound velocity v_S at arbitrary c is always less than v_F , see Fig. 2.1.

2.3 Arbitrary c

We construct in section 2.3.1 the eigenfunctions and the spectrum of the Lieb-Liniger model (2.1.11) at arbitrary coupling strength c . In section 2.3.2 we identify the ground state of the system and take the thermodynamic limit. Section 2.3.3 is closely related to the material of chapter 1: we extract the effective velocities v_S , v_N and v_J from the thermodynamics of the Lieb-Liniger model and check that they satisfy the condition (1.1.22). The fact that (1.1.22) is fulfilled indicates that the low-energy sector of the Lieb-Liniger model can be described using the Luttinger liquid theory. Another reason

for calculating the effective velocities is that they were used in section 1.4.2 to get the long-distance asymptotics of the density matrix. We study the weak (Bogoliubov) and strong (Tonks-Girardeau) coupling limits in the solution of the Lieb-Liniger model at arbitrary coupling strength in section 2.3.4.

2.3.1 Eigenfunctions and spectrum

The structure of the eigenfunctions at general c is much more complicated as compared to the eigenfunctions at $c = \infty$, considered in section 2.2.1. That particular case, however, shows a way one should proceed: The eigenfunctions $\chi_N(x_1, \dots, x_N)$ in the N -particle sector are symmetric under any interchange of coordinates, therefore it is sufficient to consider the following domain T in the coordinate space:

$$T : \quad x_1 < x_2 < \dots < x_N. \quad (2.3.1)$$

In this domain χ_N is an eigenfunction of the free Hamiltonian

$$\mathcal{H}_N^0 = - \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} \quad (2.3.2)$$

$$\mathcal{H}_N^0 \chi_N = E_N(k_1, \dots, k_N) \chi_N \quad (2.3.3)$$

The following boundary condition should be satisfied:

$$\left(\frac{\partial}{\partial x_{j+1}} - \frac{\partial}{\partial x_j} - c \right) \chi_N = 0, \quad x_{j+1} = x_j + 0. \quad (2.3.4)$$

This boundary condition obviously reduces to Eq. (2.2.1) in the Tonks-Girardeau limit. To prove Eq. (2.3.4) use the variables $z = x_{j+1} - x_j$ and $Z = (x_{j+1} + x_j)/2$.

One has

$$\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1} = 2 \frac{\partial}{\partial z}, \quad \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = 2 \frac{\partial^2}{\partial z^2} + \frac{1}{2} \frac{\partial^2}{\partial Z^2}. \quad (2.3.5)$$

Integrating Eq. (2.1.12) over z in the small domain $|z| < \epsilon \rightarrow 0$ one can see that the eigenfunction of the free-particle Hamiltonian (2.3.2) obeying the boundary condition (2.3.4) is indeed an eigenfunction of the Hamiltonian (2.1.11).

To get an explicit expression for the eigenfunctions (2.1.12) we start from the trial expression (2.2.2). We suggest the following form of the eigenfunctions:

$$\chi_N = \text{const} \left[\prod_{N \geq j > l \geq 1} \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_l} + c \right) \right] \det[\exp\{ik_j x_l\}]. \quad (2.3.6)$$

One should show that this expression satisfies Eqs. (2.3.2) and (2.3.4). Check, for instance, that

$$\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1} - c \right) \chi_N = 0, \quad x_2 = x_1 + 0. \quad (2.3.7)$$

Do do this, rewrite Eq. (2.3.6) as follows:

$$\chi_N = \left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1} + c \right) \tilde{\chi}_N, \quad (2.3.8)$$

where

$$\begin{aligned} \tilde{\chi}_N = \text{const} \prod_{j=3}^N \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_1} + c \right) \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_2} + c \right) \\ \times \prod_{N \geq j > l \geq 3} \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_l} + c \right) \det[\exp\{ik_j x_l\}]. \end{aligned} \quad (2.3.9)$$

The function $\tilde{\chi}_N$ is antisymmetric with respect to the interchange of x_1 and x_2 ,

$$\tilde{\chi}_N(x_1, x_2) = -\tilde{\chi}_N(x_2, x_1). \quad (2.3.10)$$

Writing the left hand side of the Eq. (2.3.7) in the form

$$\left[\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1} \right)^2 - c^2 \right] \tilde{\chi}_N \quad (2.3.11)$$

one sees that it is antisymmetric with respect to the interchange of x_1 and x_2 , and, therefore, equal to zero when $x_2 \rightarrow x_1$. The boundary condition (2.3.4) for the other

x_j can be checked similarly. We thus showed that the wave function (2.3.6) is indeed an eigenfunction of the Hamiltonian (2.1.11). Using the definition of the determinant

$$\det[\exp\{ik_j x_l\}] = \sum_P (-1)^{[P]} \exp\left\{i \sum_{n=1}^N x_n k_{P_n}\right\} \quad (2.3.12)$$

one gets for (2.3.6)

$$\begin{aligned} \chi_N = \text{const} & \left\{ N! \prod_{j>l} [(k_j - k_l)^2 + c^2] \right\}^{-1/2} \\ & \times \sum_P (-1)^{[P]} \exp\left\{i \sum_{n=1}^N x_n k_{P_n}\right\} \prod_{j>l} [k_{P_j} - k_{P_l} - ic \text{sign}(x_j - x_l)]. \end{aligned} \quad (2.3.13)$$

This is the desired expression for the eigenfunctions of the Hamiltonian (2.1.11). We wrote Eq. (2.3.13) in such a form that it is valid for arbitrary values of x_1, \dots, x_N , thus removing the restriction (2.3.1). One can see that the wave functions given by Eq. (2.3.13) reduce to the wave functions given by Eq. (2.2.3) in the $c \rightarrow \infty$ limit.

The eigenvalues of the Hamiltonian (2.1.11) and the momentum operator (2.1.13) on the eigenfunctions (2.3.13) are

$$E_N = \sum_{j=1}^N k_j^2, \quad P_N = \sum_{j=1}^N k_j, \quad (2.3.14)$$

respectively. These formulas look very similar to that of a free system and because of this similarity the parameters k_j are called quasi-momenta.

No boundary conditions were imposed so far. Let us do it now. Recall that we use the periodic boundary conditions exclusively in this Thesis. The condition (2.2.5) imposed on the function (2.3.13) gives

$$\exp\{ik_j L\} = \prod_{l \neq j}^N \frac{k_j - k_l + ic}{k_j - k_l - ic}, \quad j = 1, \dots, N. \quad (2.3.15)$$

These coupled nonlinear equations are called Bethe equations. In the $c \rightarrow \infty$ limit these equations reduce to Eqs. (2.2.6). It can be shown that all their solutions are real (see, for example, Ref. [33]). It is often more convenient to use the logarithmic form of Bethe equations:

$$Lk_j + \sum_{m=1}^N \theta(k_j - k_m) = 2\pi \left(n_j - \frac{N+1}{2} \right), \quad j = 1, \dots, N. \quad (2.3.16)$$

Here $\{n_j\}$ is a set of integers, and the function θ is

$$\theta(k) = i \ln \left(\frac{ic + k}{ic - k} \right), \quad \theta(\pm\infty) = \pm\pi. \quad (2.3.17)$$

An important property of the representation (2.3.16) is that the sets of n_j and the sets of quasi-momenta k_j are in one to one correspondence. It is often more convenient to parameterize a quantum state by the set n_j rather than by the set of quasi-momenta k_j .

Let us sum up Bethe equations (2.3.16) over j . The function $\theta(k)$ defined by Eq. (2.3.17) is antisymmetric, therefore

$$\sum_{j,m=1}^N \theta(k_j - k_m) = 0 \quad (2.3.18)$$

and one gets

$$L \sum_{j=1}^N k_j = 2\pi \sum_{j=1}^N \left(n_j - \frac{N+1}{2} \right). \quad (2.3.19)$$

Comparing this equation with Eq. (2.3.14) one gets a relation between the momentum of an arbitrary state and the set of quantum numbers n_j :

$$P_N = \frac{2\pi}{L} \sum_{j=1}^N \left(n_j - \frac{N+1}{2} \right). \quad (2.3.20)$$

The calculation of the normalization constant in Eq. (2.3.13) is a nontrivial task (recall that in the Tonks-Girardeau limit this constant is given by Eqs. (2.2.8) and

(2.2.9)). A closed expression for this constant was suggested by Gaudin (see [19] and references therein) and proved in Ref. [32]; a detailed discussion can be found in Ref. [33].

2.3.2 The ground state and thermodynamics at zero temperature

The ground state of the model in the N -particle sector corresponds to the following choice of the quantum numbers n_j in Eq. (2.3.16):

$$n_j = j, \quad j = 1, \dots, N, \quad (2.3.21)$$

that is the Bethe equations (2.3.16) take the form

$$Lk_j + \sum_{m=1}^N \theta(k_j - k_m) = 2\pi \left(j - \frac{N+1}{2} \right), \quad j = 1, \dots, N. \quad (2.3.22)$$

in the ground state of the model. The momentum of the ground state is equal to zero.

Consider now the thermodynamic limit of the model, defined by Eq. (2.2.11).

Write

$$\sum_{j=1}^N = \sum_{j=1}^N \frac{k_{j+1} - k_j}{k_{j+1} - k_j} = L \sum_{j=1}^N \rho(k_j)(k_{j+1} - k_j) \quad (2.3.23)$$

where we have defined $\rho(k_j)$ as follows

$$\rho(k_j) = \frac{1}{L(k_{j+1} - k_j)}. \quad (2.3.24)$$

In the thermodynamic limit one has for an arbitrary function $f(k_j)$:

$$\sum_{j=1}^N f(k_j) \rightarrow L \int_{-\Lambda}^{\Lambda} dk \rho(k) f(k). \quad (2.3.25)$$

The parameter Λ plays a role of the Fermi momentum: all the states with $|k| < \Lambda$ are occupied, and all the states with $|k| > \Lambda$ are empty. The value of Λ is given by the normalization condition

$$\int_{-\Lambda}^{\Lambda} dk \rho(k) = \frac{N}{L} = \rho. \quad (2.3.26)$$

To get the thermodynamic limit of the Bethe equations (2.3.22), take the difference of the equations with the indices $j + 1$ and j :

$$L(k_{j+1} - k_j) + \sum_{j=1}^N [\theta(k_{j+1} - k_m) - \theta(k_j - k_m)] = 2\pi. \quad (2.3.27)$$

The difference $k_{j+1} - k_j$ is small in the thermodynamic limit, therefore Eq. (2.3.27) can be written as follows

$$L(k_{j+1} - k_j) + (k_{j+1} - k_j) \sum_{m=1}^N \theta'(k_j - k_m) = 2\pi, \quad N, L \rightarrow \infty, \quad (2.3.28)$$

where $\theta'(k) \equiv \partial\theta(k)/\partial k$. Using Eqs. (2.3.24) and (2.3.25) one gets after some algebra

$$\rho(k) - \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dq \rho(q) K(k, q) = \frac{1}{2\pi}, \quad -\Lambda \leq k \leq \Lambda, \quad (2.3.29)$$

where

$$K(k, q) \equiv \theta'(k - q) = \frac{2c}{c^2 + (k - q)^2}. \quad (2.3.30)$$

The linear integral equation (2.3.29) is called Lieb equation. Its solution, together with the normalization condition (2.3.26), gives the quasi-momentum distribution function $\rho(k)$. All thermodynamic parameters can be calculated from $\rho(k)$ and Λ .

The ground state energy in the canonical ensemble is

$$E_0 = \sum_{j=1}^N k_j^2 \rightarrow L \int_{-\Lambda}^{\Lambda} dk \rho(k) k^2. \quad (2.3.31)$$

Following [35], change the variables as follows:

$$k = \Lambda z, \quad c = \Lambda \alpha, \quad \rho(\Lambda z) = g(z). \quad (2.3.32)$$

Written in these variables, Eqs. (2.3.26), (2.3.29), and (2.3.31) are, respectively,

$$\gamma \int_{-1}^1 dz g(z) = \alpha, \quad (2.3.33)$$

$$g(z) - \frac{1}{2\pi} \int_{-1}^1 dy \frac{2\alpha g(y)}{\alpha^2 + (y-z)^2} = \frac{1}{2\pi}, \quad (2.3.34)$$

and

$$e(\gamma) \equiv \frac{E_0}{N\rho^2} = \frac{\gamma^3}{\alpha^3} \int_{-1}^1 dy g(y) y^2. \quad (2.3.35)$$

Recall that γ is given by Eq. (2.1.2). The system of equations (2.3.33) and (2.3.34) is much more suitable for the numerical analysis than the system (2.3.26) and (2.3.29).

To solve the system (2.3.33) and (2.3.34) the following two steps should be done:

- (i) Solve Eq. (2.3.34) for a given α ;
- (ii) use Eq. (2.3.33) to determine γ as a function of α ;

Having solved the system (2.3.33) and (2.3.34), one can easily go back to the equations in the form (2.3.26) and (2.3.29), if necessary. The Fermi momentum Λ can be found by combining Eqs. (2.1.2) and (2.3.32):

$$\Lambda = \rho \frac{\gamma}{\alpha} = \rho \left[\int_{-1}^1 dz g(z) \right]^{-1}. \quad (2.3.36)$$

The quasi-momentum density $\rho(k)$ is related to $g(z)$ by Eq. (2.3.32).

Concluding this section, define the moments of the function $g(z)$ by the formula

$$\epsilon_m = \left(\frac{\gamma}{\alpha} \right)^{m+1} \int_{-1}^1 dz z^m g(z). \quad (2.3.37)$$

We will need these moments (actually, ϵ_2 and ϵ_4 only) in calculation of local correlation functions performed in section 3.2. Note that $\epsilon_2 = e(\gamma)$, which follows from (2.3.35).

2.3.3 Effective velocities

Discuss first the thermodynamic definition of the sound velocity, Eq. (2.2.18). Taking E_0 from the Eq. (2.3.35), and using Eq. (2.2.17) one gets

$$v_S = 2\rho \left[3e(\gamma) - 2\gamma e'(\gamma) + \frac{1}{2}\gamma^2 e''(\gamma) \right]^{1/2} = 2 \left[\mu(\gamma) - \frac{1}{2}\gamma \mu'(\gamma) \right]^{1/2}, \quad (2.3.38)$$

where “prime” denotes differentiation with respect to γ and μ is the chemical potential given by

$$\mu = \left(\frac{\partial E_0}{\partial N} \right)_L = \rho^2 [3e(\gamma) - \gamma e'(\gamma)]. \quad (2.3.39)$$

Note an important technical point: since P and μ depend on N and L only via $\rho = N/L$, the following identities are valid:

$$\left(\frac{\partial P}{\partial \rho} \right)_N = \left(\frac{\partial P}{\partial \rho} \right)_L, \quad \left(\frac{\partial \mu}{\partial \rho} \right)_N = \left(\frac{\partial \mu}{\partial \rho} \right)_L. \quad (2.3.40)$$

These identities are very useful in getting various thermodynamic relations.

Another, microscopic definition of the sound velocity is

$$v_S = \left. \frac{\partial \epsilon(k)}{\partial P(k)} \right|_{k=\Lambda}, \quad (2.3.41)$$

where $\epsilon(k)$ is the energy change of the system when a particle with the quasi-momentum k is added; $P(k)$ is the momentum change of the system. The equivalence of the definitions (2.2.18) and (2.3.41) in the Lieb-Liniger was proven in Ref. [35]; for more modern presentation one can consult Ref. [33]. We will omit this proof since it is quite long; we only present here several equivalent representations for v_S obtained in course of this proof:

$$v_S = \sqrt{2\rho \frac{\partial \mu}{\partial \rho}} = \frac{\partial \mu}{\partial \Lambda} = \frac{\rho}{2\pi\rho(\Lambda)^2}. \quad (2.3.42)$$

Consider the following excitation in the system: choose the quantum numbers n_j in Eq. (2.3.16) to be

$$\tilde{n}_j = j + \frac{J}{2}, \quad j = 1, \dots, N. \quad (2.3.43)$$

The momentum of this state is given by Eq. (2.3.20):

$$P = \pi J \frac{N}{L} = \pi J \rho. \quad (2.3.44)$$

The set of quasi-momenta \tilde{k}_j is related to the ground state quasi-momenta k_j as follows:

$$\tilde{k}_j = k_j + \frac{\pi J}{L}. \quad (2.3.45)$$

Calculate the energy difference ϵ between the energy of this state and the ground state energy:

$$\epsilon = \sum_{j=1}^N (\tilde{k}_j^2 - k_j^2) = N \left(\frac{\pi J}{L} \right)^2. \quad (2.3.46)$$

Write this expression as follows

$$\epsilon = \frac{\pi}{2L} v_J J^2, \quad (2.3.47)$$

where

$$v_J = 2\pi\rho. \quad (2.3.48)$$

The parameter v_J coincides with the Fermi velocity, Eq. (2.2.21). We stress that in getting Eq. (2.3.48) we have used the fact that thermodynamic (2.2.18) and microscopic (2.3.41) definitions of the sound velocity coincide in the Lieb-Liniger model. The parameter v_J can be attributed to the velocity of the current excitations in the low energy sector of the Lieb-Liniger model, Eq. (1.1.14). More precisely, we argue now why it is natural to *believe* that v_J defined by Eq. (2.3.47) and v_J entering Eq. (1.1.14) are equal to each other. Consider the vacuum average $\langle P_{\text{lutt}} \rangle$ of the

momentum operator (1.1.19) in the sector with $\langle N \rangle = \langle N_0 \rangle$ and $\langle J \rangle \neq 0$:

$$\langle P_{\text{lutt}} \rangle = \pi \langle J \rangle \frac{N_0}{L}. \quad (2.3.49)$$

Recall that the parameter N_0 entering the Luttinger model Hamiltonian Eq. (1.1.14) is associated with the number of particles N in the microscopic model. We thus see that Eqs. (2.3.44) and (2.3.49) have the same form. The energy of the state (2.3.49) can be easily found from Eq. (1.1.14):

$$\langle H_{\text{lutt}} \rangle = \frac{\pi}{2L} v_J \langle J \rangle^2. \quad (2.3.50)$$

Comparing Eqs. (2.3.46) and (2.3.50) one can easily convince oneself that v_J in these formulas should coincide.

Define another important velocity parameter, v_N , by the formula

$$v_N = \frac{1}{\pi} \left(\frac{\partial \mu}{\partial \rho} \right)_L. \quad (2.3.51)$$

This equation coincides with Eq. (1.1.52), defining v_N within the Luttinger model.

Comparing Eqs. (2.3.48), (2.3.51), and (2.3.42) one gets a connection formula for the microscopically defined parameters v_S , v_N , and v_J :

$$v_S = \sqrt{v_N v_J}. \quad (2.3.52)$$

It is the same constraint as given by Eq. (1.1.22) within the Luttinger model. Equation (2.3.52) provides us with a strong validation of the applicability of the Luttinger liquid description to the Lieb-Liniger model [23].

2.3.4 Small and large c limits from the general solution

In the $\gamma \rightarrow \infty$ limit of the model, known as the Tonks-Girardeau gas [45, 21], the spectrum is the same as that of a free fermion gas. When γ is reduced from this

limit to arbitrarily small values, the model exhibits no phase transition, which is an illustration that the difference between boson- and fermion-like behavior in 1D is much less pronounced than in higher dimensions.

The large γ limit is easy to work out since Eq. (2.3.34) admits a regular perturbative expansion with $1/\alpha$ being a small parameter. One has

$$g(z) = \frac{1}{2\pi} + \frac{1}{\pi^2\alpha} + \frac{2}{\pi^3\alpha^2} - \frac{C_3(z)}{3\pi^4\alpha^3} - \frac{2C_4(z)}{\pi^5\alpha^4} + \frac{C_5(z)}{15\pi^6\alpha^5} + \frac{2C_6(z)}{45\pi^7\alpha^6} + \dots, \quad \alpha \rightarrow \infty, \quad (2.3.53)$$

where

$$C_3(z) = 3\pi^2 z^2 + \pi^2 - 12, \quad (2.3.54)$$

$$C_4(z) = \pi^2 z^2 + \pi^2 - 4, \quad (2.3.55)$$

$$C_5(z) = 15\pi^4 z^4 + 30\pi^2(\pi^2 - 2)z^2 + 3\pi^4 - 100\pi^2 + 240, \quad (2.3.56)$$

$$C_6(z) = 45\pi^4 z^4 + 60\pi^2(2\pi^2 - 3) + 71\pi^4 - 420\pi^2 + 720. \quad (2.3.57)$$

Equation (2.3.33) then gives

$$\gamma = \pi\alpha - 2 + \frac{4}{3\alpha^2} - \frac{32}{15\alpha^4} - \frac{16}{45\pi\alpha^5} + \dots, \quad \alpha \rightarrow \infty. \quad (2.3.58)$$

Inverting this expression, one gets

$$\alpha = \frac{\gamma}{\pi} + \frac{2}{\pi} - \frac{4\pi}{3\gamma^2} + \frac{16\pi}{3\gamma^3} + \left(\frac{2\pi^2}{15} - 1\right) \frac{16\pi}{\gamma^4} - \left(\frac{19\pi^2}{5} - 8\right) \frac{16\pi}{3\gamma^5} + \dots, \quad \gamma \rightarrow \infty. \quad (2.3.59)$$

For the moments ϵ_2 and ϵ_4 defined by Eq. (2.3.37) one gets

$$\epsilon_2(\gamma) = \frac{\pi^2}{3} \left[1 - \frac{4}{\gamma} + \frac{12}{\gamma^2} + (\pi^2 - 15) \frac{32}{15\gamma^3} - (4\pi^2 - 15) \frac{16}{3\gamma^4} - (3\pi^4 - 140\pi^2 + 210) \frac{32}{35\gamma^5} + (131\pi^4 - 1680\pi^2 + 1260) \frac{16}{45\gamma^6} + \dots \right] \quad (2.3.60)$$

and

$$\begin{aligned} \epsilon_4(\gamma) = \frac{\pi^4}{5} \left[1 - \frac{8}{\gamma} + \frac{40}{\gamma^2} + (\pi^2 - 35) \frac{32}{7\gamma^3} - (4\pi^2 - 35) \frac{16}{\gamma^4} - (31\pi^4 - 2520\pi^2 \right. \\ \left. + 8820) \frac{64}{315\gamma^5} + (449\pi^4 - 10080\pi^2 + 17640) \frac{32}{105\gamma^6} + \dots \right]. \end{aligned} \quad (2.3.61)$$

One should write explicitly the terms up to the order of γ^{-6} in the expansions (2.3.60) and (2.3.61) to get the leading term in the large γ expansion (3.2.15) of the three-body correlation function (3.2.2). From the exact expression (2.3.38) for the sound velocity in the Lieb-Liniger model, one gets the following large γ expansion:

$$v_S = 2\pi\rho \left[1 - \frac{4}{\gamma} + \frac{12}{\gamma^2} + (\pi^2 - 6) \frac{16}{3\gamma^3} - (2\pi^2 - 3) \frac{80}{3\gamma^4} + \dots \right]. \quad (2.3.62)$$

The leading terms in all the expansions written in the present section correspond to the Tonks-Girardeau limit and were obtained in the section 2.2.2.

The opposite limit, $\gamma \rightarrow 0$, is much more difficult to analyze since the kernel in Eq. (2.3.29) becomes singular when $\alpha \rightarrow 0$. Beyond the leading order, the results were obtained only recently [47]:

$$g(z) \simeq \frac{\sqrt{1-z^2}}{2\pi\alpha} - \frac{1}{4\pi} \sum_{l=0}^{\infty} c_l z^{2l}, \quad \alpha \rightarrow 0, \quad (2.3.63)$$

where

$$c_l = \frac{(2l-1)!!}{(2l)!!} \sum_{m=0}^{\infty} \frac{(2m+1)!!(2m-1)!!}{[(2m)!!]^2(2l-2m-1)}. \quad (2.3.64)$$

For the moments ϵ_2 and ϵ_4 defined by Eq. (2.3.37) one gets

$$\epsilon_2 \simeq \gamma \left(1 - \frac{4}{3\pi} \sqrt{\gamma} \right), \quad \gamma \rightarrow 0. \quad (2.3.65)$$

$$\epsilon_4 \simeq 2\gamma^2 \left(1 - \frac{44}{15\pi} \sqrt{\gamma} \right), \quad \gamma \rightarrow 0. \quad (2.3.66)$$

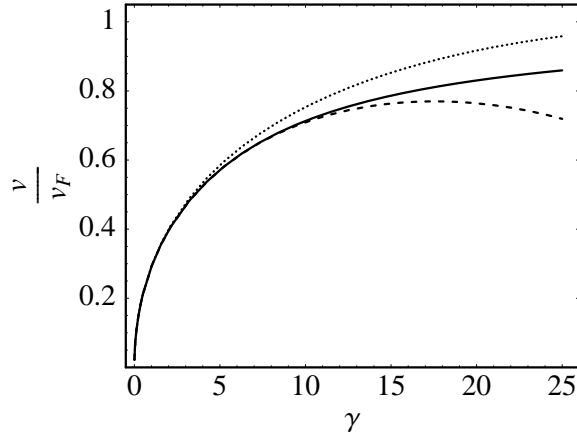


Figure 2.1: Shown is the inverse Luttinger parameter $K^{-1} = v_S/v_F$, Eqs. (1.4.7) and (2.3.38) (solid line), its small γ approximation v_{S1}/v_F , Eq. (2.3.67) (dashed line), and another small γ approximation v_{S2}/v_F , Eq. (2.3.68) (dotted line). The Fermi velocity v_F is the sound velocity at $\gamma = \infty$; it is given by Eq. (2.2.21). Note slow convergence of v_S to v_F as $\gamma \rightarrow \infty$.

Substitute the expansion (2.3.65) into Eq. (2.3.38):

$$v_{S1} = 2\rho \left(\gamma - \frac{1}{2\pi} \gamma^{3/2} \right)^{1/2}, \quad \gamma \rightarrow 0. \quad (2.3.67)$$

Since only two terms are written explicitly in the expansion (2.3.65), one should expand Eq. (2.3.67) and drop all the terms except the leading and the first subleading:

$$v_{S2} = 2\rho\sqrt{\gamma} \left(1 - \frac{\sqrt{\gamma}}{4\pi} \right), \quad \gamma \rightarrow 0. \quad (2.3.68)$$

However, the numerical analysis shows that Eq. (2.3.67) gives much better approximation to the exact expression (2.3.38) than the expression (2.3.68). We plot v_S/v_F , v_{S1}/v_F , and v_{S2}/v_F versus γ in Fig. 2.1. The velocity v_F is the sound velocity of the gas in the Tonks-Girardeau limit, Eq. (2.2.21).

Chapter 3

Correlation functions of the Lieb-Liniger model

The knowledge of the exact eigenfunctions and spectrum for the Lieb-Liniger model makes it possible to develop the microscopic approach to the calculation of its correlation functions. The procedure is straightforward: calculate the form-factors of local field operators in the basis formed by the eigenfunctions of the Hamiltonian (2.1.1) in the sector with given number of particles N , then sum up over all intermediate states, and, finally, take the thermodynamic limit. It is very easy to formulate such a program, and very difficult to implement it because of the complicated structure of the Bethe wave function (2.3.13) and Bethe equations (2.3.15). The book [33] is mostly devoted to the development of this program and is, probably, the most comprehensive monograph in this field. Among the results not included in the cited book one should mention the papers [30, 14, 31], where the local field operators were represented via the fundamental operators of the Quantum Inverse Scattering Problem (this approach works for spin chains, where the quantum space is finite-dimensional, but not for the Lieb-Liniger model; there is, however a trick [5], solving the inverse scattering problem for the Lieb-Liniger model as well).

Unfortunately, the final expressions for the correlation functions obtained within this straightforward approach are, in general, too complicated to be successfully analyzed either by analytic methods or numerically. There are, however, several important exceptions. One of them, the Tonks-Girardeau gas, is discussed in this chapter, section 3.1. Another example, two-component fermion and boson gases in the limit of infinite repulsion, will be a subject of chapter 4.

We have shown in chapters 1 and 2 that as far as the low-energy properties are concerned, the Lieb-Liniger model belongs to the Luttinger liquid universality class. In section 3.2, we give the closed expression for a *local* correlation function Eq. (3.2.2), obtained by combining the bosonization procedure with the complete integrability of the Lieb-Liniger model. We discuss mostly the physical implementation of the obtained result, while the details of calculation are given in a separate chapter 5.

The content of section 3.1 is not original; the purpose of writing this section was mostly to introduce some objects used in chapter 4. The content of section 3.2 is original and was reported in the paper [9].

3.1 Correlation functions of the Tonks-Girardeau gas

The relatively simple form of the eigenfunctions of the Tonks-Girardeau gas make it possible to calculate the correlation functions explicitly for arbitrary number of particles N . The first non-trivial correlation function which was calculated is the one-particle density matrix. The calculations were done by Lenard [34] and the result is now known as the Lenard's formula. We present this formula in section

3.1.1. Lenard's formula contains the determinant of an $N \times N$ matrix. The $N \rightarrow \infty$ limit of the determinant of an $N \times N$ matrix is called the Fredholm determinant. It is discussed in section 3.1.2. The thermodynamic limit of the Lenard's formula is studied in sections 3.1.3 and 3.1.4. The momentum distribution function for the Tonks-Girardeau gas is calculated in section 3.1.5.

3.1.1 One-particle density matrix: Lenard's formula

The one-particle density matrix in the N -particle sector is defined as follows

$$\rho_N(x) = \langle \Psi^\dagger(x) \Psi(0) \rangle, \quad (3.1.1)$$

where the average is taken over the ground state of the system in the N -particle sector. The expression for the $\rho_N(x)$ in the first-quantized representation can be obtained easily (remember the commutation relations (2.1.3) and the representation (2.1.8)):

$$\begin{aligned} \rho_N(x) &= \langle \Psi_N(k_1, \dots, k_N) | \Psi^\dagger(x) \Psi(0) | \Psi_N(k_1, \dots, k_N) \rangle = \frac{1}{N!} \int_0^L d^N z d^N z' \\ &\times \bar{\chi}_N(z_1, \dots, z_N) \chi_N(z'_1, \dots, z'_N) \langle 0 | \Psi(z_1) \dots \Psi(z_N) | \Psi^\dagger(x) \Psi(0) | \Psi^\dagger(z'_1) \dots \Psi^\dagger(z'_N) | 0 \rangle \\ &= N \int_0^L dz_2 \dots dz_N \bar{\chi}_N(x, z_2, \dots, z_N) \chi_N(0, z_2, \dots, z_N) \end{aligned} \quad (3.1.2)$$

In the Tonks-Girardeau case χ_N is given by Eqs. (2.2.3), the normalization constant C for the periodic boundary conditions is given by Eq. (2.2.9), and the ground state quasi-momentum distribution is given by Eq. (2.2.10). Therefore

$$\begin{aligned} \rho_N(x) &= \frac{1}{(N-1)! L^N} \int_0^L dz_2 \dots dz_N \\ &\times \sum_P \sum_{\tilde{P}} (-1)^{[P]+[\tilde{P}]} \exp\{-ixk_{P_1}\} \prod_{a=2}^N \exp\{iz_a(k_{\tilde{P}_a} - k_{P_a})\} \text{sgn}(z_a - x). \end{aligned} \quad (3.1.3)$$

All the integrals in this expression can be taken explicitly and give

$$\frac{1}{L} \int_0^L dz_a \exp\{iz_a(k_{\tilde{P}_a} - k_{P_a})\} \text{sgn}(z_a - x) = f(k_{\tilde{P}_a} - k_{P_a}), \quad a = 2, 3, \dots, N, \quad (3.1.4)$$

where

$$f(k_a - k_b) = \delta_{k_a k_b} - \frac{2 \exp\{ix(k_a - k_b)\} - 1}{L i(k_a - k_b)}. \quad (3.1.5)$$

One gets, therefore,

$$\rho_N(x) = \frac{1}{(N-1)!L} \sum_P \sum_{\tilde{P}} (-1)^{[P]+[\tilde{P}]} \frac{\exp\{-ixk_{P_1}\}}{f(k_{\tilde{P}_1} - k_{P_1})} \prod_{a=1}^N f(k_{\tilde{P}_a} - k_{P_a}). \quad (3.1.6)$$

Represent the permutation \tilde{P} in the internal sum as a sequence of two permutations

$$\tilde{P} = P'P \quad (3.1.7)$$

where P is the running index of the external sum. Further, introduce the notation

$$k_{P_a} \equiv P[k_a]. \quad (3.1.8)$$

Written in these notations, Eq. (3.1.6) reads

$$\rho_N(x) = \frac{1}{(N-1)!L} \sum_P \sum_{P'} (-1)^{[P']} \frac{\exp\{-ixP[k_1]\}}{f(P'[P[k_1]] - P[k_1])} \prod_{a=1}^N f(P'[P[k_a]] - P[k_a]). \quad (3.1.9)$$

One can easily notice the following identity:

$$\prod_{a=1}^N f(P'[P[k_a]] - P[k_a]) = \prod_{a=1}^N f(P'[k_a] - k_a), \quad (3.1.10)$$

therefore

$$\begin{aligned}
\rho_N(x) &= \frac{1}{(N-1)!L} \sum_{P'} (-1)^{[P']} \prod_{a=1}^N f(P'[k_a] - k_a) \sum_P \frac{\exp\{-ixP[k_1]\}}{f(P'[P[k_1]] - P[k_1])} \\
&= \frac{1}{L} \sum_{P'} (-1)^{[P']} \prod_{a=1}^N f(P'[k_a] - k_a) \sum_{j=1}^N \frac{\exp\{-ixk_j\}}{f(P'[k_j] - k_j)} \\
&= \frac{1}{L} \sum_{j=1}^N \exp\{-ixk_j\} \sum_P (-1)^{[P]} \prod_{a \neq j} f(P[k_a] - k_a). \quad (3.1.11)
\end{aligned}$$

Consider the determinant of an $N \times N$ matrix with the entries $f(k_j - k_l)$:

$$\left| \begin{array}{ccc} f(k_1 - k_1) & \cdots & f(k_1 - k_N) \\ \vdots & \ddots & \vdots \\ f(k_N - k_1) & \cdots & f(k_N - k_N) \end{array} \right| = \sum_P (-1)^{[P]} \prod_{a=1}^N f(P[k_a] - k_a). \quad (3.1.12)$$

To reproduce the last line of the Eq. (3.1.11) one should replace the l -th column of the matrix $f(k_j - k_l)$ with the expression $\exp\{-iz_1 k_l\}/L$:

$$\rho_N(x) = \left. \frac{\partial}{\partial \lambda} \det_N(I + \tilde{V}_1 + \lambda \tilde{V}_2) \right|_{\lambda=0} = \det_N(I + \tilde{V}_1 + \tilde{V}_2) - \det_N(I + \tilde{V}_1), \quad (3.1.13)$$

where

$$(\tilde{V}_1)_{ab} = -\frac{2 \exp\{ix(k_a - k_b)\} - 1}{L i(k_a - k_b)}, \quad (\tilde{V}_2)_{ab} = \frac{1}{L} \exp\{-ixk_b\}, \quad a, b = 1, \dots, N. \quad (3.1.14)$$

Equation (3.1.13) can be rewritten in the following form

$$\rho_N(x) = \left. \frac{\partial}{\partial \lambda} \det_N(I + V_1 + \lambda V_2) \right|_{\lambda=0} = \det_N(I + V_1 + V_2) - \det_N(I + V_1), \quad (3.1.15)$$

where

$$(V_1)_{ab} = -2 \frac{2\pi \sin[\frac{x}{2}(k_a - k_b)]}{L \pi(k_a - k_b)}, \quad (V_2)_{ab} = \frac{1}{L} \exp\left\{-i\frac{x}{2}(k_a + k_b)\right\}, \quad a, b = 1, \dots, N. \quad (3.1.16)$$

Equations (3.1.15) and (3.1.16) are the desired Lenard's formula for the one-particle density matrix of the Tonks-Girardeau gas at zero temperature. It can be easily generalized to a finite temperature case.

It is proper place to stress again that the Tonks-Girardeau gas is *not* equivalent to the free-fermion gas: though the spectrum of the Tonks-Girardeau gas is the same as that of the free fermion gas, the wave functions have different symmetry. This changes some observables drastically: for example, the one-particle density matrix for the free fermion gas

$$\rho_N^{ff}(x) = \frac{1}{L} \sum_{j=1}^N \exp\{-ixk_j\} \quad (3.1.17)$$

is totally different from that of the Tonks-Girardeau gas, Eqs. (3.1.15) and (3.1.16).

3.1.2 The Fredholm determinant

We have seen that in the N -particle sector the density matrix (3.1.1) is expressed via the determinants of $N \times N$ matrices, Eqs. (3.1.15) and (3.1.16). In the thermodynamic limit (2.2.11) these matrices becomes infinite-dimensional. Thus, an important object comes into play: the determinant of an infinite-dimensional matrix (it is called Fredholm determinant). This object appears regularly in the calculations of the correlation functions of the integrable models, and we will work with them throughout this Thesis.

We give in this section two alternative definitions of the Fredholm determinant. The first one is: Let V be an $N \times N$ matrix with the entries $V_{ab} = V(k_a, k_b)$. Let

$$k_a = \left(\frac{2a}{N-1} - 1 \right), \quad a = 0, 1, \dots, N-1. \quad (3.1.18)$$

Then the Fredholm determinant $\det(\hat{I} + \hat{V})$ is defined as follows:

$$\det(\hat{I} + \hat{V}) = \lim_{N \rightarrow \infty} \det_N \left(I + \frac{2}{N-1} V \right). \quad (3.1.19)$$

The matrix I on the right hand side of Eq. (3.1.19) is the $N \times N$ identity matrix. We see that with increasing number of divisions, N , the possible values of k_a , Eq. (3.1.18), fill the interval $-1 \leq k_a \leq 1$ densely. The objects \hat{I} and \hat{V} on the left hand side of the Eq. (3.1.19) should thus be some integral operators with the kernels defined on $[-1, 1] \times [-1, 1]$. This can be readily seen from an alternative definition of the Fredholm determinant:

$$\det(\hat{I} + \hat{V}) = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{-1}^1 dk_1 \dots \int_{-1}^1 dk_N \det \begin{bmatrix} V(k_1, k_1) & \dots & V(k_1, k_N) \\ \vdots & \ddots & \vdots \\ V(k_N, k_1) & \dots & V(k_N, k_N) \end{bmatrix}, \quad (3.1.20)$$

where \hat{V} is a linear integral operator with the kernel $V(k, p)$ defined on $[-1, 1] \times [-1, 1]$. The equivalence of the definitions (3.1.19) and (3.1.20) is proven, for example, in Ref. [43]. These definitions are somehow complementary: the former one is more suitable for a numerical analysis, while the latter one reveals better the analytic structure of the problem.

3.1.3 One-particle density matrix: thermodynamic limit

Let us set up a convention first. The one-dimensional density ρ is controlled by the chemical potential μ in the infinite γ limit as given by Eq. (2.2.15):

$$\sqrt{\mu} = \pi\rho. \quad (3.1.21)$$

It follows from this equation that $\sqrt{\mu}$ has a dimension of inverse length. In the rest of section 3.1 we measure distances in units of $1/\sqrt{\mu}$. This convention implies that the momentum is measured in units of the Fermi momentum k_F , Eq. (2.2.16).

The ground state and zero temperature thermodynamics of the Tonks-Girardeau gas are described in section 2.2.2. Using formulas from that section, write the thermodynamic limit of the Lenard's formula (3.1.15) in the form prescribed by the definition (3.1.19):

$$\frac{\rho(x)}{\rho(0)} = \det(\hat{I} + \hat{V}_1 + \hat{V}_2) - \det(\hat{I} + \hat{V}_1). \quad (3.1.22)$$

The kernels of the operators \hat{V}_1 and \hat{V}_2 in Eq. (3.1.22) are given by the thermodynamic limit of Eq. (3.1.16):

$$V_1(k, p) = \alpha \frac{\sin \frac{x}{2}(k-p)}{\pi(k-p)} \quad (3.1.23)$$

and

$$V_2(k, p) = \frac{1}{2} \exp\{-i \frac{x}{2}(k+p)\}. \quad (3.1.24)$$

These kernels are defined on $[-1, 1] \times [-1, 1]$, in accordance with our convention formulated below Eq. (3.1.21). The parameter α is

$$\alpha = -2. \quad (3.1.25)$$

Note that $\rho(0) = \rho$; one has $\rho = 1/\pi$ if measured in units of $\sqrt{\mu}$.

3.1.4 One-particle density matrix: asymptotics and numerics

Since $\rho(x) = \rho(-x)$, we assume $x \geq 0$ in this section. For short distances ($x \ll 1$) one can easily find from Eqs. (3.1.22) and (3.1.20) that

$$\frac{\rho(x)}{\rho(0)} = 1 - \frac{x^2}{6} - \frac{\alpha x^3}{\pi 18} + \frac{x^4}{120} + \frac{\alpha 11x^5}{\pi 2700} + \dots, \quad x \rightarrow 0. \quad (3.1.26)$$

For $\alpha = 0$ this is simply the expansion of $\sin x/x$. Notice that higher order terms that are not exhibited on the r.h.s. of Eq. (3.1.26), depend on α nonlinearly.

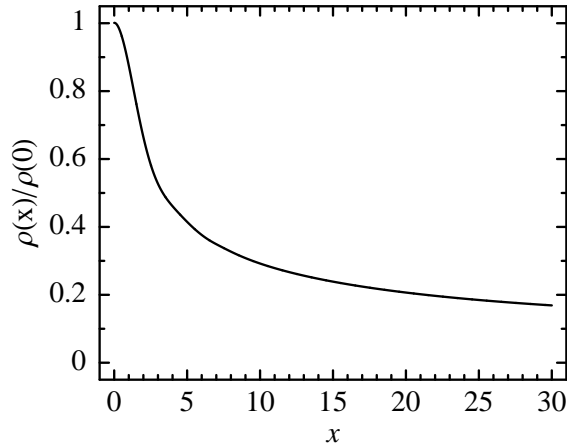


Figure 3.1: Shown is the function $\rho(x)/\rho(0)$ versus x . The density matrix $\rho(x)$ is given by Eq. (3.1.22); in getting the plot the Fredholm determinants entering Eq. (3.1.22) were approximated by finite-dimensional matrices, Eq. (3.1.19), with $N = 800$.

Next, consider the long-distance expansion of Eq. (3.1.22). Compared to the short-distance analysis, this is a more sophisticated task. The analysis carried out in Refs. [13, 46, 27] shows the power-law decay of the density matrix:

$$\frac{\rho(x)}{\rho(0)} = \frac{C}{\sqrt{x}} \left[1 + \frac{1}{8x^2} \left(\cos 2x - \frac{1}{4} \right) + \frac{3}{16x^3} \sin 2x \right], \quad x \rightarrow \infty \quad (3.1.27)$$

with relative corrections of the order of x^{-1} . The constant C is given by

$$C = \pi e^{1/2} 2^{-1/3} A^{-6} \approx 0.924, \quad (3.1.28)$$

with $A = 1.2824271 \dots$ being Glaisher's constant. The result (3.1.27) confirms confirms the Luttinger liquid theory predictions, Eq. (1.4.5).

For the curve in Fig. 3.1 we took $N = 800$ in using the representation (3.1.19) for the Fredholm determinants entering Eq. (3.1.22). For x large than 30 it is safe to use the asymptotic expansion (3.1.27) instead of the exact expression (3.1.22): the

relative difference $\Delta(x)$ between the exact expression and the leading term of the large x expansion

$$\Delta(x) = \frac{\rho(x) - C/\sqrt{x}}{\rho(x)} \quad (3.1.29)$$

is very small for $x = 30$:

$$\Delta(30) \approx 6 \cdot 10^{-4}. \quad (3.1.30)$$

so the subleading terms in expansion (3.1.27) are not relevant for most applications. On the other hand, the function $1/\sqrt{x}$ converges to zero very slowly and the knowledge of constant C entering the expansion (3.1.27) is required with a good precision. Thus, the exact expression (3.1.28) is very much appreciated.

3.1.5 The momentum distribution

The momentum distribution function $n(k)$ is given by

$$n(k) = \int dx e^{-ikx} \rho(x), \quad (3.1.31)$$

where $\rho(x)$ is the one-particle density matrix (3.1.1). Recall that x is measured in units of $1/\sqrt{\mu}$ as defined below Eq. (3.1.21), and that $\rho(x) = \rho(-x)$. The momentum distribution is also symmetric, $n(k) = n(-k)$. It is normalized as follows

$$\int_0^\infty dk n(k) = 1. \quad (3.1.32)$$

We plot $n(k)$ in Fig. 3.2. The density matrix $\rho(x)$ for $x \leq 30$ was calculated from the Fredholm determinant representation; for $x \geq 30$ we have used the asymptotic expansion (3.1.27). This momentum distribution is compared with the momentum distribution of the free fermion gas.

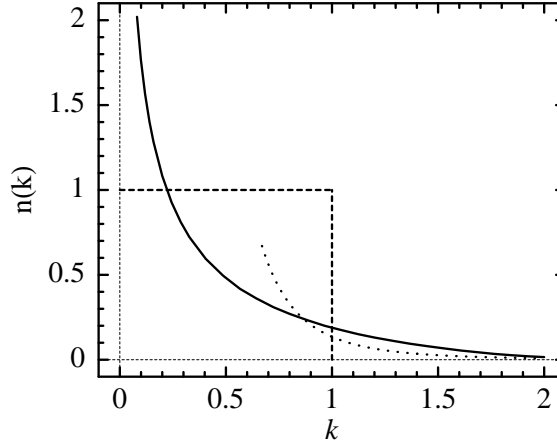


Figure 3.2: Shown is the momentum distribution for the Tonks-Girardeau gas (solid curve) and for the free fermion gas (dashed curve). Both distributions are normalized by the condition (3.1.32). Dotted curve represents the large k asymptotics of the momentum distribution, Eq. (3.1.33).

For the large momentum tail of $n(k)$ one gets

$$n(k) \simeq \frac{4}{3\pi^2} \frac{1}{k^4}, \quad k \rightarrow \infty. \quad (3.1.33)$$

The $1/k^4$ decay originates from the cusp of $\rho(x)$ at $x = 0$ seen in the third term on the right-hand side of Eq. (3.1.26). The coefficient $3/4\pi^2$ in Eq. (3.1.33) is taken from the Ref. [38]. It is useful to note that Ref. [38] contains more general result: the large momentum tail of $n(k)$ decays as $1/k^4$ at any coupling strength; the coefficient stands at $1/k^4$ term was also found in the cited reference.

It follows from the large x expansion (3.1.27) that the momentum distribution is divergent at small momenta,

$$n(k) \simeq C \sqrt{\frac{2}{\pi k}}, \quad k \rightarrow 0. \quad (3.1.34)$$

where the coefficient C is given by Eq. (3.1.28). This singularity is predicted by the

Luttinger liquid theory for any coupling strength γ , Eq. (1.4.6), and is the manifestation of quasi-long-range order, that is, a quasicondensate. The singularity is integrable for any $\gamma \neq 0$, therefore zero momentum state is not macroscopically occupied. Note, however, that a significant fraction of particles lies in low-momentum states: for the Tongs-Girardeau gas about 15% of particles have a momentum lower than 0.01 (in units of k_F , as discussed below Eq. (3.1.21)). When γ lowers this fraction is increased.

3.2 Local correlation functions

It is customary to identify the Bose-Einstein condensation phenomenon with the appearance of a non-vanishing average $\lim_{x \rightarrow \infty} \langle \Psi^\dagger(x) \Psi(0) \rangle \neq 0$. Due to strong quantum fluctuations there is no true long-range order in the 1D systems, hence there is no Bose-Einstein condensation at any temperature. The only reminiscent of the condensation phenomenon is the appearance of a quasi-long-range order at zero temperature: the zero-temperature correlation functions exhibit power-law decay. This phenomenon was studied within the Luttinger liquid approach in chapter 1, Eq. (1.4.5), and confirmed microscopically in the infinite γ limit of the Lieb-Liniger model in section 3.1, Eq. (3.1.27).

All experiments on condensation phenomenon in 1D are carried out using an ensemble of cold alkali atoms confined in a highly elongated trap. The temperature of the alkali vapor is lowered much below 1K in course of the experiment. An atomic cloud is, therefore, unstable: atoms tend to form molecules, molecules tend to form bigger clusters, and so on. Molecules do not “feel” trapping potential and escape the trap. Thus, the atomic cloud has a finite lifetime and this lifetime is an important experimental parameter.

To form a molecule atoms should drop their extra potential energy. In case of two-body collision the only way to do this is to emit photons. In case of three and more atoms colliding there is another option: two atoms form a molecule, and the released energy is taken by the other atoms participating in the collision. The probability of the photon emission could be very low and it is, therefore, not clear at all whether the two-body collisions are the dominant mechanism of the molecules formation.

The above mentioned mechanisms of the recombination processes were studied extensively from both theoretical and experimental points of view, see [28, 8, 44, 18] and references therein. It was shown, in particular, that in a quasi-one-dimensional atomic cloud of cold ^{87}Rb atoms the main losses come from one-body escape processes (atoms escape from the trap) and three-body recombination processes. In the experiment described in Ref. [44] the following differential equation were used to describe the number of atoms N in a trap as a function of time:

$$\frac{dN}{dt} = -K_1(\gamma)N - \frac{g_3(\gamma, N)}{\rho^3}\alpha(t, \gamma)N^3. \quad (3.2.1)$$

The parameter ρ is an average one-dimensional density and the parameter γ is the dimensionless coupling strength, given within the Lieb-Liniger model by Eq. (2.1.2). The inhomogeneity of the density distribution in both transverse and longitudinal directions are encoded in the parameter $\alpha(t, \gamma)$, for which an analytic expression was given, within the Thomas-Fermi approximation, in Ref. [44]. The parameter $K_1(\gamma)$ gives the escape rate of atoms from the trap; it does not depend on time. The local three-body correlation function g_3 is defined as the ground-state expectation value

$$g_3(\gamma, N) = \langle : \hat{n}^3(x) : \rangle \equiv \langle \hat{\Psi}^\dagger(x)^3 \hat{\Psi}(x)^3 \rangle. \quad (3.2.2)$$

The symbol $::$ stands for normal ordering. Assuming that $g_3(\gamma, N) \simeq g_3(\gamma, \infty)$ during

the course of the measurements, the value of $g_3(\gamma, \infty)$ was obtained in Ref. [44] for a particular value of γ (≈ 0.5).

Motivated by the experimental findings of Ref. [44] we calculate $g_3(\gamma, \infty)$ within the Lieb-Liniger model. The calculation of $g_3(\gamma, \infty)$ is nontrivial. In section 3.2.1 we discuss the function $g_2(\gamma, N)$. Its analysis is much easier technically and will provide us with some intuition about the behavior of $g_3(\gamma, N)$. The function $g_3(\gamma, N)$ is studied in section 3.2.2. The exact expression for $g_3(\gamma, \infty)$, Eq. (3.2.13), is one of the main results of the present Thesis. In section 3.2.3 we sketch the main steps of the derivation of Eq. (3.2.13), referring the reader to chapter 5 for further details.

3.2.1 Two-body correlation function

The local two-body correlation function g_2 is defined as the ground state expectation value

$$g_2(\gamma, N) = \langle : \hat{n}^2(x) : \rangle \equiv \langle \hat{\Psi}^\dagger(x)^2 \hat{\Psi}(x)^2 \rangle. \quad (3.2.3)$$

The symbol $::$ stands for normal ordering. Written in the first-quantized form, $g_2(\gamma, N)$ is

$$g_2(\gamma, N) = \frac{N!}{(N-2)!} \int_0^L dX |\chi_N(0, 0, x_3, \dots, x_N)|^2, \quad (3.2.4)$$

where $dX = dx_3 \dots dx_N$ and χ_N is given by Eq. (2.3.13). Note that the average $\langle \dots \rangle$ in Eq. (3.2.2) is taken over the ground state of the system; equivalently, it is the ground-state wave function which enters Eq. (3.2.4). This is ensured by selecting the proper set of the quasi-momenta k_j among all the solutions of the Bethe equations (2.3.15).

All quasi-momenta k_j are equal to zero for the non-interacting system, $\gamma = 0$, and

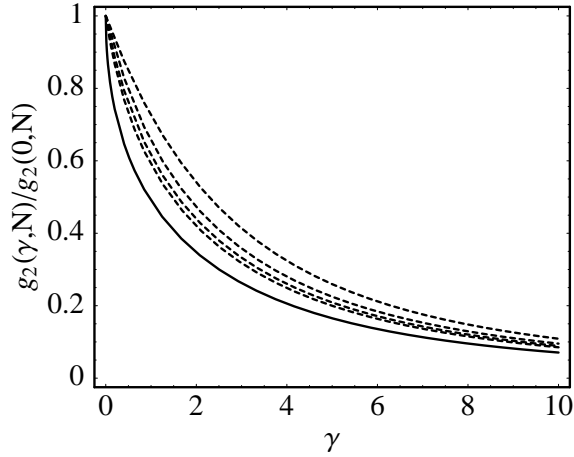


Figure 3.3: The two-body correlation function g_2 , normalized to its value in the absence of interactions, as a function of γ . The dashed lines are for particle numbers $N = 2, 3, 4$, and 5 (from top to bottom), and the full line is for $N = \infty$.

the wave function (2.3.13) is uniform in space, $\chi_N = (1/\sqrt{L})^N$, which implies that

$$g_2(0, N) = \rho^2 \frac{N-1}{N}. \quad (3.2.5)$$

We shall use this equation for normalizing our results for the correlation function $g_2(\gamma, N)$. We use the wave function (2.3.13) to get $g_2(\gamma, N)$ for $N = 2, 3, 4$, and 5 numerically. The results are plotted in Fig. 3.3.

The eigenfunctions (2.3.13) are very complicated for a general value of N , and it is not at all obvious how they could be simplified in the limit $N \rightarrow \infty$. One can, however, to calculate $g_2(\gamma, N)$ for $N = \infty$ as well as for finite N using the following trick [18]: the ground state average of the potential energy term in the Hamiltonian (2.1.1) coincides with the correlation function (3.2.3). Employing the Hellmann-Feynman theorem, one gets

$$\frac{dE_N}{dc} = \left\langle \frac{dH}{dc} \right\rangle = Lg_2(\gamma, N), \quad (3.2.6)$$

where the energy E_N is given by Eq. (2.3.14) and the ground state quasi-momentum distribution is given by Eq. (2.3.22).

To get $g_2(\gamma, \infty)$ use the expression (2.3.35) for the ground state energy

$$E_0 = N\rho^2 e(\gamma) \quad (3.2.7)$$

and the relation Eq. (2.1.2) between c and γ . The result is as follows

$$\frac{g_2(\gamma, \infty)}{\rho^2} = \frac{\partial e(\gamma)}{\partial \gamma}. \quad (3.2.8)$$

This function is plotted in Fig. 3.3.

In the large γ limit Eq. (2.3.60) gives

$$\frac{g_2(\gamma, \infty)}{\rho^2} \simeq \frac{4\pi^2}{3\gamma^2}, \quad \gamma \rightarrow \infty. \quad (3.2.9)$$

In the opposite limit, $\gamma \rightarrow 0$, one gets with the help of Eq. (2.3.65):

$$\frac{g_2(\gamma, \infty)}{\rho^2} \simeq 1 - \frac{2}{\pi}\sqrt{\gamma}, \quad \gamma \rightarrow 0. \quad (3.2.10)$$

3.2.2 Three-body correlation function

The first-quantized representation for the local three-body correlation function, Eq. (3.2.2), is

$$g_3(\gamma, N) = \frac{N!}{(N-3)!} \int_0^L dX |\chi_N(0, 0, 0, x_4, \dots, x_N)|^2, \quad (3.2.11)$$

where $dX = dx_4 \dots dx_N$ and χ_N is given by Eq. (2.3.13). Like for $g_2(\gamma, N)$, the average $\langle \dots \rangle$ in Eq. (3.2.2) is taken over the ground state of the system; equivalently, it is the ground-state wave function which enters Eq. (3.2.11). This is ensured by selecting the proper set of the quasi-momenta k_j among all the solutions of the Bethe equations (2.3.15).

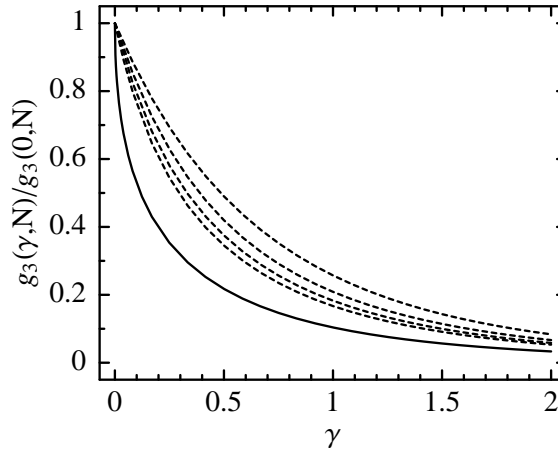


Figure 3.4: The three-body correlation function g_3 , normalized to its value in the absence of interactions, as a function of γ . The dashed lines are for particle numbers $N = 3, 4, 5$, and 6 (from top to bottom), and the full line is for $N = \infty$.

For the non-interacting system one has

$$\frac{g_3(0, N)}{\rho^3} = \frac{(N-1)(N-2)}{N^2}. \quad (3.2.12)$$

Like for $g_2(\gamma, N)$ we use the wave function (2.3.13) to get $g_2(\gamma, N)$ for $N = 3, 4, 5$, and 6 numerically. The results are plotted in Fig. 3.4.

The exact expression for $g_3(\gamma, \infty)$ is

$$\frac{g_3(\gamma, \infty)}{\rho^3} = \frac{3}{2\gamma}\epsilon'_4 - \frac{5\epsilon_4}{\gamma^2} + \left(1 + \frac{\gamma}{2}\right)\epsilon'_2 - 2\frac{\epsilon_2}{\gamma} - \frac{3\epsilon_2\epsilon'_2}{\gamma} + \frac{9\epsilon_2^2}{\gamma^2}, \quad (3.2.13)$$

where ϵ'_m is the derivative of ϵ_m with respect to γ . The detail of calculation are given in chapter 5. We stress that Eq. (3.2.13) is valid in the thermodynamic limit only, in contrast to Eq. (3.2.6).

In Fig. 3.4 we compare the normalized function $g_3(\gamma, \infty)/g_3(0, \infty)$ with that calculated for $N = 3, 4, 5$, and 6 . One can see that the convergence to the thermodynamic

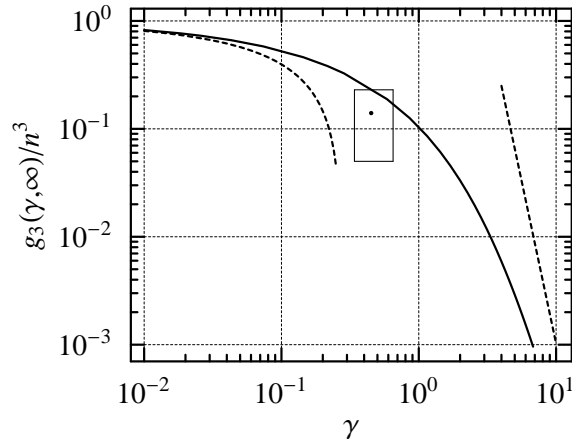


Figure 3.5: Double-logarithmic plot of the three-body correlation function $g_3(\gamma, \infty)$ as a function of γ (full line). The dashed lines indicate the asymptotic results given in Eqs. (3.2.14) and (3.2.15) for small and large γ , respectively. The measured value is shown as the dot inside a box indicating the experimental uncertainty [44].

limit is quite slow. In Fig. 3.5 we compare $g_3(\gamma, \infty)/n^3$ with the experimental data from Ref. [44]. The average value of γ , measured in [44], $\gamma_m \approx 0.45$, and the corresponding value of $g_3(\gamma_m, \infty) \approx 0.14$ are shown as the dot inside the box which represents the experimental uncertainty according to Ref. [44]: $0.34 < \gamma_m < 0.65$ and $0.05 < g_3(\gamma_m, \infty) < 0.23$. The dashed lines show the asymptotic expressions given in Eqs. (3.2.14) and (3.2.15) for small and large γ , respectively. Evidently, the asymptotic expressions do not account for the observed value, and the exact expression (3.2.13) is needed.

Using Eqs. (2.3.65) and (2.3.66) one gets for the $\gamma \rightarrow 0$ limit of Eq. (3.2.13)

$$\frac{g_3(\gamma, \infty)}{\rho^3} \simeq 1 - \frac{6}{\pi} \sqrt{\gamma}, \quad \gamma \rightarrow 0. \quad (3.2.14)$$

In the large γ limit one gets, using the expansions (2.3.60) and (2.3.61):

$$\frac{g_3(\gamma, \infty)}{\rho^3} \simeq \frac{16\pi^6}{15\gamma^6}, \quad \gamma \rightarrow \infty. \quad (3.2.15)$$

The asymptotic result (3.2.15) is nearly a factor of two large than the exact result for $\gamma = 30$.

3.2.3 How to get Eq. (3.2.13): an overview

The formal derivation of Eq. (3.2.13) is quite lengthy and is given in chapter 5. Here we shall only summarize the main steps: We are interested in the thermodynamic limit of the Lieb-Liniger model. In this limit it is natural to use a field-theoretical approach rather than the first-quantized formalism we employed for the few-particle systems. The Lieb-Liniger model is completely integrable. In the field-theoretical language complete integrability implies the existence of an infinite family of conserved currents \mathcal{J}_n , $n = 0, 1, 2, \dots$. The integrals over space of these conserved currents commute with each other. There is thus an infinite set of mutually commuting operators J_n , $n = 0, 1, 2, \dots$, for which the eigenfunctions and the spectrum are known exactly. A procedure for generating J_n is discussed, for example, in Ref. [33]. The first three currents generated by this method are $\mathcal{J}_0 = \mathcal{N}$, $\mathcal{J}_1 = \mathcal{P}$, and $\mathcal{J}_2 = \mathcal{H}$, where \mathcal{N} , \mathcal{P} , and \mathcal{H} are the number density of particles, the momentum density, and the Hamiltonian density, respectively. These three currents exist for non-integrable models as well, while the currents \mathcal{J}_n for $n > 2$ are specific to the LL model. To illustrate our approach we give the classical expression for the current \mathcal{J}_4 (its quantization will be

discussed in the next paragraph):

$$\begin{aligned} \mathcal{J}_4 = & \Psi^* \partial_x^4 \Psi + c \Psi^* \Psi^* (\partial_x \Psi) \partial_x \Psi + 8c (\partial_x \Psi^*) \Psi^* \Psi \partial_x \Psi \\ & + c (\partial_x \Psi^*) (\partial_x \Psi^*) \Psi \Psi + 2c^2 (\Psi^3)^* \Psi^3. \end{aligned} \quad (3.2.16)$$

The quantum version of the last term in Eq. (3.2.16) is $\hat{\Psi}^\dagger(x)^3 \hat{\Psi}(x)^3$. When averaged over the ground state, this term gives the desired correlation function g_3 , Eq. (3.2.2). Some of the remaining terms in (3.2.16) can be eliminated by using the Hellmann-Feynman theorem, which applies to all conserved quantities. This theorem was applied to the Hamiltonian of the LL model in Ref. [18] to calculate the correlation function $\langle \hat{\Psi}^\dagger(x)^2 \hat{\Psi}(x)^2 \rangle$ (we have discussed the corresponding procedure in section 3.2.1). In case of g_3 not all unwanted terms in \mathcal{J}_4 can be eliminated by use of the Hellmann-Feynman theorem. We obtain additional identities in chapter 5 by employing the long-distance asymptotics of correlation functions of conserved currents known from the conformal limit of the theory.

The quantization of the classical current Eq. (3.2.16) is delicate. A naive quantum-mechanical version of the expression (3.2.16) contains terms involving derivatives of field operators $\hat{\Psi}$ and $\hat{\Psi}^\dagger$. Such terms are not well-defined in general. An explicit demonstration of this was given in the paper [38], where it was shown that the expectation value $\langle \hat{\Psi}^\dagger(0) \partial_x^3 \hat{\Psi}(x) \rangle$ is discontinuous at $x = 0$. The reason for this behavior can be seen from the structure of the Bethe-ansatz wave function (2.3.13), which has a cusp whenever the coordinates of any two particles coincide. This does not cause difficulties when one considers the momentum operator and the Hamiltonian, that is, when one works with the operators $\hat{\Psi}^\dagger(x) \partial_x \hat{\Psi}(x)$ and $\hat{\Psi}^\dagger(x) \partial_x^2 \hat{\Psi}(x)$. However, for other operators we encounter difficulties which we have not been able to resolve within the continuum model (2.1.1). Therefore we have used an integrable lattice

model, the so-called q -boson hopping model [6, 7], which reduces to the Lieb-Liniger model (2.1.1) in the continuum limit. Within this model all higher quantum integrals of motion are defined unambiguously.

Chapter 4

Impenetrable fermion gas

The inclusion of internal (“spin”) degrees of freedom greatly enriches the physical properties of the systems. We show in this chapter that the new temperature scale appears for the strongly repulsive fermion (and boson) systems, if the spin degrees of freedom are taken into account. This new temperature scale, whose characteristic temperature T_0 equals to the Fermi temperature divided by the dimensionless coupling strength, Eq. (4.1.29), is a feature of the two-component model and does not exist in the one-component case. We show rigorously the appearance of this new temperature scale for the quantum strongly repulsive δ -interacting fermion gas in section 4.1. In section 4.2 we compare correlation properties of this fermion gas above and below T_0 . In particular, we show in section 4.2.3 that a pronounced reconstruction of the momentum distribution function takes place when the temperature changes from $T \ll T_0$ to $T \gg T_0$, see Fig. 4.1. This momentum reconstruction was not measured experimentally so far, but we expect the corresponding experiment with an ensemble of the ultracold trapped atoms will be done in the nearest future. The results of the section 4.2.3 were reported in Ref. [10]. In section 4.2.4 we compare the asymptotics of dynamical correlation functions calculated at $T \ll T_0$ (Luttinger regime) with

those calculated at $T \ll T_0$ (“spin-disordered” regime). The results for the “spin-disordered” regime are original and were reported in Refs. [11, 12].

4.1 The model

Similarly to the one-component case, studied in chapter 2, one can consider a two-component gas of fermions (bosons). Such gases consist of particles with some internal quantum parameter σ , which can take two values. For brevity we refer to this parameter as “spin” (regardless of its physical nature), and label the corresponding states by $\sigma = \uparrow, \downarrow$. In the situation where the interaction constant is tuned to be independent of the “spin” index, we take the first-quantized Hamiltonian of the system to be the same as for the one-component case, Eq. (2.1.11):

$$\mathcal{H}_N = - \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + 2c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j). \quad (4.1.1)$$

The information about the system being two-component is encoded in the symmetry of the coordinate wave function. To introduce the “spin” indices in the Hamiltonian explicitly one can recast it in the second quantized form using the particle creation and annihilation operators $\Psi_\sigma^\dagger(x)$ and $\Psi_\sigma(x)$. Written in the second-quantized form the Hamiltonian (2.1.11) is

$$H_F = \int dx \left[- \sum_{\sigma=\uparrow,\downarrow} \Psi_\sigma^\dagger(x) \partial_x^2 \Psi_\sigma(x) + c n_\uparrow(x) n_\downarrow(x) \right] \quad (4.1.2)$$

for two-component fermions, and

$$H_B = \int dx \left[- \sum_{\sigma=\uparrow,\downarrow} \Psi_\sigma^\dagger(x) \partial_x^2 \Psi_\sigma(x) + \frac{c}{2} : n(x)^2 : \right] \quad (4.1.3)$$

for two-component bosons. Here

$$n_\sigma(x) = \Psi_\sigma^\dagger(x) \Psi_\sigma(x), \quad \sigma = \uparrow, \downarrow \quad (4.1.4)$$

is the density of particles of type σ ; the total density $n(x)$ is

$$n(x) = n_{\uparrow}(x) + n_{\downarrow}(x), \quad (4.1.5)$$

and the symbol $::$ stands for the normal ordering. For interactions of this form the total number of particles N and the total number $N_{\uparrow(\downarrow)}$ of “spin” up (down) particles are good quantum numbers. Note that the interaction term in Eq. (4.1.2) is the most generic form of contact interaction for fermions, which is a consequence of the Pauli principle. In the bosonic case, however, the interaction term in Eq. (4.1.3) represents a special situation where the three coupling constants are tuned to be equal.

The interaction constant $c \geq 0$ has the dimension of inverse length. Like in one-component model, define the dimensionless coupling strength γ :

$$\gamma = \frac{c}{\rho}. \quad (4.1.6)$$

The structure of the Bethe ansatz solution is similar for the fermion (4.1.2) and boson (4.1.3) cases. All the new physics we discuss in this chapter is also applied to both fermion and boson model. We thus limit ourselves by considering the fermion case exclusively. The reader interested in the extension of the results to the boson case is referred to the paper [10].

4.1.1 Eigenfunctions and spectrum

The basis in the Fock space of the model is constructed by acting with operators $\Psi_{\sigma}(x)$ onto the vacuum $|0\rangle$ defined as

$$\Psi_{\sigma}(x)|0\rangle = 0, \quad \langle 0|\Psi_{\sigma}(x) = 0, \quad \langle 0|0\rangle = 1. \quad (4.1.7)$$

We say that a state belongs to the sector (N, M) of the Fock space if it contains $N - M$ spin-up particles ($\sigma = 1$) and M spin-down particles ($\sigma = 2$), so that the total number

if particles is N . In the sector (N, M) the eigenfunctions are enumerated by two sets, $\{k\} = k_1, \dots, k_N$ and $\{\lambda\} = \lambda_1, \dots, \lambda_M$, of unequal (in each set separately) real numbers. Thus, the eigenfunctions can be written in the form

$$|\psi_{N,M}(\{k\}; \{\lambda\})\rangle = \int_0^L dx_1 \dots \int_0^L dx_N \sum_{\sigma_1, \dots, \sigma_N=1,2} \chi_{N,M}^{\sigma_1, \dots, \sigma_N}(x_1, \dots, x_N | \{k\}; \{\lambda\}) \times \Psi_{\sigma_1}^\dagger(x_1) \dots \Psi_{\sigma_N}^\dagger(x_N) |0\rangle, \quad (4.1.8)$$

where the wave function $\chi_{N,M}^{\sigma_1, \dots, \sigma_N}(x_1, \dots, x_N | \{k\}, \{\lambda\})$ is not equal to zero only if $N-M$ elements are equal to 1 and M elements are equal to 2 in the set $\sigma_1, \dots, \sigma_N$, that is $\sum_{j=1}^N \alpha_j = N + M$.

$$\chi_{N,M}^{\sigma_1, \dots, \sigma_N}(x_1, \dots, x_N | \{k\}; \{\lambda\}) = \frac{1}{N!} \det_N[\exp\{ik_a x_b\}] \times \left[\sum_P \xi_{N,M}^{\sigma_{P_1} \dots \sigma_{P_N}}(\{\lambda\}) \theta(x_{P_1} < \dots < x_{P_N}) \right], \quad (4.1.9)$$

where $\xi_{N,M}^{\sigma_1, \dots, \sigma_N}(\{\lambda\})$ are components of a 2^N dimensional vector $\xi_{N,M}(\{\lambda\})$. The sum is taken over all permutations P of N numbers; $[P]$ denotes the parity of the permutation.

The periodic boundary conditions imposed on $\chi_{N,M}$ give rise the following condition on $\xi_{N,M}$, called the auxiliary lattice problem:

$$\xi_{N,M}^{\sigma_1 \sigma_2 \dots \sigma_N}(\{\lambda\}) = e^{ik_j L} \xi_{N,M}^{\zeta \sigma_2 \dots \sigma_N \sigma_1}(\{\lambda\}), \quad j = 1, \dots, N. \quad (4.1.10)$$

These equations should hold for all quasi-momenta k_1, \dots, k_N . They can be regarded as the eigenvalue problem for the cyclic shift operator C_N acting in a 2^N -dimensional vector space:

$$\zeta^{\sigma_1 \sigma_2 \dots \sigma_N} = C_N \zeta^{\sigma_2 \dots \sigma_N \sigma_1}. \quad (4.1.11)$$

Izergin and Pronko [26] have used the fact that the eigenvectors of C_N can be chosen as the eigenvectors of the XX0 Heisenberg spin 1/2 chain with the periodic boundary conditions:

$$\xi_{N,M}(\{\lambda\}) = \sum_{n_1=1}^N \cdots \sum_{n_M=1}^N \varphi_{N,M}(n_1, \dots, n_M | \{\lambda\}) \sigma_{n_1}^- \cdots \sigma_{n_M}^- | \uparrow_N \rangle, \quad (4.1.12)$$

where

$$\varphi_{N,M}(n_1, \dots, n_M | \{\lambda\}) = \frac{1}{M!} \left[\prod_{1 \leq j < l \leq N} \text{sgn}(n_l - n_j) \right] \det_M [\exp\{i\lambda_a n_b\}]. \quad (4.1.13)$$

The state $|\uparrow_N\rangle = \otimes_{n=1}^N |\uparrow_n\rangle$ (all spins up) is the vacuum state for the chain with N sites. The matrix σ_j^- is a Pauli spin matrix; the subscript j labels a site of the chain where this matrix acts. The XX0 Heisenberg spin 1/2 chain is the lattice regularization for the TG gas, and one can easily recognize in the expression (4.1.13) an eigenfunction of the TG gas, Eq. (2.2.3). The periodic boundary conditions imposed on $\varphi_{N,M}$, give the quantization condition for the set of quasi-momenta λ_j :

$$e^{i\lambda_b N} = (-1)^{M+1}, \quad b = 1, \dots, M. \quad (4.1.14)$$

Vectors (4.1.12) are the eigenvectors of the operator C_N :

$$C_N \xi_{N,M}(\{\lambda\}) = e^{i\sum_{b=1}^M \lambda_b} \xi_{N,M}(\{\lambda\}) \quad (4.1.15)$$

Comparing this equation with Eqs. (4.1.10) and (4.1.11) one gets

$$e^{ik_a L} = e^{i\sum_{b=1}^M \lambda_b}, \quad a = 1, \dots, N. \quad (4.1.16)$$

Equations (4.1.14) and (4.1.16) are the desired Bethe equations for the fermion gas in the limit $c \rightarrow \infty$. These equations can be solved easily: The permitted values $(k_a)_j$ of the quasi-momenta k_a , ($a = 1, \dots, N$) are

$$(k_a)_j = \frac{2\pi}{L} j + \frac{1}{L} \sum_{b=1}^M \lambda_b, \quad j = 0, \pm 1, \pm 2, \dots \quad (4.1.17)$$

and the permitted values $(\lambda_b)_l$ of the quasi-momenta λ_b , ($b = 1, \dots, M$) are

$$(\lambda_b)_l = \frac{2\pi}{N} \left(-\frac{N}{2} - \frac{1 + (-1)^{N-M}}{4} + l \right), \quad l = 1, \dots, N. \quad (4.1.18)$$

The eigenvalues $E_{N,M}(\{k\}, \{\lambda\})$ of the Hamiltonian are

$$E_{N,M}(\{k\}, \{\lambda\}) = \sum_{j=1}^N k_j^2. \quad (4.1.19)$$

This formula resembles the expression (2.2.4) for the spectrum of the TG gas; one should, however, bear in mind that k_j depend on the total quasi-momentum of the auxiliary lattice problem, as given by Eqs. (4.1.17) and (4.1.18).

The idea to represent the solutions of the Eq. (4.1.10) in the form (4.1.12) have played an important role in Izergin and Pronko calculations [26] of the determinant representation for dynamical correlation functions. It is interesting to note that taking the limit of infinite repulsion in the Bethe ansatz solution (found in Ref. [49]) of the model at an arbitrary coupling, one gets $\xi_{N,M}$ in the form of the eigenfunctions of the XXX Heisenberg spin 1/2 chain. These functions have much more complicated structure than that ones given by Eq. (4.1.12). We will continue this discussion in the section 4.2.1.

4.1.2 The ground state and thermodynamics at zero temperature

One can see by comparing Eqs. (4.1.19) and (2.2.4) that the expression for the energy of the impenetrable two-component fermion gas resembles that one of the TG gas. What is different, however, is the multiplicity of states, and we will see that this difference gives rise to a physical phenomenon non-existing in the one-component models.

Let us find the ground state of the model and calculate its degeneracy. Assume N is odd for simplicity. One can easily see that the absolute minimum of the Eq. (4.1.19) in the N -particle sector is

$$E_N^{\min} = \left(\frac{2\pi}{L}\right)^2 \sum_{j=1}^N \left(j - \frac{N+1}{2}\right)^2. \quad (4.1.20)$$

This minimum is clearly degenerate since all the states with $\sum_{b=1}^M \lambda_b = 0$ have the same E_N^{\min} regardless of the value of M . All the states with $E = E_N^{\min}$ should be counted, but not only these states: The degeneracy is partially lifted by the periodic boundary conditions imposed on the system; in the thermodynamic limit the boundary conditions should play no role and some states with the energy $E_N \rightarrow E_N^{\min}$ as $N \rightarrow \infty$ could influence ground state properties. To identify such states note that the energy difference between the ground state and first excited state is of the order of $1/L$ in the free fermion gas. We say that all the states with

$$E_N - E_N^{\min} \rightarrow 0 \quad \text{faster than} \quad \frac{1}{L} \quad \text{as} \quad L \rightarrow \infty \quad (4.1.21)$$

equally contribute to the ground state properties in the thermodynamic limit. With the definition (4.1.21) in mind, one can see that the ground state degeneracy is 2^N when N is large. In a sector with given M the degeneracy is C_N^M , where C_N^M is the binomial coefficient.

The following arguments can be employed to explain this 2^N degeneracy: in the $c \rightarrow \infty$ the interaction between the particles is spin-independent. Indeed, the many-body wave functions become equal to zero when any two particles collide regardless of their spin orientations. The words ‘‘spin-disordered’’ seem to be well describing this regime.

For a given spin orientation the ground state properties are the same at those of

the TG gas (or of the one-component free fermion gas), so one can use results of the section (2.2.4). The only difference from the section (2.2.4) is in the definition of the Fermi momentum. For the two-component systems it is given by

$$k_F = \frac{\pi\rho}{2}. \quad (4.1.22)$$

The Fermi energy is defined as

$$E_F = k_F^2. \quad (4.1.23)$$

The relation between the density and the chemical potential is the same as in the one-component case Eqs. (2.2.15) and (3.1.21):

$$\sqrt{\mu} = \pi\rho. \quad (4.1.24)$$

The same convention as in one-component case will be used: the distances will be measured in units of $1/\sqrt{\mu}$.

4.1.3 Separation of spin and charge energy scales in the $c \rightarrow \infty$ limit

In this section we discuss the energy scales associated with the two-component model. Let the temperature T and coupling c be arbitrary. First scale is defined by the Fermi temperature $T_F = E_F$, Eq. (4.1.23). Low temperatures $T < T_F$ correspond to the degenerate quantum gas; when $T > T_F$ the gas becomes classical. In case of the one-component gas this is the only energy (or temperature) scale. The inclusion of “spin” gives rise to another energy scale for the strong coupling regime $\gamma \gg 1$. The characteristic energy for this scale is E_F/γ .

Why this “spin” energy scale appears? Take Yang’s solution for the fermion gas at arbitrary γ [49]. Consider the sector of the Hilbert space with a given number

of particles N and a polarization $M = N_\uparrow - N_\downarrow$. Denote by $E(M)$ the ground state energy in this sector. The function $E(M)$ has its absolute minimum at $M = 0$ [49] in agreement with the Lieb-Mattis theorem [36]. The energy difference per particle between the polarized and unpolarized phases

$$\epsilon = \frac{E(N) - E(0)}{N} \quad (4.1.25)$$

gives an estimate for the bandwidth of “spin” excitations. From the Bethe ansatz solution given in [49] one can find that

$$\epsilon \simeq \frac{8 \ln 2}{3} \frac{E_F}{\gamma}, \quad \gamma \rightarrow \infty. \quad (4.1.26)$$

This expression proves the appearance of the “spin” energy scale. The calculation of Eq. (4.1.26) is rather straightforward and we do not give it here; to get an intuition how to work with the large γ limit in the Bethe equations for two-component models one can consult Appendix A of the paper [37].

It is clear now that the results of the sections 4.1.1 and 4.1.2 are relevant for the region

$$\frac{E_F}{\gamma} \ll T \ll E_F. \quad (4.1.27)$$

Indeed, we have calculated the eigenfunctions and spectrum assuming $\gamma = \infty$ in these sections and have found 2^N degeneracy of the ground state: states with every possible spin configuration contribute equally to the ground state. Therefore, the “spin-flip” energy is exactly zero and any finite temperature is larger than the bandwidths of the spin excitations. This is exactly the condition (4.1.27). We call the regime (4.1.27) “spin-disordered”. Another quantum regime is “conventional” (Luttinger) regime

$$T \ll \frac{E_F}{\gamma} \ll E_F. \quad (4.1.28)$$

The crossover from one regime to another happens at a temperature of the order of

$$T_0 = \frac{E_F}{\gamma}. \quad (4.1.29)$$

The regimes $T \ll T_0$ and $T_0 \ll T$ are accessed by choosing the appropriate order of limits $T \rightarrow 0$ and $\gamma \rightarrow \infty$: To ensure $T_0 \ll T$ one should take the limit $\gamma \rightarrow \infty$ first and then set $T = 0$ in the wave functions and spectra of the model, while for the Luttinger regime $T \ll T_0$ one should take the limit $T \rightarrow 0$ first.

The behavior of the correlation functions in the Luttinger regime was studied in section 1.5. An exact representation for the correlation functions in the “spin-disordered” regime is given in section 4.2.1.

4.2 Correlation functions

We consider two different time-dependent correlation functions

$$G_h(x, t) = \langle \Psi_{\uparrow}^{\dagger}(x, t) \Psi_{\uparrow}(0, 0) \rangle \quad (4.2.1)$$

and

$$G_e(x, t) = \langle \Psi_{\uparrow}(x, t) \Psi_{\uparrow}^{\dagger}(0, 0) \rangle. \quad (4.2.2)$$

Due to the parity and the time reversal symmetry of the Hamiltonian (4.1.2) the functions (4.2.1) and (4.2.2) obey the following symmetry properties

$$G_{h(e)}(x, t) = G_{h(e)}(-x, t) \quad (4.2.3)$$

and

$$G_{h(e)}(x, t) = G_{h(e)}^*(x, -t). \quad (4.2.4)$$

Here the asterisk stands for the complex conjugation. Furthermore, the following relation between G_h and G_e can be written

$$G_h(x, t) = -G_e^*(x, t) + \left\langle \left\{ \Psi_\uparrow^\dagger(x, t), \Psi_\uparrow(0, 0) \right\} \right\rangle \quad (4.2.5)$$

where $\{, \}$ stands for the anticommutator. At $t = 0$ this relation becomes

$$G_h(x, 0) = -G_e^*(x, 0) + \delta(x). \quad (4.2.6)$$

Note also the scaling relation

$$G_{e(h)}(x, t; \mu) = \sqrt{\mu} G_{e(h)}(\sqrt{\mu}x, \mu t; 1) \quad (4.2.7)$$

where $G_{e(h)}(x, t; \mu)$ is the correlation function taken at a given chemical potential μ .

In the following we will assume that $x \geq 0$, $t \geq 0$ and $\mu = 1$ (We thus measure distances in units of $1/\sqrt{\mu}$.) The correlation functions for all other cases will follow from the relations (4.2.3), (4.2.4) and (4.2.7).

4.2.1 Determinant representation for G_h and G_e

In this section, following [26], we write down the determinant representation for the correlation functions (4.2.1) and (4.2.2) in the “spin-disordered” regime. We use the notations of Ref. [12].

In paper [26] the following representation of the correlation functions (4.2.1) and (4.2.2) was derived

$$G_h(x, t) = \frac{e^{-it}}{8\pi} \int_{-\pi}^{\pi} d\eta F(\eta) B_{--}(x, t, \eta) \det(\hat{I} + \hat{V})(x, t, \eta) \quad (4.2.8)$$

and

$$G_e(x, t) = -\frac{e^{it}}{2\pi} \int_{-\pi}^{\pi} d\eta \frac{F(\eta)}{1 - \cos \eta} b_{++}(x, t, \eta) \det(\hat{I} + \hat{V})(x, t, \eta). \quad (4.2.9)$$

To shorten notations we will henceforth omit x , t and η in the arguments of all functions unless they are necessary.

The objects entering the representations (4.2.8) and (4.2.9) are defined as follows. The function $F(\eta)$ is

$$F(\eta) = 1 + \frac{e^{i\eta}}{2 - e^{i\eta}} + \frac{e^{-i\eta}}{2 - e^{-i\eta}}. \quad (4.2.10)$$

The determinant $\det(\hat{I} + \hat{V})$ is the Fredholm determinant of a linear integral operator \hat{V} . Two equivalent definitions of the Fredholm determinant are given by Eqs. (3.1.19) and (3.1.20). The kernel $V(k, p)$ is defined on $[-1, 1] \times [-1, 1]$ and is given by the following formula

$$V(k, p) = \frac{e_+(k)e_-(p) - e_+(p)e_-(k)}{k - p}. \quad (4.2.11)$$

The functions entering this kernel are defined as follows:

$$e_+(k) = \frac{1}{2\sqrt{\pi}} e^{-\tau(k)/2} [(1 - \cos \eta)e^{\tau(k)} E_0(k) + \sin \eta] \quad (4.2.12)$$

$$e_-(k) = \frac{1}{\sqrt{\pi}} e^{\tau(k)/2} \quad (4.2.13)$$

$$\tau(k) = ik^2 t - ikx \quad (4.2.14)$$

$$E_0(k) = \text{p.v.} \int_{-\infty}^{\infty} dp \frac{e^{-\tau(p)}}{\pi(p - k)} \quad (4.2.15)$$

Note that $V(k, p)$ is symmetric, $V(k, p) = V(p, k)$, and nonsingular at $k = p$.

To define B_{--} and b_{++} introduce the resolvent operator \hat{R} :

$$\hat{I} - \hat{R} = (\hat{I} + \hat{V})^{-1}. \quad (4.2.16)$$

Due to the specific form (4.2.11) of the kernel $V(k, p)$ the resolvent kernel $R(k, p)$ may be written as

$$R(k, p) = \frac{f_+(k)f_-(p) - f_+(p)f_-(k)}{k - p} \quad (4.2.17)$$

where functions f_{\pm} are the solutions to the integral equations

$$f_{\pm}(k) + \int_{-1}^1 dp V(k, p) f_{\pm}(p) = e_{\pm}(k). \quad (4.2.18)$$

For the pairs $e_{\pm}(k)$ and $f_{\pm}(k)$ we will often use the vector notation:

$$\vec{e}(k) = \begin{pmatrix} e_+(k) \\ e_-(k) \end{pmatrix} \quad \vec{f}(k) = \begin{pmatrix} f_+(k) \\ f_-(k) \end{pmatrix}. \quad (4.2.19)$$

Now define the functions B_{ab} and C_{ab}

$$B_{ab} = \int_{-1}^1 dk e_a(k) f_b(k) \quad C_{ab} = \int_{-1}^1 dk k e_a(k) f_b(k) \quad (4.2.20)$$

where a and b run through two values: $a, b = \pm$. In particular,

$$B_{--} = \int_{-1}^1 dk e_-(k) f_-(k). \quad (4.2.21)$$

The function b_{++} in (4.2.9) is

$$b_{++} = B_{++} - (1 - \cos \eta) G_0 \quad (4.2.22)$$

where $G_0(x, t)$ is the vacuum Green function

$$G_0(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-\tau(k)} = \begin{cases} \frac{e^{-i\pi/4}}{2\sqrt{\pi t}} e^{ix^2/4t} & t \neq 0 \\ \delta(x) & t = 0 \end{cases}. \quad (4.2.23)$$

4.2.2 Density matrix

For two-component systems the one-particle density matrix is defined as

$$\rho(x) = \frac{1}{2} \left[\langle \Psi_{\uparrow}^{\dagger}(x) \Psi_{\uparrow}(0) \rangle + \langle \Psi_{\downarrow}^{\dagger}(x) \Psi_{\downarrow}(0) \rangle \right]. \quad (4.2.24)$$

The symmetry of the interaction term (4.1.2) with respect to spin rotation implies

$$\langle \Psi_{\uparrow}^{\dagger}(x) \Psi_{\uparrow}(0) \rangle = \langle \Psi_{\downarrow}^{\dagger}(x) \Psi_{\downarrow}(0) \rangle, \quad (4.2.25)$$

therefore

$$\rho(x) = G_h(x, 0). \quad (4.2.26)$$

where $G_h(x, 0)$ is defined by Eq. (4.2.1). The density of particles is

$$n_{1D} = 2\rho(0). \quad (4.2.27)$$

An exact expression for $\rho(x)$ can be obtained by taking the limit $t \rightarrow 0$ in the formulas of section 4.2.1. This requires some efforts and we refer the reader to the Ref. [12], section 2.3 for details. Surprisingly, the result looks very similar to the Lenard's formula given in section 3.1.3. Namely, equations (3.1.22)–(3.1.24) hold true, and the only difference is that

$$\alpha = -\frac{1}{2}, \quad (4.2.28)$$

while for the one-component boson gas α is given by Eq. (3.1.25).

The short-distance expansion of the density matrix in our case is very similar to that one for the Tonks-Girardeau gas: one should simply substitute α given by Eq. (4.2.28) into Eq. (3.1.26). The long-distance asymptotics is, however, very non-trivial:

$$\rho(x) \simeq C_1 e^{-\nu x} x^{\Delta_F} \sin(x - \nu \ln x + C_2), \quad x \rightarrow \infty. \quad (4.2.29)$$

Here C_1 and C_2 are some constants which are given explicitly in Ref. [2], while

$$\Delta_F = \frac{\nu^2}{2} - 1, \quad \nu = \frac{\ln 2}{\pi}. \quad (4.2.30)$$

The parameters Δ_F and ν , Eq. (4.2.30), were calculated first in Ref. [3].

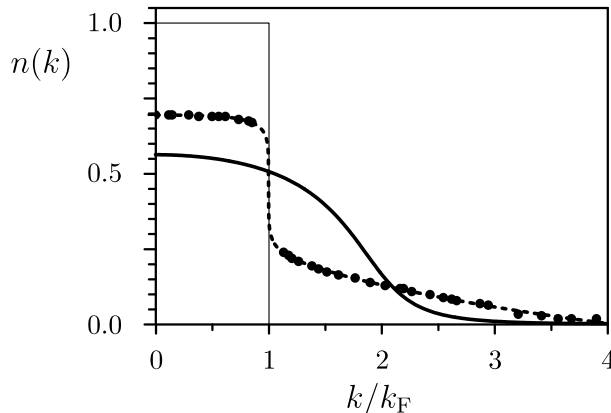


Figure 4.1: The momentum distribution function $n(k)$ for two-component fermions. The thick solid curve is our result for $T \gg T_0$. The dashed curve corresponds to $T \ll T_0$ and is obtained by the interpolation of the data (filled circles) from Ref. [37]. The thin solid line shows the distribution function for non-interacting two-component fermions.

4.2.3 Low-temperature reconstruction of the momentum distribution

In this section we study the momentum distribution function of two-component fermions in the two temperature regimes $T \ll T_0$ and $T \gg T_0$ discussed in Section 4.1.3, assuming always that the temperature is much less than the Fermi temperature.

The momentum distribution for different temperature regimes is shown in Fig. 4.1. The $T \gg T_0$ curve was obtained using the results of Sections 3.1.3 and 4.2.2: We calculated $\rho(x)$ numerically from the determinant representation Eq. (3.1.22) and then made use of Eq. (1.3.12). Due to the exponential decay of $\rho(x)$ as $x \rightarrow \infty$, Eq. (4.2.29), one needs to calculate it in a relative small vicinity of the point $x = 0$. For the curve in Fig. 4.1 we took $-20 \leq x \leq 20$ and $L = 800$ in using the representation (3.1.19) for the Fredholm determinants entering Eq. (3.1.22). For the large momentum tail of

$n(k)$ one gets

$$n(k) \propto \frac{1}{k^4}, \quad k \rightarrow \infty \quad (4.2.31)$$

which originates from the cusp of $\rho(x)$ at $x = 0$ seen in the third term on the right hand side of Eq. (3.1.26).

The $T \ll T_0$ curve exhibits power-law singularity near $k = k_F$, as can be seen from Eq. (1.5.18). The large- k asymptotics is given by Eq. (4.2.31). This can be shown using the same arguments as were applied to the one-component boson gas in Ref. [38]. There are, however, no other analytical results on $n(k)$ in this regime. To plot $n(k)$ we used the interpolation of the numerical data given in Ref. [37] for the Hubbard model at low filling.

One can see a dramatic change in the shape of the momentum distribution function as the temperature increases from $T \ll T_0$ to $T \gg T_0$. The singularity of the momentum distribution at $k = k_F$ predicted by the Luttinger theorem [4, 48] disappears. This is a direct consequence of the exponential decay of $\rho(x)$ as $x \rightarrow \infty$, Eq. (4.2.29). Another very pronounced effect, seen in Fig. 4.1, is that the large-momentum tail of the momentum distribution gets strongly suppressed as the temperature increases. This is a counterintuitive result from the point of view of physics of weakly interacting systems, where the fraction of large-momentum particles grows monotonously with increasing temperature. In strongly correlated systems, however, bare particles have little overlap with elementary excitations of the system and the momentum distribution of bare particles is not directly related to the distribution of energy between the eigenmodes of the system.

4.2.4 Asymptotics of G_h and G_e is the “spin-disordered” regime

Concluding this chapter, we present the long time and distance asymptotics of the correlation functions (4.2.1) and (4.2.2) in the “spin-disordered” regime. The physics of these results is discussed in Ref. [11], and the detailed calculations are presented in Ref. [12].

One has in the limit $x, t \rightarrow \infty$ and $x/t = \text{const} \neq 0$:

$$G_h(x, t) \simeq \frac{\Xi e^{-2k_F x \ln 2/\pi} e^{i(2k_F x - \phi_+)}}{(x - 4k_F t)^{2\Delta} (x + 4k_F t)^{2\bar{\Delta}}} - \frac{\Xi e^{-2k_F x \ln 2/\pi} e^{-i(2k_F x - \phi_-)}}{(x + 4k_F t)^{2\Delta} (x - 4k_F t)^{2\bar{\Delta}}}. \quad (4.2.32)$$

Here Ξ is a constant, which is calculated explicitly in [12]. The phases ϕ_{\pm} are given by

$$\phi_{\pm} = -2\text{Im} \left[\ln \Gamma \left(\frac{i \ln 2}{2\pi} \right) \right] + \frac{\ln 2}{\pi} \ln [4k_F x \pm (4k_F)^2 t] \quad (4.2.33)$$

and the anomalous dimensions are given by

$$\Delta = \frac{1}{2} - \frac{1}{8} \left(\frac{\ln 2}{\pi} \right)^2, \quad \bar{\Delta} = -\frac{1}{8} \left(\frac{\ln 2}{\pi} \right)^2. \quad (4.2.34)$$

The Fermi momentum k_F is defined by Eq. (4.1.22). The result (4.2.32) is consistent with the previously calculated equal time correlation function [2]. The case $x = 0$ is special. In this case

$$G(0, t) = \frac{\Xi'}{\sqrt{t \ln(k_F^2 t)}} \quad (4.2.35)$$

where Ξ' is a constant. The asymptotics of $G_e(x, t)$ is easily related to the asymptotics of G_h : one should use the involution (4.2.5) neglecting the contribution from the anticommutator.

The exponentially decaying part of the correlation function (4.2.32) can be attributed to the non-propagating spin mode. The algebraically decaying part of the

correlation function only depends on combinations $x - 4k_F t$ and $x + 4k_F t$ which correspond to propagation with velocity $4k_F$. This is exactly the velocity of the charge propagation in the model. One can see that the behavior of G_h in the “spin-disordered” regime is strikingly different from that of in the Luttinger regime, Eqs. (1.5.9)–(1.5.13).

Note that the result (4.2.32) was reproduced (and generalized) by the authors of Ref. [15] using a combination of the functional integral and bosonization techniques.

Chapter 5

Calculations of Eq. (3.2.13)

5.1 q -Boson lattice model

In this section an integrable lattice regularization of the Lieb-Liniger model (2.1.1) is described. The lattice regularization is a natural tool to circumvent the short-range singularities discussed in section 3.2.3 and can be, in principle, done in many different ways. This number of ways diminishes when one requires for the lattice model remain integrable. Several different regularization schemes preserving integrability are discussed in Ref. [1]. We will use the so-called q -boson hopping model [6, 7] as a lattice regularization of the Lieb-Liniger model (2.1.1). This choice was motivated by the simplicity of this model (to be more precise, this is the simplest lattice integrable regularization of the Lieb-Liniger model among all known).

5.1.1 The q -boson algebra

Consider the operators B_n , B_n^\dagger and $N_n = N_n^\dagger$ satisfying the so-called q -boson algebra

$$B_n B_n^\dagger - q^{-2} B_n^\dagger B_n = 1 \quad (5.1.1)$$

$$[N_n, B_n] = -B_n \quad [N_n, B_n^\dagger] = B_n^\dagger \quad (5.1.2)$$

The index n is reserved to label the sites of a lattice; it will be “frozen” in this subsection: we will work with the operators on a given lattice site. The c -number q is taken to be

$$q = e^\gamma \tag{5.1.3}$$

and it is enough for our purposes to work with real $\gamma > 0$.

Define now the Fock space for the q -boson algebra (5.1.2). Let, by default, the local basis states $|m_j\rangle_j$ for this algebra be the states of the canonical lattice bosons b_n^\dagger, b_n :

$$[b_n, b_n^\dagger] = 1, \quad b_n^\dagger b_n = N_n. \tag{5.1.4}$$

that is, harmonic oscillator states:

$$b_n |m_n\rangle_n = m_n^{1/2} |m_n - 1\rangle_n, \quad b_n^\dagger |m_n\rangle_n = (m_n + 1)^{1/2} |m_n + 1\rangle_n. \tag{5.1.5}$$

The q -boson algebra (5.1.2) has the following representation in thus defined Fock space:

$$\begin{aligned} B_n |0\rangle_n &= 0 & N_n |0\rangle_n &= 0 \\ B_n |m_n\rangle_n &= [m_n]_q^{1/2} |m_n - 1\rangle_n & B_n^\dagger |m_n\rangle_n &= [m_n + 1]_q^{1/2} |m_n + 1\rangle_n \end{aligned} \tag{5.1.6}$$

where a notation is used

$$[x]_q = \frac{1 - q^{-2x}}{1 - q^{-2}}. \tag{5.1.7}$$

One has $[x]_q \rightarrow x$ as $q \rightarrow 1$: the q -bosons become canonical lattice bosons in this limit.

It should be stressed that the operator counting the number of the q -bosons is the operator N_n defined by Eq. (5.1.4), while the operator $B_n^\dagger B_n$ (which one would expect to be the number operator) have the spectrum

$$B_n^\dagger B_n |m_n - 1\rangle_n = [m_n]_q |m_n\rangle_n \tag{5.1.8}$$

which follows from the Eqs. (5.1.6) immediately. We thus have

$$B_n^\dagger B_n = [N_n]_q = \frac{1 - q^{-2N_n}}{1 - q^{-2}} \quad (5.1.9)$$

in accordance with Eq. (5.1.7). The q -boson operators B_n^\dagger, B_n can also be expressed via canonical lattice bosons:

$$B_n = (B_n^\dagger)^\dagger = \sqrt{\frac{[N_n + 1]_q}{N_n + 1}} b_n. \quad (5.1.10)$$

Note an alternative representation for the commutation relation (5.1.1):

$$[B_n, B_n^\dagger] = q^{-2N_n}. \quad (5.1.11)$$

Eqs. (5.1.10) and (5.1.11) can be easily checked using Eqs. (5.1.5) and (5.1.6).

5.1.2 Hamiltonian of the q -boson hopping model

To go from the quantum mechanical (one site) model considered in the previous subsection to a quantum field theory model introduce a lattice with M sites; let the index n in B_n, B_n^\dagger and N_n number the lattice sites. Let $B_{n_1}, B_{n_2}^\dagger$ and N_{n_3} commute at different lattice sites. The normalized states on the whole lattice are constructed as a tensor product of the local Fock states:

$$|0\rangle = \otimes_{j=1}^M |0\rangle_j, \quad |m\rangle = \otimes_{j=1}^M |m_j\rangle_j. \quad (5.1.12)$$

Consider the following lattice Hamiltonian

$$H_q = -\frac{1}{\delta^2} \sum_{n=1}^M (B_n^\dagger B_{n+1} + B_{n+1}^\dagger B_n - 2N_n) \quad (5.1.13)$$

with the periodic boundary conditions $M + 1 = 1$ imposed. Here δ is the lattice spacing. The factor δ^{-2} is taken to ensure the proper continuum limit. This Hamiltonian

commutes with the number operator

$$N = \sum_n N_n. \quad (5.1.14)$$

Note that H_q is highly nonlinear in terms of the fields B_n^\dagger, B_n since the operator N is highly nonlinear (it is obvious from Eq. (5.1.9)); H_q is nonlinear in terms of the canonical lattice bosons (5.1.4) too, now due to the nonlinearity of the hopping term. Thus the model is interacting and the interaction is encoded in the deformation parameter q : if one take $q \rightarrow 1$ the result will be quadratic boson Hamiltonian on a lattice, describing nearest-neighbor hoppings.

If one, however, take the limits $q \rightarrow 1$ and $\delta \rightarrow 0$ simultaneously, the result appears to be highly nontrivial. To be more precise, define the continuum limit of the model by

$$\delta \rightarrow 0, \quad M\delta = L, \quad \gamma = \frac{c\delta}{2}, \quad (5.1.15)$$

where γ is related to q by Eq. (5.1.3). The sums in the continuum limit are converted into the integrals in the usual way

$$\delta \sum_{n=1}^M \rightarrow \int_0^L dx. \quad (5.1.16)$$

Define the continuum boson fields $\Psi(x)$ and $\Psi^\dagger(x)$

$$\Psi(x) = \delta^{-1/2} b_n, \quad (5.1.17)$$

$$\Psi^\dagger(x) = \delta^{-1/2} b_n^\dagger \quad (5.1.18)$$

where $x = \delta n$. The fields $\Psi(x)$, $\Psi^\dagger(x)$ satisfy canonical commutation relations

$$[\Psi(x), \Psi^\dagger(x')] = \delta(x - x'). \quad (5.1.19)$$

In the continuum limit (5.1.15) the q -boson Hamiltonian (5.1.13) of the model becomes

$$H = \int_0^L dx [\partial_x \Psi^\dagger(x) \partial_x \Psi(x) + c \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x)] \quad (5.1.20)$$

which is the Hamiltonian of the Lieb-Liniger model, Eq. (2.1.1). Note that we do not change the chemical potential of the system: the term $-2N_n$ in Eq. (5.1.13) is absorbed by the kinetic and potential terms of the Hamiltonian (5.1.20) in the continuum limit.

Conclude this subsection with the discussion on the number and momentum operators in the q -boson model (5.1.13) and their continuum limit. We have defined the number operator by Eq. (5.1.14). In the continuum limit (5.1.15) the number operator becomes

$$N = \int_0^L dx \Psi^\dagger(x) \Psi(x). \quad (5.1.21)$$

The momentum operator is defined as follows:

$$P_q = -\frac{i}{2} \frac{1}{\delta} \sum_{n=1}^M (B_n^\dagger B_{n+1} - B_{n+1}^\dagger B_n). \quad (5.1.22)$$

In the continuum limit it becomes

$$P = -\frac{i}{2} \int_0^L \{ \Psi^\dagger(x) \partial_x \Psi(x) - [\partial_x \Psi^\dagger(x)] \Psi(x) \}. \quad (5.1.23)$$

5.1.3 Generating function for higher currents

We generate an infinite set of the integrals of motion in the q -boson model using the Quantum Inverse Scattering Method (QISM) in this subsection. A general description of the QISM is given, for example, in the book [33]. To treat the q -boson model this method was applied, in particular, in the paper [7], to which we refer the reader for a more extensive references.

Introduce the so-called L -operator for the q -boson model:

$$L_n(\lambda) = \begin{pmatrix} e^\lambda & \chi B_n^\dagger \\ \chi B_n & e^{-\lambda} \end{pmatrix}, \quad \chi = \sqrt{1 - q^{-2}}. \quad (5.1.24)$$

The L -operator (5.1.24) is a 2×2 matrix with the entries being quantum operators acting in the infinite dimensional Fock space defined by Eqs. (5.1.6). The QISM is based on the existence of the intertwining relation for the L -operator:

$$R(\lambda, \mu) (L_n(\lambda) \otimes L_n(\mu)) = (L_n(\mu) \otimes L_n(\lambda)) R(\lambda, \mu) \quad (5.1.25)$$

with the R -matrix defined by

$$R(\lambda, \mu) = \begin{pmatrix} f(\mu, \lambda) & 0 & 0 & 0 \\ 0 & g(\mu, \lambda) & q & 0 \\ 0 & q^{-1} & g(\mu, \lambda) & 0 \\ 0 & 0 & 0 & f(\mu, \lambda) \end{pmatrix}, \quad (5.1.26)$$

where

$$f(\lambda, \mu) = \frac{\sinh(\lambda - \mu + \gamma)}{\sinh(\lambda - \mu)}, \quad g(\lambda, \mu) = \frac{\sinh(\gamma)}{\sinh(\lambda - \mu)}. \quad (5.1.27)$$

Recall that the tensor product for 2×2 matrices a and b is defined as follows:

$$a \otimes b = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}. \quad (5.1.28)$$

The monodromy matrix is defined as a matrix product of the L -operators taken over all lattice sites

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = L_M(\lambda) \cdots L_1(\lambda). \quad (5.1.29)$$

The entries $A(\lambda), \dots, D(\lambda)$ of the monodromy matrix are quantum operators acting in the tensor product of the local Fock spaces over all sites of the lattice. Due to

relation (5.1.25) and the commutativity of the entries of the L -operator (5.1.24) at different lattice sites one has the intertwining relation for the monodromy matrix

$$R(\lambda, \mu) (T(\lambda) \otimes T(\mu)) = (T(\mu) \otimes T(\lambda)) R(\lambda, \mu). \quad (5.1.30)$$

Equation (5.1.30) defines 16 commutation relations for the operators entering the monodromy matrix. Write explicitly those which we will use in deriving the Bethe equations in the next subsection:

$$qA(\lambda)B(\mu) = f(\lambda, \mu)B(\mu)A(\lambda) + g(\mu, \lambda)B(\lambda)A(\mu), \quad (5.1.31)$$

$$qD(\lambda)B(\mu) = f(\mu, \lambda)B(\mu)D(\lambda) + g(\lambda, \mu)B(\lambda)D(\mu), \quad (5.1.32)$$

$$[B(\lambda), B(\mu)] = 0. \quad (5.1.33)$$

The transfer matrix is defined as a trace in the matrix space of the monodromy matrix

$$\tau(\lambda) = \text{Tr} T(\lambda) = A(\lambda) + D(\lambda). \quad (5.1.34)$$

It can be proven (see chapter VI of the book [33] for details) that for any λ and μ

$$[\tau(\lambda), \tau(\mu)] = 0 \quad (5.1.35)$$

which implies that τ is a generating function of the integrals of motion of the problem: expanding $\tau(\lambda)$ in λ one gets a set of commuting integrals of motion. This set can be chosen in many different ways since any analytic function of $\tau(\lambda)$ can play a role of the generating functional. We, however, impose an additional very restrictive locality condition on these integrals: we require they are written in the following form

$$I_m = \sum_{n=1}^M J_\tau^{(m)}(n), \quad (5.1.36)$$

where operators $J_\tau^{(m)}(n)$ act nontrivially in m neighboring lattice sites only. The subscript τ appears in $J_\tau^{(m)}(n)$ since these local operators are τ -components of the corresponding Noether currents, which will be discussed in the next Section. To get the set $\{I_m\}$ introduce the variable

$$\zeta = e^\lambda \quad (5.1.37)$$

and consider the expression

$$I_m = \frac{1}{(2m)!} \frac{d^{2m}}{d\zeta^{2m}} \ln [\zeta^M \tau(\zeta)] \Big|_{\zeta \rightarrow 0}, \quad m = 1, 2, 3, \dots \quad (5.1.38)$$

We assume that λ is real here and below, therefore ζ is real and nonnegative. The local operators $J_\tau^{(1)}(n)$, $J_\tau^{(2)}(n)$ and $J_\tau^{(3)}(n)$ are

$$J_\tau^{(1)}(n) = \chi^2 B_n^\dagger B_{n+1} \quad (5.1.39)$$

$$J_\tau^{(2)}(n) = \chi^2 \left(1 - \frac{\chi^2}{2}\right) \left(B_n^\dagger B_{n+2} - \frac{\chi^2}{2 - \chi^2} B_n^\dagger B_n^\dagger B_{n+1} B_{n+1} - \chi^2 B_n^\dagger B_{n+1}^\dagger B_{n+1} B_{n+2} \right) \quad (5.1.40)$$

and

$$\begin{aligned} J_\tau^{(3)}(n) = & \chi^2 \left(1 - \chi^2 + \frac{\chi^4}{3}\right) \{ B_n^\dagger B_{n+3} - \chi^2 B_n^\dagger B_n^\dagger B_{n+1} B_{n+2} - \chi^2 B_n^\dagger B_{n+1}^\dagger B_{n+1} B_{n+3} \\ & - \chi^2 B_n^\dagger B_{n+1}^\dagger B_{n+2} B_{n+2} - \chi^2 B_n^\dagger B_{n+2}^\dagger B_{n+2} B_{n+3} + \frac{\chi^4}{3 - 3\chi^2 + \chi^4} B_n^\dagger B_n^\dagger B_n^\dagger B_{n+1} B_{n+1} B_{n+1} \\ & + \chi^4 B_n^\dagger B_n^\dagger B_{n+1}^\dagger B_{n+1} B_{n+1} B_{n+2} + \chi^4 B_n^\dagger B_{n+1}^\dagger B_{n+1}^\dagger B_{n+1} B_{n+2} B_{n+2} \\ & + \chi^4 B_n^\dagger B_{n+1}^\dagger B_{n+2}^\dagger B_{n+1} B_{n+2} B_{n+3} \} \quad (5.1.41) \end{aligned}$$

respectively.

Note that the integrals I_m are not Hermitian. They contain both real and imaginary part. Using the involution

$$[\tau(\zeta)]^\dagger = \tau(\zeta^{-1}) \quad (5.1.42)$$

it is easily shown that

$$[I_m^\dagger, I_n] = 0 \quad \text{for any } m, n. \quad (5.1.43)$$

For the integrals of motion I_m^\dagger , use the following notation

$$I_{-m} \equiv I_m^\dagger, \quad m = 1, 2, 3, \dots \quad (5.1.44)$$

The Hamiltonian (5.1.13) can thus be written as

$$H_q = -\frac{1}{\chi^2 \delta^2} (I_1 + I_{-1} - 2\chi^2 N). \quad (5.1.45)$$

5.1.4 Thermodynamics of the q -boson model

The algebraic Bethe ansatz is a very powerful tool in calculating the thermodynamic properties of the integrable models. We calculate using algebraic Bethe ansatz the spectrum and various thermodynamic functions of the integrals of motion (5.1.38) in this section.

The cornerstone of the algebraic Bethe Ansatz method is the fact that the vacuum $|0\rangle$ defined by Eq. (5.1.12) annihilates the $C(\lambda)$ entry of the monodromy matrix (5.1.29) and is an eigenfunction for the $A(\lambda)$ and $D(\lambda)$ entries:

$$A(\lambda)|0\rangle = e^{M\lambda}|0\rangle, \quad D(\lambda)|0\rangle = e^{-M\lambda}|0\rangle. \quad (5.1.46)$$

Since N is a good quantum number, one can work in the N -particle sector. Define a set of states by the following formula

$$|\psi_N(\lambda_1, \dots, \lambda_N)\rangle = \prod_{j=1}^N B(\lambda_j)|0\rangle. \quad (5.1.47)$$

(these states are often called Bethe states). These states are eigenstates of the transfer matrix and hence of the Hamiltonian if the parameters $\lambda_1, \dots, \lambda_M$ satisfy a system

of coupled nonlinear equations (called Bethe equations):

$$e^{2M\lambda_j} = \prod_{\substack{k=1 \\ k \neq j}}^N \frac{f(\lambda_k, \lambda_j)}{f(\lambda_j, \lambda_k)}, \quad j = 1, \dots, N. \quad (5.1.48)$$

The eigenvalues θ_N of the transfer matrix on the Bethe states (5.1.47) are

$$\begin{aligned} \tau(\lambda)|\psi_N\rangle &= \theta_N(\lambda, \{\lambda_j\})|\psi_N\rangle, \\ q^N \theta_N(\lambda, \{\lambda_j\}) &= e^{M\lambda} \prod_{j=1}^N f(\lambda, \lambda_j) + e^{-M\lambda} \prod_{j=1}^N f(\lambda_j, \lambda). \end{aligned} \quad (5.1.49)$$

Using (5.1.49) one gets the spectrum of the operator (5.1.38)

$$I_m|\psi_N\rangle = (1 - q^{-2m}) \frac{1}{m} \sum_{j=1}^N e^{-2m\lambda_j} |\psi_N\rangle. \quad (5.1.50)$$

It will be convenient to use instead of $\lambda_1, \dots, \lambda_N$ a set of quasimomenta p_1, \dots, p_N :

$$\lambda_j = i \frac{p_j}{2}, \quad j = 1, \dots, N. \quad (5.1.51)$$

Written in these variables, Bethe equations (5.1.48) are

$$e^{iMp_j} = \prod_{\substack{k=1 \\ k \neq j}}^N \frac{\sin[\frac{1}{2}(p_j - p_k) + i\gamma]}{\sin[\frac{1}{2}(p_j - p_k) - i\gamma]}, \quad j = 1, \dots, N. \quad (5.1.52)$$

Using the representation (5.1.45) one gets for the spectrum of the Hamiltonian (5.1.13):

$$E_N = \frac{4}{\delta^2} \sum_{j=1}^N \sin^2 \frac{p_j}{2}. \quad (5.1.53)$$

In the continuum limit (5.1.15) one should use renormalized quasimomenta $k_j = \delta^{-1}p_j$ and for these quasimomenta one reproduces the spectrum and Bethe equations of the Lieb-Liniger Hamiltonian (5.1.20). All the solutions p_j of the Bethe equations (5.1.52) are real. The proof is the same as given in the book [33] (page 11, Theorem 1) for

the continuum case, and we refer the reader to this book for details. It follows from Eq. (5.1.53) that the one-particle dispersion is a periodic function of the quasimomenta p_j with the period 2π . We will work with the quasimomenta lying in the interval

$$-\pi < p_j < \pi, \quad j = 1, \dots, N. \quad (5.1.54)$$

The condition (5.1.54) will be assumed in all subsequent formulas.

Write Bethe equations (5.1.52) in the logarithmic form

$$Mp_j + \sum_{k=1}^N \theta(p_j - p_k) = 2\pi \left(n_j + \frac{N-1}{2} \right), \quad j = 1, \dots, N. \quad (5.1.55)$$

Parameters n_j take arbitrary integer values. From now on we will work with odd N .

The function $\theta(p)$ is

$$\theta(p) = i \ln \frac{\sin(i\gamma + \frac{1}{2}p)}{\sin(i\gamma - \frac{1}{2}p)}, \quad \theta(-p) = -\theta(p). \quad (5.1.56)$$

The derivative of $\theta(p)$ is positive,

$$\theta'(p) = \frac{i \sin(2i\gamma)}{2 \sin(i\gamma + \frac{1}{2}p) \sin(i\gamma - \frac{1}{2}p)} = \frac{\sinh(2\gamma)}{\cosh(2\gamma) - \cos p} > 0, \quad (5.1.57)$$

therefore $\theta(p)$ grows monotonously on the interval $-\pi < p < \pi$ (recall that the condition (5.1.54) is assumed). The property (5.1.57) is crucial for finding the ground state of the model: Write, using Eq. (5.1.55),

$$M(p_j - p_s) + \sum_{k=1}^N [\theta(p_j - p_k) - \theta(p_s - p_k)] = 2\pi(n_j - n_s), \quad j, k = 1, \dots, N. \quad (5.1.58)$$

Since $\theta'(p) > 0$, Eq. (5.1.57), the left hand side of the Eq. (5.1.58) is a monotonously growing function of the parameter $p_j - p_k$. The following condition follows immediately: If $n_j > n_s$ then $p_j > p_s$; if $n_j = n_s$ then $p_j = p_s$. We thus shown that the set of quasimomenta $\{p_j\}$ is uniquely characterized by the set $\{n_j\}$, and vice versa.

Consider the sector with the fixed number of particles, N . It can be shown [?] that the energy functional (5.1.53) has the minimum in this sector if n_j take the values

$$n_j = -N + j, \quad j = 1, \dots, N. \quad (5.1.59)$$

The Bethe equations (5.1.55) for the ground state are, therefore,

$$Mp_j + \sum_{k=1}^N \theta(p_j - p_k) = 2\pi \left(j - 1 - \frac{N-1}{2} \right), \quad j = 1, \dots, N. \quad (5.1.60)$$

Having Eqs. (5.1.60) we are ready to study the ground state properties of the model in the thermodynamic limit. By “thermodynamic limit” we mean the limit

$$N \rightarrow \infty, \quad M \rightarrow \infty, \quad \frac{N}{M} = \text{const} \neq 0, \infty. \quad (5.1.61)$$

The thermodynamic limit should not be confused with the continuum limit, Eq. (5.1.15).

The quasimomenta p_j scales inversely proportional to M in the thermodynamic limit: $p_j - p_k \sim M^{-1}$ (see the book [33] for the detailed analysis) and fill the symmetric interval $-\Lambda \leq p_j \leq \Lambda$. The parameter Λ thus plays a role of the Fermi momentum. The density of the quasiparticles in the momentum space is given by the function

$$\rho(p_k) = \frac{1}{M(p_{k+1} - p_k)}. \quad (5.1.62)$$

The occupation number N/M is related in the thermodynamic limit to ρ and Λ as follows

$$\frac{N}{M} = \int_{-\Lambda}^{\Lambda} dp \rho(p). \quad (5.1.63)$$

The Bethe equations on the ground state quasimomentum distribution (5.1.60) transform in the thermodynamic limit to a linear integral equation called Lieb equation

$$\rho(p) - \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dk \theta'(p-k) \rho(k) = \frac{1}{2\pi}, \quad (5.1.64)$$

where the kernel $\theta'(p - k)$ is given by Eq. (5.1.57).

Having defined $\rho(p)$ and Λ one can easily get all thermodynamic quantities in the ground state. The energy Eq. (5.1.53) of the system in the ground state is given by

$$E_N = \frac{4}{\delta^2} \int_{-\Lambda}^{\Lambda} dp \rho(p) \sin^2 \frac{p}{2}. \quad (5.1.65)$$

5.1.5 The Hellmann-Feynman identity

In this subsection we use the Hellmann-Feynman theorem to derive an identity for the average of local fields B_n, B_n^\dagger .

The eigenstates (5.1.47) of the q -boson Hamiltonian are the eigenstates for the integrals of motion Eq. (5.1.38) and their Hermitian conjugated Eq. (5.1.44). Take the ground state wave function in an N -particle sector and write the Hellmann-Feynman theorem for the integral I_1 :

$$\frac{d}{dq} \langle I_1 \rangle = \left\langle \frac{d}{dq} I_1 \right\rangle. \quad (5.1.66)$$

Here q is the deformation parameter introduced in Eq. (5.1.1). Note that the Hellmann-Feynman theorem cannot be written in the form (5.1.66) for an arbitrary non-Hermitian operator; In getting (5.1.66) we have used the commutativity of I_1 with its Hermitian conjugated, Eq. (5.1.43). The explicit expression for I_1 via local fields B_n, B_n^\dagger is given by Eqs. (5.1.36) and (5.1.39). Using the translational invariance of the ground state, implying that

$$\langle B_{j_1}^\dagger \dots B_{j_m} \rangle = \langle B_{j_1+k}^\dagger \dots B_{j_m+k} \rangle \quad (5.1.67)$$

equation (5.1.66) is rewritten in terms of local operators:

$$\frac{d}{dq} \langle \chi^{-2} I_1 \rangle = M \left\langle \frac{d}{dq} (B_j^\dagger B_{j+1}) \right\rangle. \quad (5.1.68)$$

The dependence of the operators B_j and B_j^\dagger on the parameter q can be analyzed using the representation (5.1.10). One gets, in particular, from this representation

$$\frac{d}{dq}B_j = [N_j + 1]_q^{-1/2} \frac{d}{dq}[N_j + 1]_q^{1/2} B_j, \quad (5.1.69)$$

$$\frac{d}{dq}B_j^\dagger = B_j^\dagger [N_j + 1]_q^{-1/2} \frac{d}{dq}[N_j + 1]_q^{1/2}. \quad (5.1.70)$$

Introduce the notation

$$g(q, x) = [x + 1]_q^{-1/2} \frac{d}{dq}[x + 1]_q^{1/2} = \frac{1}{q} \left(\frac{x + 1}{q^{2x+2} - 1} - \frac{1}{q^2 - 1} \right). \quad (5.1.71)$$

Then combining Eqs. (5.1.68), (5.1.70) and (5.1.69) one finds

$$\frac{d}{dq}(B_j^\dagger B_{j+1}) = B_j^\dagger g(q, N_j) B_{j+1} + B_j^\dagger g(q, N_{j+1}) B_{j+1}. \quad (5.1.72)$$

5.1.6 Noether currents in q -boson model

The q -Boson model is completely integrable and contains an infinite hierarchy of Noether currents. In this section we describe the currents needed for our purposes.

The lattice version of the continuity equation for a conserved current (J_τ, J_x) is

$$\frac{\partial}{\partial \tau} J_\tau(n, \tau) + \frac{1}{\delta} [J_x(n, \tau) - J_x(n - 1, \tau)] = 0 \quad (5.1.73)$$

where

$$\frac{\partial}{\partial \tau} J = [H, J] \quad (5.1.74)$$

The U(1) current

The Hamiltonian (5.1.75) commutes with the total number of particles

$$N = \sum_{n=1}^M N_n \quad (5.1.75)$$

This operator generates global $U(1)$ rotation of the B_n fields

$$e^{-i\phi N} B_n e^{i\phi N} = B_n e^{i\phi} \quad (5.1.76)$$

which leaves the Hamiltonian invariant. The corresponding local current is given by

$$J_\tau^e = \frac{1}{\delta} N_n \quad (5.1.77)$$

$$J_x^e = \frac{1}{\delta^2} (B_{n+1}^\dagger B_n - B_n^\dagger B_{n+1}) \quad (5.1.78)$$

We shall extensively use this operator for bosonization.

The local gauge transformations

The charge density operator (5.1.77) generates local gauge transformations. Consider an infinitesimal gauge transformation generated by a unitary operator

$$Q_\epsilon = \exp \left[i\epsilon \delta \sum_{n=1}^M \eta(n) J_\tau^e(n) \right] \quad (5.1.79)$$

Here ϵ is the infinitesimal parameter, η is some arbitrary function. The operator (5.1.79) induces the following transformation of the lattice q-Boson fields

$$Q_\epsilon^{-1} B_n Q_\epsilon = e^{i\epsilon\eta(n)} B_n \quad Q_\epsilon^{-1} B_n^\dagger Q_\epsilon = e^{-i\epsilon\eta(n)} B_n^\dagger \quad (5.1.80)$$

Using this equation one can easily see that for the integral of motion densities we have

$$Q_\epsilon^{-1} J_\tau^{(m)}(n) Q_\epsilon = e^{i\epsilon\eta(n+m) - i\epsilon\eta(n)} J_\tau^{(m)}(n) \quad (5.1.81)$$

In the differential form this equation is written as

$$\frac{d}{d\epsilon} j_\tau^{(m)}(n) = \frac{d}{d\epsilon} J_\tau^{(m)}(n) = i [\eta(n+m) - \eta(n)] J_\tau^{(m)}(n) \quad (5.1.82)$$

Similar equation can be written for the components of the electric current (5.1.77) and (5.1.78):

$$\frac{d}{d\epsilon} j_\tau^e(n) = 0 \quad (5.1.83)$$

$$\frac{d}{d\epsilon} j_x^e(n) = -\frac{i}{\delta^2 \chi^2} \{ [\eta(n+1) - \eta(n)] J_\tau^{(1)}(n) - [\eta(n-1) - \eta(n)] J_\tau^{(-1)}(n) \} \quad (5.1.84)$$

5.2 Bosonization of the model.

The model (5.1.20) is integrable which implies that there is an infinite number of conserved currents J^ν :

$$\partial^\nu J_\nu^{(m)} = 0. \quad (5.2.1)$$

Conformal limit is defined with respect to some fixed vacuum $|0\rangle$. Introduce the "normal ordered" currents

$$j_\mu^{(m)} = J_\mu^{(m)} - \langle 0 | J_\mu^{(m)} | 0 \rangle \quad (5.2.2)$$

whose vacuum expectation value vanishes. The currents (5.2.2) can be expanded in local fields of the effective theory (1.1.4).

Consider first the conserved $U(1)$ current. To fix the normalization of the effective action chose

$$j_\tau^e = \frac{1}{\pi} \partial_x \phi + \text{h.o.t} \quad (5.2.3)$$

$$j_x^e = -\frac{1}{\pi} \partial_\tau \phi + \text{h.o.t}. \quad (5.2.4)$$

Generally, other conserved currents can be expanded in local operators of the effective theory as follows

$$j_\tau^{(m)} = \frac{\alpha_m}{\pi} \partial_x \phi - \frac{\beta_m}{\pi} \partial_\tau \phi + \text{h.o.t}. \quad (5.2.5)$$

and

$$j_x^{(m)} = -\frac{\alpha_m}{\pi} \partial_\tau \phi - \frac{\beta_m v^2}{\pi} \partial_x \phi + \text{h.o.t.} \quad (5.2.6)$$

where h.o.t. stands for the subleading terms (terms of dimension higher than unity) and α_m, β_m are some constants (possibly zero). Note, that conservation of the current $j^{(m)}$ is ensured by the equation of motion for the field ϕ :

$$\partial_\tau^2 \phi + v^2 \partial_x^2 \phi = 0. \quad (5.2.7)$$

Consider an operator

$$W(x, \tau) = \int_{-\infty}^x dx' j_\tau^{(m)}(x', \tau) + \text{h.o.t.} \quad (5.2.8)$$

where h.o.t. stands for operators of dimension higher than 0. Then the following identity involving the current (5.2.2) is true

$$\int_\gamma dx_\mu \epsilon^{\mu\nu} \langle T j_\nu^{(n)}(x, \tau) W(0, 0) \rangle = -\frac{Kv}{\pi} (\alpha_n \beta_m + \alpha_m \beta_n) \quad (5.2.9)$$

where the contour goes counterclockwise around the origin. The proof of this identity is simple. By virtue of equation (5.2.1) and the Stokes theorem the integral in equation (5.2.9) is contour-independent. Make the contour a circle of infinitely large radius, then, the contribution of h.o.t. to the correlation function will vanish and only the leading term, which is of the order of radius will stay. Replace the correlation function by the asymptotic expression obtained from the leading terms in Eqs. (5.2.5) and (5.2.6), use the expression

$$G(x, \tau) = \langle T \phi(x, \tau) \phi(0, 0) \rangle = -\frac{K}{4} \ln(x^2 + v^2 \tau^2) \quad (5.2.10)$$

and take all the integrals. The answer will be given by the right hand side of (5.2.9).

5.3 Bosonization of the Noether currents

In this subsection we investigate the leading terms in the conformal limit of the Noether currents. First, write the conformal expansion for the equations (5.1.82) and (5.1.84) assuming that η is a slowly varying function (formally it means that $\eta(n) = \tilde{\eta}(n\delta/L)$, where $\tilde{\eta}$ is fixed and L is arbitrarily large):

$$\frac{d}{d\epsilon} j_\tau^{(m)}(x) = im\delta \eta'(x) \langle J_\tau^{(m)} \rangle + \text{h.o.t.} \quad (5.3.1)$$

and

$$\frac{d}{d\epsilon} j_\tau^e(x) = -\frac{2i}{\delta^2 \chi^2} \delta \eta'(x) \langle J_\tau^{(1)} \rangle + \text{h.o.t.} \quad (5.3.2)$$

Here we used the even parity of the ground state, which implies $\langle J_\tau^{(-m)} \rangle = \langle J_\tau^{(m)} \rangle$.

Now rewrite equation (5.2.5) in terms of the normal ordered electric current:

$$j_\tau^{(m)} = \alpha_m j_\tau^e + \beta_m j_x^e + \text{h.o.t.} \quad (5.3.3)$$

From this representation it is evident, that the coefficients α_m and β_m can be found by investigating the response of the system to large scale (smooth) variations of charge density and the electric current. In particular, from the homogenous density variation in the whole chain one finds

$$\alpha_m = \frac{d}{d\rho} \langle J_\tau^{(m)} \rangle \quad (5.3.4)$$

where

$$\rho = \langle J_\tau^e \rangle = \frac{N}{\delta M} \quad (5.3.5)$$

is the vacuum expectation value of the density of particles on the lattice. To find β_m use eqs (5.3.1), (5.3.2) and (5.1.83):

$$\beta_m = -\frac{\delta^2 \chi^2 m}{2} \frac{\langle J_\tau^{(m)} \rangle}{\langle J_\tau^{(1)} \rangle} \quad (5.3.6)$$

The relations (5.3.4) and (5.3.6) can be rewritten in terms of the integrals of motion:

$$\alpha_m = \frac{d}{d\rho} \frac{I_m}{M} \quad (5.3.7)$$

$$\beta_m = -\frac{\delta^2 \chi^2}{2} \frac{m I_m}{I_1} \quad (5.3.8)$$

Next, we use the local gauge transformations to calculate the quantity Kv . The leading term in the conformal expansion of the electric current is given in eqs. (5.2.3), (5.2.4). From the kinetic term in the action (1.1.4) one can see that

$$\Pi = \frac{1}{i\pi Kv} \partial_\tau \phi. \quad \text{check the sign!} \quad (5.3.9)$$

Comparing this expression with Eq. (5.2.4) we find

$$j_x^e(x) = iKv\Pi(x) \quad (5.3.10)$$

This gives the following commutation relation

$$[j_x^e(x), j_\tau^e(y)] = \frac{iKv}{\pi} \frac{d}{dy} [\Pi(x), \phi(y)] = -\frac{Kv}{\pi} \delta'(x-y). \quad (5.3.11)$$

On the other hand, the boost operator (5.1.79) is written in the conformal limit as

$$Q_\epsilon = \exp \left(i\epsilon \int dx \eta(x) j_\tau^e(x) \right) \quad (5.3.12)$$

Using the definitions (5.2.3) and (5.2.4) and the canonical commutation relation (1.1.6) one finds

$$\frac{d}{d\epsilon} j_x^e(x) = i \int dy \eta(y) [j_x^e(x), j_\tau^e(y)] = -\frac{iKv}{\pi} \eta'(x) \quad (5.3.13)$$

Comparing (5.3.13) with equation (5.3.2) we find

$$Kv = \frac{2\pi}{\delta\chi^2} \frac{I_1}{M}. \quad (5.3.14)$$

5.4 The String Identities.

In this section we derive a general identity exploiting both integrability and the conformal field theory description of the model. For the simplicity of presentation we demonstrate the idea of the derivation on an example of a continuous model.

Integrability implies that there exists an infinite number of conserved Noether currents (5.2.2) satisfying

$$\partial^\mu j_\mu^{(m)} = 0 \quad (5.4.1)$$

Take any local operator $\mathcal{O}(x, \tau)$, then equation (5.4.1) implies that by virtue of the Stokes theorem

$$\int_\gamma dx_\mu \epsilon^{\mu\nu} \langle T j_\nu^{(n)}(x, \tau) \mathcal{O}(x', \tau') \rangle = 0 \quad (5.4.2)$$

for any contractible contour γ (that is the contour which can be contracted without crossing the singularity point (x', τ')) In equation (5.4.2) $\epsilon_{\mu\nu}$ is the antisymmetric tensor and T is the imaginary time-ordering operator. Interesting identities can be derived if one considers the string operator

$$W(x, \tau) = \int_{-\infty}^x dx' j_\tau^{(n)}(x', \tau') \quad (5.4.3)$$

For $W(0, 0)$ the integral in (5.4.3) runs along the x axis as shown in figure. The contractible contour γ in this case can be chosen as shown in the same figure. Splitting the integration along γ in the part running along the real axis and the circle we obtain the formula

$$\oint_{|x|=R} dx_\mu \epsilon^{\mu\nu} \langle T j_\nu^{(m)}(x, \tau) W(0, 0) \rangle + \int_{-R}^{\infty} dx \langle [j_\tau^{(m)}(x, 0), W(0, 0)] \rangle = 0 \quad (5.4.4)$$

where the radius R of the circle in the first integral can be made arbitrarily big. Due to the choice of the string operator (5.4.3) it is possible to evaluate the integral along

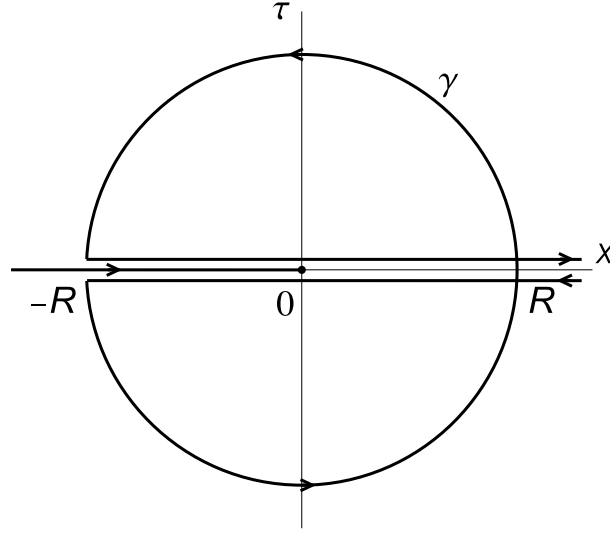


Figure 5.1: Integration contour

the big circle using the conformal field theory. Indeed, write

$$W(x, \tau) = \tilde{W}(x, \tau) + I_m - \langle I_m \rangle, \quad (5.4.5)$$

where

$$\tilde{W}(x, \tau) = - \int_x^\infty dx' j_\tau^{(n)}(x', \tau') \quad (5.4.6)$$

Then split the first integral in (5.4.4) as follows

$$\begin{aligned} \oint_{|x|=R} dx_\mu \epsilon^{\mu\nu} \langle T j_\nu^{(m)}(x, \tau) W(0, 0) \rangle &= \int_{|x|=R, x>0} dx_\mu \epsilon^{\mu\nu} \langle T j_\nu^{(m)}(x, \tau) W(0, 0) \rangle + \\ &\int_{|x|=R, x<0} dx_\mu \epsilon^{\mu\nu} \langle T j_\nu^{(m)}(x, \tau) \tilde{W}(0, 0) \rangle + \int_{|x|=R, x<0} dx_\mu \epsilon^{\mu\nu} \langle T j_\nu^{(m)}(x, \tau) (I_m - \langle I_m \rangle) \rangle \end{aligned} \quad (5.4.7)$$

The last integral vanishes because the vacuum is an eigenstate of I_m . In the remaining integrals the distance between local operators in the average is always bigger than R and therefore the conformal asymptotics can be used! Assuming that the current j^μ

are bosonized as described in eqs. (5.2.5) and (5.2.6) we find with the help of Eq.

(5.2.9)

$$\int_{-R}^{\infty} dx \int_{-\infty}^0 dx' \langle [j_{\tau}^{(n)}(x), j_{\tau}^{(m)}(x')] \rangle = \frac{Kv}{\pi} (\alpha_m \beta_n + \alpha_n \beta_m) \quad (5.4.8)$$

The same calculation can be performed with lattice regularization resulting in

$$\delta^2 \sum_{k=-L}^{\infty} \sum_{p=-\infty}^0 \langle [j_{\tau}^{(n)}(k), j_{\tau}^{(m)}(p)] \rangle = \frac{Kv}{\pi} (\alpha_m \beta_n + \alpha_n \beta_m) \quad (5.4.9)$$

where L is an arbitrary number.

Now, consider the following string operator

$$W(n, \tau) = \sum_{m=-\infty}^n j_{\tau}^{(2)}(m) \quad (5.4.10)$$

Then it is straightforward to see that

$$\frac{1}{\chi^2} \left(1 - \frac{\chi^2}{2}\right)^{-1} \sum_{n=-L}^{\infty} \sum_{m=-\infty}^0 \langle [j_{\tau}^{(-1)}(n), j_{\tau}^{(2)}(m)] \rangle = \langle -\chi^2 B_j^{\dagger} B_{j+1} + \chi^4 B_j^{\dagger} B_j^{\dagger} B_j B_{j+1} \rangle \quad (5.4.11)$$

On the other hand, by comparison with (5.4.9) we find

$$\langle -\chi^2 B_j^{\dagger} B_{j+1} + \chi^4 B_j^{\dagger} B_j^{\dagger} B_j B_{j+1} \rangle = \frac{1}{\chi^2} \left(1 - \frac{\chi^2}{2}\right)^{-1} \frac{Kv}{\pi \delta^2} (\alpha_{-1} \beta_2 + \alpha_2 \beta_{-1}) \quad (5.4.12)$$

Equation (5.4.12) is one of the many string identities, relating expectation values of

local operators and the thermodynamic information encoded in eqs (5.3.7) (5.3.8). It

is convenient to combine (5.4.12) with (5.1.39). This gives

$$\langle B_j^{\dagger} B_j^{\dagger} B_j B_{j+1} \rangle = \frac{1}{M \chi^4} I_1 + \frac{1}{\chi^6} \left(1 - \frac{\chi^2}{2}\right)^{-1} \frac{Kv}{\pi \delta^2} (\alpha_{-1} \beta_2 + \alpha_2 \beta_{-1}) \quad (5.4.13)$$

Substituting in this expression the results (5.3.7), (5.3.8) and (5.3.14) one finds

$$\langle B_j^{\dagger} B_j^{\dagger} B_j B_{j+1} \rangle = \frac{I_1}{M \chi^4} \left[1 - \frac{2}{(2 - \chi^2) \chi^2 \delta} \left(\frac{2I_2}{I_1} \frac{d}{d\rho} \frac{I_1}{M} - \frac{d}{d\rho} \frac{I_2}{M} \right) \right] \quad (5.4.14)$$

5.5 Thermodynamic Bethe Ansatz close to the continuum limit

As discussed earlier the continuum limit is defined as

$$\delta \rightarrow 0, \quad \frac{N}{\delta M} \rightarrow n, \quad \gamma = \frac{c\delta}{2} \quad (5.5.1)$$

where n is the density of particles in the 1d bose gas. The equations of TBA close to this limit have the form

$$\rho(p) - \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dk K(p-k)\rho(k) = \frac{1}{2\pi} \quad (5.5.2)$$

$$n = \int_{-\Lambda}^{\Lambda} dk \rho(k) \quad (5.5.3)$$

where

$$K(p) = \frac{2c}{c^2 + p^2} + \frac{c\delta^2}{6} + \frac{1}{360}(3cp^2 - c^3)\delta^4 + \dots \quad (5.5.4)$$

For the purposes of this paper we shall only need the δ^2 order in equation (5.5.4).

The integrals of motion can also be expanded as follows:

$$\begin{aligned} \frac{I_1}{M} &= \chi^2 \delta \left(1 - \frac{\delta^2}{2!} h_2 + \frac{\delta^4}{4!} h_4 + \dots \right) \\ \frac{I_2}{M} &= \chi^2 \left(1 - \frac{\chi^2}{2} \right) \delta \left(1 - \frac{(2\delta)^2}{2!} h_2 + \frac{(2\delta)^4}{4!} h_4 + \dots \right), \end{aligned} \quad (5.5.5)$$

where

$$h_m = \int_{-\Lambda}^{\Lambda} dk \rho(k) k^m \quad (5.5.6)$$

The functions h_m depend on the small parameter δ through the quasimomentum distribution ρ and through Λ . These functions can be expanded in series

$$h_m = h_m^{(0)} + \delta^2 h_m^{(1)} + \dots \quad (5.5.7)$$

First, consider the solution of the TBA for $\delta = 0$. In this case on dimensional grounds we find

$$h_m^{(0)}(n, c) = n^{m+1} \epsilon_m \left(\frac{c}{n} \right) \quad (5.5.8)$$

where $\epsilon_m(x)$ is some parameter-free function. This can be rewritten as the differential relation

$$n \frac{\partial}{\partial n} h_2^{(0)} + c \frac{\partial}{\partial c} h_2^{(0)} - 3h_2^{(0)} = 0 \quad (5.5.9)$$

Next, consider the solution of the TBA equations to the second order in δ . Substituting (5.5.4) in (5.5.2) and keeping only the second order term in δ we find

$$\rho(p) - \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dk \frac{2c}{c^2 - (p-k)^2} \rho(k) = \frac{1}{2\pi} \left(1 + \frac{c\delta^2 n}{6} \right) \quad (5.5.10)$$

By rescaling the quasimomentum distribution function

$$\rho(k) = \left(1 + \frac{c\delta^2 n}{6} \right) \tilde{\rho}(k) \quad (5.5.11)$$

We find that $\tilde{\rho}$ satisfies the Lieb Liniger equation with the renormalized density constraint

$$n \left(1 - \frac{c\delta^2 n}{6} \right) = \int_{-\Lambda}^{\Lambda} dk \tilde{\rho}(k) \quad (5.5.12)$$

At the same time the function h_2 is given by

$$h_2(n, c) = \left(1 + \frac{c\delta^2 n}{6} \right) \int_{-\Lambda}^{\Lambda} dk k^2 \tilde{\rho}(k) = \left(1 + \frac{c\delta^2 n}{6} \right) h_2^{(0)} \left(n - \frac{c\delta^2 n^2}{6}, c \right) \quad (5.5.13)$$

Expanding this equation and keeping the first subleading term we find

$$h_2 = h_2^{(0)} + \frac{c\delta^2 n}{6} \left(h_2^{(0)} - n \frac{\partial}{\partial n} h_2^{(0)} \right) \quad (5.5.14)$$

To finish this section derive the correction to the homogeneity condition (5.5.9):

$$n \frac{\partial}{\partial n} h_2 + c \frac{\partial}{\partial c} h_2 - 3h_2 = \frac{c\delta^2 n}{3} \left(h_2 - n \frac{\partial}{\partial n} h_2 \right) + o(\delta^2) \quad (5.5.15)$$

5.6 Three body recombination rate

Consider the continuum limit in the following average:

$$\left\langle (1 - \chi^2) B_j^\dagger B_j^\dagger B_j B_{j+1} + \left(1 - \frac{\chi^2}{2}\right) q \frac{d}{dq} B_j^\dagger B_{j+1} \right\rangle = -\frac{c\delta^4}{3} \langle \bar{\psi}^3(x) \psi^3(x) \rangle + o(\delta^4), \quad (5.6.1)$$

where we used eq. (5.1.72). Combining this with eqs. (5.4.14) and (5.1.68) one finds

$$\begin{aligned} & \frac{c\delta^4}{3} \langle \bar{\psi}^3(x) \psi^3(x) \rangle = \\ & - \left\{ (1 - \chi^2) \frac{I_1}{M\chi^4} \left[1 - \frac{2}{(2 - \chi^2)\chi^2\delta} \left(\frac{2I_2}{I_1} \frac{d}{d\rho} \frac{I_1}{M} - \frac{d}{d\rho} \frac{I_2}{M} \right) \right] + \left(1 - \frac{\chi^2}{2}\right) q \frac{d}{dq} \frac{I_1}{\chi^2 M} \right\} \end{aligned} \quad (5.6.2)$$

Using the expansion (5.5.5) we find

$$\begin{aligned} & \frac{c\delta^4}{3} \langle \bar{\psi}^3(x) \psi^3(x) \rangle = \left(\frac{\delta^2}{c} - \frac{\delta^3}{2} \right) \left(n \frac{\partial}{\partial n} h_2 + c \frac{\partial}{\partial c} h_2 - 3h_2 \right) \\ & + \delta^4 \left\{ \frac{c}{12} \left(n \frac{\partial}{\partial n} h_2 + 3c \frac{\partial}{\partial c} h_2 - 3h_2 \right) - \frac{1}{12c} \left(7n \frac{\partial}{\partial n} h_4 + c \frac{\partial}{\partial c} h_4 - 15h_4 \right) + \frac{h_2}{c} \frac{\partial}{\partial n} h_2 \right\} \end{aligned} \quad (5.6.3)$$

using equation (5.5.15) and replacing the functions h_m by their Lieb-Liniger limit we find

$$\langle \bar{\psi}^3(x) \psi^3(x) \rangle = n^3 g \left(\frac{c}{n} \right) \quad (5.6.4)$$

where

$$g(x) = \frac{3}{2x} \frac{d\epsilon_4}{dx} - \frac{5\epsilon_4}{x^2} + \left(1 + \frac{x}{2}\right) \frac{d\epsilon_2}{dx} - \frac{2\epsilon_2}{x} - \frac{3\epsilon_2}{x} \frac{d\epsilon_2}{dx} + \frac{9\epsilon_2^2}{x^2} \quad (5.6.5)$$

Thus, we got the desired expression (3.2.13).

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