Quantum gravity, effective fields and string theory
Abstract

In this thesis we will look into some of the various aspects of treating general relativity as a quantum theory. The thesis falls in three parts. First we briefly study how general relativity can be consistently quantized as an effective field theory, and we focus on the concrete results of such a treatment. As a key achievement of the investigations we present our calculations of the long-range low-energy leading quantum corrections to both the Schwarzschild and Kerr metrics. The leading quantum corrections to the pure gravitational potential between two sources are also calculated, both in the mixed theory of scalar QED and quantum gravity and in the pure gravitational theory. Another part of the thesis deals with the (Kawai-Lewellen-Tye) string theory gauge/gravity relations. Both theories are treated as effective field theories, and we investigate if the KLT operator mapping is extendable to the case of higher derivative operators. The constraints, imposed by the KLT-mapping on the effective coupling constants, are also investigated. The KLT relations are generalized, taking the effective field theory viewpoint, and it is noticed that some remarkable tree-level amplitude relations exist between the field theory operators. Finally we look at effective quantum gravity treated from the perspective of taking the limit of infinitely many spatial dimensions. The results are here mostly phenomenological, but have some practical and theoretical implications as it is verified that only a certain class of planar graphs will in fact contribute to the $n$-point functions at $D = \infty$. This limit is somewhat an analogy to the large-$N$ limit of gauge theories although the interpretation of such a graph limit in a gravitational framework is quite different. We will end the thesis with a summary of what have been achieved and look into the perspectives of further investigations of quantum gravity.
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Chapter 1

Introduction

Everywhere in Nature gravitational phenomena are present and daily experiences teach us to recognize, appreciate and live with the basic effects and constraints of gravity. A life completely without earthly gravity would indeed be very different!

Newton realized that tides, falling apples, and the motion of the planets all originate from the same force of Nature – the universal gravitational attraction. Everyone knows that things are supposed to fall if you drop them. The great invention of Newton was to put the experimental facts of Nature into a mathematical formalism which predicts both the motion of a falling apple and the orbit of the Moon. The fundamental equation is his gravitational law, also known as the Newtonian potential, which predicts the potential energy of the gravitational attraction between two bodies:

$$V(r) = -\frac{G m_1 m_2}{r}$$

(1.1)

here \(V(r)\) is a measure for the potential energy, \((m_1)\) and \((m_2)\) are the masses of the two particles, \((r)\) is the distance between the masses and \((G)\) is the gravitational constant. The gravitational potential energy thus is predicted by Newton to be proportional to masses and inverse proportional to distances. Newton’s law together with his mechanic principles (ref. [1]) form the foundation of theoretical physics and create the basis for understanding and predicting theoretical effects of gravity.

Einstein realized that the gravitational attraction is in fact really proportional to the total energy of matter and not only to rest masses. At relativistic velocities this can be observed – as an extra gravitational attraction caused by the kinetic energy of particles. Thus relativistic particles such as light rays are in fact also sources of gravity. The
first confirmation of Einstein’s proposal was through the direct observation of light rays bending around the Sun.

![Figure 1.2: Illustration of light rays bending around the Sun.](image)

The theory of general relativity (ref. [2]) provides a framework for extending Newton’s theory to objects with no rest mass and relativistic velocities. In general relativity one solves the basic field equation, the Einstein equation:

\[ R_{\mu\nu}(g_{\mu\nu}) - \frac{1}{2}R(g_{\mu\nu})g_{\mu\nu} = 16\pi G T_{\mu\nu} - \lambda g_{\mu\nu} \]  

(1.2)

where \((g_{\mu\nu})\) is the gravitational metric, \((R^\alpha_{\beta\mu\nu})\) is a measure for the curvature of space\(^1\) and \((T_{\mu\nu})\) is the total energy-momentum tensor for the sources of gravity. The cosmological constant \((\lambda)\) is needed on cosmological scales, and is today believed to have a non-zero expectation value in the Universe. Whenever we have a concrete gravitational problem (a specified energy distribution, \(i.e.,\) a mathematical expression for the energy-momentum tensor) we solve the Einstein equation for the metric field. The metric will be a local object and depend on the geometry of space. In this way a solution of the gravitational problem is found. General relativity relates the geometry of space and gravity. Einstein’s description holds in the fully relativistic regime, and its low-energy and nonrelativistic predictions fit the expectations of Newtonian mechanics.

The mathematical theories for gravitational phenomena have been very successful and calculations of orbits for planets in the solar system fit experimental data with great precision. Classical gravitational models for the Universe are being investigated and observations so far support the theories up to the precision of the measurements. It must be concluded that general relativity is a very successful theory for the classical gravitational phenomena of the Universe.

Quantum mechanics and quantum field theory are the great other fundamental physical theories. They essentially explain to us how to interpret physics at the atomic and subatomic scales. Quantum mechanics provide the theoretical basis for most solid state physics. Without a proper knowledge of quantum mechanics we could in fact neither build semi-conductor chips nor lasers. Most chemistry and atomic physics rely heavily on quantum physics. Quantum mechanics do not only give models for atoms, but also for nuclear physics and for the interactions of the elementary particles in the Standard

\(^1\)\((R_{\mu\nu} = R^\alpha_{\mu\nu\beta}), (R \equiv g^{\mu\nu} R_{\mu\nu})\)
Model. This is the theory uniting the electro-weak and strong interactions of elementary particles in a
\[
U(1) \times SU(2) \times SU(3)
\] gauge theory. It is believed to form the basis of particle physics and to be valid until energies around \(\sim 1 \text{ TeV}\).

The crucial distinctions between classical and quantum theories, \textit{e.g.}, the replacement of \(c\)-numbers by operators etc, rule out the possibility that the Universe fundamentally could have a combined classical and quantum mechanical description. As strong experimental evidence (\textit{e.g.}, ref. [4]) are in favor of a quantum description of Nature at the fundamental Planckian scale, \textit{i.e.}, \((E_{\text{Planck}} \sim 10^{19} \text{ GeV})\), the logical consequence is that an elementary theory for gravity will have to be a quantum theory.

So the problem can be stated in the following way. As general relativity is a classical theory – it cannot be the whole story of gravity and there thus must exist a fundamental theory that in the limit of classical energy scales provides us with the classical expectations.

![Figure 1.3: Feynman's double slit experiment, see ref. [3]. An experiment to illustrate intuitively why the Universe fundamentally must be of a classical or of a quantum nature, \textit{i.e.}, a combined description is not possible. If gravitational waves from the quantum particle in the experiment are \textit{classical}, there is something wrong from a \textit{quantum} mechanical perspective, because when we in principle could detect which slit the quantum particle went. This would disagree with the principles of quantum mechanics. On the other hand if the wave-detector senses gravitational waves, with a \textit{quantum} nature, \textit{i.e.}, wave amplitudes, surely gravity self must be a \textit{quantum} theory for the experiment to be consistent! Thus a fundamental theory for gravity has to be a \textit{quantum} theory, if we assume that all other theories of Nature are \textit{quantum} theories.](image)

It is not obvious in any way to understand general relativity as a quantum theory – and in fact general relativity and quantum mechanics seem to be based on completely different perceptions of physics – nevertheless this question is one of the most pressing questions of modern theoretical physics and has been the subject of many authors, \textit{e.g.}, see refs. De Witt [5], Feynman [3, 6], Gupta et al [7], Faddeev and Popov [8], Mandelstam [9], Schwinger [10, 11], Weinberg [12], Iwasaki [13], van Dam [14, 15], 't Hooft and Veltman [16, 17], Zakharov [18], Capper et al [19, 20, 21], Duff [22, 23, 24], Deser [25, 26, 27], Berends and Gastmans [28] and Goroff and Sagnotti [29], just to mention a few. All sorts of interpretational complications arise promoting general relativity to be a quantum theory. The adequate starting point for such a theory appears to be to interpret general relativity as a quantum field theory, to let the metric be the basic gravitational field, and to quantize
the Einstein-Hilbert action:

\[ \mathcal{L}_{EH} = \int d^4x \sqrt{-g} \frac{R}{16\pi G} \]  

(1.3)

where \((g = \det(g_{\mu\nu}))\) and \((R)\) is the scalar curvature. However the above action is not self contained as loop diagrams will generate new terms not present in the original action, \(i.e.,\) the action cannot be renormalized in a traditional way, see refs. [16, 17, 25, 26, 27, 29, 30]. This is the renormalization problem of general relativity.

We will here investigate quantum gravity from the point of view of treating it as an effective field theory below the energies of the Planck scale, see refs. [31, 32, 33]. As we will observe as an effective field theory there are no problems with the renormalization of the action. For a recent review of general relativity as an effective field theory, see ref. [34]. The thesis work can be divided into three parts. First it was investigated how to make quantum predictions for observables by treating gravity as an effective field theory. This part of the thesis is covered by the research papers, see refs. [35, 37, 38]. These papers explain the effective field theory treatment of gravity and show that below the Planck scale, quantum and post-Newtonian corrections to the Schwarzschild and Kerr metrics and to scattering matrix potentials can be calculated. The thesis work covered by the research papers, see refs. [39, 40] is next discussed. The discussion is the Kawai-Lewellen-Tye string relations, which is treated from the viewpoint of quantum gravity and Yang-Mills theory as effective field theories. Generalized gauge-gravity relations for tree amplitudes are derived from this investigation, and it is observed that the KLT-mapping is extendable in the framework of effective field theories. Finally we look at the paper ref. [41]. Here we take up the old idea by Strominger, see ref. [42], which discusses quantum gravity as a theory in the limit of an infinite number of dimensions. We extend this idea to quantum gravity seen as an effective field theory.

In the thesis we will everywhere work in harmonic gauge, use units \((c = \hbar = 1)\) and metric \((+1, -1, -1, -1)\) when anything else is not explicitly stated.
Chapter 2

Quantum gravity as an effective field theory

2.1 Introduction

We will in this section consider general relativity as an effective quantum field theory. In the concept of effective field theories, see ref. [31], the couplings which should be included in a particular Lagrangian are perturbatively determined by the energy scale. It is traditional to require strict renormalization conditions in a quantum field theory, however this has no meaning for effective field theories. Every term consistent with the underlying symmetries of the fields of the theory has to be included in the action, and by construction any effective field theory is hence trivially renormalizable and has an infinite number of field couplings. The various terms of the action correspond to different energy scales for the theory in a perturbative manner. At each loop order only a finite number of terms of the action need to be accounted for. In non-effective field theories the Lagrangian is believed to be fundamental and valid at every energy scale – from this viewpoint the effective action is less fundamental, i.e., at sufficiently high energies it has to be replaced by some other theory, perhaps a string theory description etc. However at sufficiently low energies the effective Lagrangian presents both an interesting path to avoid the traditional renormalization problems of non-renormalizable theories and a way to carry out explicit field theory calculations.

It is a known fact that quantum field theories of pure general relativity, as well as quantum theories for general relativity including scalar, see refs. [16, 17, 29, 30], fermion or photon fields, see refs. [25, 26, 27], suffer from severe problems with renormalizability in the traditional meaning of the word.

This apparent obstacle can be avoided if we treat general relativity as an effective field theory. We consider then the gravitational action as consisting not solely of the Einstein curvature term plus the minimal couplings of the matter terms, but as an action consisting of all higher derivative terms which are generally covariant. As an effective field theory, general relativity thus can be dealt with as any other quantum field theory up to the limiting scale of the Planck energy $\sim 10^{19}$ GeV.
Actions for general relativity where a finite number of additional derivative couplings have been allowed for are discussed in the literature, see refs. [43, 44, 45, 46], and various issues concerning general relativity as a classical or a quantum theory with higher derivative terms have been dealt with. Chiral perturbation theories have many uses for effective action approaches, see ref. [47]. Treating general relativity as an effective field theory in order to calculate leading 1-loop pure gravitational corrections to the Newtonian potential was first done in refs. [32, 33]. This calculation has since been followed by the various calculations, see refs. [48, 49, 50, 51, 52]. In ref. [53] effective action techniques were used to calculate the quantum corrections to the Reissner-Nordström and Kerr-Newman metrics. Here we will present the results of the three thesis papers, see refs. [35, 37, 38] which sort out the numerical inconsistencies of previous calculations. The first ref. [35] considers the combined theory of general relativity and scalar QED. In this paper the quantum and post-Newtonian corrections to the mixed scattering matrix potential of these two theories are calculated.\(^1\) In the other two papers refs. [37, 38] the quantum and post-Newtonian corrections to the Schwarzschild and Kerr metric as well as to the Newtonian potential are calculated correctly for the first time.

The background field method first introduced in ref. [5] will be employed here.

The results will be discussed in relation to, see refs. [13, 53, 54, 55, 56, 57]. The used vertex rules are presented in appendix A.1.

### 2.2 General relativity as an effective field theory

In this section we will consider in a more detailed manner how to treat general relativity as an effective field theory.

The Einstein action for general relativity has the form:

\[
L = \sqrt{-g} \left[ \frac{2R}{\kappa^2} + L_{\text{matter}} \right],
\]

where \((\kappa^2 = 32\pi G)\) is defined as the gravitational coupling, the curvature tensor is defined as \((R^\mu_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\sigma\alpha} \Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta} \Gamma^\sigma_{\nu\alpha})\) and the determinant of the metric field \((\det(g_{\mu\nu}))\) is denoted as \((g)\). The term \((\sqrt{-g}L_{\text{matter}})\) is a covariant expression for the inclusion of matter into the theory. We can include any type of matter. As a classical theory the above Lagrangian defines the theory of general relativity.

Any effective field theory can be seen as an expansion in energies of the light fields of the theory below a certain scale. Above the scale transition energy there will be additional heavy fields that will manifest themselves. Below the transition the heavy degrees of freedom will be integrated out and will hence not contribute to the physics. Any effective field theory is built up from terms with higher and higher numbers of derivative couplings on the light fields and obeying the gauge symmetries of the basic theory. This gives us a precise description of how to construct effective Lagrangians from the gauge invariants of the theory. We expand the effective Lagrangian in the invariants ordered in magnitude of their derivative contributions. Derivatives of light fields \((\tilde{\partial})\) will essentially go as

\(^1\)In ref. [36] we will consider the corresponding mixed calculation with QED and general relativity.
2.2. GENERAL RELATIVITY AS AN EFFECTIVE FIELD THEORY

powers of momentum, while derivatives of massive fields (\(\partial\)) will generate powers of the interacting masses. As the interacting masses usually are orders of magnitude higher than the momentum terms — the derivatives on the massive fields will normally generate the leading contributions.

An effective treatment of general relativity results in the following Lagrangian:

\[
\mathcal{L} = \sqrt{-g} \left[ \frac{2R}{\kappa^2} + c_1 R^2 + c_2 R^{\mu\nu} R_{\mu\nu} + \ldots \right],
\]

(2.2)

the ellipses in the equation express that we are dealing with an infinite series — additional higher derivative terms have to be included at each loop order. The coefficients \((c_1, c_2, \ldots)\) represent the appearance of the effective couplings in the action. Through cosmology we can bound the effective coupling constants, we should have \((c_1, c_2 > 10^{70})\) in order to generate measurable deviations from the predictions of traditional general relativity, e.g., see ref. [45].

As an example of how to construct an effective Lagrangian for a theory we can look at the case of QED with charged scalars and photons. A general covariant version of the scalar QED Lagrangian is:

\[
\mathcal{L} = \sqrt{-g}\left[ -\frac{1}{4} \left( g^{\alpha\mu} g^{\beta\nu} F_{\alpha\nu} F_{\mu\beta} \right) + (D_\mu \phi + ieA_\mu)^* (D_\nu \phi + ieA_\nu) - m^2 |\phi|^2 \right],
\]

(2.3)

where \((F_{\mu\nu} \equiv D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu)\), and \((D_\mu)\) denotes the covariant derivative with respect to the gravitational field, \((g_{\mu\nu})\). As \((\phi)\) is a scalar \((D_\mu \phi = \partial_\mu \phi)\).

Counting the number of derivatives in each term of the above Lagrangian we see that the term with \((g^{\alpha\mu} g^{\beta\nu} F_{\alpha\nu} F_{\mu\beta})\) goes as \((\sim \partial^2 \partial^2)\), while the scalar field terms go as \((\sim \partial \partial)\) and \((\sim 1)\) respectively. Thus seen in the light of effective field theory, the above Lagrangian represents the minimal derivative couplings of the gravitational fields to the photon and complex scalar fields.

Typical 1-loop field singularities for the mixed graviton and photon fields in the minimal theory are known to take the form ref. [25]:

\[
\sqrt{-g} T^2_{\mu\nu}, \quad \sqrt{-g} R_{\mu\nu} T^{\mu\nu},
\]

(2.4)

where:

\[
T_{\mu\nu} = F_{\mu\alpha} F^{\alpha}_\nu - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta},
\]

is the Maxwell stress tensor, and where:

\[
R^{\mu\nu}_{\alpha\beta} \equiv \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\nu \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\sigma\alpha} \Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta} \Gamma^\sigma_{\nu\alpha},
\]

is the Einstein curvature tensor. Examples of similar 1-loop divergences for the mixed graviton and scalar fields are:

\[
\sqrt{-g} R^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi, \quad \sqrt{-g} R |\partial_\mu \phi|^2, \quad \sqrt{-g} R m^2 |\phi|^2.
\]

(2.5)

The two photon contributions are seen to go as \(\sim \partial \partial \partial \partial \partial \partial \partial \), while the scalar contributions go as \(\sim \partial \partial \partial \partial \partial \partial \) and \(\partial \partial \partial \partial \partial \partial \) respectively. So clearly the 1-loop singularities correspond to higher derivative couplings of the fields.
Of course there will also be examples of mixed terms with both photon, graviton and complex scalar fields. We will not consider any of these terms explicitly.

As we calculate the 1-loop diagrams using the minimal theory, singular terms with higher derivative couplings of the fields will unavoidably appear. None of these singularities need to be worried about explicitly, because the combined theory is seen as an effective field theory.

To make the combined theory an effective field theory we thus need to include into the minimal derivative coupled Lagrangian a piece as:

\[ \mathcal{L}_{\text{photon}} = \sqrt{-g} \left[ c_1 T_{\mu\nu}^2 + c_2 R_{\mu\nu} T^{\mu\nu} + \ldots \right], \]  
\[ \text{(2.6)} \]

for the photon field, and:

\[ \mathcal{L}_{\text{scalar}} = \sqrt{-g} \left[ d_1 R^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi + d_2 R |\partial_\mu \phi|^2 + d_3 R m^2 |\phi|^2 + \ldots \right], \]  
\[ \text{(2.7)} \]

for the scalar field. The ellipses symbolize other higher derivative couplings at 1-loop order which are not included in the above equations, e.g., other higher derivative couplings and mixed contributions with both photon, graviton and scalar couplings.

The coefficients \((c_1, c_2, d_1, d_2, d_3, \ldots)\) in the above equations are in the effective theory seen as energy scale dependent couplings constants to be measured experimentally. All singular field terms from the lowest order Lagrangian are thus absorbed into the effective action, leaving us with a theory at 1-loop order, with a finite number of coupling coefficients to be determined by experiment.

The effective combined theory of scalar QED and general relativity is thus in some sense a traditional renormalizable theory at 1-loop order. At low energies the theory is determined only by the minimal derivative coupled Lagrangian, however at very high energies, higher derivative terms will manifest themselves in measurable effects, and the unknown coefficients \((c_1, c_2, \ldots), (d_1, d_2, \text{and } d_3, \ldots)\) will have to be determined explicitly by experiment. This process of absorbing generated singular field terms into the effective action will of course have to continue at every loop order.

### 2.2.1 Analytic and non-analytic contributions

Computing the leading long-range and low-energy quantum corrections of an effective field theory, a useful distinction is between non-analytical and analytical contributions from the diagrams. Non-analytical contributions are inherently non-local effects not expandable in a power series in momentum. Such effects come from the propagation of massless particle modes, e.g., gravitons and photons. The difference between massive and massless particle modes originates from the impossibility of expanding a massless propagator \(\left(\sim \frac{1}{q^2}\right)\) while we have the well known:

\[ \frac{1}{q^2 - m^2} = - \frac{1}{m^2} \left( 1 + \frac{q^2}{m^2} + \ldots \right), \]  
\[ \text{(2.8)} \]

expansion for the massive propagator. No \(\left(\sim \frac{1}{q^2}\right)\)-terms are generated in the above expansion of the massive propagator, thus such terms all arise from the propagation of massless modes.
As will be explicitly seen, to leading order the non-analytic contributions are governed only by the minimally coupled Lagrangian.

The analytical contributions from the diagrams are local effects and thus expandable in power series.

Non-analytical effects are typically originating from terms which in the S-matrix go as, e.g., \((\sim \ln(-q^2))\) or \((\sim \frac{1}{\sqrt{-q^2}})\), while the generic example of an analytical contribution is a power series in momentum \((q)\). Our interest is only in the non-local effects, thus we will only consider the non-analytical contributions of the diagrams.

The high-energy renormalization of the effective field theory thus does not concern us — we focus only on finding the low-energy leading non-analytical momentum terms from the 1-loop diagrams. Singular analytical momentum effects, which are to be absorbed into the coefficients of the higher derivative couplings of the effective theory, are of no particular interest to us as they have no low-energy manifestation.

## 2.2.2 The quantization of general relativity

The procedure of the background field quantization is as follows. We define the metric as the sum of a background part \((\bar{g}_{\mu\nu})\) and a quantum contribution \((\kappa h_{\mu\nu})\) where \((\kappa)\) is defined from \((\kappa^2 = 32G\pi)\).

The expansion of \((g_{\mu\nu})\) is done as follows, we define:

\[
g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}. \tag{2.9}\]

From this equation we get the expansions for the upper metric field \((g^{\mu\nu})\) (it is defined to be the inverse matrix), and for \((\sqrt{-g})\) (where \((\det(g_{\mu\nu}) = g)\)):

\[
g^{\mu\nu} = \bar{g}^{\mu\nu} - \kappa h^{\mu\nu} + \frac{1}{2} h_{\alpha} h^{\alpha\beta} + \ldots \]

\[
\sqrt{-g} = \sqrt{-\bar{g}} \left[ 1 + \frac{1}{2} \kappa h - \frac{1}{4} h_{\alpha} h^{\beta} + \frac{1}{8} h^2 + \ldots \right], \tag{2.10}\]

in the above equation we have made the definitions:

\[
h_{\mu\nu} \equiv \bar{g}^{\mu\alpha} \bar{g}^{\nu\beta} h_{\alpha\beta} \tag{2.11}\]

and

\[
h \equiv \bar{g}^{\mu\nu} h_{\mu\nu}. \tag{2.12}\]

In the quantization, the Lagrangian is expanded in the gravitational fields, separated in quantum and background parts, and the vertex factors as well as the propagator are derived from the expanded action. Details of these calculations can be found in the literature, for a detailed derivation, see ref. [51].

The matter Lagrangian is also expanded. In the scalar QED case we expand the action in the scalar field \((\phi)\) in the vector field \((A_\mu)\) as well as in the gravitational field \((h_{\mu\nu})\). The expanded result for the photon parts reads:

\[
\mathcal{L} = -\frac{1}{4} \kappa h \left( \partial_\mu A_\alpha \partial^\mu A^\alpha - \partial_\mu A_\alpha \partial^\alpha A^\mu \right) + \frac{1}{2} \kappa h^{\mu\nu} \left( \partial_\mu A_\alpha \partial_\nu A^\alpha + \partial_\alpha A_\mu \partial^\alpha A_\nu - \partial_\alpha A_\mu \partial_\nu A^\alpha - \partial_\alpha A_\nu \partial_\mu A^\alpha \right), \tag{2.13}\]
while the complex scalar part can be quoted as:

\[
\mathcal{L} = \frac{1}{2} \kappa h \left( |\partial_\mu \phi|^2 - m^2 |\phi|^2 \right) - \kappa h^{\mu \nu} (\partial_\mu \phi^* \partial_\nu \phi) + (i e A_\mu \partial^\mu \phi^* \phi - i e A_\mu \phi^* \partial^\mu \phi) + e^2 A_\mu A^\mu |\phi|^2 \\
+ \frac{1}{2} \kappa h \partial_\mu \phi^* (i e A^\mu) \phi - \frac{1}{2} \kappa h (i e A^\mu) \phi^* \partial_\mu \phi - \kappa h^{\mu \nu} \partial_\mu \phi^* (i e A_\nu) \phi + \kappa h^{\mu \nu} (i e A_\mu) \phi^* \partial_\nu \phi.
\]  

(2.14)

One can immediately find the vertex rules for the lowest order interaction vertices of photons, complex scalars and gravitons from the above equations.

Another necessary issue to consider is the gravitational coupling of fermions, an area which is not very investigated in the literature. See ref. [17, 26] for some useful considerations and calculations.

To couple a fermion to a gravitational field an additional formalism is needed. A coupling through a metric is not possible, see e.g. ref. [59, 58], so a new object the vierbein thus has to be introduced. Resembling the matrix square root of the metric, it allows the Dirac equation in curved space-time to be written consistently in the following way:

\[
\sqrt{e} \mathcal{L}_m = \sqrt{e} \bar{\psi} (i \gamma^\mu e_\mu D_\mu - m) \psi,
\]

(2.15)

here \((e_\mu^a)\) is the vierbein field which connects the global curved space-time with a space-time which is locally flat. The vierbein is connected to the metric through the following set of relations:

\[
\begin{align*}
    e^a_\mu e^b_\nu \eta_{ab} &= g_{\mu \nu}, \\
    e^a_\mu e_{a \nu} &= g_{\mu \nu}, \\
    e^a_\mu e_{b \mu} &= \delta^a_b, \\
    e^a_\mu e^a_\nu &= g^{\mu \nu},
\end{align*}
\]

(2.16)

and it is thus indeed seen to be the matrix square root of the metric.

We also have to define a covariant derivative which acts on fermion fields:

\[
D_\mu \psi = \partial_\mu \psi + i \frac{1}{4} \sigma^{ab} \omega_{\mu ab},
\]

(2.17)

Insisting that (a torsion-free scenario):

\[
e^a_\mu \omega_{\mu \nu} = \partial_\nu e^a_\mu - \Gamma^\sigma_{\mu \nu} e^a_\sigma + \omega^a_\nu e^b_\mu = 0,
\]

(2.18)

one can relate \((\omega_{\mu ab})\) to the vierbein field and derive that:

\[
\omega_{\mu ab} = \frac{1}{2} e^a_\nu (\partial_\mu e_{b \nu} - \partial_\nu e_{b \mu}) - \frac{1}{2} e^b_\nu (\partial_\mu e_{a \nu} - \partial_\nu e_{a \mu}) + \frac{1}{2} e^a_\sigma e^b_\nu (\partial_\mu e_{c \sigma} - \partial_\sigma e_{c \mu}) e^c_\mu.
\]

(2.19)

In order to quantize the vierbein fields we make the expansion:

\[
e^a_\mu = \delta^a_\mu + c^{(1)a}_\mu + c^{(2)a}_\mu + \ldots,
\]

(2.20)

the superscript indices in the above equation correspond to the number of powers of the gravitational coupling \((\kappa)\). This notation will useful later, dealing with the corrections to the metric.
The inverse vierbein matrix is:
\[
e_a^\mu = \delta_a^\mu - c_a^{(1)\mu} - c_b^{(2)\mu} + c_b^{(1)\mu} e^{(1)b} + \ldots,
\]
so that the metric becomes:
\[
g_{\mu\nu} = \eta_{\mu\nu} + c_a^{(1)\mu} + c_a^{(1)\nu} + c_a^{(1)\alpha} c_a^{(1)\mu} + \ldots,
\]
\[
g^{\mu\nu} = \eta^{\mu\nu} - c_a^{(1)\mu} - c_a^{(1)\nu} - c_a^{(1)\alpha} c_a^{(1)\nu} + c_a^{(1)\mu} c_a^{(1)\nu} + \ldots,
\]
in terms of the vierbein fields. Only the symmetric part of the \(c\)-field is necessary to keep in expansions and derivations of the vertex rules, this is because the non-analytic (long-range) effects only arise from such contributions. Antisymmetric field components will not generate non-analytic effects and are associated with the freedom of transforming among local Lorentz frames.

To first order in \((\kappa)\) the metric is connected to the vierbein fields in the following way:
\[
c_a^{(1)\mu} \rightarrow \frac{1}{2}(c_a^{(1)\mu} + c_a^{(1)\nu}) = \frac{1}{2} h_a^{(1)\mu},
\]
and we can rewrite all vierbein equations in terms of the metric fields:
\[
det e = 1 + c + \frac{1}{2} c^2 - \frac{1}{2} c_a^b c_a^a + \ldots
\]
\[
= 1 + \frac{1}{2} h + \frac{1}{8} h^2 - \frac{1}{8} h_b^b h_a^a + \ldots,
\]
and so forth for the other equations.

Making all expansions we arrive at:
\[
\sqrt{e} \mathcal{L}_m^{(0)} = \bar{\psi} \left( \frac{i}{2} \gamma^\alpha \delta_\mu^\alpha \partial_\mu^{LR} - m \right) \psi,
\]
\[
\sqrt{e} \mathcal{L}_m^{(1)} = -\frac{1}{2} h^{(1)\alpha\beta} \bar{\psi} i \gamma_\alpha \partial_\beta^{LR} \psi - \frac{1}{2} h^{(1)} \bar{\psi} \left( \frac{i}{2} \gamma \partial_\alpha^{LR} - m \right) \psi,
\]
\[
\sqrt{e} \mathcal{L}_m^{(2)} = -\frac{1}{2} h^{(2)\alpha\beta} \bar{\psi} i \gamma_\alpha \partial_\beta^{LR} \psi - \frac{1}{2} h^{(2)} \bar{\psi} \left( \frac{i}{2} \gamma \partial_\alpha^{LR} - m \right) \psi
\]
\[
- \frac{1}{8} h^{(1)\alpha\beta} h^{(1)\gamma} \bar{\psi} i \gamma_\gamma \partial_\alpha^{LR} \psi + \frac{1}{16} (h^{(1)})^2 \bar{\psi} i \gamma \partial_\gamma^{LR} \psi
\]
\[
- \frac{1}{8} h^{(1)\alpha} h^{(1)\beta} \bar{\psi} i \gamma_\alpha \partial_\beta^{LR} \psi + \frac{3}{16} h^{(1)\alpha} h^{(1)\beta} \bar{\psi} i \gamma \partial_\alpha^{LR} \psi
\]
\[
+ \frac{1}{4} h^{(1)\alpha\beta} h^{(1)\gamma} \bar{\psi} m \psi - \frac{1}{8} (h^{(1)})^2 \bar{\psi} m \psi
\]
\[
+ \frac{i}{16} h^{(1)\alpha} \partial_\beta h^{(1)\nu} \partial_\alpha h^{(1)\mu} \epsilon^{\alpha\beta\delta\epsilon} \bar{\psi} \gamma_\epsilon \gamma_5 \psi,
\]
where
\[
\bar{\psi} \partial_\alpha^{LR} \psi \equiv \bar{\psi} \partial_\alpha \psi - (\partial_\alpha \bar{\psi}) \psi.
\]
as the the expanded Dirac Lagrangian.

From these equations vertex forms for fermions can be derived. A summary of the vertex rules needed for our survey of the scattering amplitudes and the metric is presented in appendix A.1.
2.3 Quantum Corrections to the Schwarzschild and Kerr Metrics

In this section we will discuss the results of the paper ref. [37], which was devoted to a study of the quantum and post-Newtonian corrections to the Schwarzschild and Kerr metrics. Looking at the 1-loop lowest order radiative corrections to the interactions of massive particles, it was shown that long-range corrections to the Schwarzschild and Kerr metrics was possible to obtain, through the calculation of the non-analytic gravitational radiative corrections for a massive source. The results for the Kerr metric was obtained using a massive fermion as the source, while the Schwarzschild result was found using a massive scalar source.

The classical result for the Schwarzschild metric is refs. [60, 61]:

\[
\begin{align*}
  g_{00} &= \left(1 - \frac{2Gm}{r}\right) = 1 - 2\frac{Gm}{r} + 2\frac{G^2m^2}{r^2} + \ldots, \\
  g_{0i} &= 0, \\
  g_{ij} &= -\delta_{ij}\left(1 + \frac{Gm}{r}\right)^2 = \frac{G^2m^2}{r^2} \left(1 + \frac{Gm}{r} \frac{r_i r_j}{r^2}\right) \\
  &= -\delta_{ij} \left(1 + 2\frac{Gm}{r} + \frac{G^2m^2}{r^2}\right) - \frac{r_i r_j G^2m^2}{r^2} + \ldots, \quad (2.27)
\end{align*}
\]

while the corresponding result for the Kerr metric is ref. [62]:

\[
\begin{align*}
  g_{00} &= \left(1 - \frac{2Gm}{r}\right) + \ldots = 1 - 2\frac{Gm}{r} + 2\frac{G^2m^2}{r^2} + \ldots, \\
  g_{0i} &= \frac{2G}{r^2(r + mG)}(\vec{S} \times \vec{r})_i + \ldots = \left(\frac{2G}{r^3} - \frac{2G^2m}{r^4}\right) (\vec{S} \times \vec{r})_i + \ldots, \\
  g_{ij} &= -\delta_{ij}\left(1 + \frac{Gm}{r}\right)^2 - \frac{G^2m^2}{r^2} \left(1 + \frac{Gm}{r} \frac{r_i r_j}{r^2}\right) + \ldots \\
  &= -\delta_{ij} \left(1 + 2\frac{Gm}{r} + \frac{G^2m^2}{r^2}\right) - \frac{r_i r_j G^2m^2}{r^2} + \ldots \quad (2.28)
\end{align*}
\]

We note that in both the above two expressions for the metrics we have kept only terms linear in momentum.

One of the key result of this paper was the reproduction of the expressions for the classical parts of the metric using Feynman diagrams. Together with these classical post-Newtonian corrections we produce some additional corrections not present in general relativity. These are interpreted as the long-distance quantum corrections to the metric.
The results for the metric including these corrections are in the Schwarzschild case:

\[ g_{00} = 1 - 2 \frac{G m}{r} + 2 \frac{G^2 m^2}{r^2} + 62 \frac{G^2 m \hbar}{15 \pi r^3} + \ldots, \]
\[ g_{0i} = 0, \]
\[ g_{ij} = -\delta_{ij} \left( 1 + 2 \frac{G m}{r} + \frac{G^2 m^2}{r^2} + \frac{14G^2 m \hbar}{15 \pi r^3} - \frac{76G^2 m \hbar (1 - \log r)}{15 \pi r^3} \delta_{ij} \right) \]
\[ - \frac{r_i r_j}{r^2} \left( \frac{G^2 m^2}{r^2} + \frac{76G^2 m \hbar}{15 \pi r^3} + \frac{76G^2 m \hbar (1 - \log r)}{5 \pi r^3} \right) + \ldots, \]  

(2.29)

while for the Kerr metric:

\[ g_{00} = 1 - 2 \frac{G m}{r} + 2 \frac{G^2 m^2}{r^2} + 62 \frac{G^2 m \hbar}{15 \pi r^3} + \ldots, \]
\[ g_{0i} = \left( \frac{2G}{r^3} - \frac{2G^2 m}{r^4} + \frac{36G^2 \hbar}{15 \pi r^5} \right) (\vec{S} \times \vec{r})_i + \ldots, \]
\[ g_{ij} = -\delta_{ij} \left( 1 + 2 \frac{G m}{r} + \frac{G^2 m^2}{r^2} + \frac{14G^2 m \hbar}{15 \pi r^3} - \frac{76G^2 m \hbar (1 - \log r)}{15 \pi r^3} \delta_{ij} \right) \]
\[ - \frac{r_i r_j}{r^2} \left( \frac{G^2 m^2}{r^2} + \frac{76G^2 m \hbar}{15 \pi r^3} + \frac{76G^2 m \hbar (1 - \log r)}{5 \pi r^3} \right) + \ldots. \]  

(2.30)

The quantum corrections in the above expressions carry an explicit factor of \((\hbar)\) put in by dimensional analysis. The spin independent classical parts are the same for both the Kerr and the Schwarzschild metrics. This is as expected from general relativity. That also the quantum contributions satisfy this property of universality of fermions and bosons is another interesting achievement of our calculations.

### 2.4 How to derive a metric from a static source

The metric for a static source can be extracted using the Einstein equation. The metric will be a function of the energy-momentum tensor, that will have to be derived for the particular source. Next we solve for the metric order by order in the gravitational coupling \((\kappa)\).

To lowest approximation a source is just a point particle, and depending on the type of matter, with or without a spin. To higher orders of approximation the picture of a point particle is no longer correct, because there will be a surrounding gravitational field which effects have to be included in the calculations, e.g., the gravitational self-coupling. In order to add such effects into the expression for the energy-momentum tensor it has to be corrected so that it includes the energy-momentum carried by the gravitational fields.

The masslessness of the propagating graviton assures us that there will be long-ranged fields around the source, and that the gravitational fields will dependent on the transverse momentum, i.e., the exchanged momentum between sources.

Calculating the 1-loop radiative corrections to the source, non-analytic terms like \(\sim \sqrt{-q^2}\) and \(\sim q^2 \ln -q^2\), together with analytic terms like \(q^2, q^4, ...\) are generated. Here \(q\) defines the transferred momentum. As we previously have discussed, the analytic terms have
a short-ranged nature, while the long-ranged terms always will be non-analytic. The term \( \sim \sqrt{-q^2} \) will carry the long-ranged post-Newtonian effects, while the term \( (q^2 \ln -q^2) \) will give rise to a quantum contribution and be proportional to \( \hbar \).

Including the 1-loop radiative corrections to the energy-momentum tensor we can solve the equation of motion for the post-Newtonian and quantum corrections to the metric. The metric will take the generic form:

\[
\text{metric} \sim Gm \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} \left[ 1 - aGq^2 \sqrt{\frac{m^2}{q^2}} - bGq^2 \log(q^2) - cGq^2 + \ldots \right] \\
\sim Gm \left[ \frac{1}{r} + \frac{aGm}{r^2} + \frac{bG\hbar}{r^3} + cG\delta^3(x) + \ldots \right],
\]

(2.31)

where \( (a) \), \( (b) \) and \( (c) \) are dimensionless factors. The non-analytic terms are completely unrelated to the high-energy scale of gravity. The predictions from such terms are only related to the low-energy scale of general relativity. The effective terms of the Lagrangian do not provide long-distance corrections of the metric at 1-loop leading order and will hence be neglected in the calculations.

### 2.5 The form factors

In this section we will discuss the generic expressions for the energy-momentum tensors of scalars and fermions.

The transition density:

\[
\langle p_2 | T_{\mu\nu}(x) | p_1 \rangle,
\]

(2.32)

expresses the energy-momentum tensor in quantum mechanics. If we make the requirement that \( (T_{\mu\nu}) \) is a second rank tensor and that \( (T_{\mu\nu}) \) is conserved, \( i.e., \) \( (\partial^\mu T_{\mu\nu} = 0) \), we arrive at:

\[
\langle p_2 | T_{\mu\nu}(x) | p_1 \rangle = \frac{e^{i(p_2 - p_1) \cdot x}}{\sqrt{4E_2 E_1}} \left[ 2P_\mu P_\nu F_1(q^2) + (q_\mu q_\nu - \eta_{\mu\nu}q^2)F_2(q^2) \right],
\]

(2.33)

as the most general scalar form for \( (T_{\mu\nu}) \). In the above expression the following definitions have been made:

\[
P_\mu = \frac{1}{2}(p_1 + p_2)_\mu,
\]

(2.34)

and

\[
q_\mu = (p_1 - p_2)_\mu,
\]

(2.35)

as well as we have taken the traditional normalization:

\[
\langle p_2 | p_1 \rangle = 2E_1(2\pi)^3 \delta^3(\vec{p}_2 - \vec{p}_1).
\]

(2.36)

For fermions the corresponding expression for the energy-momentum tensor is, see ref. [64]:

\[
\langle p_2 | T_{\mu\nu} | p_1 \rangle = \bar{u}(p_2) \left[ F_1(q^2)P_\mu P_\nu \frac{1}{m} - F_2(q^2)(\frac{i}{4m} \sigma_{\mu\lambda}q^\lambda P_\nu + \frac{i}{4m} \sigma_{\nu\lambda}q^\lambda P_\mu) \right. \\
\left. + F_3(q^2)(q_\mu q_\nu - \eta_{\mu\nu}q^2) \frac{1}{m} \right] u(p_1).
\]

(2.37)
In the above expression for the fermions the notation of Bjorken and Drell, ref. [63], is taken. The condition \((F_1(q^2 = 0) = 1)\) is related to the conservation of energy and momentum as in the scalar case. Imposing that angular momentum is conserved:

\[
\hat{M}_{12} = \int d^3x (T_{01} x_2 - T_{02} x_1) \quad \overset{q \to 0}{\longrightarrow} -i(\nabla_q)_{1} \int d^3x e^{i\vec{q} \cdot \vec{r}} T_{01}(\vec{r}) + i(\nabla_q)_{1} \int d^3x e^{i\vec{q} \cdot \vec{r}} T_{02}(\vec{r}),
\]

(2.38)

requires that:

\[
\lim_{q \to 0} \langle p_2 | \hat{M}_{12} | p_1 \rangle = \frac{1}{2} = \frac{1}{2} \bar{u}_1(p) \sigma_3 u_1(p) F_2(q^2).
\]

(2.39)

The constraint of angular momentum conservation thus implies: \((F_2(q^2 = 0) = 1)\).

Calculating the energy-momentum tensor for massive scalars and fermions we have to lowest order the expressions:

\[
\langle p_2 | T_\mu^\nu(0) | p_1 \rangle = \frac{1}{\sqrt{4E_2 E_1}} \left[ 2P_\mu P_\nu - \frac{1}{2} (q_\mu q_\nu - \eta_\mu\nu q^2) \right],
\]

(2.40)

for the scalar, together with:

\[
\langle p_2 | T_\mu^\nu(0) | p_1 \rangle = \bar{u}(p_2) \frac{1}{2} (\gamma_\mu P_\nu + \gamma_\nu P_\mu) u(p_1)
\]

\[
= \bar{u}(p_2) \left[ \frac{1}{m} P_\mu P_\nu - \frac{i}{4m} (\sigma_\mu\lambda q^\lambda P_\nu + \sigma_\nu\lambda q^\lambda P_\mu) \right] u(p_1),
\]

(2.41)

for the fermion. These expressions can be read off from the vertex rules.

### 2.6 How to derive the metric from the equations of motion

In this section we describe how to calculate the metric from the Einstein equation of motion. The starting point is the basic Einstein-Hilbert action which is given by:

\[
S_g = \int d^4x \sqrt{-\tilde{g}} \left( \frac{1}{16\pi G} R + L_m \right),
\]

(2.42)

where \((L_m)\) defines a Lagrangian density for a specific type of matter. Here we will consider both fermionic and bosonic matter.

Through the variational principle we can generate the equation of motion:

\[
R_\mu^\nu - \frac{1}{2} g_\mu^\nu R = -8\pi G T_\mu^\nu.
\]

(2.43)

This is the Einstein equation. The energy-momentum tensor \((T_\mu^\nu)\) can formally be expressed as:

\[
T_\mu^\nu = \frac{2}{\sqrt{-g}} \frac{\partial}{\partial g_\mu^\nu} (\sqrt{-g} L_m).
\]

(2.44)
To solve the field equation we have to expand the Einstein equation in terms of the metric. Hence we make a weak field limit expansion of \( (g_{\mu\nu}) \) in powers of the gravitational coupling \((G)\):

\[
g_{\mu\nu} \equiv \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \ldots,
\]

\[
g^{\mu\nu} = \eta^{\mu\nu} - h^{(1)\mu\nu} - h^{(2)\mu\nu} + h^{(1)\mu\lambda} h^{(1)\lambda\nu} + \ldots, \tag{2.45}\]

the superscripts in these expressions count the number of powers of \((\kappa)\) which go into the field. All indices in the expressions are raised or lowered by \((\eta_{\mu\nu})\).

The determinant can be expanded as:

\[
\sqrt{-g} = \exp \frac{1}{2} \text{tr} \log g = 1 + \frac{1}{2} (h^{(1)} + h^{(2)}) - \frac{1}{4} h^{(1)}_{\alpha\beta} h^{(1)\alpha\beta} + \frac{1}{8} h^{(1)2} + \ldots, \tag{2.46}\]

Together with the curvatures:

\[
R^{(1)}_{\mu\nu} = \frac{1}{2} \left[ \partial_\mu \partial_\nu h^{(1)} + \partial_\lambda \partial^\lambda h^{(1)}_{\mu\nu} - \partial_\mu \partial_\lambda h^{(1)}_{\nu\lambda} - \partial_\nu \partial_\lambda h^{(1)}_{\mu\lambda} \right],
\]

\[
R^{(1)} = \Box h^{(1)} - \partial_\mu \partial_\nu h^{(1)\mu\nu},
\]

\[
R^{(2)}_{\mu\nu} = \frac{1}{2} \left[ \partial_\mu \partial_\nu h^{(2)} + \partial_\lambda \partial^\lambda h^{(2)}_{\mu\nu} - \partial_\mu \partial_\lambda h^{(2)}_{\nu\lambda} - \partial_\nu \partial_\lambda h^{(2)}_{\mu\lambda} \right]
\]

\[
\left( - \frac{1}{4} \partial_\mu h^{(1)}_{\alpha\beta} \partial_\nu h^{(1)\alpha\beta} - \frac{1}{2} \partial_\alpha h^{(1)}_{\mu\lambda} \partial^\alpha h^{(1)}_{\nu\lambda} + \frac{1}{2} \partial_\alpha h^{(1)}_{\mu\lambda} \partial^{\alpha} h^{(1)}_{\nu\lambda} \right)
\]

\[
+ \frac{1}{2} h^{(1)}_{\lambda\alpha} \left[ \partial_\mu \partial_\nu h^{(1)}_{\mu\alpha} + \partial_\lambda \partial_\mu h^{(1)}_{\nu\lambda} - \partial_\mu \partial_\alpha h^{(1)}_{\nu\lambda} - \partial_\lambda \partial_\alpha h^{(1)}_{\mu\nu} \right]
\]

\[
+ \frac{1}{2} \left( h^{(1)\beta\alpha} \partial^{\beta} h^{(1)} - \frac{1}{2} h^{(1)\beta\alpha} \right) \left( \partial^\sigma h^{(1)}_{\sigma\alpha} - \frac{1}{2} h^{(1)}_{\sigma\alpha} \right), \tag{2.47}\]

for a detailed calculation of these quantities, e.g., see ref. [51].

The propagator is defined making a gauge choice. Harmonic gauge correspond to, \((g^{\mu\nu} \Gamma^\lambda_{\mu\nu} = 0)\).

As a field expansion this the same as:

\[
0 = \partial^\beta h^{(1)}_{\beta\alpha} - \frac{1}{2} \partial_\alpha h^{(1)}
\]

\[
= \left( \partial^\beta h^{(2)}_{\beta\alpha} - \frac{1}{2} \partial_\alpha h^{(2)} - \frac{1}{2} h^{(1)\lambda\sigma} \left( \partial_\lambda h^{(1)}_{\sigma\alpha} + \partial_\sigma h^{(1)}_{\lambda\alpha} - \partial_\alpha h^{(1)}_{\lambda\sigma} \right) \right). \tag{2.48}\]

Using all these results we can write the Einstein equation as:

\[
\Box h^{(1)}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \Box h^{(1)} - \partial_\mu \left( \partial^\beta h^{(1)}_{\beta\nu} - \frac{1}{2} \partial_\nu h^{(1)} \right) - \partial_\nu \left( \partial^\beta h^{(1)}_{\beta\mu} - \frac{1}{2} \partial_\mu h^{(1)} \right) = -16\pi G T^{\text{matt}}_{\mu\nu}. \tag{2.49}\]
or using the gauge condition:

\[ \square \left( h_{\mu\nu}^{(1)} - \frac{1}{2} \eta_{\mu\nu} h^{(1)} \right) = -16\pi G T_{\mu\nu}^{\text{matt}}, \]  

(2.50)

which is equivalent to:

\[ \square h_{\mu\nu}^{(1)} = -16\pi G \left( T_{\mu\nu}^{\text{matt}} - \frac{1}{2} \eta_{\mu\nu} T^{\text{matt}} \right). \]  

(2.51)

In this form the Einstein equation relates the gravitational field \( h_{\mu\nu}^{(1)} \) to the energy-momentum tensor \( T_{\mu\nu} \), where the trace is expressed as: \( T = \eta^{\mu\nu} T_{\mu\nu} \).

We can then calculate the metric as:

\[ h_{\mu\nu}(x) = -16\pi G \int d^3 y D(x - y) \left( T_{\mu\nu}(y) - \frac{1}{2} \eta_{\mu\nu} T(y) \right) \]

\[ = -16\pi G \int \frac{d^3 q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} \frac{1}{q^2} \left( T_{\mu\nu}(q) - \frac{1}{2} \eta_{\mu\nu} T(q) \right). \]  

(2.52)

As an explicit example we have in limit where \( q^2 \ll m^2 \) for both fermions and scalars:

\[ \langle p_2 | T_{\mu\nu}^{(0)}(0) | p_1 \rangle \simeq m \delta_{\mu0} \delta_{\nu0}. \]  

(2.53)

Using the above form of the Einstein equation we end up with:

\[ h_{\mu\nu}^{(1)}(\vec{q}) = -\frac{8\pi G m}{q^2} \times \begin{cases} 1 & \mu = \nu = 0 \\ 0 & \mu = 0, \nu = i + \ldots, \\ \delta_{ij} & \mu = i, \nu = j \end{cases} \]  

(2.54)

which in coordinate space results in:

\[ h_{\mu\nu}^{(1)}(\vec{r}) = f(r) \times \begin{cases} 1 & \mu = \nu = 0 \\ 0 & \mu = 0, \nu = i + \ldots, \\ \delta_{ij} & \mu = i, \nu = j \end{cases} \]  

(2.55)

with: \( f(r) = -\frac{2Gm}{r} \). This is the leading order Schwarzschild result ref. [61]. A list of the used Fourier transforms is given in appendix A.3.

The fermion will to lowest order also have a spin component:

\[ \langle p_2 | T_{\mu\nu}^{(0)}(0) | p_1 \rangle \simeq \chi_2^i \hat{\sigma} \chi_1 \times \vec{q}, \]  

(2.56)

This generates the following off-diagonal result for the metric:

\[ h_{0i}^{(1)}(\vec{q}) = -8\pi i G \frac{1}{q^2} (\vec{S} \times \vec{q})_i, \]  

(2.57)

which in coordinate space is:

\[ h_{0i}^{(1)}(\vec{r}) = \frac{2G}{r^3} (\vec{S} \times \vec{r})_i, \]  

(2.58)
This result for the off-diagonal metric is in complete agreement with the Kerr result ref. [62].

Thus to first order in the gravitational coupling we can solve for the metric given a specific energy-momentum distribution. We will now extend these results to second order in the gravitational coupling.

The second order Einstein equation has the form:

\[
R^{(2)}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R^{(2)} - \frac{1}{2} h^{(1)}_{\mu\nu} R^{(1)} = 0.
\]

Expanding the above expression we get:

\[
\square h^{(2)}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \square h^{(2)} - \partial_{\mu} \left( \partial^{\beta} h^{(2)}_{\beta\nu} - \frac{1}{2} \partial_{\nu} h^{(2)} \right) - \partial_{\nu} \left( \partial^{\beta} h^{(2)}_{\beta\mu} - \frac{1}{2} \partial_{\mu} h^{(2)} \right) \equiv -16\pi G T^{(2)}_{\mu\nu}.
\]

The gravitational field itself generates an energy-momentum distribution which we will write as \((T^{(2)}_{\mu\nu})\). It takes the form:

\[
8\pi G T^{(2)}_{\mu\nu} = -\frac{1}{2} h^{(1)}_{\lambda\nu} \left[ \partial_{\mu} \partial_{\nu} h^{(1)}_{\lambda\rho} + \partial_{\lambda} \partial_{\rho} h^{(1)}_{\mu\nu} - \partial_{\nu} \left( \partial_{\rho} h^{(1)}_{\mu\lambda} + \partial_{\lambda} h^{(1)}_{\rho\mu} \right) \right] - \frac{1}{2} \partial_{\lambda} h^{(1)}_{\sigma\nu} \partial^{\lambda} h^{(1)}_{\mu\sigma} - \frac{1}{2} \partial_{\lambda} h^{(1)}_{\sigma\mu} \partial^{\lambda} h^{(1)}_{\nu\sigma} - \frac{1}{4} \partial_{\lambda} h^{(1)}_{\sigma\lambda} \partial_{\mu} h^{(1)}_{\nu\sigma} - \frac{1}{4} \partial_{\lambda} h^{(1)}_{\sigma\lambda} \partial_{\nu} h^{(1)}_{\mu\sigma} - \frac{1}{4} \eta_{\mu\nu} h^{(1)}_{\alpha\beta} \square h^{(1)}_{\alpha\beta}.
\]

Employing the gauge condition in the above expression, and isolating for \((h^{(2)}_{\mu\nu})\) we end up with:

\[
\square \left( h^{(2)}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h^{(2)} \right) = -16\pi G T^{(2)}_{\mu\nu}
\]

\[
+ \partial_{\mu} \left( h^{(1)}_{\lambda\sigma} \left[ \partial_{\lambda} h^{(1)}_{\sigma\nu} - \frac{1}{2} \partial_{\nu} h^{(1)}_{\lambda\sigma} \right] \right) + \partial_{\nu} \left( h^{(1)}_{\lambda\sigma} \left[ \partial_{\lambda} h^{(1)}_{\sigma\mu} - \frac{1}{2} \partial_{\mu} h^{(1)}_{\lambda\sigma} \right] \right)
\]

or using the solution we found for \((h^{(1)}_{\mu\nu})\):

\[
\square h^{(2)}_{\mu\nu} = -16\pi G \left( T_{\mu\nu}^{(2)} - \frac{1}{2} \eta_{\mu\nu} T^{(2)} \right) - \partial_{\mu} \left( f(r) \partial_{\nu} f(r) \right) - \partial_{\nu} \left( f(r) \partial_{\mu} f(r) \right).
\]

For a fermion there will also be a corresponding off-diagonal equation:

\[
\square h^{(2)}_{0i} = -16\pi G T_{0i}^{(2)} - \nabla_i (h^{(1)}_{0j} \nabla_j h^{(1)}_{00}) + \nabla_i (h^{(1)}_{jk} \nabla_j h^{(1)}_{0k}),
\]

As it is easily verified that:

\[
\nabla_i (h^{(1)}_{0j} \nabla_j h^{(1)}_{00}) = \nabla_i (h^{(1)}_{jk} \nabla_j h^{(1)}_{0k}) = 0,
\]
the second order off-diagonal Einstein equation has the form:

\[ \Box h_{0i}^{(2)} = -16\pi G T_{0i}^{grav}. \] (2.66)

This together with the above equation concludes our investigations of how to calculate the metric from the energy-momentum tensor. What has to be done in order to derive the metric is to use the above equations and to Fourier transformation to get the coordinate results. The used Fourier transforms are given in appendix A.3.

### 2.7 Loop corrections and the energy-momentum tensor

In the case of scalars the 1-loop non-analytic radiative vertex corrections to the energy-momentum tensor were first calculated in refs. [32, 33], and an error in these results was corrected in ref. [51]. In order to determine the radiative corrections to the vertex form factors we have to calculate the diagrams, see figure 2.1.

![Figure 2.1: The only gravitational radiative diagrams which carry non-analytic contributions.](image)

These diagrams have the following expressions:

\[
A_{(a)}^{\mu\nu} = i P^{\alpha,\lambda\kappa} i P^{\gamma\delta,\rho\sigma} \int \frac{d^4\ell}{(2\pi)^4} \frac{\tau_{6\alpha\beta,m}(p,p' - \ell)\tau_{6\gamma\delta}(p' - \ell, p', m') \tau_{10\rho\sigma,\lambda\kappa}^{\mu\nu}(\ell, q)}{\ell^2 (\ell - q)^2 (\ell - p')^2 - m^2}, \tag{2.67}
\]

and

\[
A_{(b)}^{\mu\nu} = \frac{i}{2} D^{\alpha\beta,\lambda\kappa} i P^{\gamma\delta,\rho\sigma} \tau_{8\alpha\beta,\gamma\delta}(p,p') \int \frac{d^4\ell}{(2\pi)^4} \frac{\tau_{10\lambda\kappa,\rho\sigma}^{\mu\nu}(\ell, q)}{\ell^2 (\ell - q)^2}. \tag{2.68}
\]

The formal expressions for the vertex factors can be found in appendix A.1.

In terms of the form factors the results for the diagrams are:

\[
F_1(q^2) = 1 + \frac{Gq^2}{\pi} (-\frac{3}{4} \log \frac{-q^2}{m^2} + \frac{1}{16} \frac{\pi^2 m}{\sqrt{-q^2}}) + \ldots
\]

\[
F_2(q^2) = -\frac{1}{2} + \frac{Gm^2}{\pi} (-2 \log \frac{-q^2}{m^2} + \frac{7}{8} \frac{\pi^2 m}{\sqrt{-q^2}}) + \ldots
\]
These corrections lead to the results for the energy-momentum tensor, using the integrals listed in appendix A.3.

\[
T_{00}(\vec{r}) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \left( mF_1(q^2) + \frac{q^2}{2m}F_2(q^2) \right)
= \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \left[ m + \pi Gm^2(-\frac{1}{16} + \frac{7}{16})q + \frac{Gm}{\pi}q^2 \log q^2(\frac{3}{4} - 1) \right]
= m\delta^3(r) - \frac{3Gm^2}{8\pi r^4} - \frac{3Gmh}{4\pi^2 r^5},
\]

\[
T_{0i}(\vec{r}) = 0,
\]

\[
T_{ij}(\vec{r}) = \frac{1}{2m} \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} (q_iq_j - \delta_{ij}q^2)F_2(q^2)
= \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \left[ \frac{7\pi Gm^2}{16q} (q_iq_j - \delta_{ij}q^2) - (q_iq_j - \delta_{ij}q^2) \frac{Gm}{\pi} \log q^2 \right]
= -\frac{7Gm^2}{4\pi r^4} \left( \frac{r_ir_j}{r^2} - \frac{1}{2} \delta_{ij} \right) + \frac{Gmh}{2\pi^2 r^5} \left( 9\delta_{ij} - 15\frac{r_ir_j}{r^2} \right).
\]

(2.69)

For fermions we have to calculate the diagrams, see figure 2.2.

Figure 2.2: The only gravitational radiative diagrams which carry non-analytic contributions.

They lead to the results in terms of form factors:

\[
F_1(q^2) = 1 + \frac{Gq^2}{\pi} \left( \frac{\pi^2 m}{16\sqrt{-q^2}} - \frac{3}{4} \log \frac{-q^2}{m^2} \right) + \ldots,
\]

\[
F_2(q^2) = 1 + \frac{Gq^2}{\pi} \left( \frac{\pi^2 m}{4\sqrt{-q^2}} + \frac{1}{4} \log \frac{-q^2}{m^2} \right) + \ldots,
\]

(2.70)

\[
F_3(q^2) = \frac{Gm^2}{\pi} \left( \frac{7\pi^2 m}{16\sqrt{-q^2}} - \log \frac{-q^2}{m^2} \right) + \ldots,
\]

which were calculated in the paper, ref. [37].
Again these results are used to calculate the energy-momentum tensor:

\[
T_{00}(\vec{r}) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} \left( mF_1(-q^2) + \frac{q^2}{m} F_3(-q^2) \right),
\]

\[
T_{0i}(\vec{r}) = i \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} \frac{1}{2} (\vec{S} \times \vec{q})_i F_2(-q^2),
\]

\[
T_{ij}(\vec{r}) = \frac{1}{m} \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} (q_i q_j - \delta_{ij} q^2) F_3(-q^2),
\] (2.71)

where we have defined \((\vec{S} = \vec{\sigma}/2)\).

The form factors then become:

\[
T_{00}(\vec{r}) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} \left( m + \frac{3Gm^2\pi}{8} |\vec{q}| - \frac{Gm^2}{4\pi} q^2 \log q^2 \right) + \ldots
\]

\[
= m\delta^3(\vec{r}) - \frac{3Gm^2}{8\pi r^4} - \frac{Gmh}{4\pi r^5} + \ldots,
\]

\[
T_{0i}(\vec{r}) = \frac{i}{2} \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} (\vec{S} \times \vec{q})_i \left( 1 - \frac{Gm\pi}{4} |\vec{q}| - \frac{G}{4\pi} q^2 \log q^2 \right) + \ldots,
\]

\[
= \frac{1}{2} (\vec{S} \times \vec{\nabla})_i \delta^3(\vec{r}) + \left(-\frac{Gm}{2\pi r^6} + \frac{15Gh}{4\pi^2 r^7}\right) (\vec{S} \times \vec{r})_i + \ldots
\]

\[
T_{ij}(\vec{r}) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} \left( \frac{7Gm^2\pi}{16|\vec{q}|} - \frac{Gm}{\pi} \log q^2 \right) (q_i q_j - \delta_{ij} q^2) + \ldots
\]

\[
= -\frac{7Gm^2}{4\pi r^4} \left( \frac{r_i r_j}{r^2} - \frac{1}{2} \delta_{ij} \right) + \frac{Gmh}{2\pi^2 r^5} \left( 9\delta_{ij} - 15 \frac{r_i r_j}{r^2} \right) + \ldots
\] (2.72)

Together with the vertex correction there is an additional effect due to the vacuum polarization of the graviton. These additional correction terms come from the vacuum polarization corrections to the propagator and can be graphically depicted, see figure 2.3.

![Vacuum polarization diagrams including the ghost diagram](image)

Figure 2.3: The vacuum polarization diagrams including the ghost diagram.

To understand this better we have to take a look at the effective action. At 1-loop level it can be written in the following way:

\[
Z[h] = -\int d^4 x d^4 y \frac{1}{2} [h_{\mu\nu}(x) \Delta^{\mu\nu,\alpha\beta}(x-y) h_{\alpha\beta}(y) + O(h^3)] + Z_{\text{matter}}[h, \phi].
\] (2.73)

Properly renormalized, the operator: \((\Delta^{\mu\nu,\alpha\beta}(x-y))\), will contain a contribution from the vacuum polarization.

This can be mathematically expressed as:

\[
\Delta^{\mu\nu,\alpha\beta}(x-y) = \delta^4(x-y) D^{\mu\nu,\alpha\beta}_2 + \hat{\Pi}^{\mu\nu,\alpha\beta}(x-y) + O(\partial^4),
\] (2.74)
where the differential operator \( D_2^{\mu\nu,\alpha\beta} \) will give the traditional harmonic gauge propagator and \( \hat{\Pi}^{\mu\nu,\alpha\beta}(x - y) \) contains the vacuum polarization correction. To generate the effect on the metric we solve the equation of motion once again:

\[
\square h_{\mu\nu}(x) + P_{\mu\nu\alpha\beta} \int d^4 y \hat{\Pi}^{\alpha\beta,\gamma\delta}(x - y) h_{\gamma\delta}(y) = -16\pi G(T_{\mu\nu}^{\text{grav}} - \frac{1}{2} \eta_{\mu\nu} T^{\text{grav}}) - \partial_{\mu}(f(r)\partial_{\nu} f(r)) - \partial_{\nu}(f(r)\partial_{\mu} f(r)),
\]

(2.75)

here \((P_{\mu\nu\alpha\beta})\) and \((\Pi_{\mu\nu\alpha\beta})\) are defined as:

\[
P_{\mu\nu\alpha\beta} = \Pi_{\mu\nu\alpha\beta} - \frac{1}{2} \eta_{\mu\nu} \eta_{\alpha\beta}
\]

\[
\Pi_{\mu\nu\alpha\beta} = \frac{1}{2} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\nu\alpha} \eta_{\mu\beta}).
\]

(2.76)

Using the harmonic gauge condition, we arrive at:

\[
\square h_{\mu\nu} = -16\pi G(T_{\mu\nu}^{\text{grav}} - \frac{1}{2} \eta_{\mu\nu} T^{\text{grav}}) - \partial_{\mu}(f(r)\partial_{\nu} f(r)) - \partial_{\nu}(f(r)\partial_{\mu} f(r))
\]

\[
+ 16\pi G \int d^4 y d^4 z P_{\mu\nu\alpha\beta} \hat{\Pi}^{\alpha\beta,\gamma\delta}(x - y) D(y - z) (T_{\gamma\delta}^{\text{matt}}(z) - \frac{1}{2} \eta_{\gamma\delta} T^{\text{matt}}(z)).
\]

(2.77)

Figure 2.4: Vacuum polarization modification of the energy-momentum tensor for a scalar and a fermion.

't Hooft and Veltman ref. [16] were the first to calculate the vacuum polarization diagram in gravity. The source of the non-locality of the diagram is the factor of \((q^4 \log(-q^2))\) and the specific form is:

\[
\hat{\Pi}_{\alpha\beta,\gamma\delta} = -\frac{2G}{\pi} \log(-q^2) \left[ \frac{21}{120} q^4 I_{\alpha\beta,\gamma\delta} + \frac{23}{120} q^4 \eta_{\alpha\beta} \eta_{\gamma\delta} - \frac{23}{120} q^2 \eta_{\alpha\beta} q_\gamma q_\delta + \eta_{\gamma\delta} q_\alpha q_\beta \right]
\]

\[
- \frac{21}{240} q^2 (q_\alpha q_\delta \eta_{\gamma\beta} + q_\beta q_\delta \eta_{\alpha\gamma} + q_\alpha q_\gamma \eta_{\beta\delta} + q_\beta q_\gamma \eta_{\alpha\delta}) + \frac{11}{30} q_\alpha q_\beta q_\gamma q_\delta \right].
\]

(2.78)

Using this result we see that the vacuum polarization introduces a shift in the metric by:

\[
\delta h^{(2)\text{vacpol}}_{\mu\nu}(x) = 32G^2 \int \frac{d^3 q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} \log(q^2) \left[ \frac{21}{120} T^{\text{matt}}_{\mu\nu}(q) + \left( \frac{1}{240} \eta_{\mu\nu} - \frac{11 q^\mu q^\nu}{60 q^2} \right) T^{\text{matt}}(q) \right].
\]

(2.79)
This is in terms of components:

\[
\begin{align*}
\delta h^{(2)}_{00}^{\text{vacpol}} &= -\frac{43G^2m\hbar}{15\pi r^3}, \\
\delta h^{(2)}_{ij}^{\text{vacpol}} &= \frac{G^2m\hbar}{15\pi r^3}(\delta_{ij} + 44\frac{r_ir_j}{r^2}) - \frac{44G^2m(1 - \log r)\hbar}{15\pi r^3}(\delta_{ij} - 3\frac{r_ir_j}{r^2}). \\
\end{align*}
\] (2.80)

Adding all logarithmic corrections together we find:

\[
\begin{align*}
\delta h^{(2)}_{00}^{\text{vertex}}(r) &= -16\pi G \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\vec{r}} \frac{1}{q^2} \left( \frac{m}{2} F_1(-\vec{q}^2) - \frac{q^2}{4m} F_2(-\vec{q}^2) \right) \\
\delta h^{(2)}_{ii}^{\text{vertex}}(r) &= 0, \\
\delta h^{(2)}_{ij}^{\text{vertex}}(r) &= -16\pi G \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\vec{r}} \frac{Gm}{\pi} \left( \frac{3}{8} + \frac{1}{2} \log q^2 \right) = \frac{7G^2m\hbar}{\pi r^3}, \\
\delta h^{(2)}_{ij}^{\text{vertex}}(r) &= -\frac{G^2m\hbar}{\pi r^3}(\delta_{ij} + 8\frac{r_ir_j}{r^2}) + \frac{8G^2m(1 - \log r)}{\pi r^3}(\delta_{ij} - 3\frac{r_ir_j}{r^2}). \\
\end{align*}
\] (2.81)

These results lead to the metric:

\[
\begin{align*}
g_{00} &= 1 - \frac{2Gm}{r} + \frac{G^2m^2}{r^2} + \frac{62G^2m\hbar}{15\pi r^3} + \ldots, \\
g_{0i} &= 0, \\
g_{ij} &= -\delta_{ij} \left( 1 + \frac{2Gm}{r} + \frac{G^2m^2}{r^2} + \frac{14G^2m\hbar}{15\pi r^3} - \frac{76G^2m(1 - \log r)}{15\pi r^3} \right) \\
&\quad - \frac{r_ir_j}{r^2} \left( \frac{G^2m^2}{r^2} + \frac{76G^2m\hbar}{15\pi r^3} + \frac{76G^2m(1 - \log r)}{5\pi r^3} \right) + \ldots. \\
\end{align*}
\] (2.82)

which clearly is identical to the classical Schwarzschild metric plus a unique set of quantum corrections, caused by the quantum nature of the theory of general relativity.

For fermions we make the observation that all diagonal components of the vacuum polarization are completely identical to the bosonic case. For the off-diagonal terms there is an extra component associated with the spin. It can be written:

\[
\begin{align*}
 h^{(2)}_{0i}^{\text{vacpol}} &= 32G^2 \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\vec{r}} \log q^2 \frac{21}{240} i F_2(q^2)(\vec{S} \times \vec{q})_i \\
&= \frac{21G^2\hbar}{5\pi r^5} (\vec{S} \times \vec{r})_i. \\
\end{align*}
\] (2.83)
Adding all corrections together we end up with:

\[
\begin{align*}
    g_{00} &= 1 - 2\frac{Gm}{r} + 2\frac{G^2m^2}{r^2} + \frac{62G^2mh}{15\pi r^3} + \ldots \\
    g_{0i} &= \left(\frac{2G}{r^3} - \frac{2G^2m}{r^4} + \frac{36G^2h}{15\pi r^5}\right) (\vec{S} \times \vec{r})_i + \ldots \\
    g_{ij} &= -\delta_{ij} \left(1 + 2\frac{Gm}{r} + \frac{G^2m^2}{r^2} + \frac{14G^2mh}{15\pi r^3} - \frac{76G^2mh(1 - \log r)}{15\pi r^3}\right) \\
    &\quad - \frac{r_i r_j}{r^2} \left(\frac{G^2m^2}{r^2} + \frac{76G^2mh}{15\pi r^3} + \frac{76G^2mh(1 - \log r)}{5\pi r^3}\right) + \ldots .
\end{align*}
\] (2.84)

This is the result for the classical Kerr metric together with unique quantum corrections.

The reproduced classical metric and the results for the quantum corrections are essentially the main results of the paper ref. [37]. In the next section we will discuss the results of the papers refs. [35, 38], here the diagrammatic 1-loop quantum corrections will be discussed from the point of view of defining a potential.

### 2.8 Calculations of the scattering matrix potential

In this section we will discuss on more general grounds how to define a potential from the scattering matrix and how to derive the 1-loop scattering matrix potential for a pure gravitational interaction and in the mixed theory of scalar QED and gravity.

The general form for any diagram contributing to the scattering matrix is, e.g., in the mixed theory of gravity scalar QED:

\[
\mathcal{M} \sim \left( A + Bq^2 + \ldots + (\alpha_1\kappa^2 + \alpha_2\kappa^2) \frac{1}{q^2} + \beta_1\kappa^2 \ln(-q^2) + \beta_2\kappa^2 \frac{m}{\sqrt{-q^2}} + \ldots \right),
\] (2.85)

where \(A, B, \ldots\) correspond to the local analytical interactions and \(\alpha_1, \alpha_2\) and \(\beta_1, \beta_2, \ldots\) correspond to the leading non-analytical, non-local, long-range interactions.

The space parts of the non-analytical terms Fourier transform as:

\[
\begin{align*}
    \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} \frac{1}{|\vec{q}|^2} &= \frac{1}{4\pi r}, \\
    \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} \frac{1}{|\vec{q}|} &= \frac{1}{2\pi^2 r^2}, \\
    \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} \ln(q^2) &= \frac{-1}{2\pi^3 r^3},
\end{align*}
\] (2.86)

so clearly these terms will contribute to the long-range corrections.

The non-analytical contribution, corresponding to the \(\left(\frac{1}{q^2}\right)\)-part, gives as seen the Newtonian and Coulomb potentials respectively. The other non-analytical contributions generate the leading quantum and classical corrections to the Coulomb and Newtonian potentials in powers of \(\left(\frac{1}{r}\right)\). It is necessary to have non-analytic contributions in the matrix element, to ensure that the S-matrix is unitary.
The analytic contributions will not be considered. As noted previously these corrections correspond to local interactions, and are thus only needed for the high-energy manifestation of the theory. Many of the analytical corrections will be divergent, and hence have to be carefully absorbed into the appropriate terms of the effective action of the theory.

We will not consider the radiative corrections due to soft bremsstrahlung in this approach. In some of the diagrams in gravity, as well as in QED, there is a need for introducing soft bremsstrahlung radiative corrections to the sum of the diagrams constituting the vertex corrections. We will not consider this aspect of the theory in this approach, as we are not computing the full amplitude of the S-matrix. Furthermore certain effects have been included in the recent work of ref. [53], where the gravitational vertex corrections are treated. This issue should be dealt with at some stage refining the theories, however here we will carry on and simply concentrate on the leading post-Newtonian and quantum corrections to the scattering matrix, leaving this concern for future investigations.

The definition of the potential

The various definitions of the potential have been discussed at length in the literature. We will here define the potential directly from the scattering matrix amplitude.

In the quantization of general relativity the definition of a potential is certainly not obvious. One can choose between several definitions of the potential depending on, e.g., the physical situation, how to define the energy of the fields, the diagrams included etc.

Clearly a valid choice of potential should be gauge invariant to be physically reasonable, but while other gauge theories like QCD allow a gauge invariant Wilson loop definition — e.g., for a quark-anti-quark potential, this is not directly possible in general relativity.

It has however been attempted to make the equivalent of a Wilson loop potential for quantum gravity. A Wilson-like potential seems to be possible to construct in general relativity using the Arnowitt-Deser-Misner formula for the total energy of the system ref. [65]. This choice of potential has been discussed in ref. [48] in the case of pure gravity coupled to scalar fields.

A recent suggestion ref. [66] is that one should use the full set of diagrams constituting the scattering matrix, and decide the nonrelativistic potential from the total sum of the 1-loop diagrams. As the full 1-loop scattering matrix is involved, this choice of potential gives a gauge invariant definition.

This choice of potential is equivalent to that of ref. [49], where the scalar source pure gravitational potential was treated.

This choice of potential, which includes all 1-loop diagrams, seems to be the simplest, gauge invariant definition of the potential.

We will calculate the nonrelativistic potential using the the full amplitude. We will simply relate the expectation value for the \((iT)\) matrix to the Fourier transform of the potential \((\tilde{V}(q))\) in the nonrelativistic limit in the following way:

\[
\langle k_1, k_2 | iT | k'_1, k'_2 \rangle = -i \tilde{V}(q)(2\pi)\delta(E - E'),
\]

(2.87)

where \((k_1), (k_2)\) and \((k'_1), (k'_2)\) are the incoming and outgoing momentum respectively, and \((E - E')\) is the energy difference between the incoming and outgoing states. Comparing
this to the definition of the invariant matrix element \((i\mathcal{M})\) we get from the diagrams:

\[
\langle k_1, k_2|i T|k_1', k_2'\rangle = (2\pi)^4\delta^{(4)}(k_1 - k_1' + k_2 - k_2')(i\mathcal{M}),
\]  

(2.88)

we see that (we have divided the above equation with \((2m_12m_2)\) to obtain the nonrelativistic limit):

\[
\tilde{V}(q) = -\frac{1}{2m_1} \frac{1}{2m_2} \mathcal{M},
\]  

(2.89)

so that

\[
V(x) = -\frac{1}{2m_1} \frac{1}{2m_2} \int \frac{d^3 k}{(2\pi)^3} e^{i k \cdot x} \mathcal{M}.
\]  

(2.90)

This is how we define the nonrelativistic potential generated by the considered nonanalytic parts. In the above equation \((\mathcal{M})\) is the non-analytical part of the amplitude of the scattering process to a given loop order. This definition of the potential is also used in ref. [49].

We will note here that the above definition of a potential \((V(q))\) using the scattering amplitude is not the only possibility. The nonrelativistic bound state quantum mechanics potential is arrived at, if we subtract off the second order Born contribution. This can be done using the following prescription:

\[
i\langle f|T|i \rangle = -2\pi i\delta(E - E')
\]

\[
\times \left[ \langle f|\tilde{V}_{bs}(q)|i \rangle + \sum_n \frac{\langle f|\tilde{V}_{bs}(q)|n \rangle \langle n|\tilde{V}_{bs}(q)|i \rangle}{E - E_n + i\epsilon} + \ldots \right].
\]  

(2.91)

For a detailed definition of the bound state potential, see ref. [13]. In a treatment involving the Hamiltonian there will be terms such as \((\alpha^2r^2)\) contributing to the potential at the same order. To lowest order in Einstein-Infeld-Hoffmann coordinates the the bound state potential \((\tilde{V}_{bs}(q))\) will be:

\[
\tilde{V}_{bs}(r) = V(r) + \frac{7Gm_1m_2(m_1 + m_2)}{2\epsilon^2r^2}.
\]  

(2.92)

### 2.9 The results for the Feynman diagrams

#### 2.9.1 The diagrams contribution to the non-analytical parts of the scattering matrix in the combined theory of scalar QED and general relativity

Of the diagrams contributing to the scattering matrix only a certain class of diagrams will actually contribute to the sum of the non-analytical terms considered here — the logarithmic and square-root parts. In this treatment we will only in detail consider the diagrams which contribute with non-analytical contributions. Diagrams with many massive propagators will usually only contribute with analytical terms. Due to the involved vertex rules, some of the diagrams will have a somewhat complicated algebraic structure.
To do the diagrams we developed an algebraic program for Maple 7™\textsuperscript{2}. The program contracts the various indices and performs the loop integrations. In the following we will go through the diagrams and discuss how they are calculated in detail. We will begin with the tree diagrams.

The tree diagrams

The set of tree diagrams contributing to the scattering matrix are those of figure 2.5. The formal expression for these diagrams are:

\( i\mathcal{M}_{1(a)} = \tau_{2}^{\mu\nu}(k_1, k_2, m_1) \left[ \frac{i\mathcal{P}_{\mu\nu\alpha\beta}}{q^2} \right] \tau_{2}^{\alpha\beta}(k_3, k_4, m_2), \) \hspace{1cm} (2.93)

and

\( i\mathcal{M}_{1(b)} = \tau_{1}^{\mu}(k_1, k_2, e_1) \left[ \frac{-i\eta_{\mu\nu}}{q^2} \right] \tau_{1}^{\nu}(k_3, k_4, e_2). \) \hspace{1cm} (2.94)

These diagrams yield no complications. Contracting all indices and preforming the Fourier transforms one ends up with:

\( V_{1(a)}(r) = -\frac{Gm_1m_2}{r}, \) \hspace{1cm} (2.95)

\( V_{1(b)}(r) = \frac{e_1e_2}{4\pi r}, \) \hspace{1cm} (2.96)

where \((e_1, m_1)\) and \((e_2, m_2)\) are the two charges and masses of the system respectively. This is of course the expected results for these diagrams. One gets the Newtonian and Coulomb terms for the potential of two charged scalars.

The next class of diagrams we will consider is that of the box diagrams.

The box diagrams and crossed box diagrams

There are four distinct diagrams. See figure 2.6. Two crossed box and two box diagrams. We will not treat all diagrams separately but rather discuss one of the diagrams in detail.

\hspace{1cm} \textsuperscript{2}Maple and Maple V are registered trademarks of Waterloo Maple Inc.
and then present the total result. The diagram 2(a) is defined in the following way:

\[
iM_{2(a)} = \int \frac{d^4l}{(2\pi)^4} \frac{i}{(l + k_1)^2 - m_1^2} \frac{i}{(l - k_3)^2 - m_2^2} \times \tau_1^\gamma(k_1, k_1 + l, e_1) \left[ -\frac{i\eta_\delta}{l^2} \right] \tau_1^\delta(k_3, -l + k_3, e_2) \\
\times \tau_2^{\mu\nu}(l + k_1, k_2, m_1) \left[ \frac{iP_{\mu\nu\sigma\rho}}{(l + q)^2} \right] \tau_2^{\sigma\rho}(k_3 - l, k_4, m_2).
\]

(2.97)

where we have chosen a certain parametrization for the momenta in the diagram, the side with mass \((m_1)\) and charge \((e_1)\) has \((k_1), (k_2)\) as incoming and outgoing momentum respectively. Correspondingly the other side with mass \((m_2)\) and charge \((e_2)\) has \((k_3), (k_4)\) as incoming and outgoing momentum respectively. These diagrams are quite hard to do. In appendix we will present all the needed integrals as well as we will discuss some calculational simplifications which are useful in doing the box-integrals, see appendix A.2.

The final sum for these diagrams is:

\[
V_{2(a)+2(b)+2(c)+2(d)}(r) = \frac{10Ge_1e_2}{3\pi^2r^3}.
\]

(2.98)

The triangular diagrams

The following triangular diagrams contribute with non-analytic contributions to the potential. See figure 2.7. As for the box diagrams we will only consider one of the diagrams — here again the first namely 3(a). The formal expression for this particular diagram is,
2.9. THE RESULTS FOR THE FEYNMAN DIAGRAMS

Figure 2.7: The set of triangular diagrams contributing to the non-analytical terms of the potential.

We just apply the vertex rules:

\[
\begin{align*}
i \mathcal{M}_{3(a)} &= \int \frac{d^4 l}{(2\pi)^4} \frac{i}{(l + k_1)^2 - m_1^2} \\
&\quad \times \tau_1^\gamma(k_1, k_1 + l, e_1) \left[ -i \eta_\gamma^\delta \right] \tau_2^{\mu \nu}(l + k_1, k_2, m_1) \\
&\quad \times \left[ \frac{i \mathcal{P}_{\mu \nu \rho \sigma}}{(l + q)^2} \right] \tau_5^{(\delta)\sigma \rho}(k_3, k_4, e_2).
\end{align*}
\]  

Again all the needed integrals are of the type discussed in the appendix A.2. Applying our contraction program and doing the integrations leave us with a result, which Fourier transformed yields the following contribution to the potential:

\[
V_{3(a)+3(b)+3(c)+3(d)}(r) = \frac{G e_1 e_2 (m_1 + m_2)}{\pi r^2} - \frac{4 e_1 e_2 G}{\pi^2 r^3}. 
\]  

As seen these diagrams yield both a classical, the \( \sim \frac{1}{r^2} \) contribution, as well as a quantum correction \( \sim \frac{1}{r^3} \).

The circular diagram

The circular diagram, see figure 2.8, has the following formal expression:

\[
\begin{align*}
i \mathcal{M}_{4(a)} &= \int \frac{d^4 l}{(2\pi)^4} \tau_5^{\mu \nu (\gamma)}(k_1, k_2, e_1) \left[ -i \eta_\gamma^\delta \right] \\
&\quad \times \left[ \frac{i \mathcal{P}_{\mu \nu \rho \sigma}}{(l + q)^2} \right] \tau_5^{\rho (\delta)}(k_3, k_4, e_2).
\end{align*}
\]  

Doing the contractions and integrations give the following contribution to the potential:

\[
V_{4(a)}(r) = \frac{2 Ge_1 e_2}{\pi^2 r^3}. 
\]
1PR-diagrams

The following class of the set of 1PR-diagrams corresponding to the gravitational vertex correction will contribute to the potential, see figure 2.9.

![Figure 2.8: The circular diagram with non-analytic contributions.](image)

![Figure 2.9: The class of the graviton 1PR vertex corrections which yield non-analytical corrections to the potential.](image)

Again we will not treat all diagrams separately. Instead we will consider two of the diagrams in details — namely the diagrams 5(a) and 5(c). First we will present the formal expressions for the diagrams using the Feynman vertex rules. Next we will briefly consider the calculations and finally we will present the results.

The formal expression for 5(a) is:

\[
\begin{align*}
    i M_{5(a)} &= \int \frac{d^4 l}{(2\pi)^4} \frac{i}{(l - k_3)^2 - m_2^2} \tau_2^{\mu \nu}(k_1, k_2, m_1) \\
    &\times \left[ \frac{iP_{\mu \nu \rho \sigma}}{q^2} \right] \tau_3^{\rho \sigma}(l, l + q) \tau_1^{\alpha}(k_3, k_3 - l, e_2) \tau_1^{\beta}(k_3 - l, k_4, e_2),
\end{align*}
\]  

(2.103)
while the expression for $5(c)$ reads:

$$i\mathcal{M}_{5(c)} = \int \frac{d^4 l}{(2\pi)^4} \tau_2^{\mu\nu}(k_1, k_2, m_1) \left[ \frac{i\mathcal{P}_{\mu\nu\rho\sigma}}{q^2} \right] \times \tau_3^{\rho\sigma(\gamma\delta)}(l, l + q) \tau_4^{\alpha\beta}(k_3, k_4, e_2) \left[ -i\eta_{\gamma\alpha} \right] \left[ -i\eta_{\delta\beta} \right].$$

Again the calculations of these diagrams yield no real complications using our algebraic program.

The result for the diagrams $5(a - d)$ are in terms of the corrections to the potential:

$$V_{5(a)+5(b)+5(c)+5(d)}(r) = \frac{G(e_2^2m_1 + e_1^2m_2)}{8\pi r^2} - \frac{G(\frac{m_1}{m_2}e_2^2 + \frac{m_2}{m_1}e_1^2)}{3\pi^2 r^3},$$

where we have associated a factor of one-half due to the symmetry of the diagrams $5(c-d)$.

We have checked explicitly, that the above result for correction to the potential is in complete agreement, with the result for the gravitational vertex correction calculated in ref. [53].

For the photonic vertex correction we consider the following diagrams. See figure 2.10.

![Figure 2.10: The first class of the photon vertex 1PR corrections which yield non-analytical corrections to the potential.](image)

Figure 2.10: The first class of the photon vertex 1PR corrections which yield non-analytical corrections to the potential.

Together with the diagrams. See figure 2.11.

Again we look upon the formal expression for only two of the diagrams — namely 6(a)
Figure 2.11: The remaining photonic vertex 1PR diagrams which yield non-analytical corrections to the potential.

and 7(a):

$$iM_{6(a)} = \int \frac{d^4l}{(2\pi)^4} \frac{i}{(l - k_3)^2 - m_2^2} \tau_1^\gamma(k_1, k_2, e_1) \times \left[ \frac{-i\eta_{\gamma\delta}}{q^2} \right] \tau_3^{\sigma\rho(\delta\alpha)}(q, l + q) \tau_2^{\rho\nu}(k_3, k_3 - l, m_2) \times \left[ \frac{iP_{\mu\nu\sigma\rho}}{l^2} \right] \left[ \frac{-i\eta_{\beta\alpha}}{(l + q)^2} \right] \tau_1^\beta(k_3 - l, k_4, e_2).$$  \hfill (2.106)

$$iM_{7(a)} = \int \frac{d^4l}{(2\pi)^4} \tau_1^\gamma(k_1, k_2, e_1) \left[ \frac{-i\eta_{\gamma\delta}}{q^2} \right] \tau_3^{\mu\nu(\delta\alpha)}(q, l + q) \times \left[ \frac{iP_{\mu\nu\sigma\rho}}{l^2} \right] \left[ \frac{-i\eta_{\beta\alpha}}{(l + q)^2} \right] \tau_5^{\sigma\rho(\beta\delta)}(k_3, k_4, e_2).$$  \hfill (2.107)

The result for the diagrams $6(a - d) + 7(a - b)$ are in terms of the corrections to the potential:

$$V_{6(a)+6(b)+6(c)+6(d)+7(a)+7(b)}(r) = -\frac{G\epsilon_1\epsilon_2(m_1 + m_2)}{4\pi r^2}. \hfill (2.108)$$

The vacuum polarization diagram

Figure 2.12: The only mixed vacuum polarization diagram to contribute to the potential. There is no mixed corresponding ghost diagram associated with this diagram.
Figure 2.13: Diagrams which will only give contributions to the analytical parts of the potential.

The vacuum diagram, see figure 2.12, has the following formal expression:

\[
\begin{align*}
  i\mathcal{M}_{8(a)} = \int \frac{d^4l}{(2\pi)^4} \gamma_1(k_1, k_2, e_1) \left[ \frac{-i\eta_\gamma\delta}{q^2} \right] \tau_3^{\sigma\rho(\delta\alpha)}(q, -l) \\
  \times \tau_3^{\mu\nu}(l, q) \left[ \frac{-i\eta_\rho\beta}{l^2} \right] \left[ \frac{-i\eta_\alpha}{q^2} \right] \\
  \times \tau_1^{\phi}(k_3, k_4, e_2).
\end{align*}
\]  

(2.109)

It gives the following contribution to the potential:

\[V_{8(a)}(r) = \frac{Ge_1e_2}{6\pi^2r^3}.\]  

(2.110)

The exact photon contributions for the 1-loop divergences of the minimal theory can be found in ref. [25]. Using that the pole singularity \((\frac{1}{\epsilon})\) always is followed by a \((\ln(-q^2))\)-contribution, one can read off the non-analytic result for the loop diagram using the coefficient of the singular pole term. We have explicitly checked our result for this diagram with the result derived in this fashion.

The above diagrams generate all the non-analytical contributions to the potential. There are other diagrams contributing to the 1-loop scattering matrix, but those diagrams will only give analytical contributions, so we will not discuss them here in much detail. Examples of such diagrams are shown, see figure 2.13.

The (diagram A) is a tadpole. Tadpole diagrams will never depend on the transverse momentum of the diagrams, and will thus never contribute with a non-analytical term. In fact, massless tadpoles will be zero in dimensional regularization. The (diagram B) is interesting — as it is of the same type as the diagrams 6(a-d), however with two massive propagators and one massless instead of two massless and one massive propagator. One can show that this diagram will not contribute with non-analytic terms, because such an integral with two massive denominators and one massless only will give analytical contributions. In the case of (diagram C), the loop is on one of the external legs. Hence the loop integrations will not depend on the interchanged momentum of the diagram. Thus it cannot give any non-analytical contributions to the potential.

2.9.2 The results for the diagrams in general relativity

The results in the pure general relativity case are more complicated to calculate than in the combined results of general relativity and scalar QED. This is basically because the
vertex rules are more complicated. The integrals and the algebraic contractions follow exactly the same principles. The 1-loop scattering diagrams for the pure gravitational theory will be presented here. Calculations have been done both by hand and by the computer algorithm for Maple. As most calculations follow the same principles as in the preceding section we will be more brief here and proceed rather directly to the results.

The discussion for the tree diagram is identical to the discussion in the previous section. So we will continue straight ahead with the results for the 1-loop diagrams.

The box and crossed box diagrams

In this subsection we will present the results for the box and crossed box diagrams. Graphically these diagrams can be depicted as, see figure 2.14. Formally we can write the

![Diagram](image)

Figure 2.14: The box and crossed box diagrams which go in to a calculation of the 1-loop scattering matrix potential.

contributions from these two diagrams in the following manner:

\[
M_{2a} = \int \frac{d^4l}{(2\pi)^4} \tau_6^{\mu\nu}(k_1, k_1 + l, m_1) \tau_6^\rho(1+l, k_2, m_1) \\
\times \tau_6^{\gamma\delta}(k_3, k_3 - l, m_2) \tau_6^\rho (k_3 - l, k_4, m_2) \\
\times \frac{i}{(k_1 + l)^2 - m_1^2} \left[ \frac{i}{(k_3 - l)^2 - m_2^2} \right] \frac{i}{l^2} \left[ \frac{i}{(l + q)^2} \right],
\]

(2.111)

for the box and:

\[
M_{2b} = \int \frac{d^4l}{(2\pi)^4} \tau_6^{\mu\nu}(k_1, k_1 + l, m_1) \tau_6^\rho(1+l, k_2, m_1) \\
\times \tau_6^{\gamma\delta}(k_3, l + k_4, m_2) \tau_6^\rho (l + k_4, k_4, m_2) \\
\times \frac{i}{(k_1 + l)^2 - m_1^2} \left[ \frac{i}{(l + k_4)^2 - m_2^2} \right] \frac{i}{l^2} \left[ \frac{i}{(l + q)^2} \right].
\]

(2.112)

Preforming all integrals by the same principles as in the the previous section, and taking the nonrelativistic limit, we end up with:

\[
V_{2(a)+2(b)}^{\text{tot}}(r) = -\frac{47}{3} \frac{m_1 m_2 G^2}{\pi r^3}.
\]

(2.113)

as the total contribution to the potential from the box and cross box diagrams. It is seen that we have agreement with the result of ref. [52].
2.9. THE RESULTS FOR THE FEYNMAN DIAGRAMS

The triangle diagrams

The triangle diagram we graphically express as, see figure 2.15. Formally we can write

\[
M_{3(a)}(q) = \int \frac{d^4l}{(2\pi)^4} \tau_6^{\mu\nu}(k_1, l + k_1, m_1) \tau_6^{\alpha\beta}(l + k_1, k_2, m_1) \tau_8^{\rho\gamma\delta}(k_3, k_4, m_2) \\
\times \left[ \frac{i\mathcal{P}_{\alpha\beta\gamma\delta}}{(l + q)^2} \right] \left[ \frac{i\mathcal{P}_{\mu\nu\rho}}{l^2} \right] \left[ \frac{i}{(l + k_1)^2 - m_1^2} \right],
\]

(2.114)

and

\[
M_{3(b)}(q) = \int \frac{d^4l}{(2\pi)^4} \tau_6^{\rho}(k_3, k_3 - l, m_2) \tau_8^{\gamma\delta}(k_3 - l, k_4, m_2) \tau_8^{\mu\alpha\beta}(k_1, k_2, m_1) \\
\times \left[ \frac{i\mathcal{P}_{\mu\nu\rho}}{l^2} \right] \left[ \frac{i\mathcal{P}_{\alpha\beta\gamma\delta}}{(l + q)^2} \right] \left[ \frac{i}{(l - k_3)^2 - m_2^2} \right].
\]

(2.115)

Again using the same type of calculations as previously and employing simplifications such as:

\[
\mathcal{P}_{\gamma\delta\rho\sigma} \mathcal{P}_{\alpha\beta\mu\nu} \tau_8^{\rho\mu\nu}(k_1, k_2, m_1) = \tau_8\gamma\delta\alpha\beta(k_1, k_2, m_1),
\]

(2.116)

we end up with the result:

\[
V_{3(a)+3(b)}(r) = -4 \frac{G^2 m_1 m_2 (m_1 + m_2)}{r^2} + 28 \frac{m_1 m_2 G^2}{\pi r^3},
\]

(2.117)

if we take the nonrelativistic limit and Fourier transform. The result match ref. [52].

The double-seagull diagram

The double-seagull diagram can be depicted graphically as in figure 2.16. Formally we can write this contribution as:

\[
M_{4(a)}(q) = \frac{1}{2!} \int \frac{d^4l}{(2\pi)^4} \tau_8^{\alpha\beta\gamma\delta}(k_1, k_2, m_1) \tau_8^{\rho\mu\nu}(k_3, k_4, m_2) \\
\times \left[ \frac{i\mathcal{P}_{\alpha\beta\mu\nu}}{(l + q)^2} \right] \left[ \frac{i\mathcal{P}_{\gamma\delta\rho\sigma}}{l^2} \right].
\]

(2.118)

It should be remembered that there is a symmetry factor of \(\frac{1}{2!}\) in the diagram. The result for the diagram is found rather straightforwardly and the resulting Fourier transformed amplitude is:

\[
V_{4(a)}(r) = -22 \frac{m_1 m_2 G^2}{\pi r^3}
\]

(2.119)

Also this result match ref. [52].
The vertex correction diagrams

The vertex correction diagrams are graphically represented, see figure 2.17. Two classes of diagrams go into the set of vertex corrections. There are two massive loop diagrams, which are shown in Figures 5(a) and 5(b). For these diagrams we can write:

\[
M_{5(a)}(q) = \frac{d^4l}{(2\pi)^4} \tau_6^{\alpha \beta}(k_1, k_2, m_1) \tau_6^{\mu \nu}(k_3, k_3 - l, m_2) \tau_6^{\rho \sigma}(k_3 - l, k_4, m_2) \tau_{10}^{\lambda \delta \phi \gamma}(l, -q) \times \left[ \frac{iP_{\lambda \rho \mu \nu}}{l^2} \right] \left[ \frac{iP_{\phi \epsilon \rho \sigma}}{(l + q)^2} \right] \left[ \frac{iP_{\alpha \beta \gamma \delta}}{q^2} \right] \left[ \frac{i}{(l - k_3)^2 - m_2^2} \right],
\]

and

\[
M_{5(b)}(q) = \int \frac{d^4l}{(2\pi)^4} \tau_6^{\alpha \beta}(k_1, l + k_1, m_1) \tau_6^{\mu \nu}(l + k_1, k_2, m_1) \tau_6^{\lambda \kappa}(k_3, k_4, m_2) \tau_{10}^{\gamma \delta \rho \phi}(l, -q) \times \left[ \frac{iP_{\alpha \beta \gamma \delta}}{l^2} \right] \left[ \frac{iP_{\mu \nu \rho \sigma}}{(l + q)^2} \right] \left[ \frac{iP_{\phi \epsilon \lambda \kappa}}{q^2} \right] \left[ \frac{i}{(l + k_1)^2 - m_1^2} \right].
\]
The other set of diagrams is shown in Figures 5(c) and 5(d), and can be written as:

\[ M_{5(c)}(q) = \frac{1}{2!} \int \frac{d^4l}{(2\pi)^4} \epsilon_8^{\lambda\kappa\nu\phi}(k_3, k_4, m_2) \tau_6^{\alpha\beta}(k_1, k_2, m_1) \epsilon_{10}^{\mu\nu\rho\sigma}(l, -q) \]
\[ \times \left[ \frac{i\mathcal{P}_{\mu\nu\lambda\kappa}}{l^2} \right] \left[ \frac{i\mathcal{P}_{\rho\sigma\epsilon\phi}}{(l + q)^2} \right] \left[ \frac{i\mathcal{P}_{\alpha\beta\gamma\delta}}{q^2} \right], \tag{2.122} \]

and

\[ M_{5(d)}(q) = \frac{1}{2!} \int \frac{d^4l}{(2\pi)^4} \epsilon_8^{\rho\sigma\mu\nu}(k_1, k_2, m_1) \epsilon_6^{\lambda\kappa}(k_3, k_4, m_2) \epsilon_{10}^{\alpha\beta\gamma\delta}(l, -q) \]
\[ \times \left[ \frac{i\mathcal{P}_{\mu\nu\gamma\delta}}{l^2} \right] \left[ \frac{i\mathcal{P}_{\rho\sigma\alpha\beta}}{(l + q)^2} \right] \left[ \frac{i\mathcal{P}_{\lambda\kappa\phi}}{q^2} \right]. \tag{2.123} \]

These diagrams are among the most complicated to calculate. The first results for these diagrams date back to the original calculation of refs. [32, 33] — but because of an algebraic error in the calculation, the original result was in error and despite various checks of the calculation refs. [50, 52] the correct result has not been given until ref. [37, 51]. Quoting ref. [37], taking the nonrelativistic limit and Fourier transforming, we get:

\[ V_{5(a)+5(b)}(r) = \frac{G^2 m_1 m_2 (m_1 + m_2)}{r^2} - \frac{5 m_1 m_2 G^2}{3 \pi r^3}, \tag{2.124} \]

and

\[ V_{5(c)+5(d)}(r) = \frac{26 m_1 m_2 G^2}{3 \pi r^3}. \tag{2.125} \]

For the diagrams 5(c) and 5(d) a symmetry factor \( \frac{1}{2!} \) have been included. These results have recently been confirmed in the independent paper ref. [57].

### 2.10 The result for the potential

We will first discuss the results for the scattering potentials. Adding everything up in the scalar QED case, the final result for the potential reads:

\[ V(r) = -\frac{G m_1 m_2}{r} + \frac{\tilde{\alpha} \tilde{e}_1 \tilde{e}_2}{r} + \frac{1}{2} \left( \frac{m_1 \tilde{e}_2^2 + m_2 \tilde{e}_1^2}{c^2 r^2} \right) G \tilde{\alpha} + \frac{3 \tilde{e}_1 \tilde{e}_2 (m_1 + m_2) G \tilde{\alpha}}{c^2 r^2} \]
\[ - \frac{4}{3} \left( \frac{m_1^2 \tilde{e}_2^2 + m_2^2 \tilde{e}_1^2}{m_1 m_2} \right) G \tilde{h} \pi c^3 r^3 - \frac{8 \tilde{e}_1 \tilde{e}_2 G \tilde{\alpha} \tilde{h}}{\pi c^3 r^3}. \tag{2.126} \]

In the above expression we have included the appropriate physical factors of \( \tilde{h} \) and \( \tilde{c} \), and rescaled the equation in terms of \( \tilde{\alpha} = \frac{h \pi}{137} \). The charges \( \tilde{e}_1 \) and \( \tilde{e}_2 \) are normalized in units of the elementary charge.

It is seen that in the above expression for the potential there many different terms. The first two terms are, as noted before, the well-known Newtonian and Coulomb terms. They represent the lowest order interactions of the two sources without the inclusion of radiative
corrections. These terms come out of the formalism as expected, and they will dominate the potential at sufficiently low energies.

The classical post-Newtonian corrections to the potential are represented by the two next terms, and they can also be derived from general relativity with the inclusion of charged particles.

The last two contributions are the most interesting from a quantum point of view. They represent the leading 1-loop quantum corrections to the mixed theory of general relativity and scalar QED, and they were computed in ref. [35] for the first time. It is seen in SI units \( \left( \frac{\hbar c^2}{\pi} \sim 10^{-70} \text{ meters}^2 \right) \). Therefore these corrections are thus indeed very small. Seemingly they will be impossible to detect experimentally. This is especially due to the large numerical contributions usually coming from the Coulomb and Newtonian terms.

When looking at the expression for the potential, it is noticed that the corrections to the potential can be divided into two classes of terms. There are terms with the two charges multiplied together, and terms which contain a sum of squares of each charge. For identical particles the two types of terms must be exactly the same in form and in this case one should also include the appropriate diagrams with crossed particle lines. Our result hold for nonidentical particles only.

In the case of one of the masses or charges being either very large or very high respectively, some of the contributions in the potential will dominate over others. The terms with separated charges will correspond to the dominating terms, if one of the scattered masses or charges were much larger than other. In this case the gravitational vertex corrections will generate the dominating leading contribution to the potential. For example with a very high charge for one of the particles — the probing particle will fell most of the gravitational effect coming from the electromagnetic field surrounding the heavily charged particle.

For a very large mass: \( (M \sim 10^{30} \text{ kg}) \) but a very low charged particle: \( (\tilde{e}_1 \sim 0) \) (the Sun), and a charged: \( (\tilde{e}_2 \sim 1) \) but very low mass particle: \( (m \sim 10^{-31} \text{ kg}) \) (an electron), one could perhaps test this effect experimentally, because then the Newton effect is: \( \left( G m M \sim \frac{10^{-10}}{r} \text{ J-meters} \right) \), while the quantum effect is: \( \left( G \hbar \tilde{e}_2^2 M c^3 = \frac{10^{-37}}{r^3} \text{ J-meters}^3 \right) \), and the classical contribution: \( \left( G M \tilde{e}_2^2 c^2 = \frac{10^{-25}}{r^4} \text{ J-meters}^2 \right) \). The ratio between the post-Newtonian effects and the quantum correction is for this experimental setup still very large, but not quite as impossible as often seen in quantum gravity.

We also have a result for the total potential in the case of pure gravitational interactions. If we add up all the corrections to the scattering potential, we arrive at:

\[
V(r) = -\frac{G m_1 m_2}{r} \left[ 1 + 3 \frac{G(m_1 + m_2)}{r} + \frac{41 \ G \hbar}{10\pi \ r^2} \right].
\]  

(2.127)

It should be noticed that the classical post-Newtonian term in this expression is in an agreement with (Eq. 2.5) of Iwasaki ref. [13]. This result was first published in ref. [38].

The quantum correction is not in agreement with the results refs. [32, 33, 48, 49, 50, 52] but it is believed that the above result presents the correct result for the scattering matrix potential, because all calculations have been carried out extremely carefully and because all results are done both by hand and by computer. The recently published paper ref. [57] confirms all calculations.
We can here again easily identify the post-Newtonian correction as well as the quantum correction and it is seen that a direct experimental verification of the quantum corrections will indeed be very difficult.

### 2.10.1 A Potential Ambiguity

A result for the post-Newtonian corrections derived from classical considerations can be found in ref. [54]. In the scalar QED case, they read in our notation:

\[
V_{\text{post-Newtonian}} = \frac{1}{2} \left( \frac{m_1 \tilde{e}_2^2 + m_2 \tilde{e}_1^2}{c^2 r^2} \right) G \tilde{\alpha} + (\alpha_p + \alpha_g - 1) \frac{\tilde{e}_1 \tilde{e}_2 (m_1 + m_2) G \tilde{\alpha}}{c^2 r^2} .
\]  

(2.128)

In the pure gravitational case the expected post-Newtonian terms are:

\[
V_{\text{post-Newtonian}} = \left( \frac{1}{2} - \alpha_g \right) \frac{G^2 m_1 m_2 (m_1 + m_2)}{c^2 r^2} .
\]  

(2.129)

It is seen that the expectations for the classical post-Newtonian correction terms exactly match the ones derived, and that the coefficient in front of the first term is equal to ours. The result for the second term is equivalent in form to the term we have derived, and the coefficient can be made to match our result for particular values of \((\alpha_p)\) and \((\alpha_g)\). The physical significance of the arbitrary parameters \((\alpha_g)\) and \((\alpha_p)\), however requires some explanation.

The parameter \((\alpha_g)\) is related to the gravitational propagator, while the coefficient \((\alpha_p)\) is related to the photonic propagator and the values of \((\alpha_p)\) and \((\alpha_g)\) can be seen as coordinate dependent coefficients for the potential. They can take on arbitrary values depending on the coordinate system chosen to represent the potential. This ambiguity is built into the nonrelativistic potential refs. [13, 54, 55, 56].

Despite the above references the issue concerning the ambiguity of the classical potential have so far not created much attention. The references [54, 55, 56], deal directly with the post-Newtonian contributions and their arguments for these terms can be used to look at the ambiguity of the quantum terms as well. We will here discuss the ambiguity of the potential and observe that the quantum corrections at 1-loop order are well defined and unambiguous.

In ref. [56] the terms present in the classical post-Newtonian potential are recast under coordinate transformations such as:

\[
r \rightarrow r \left[ 1 + \alpha \frac{G(m_1 + m_2)}{r} \right] .
\]  

(2.130)

Such a coordinate change can always freely be made, and the classical terms will be modified in the following manner:

\[
\frac{G m_1 m_2}{r} \left[ 1 + \frac{c G(m_1 + m_2)}{r} \right] \rightarrow \frac{G m_1 m_2}{r} \left[ 1 + \left( c - \alpha \right) \frac{G(m_1 + m_2)}{r} \right] .
\]  

(2.131)
If we follow refs. [56, 55] the ambiguity in defining the coordinates arises from a freedom in the propagator:

\[
\frac{1}{q^2} = \frac{1}{q_0^2 - q^2}.
\] (2.132)

Using the energy-conservation, we can namely redefine it for a general \( x \):

\[
\frac{1}{2}(1 - x)((p_{20} - p_{10})^2 + (p_{40} - p_{30})^2) - x(p_{20} - p_{10})(p_{40} - p_{30}) - q^2,
\] (2.133)

and using this propagator in the derivations, affects the gravitational potential, so that it will generally depend on \( x \). The transformation coefficient \( \alpha \) will depend on \( x \), and in our notation, \( \alpha = -\frac{1}{4}(1 - x) \). The terms \( \left( \frac{Gm}{r^2} \right) \) as well as \( \left( \frac{Gm}{r} \right) \left( \frac{Gm}{r} \right) \), will therefore generically in the potential have a dependence on \( x \). The post-Newtonian parts in the static gravitational potential are hence not well-defined.

Next we can look at the coordinate ambiguity from the point of view of the quantum corrections. The coordinate redefinition:

\[
r \rightarrow r \left[ 1 + \beta \frac{G\hbar}{r^2} \right],
\] (2.134)

can be considered. Such a transformation will change the quantum coefficient in the potential in the following way:

\[
\frac{Gm_1m_2}{r} \left[ 1 + d \frac{G\hbar}{r^2} \right] \rightarrow \frac{Gm_1m_2}{r} \left[ 1 + (d - \beta) \frac{G\hbar}{r^2} \right].
\] (2.135)

It is hence seen that quantum modifications of the potential can arise, so is the quantum correction of the potential unique? That is what we will address next.

General relativity is a theory based on the invariance under coordinate transformations. If we make a coordinate shift of the potential which generates a modification, a compensating modification of another term must be present to leave the physics invariant. In other words in a Hamiltonian formulation a modification of the post-Newtonian terms in the potential must be compensated by a change of the kinetic pieces in the Hamiltonian.

We look first at the classical ambiguity. Two dimensionless variables can be constructed as possible generators of coordinate transformations:

\[
\frac{p^2}{m^2}, \quad \frac{Gm}{r}.
\] (2.136)

At leading order these pieces appear at same order in the Hamiltonian:

\[
H_1 \sim \frac{p^2}{m}, \quad \frac{Gm^2}{r}.
\] (2.137)

To leading order these two terms are unambiguous. At next order we have:

\[
H_2 \sim \frac{p^4}{m^4}, \quad \frac{Gm^2 Gm}{r^3}, \quad \left( \frac{Gm}{r} \right) \left( \frac{p^2}{m} \right).
\] (2.138)
and now interestingly a coordinate change ambiguity between the two last terms arises. As shown explicitly in ref. [56] this ambiguity disappear for physical observables. In the center of mass frame we can write the Hamiltonian to post-Newtonian order in the following way:

\[
H = \left( \frac{p^2}{2m_1} + \frac{p^2}{2m_2} \right) - \frac{Gm_1m_2}{r} \left[ 1 + a \frac{p^2}{m_1m_2} + b \left( \frac{p \cdot \hat{r}}{m_1m_2} \right)^2 + c \frac{G(m_1 + m_2)}{r} \right].
\] (2.139)

The standard Einstein-Infeld-Hoffmann coordinates \((a, b, c)\) will have the values:

\[
a = \frac{1}{2} \left[ 1 + 3 \left( \frac{m_1 + m_2}{m_1m_2} \right)^2 \right], \\
b = \frac{1}{2}, \\
c = -\frac{1}{2}.
\] (2.140)

and making coordinate transformations such as of the type in Eq. 2.130, one arrives at:

\[
a \rightarrow \frac{1}{2} \left[ 1 + (3 + 2\alpha) \left( \frac{m_1 + m_2}{m_1m_2} \right)^2 \right], \\
b \rightarrow \frac{1}{2} - \alpha \left( \frac{m_1 + m_2}{m_1m_2} \right)^2, \\
c \rightarrow -\frac{1}{2} - \alpha.
\] (2.141)

Hence in in the case of the classical corrections to a static potential, it is necessary to specify the momentum coordinates as well. In particular if we look at the potential presented by Iwasaki, it will be appropriate for the choice of Einstein-Infeld-Hoffmann coordinates.

At the next order there will be the terms:

\[
H_3 \sim \frac{p^6}{m^5}, \quad \frac{p^4}{m^3} \left( \frac{Gm}{r} \right), \quad \frac{p^2}{m} \left( \frac{Gm}{r} \right)^2, \quad \frac{Gm^2}{r} \left( \frac{Gm}{r} \right)^2,
\] (2.142)

in the Hamiltonian. The term which goes as \((\frac{1}{r^3})\) has the same radial dependence as the quantum term. However its mass dependence is wrong and it is of order \((G^3)\). Without going into details we note that such terms will also be ambiguous under coordinate transformations. However cancellations have to appear among the terms of the same order. Hence a coordinate change affects the \((\frac{1}{r^3})\) term in the potential, but classical terms will cancel among themselves and not influence the quantum terms.

The quantum effects are next dealt with. The generator of coordinate transformations for the quantum contributions is:

\[
\frac{G\hbar}{r^2}.
\] (2.143)
Using the same principles as for the classical terms we can generate theoretical quantum contributions to the Hamiltonian, and such terms will be:

\[ H_q = \frac{G m^2 G h}{r^2}, \]

(2.144)

together with:

\[ H_{qp} \sim \frac{p^2 G h}{m r^2}, \]

(2.145)

to first order in \( \hbar \). A quantum change of coordinates, \textit{i.e.}, making a coordinate transformation as in Eq. 2.134 will definitely generate a term such as \( H_q \) in the Hamiltonian, but in order to cancel this effect for physical observables a term \( H_{qp} \) must be introduced as well. However a term such as \( H_{qp} \) cannot exist in a two-particle potential for any choice of coordinates, simply because it involves only a single power of \( G \). A single power of \( G \) is possible for one particle quantum observables, but never for observables involving two particles.

Since no term are of the form \( H_{qp} \), the quantum contribution must be unique and specified in coordinates which make \( H_{qp} \) vanish. Hence at 1-loop order the quantum potential is a well defined quantity. Without going into details we note that this must too be the case for the quantum corrections to the scalar QED potential.

### 2.11 Quantum corrections to the metric

Another way to look at the 1-loop quantum corrections is to consider the metric instead of a potential.

In ref. [23] quantum corrections to the Schwarzschild solution are discussed from the viewpoint of including only the vacuum polarization diagram. This is correct if the source is only classical, meaning that radiative corrections to the vertex is not included. However, any matter source will emit gravitons and these fields will have quantum components. An effective treatment of the 1-loop scattering diagrams illustrates the importance of including the radiative quantum corrections caused by the vertex diagrams.

The diagrams which contribute to the metric corrections are only a subset of the full scattering matrix.

We can use this set of the one-particle reducible diagrams to define a potential. It results...
in:
\[ V(r) = -\frac{Gm_1m_2}{r} \left( 1 - \frac{G(m_1 + m_2)}{r} - \frac{167 G\hbar}{30\pi r^2} + \ldots \right). \]
(2.146)

The potential obtained this way is not a scattering matrix potential, but it suggests a way to define a harmonic gauge running gravitational coupling constant:
\[ G(r) = G \left( 1 - \frac{167 G\hbar}{30\pi r^2} \right). \]
(2.147)

The running gravitational coupling is seen to be independent of the masses of the involved objects. Hence it has the expected universal character which a running charge should have. Its independence of spin as can be seen from the fermionic data.

At short distances the gravitational running constant will be weaker. This can be seen as a sort of gravitational screening. At large distances the source, when probed, will occur more point-like, while at small distances the gravitational field smears out.

Experimental verifications of general relativity as an effective field will perhaps be a very difficult task. The problem is caused by the normally very large classical expectations of the theory. These expectations imply that nearly any quantum effect in powers of \((G\hbar)\) will be nearly neglectable compared to \((G)\). Therefore the quantum effects will be very hard to extract, using measurements where classical expectations are involved. The solution to this obstacle could be to magnify a certain quantum effect. This could be in cases where the classical effects were independent of the energy scale, but where the quantum effects were largely effected by the energy scale. Such effects would only be observed, when very large interaction energies are involved.

Another way to observe a quantum effect could be when a certain classical expectation is zero, but the quantum effect would yield a contribution. A quantum gravitational "anomaly". Such 'null' experiments maybe used to test a quantum theory for gravity.

### 2.12 Discussion

Normally general relativity is viewed as a non-renormalizable theory, and consequently a quantum theory for general relativity is believed to be an inconsistent theory. However, treated as an effective field theory, the renormalization inconsistency of general relativity is not an issue — as the theory can be explicitly renormalized at any given loop order. This fact was first explored in refs. [32, 33]. Here we have presented three recent works which address general relativity from the view point of treating it as an effective field theory. We have derived the quantum and post-Newtonian corrections to the Schwarzschild and Kerr metric, discussed general relativity and the combined theory of general relativity and scalar QED, and extracted useful information about the quantum and post-Newtonian long-distance behavior of the gravitational attraction. Most important we have directly observed that it is possible in this perspective to extract information about quantum gravity. The found quantum corrections to the metric and to the scattering matrix potential are unique and exact signatures of the quantum nature of gravity.

The effective field theory approach is only valid at sufficiently low energies, \(i.e.,\) below the Planck scale \(\sim 10^{19}\) GeV, and at long distances. Below these scales the effective field
theory approach present a nice and good way to handle the challenges of quantum gravity. At higher energies the effective action will break down and have to be replaced by a new unknown theory. It will govern the quantum gravitational effects and might introduce completely new aspects of physics. Present thoughts circle around a string or $M$-theory, which is compactified at low energies. However it is important to notice that the Planck energy scale is much larger than ‘traditional’ high-energy scales. Optimistic predictions expect a limiting scale for the Standard Model around $\sim 10^3 \text{ GeV}$. In this light the effective field theory approach is seemingly good for all energies presently dealt with in high energy physics.

Experiments to verify quantum gravity are difficult to propose. This is both because quantum gravity effects are so small at normal energies and because classical effects normally are rather large. The nonrelativistic potential may not be the best offset for an experimental verification of quantum gravity – however in the quest for an experimental verification of quantum gravity it is very interesting that quantum effects can be calculated and that such corrections are unique quantum fingerprints of the nature of gravity.
Chapter 3

String theory and effective quantum gravity

3.1 Introduction

In this section we are discussing the thesis work which resulted in the two papers refs. [39, 40]. They survey the implications of imposing the Kawai, Lewellen and Tye (KLT) string relations ref. [67] in the effective field theory limit of open and closed strings. At sufficiently low energy scales the tree scattering amplitudes derived from string theory are reproduced exactly by the tree scattering amplitudes, generated from an effective field theory description with an infinite number of higher derivative terms ordered in a series in powers of the string tension $T \sim \frac{1}{\alpha'}$, see refs. [68, 69, 70, 73, 74, 75, 76, 77]. As it probably is well known that closed strings have a natural interpretation as fundamental theories of gravity – while open string theories on the other hand turn up as non-abelian Yang-Mills vector field theories, the imposed KLT gravity/gauge correspondence between the tree-amplitudes will create a link between gravity and Yang-Mills theories. The induced connection between the two effective Lagrangians is the turning point of the investigations.

At the tree level the KLT string relations connect on-shell scattering amplitudes for closed strings with products of left/right moving amplitudes for open strings. This was first applied in ref. [78] to gain useful information about tree gravity scattering amplitudes from the much simpler Yang-Mills tree amplitudes. The KLT-relations between scattering amplitudes alone give a rather remarkable non-trivial link between string theories, but in the field theory regime the existence of such relationships is almost astonishing, and nonetheless valid.

Yang-Mills theory and general relativity being both non-abelian gauge theories have some resemblance, but their dynamical and limiting behaviors at high-energy scales are in fact quite dissimilar. One of the key results of the paper ref. [39] was to demonstrate that once the KLT-relations were imposed, they uniquely relate the tree limits of the generic effective Lagrangians of the two theories. The generic effective Lagrangian for gravity will have a restricting tree level connection to that of Yang-Mills theory, through the KLT-relations. In the language of field theories the KLT-relations present us with a link between the on-shell field operators in a gravitational theory as a product of Yang-Mills
field theory operators.

Even though the KLT-relations hold only explicitly at the tree level the factorization of gravity amplitudes into Yang-Mills amplitudes can be applied with great success in loop calculations. This has been the focus of various investigations, and much work have been carried out, e.g., see refs. [79, 80, 81, 82]. Loop diagram cuts together with properties such as unitarity of the S-matrix are used to calculate the results of loop diagrams. A good review and a demonstration of such calculations in QCD is ref. [83]. See also refs. [84, 85]. These methods can be employed in many ways, and as the methods are quite general they apply in many cases, e.g., complicated gluon amplitudes are calculated through cuts of diagrams, and employing the KLT-relations this can be used to derive results for gravity loop diagrams. An important result derived this way was that N=8 supergravity, is a less divergent theory in the ultraviolet than has previously been appreciated. The KLT formalism allows for the presence of matter sources. They can be introduced in the KLT formalism as done in ref. [86]. In the papers ref. [39, 40] we did not consider loop extensions, neither the inclusion of matter sources, instead our focus was on the actual factorization properties of tree amplitudes.

At infinite string tension the KLT-mapping simply factorizes the amplitudes of the Einstein-Hilbert action:

\[ \mathcal{L}_{EH} = \int d^4x \left[ \sqrt{-g} R \right], \]

into a sum of amplitude products of a 'left' and a 'right' moving Yang-Mills action:

\[ \mathcal{L}_{YM} = \int d^4x \left[ -\frac{1}{8g^2} \text{tr} F_{\mu\nu} F_{\mu\nu} \right], \]

where \((A_\mu)\) is the vector field and \((F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu])\). The factorizations at the amplitude level have been investigated as a method to explore if gravity vertices might be factorizable into vector vertices at the Lagrangian level, see ref. [87]. Remarkable factorizations of gravity vertices have already been presented in ref. [88] using a certain vierbein formalism. The idea that gravity could be factorizable into a product of independent Yang-Mills theories is indeed very beautiful, the product of two independent Yang-Mills vectors \((A_\mu^L)\) and \((A_\mu^R)\) (without internal contractions and interchanges of vector indices) should then be related to the gravitational tensor field \((g_{\mu\nu})\) through KLT:

\[ A_\mu^L \times A_\mu^R \sim g^{\mu\nu}, \]

however such a direct relationship, has so far only been realized at the tree amplitude level. At the Lagrangian (vertex) level gauge issues mix in and complicate the matter. Depending on the particular gauge choice, e.g., the vertices can take different forms. Whether this can be resolved or not – and how – remains essentially to be understood. We did not go into the issues about factorizations of vertices in the papers refs. [39, 40], where the focus was more about gaining insight into the tree level factorizations of the amplitudes derived from the effective actions of general relativity and Yang-Mills at order \(O(\alpha'^3)\).

As we have seen in the previous chapter, despite common belief, string theory is not the only possible starting point for the safe arrival at a successful quantum theory for gravity below the Planck scale.
An effective field theory appears to be the obvious scene for quantum gravity at normal energies—i.e., below $10^{19}$ GeV. The idea behind the effective field theory approach is so broad and generic that it appears to be the obvious extension of general relativity at low-energy scales. At the Planck energy a phase transition is believed to take place, leaving us in dark about what takes place at very high energies. String theories can be the answer here but other theories might be possible to construct. From the effective field theory point of view there is no need—however—to assume that the high-energy theory a priori should be a string theory. The effective field theory viewpoint allows for a broader range of possibilities, than that imposed by a specific field theory limit of a string theory. We will see that this also holds true in respect to imposing the KLT-relations. The effective field approach for the Yang-Mills theory in four dimensions is not really necessary, because it is already a renormalizable theory—but no principles are forbidding us to include additional terms in the non-abelian action and this is anyway needed for a D-dimensional ($D > 4$) Yang-Mills Lagrangian. By modern field theory principles the non-abelian Yang-Mills action can always be regarded as an effective field theory and treated as such in calculations.

Our aim was to gain additional insight in the mapping process and to investigate the options of extending the KLT relations between scattering amplitudes at the effective field theory level. Profound relations between effective gravity and Yang-Mills diagrams and between pure effective Yang-Mills diagrams was found in this process and will be presented in this chapter. To which level string theory actually is needed in the mapping process will be another aspect of the investigations.

The KLT-relations also work for mixed closed and open string amplitudes, the so called heterotic string modes. In order to explore such possibilities an antisymmetric term will be augmented to the gravitational effective action. It is needed to allow for mixed string modes ref. [75].

This chapter is organized as follows. First we review some material about scattering amplitudes. We will next briefly discuss the theoretical background for the KLT-relations in string theory and make a generalization of the mapping. The mapping solution is then presented, and we give some beautiful diagrammatic relationships generated by the mapping solution. These relations not only hold between gravity and Yang-Mills theory, but also in pure Yang-Mills theory itself. We end the chapter with a discussion about the implications of such mapping relations between gravity and gauge theory and try to look ahead. The same conventions as in the papers refs. [39, 40] will be used here, i.e., metric $(+ − − −)$ and units $(c = \hbar = 1)$.

### 3.2 The effective actions

In this section we will review the main results of refs. [39, 40, 68, 69, 70]. The Einstein-Hilbert action is defined as previously and reads:

$$\mathcal{L} = \sqrt{-g}\left[\frac{2R}{\kappa^2}\right],$$

where $(g_{\mu\nu})$ denotes the gravitational field, $(g \equiv \text{det}(g_{\mu\nu}))$ and $(R)$ is the scalar curvature tensor. $(\kappa^2 \equiv 32\pi G)$, where $(G)$ is the gravitational constant. The convention,
For the theory of Yang-Mills non-abelian vector field theory, the pure Lagrangian for the non-abelian vector field can be written:

$$\mathcal{L} = -\frac{1}{8g^2} \text{tr} F_{\mu\nu}^2,$$

where \((A_\mu)\) is the vector field and \((F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu])\). The trace in the Lagrangian is over the generators of the non-abelian Lie-algebra, e.g., \((\text{tr}(T^a T^b T^c) F_{\mu\lambda}^a F_{\nu\rho}^b F_{\gamma\delta}^c)\), where the \((T^a)\)'s are the generators of the algebra.' The convention \((g = 1)\) will also be employed, as this is convenient when working with the KLT-relations. The above non-abelian Yang-Mills action is of course renormalizable at \((D = 4)\), but it still makes sense to treat it as the minimal Lagrangian in an effective field theory. In the modern view of field theory, any invariant obeying the underlying basic field symmetries, such as local gauge invariance, should in principle always be allowed into the Lagrangian.

In order to generate a scattering amplitude we need expressions for the generic effective Lagrangians in gravity and Yang-Mills theory. In principle any higher derivative gauge or reparametrization invariant term could be included – but it turns out that some terms will be ambiguous under field redefinitions such as:

$$\delta g_{\mu\nu} = a_1 R_{\mu\nu} + a_2 g_{\mu\nu} R \ldots$$

$$\delta A_\mu = \tilde{a}_1 D_\nu F_{\nu\mu} + \ldots,$$

where \((\mathcal{D})\) is the gauge theory covariant derivative.

Following ref. [68] the effective non-abelian Yang-Mills action for a massless vector field takes the generic form including all independent invariant field operators:

$$\mathcal{L} = -\frac{1}{8} \text{tr} \left[ F_{\mu\nu}^2 + \alpha' (a_1 F_{\mu\lambda} F_{\lambda\nu} F_{\nu\mu} + a_2 D_\lambda F_{\lambda\mu} D_\rho F_{\rho\mu}) \right.$$ 

$$\left. + (\alpha')^2 (a_3 F_{\mu\lambda} F_{\nu\lambda} F_{\nu\mu} + a_4 F_{\mu\lambda} F_{\nu\rho} F_{\rho\lambda} F_{\nu\mu} + a_5 F_{\mu\nu} F_{\lambda\rho} F_{\lambda\rho} + a_6 F_{\mu\nu} F_{\lambda\rho} F_{\nu\rho} F_{\lambda\mu} + a_7 F_{\mu\nu} D_\lambda F_{\nu\lambda} D_\rho F_{\rho\lambda} F_{\lambda\mu}) \right. 
$$

$$\left. + a_8 D_\lambda F_{\mu\lambda} D_\rho F_{\rho\mu} F_{\lambda\mu} + a_9 D_\rho D_\lambda F_{\lambda\mu} D_\rho D_\sigma F_{\rho\sigma} \right) + \ldots \right], \tag{3.7}$$

where \((\alpha')\) orders the derivative expansion. In string theory \((-\frac{1}{\alpha'})\) will of course be the string tension. \((F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])\) and \((\mathcal{D})\) is the non-abelian covariant derivative.

In general relativity a similar situation occurs for the effective action, following ref. [69] the most general Lagrangian one can consider is:

$$\mathcal{L} = \frac{2\sqrt{-g}}{\kappa^2} \left[ R + \alpha' (a_1 R_{\lambda\mu\nu\rho}^2 + a_2 R_{\mu\nu}^2 + a_3 R^2) \right.$$ 

$$+ (\alpha')^2 (b_1 R_{\alpha\beta\lambda\rho} R_{\alpha\beta\lambda\rho} R_{\lambda\rho}^2 + b_2 R_{\alpha\beta\lambda\rho} R_{\lambda\rho}^2 + b_3 R_{\mu\lambda\nu\gamma} R_{\mu\lambda\nu\gamma} + b_4 R_{\mu\lambda\nu\rho} R_{\mu\lambda\nu\rho} R_{\mu\lambda\nu\rho} \right.$$ 

$$- 2 R_{\mu\nu\beta\gamma} R_{\mu\nu\beta\gamma}^2 + b_6 R_{\mu\lambda\nu\rho} R_{\mu\lambda\nu\rho} + b_7 R_{\mu\lambda\nu\rho} R_{\mu\lambda\nu\rho} R_{\mu\lambda\nu\rho} + b_8 R_{\mu\lambda\nu\rho} R_{\mu\lambda\nu\rho} R_{\mu\lambda\nu\rho} \right.$$ 

$$+ b_9 R_{\mu\lambda\nu\rho} R_{\mu\lambda\nu\rho} + b_{10} R_{\mu\lambda\nu\rho} R_{\mu\lambda\nu\rho} + b_{11} R_{\mu\lambda\nu\rho} R_{\mu\lambda\nu\rho} R_{\mu\lambda\nu\rho} \right] \right). \tag{3.8}$$

---

1The notation for the Yang-Mills action, i.e., the \(-\frac{1}{8} \ldots\) follows the non-standard conventions of ref. [68]. Please note that in this letter the action is not multiplied with \((2\pi\alpha')^2\).
As the S-matrix is manifestly invariant under a field redefinition such ambiguous terms need not to be included in the generic Lagrangian which give rise to the scattering amplitude, see also refs. [71, 73]. The Lagrangian obtained from the reparametrization invariant terms alone will be sufficient, because we are solely considering scattering amplitudes. Including only field redefinition invariant contributions in the action one can write for the on-shell action of the gravitational fields (to order \(O(\alpha'^3)\)):

\[
\mathcal{L} = \frac{2\sqrt{-g}}{\kappa^2} \left[ R + \alpha' (a_1 R_{\mu
u}^2) + (\alpha')^2 \left( b_1 R_{\alpha\beta}^\mu R_{\lambda\rho}^\nu R_{\mu\nu}^\alpha R_{\lambda\rho}^\beta + b_2 (R_{\alpha\beta}^\mu R_{\lambda\rho}^\nu R_{\mu\nu}^\alpha R_{\lambda\rho}^\beta) - 2 R_{\alpha\beta}^\mu R_{\lambda\rho}^\nu R_{\lambda\rho}^\beta R_{\mu
u}^\alpha + \ldots \right) \right],
\]  

(3.9)

while the on-shell effective action in the Yang-Mills case (to order \(O(\alpha'^3)\)) has the form:

\[
\mathcal{L}_{YM}^L = -\frac{1}{8} \text{tr} \left[ F_{\mu\nu}^L F_{\mu\nu}^L + \alpha' (a_1 F_{\mu\lambda}^L F_{\nu\rho}^L F_{\mu\nu}^L) + (\alpha')^2 (a_3 F_{\mu\lambda}^L F_{\nu\rho}^L F_{\mu\nu}^L F_{\nu\rho}^L + a_4 F_{\mu\lambda}^L F_{\nu\rho}^L F_{\nu\rho}^L F_{\mu\nu}^L + a_5 F_{\mu\lambda}^L F_{\nu\rho}^L F_{\nu\rho}^L F_{\mu\nu}^L + \ldots) \right].
\]

(3.10)

The \((L)\) in the above equation states that this is the effective Lagrangian for a 'left' moving vector field. To the 'left' field action is associated an independent 'right' moving action, where 'left' and 'right' coefficients are treated as generally dissimilar. The 'left' and the 'right' theories are completely disconnected theories, and have no interactions. The Yang-Mills coupling constant in the above equation is left out for simplicity. Four additional double trace terms at order \(O(\alpha'^3)\) have been neglected, see ref. [89]. These terms have a different Chan-Paton structure than the single trace terms. We need to augment the gravitational effective Lagrangian in order to allow for the heterotic string ref. [75]. As shown in ref. [69], one should in that case add an antisymmetric tensor field coupling:

\[
\mathcal{L}_{\text{Anti}} = \frac{2\sqrt{-g}}{\kappa^2} \left[ -\frac{3}{4} (\partial_{\mu} B_{\nu\rho} + \alpha' \omega_{\mu}^{ab} R_{\nu\rho}^{ab} + \ldots) (\partial_{(\mu} B_{\nu\rho)} + \alpha' \omega_{\mu}^{cd} R_{\nu\rho}^{cd} + \ldots) \right],
\]

(3.11)

where \((\omega_{\mu}^{ab})\) represents the spin connection.

This term will contribute to the 4-graviton amplitude. We will observe how such a term affects the mapping solution, and forces the 'left' and 'right' coefficients for the Yang-Mills Lagrangians to be different.\(^3\)

From the effective Lagrangians above one can expand the field invariants and thereby derive the scattering amplitudes. The results are:

\[
R = -\frac{\kappa^3}{4} h_{\alpha\beta} (h_{\lambda\rho} \partial_\alpha \partial_\beta h_{\lambda\rho} + 2 \partial_\alpha h_{\lambda\rho} \partial_\rho h_{\beta\lambda}) + \ldots,
\]

(3.12)

\(^2\)For completeness we note; in, e.g., six-dimensions there exist an integral relation linking the term \((R_{\alpha\beta}^\mu R_{\lambda\rho}^\nu R_{\mu\nu}^\alpha R_{\lambda\rho}^\beta)\) with the term \((R_{\alpha\beta}^\mu R_{\lambda\rho}^\nu R_{\nu\rho}^\alpha R_{\lambda\gamma}^\beta)\), in four-dimensions there exist an algebraic relation also linking these terms, see ref. [90]. For simplicity we have included all the reparametrization invariant terms in our approach and not looked into the possibility that some of these terms actually might be further related through an algebraic or integral relation.

\(^3\)See, e.g., ref. [91], for a discussion of field redefinitions and the low-energy effective action for Heterotic strings.
\[ R^2_{\mu\nu\rho} - 4R^2_{\mu\nu} + R^2 = -\kappa^3 h_{\mu\nu} \partial_\mu h_{\alpha\beta} \partial_\alpha h_{\nu\rho} + \ldots, \]  
\[ R^\alpha_{\mu\nu} R^\beta_{\lambda\rho} R^\lambda_{\mu\nu} = -\kappa^3 \partial_\rho h_{\beta\nu} \partial_\lambda h_{\mu\rho} \partial_\mu h_{\nu\rho} + \ldots, \]

where \( (g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}) \), and \( (\eta_{\mu\nu}) \) is the flat metric. Because of the flat background metric no distinctions are made of up and down indices on the right hand side of the equations, so indices can be put where convenient. The results are identical to those of ref. [69]. This gives the following generic 3-point scattering amplitude for the effective action of general relativity:

\[
M_3 = \kappa \left[ \zeta^{\mu\sigma} \xi^{\rho\lambda} (\zeta_1^{\alpha\beta} k_1^{\gamma} \delta^{\sigma\rho} + \zeta_1^{\alpha\gamma} k_1^{\beta} \delta^{\sigma\rho} + \zeta_1^{\beta\gamma} k_1^{\alpha} \delta^{\rho\sigma}) + \zeta_1^{\mu\rho} \xi^{\lambda\sigma} (\zeta_2^{\alpha\beta} k_2^{\gamma} \delta^{\rho\sigma} + \zeta_2^{\alpha\gamma} k_2^{\beta} \delta^{\rho\sigma} + \zeta_2^{\beta\gamma} k_2^{\alpha} \delta^{\rho\sigma}) \right. \\
+ \zeta_2^{\alpha\beta} k_2^{\gamma} + \zeta_2^{\alpha\gamma} k_2^{\beta} + \zeta_2^{\beta\gamma} k_2^{\alpha} + \zeta_2^{\mu\rho} (\zeta_3^{\alpha\beta} k_3^{\gamma} \delta^{\rho\sigma} + \zeta_3^{\alpha\gamma} k_3^{\beta} \delta^{\rho\sigma} + \zeta_3^{\beta\gamma} k_3^{\alpha} \delta^{\rho\sigma}) \\
+ \zeta_3^{\alpha\beta} k_3^{\gamma} + \zeta_3^{\alpha\gamma} k_3^{\beta} + \zeta_3^{\beta\gamma} k_3^{\alpha} + \zeta_3^{\mu\rho} (\zeta_4^{\alpha\beta} k_4^{\gamma} \delta^{\rho\sigma} + \zeta_4^{\alpha\gamma} k_4^{\beta} \delta^{\rho\sigma} + \zeta_4^{\beta\gamma} k_4^{\alpha} \delta^{\rho\sigma}) \left. \\
+ \alpha' [4a_1 \zeta_2^{\mu\rho} \xi^{\lambda\sigma} (\zeta_3^{\alpha\beta} k_3^{\gamma} \delta^{\rho\sigma} + \zeta_3^{\alpha\gamma} k_3^{\beta} \delta^{\rho\sigma} + \zeta_3^{\beta\gamma} k_3^{\alpha} \delta^{\rho\sigma}) + 4a_1 \zeta_3^{\mu\rho} \xi^{\lambda\sigma} (\zeta_4^{\alpha\beta} k_4^{\gamma} \delta^{\rho\sigma} + \zeta_4^{\alpha\gamma} k_4^{\beta} \delta^{\rho\sigma} + \zeta_4^{\beta\gamma} k_4^{\alpha} \delta^{\rho\sigma})] + (\alpha')^2 [12b_1 \zeta_2^{\mu\rho} \xi^{\lambda\sigma} (\zeta_3^{\alpha\beta} k_3^{\gamma} \delta^{\rho\sigma} + \zeta_3^{\alpha\gamma} k_3^{\beta} \delta^{\rho\sigma} + \zeta_3^{\beta\gamma} k_3^{\alpha} \delta^{\rho\sigma}) + 12b_1 \zeta_3^{\mu\rho} \xi^{\lambda\sigma} (\zeta_4^{\alpha\beta} k_4^{\gamma} \delta^{\rho\sigma} + \zeta_4^{\alpha\gamma} k_4^{\beta} \delta^{\rho\sigma} + \zeta_4^{\beta\gamma} k_4^{\alpha} \delta^{\rho\sigma})], \tag{3.15} \]

where \( (\zeta_i^{\mu\nu}) \) and \( (k_i, \ i = 1, \ldots, 3) \) denote the polarization tensors and momenta for the external graviton legs.

For the gauge theory 3-point amplitude one finds:

\[
A_{3L} = - [(\zeta_1 \cdot k_1 \zeta_2 \cdot k_2 \zeta_3 \cdot \zeta_3 + \zeta_1 \cdot k_2 \zeta_2 \cdot \zeta_3) + \frac{3}{4} \alpha' a_1^L \zeta_1 \cdot k_2 \zeta_2 \cdot k_3 \zeta_3 \cdot k_1]. \tag{3.16} \]

For the general 4-point amplitude matters are somewhat more complicated. The scattering amplitude will consist both of direct contact terms as well as 3-point contributions combined with a propagator – the interaction parts. Furthermore imposing on-shell constraints will still leave many non-vanishing terms in the scattering amplitude. A general polynomial expression for the 4-point scattering amplitude has the form, see refs. [71, 72]:

\[
A_4 \sim \sum_{0 \leq n + m + k \leq 2} b_{nmk} s^n t^m u^k, \tag{3.17} \]

where \( (s), (t) \) and \( (u) \) are normal Mandelstam variables and the factor \( (b_{nmk}) \) consist of scalar contractions of momenta and polarizations for the external lines. The case \( (n + m + k = 2) \) corresponds to a particular simple case, where the coefficients \( (b_{nmk}) \) consist only of momentum factors contracted with momentum factors, and polarization indices contracted with other polarization indices. In the process of comparing amplitudes, we need only to match coefficients for a sufficient part of the amplitude; i.e., for a specific choice of factors \( (b_{nmk}) \), – gauge symmetry will then do the rest and dictate that once adequate parts of the amplitudes match, we have achieved matching of the full amplitude. We will choose the case where \( (b_{nmk}) \) has this simplest form. Expanding the Lagrangian, one finds the following contributions:

\[
R = -\frac{\kappa^3}{4} h_{\mu\nu} h_{\nu\rho} \partial^2 h_{\rho\mu} + \ldots, \tag{3.18} \]

\[
R^2_{\lambda\mu\nu\rho} - 4R^2_{\mu\nu} + R^2 = -\frac{\kappa^3}{2} \partial_\alpha h_{\mu\nu} \partial_\alpha h_{\nu\rho} \partial^2 h_{\rho\mu} + \ldots, \tag{3.19} \]
3.2. THE EFFECTIVE ACTIONS

\[ R^{\alpha \beta} R_{\alpha \beta} R_{\lambda \rho} R^{\lambda \rho} - \frac{3 \kappa^2}{2} \partial_\alpha \partial_\beta h_{\mu \nu} \partial_\alpha \partial_\beta h_{\nu \rho} \partial_\rho h_{\mu \mu} + 3 \kappa^4 (h_{\mu \nu} \partial_\alpha \partial_\beta h_{\nu \rho} \partial_\gamma \partial_\beta h_{\gamma \mu} h_{\mu \rho} \partial_\sigma h_{\rho \mu} \partial_\gamma h_{\sigma \mu} + \ldots, \]

\[ \left(R^{\mu \alpha} R_{\alpha \beta} R^{\beta \gamma} R_{\gamma \lambda} R^{\lambda \alpha} \right) = -\frac{3 \kappa^4}{8} \partial_\alpha h_{\mu \nu} \partial_\alpha h_{\nu \mu} \partial_\beta h_{\nu \rho} \partial_\beta h_{\gamma \mu} \partial_\gamma h_{\rho \mu} + \ldots \]

The same conventions as for 3-point terms are used in these equations. The above results simply verify ref. [69] and they result the following expression for 4-point amplitude:

\[ M_4 = \frac{1}{2 \alpha'} \left[ \zeta_1 \zeta_2 \zeta_3 \zeta_4 (z+a_1 z^2 - 3 b_1 (z^3 - 4 x y z) - 3 b_2 (z^3 - \frac{7}{2} x y z) \right. \]

\[ + (a_1^2 + 3 b_1 + 3 b_2) (z^3 - 3 x y z) + \frac{1}{4} c^2 (z^3 - x y z) \]

\[ + \zeta_1 \zeta_2 \zeta_4 \zeta_3 (y + a_1 y^2 - 3 b_1 (y^3 - 4 x y z) - 3 b_2 (y^3 - \frac{7}{2} x y z) \]

\[ + (a_1^2 + 3 b_1 + 3 b_2) (y^3 - 3 x y z) + \frac{1}{4} c^2 (y^3 - x y z) \]

\[ + \zeta_1 \zeta_3 \zeta_2 \zeta_4 (x + a_1 x^2 - 3 b_1 (x^3 - 4 x y z) - 3 b_2 (x^3 - \frac{7}{2} x y z) \]

\[ + (a_1^2 + 3 b_1 + 3 b_2) (x^3 - 3 x y z) + \frac{1}{4} c^2 (x^3 - x y z) \right] \]

where we use the definitions: \( \zeta_1 \zeta_2 \zeta_3 \zeta_4 = \zeta_1^{\alpha \beta} \zeta_2^{\beta \gamma} \zeta_3^{\gamma \delta} \zeta_4^{\delta \alpha} \) as well as \( (x = -2 \alpha' (k_1 \cdot k_2)), \)

\( (y = -2 \alpha' (k_1 \cdot k_4)) \) and \( (z = -2 \alpha' (k_1 \cdot k_3)) \). In the above expression the antisymmetric term (with generic coefficient \( c \)) needed in the heterotic string scattering amplitude has been included refs. [69, 75].

The corresponding 4-point gauge amplitude can be written:

\[ A_{4L} = \left[ (\zeta_{1324} + \frac{z}{x} \zeta_{1234} + \frac{z}{y} \zeta_{1423} \right] + \left[ -\frac{3 a_1^4}{8} z (\zeta_{1324} + \zeta_{1234} + \zeta_{1423} \right] \]

\[ + \left[ \frac{9 (a_1^4)^2}{128} (x z - y) \zeta_{1234} + y z - x) \zeta_{1423} \right] \]

\[ - \frac{1}{4} \left[ \left( \frac{1}{2} a_1^4 \right) x y z \zeta_{1234} - \left( \frac{1}{4} a_1^4 + 2 a_2^4 \right) z^2 \zeta_{1324} + \left( \frac{1}{4} a_3^4 + \frac{1}{2} a_4^4 \right) y z \zeta_{1234} + \left( \frac{1}{4} a_3^4 + a_5^4 \right) x z \zeta_{1234} \right. \]

\[ + \left( \frac{1}{2} a_1^4 + a_5^4 \right) y x \zeta_{1234} + \left( \frac{1}{4} a_3^4 + a_5^4 \right) y z \zeta_{1423} + \left( \frac{1}{4} a_3^4 + \frac{1}{2} a_4^4 \right) y z \zeta_{1234} + \left( \frac{1}{4} a_4^4 + a_5^4 \right) x y \zeta_{1423} \right], \]

where \( (x), (y) \) and \( (z) \) are defined as above, and, e.g., \( (\zeta_{1324} = (\zeta_1 \cdot \zeta_2)(\zeta_3 \cdot \zeta_4) \) and etc. The contact terms in the above expression were calculated explicitly and the results agree with those of ref. [69]. The non-contact terms were adapted from ref. [69].

The expressions for the scattering amplitudes are quite general. They relate the general generic effective Lagrangians in the gravity and Yang-Mills case to their 3- and 4-point scattering amplitudes respectively in any dimension. The next section will be dedicated to showing how the 3-point amplitudes and 4-point amplitudes in gravity and Yang-Mills have to be related through the KLT-relations.
3.3 The open-closed string relations

The KLT-relations between closed and open string are discussed in literature on string theory, but let us recapitulate the essentials here. Following conventional string theory ref. [92] the general $M$-point scattering amplitude for a closed string is related to that of an open string in the following manner:

$$A_{\text{closed}}^M \sim \sum_{\Pi, \tilde{\Pi}} e^{i\pi \Phi(\Pi, \tilde{\Pi})} A_{\text{left open}}^M (\Pi) A_{\text{right open}}^M (\tilde{\Pi}),$$

in this expression $(\Pi)$ and $(\tilde{\Pi})$ corresponds to particular cyclic orderings of the external lines of the open string. While the $(\Pi)$ ordering corresponds to a left-moving open string, the $(\tilde{\Pi})$ ordering corresponds to the right-moving string. The factor $(\Phi(\Pi, \tilde{\Pi}))$ in the exponential is a phase factor chosen appropriately with the cyclic permutations $(\Pi)$ and $(\tilde{\Pi})$. In the cases of 3- and 4-point amplitudes, the following specific KLT-relations can be adapted from the $M$-point amplitude:

$$M_{3 \text{ gravity}}^{\mu\nu\rho\sigma}(1, 2, 3) = \kappa A_{3 \text{ L-gauge}}^{\mu\nu\rho}(1, 2, 3) \times A_{3 \text{ R-gauge}}^{\sigma}(1, 2, 3)$$

$$M_{4 \text{ gravity}}^{\mu\nu\rho\sigma\tilde{\rho}\tilde{\sigma}}(1, 2, 3, 4) = \frac{\kappa^2}{4\pi\alpha'} \sin(\pi x) \times A_{4 \text{ L-gauge}}^{\mu\nu\rho\sigma}(1, 2, 3, 4) \times A_{4 \text{ R-gauge}}^{\tilde{\rho}\tilde{\sigma}}(1, 2, 4, 3),$$

where $(M)$ is a tree amplitude in gravity, and we have the color ordered amplitude $(A)$ for the gauge theory with coupling constant $(g = 1)$. As previously the definition $(x = -2\alpha'(k_1 \cdot k_2), \ldots)$ is employed, where $(k_1)$ and $(k_2)$ are particular momenta of the external lines.

In string theory the specific mapping relations originate through the comparison of open and closed string amplitudes. However considering effective field theories there is no need to assume anything a priori about the tree amplitudes. It makes sense to investigate the mapping of scattering amplitudes in the broadest possible setting. We will thus try to generalize the above mapping relations. In the case of the 4-amplitude mapping it seems that a possibility is to replace the specific sine function with a general Taylor series, e.g.:

$$\frac{\sin(\pi x)}{\pi} \to xf(x) = x(1 + P_1 x + P_2 x^2 + \ldots)$$

If this is feasible and what it means for the mapping, will be explored below.

3.4 Generalized mapping relations

Insisting on the ordinary mapping relation for the 3-point amplitude and replacing in the 4-point mapping relation the sine function with a general polynomial the following relations between the coefficients in the scattering amplitudes are found to be necessary
in order for the generalized KLT-relations to hold. From the 3-amplitude at order \((\alpha')\) we have:

\[
3a_1^L + 3a_1^R = 16a_1, \quad (3.27)
\]

while from the 3-amplitude at order \((\alpha'^2)\) one gets:

\[
3a_1^L a_1^R = 64b_1 \quad (3.28)
\]

From the 4-amplitude at order \((\alpha')\) we have:

\[
16a_1 = 3a_1^L + 3a_1^R, \quad P_1 = 0 \quad (3.29)
\]

while the 4-amplitude at order \((\alpha'^2)\) states:

\[
6a_5^L + 3a_4^L + \frac{27(a_1^L)^2}{16} = 0, \quad 6a_5^R + 3a_4^R + \frac{27(a_1^R)^2}{16} = 0, \quad (3.30)
\]

together with the equations:

\[
24c^2 + 96a_1^2 = 6a_3^L + 3a_3^R + 18a_4^L + 12a_5^L + 24a_6^R + \frac{81(a_1^L)^2}{8} + 96P_2 + 18(a_1^R + a_1^L)P_1, \quad (3.31)
\]

and furthermore:

\[
96b_1 + 48b_2 + 16c^2 = 4a_3^L + 5a_3^R + 2a_4^L + 6a_4^R + 8a_6^L + \frac{9(a_1^R)^2}{4} + \frac{9a_1^L a_1^R}{2} + 96P_2 + 18(a_1^R + a_1^L)P_1, \quad (3.32)
\]
as well as:

\[
96a_1^2 - 96b_1 - 48b_2 + 8c^2 = a_3^L + 2a_4^R - 12a_5^L - 12a_5^R \quad (3.34)
\]
\[
+ 64P_2 - 6(a_1^R + a_1^L)P_1,
\]
\[
96a_1^2 - 96b_1 - 48b_2 + 8c^2 = a_4^R - 4a_5^L - 12a_5^R + 8a_6^L \quad (3.35)
\]
\[
+ 32P_2,
\]
\[
96a_1^2 - 96b_1 - 48b_2 + 8c^2 = 2a_4^R - 4a_5^L - 12a_5^R + 8a_6^L \quad (3.36)
\]
\[
+ 32P_2,
\]
\[
96a_1^2 - 96b_1 - 48b_2 + 8c^2 = a_3^L + 2a_4^R - 12a_5^L - 12a_5^R \quad (3.37)
\]
\[
+ 64P_2 - 6(a_1^R + a_1^L)P_1.
\]

These equations are found by relating similar scattering components, \textit{e.g.}, the product resulting from the generalized KLT relations: \((\zeta_{1234}\zeta_{1234}y^2x)\) on the gauge side, with \((\zeta_1\zeta_2\zeta_3\zeta_4y^2x)\) on the gravity side. The relations can be observed to be a generalization of the mapping equations we found in ref. [39]. Allowing for a more general mapping and including the antisymmetric term needed in the case of an heterotic string generates additional freedom in the mapping equations. The generalized equations can still be solved and the solution is unique. One one ends up with the following solution:\footnote{We have employed an algebraic equation solver \textit{Maple}, to solve the equations, (Maple and Maple V are registered trademarks of Maple Waterloo Inc.)}

\[
a_1^L = \frac{8}{3}a_1 \pm \frac{4}{3}c, \quad a_1^R = \frac{8}{3}a_1 \mp \frac{4}{3}c, \quad (3.34)
\]
\[
a_3^L = -8P_2, \quad a_3^R = -8P_2, \quad (3.35)
\]
\[
a_4^L = -4P_2, \quad a_4^R = -4P_2, \quad (3.36)
\]
\[
a_5^L = \mp 2a_1c - 2a_1^2 - \frac{1}{2}c^2 + 2P_2, \quad a_5^R = \pm 2a_1c - 2a_1^2 - \frac{1}{2}c^2 + 2P_2, \quad (3.37)
\]
\[
a_6^L = \pm 2a_1c + 2a_1^2 + \frac{1}{2}c^2 + P_2, \quad a_6^R = \mp 2a_1c + 2a_1^2 + \frac{1}{2}c^2 + P_2, \quad (3.38)
\]
\[
b_1 = \frac{1}{3}a_1^2 - \frac{1}{12}c^2, \quad b_2 = \frac{2}{3}a_1^2 - \frac{1}{6}c^2. \quad (3.39)
\]

This is the unique solution to the generalized mapping relations. As seen, the original KLT solution ref. [39] is still contained but the generalized mapping solution is not as constraining as the original KLT solution was. In fact now one can freely choose \((c)\), and \((P_2)\) as well as \((a_1)\). Given a certain gravitational action with or without the possibility of terms needed for heterotic strings, one can choose between different mappings from the gravitational Lagrangian to the given Yang-Mills action. Traditional string solutions are contained in this and are possible to reproduce – but the solution space for the generalized solution is broader and allows seemingly for a wider range of possible effective actions on the gravity and the Yang-Mills side. It is important to note that this does not imply that the coefficients in the effective actions can be chosen freely – the generalized
KLT-relations still present rather restricting constraints on the effective Lagrangians. To which extent one may be able to reproduce the full solution space by string theory is not definitely answered. Clearly superstrings cannot reproduce the full solution space because of space-time supersymmetry which does not allow for terms in the effective action like \((\text{tr}(F_{\mu\nu}F_{\nu\alpha}F_{\alpha\mu}))\) on the open string side and \(\left(R^{\mu\nu}_{\alpha\beta}R^{\alpha\beta}_{\gamma\lambda}R^{\gamma\lambda}_{\mu\nu}\right)\) on the gravity side ref. [93, 94, 95]. For non-supersymmetric string theories constraints on the effective actions such as the above do not exist, and it is therefore possible that in this case some parts or all of the solution space might be reproduced by the variety of non-supersymmetric string theories presently known. It is, e.g., observed that the bosonic non-supersymmetric string solution in fact covers parts of the solution space not covered by the supersymmetric string solution.

The possibility of heterotic strings on the gravity side, i.e., a non-vanishing \((c)\), will as observed always generate dissimilar 'left' and 'right' Yang-Mills coefficients. That is for nonzero \((c)\), e.g., \((a_1^L \neq a_1^R)\), \((a_2^L \neq a_2^R)\) and \((a_0^L \neq a_0^R)\). It is also seen that for \((c = 0)\) that 'left' is equivalent to 'right'. The coefficients \((a_1^L)\), \((a_3^R)\) and \((a_1^R)\) are completely determined by the coefficient \((P_1)\). In the generalized mapping relation the only solution for the coefficient \((P_1)\) is zero.

The KLT or generalized KLT-mapping equations can be seen as coefficient constraints linking the generic terms in the gauge/gravity Lagrangians. To summarize the gravitational Lagrangian has to take the following form, dictated by the generalized KLT-relations:

\[
\mathcal{L} = \frac{2\sqrt{-g}}{\kappa^2} \left[ R + \alpha' \left( a_1 R_{\lambda\mu\nu\rho}^2 \right) + \left( \alpha' \right)^2 \left( \left( \frac{1}{3} a_1^2 - \frac{1}{12} c^2 \right) R^{\mu\nu}_{\alpha\beta} R^{\alpha\beta}_{\lambda\rho} R^{\lambda\rho}_{\mu\nu} + \left( \frac{2}{3} a_1^2 - \frac{1}{6} c^2 \right) \left( R^{\mu\nu}_{\alpha\beta} R^{\alpha\beta}_{\lambda\rho} R^{\lambda\rho}_{\mu\nu} - 2 R^{\mu\nu}_{\alpha\beta} R^{\alpha\beta}_{\lambda\mu} R^{\lambda\mu}_{\rho\gamma} \right) \right) - \frac{3}{4} \left( \partial_{[\mu} B_{\nu\rho]} + \alpha' c_{\omega[\mu} R_{\nu\rho\omega]} + \ldots \right)^2 + \ldots \right],
\]

and the corresponding 'left' or 'right' Yang-Mills action are then forced to be:

\[
\mathcal{L}_{YM} = -\frac{1}{8} \text{tr} \left[ F_{\mu\nu}^{LL} F_{\mu\nu}^{LL} + \alpha' \left( \left( \frac{1}{3} a_1 \pm \frac{1}{3} c \right) F_{\mu\lambda}^{LL} F_{\lambda\nu}^{LL} F_{\nu\mu}^{LL} \right) + \left( \alpha' \right)^2 \left( - 8 P_2 F_{\mu\lambda}^{LL} F_{\nu\beta}^{LL} F_{\rho\nu}^{LL} F_{\mu\rho}^{LL} - 4 P_2 F_{\mu\lambda}^{LL} F_{\nu\lambda}^{LL} F_{\nu\rho}^{LL} F_{\mu\rho}^{LL} \right) + \left( \pm 2 a_1 c - 2 a_1^2 + 2 P_2 - \frac{1}{2} c^2 \right) F_{\mu\nu}^{LL} F_{\mu\nu}^{LL} F_{\lambda\rho}^{LL} F_{\lambda\rho}^{LL} \right) + \ldots \right],
\]

where 'left' and 'right,' reflect opposite choices of signs, in the above equation. One sees that the Yang-Mills Lagrangian is fixed once a gravitational action is chosen; the only remaining freedom in the Yang-Mills action is then the choice of \((P_2)\), corresponding to different mappings.

It is directly seen that the graviton 3-amplitude to order \((\alpha')\) is re-expressible in terms of 3-point Yang-Mills amplitudes. This can be expressed diagrammatically in figure 3.1. At order \((\alpha'^2)\) we have the expression as shown in figure 3.2. At the amplitude level this factorization is no surprise; this is just what the KLT-relations tell us. At the Lagrangian
Figure 3.1: A diagrammatical expression for the generalized mapping of the 3-point gravity amplitude into the product of Yang-Mills amplitudes at order $\alpha'$. 

\[ \left[ (\sim \zeta)_L^{\mu \nu \beta} \right] \otimes \left[ (\sim \zeta)_R^{\mu \nu \beta} \right] + \left[ (\sim \zeta)_L^{\mu \nu \beta} \right] \otimes \left[ (\sim \zeta)_R^{\mu \nu \beta} \right] = \frac{16}{3\kappa} \left[ (\sim \zeta)_L^{\mu \nu \beta} \right] \]

Figure 3.2: A diagrammatical expression for the generalized mapping of the 3-point gravity amplitude into the product of Yang-Mills amplitudes at order $\alpha'^2$. 

level things usually get more complicated and no factorizations of gravity vertex rules are readily available. At order ($O(\alpha')$) the factorization of gravity vertex rules was investigated in ref. [87]. An interesting task would be to continue this analysis to order ($O(\alpha'^3)$), and investigate the KLT relations for effective actions directly at the Lagrangian vertex level. Everything is more complicated in the 4-point case as contact and non-contact terms mix in the mapping relations. This originates from the fact that we are actually relating S-matrix elements. Diagrammatically the 4-point generalized KLT-relation at order ($\alpha'$) is depicted in figure 3.3. This relation is essentially equivalent to the 3-vertex relation at order ($\alpha'$). The 4-point relation at order ($\alpha'$) rules out the possibility of a ($P_1$) term in the generalized mapping. The full KLT-relation at order ($\alpha'^2$) is shown in figure 3.4. The coefficients in the above expression need to be taken in agreement with the generalized KLT solution in order for this identity to hold. A profound consequence of the KLT-relations is that they link the sum of a certain class of diagrams in Yang-Mills theory with a corresponding sum of diagrams in gravity, for very specific values of the constants in the Lagrangian. At glance we do not observe anything manifestly about the decomposition of, e.g., vertex rules, – however this should be investigated more carefully before any conclusions can be drawn. The coefficient equations cannot directly be transformed into relations between diagrams, – this can be seen by calculation. However reinstating the solution for the coefficients it is possible to turn the above equation for the 4-amplitude at order ($\alpha'^2$) into interesting statements about diagrams. Relating all ($P_2$) terms give, e.g., the following remarkable diagrammatic statement as shown in figure 3.5. The surprising fact is that from the relations which apparently link gravity and Yang-Mills theory only, one can eliminate the gravity part to obtain relations entirely in pure Yang-Mills theory. Similarly, relating all contributions with $a^2_1$ give the result shown...
3.4. GENERALIZED MAPPING RELATIONS

\[\left[\alpha_3^L(\mathcal{X})_{(a_3^L)} + a_1^L(\mathcal{X})_{(a_1^L)} + a_2^L(\mathcal{X})_{(a_2^L)} + a_4^L(\mathcal{X})_{(a_4^L)}\right]^{\mu\nu\rho\sigma} \otimes \left[\mathcal{X}^{R}(t + s + u) + \mathcal{X}^{R}\right]^{\mu'\nu'\rho'\sigma'} \]

\[\quad \quad \quad + x\left[\mathcal{X}^{L}(t + s + u) + \mathcal{X}^{L}\right]^{\mu\nu\rho\sigma} \otimes \left[\alpha_3^L(\mathcal{X})_{(a_3^L)} + a_1^L(\mathcal{X})_{(a_1^L)} + a_2^L(\mathcal{X})_{(a_2^L)} + a_4^L(\mathcal{X})_{(a_4^L)}\right]^{\mu'\nu'\rho'\sigma'} \]

\[\quad \quad \quad + x\left[\alpha_1^L(\mathcal{X})_{(a_1^L)}(t + s + u) + \alpha_2^L(\mathcal{X})_{(a_2^L)}\right]^{\mu\nu\rho\sigma} \otimes \left[\alpha_3^L(\mathcal{X})_{(a_3^L)} + a_1^L(\mathcal{X})_{(a_1^L)} + a_2^L(\mathcal{X})_{(a_2^L)} + a_4^L(\mathcal{X})_{(a_4^L)}\right]^{\mu'\nu'\rho'\sigma'} \]

\[\quad \quad \quad + x\left[\alpha_1^L(\mathcal{X})_{(a_1^L)}(t + s + u) + \alpha_2^L(\mathcal{X})_{(a_2^L)}\right]^{\mu\nu\rho\sigma} \otimes \left[\alpha_3^L(\mathcal{X})_{(a_3^L)} + a_1^L(\mathcal{X})_{(a_1^L)} + a_2^L(\mathcal{X})_{(a_2^L)} + a_4^L(\mathcal{X})_{(a_4^L)}\right]^{\mu'\nu'\rho'\sigma'} \]

\[\quad \quad \quad + x^2 P_2 \left[\mathcal{X}^{L}(t + s + u) + \mathcal{X}^{L}\right]^{\mu\nu\rho\sigma} \otimes \left[\mathcal{X}^{R}(t + s + u) + \mathcal{X}^{R}\right]^{\mu'\nu'\rho'\sigma'} \]

\[= \frac{4\alpha'}{\kappa^2} \left\{ a_1^L(\mathcal{X})_{(a_1^L)}(t + s + u) + \alpha_2^L(\mathcal{X})_{(a_2^L)} \right\}^{\mu\nu\rho\sigma} \otimes \left[\mathcal{X}^{R}(t + s + u) + \mathcal{X}^{R}\right]^{\mu'\nu'\rho'\sigma'} \]

\[+ b_2 \left[\mathcal{X}^{L}(t + s + u) + \mathcal{X}^{L}\right]^{\mu\nu\rho\sigma} \otimes \left[\mathcal{X}^{R}(t + s + u) + \mathcal{X}^{R}\right]^{\mu'\nu'\rho'\sigma'} \]

\[+ c^2 \left[\mathcal{X}^{L}(t + s + u) + \mathcal{X}^{L}\right]^{\mu\nu\rho\sigma} \otimes \left[\mathcal{X}^{R}(t + s + u) + \mathcal{X}^{R}\right]^{\mu'\nu'\rho'\sigma'} \}

Figure 3.4: The diagrammatic expression for the generalized mapping of the 4-point gravity amplitude into the product of Yang-Mills amplitudes at order $\alpha'^2$.

\[\left[-8(\mathcal{X})_{(a_3^L)} + 4(\mathcal{X})_{(a_1^L)} + 2(\mathcal{X})_{(a_2^L)} + (\mathcal{X})_{(a_4^L)}\right]^{\mu\nu\rho\sigma} \otimes \left[\mathcal{X}^{R}(t + s + u) + \mathcal{X}^{R}\right]^{\mu'\nu'\rho'\sigma'} \]

\[+ \left[(\mathcal{X})^{L}(t + s + u) + (\mathcal{X})^{L}\right]^{\mu\nu\rho\sigma} \otimes \left[-8(\mathcal{X})_{(a_3^L)} + 4(\mathcal{X})_{(a_1^L)} + 2(\mathcal{X})_{(a_2^L)} + (\mathcal{X})_{(a_4^L)}\right]^{\mu'\nu'\rho'\sigma'} \]

\[= -x^2 \left[(\mathcal{X})^{L}(t + s + u) + (\mathcal{X})^{L}\right]^{\mu\nu\rho\sigma} \otimes \left[(\mathcal{X})^{R}(t + s + u) + (\mathcal{X})^{R}\right]^{\mu'\nu'\rho'\sigma'} \]

Figure 3.5: A diagrammatic relationship on the Yang-Mill side for the tree operators in the effective action between operators of unit order and order $\alpha'^2$.

in figure 3.6. Furthermore we have a relationship generated by the $ca_1$ parts as shown in figure 3.7.

These diagrammatical relationships can be readily checked by explicit calculations. It is quite remarkable that the KLT relations provide such detailed statements about pure Yang-Mills effective field theories, without any reference to gravity at all. To explain the notation in the above diagrammatic statements. An uppercase $(L)$ or $(R)$ states that the scattering amplitude originates from a 'left' or a 'right' mover respectively. The lower indices, e.g., $(\alpha')$ and the coupling constants, e.g., $(a_3^L)$ or $(a_1^L)^2$ denote respectively the order of $(\alpha')$ in the particular amplitude and the coupling constant prefactor obtained when this particular part of the amplitude is generated from the generic Lagrangian. The parentheses with $[(t + s + u)]$ denote that we are supposed to sum over all $(t)$, $(s)$ and $(u)$ channels for this particular amplitude.
\[\begin{align*}
-2(\mathcal{L}_{(a\xi)}^{L} + 2(\mathcal{L}_{(a\xi)}^{R})_{(a\xi)^{2}} \sigma_{\mu\nu} & \otimes x (\mathcal{L}_{(a\xi)}^{R})(t + s + u) + (\mathcal{L}_{(a\xi)}^{R})_{(a\xi)^{2}}^{\nu\lambda} \sigma^{\lambda}\sigma^{\mu} \\
+ x[(\mathcal{L}_{(a\xi)}^{L})(t + s + u) + (\mathcal{L}_{(a\xi)}^{L})_{(a\xi)^{2}} \sigma_{\mu\nu} & \otimes -2(\mathcal{L}_{(a\xi)}^{R})_{(a\xi)^{2}}^{\nu\lambda} \sigma^{\lambda}\sigma^{\mu} \\
+ x\left[\frac{64}{9}(\mathcal{L}_{(a\xi)}^{L})(t + s + u) + (\mathcal{L}_{(a\xi)}^{L})_{(a\xi)^{2}} \sigma_{\mu\nu} & \otimes \left[\frac{64}{9}(\mathcal{L}_{(a\xi)}^{R})(t + s + u) + (\mathcal{L}_{(a\xi)}^{R})_{(a\xi)^{2}}^{\nu\lambda} \sigma^{\lambda}\sigma^{\mu} \\
+ x\left[\frac{8}{3}((\mathcal{L}_{(a\xi)}^{L})(t + s + u) + (\mathcal{L}_{(a\xi)}^{L})_{(a\xi)^{2}} \sigma_{\mu\nu} & \otimes \left[\frac{8}{3}((\mathcal{L}_{(a\xi)}^{R})(t + s + u) + (\mathcal{L}_{(a\xi)}^{R})_{(a\xi)^{2}}^{\nu\lambda} \sigma^{\lambda}\sigma^{\mu} \\
= \frac{\alpha'}{h^{2}} \left\{ [(\mathcal{L}_{(a\xi)}^{L})(t + s + u)_{(a\xi)^{2}}^{\nu\lambda} \sigma^{\lambda}\sigma^{\mu} + \frac{1}{3}[(\mathcal{L}_{(a\xi)}^{R})(t + s + u)_{(a\xi)^{2}}^{\nu\lambda} \sigma^{\lambda}\sigma^{\mu} \\
+ \frac{2}{3}((\mathcal{L}_{(a\xi)}^{L})(t + s + u) + (\mathcal{L}_{(a\xi)}^{R})_{(a\xi)^{2}}^{\nu\lambda} \sigma^{\lambda}\sigma^{\mu} \right\}
\end{align*}\]

Figure 3.6: The essential diagrammatic relationship between gauge and gravity diagrams.

\[\begin{align*}
2(\mathcal{L}_{(a\xi)}^{L})_{(a\xi)^{2}}^{\nu\lambda} & \otimes (\mathcal{L}_{(a\xi)}^{R})(t + s + u) + (\mathcal{L}_{(a\xi)}^{R})_{(a\xi)^{2}}^{\nu\lambda} \sigma^{\lambda}\sigma^{\mu} \\
+ (\mathcal{L}_{(a\xi)}^{L})(t + s + u) + (\mathcal{L}_{(a\xi)}^{L})_{(a\xi)^{2}} \sigma_{\mu\nu} & \otimes -2(\mathcal{L}_{(a\xi)}^{R})_{(a\xi)^{2}}^{\nu\lambda} \sigma^{\lambda}\sigma^{\mu} \\
= \left[\frac{64}{9}(\mathcal{L}_{(a\xi)}^{L})(t + s + u) + (\mathcal{L}_{(a\xi)}^{L})_{(a\xi)^{2}} \sigma_{\mu\nu} \otimes \left[\frac{64}{9}(\mathcal{L}_{(a\xi)}^{R})(t + s + u) + (\mathcal{L}_{(a\xi)}^{R})_{(a\xi)^{2}}^{\nu\lambda} \sigma^{\lambda}\sigma^{\mu} \\
- (\mathcal{L}_{(a\xi)}^{L})(t + s + u) + (\mathcal{L}_{(a\xi)}^{L})_{(a\xi)^{2}} \sigma_{\mu\nu} & \otimes \left[\frac{64}{9}(\mathcal{L}_{(a\xi)}^{R})(t + s + u) + (\mathcal{L}_{(a\xi)}^{R})_{(a\xi)^{2}}^{\nu\lambda} \sigma^{\lambda}\sigma^{\mu} \right]\end{align*}\]

Figure 3.7: Another diagrammatic relationship on the Yang-Mills side between order \(\alpha'^{2}\) and \(\alpha'\) operators.

### 3.5 Discussion

The solution space of the above equations suggests the following interpretation. For a given coefficient \((a_{1})\) in the gravitational effective field theory action through order \((\alpha'^{2})\) and the string value of \((P = \frac{-\pi^{2}}{6})\), there is a set of unique ('left'-'right') effective Yang-Mills Lagrangians, up to a field redefinition, that satisfies the constraint of the KLT-relations. The set of solutions spans out a space of operator solutions, where the different types of strings correspond to certain points along the path. For string theories with similar 'left' and 'right' Lagrangians, one can find a bosonic string corresponding to the solution: \((a_{1}^{L} = a_{1}^{R} = \frac{8}{3})\), \((a_{3}^{L} = a_{3}^{R} = \frac{4\pi^{2}}{3})\), \((a_{4}^{L} = a_{4}^{R} = \frac{3\pi^{2}}{2})\), \((a_{5}^{L} = a_{5}^{R} = -\frac{\pi^{2}}{3} - 2)\), \((a_{6}^{L} = a_{6}^{R} = -\frac{\pi^{2}}{6} + 2)\), on the open string side, and \((a_{1} = 1)\), \((b_{1} = \frac{1}{3})\) and \((b_{2} = \frac{2}{3})\), on the closed string side, while the superstring correspond to: \((a_{1}^{L} = a_{1}^{R} = 0)\), \((a_{3}^{L} = a_{3}^{R} = \frac{4\pi^{2}}{3})\), \((a_{4}^{L} = a_{4}^{R} = \frac{2\pi^{2}}{3})\), \((a_{5}^{L} = a_{5}^{R} = -\frac{\pi^{2}}{3})\) and \((a_{6}^{L} = a_{6}^{R} = -\frac{\pi^{2}}{6})\), for the open string, and \((a_{1} = b_{1} = b_{2} = 0)\) for the closed string. So the string solutions are contained in the above solution space.
However it has been observed that it is possible to generalize the KLT open/closed string relations in the effective field theory framework and that the solution space is somewhat bigger than the space justified by string theory. The KLT-relations are seen to serve as mapping relations between the tree effective field theories for Yang-Mills and gravity. The belief, that general relativity is in fact an effective field theory at loop orders, makes investigations of its tree level manifestations and connections to other effective actions interesting. Links such as KLT, which are applicable and very simplifying in actual calculations should be exploited and investigated at the effective Lagrangian level. An important result of our investigations is the generalization of the KLT-relations. It is found that one cannot completely replace the sine function in the KLT-relations by an arbitrary function. To order $O(\alpha'^3)$ it is possible to replace the sine with an odd third order polynomial in $x$. However, the degree of freedom represented by $P_2$ can be completely absorbed into a rescaling of $\alpha'$ at this order, so additional investigations are required before any conclusions can be drawn. The mapping relations between the effective theories are found to be broader than those given completely from the KLT-relations as the coefficient $(P_2)$ in the generalized framework can be chosen freely. Such a rescaling of $(P_2)$ represents an additional freedom in the mapping. It has been shown that despite this generalization, the effective extension of the KLT-relations is still rather restrictive.

The possibility of an antisymmetric coupling of gravitons needed in the effective action of a heterotic string has also been allowed for and is seen to be consistent with the original and generalized KLT-relations. We have learned that detailed diagrammatic statements can be deduced from the KLT-relations. This presents very interesting aspects which perhaps can be used to gain additional insight in issues concerning effective Lagrangian operators. Furthermore we expect this process to continue, – at order $O(\alpha'^4)$ we assume that the KLT-relations will tell us about new profound diagrammatic relationships between effective field theory on-shell operators in gauge theory and gravity.

The generalized mapping relations represent an effective field theory version of the well known KLT-relations. We have used what we knew already from string theory about the KLT-relations to produce a more general description of mapping relations between gravity and Yang-Mills theory. String theory is not really needed in the effective field theory setting. All that is used here is the tree scattering amplitudes. Exploring if or if not a mapping from the general relativity side to the gauge theory side is possible produces the generalized KLT-relations. The KLT-relations could also be considered in the case of external matter. In this case, operators as the Ricci tensor and the scalar curvature will not vanish on-shell. Such an approach is presently the subject for an ongoing research project, see ref. [96].

KLT-relations involving loops are not yet resolved. One can perform some calculations by making cuts of the diagrams using the unitarity of the S-matrix but no direct factorizations of loop scattering amplitudes have been seen to support a true loop extension of the KLT-relations. Progress in this direction still waits to be seen, and perhaps such loop extensions are not possible. Loops in string theory and in field theory are not directly comparable, and complicated issues with additional string theory modes in the loops seem to be unavoidable in extensions of the KLT-relations beyond tree level. Perhaps since 4-point scattering amplitudes are much less complicated compared to 5-point amplitudes, it would be actually more important first to check the mapping solutions at the 5-point
level before considering the issue of loop amplitudes.
Chapter 4

Quantum gravity at large numbers of dimensions

4.1 Introduction

In this chapter we will discuss the last part of the thesis work which is described in the paper, ref. [41].

The idea is to look upon quantum gravity at a large number of dimensions. An intriguing paper by Strominger ref. [42] deals with the perspectives of using the basic Einstein-Hilbert action to define a theory of quantum gravity but to let the theory have an infinite number of dimensions. The fundamental idea in this approach is to let the spatial dimension be a parameter in which one is allowed to expand the theory. Each graph of the theory is then associated with a dimensional factor. Formally one can expand every Greens function of the theory as a series in $\left(\frac{1}{D}\right)$ and the gravitational coupling constant ($\kappa$):

$$G = \sum_{i,j} (\kappa)^i \left(\frac{1}{D}\right)^j \mathcal{G}_{i,j}.$$ \hspace{1cm} (4.1)

The contributions with highest dimensional dependence will be the leading ones in the large-$D$ limit. Concentrating on these graphs only simplifies the theory, and makes explicit calculations easier. In the effective field theory we expect an expansion of the theory of the type:

$$G = \sum_{i,j,k} (\kappa)^i \left(\frac{1}{D}\right)^j (\mathcal{E})^{2k} \mathcal{G}_{i,j,k},$$ \hspace{1cm} (4.2)

where $(\mathcal{E})$ represents a parameter for the effective expansion of the theory in terms of the energy, i.e., each derivative acting on a massless field will correspond to a factor of $(\mathcal{E})$. The basic Einstein-Hilbert scale will correspond to the $(\mathcal{E}^2 \sim \partial^2 g)$ contribution, while higher order effective contributions will be of order $(\mathcal{E}^4 \sim \partial^4 g), (\mathcal{E}^6 \sim \partial^6 g), \ldots$ etc.

The idea of making a large-$D$ expansion in gravity is somewhat similar to the large-$N$ Yang-Mills planar diagram limit considered by ’t Hooft ref. [97]. In large-$N$ gauge theories, one expands in the internal symmetry index ($N$) of the gauge group, e.g., $(SU(N))$ or $(SO(N))$. The physical interpretations of the two expansions are of course completely
dissimilar, e.g., the number of dimensions in a theory cannot really be compared to the internal symmetry index of a gauge theory. A comparison of the Yang Mills large-$N$ limit (the planar diagram limit), and the large-$D$ expansion in gravity is however still interesting to perform, and as an example of a similarity between the two expansions; the leading graphs in the large-$D$ limit in gravity consist of a subset of planar diagrams.

Higher dimensional models for gravity are well known from string and supergravity theories. The mysterious $M$-theory has to exist in an 11-dimensional space-time. So there are many good reasons to believe that on fundamental scales, we might experience additional space-time dimensions. Additional dimensions could be treated as free or as compactified below the Planck scale, as in the case of a Kaluza-Klein mechanism. One application of the large-$D$ expansion in gravity could be to approximate Greens functions at finite dimension, e.g., $(D = 4)$. The successful and various uses of the planar diagram limit in Yang-Mills theory might suggest other possible scenarios for the applicability of the large-$D$ expansion in gravity. For example, is effective quantum gravity renormalizable in its leading large-$D$ limit, i.e., is it renormalizable in the same way as some non-renormalizable theories are renormalizable in their planar diagram limit? It could also be, that quantum gravity at large-$D$ is a completely different theory, than Einstein gravity. A planar diagram limit is essentially a string theory at large distances, i.e., could gravity at large-$D$ be interpreted as a large distance truncated string limit?

We will here discuss the results of ref. [41], i.e., we will combine the successful effective field theory approach which holds in any dimension, with the expansion of gravity in the large-$D$ limit. The treatment will be mostly conceptual, but we will also address some of the phenomenological issues of this theory. The structure of this chapter will be as follows. First we will briefly review the basic quantization of the Einstein-Hilbert action, and then we will go on with the large-$D$ behavior of gravity, i.e., we will show how to derive a consistent limit for the leading graphs. The effective extension of the theory will next be taken up in the large-$D$ framework, and we will especially focus on the implications of the effective extension of the theory in the large-$D$ limit. The $D$-dimensional space-time integrals will also be looked upon, and we will make a conceptual comparison of the large-$D$ in gravity and the large-$N$ limit in gauge theory. Here we will point out some similarities and some dissimilarities of the two theories. We will also discuss some issues in the original paper by Strominger, ref. [42].

We will work in units ($\hbar = c = 1$), and metric (diag($\eta_{\mu\nu}$) = (1, $-1, -1, -1, \ldots , -1$)), i.e., with $(D - 1)$ minus signs.

### 4.2 Review of the large-$D$ limit of Einstein gravity

In this section we will review the main idea of the large-$D$ expansion of quantum gravity. We have already discussed how to quantize a gravitational theory at $(D = 4)$ using the background field method. Here we will apply a slightly different expansion of the metric field and we will keep an arbitrary number of dimensions $(D)$ at every step in the quantization. For completeness we quickly summarize the essentials of the quantization here.
4.2. REVIEW OF THE LARGE-\(D\) LIMIT OF EINSTEIN GRAVITY

As is well known that the \(D\)-dimensional Einstein-Hilbert Lagrangian has the form:

\[
\mathcal{L} = \int d^D x \sqrt{-g} \left( \frac{2R}{\kappa^2} \right).
\]  

(4.3)

The notation in the above equation is identical to what previously have been used. If we neglect the renormalization problems of this action, it is possible to make a formal quantization of the theory using the path integral. To make a diagram expansion of this theory we have to introduce a gauge breaking term in the action to fix the propagator, and because gravity is a non-abelian theory we have to introduce a proper Faddeev-Popov ref. [8] ghost action as well.

The action for the quantized theory will consequently be:

\[
\mathcal{L} = \int d^D x \sqrt{-g} \left( \frac{2R}{\kappa^2} + \mathcal{L}_{\text{gauge fixing}} + \mathcal{L}_{\text{ghosts}} \right),
\]  

(4.4)

where \((\mathcal{L}_{\text{gauge fixing}})\) is the gauge fixing term, and \((\mathcal{L}_{\text{ghosts}})\) is the ghost contribution. In order to generate vertex rules for this theory an expansion of the action has to be carried out, and vertex rules have to be extracted from the gauge fixed action. The vertices will depend on the gauge choice and on how we define the gravitational field.

It is conventional to define:

\[
g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu},
\]  

(4.5)

and work in harmonic gauge \((\partial^\lambda h_{\mu\lambda} = \frac{1}{2} \partial_\mu h_\lambda^\lambda). For this gauge choice the vertex rules for the 3- and 4-point Einstein vertices can be found in refs. [5, 77].

Yet another possibility is to use the background field method as we did previously:

\[
g_{\mu\nu} \equiv \tilde{g}_{\mu\nu} + \kappa h_{\mu\nu},
\]  

(4.6)

and expand the quantum fluctuation \((h_{\mu\nu})\) around an external source, the background field \((\tilde{g}_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa H_{\mu\nu})\). In the background field approach we have to differ between vertices with internal lines (quantum lines: \(\sim h_{\mu\nu}\)) and external lines (external sources: \(\sim H_{\mu\nu}\)). There are no real problems in using either method; it is mostly a matter of notation and conventions. Here we will concentrate our efforts on the conventional approach.

Another possibility is to define:

\[
\tilde{g}^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu},
\]  

(4.7)

this field definition makes a transformation of the Einstein-Hilbert action into the following form:

\[
\mathcal{L} = \int d^D x \frac{1}{2\kappa^2} (\tilde{g}^{\rho\sigma} \tilde{g}_{\lambda\alpha} \tilde{g}_{\kappa\tau} \tilde{g}^{\alpha\kappa}_{\cdot \rho} \tilde{g}^{\lambda\tau}_{\cdot \sigma} - \frac{1}{(D-2)} \tilde{g}^{\rho\sigma} \tilde{g}_{\alpha\kappa} \tilde{g}^{\lambda\tau}_{\cdot \sigma} \tilde{g}^{\alpha\kappa}_{\cdot \rho} + 2 g^{\rho\sigma} \tilde{g}_{\alpha\kappa}_{\cdot \rho} \tilde{g}^{\lambda\tau}_{\cdot \sigma} \tilde{g}^{\alpha\kappa}_{\cdot \rho} \tilde{g}^{\lambda\tau}_{\cdot \sigma}),
\]  

(4.8)

in, \((D = 4)\), this form of the Einstein-Hilbert action is known as the Goldberg action ref. [19]. In this description of the Einstein-Hilbert action its dimensional dependence is more obvious. It is however more cumbersome to work with in the course of practical large-\(D\) considerations.
In the standard expansion of the action the propagator for gravitons take the form:

\[ D_{\alpha\beta\gamma\delta}(k) = \frac{-i}{2} \left[ \eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - \frac{2}{D-2} \delta_{\alpha\beta}\eta_{\gamma\delta} \right] \frac{1}{k^2 - i\epsilon}. \] (4.9)

The 3- and 4-point vertices for the standard expansion of the Einstein-Hilbert action can be found in ref. [5, 77] and has the following form:

\[ V_{\mu\alpha,\nu\beta,\sigma\gamma}(k_1, k_2, k_3) = \kappa \text{sym} \left[ -\frac{1}{2} P_5(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\sigma\gamma}) - \frac{1}{2} P_6(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\sigma\gamma}) + \frac{1}{2} P_3(k_1 \cdot k_2 \eta_{\alpha\beta} \eta_{\gamma\delta}) + P_6(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\sigma\gamma}) + 2P_3(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\sigma\gamma}) - P_3(k_1 \cdot k_2 \eta_{\alpha\beta} \eta_{\gamma\delta}) + P_6(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\sigma\gamma}) + 2P_3(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\sigma\gamma}) - 2P_3(k_1 \cdot k_2 \eta_{\alpha\beta} \eta_{\gamma\delta}) \right]. \] (4.10)

\[ V_{\mu\alpha,\nu\beta,\sigma\gamma,\rho\lambda}(k_1, k_2, k_3, k_4) = \kappa^2 \text{sym} \left[ -\frac{1}{4} P_6(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\sigma\gamma} \eta_{\rho\lambda}) - \frac{1}{4} P_6(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\sigma\gamma} \eta_{\rho\lambda}) + \frac{1}{2} P_6(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\rho\lambda}) + \frac{1}{2} P_6(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\rho\lambda}) \right]. \] (4.11)

in the above two expressions, 'sym', means that each pair of indices: \((\mu\alpha), (\nu\beta), \ldots\) will have to be symmetrized. The momenta factors: \((k_1, k_2, \ldots)\) are associated with the index pairs: \((\mu\alpha, \nu\beta, \ldots)\) correspondingly. The symbol: \((P_#)\) means that a \(\#\)-permutation of indices and corresponding momenta has to carried out for this particular term. As seen the algebraic structures of the 3- and 4-point vertices are already rather involved and complicated. 5-point vertices will not be considered explicitly here.
4.2. REVIEW OF THE LARGE-D LIMIT OF EINSTEIN GRAVITY

The explicit prefactors of the terms in the vertices will not be essential in this treatment. The various algebraic structures which constitute the vertices will be more important. Different algebraic terms in the vertex factors will generate dissimilar traces in the final diagrams. It is useful to adopt an index line notation for the vertex structure similar to that used in large-$N$ gauge theory.

In this notation we can represent the different algebraic terms of the 3-point vertex (see figure 4.1).

\[
\begin{align*}
3A & \sim \text{sym}\left[\frac{1}{2} P_3(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\sigma\gamma})\right], \\
3B & \sim \text{sym}\left[\frac{1}{2} P_3(k_1 \cdot k_2 \eta_{\mu\nu} \eta_{\lambda\beta} \eta_{\sigma\gamma}) + P_6(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\sigma} \eta_{\beta\gamma})\right], \\
3C & \sim \text{sym}\left[P_3(k_1 \cdot k_2 \eta_{\mu\nu} \eta_{\alpha\beta} \eta_{\sigma\gamma})\right], \\
3D & \sim \text{sym}\left[2P_6(k_1 \nu k_2 \gamma \eta_{\beta\mu} \eta_{\alpha\sigma}) + 2P_3(k_1 \eta_{\mu\nu} \eta_{\beta\gamma} \eta_{\sigma\alpha})\right], \\
3E & \sim \text{sym}\left[2P_3(k_1 \eta_{\nu\alpha} \eta_{\beta\sigma} \eta_{\gamma\mu})\right], \\
3F & \sim \text{sym}\left[2P_3(k_1 \eta_{\nu\alpha} \eta_{\beta\sigma} \eta_{\gamma\mu}) - P_3(k_1 \eta_{\nu\alpha} \eta_{\beta\gamma} \eta_{\sigma\mu})\right], \\
3G & \sim \text{sym}\left[-\frac{1}{2} P_6(k_1 \nu k_1 \eta_{\mu\alpha} \eta_{\gamma\sigma})\right].
\end{align*}
\]

Figure 4.1: A graphical representation of the various terms in the 3-point vertex factor. A dashed line represents a contraction of an index with a momentum line. A full line means a contraction of two index lines. The above vertex notation for the indices and momenta also apply here.

For the 4-point vertex we can use a similar diagrammatic notation (see figure 4.2).

There are only two sources of factors of dimension in a diagram. A dimension factor can arise either from the algebraic contractions in that particular diagram or from the space-time integrals. No other possibilities exists. In order to arrive at the large-$D$ limit in gravity both sources of dimension factors have to be considered and the leading contributions will have to be derived in a systematic way.

Comparing to the case of the large-$N$ planar diagram expansion, only the algebraic structure of the diagrams has to be considered. The symmetry index ($N$) of the gauge group does not go into the evaluation of the integrals. Concerns with the $D$-dimensional integrals only arise in gravity. This is a great difference between the two expansions.

A dimension factor will arise whenever there is a trace over a tensor index in a diagram, e.g., ($\eta_{\mu\alpha} \eta^{\mu\nu}$, $\eta_{\mu\alpha} \eta^{\alpha\beta} \eta_{\beta\gamma} \eta^{\gamma\mu}$, ... $\sim D$). The diagrams with the most traces will dominate in the large-$D$ limit in gravity. Traces of momentum lines will never generate a factor of $(D)$.

It is easy to be assured that some diagrams naturally will generate more traces than other diagrams. The key result of ref. [42] is that only a particular class of diagrams will constitute the large-$D$ limit, and that only certain contributions from these diagrams will be important there. The large-$D$ limit of gravity will correspond to a truncation of the Einstein theory of gravity, where in fact only a subpart, namely the leading contributions of the graphs will dominate. We will use the conventional expansion of the gravitational
Figure 4.2: A graphical representation of the terms in the 4-point vertex factor. The notation here is the same as the above for the 3-point vertex factor.

field, and we will not separate conformal and traceless parts of the metric tensor.

To justify this we begin by looking on how traces occur from contractions of the propagator.

We can graphically write the propagator in the way shown in figure 4.3.

\[ \begin{align*}
& \sim \text{sym}[-\frac{1}{4} P_6(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\sigma\gamma} \eta_{\rho\lambda})], \\
& \sim \text{sym}[\frac{1}{4} P_{12}(k_{1\nu} k_{1\beta} \eta_{\mu\alpha} \eta_{\sigma\gamma} \eta_{\rho\lambda})], \\
& \sim \text{sym}[\frac{1}{4} P_6(k_{1\nu} k_{2\mu} \eta_{\alpha\beta} \eta_{\gamma\delta} \eta_{\rho\lambda})], \\
& \sim \text{sym}[\frac{1}{4} P_6(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\gamma\delta} \eta_{\rho\lambda}) + \frac{1}{2} P_6(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\gamma\rho} \eta_{\lambda\delta}) + \frac{1}{2} P_6(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\gamma\sigma} \eta_{\rho\lambda})], \\
& \sim \text{sym}[\frac{1}{4} P_{12}(k_{1\nu} k_{1\beta} \eta_{\mu\alpha} \eta_{\gamma\delta} \eta_{\rho\lambda}) + \frac{1}{2} P_{12}(k_{1\nu} k_{1\beta} \eta_{\mu\alpha} \eta_{\gamma\rho} \eta_{\lambda\delta}) + \frac{1}{2} P_{12}(k_{1\nu} k_{1\beta} \eta_{\mu\alpha} \eta_{\gamma\sigma} \eta_{\rho\lambda})], \\
& \sim \text{sym}[\frac{1}{4} P_{12}(k_{1\nu} k_{1\beta} \eta_{\mu\alpha} \eta_{\gamma\rho} \eta_{\lambda\delta}) + \frac{1}{2} P_{12}(k_{1\nu} k_{1\beta} \eta_{\mu\alpha} \eta_{\gamma\rho} \eta_{\lambda\sigma})], \\
& \sim \text{sym}[-2P_{12}(k_{1\nu} k_{1\beta} \eta_{\mu\alpha} \eta_{\gamma\rho} \eta_{\lambda\delta}) - 2P_{12}(k_{1\nu} k_{1\beta} \eta_{\mu\alpha} \eta_{\gamma\rho} \eta_{\lambda\sigma})], \\
& \sim \text{sym}[2P_6(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\gamma\delta} \eta_{\rho\lambda}) + 4P_6(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\gamma\sigma} \eta_{\rho\lambda})], \\
& \sim \text{sym}[-2P_{24}(k_{1\nu} k_{1\beta} \eta_{\mu\alpha} \eta_{\gamma\rho} \eta_{\lambda\delta}) - 2P_{24}(k_{1\nu} k_{1\beta} \eta_{\mu\alpha} \eta_{\gamma\rho} \eta_{\lambda\sigma})], \\
& \sim \text{sym}[-2P_{24}(k_{1\nu} k_{2\mu} \eta_{\mu\alpha} \eta_{\gamma\rho} \eta_{\lambda\delta}) - 2P_{24}(k_{1\nu} k_{2\mu} \eta_{\mu\alpha} \eta_{\gamma\rho} \eta_{\lambda\sigma})], \\
& \sim \text{sym}[P_{24}(k_{1\nu} k_{2\mu} \eta_{\alpha\beta} \eta_{\gamma\rho} \eta_{\lambda\delta}) + P_{24}(k_{1\nu} k_{2\mu} \eta_{\alpha\beta} \eta_{\gamma\rho} \eta_{\lambda\sigma}) + P_{24}(k_{1\nu} k_{2\mu} \eta_{\alpha\beta} \eta_{\gamma\rho} \eta_{\lambda\sigma})], \\
& \sim \text{sym}[P_{24}(k_{1\nu} k_{2\mu} \eta_{\alpha\beta} \eta_{\gamma\rho} \eta_{\lambda\delta}) + P_{24}(k_{1\nu} k_{2\mu} \eta_{\alpha\beta} \eta_{\gamma\rho} \eta_{\lambda\sigma}) + P_{24}(k_{1\nu} k_{2\mu} \eta_{\alpha\beta} \eta_{\gamma\rho} \eta_{\lambda\sigma})]. \\
\end{align*} \]

Figure 4.3: A graphical representation of the graviton propagator, where a full line corresponds to a contraction of two indices.

The only way we can generate index loops is through a propagator. We can graphically represent a propagator contraction of two indices in a particular amplitude in the following way (see figure 4.4).

Essentially (disregarding symmetrizations of indices etc) only the following contractions of indices in the conventional graviton propagator can occur in an given amplitude, we can contract, \textit{e.g.}, \((\alpha \text{ and } \beta)\) or both \((\alpha \text{ and } \beta)\) and \((\gamma \text{ and } \delta)\). The results of such contractions
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Figure 4.4: A graphical representation of a propagator contraction in an amplitude. The black half dots represent an arbitrary internal structure for the amplitude, the full lines between the two half dots are internal contractions of indices, the outgoing lines represents index contractions with external sources.

are shown below and also graphically depicted in figure 4.5.

\[
\frac{1}{2} \left[ \eta_{\alpha \gamma} \eta_{\beta \delta} + \eta_{\alpha \delta} \eta_{\beta \gamma} - \frac{2}{D-2} \eta_{\alpha \beta} \eta_{\gamma \delta} \right] \eta^{\alpha \beta} = \eta_{\gamma \delta} - \frac{D}{D-2} \eta_{\gamma \delta} \\
\frac{1}{2} \left[ \eta_{\alpha \gamma} \eta_{\beta \delta} + \eta_{\alpha \delta} \eta_{\beta \gamma} - \frac{2}{D-2} \eta_{\alpha \beta} \eta_{\gamma \delta} \right] \eta^{\alpha \beta} \eta^{\gamma \delta} = - \frac{2D}{D-2}
\]

(4.12)

We can also contract, e.g., $(\alpha$ and $\gamma)$ or both $(\alpha$ and $\gamma)$ and $(\beta$ and $\delta)$ again the results are shown below and depicted graphically in figure 4.6.

\[
\frac{1}{2} \left[ \eta_{\alpha \gamma} \eta_{\beta \delta} + \eta_{\alpha \delta} \eta_{\beta \gamma} - \frac{2}{D-2} \eta_{\alpha \beta} \eta_{\gamma \delta} \right] \eta^{\alpha \gamma} = \frac{1}{2} \eta_{\beta \delta} D + \eta_{\beta \delta} \left( \frac{1}{2} + \frac{1}{D-2} \right) \\
\frac{1}{2} \left[ \eta_{\alpha \gamma} \eta_{\beta \delta} + \eta_{\alpha \delta} \eta_{\beta \gamma} - \frac{2}{D-2} \eta_{\alpha \beta} \eta_{\gamma \delta} \right] \eta^{\alpha \gamma} \eta^{\beta \delta} = D^2 - \frac{D}{D-2}
\]

(4.13)

Figure 4.5: Two possible types of contractions for the propagator. Whenever we have an index loop we have a contraction of indices. It is seen that none of the above contractions will generate something with a positive power of $(D)$.

Figure 4.6: Two other types of index contractions for the propagator. Again it is seen that only the $(\eta_{\alpha \gamma} \eta_{\beta \delta} + \eta_{\alpha \delta} \eta_{\beta \gamma})$-part of the propagator is important in large-$D$ considerations.

As it is seen, only the $(\eta_{\alpha \gamma} \eta_{\beta \delta} + \eta_{\alpha \delta} \eta_{\beta \gamma})$ part of the propagator will have the possibility to contribute with a positive power of $(D)$. Whenever the $(\eta_{\alpha \beta} \eta_{\gamma \delta})$-term in the propagator is in play, e.g., what remains is something that goes as $(\sim \frac{D}{D-2})$ and which consequently will have no support to the large-$D$ leading loop contributions.
We have seen that different index structures go into the same vertex factor. Below (see figure 4.7), we depict the same 2-loop diagram, but for different index structures of the external 3-point vertex factor (i.e., (3C) and (3E)) for the external lines. Different types of trace structures will hence occur in the same loop diagram. Every diagram will consist of a sum of contributions with a varying number of traces. It is seen that the graphical depiction of the index structure in the vertex factor is a very useful tool in determining the trace structure of various diagrams.

The following type of two-point diagram (see figure 4.8) build from the (3B) or (3C) parts of the 3-point vertex will generate a contribution which carries a maximal number of traces compared to the number of vertices in the diagram. The other parts of the 3-point vertex factor, e.g., the (3A) or (3E) parts respectively will not generate a leading contribution in the large-\( D \) limit, however such contributions will go into the non-leading contributions of the theory (see figure 4.9). The diagram built from the (3A) index structure will, e.g., not carry a leading contribution due to our previous considerations regarding contractions of the indices in the propagator (see figure 4.5).

It is crucial that we obtain a consistent limit for the graphs, when we take (\( D \to \infty \)). In order to cancel the factor of (\( D^2 \)) from the above type of diagrams, i.e., to put it on equal footing in powers of (\( D \)) as the simple propagator diagram, it is seen that we need to rescale (\( \kappa \to \frac{\kappa}{D} \)). In the large-\( D \) limit (\( \frac{\kappa}{D} \)) will define the ‘new’ redefined gravitational coupling. A finite limit for the 2-point function will be obtained by this choice, i.e., the leading \( D \)-contributions will not diverge, when the limit (\( D \to \infty \)) is imposed. By this choice all leading \( n \)-point diagrams will be well defined and go as: (\( \sim (\frac{\kappa}{D})^{(n-2)} \)) in a (\( \frac{1}{D} \)) expansion. Each external line will carry a gravitational coupling\(^1\). However, it is important to note that our rescaling of (\( \kappa \)) is solely determined by the requirement of creating a finite consistent large-\( D \) limit, where all \( n \)-point tree diagrams and their loop

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\(^1\)A tadpole diagram will thus not be well defined by this choice but instead go as (\( \sim D \)) however because we can dimensionally regularize any tadpole contribution away, any idea to make the tadpoles well defined in the large-\( D \) limit would occur to be strange.
corrections are on an equal footing. The above rescaling seems to be the obvious to do from the viewpoint of creating a physical consistent theory. However, we note that other limits for the graphs are in fact possible by other redefinitions of \((\kappa)\) but such rescalings will, \textit{e.g.}, scale away the loop corrections to the tree diagrams, thus such rescalings must be seen as rather unphysical from the viewpoint of traditional quantum field theory.

For any loop with two propagators the maximal number of traces are obviously two. However, we do not know which diagrams will generate the maximal number of powers of \(D\) in total. Some diagrams will have more traces than others, however because of the rescaling of \((\kappa)\) they might not carry a leading contribution after all. For example, if we look at the two graphs depicted in figure 4.10, they will in fact go as: \((D^3 \times \frac{\kappa^4}{\pi^4} \times D^3 \sim \frac{1}{D})\) and \((D^5 \times \frac{\kappa^8}{\pi^8} \times D^5 \sim \frac{1}{D^3})\) respectively and thus not be leading contributions.

Figure 4.10: Two multi-loop diagrams with many traces. The rescaling of \((\kappa)\) will however render both diagrams as non-leading contributions compared to diagrams with separate loops. Here and in the next coming figures we will employ the following notation for the index structure of the lines originating from loop contributions. Whenever a line originates from a loop contribution it should be understood that there are only two possibilities for its index structure, \textit{i.e.}, the lines have to be of the same type as in the contracted index lines in the index structure (3B) or as the dashed lines in (3C). No assumptions on the index structures for tree external lines are made, here any viable index structures are present, but we still represent such lines with a single full line for simplicity.

In order to arrive at the result of ref. [42], we have to consider the rescaling of \((\kappa)\) which will generate negative powers of \((D)\) as well. Let us now discuss the main result of ref. [42].

Let \((P_{\text{max}})\) be the maximal power of \((D)\) associated with a graph. Next we can look at the function \((\Pi)\):

\[
\Pi = 2L - \sum_{m=3}^{\infty} (m - 2)V_m. \tag{4.14}
\]

Here \((L)\) is the number of loops in a given diagram, and \((V_m)\) counts the number of \(m\)-point vertices. It is obviously true that \((P_{\text{max}})\) must be less than or equal to \((\Pi)\).

The above function counts the maximal number of positive factors of \((D)\) occurring for a given graph. The first part of the expression simply counts the maximal number of traces \((\sim D^2)\) from a loop, the part which is subtracted counts the powers of \((\kappa \sim \frac{1}{D})\) arising from the rescaling of \((\kappa)\).

Next, we employ the two well-known formulas for the topology of diagrams:

\[ L = I - V + 1, \tag{4.15} \]

\[ \sum_{m=3}^{\infty} mV_m = 2I + E, \tag{4.16} \]

where \((I)\) is the number of internal lines, \((V = \sum_{m=3}^{\infty} V_m)\) is total number of vertices and \((E)\) is the number of external lines.
Putting these three formulas together one arrives at:

\[ P_{\text{max}} \leq 2 - E - 2 \sum_{m=3}^{\infty} V_m. \] (4.17)

Hence, the maximal power of \( (D) \) is obtained in the case of graphs with a minimal number of external lines, and two traces per diagram loop. In order to generate the maximum of two traces per loop, separated loops are necessary, because whenever a propagator index line is shared in a diagram we will not generate a maximal number of traces. Considering only separated bubble graphs, thus the following type of \( n \)-point (see figure 4.11) diagrams will be preferred in the large-\( D \) limit:

![Figure 4.11: Examples of multi-loop diagrams with leading contributions in the large-\( D \) limit.](image)

Vertices, which generate unnecessary additional powers of \( (\frac{1}{D}) \), are suppressed in the large-\( D \) limit; multi-loop contributions with a minimal number of vertex lines will dominate over diagrams with more vertex lines. That is, in figure 4.12 the first diagram will dominate over the second one.

![Figure 4.12: Of the following two diagrams with double trace loops the first one will be leading.](image)

Thus, separated bubble diagrams will contribute to the large-\( D \) limit in quantum gravity. The 2-point bubble diagram limit is shown in figure 4.13.

![Figure 4.13: Leading bubble 2-point diagrams.](image)

Another type of diagrams, which can make leading contributions, are the diagrams depicted in figure 4.14. These contributions are of a type where a propagator directly combines different legs in a vertex factor with a double trace loop. Because of the index structure (4J) in the 4-point vertex such contributions are possible. These types of diagrams will have the same dimensional dependence: \( \left( \frac{4}{D^4} \times D^4 \sim 1 \right) \), as the 2-loop separated bubble diagrams. It is claimed in ref. [42], that such diagrams will be non-leading. We have however found no evidence of this in our investigations of the large-\( D \) limit.
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Figure 4.14: Another type of 2-point diagram build from the index structure (4J), which will be leading in the large-\(D\) limit.

Higher order vertices with more external legs can generate multi-loop contributions of the vertex-loop type, e.g., a 6-point vertex can in principle generate a 3-loop contribution to the 3-point function, a 8-point vertex a 4-loop contribution the the 4-point function etc (see figure 4.15).

Figure 4.15: Particular examples of leading vertex-loop diagrams build from 6-point and 8-point vertex factors respectively.

The 2-point large-\(D\) limit will hence consist of the sum of the propagator and all types of leading 2-point diagrams which one considered above, i.e., bubble diagrams and vertex-loops diagrams and combinations of the two types of diagrams. Ghost insertions will be suppressed in the large-\(D\) limit, because such diagrams will not carry two traces, however they will go into the theory at non-leading order. The general \(n\)-point function will consist of diagrams built from separated bubbles or vertex-loop diagrams. We have devoted appendix B.2 to a study of the generic \(n\)-point functions.

4.3 The large-\(D\) limit and the effective extension of gravity

So far we have avoided the renormalization difficulties of Einstein gravity. To obtain an effective renormalizable theory up to the Planck scale we will have to introduce the effective field theory description.

The most general effective Lagrangian of a \(D\)-dimensional gravitational theory has the form:

\[
\mathcal{L} = \int d^Dx \sqrt{-g} \left( \frac{2}{\kappa^2} R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + \ldots \right) + \mathcal{L}_{\text{eff. matter}},
\]

(4.18)

where \(R^\alpha_{\mu\nu\beta}\) is the curvature tensor \((g = -\det(g_{\mu\nu}))\) and the gravitational coupling is defined as before, i.e., \((\kappa^2 = 32\pi G_D)\). The matter Lagrangian includes in principle everything which couple to a gravitational field, i.e., any effective or higher derivative couplings of gravity to bosonic and or fermionic matter as has been discussed previously. Here we will however look solely at pure gravitational interactions. The effective Lagrangian for the theory is then reduced to invariants built from the Riemann tensors:

\[
\mathcal{L} = \int d^Dx \sqrt{-g} \left( \frac{2R}{\kappa^2} + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + \ldots \right).
\]

(4.19)
In an effective field theory higher derivative couplings of the fields are allowed for, while the underlying physical symmetries of the theory are kept manifestly intact. In gravity the general action of the theory has to be covariant with respect to the external gravitational fields.

The renormalization problems of traditional Einstein gravity are trivially solved by the effective field theory approach. The effective treatment of gravity includes all possible invariants and hence all divergences occurring in the loop diagrams can be absorbed in a renormalized effective action.

The effective expansion of the theory will be an expansion of the Lagrangian in powers of momentum, and the minimal powers of momentum will dominate the effective field theory at low-energy scales. In gravity this means that the \( R \sim (\text{momentum})^2 \) term will be dominant at normal energies, i.e., gravity as an effective field theory at normal energies will essentially still be general relativity. At higher energy scales the higher derivative terms \( R^2 \sim (\text{momentum})^4 \), \( \ldots \), \( R^3 \sim (\text{momentum})^6 \) \ldots, corresponding to higher powers of momentum will mix in and become increasingly important.

The large-\( D \) limit in Einstein gravity is arrived at by expanding the theory in \( (D) \) and subsequently taking \( (D \to \infty) \). This can be done in its effective extension too. Treating gravity as an effective theory and deriving the large-\( D \) limit results in a double expansion of the theory, i.e., we expand the theory both in powers of momentum and in powers of \( (\frac{1}{D}) \). In order to understand the large-\( D \) limit in the case of gravity as an effective field theory, we have to examine the vertex structure for the additional effective field theory terms. Clearly any vertex contribution from, e.g., the effective field theory terms, \( (R^2) \), and, \( (R^2_{\mu\nu}) \), will have four momentum factors and hence new index structures are possible for the effective vertices.

We find the following vertex structure for the \( (R^2) \) and the \( (R^2_{\mu\nu}) \) terms of the effective action (see figure 4.16 and appendix B.1).

\[
V_{3}^{\text{eff}} = \kappa^3 \left( \begin{array}{c} \text{3A} \\ \text{3B} \end{array} \right)^{\text{eff}} + \kappa^3 \left( \begin{array}{c} \text{3C} \\ \text{3D} \end{array} \right)^{\text{eff}} + \kappa^3 \left( \begin{array}{c} \text{3E} \\ \text{3F} \end{array} \right)^{\text{eff}} + \kappa^3 \left( \begin{array}{c} \text{3G} \\ \text{3H} \end{array} \right)^{\text{eff}}.
\]

Figure 4.16: A graphical representation of the \( (R^2) \) and the \( (R^2_{\mu\nu}) \) index structures in the effective 3-point vertex factor. A dashed line represents a contraction of an index with a momentum line. A full line symbolizes a contraction of two index lines.

We are particularly interested in the terms which will give leading trace contributions. For the \( (3B)^{\text{eff}} \) and \( (3C)^{\text{eff}} \) index structures of we have shown some explicit results (see figure 4.17).

It is seen that also in the effective theory there are terms which will generate double trace structures. Hence an effective 1-loop bubble diagram with two traces is possible. Such diagrams will go as \( \left( \frac{\kappa^4}{D^4} c_1 \times D^2 \sim \frac{c_1}{D^2} \right) \). In order to arrive at the same limit for the effective tree diagrams as the tree diagrams in Einstein gravity we need to rescale \( (c_1) \)
4.3. THE LARGE-D LIMIT AND THE EFFECTIVE EXTENSION OF GRAVITY

and \((c_2)\) by \((D^2)\), i.e., \((c_1 \rightarrow c_1 D^2)\) and \((c_2 \rightarrow c_2 D^2)\). Thereby every \(n\)-point function involving the effective vertices will go into the \(n\)-point functions of Einstein gravity. In fact this will hold for any \((c_i)\) provided that the index structures giving double traces also exist for the higher order effective terms. Other rescalings of the \((c_i)\)’s are possible. But such other rescalings will give a different tree limit for the effective theory at large-\(D\).

The effective tree-level graphs will be thus be scaled away from the Einstein tree graph limit.

The above rescaling with \((c_i \rightarrow c_i D^2)\) corresponds to the maximal rescaling possible. Any rescaling with, e.g., \((c_i \rightarrow c_i D^n, n > 2)\) will not be possible, if we still want to obtain a finite limit for the loop graphs at \((D \rightarrow \infty)\). A rescaling with \((D^2)\) will give maximal support to the effective terms at large-\(D\).

The effective terms in the action are needed in order to renormalize infinities from the loop diagrams away. The renormalization of the effective field theory can be carried out at any particular \((D)\). Of course the exact cancellation of pole terms from the loops will depend on the integrals, and the algebra will change with the dimension, but one can take this into account for any particular order of \((D = D_p)\) by an explicit calculation of the counter-terms at dimension \((D_p)\) followed by an adjustment of the coefficients in the effective action. The exact renormalization will in this way take place order by order in \((\frac{1}{D})\). For any rescaling of the \((c_i)\)’s it is thus always possible carry out a renormalization of the theory. Depending on how the rescaling of the coefficients \((c_i)\) are done, there will be different large-\(D\) limits for the effective field theory extension of the theory.

The effective field theory treatment does in fact not change the large-\(D\) of gravity as much as one could expect. The effective enlargement of gravity does not introduce new exciting leading diagrams. Most parts of our analysis of the large-\(D\) limit depend only on the index structure for the various vertices, and the new index structures provided by the effective field theory terms only add new momentum lines, which do not give additional traces over index loops. Therefore the effective extension of Einstein gravity is not a very radical change of its large-\(D\) limit. In the effective extension of gravity the action is trivially renormalizable up to a cut-off scale at \((M_{\text{Planck}})\) and the large-\(D\) expansion just as well defined as an large-\(N\) expansion of a renormalizable planar expansion of a Yang-Mills action.

As a subject for further investigations, it might be useful to employ more general considerations about the index structures. Seemingly all possible types of index-structures
are present in the effective field theory extension of gravity in the conventional approach. As only certain index structures go into the leading large-$D$ limit, the *interesting* index structures can be classified at any given loop order. In fact one could consider loop amplitudes with only *interesting* index structures present. Such applications might useful in very complicated quantum gravity calculations, where only the leading large-$D$ behavior is interesting.

A comparison of the index structures present in other field expansions and the conventional expansion might also be a working area for further investigations. Every index structure in the conventional expansion of the field can be build up from the following types of index structures, see figure 4.18.

![Figure 4.18: The first 8 basic index structures which builds up any conventional vertex factor.](image)

Thus, it is not possible to consider any index structures in a 3- or 4-point vertex, which cannot be decomposed into some product of the above types of index contractions. It might be possible to extend this to something useful in the analysis of the large-$D$ limit, *e.g.*, a simpler description of the large-$D$ diagrammatic truncation of the theory. This is another interesting working area for further research.

### 4.4 Considerations about the space-time integrals

The space-time integrals pose certain fundamental restrictions to our analysis of the large-$D$ limit in gravity. So far we have considered only the algebraic trace structures of the $n$-point diagrams. This analysis lead us to a consistent large-$D$ limit of gravity, where we saw that only bubble and vertex-loop diagrams will carry the leading contributions in $(\frac{1}{D})$.

In a large-$N$ limit of a gauge theory this would have been adequate, however the space-time dimension is much more that just a symmetry index for the gauge group, and has a deeper significance for the physical theory than just that of a symmetry index. Space-time integrations in a $D$-dimensional world will contribute extra dimensional dependencies to the graphs, and it thus has to be investigated if the $D$-dimensional integrals could upset the algebraic large-$D$ limit. The issue about the $D$-dimensional integrals is not resolved in ref. [42], only explicit examples are discussed and some conjectures about the contributions from certain integrals in the graphs at large-$D$ are stated. It is clear that in the case of the 1-loop separated integrals the dimensional dependence will go as $\sim \Gamma(\frac{D-1}{2})$. Such a dimensional dependence can always be rescaled into an additional redefinition of $(\kappa)$, *i.e.*, we can redefine $\left(\kappa \rightarrow \frac{\kappa}{\sqrt{C}}\right)$ where $C \sim \Gamma(\frac{D-1}{2})$. The dimensional dependence from both bubble graphs and vertex-loop graphs will be possible to scale away in this manner. Graphs which have a nested structure, *i.e.*, which have shared propagator lines, will be however be suppressed in $(D)$ compared to the separated loop diagrams. Thus the rescaling of $(\kappa)$ will make the dimensional dependence of such diagrams even worse.

In the lack of rigorously proven mathematical statements, it is hard to be completely
4.5 Comparison of the large-$N$ limit and the large-$D$ limit

The large-$N$ limit of a gauge theory and the large-$D$ limit in gravity have some similarities and some dissimilarities which we will look upon here.

The large-$N$ limit in a gauge theory is the limit where the symmetry indices of the gauge theory can go to infinity. The idea behind the gravity large-$D$ limit is to do the same, where in this case $(D)$ will play the role of $(N)$, we thus treat the spatial dimension $(D)$ as if it is only a symmetry index for the theory, i.e., a physical parameter, in which we are allowed to make an expansion.

As it was shown by 't Hooft in ref. [97] the large-$N$ limit in a gauge theory will be a planar diagram limit, consisting of all diagrams which can be pictured in a plane. The leading diagrams will be of the type depicted in figure 4.19.

The arguments for this diagrammatic limit follow similar arguments to those we applied in the large-$D$ limit of gravity. However the index loops in the gauge theory are index loops of the internal symmetry group, and for each type of vertex there will only be one
Figure 4.19: A particular planar diagram in the large-$N$ limit. A $(\times)$ symbolizes an external source, the full lines are index lines, a full internal index loop gives a factor of $(N)$.

index structure. Below we show the particular index structure for a 3-point vertex in a gauge theory.

This index structure is also present in the gravity case, \textit{i.e.}, in the (3E) index structure, but various other index structure are present too, \textit{e.g.}, (3A), (3B), \ldots, and each index structure will generically give dissimilar traces, \textit{i.e.}, dissimilar factors of $(D)$ for the loops.

In gravity only certain diagrams in the large-$D$ limit carry leading contributions, but only particular parts of the graph’s amplitudes will survive when $(D \to \infty)$ because of the different index structures in the vertex factor. The gravity large-$D$ limit is hence more like a truncated graph limit, where only some parts of the graph amplitude will be important, \textit{i.e.}, every leading graph will also carry contributions which are non-leading.

In the Yang-Mills large-$N$ limit the planar graph’s amplitudes will be equally important as a whole. No truncation of the graph’s amplitudes occur. Thus, in a Yang-Mills theory the large-$N$ amplitude is found by summing the full set of planar graphs. This is a major dissimilarity between the two expansions.

The leading diagram limits in the two expansions are not identical either. The leading graphs in gravity at large-$D$ are given by the diagrams which have a possibility for having a closed double trace structure, hence the bubble and vertex-loop diagrams are favored in the large-$D$ limit. The diagrams contributing to the gravity large-$D$ limit will thus only be a subset of the full planar diagram limit.
4.6 Other expansions of the fields

In gravity it is a well-known fact that different definitions of the fields will lead to different results for otherwise similar diagrams. Only the full amplitudes will be gauge invariant and identical for different expansions of the gravitational field. Therefore comparing the large-$D$ diagrammatical expansion for dissimilar definitions of the gravitational field, diagram for diagram, will have no meaning. The analysis of the large-$D$ limit carried out in this chapter is based on the conventional definition of the gravitational field, and the large-$D$ diagrammatic expansion should hence be discussed in this light.

Other field choices may be considered too, and their diagrammatic limits should be investigated as well. It may be worth some efforts and additional investigations to see, if the diagrammatic large-$D$ limit considered here is an unique limit for every definition of the field, i.e., to see if the same diagrammatic limit always will occur for any definition of the gravitational field. A scenario for further investigations of this, could, e.g., be to look upon if the field expansions can be dressed in such a way that we reach a less complicated diagrammatic expansion at large-$D$, at the cost of having a more complicated expansion of the Lagrangian. For example, it could be that vertex-loop corrections are suppressed for some definitions of the gravitational field, but that more complicated trace structures and cancellations between diagrams occurs in the calculations. In the case of the Goldberg definition of the gravitational field the vertex index structure is less complicated ref. [42], however the trace contractions of diagrams appears to much more complicated than in the conventional method. It should be investigated more carefully if the Goldberg definition of the field may be easier to employ in calculations. Indeed the diagrammatical limit considered by ref. [42] is easier, but it is not clear if results from two different expansions of the gravitational field are used to derive this. Further investigations should shed more light on the large-$D$ limit, and on how a field redefinition may or may not change its large-$D$ diagrammatic expansion.

Furthermore one can consider the background field method in the context of the large-$D$ limit as well. This is another working area for further investigations. The background field method is very efficient in complicated calculations and it may be useful to employ it in the large-$D$ analysis of gravity as well. No problems in using background field methods together with a large-$D$ quantum gravity expansion seem to be present.

4.7 Discussion

In this chapter we have discussed the large-$D$ limit of effective quantum gravity. In the large-$D$ limit, a particular subset of planar diagrams will carry all leading $(\frac{1}{D})$ contributions to the $n$-point functions. The large-$D$ limit for any given $n$-point function will consist of the full tree $n$-point amplitude, together with a set of loop corrections which will consist of the leading bubble and vertex-loop graphs we have considered. The effective treatment of the gravitational action insures that a renormalization of the action is possible and that none of the $n$-point renormalized amplitudes will carry uncanced divergent pole terms. An effective renormalization of the theory can be performed at any particular dimensionality.
The leading \( (\frac{1}{D}) \) contributions to the theory will be algebraically less complex than the graphs in the full amplitude. Calculating only the \( (\frac{1}{D}) \) leading contributions simplify explicit calculations of graphs in the large-\( D \) limit. The large-\( D \) limit we have found is completely well defined as long as we do not extend the space-time integrations to the full \( D \)-dimensional space-time. That is, we will always have a consistent large-\( D \) theory as long as the integrations only gain support in a finite dimensional space-time and the remaining extra dimensions in the space-time are left as compactified, \( e.g. \), below the Planck scale.

We hence have a renormalizable, definite and consistent large-\( D \) limit for effective quantum gravity – good below the Planck scale!

No solution to the problem of extending the space-time integrals to a full \( D \)-dimensional space-time has been presented in this paper, however it is clear that the effective treatment of gravity do not impose any additional problems in such investigations. The considerations about the \( D \)-dimensional integrals discussed in ref. [42] hold equally well in an effective field theory, however as well as we will have to make an additional rescaling of \( (\kappa) \) in order to account for the extra dimensional dependence coming from the \( D \)-dimensional integrals, we will have make an extra rescaling of the coefficients \( (c_i) \) in the effective Lagrangian too.

Our large-\( D \) graph limit is not in agreement with the large-\( D \) limit of ref. [42], the bubble graph are present in both limits, but vertex-loops are not allowed and claimed to be down in \( (\frac{1}{D}) \) the latter. We do not agree on this point, and believe that this might be a consequence of comparing similar diagrams with different field definitions. It is a well-known fact in gravity that dissimilar field definitions lead to different results for the individual diagrams. Full amplitudes are of course unaffected by any particular definition of the field, but results for similar diagrams with different field choices cannot be immediately compared.

Possible extensions of the large-\( D \) considerations discussed here would be to allow for external matter in the Lagrangian. This might present some new interesting aspects in the analysis, and would relate the theory more directly to external physical observables, \( e.g. \), scattering amplitudes, and corrections to the Newton potential at large-\( D \) or to geometrical objects such as a space-time metric. The investigations carried out in refs. [35, 37, 38] could be discussed from a large-\( D \) point of view.

Investigations in quantum gravitational cosmology may also present an interesting work area for applications of the large-\( D \) limit, \( e.g. \), quantum gravity large-\( D \) big bang models etc.

The analysis of the large-\( D \) limit in quantum gravity have prevailed that using the physical dimension as an expansion parameter for the theory, is not as uncomplicated as expanding a Yang-Mills theory at large-\( N \). The physical dimension goes into so many aspects of the theory, \( e.g., \) the integrals, the physical interpretations etc, that it is far more complicated to understand a large-\( D \) expansion in gravity than a large-\( N \) expansion in a Yang-Mills theory. In some sense it is hard to tell what the large-\( D \) limit of gravity really physically relates to, \( i.e., \) what describes a gravitational theory with infinitely many spatial dimensions? One way to look at the large-\( D \) limit in gravity, might be to see the large-\( D \) as a physical phase transition of the theory at the Planck scale. This suggests that the large-\( D \)
expansion of gravity should be seen as a very high energy scale limit for a gravitational
theory.

Large-\( N \) expansions of gauge theories ref. [97] have many interesting calculational and
fundamental applications in, \( e.g. \), string theory, high energy, nuclear and condensed mat-
ter physics. A planar diagram limit for a gauge theory will be a string theory at large
distance, this can be implemented in relating different physical limits of gauge theory
and fundamental strings. In explicit calculations, large-\( N \) considerations have many im-
portant applications, \( e.g. \), in the approximation of QCD amplitudes from planar diagram
considerations or in the theory of phase-transitions in condensed matter physics, \( e.g. \), in
theories of superconductivity, where the number of particles will play the role of \( \langle N \rangle \).

High energy physics and string theory are not the only places where large symmetry
investigations can be employed to obtain useful information. As a calculational tool
large-\( D \) considerations can be imposed in quantum mechanics to obtain useful information
about eigenvalues and wave-functions of arbitrary quantum mechanical potentials with a
extremely high precision ref. [98].

Practical applications of the large-\( D \) limit in explicit calculations of \( n \)-point functions
could be the scope for further investigations as well as certain limits of string theory
might be related to the large-\( D \) limit in quantum gravity. The planar large-\( D \) limit
found in our investigation might resemble that of a truncated string limit at large dis-
tances? Investigations of the (Kawai-Lewellen-Tye ref. [67]) open/closed string, gauge
theory/gravity relations, could also be interesting to perform in the context of a large-\( D \)
expansion. Such questions are outside the scope of this investigation, however knowing
the huge importance of large-\( N \) considerations in modern physics, large-\( D \) considerations
of gravity might pose very interesting tasks for further research.
Chapter 5

Summary and outlook

In this thesis we have looked at different aspects of quantum gravity from the viewpoint of general relativity being an effective field theory. We have discussed in detail how to calculate quantum gravitational corrections, and presented results of such calculations of the quantum corrections to both the Schwarzschild and Kerr metrics as well as to the scattering matrix potentials. The uniqueness of the quantum corrections has also been dealt with.

Low-energy string limits have also been discussed from the natural point of view of being specific effective actions at low energies. Using the Kawai-Lewellen-Tye string relations as a proposal for a connection between gauge theories and gravity we have derived a mysterious diagrammatic relationship between higher derivative operators. Quantum gravity with an infinitely number of spatial dimensions have also been looked at. In this case it has been shown that it is possible to treat general relativity as an effective field theory even in a large-$D$ limit. As a spin-off of this investigation the less-complicated diagrammatic limit at ($D = \infty$) might be used to approximate $n$-point gravity loop amplitudes at ($D = 4$).

It has been illustrated quite clearly that general relativity in its effective extension propose a rather good way to deal with the challenges of quantum gravity. The theory make perfect sense below the energies at the Planck scale, meaning that we have excellent theory for quantum gravity at all energies that can be reached by present experiments.

The effective approach is in some sense the most general and natural extension of general relativity one can think of. It is hard to think of any principles which should prevent the gravitational action from containing terms which are only of first order in the curvature tensor. The effective approach clearly is a very obvious way to deal with non-renormalizable theories. Any high-energy theory should in principle correspond to a specific effective theory and perhaps that is the most problematic issue in the effective field theory approach. Because effective coupling constants are experimental quantities they lack predictability, where the energies scales are to high to carry out experiments.

However, as it hopefully has been shown in this thesis, taking on the effective approach and investigating various issues concerning quantum gravity lead in fact to important results, which are unique and interesting from the point of view of understanding any theory of quantum gravity.
At very high energy scales the quantum picture of the effective field theory breaks down. About this theory we still do not know much. Further investigations are needed before we truly understand quantum gravity at such energy scales. However until a complete theory for quantum gravity is proposed we should use every aspect of quantum gravity as an effective field theory to uncover Nature’s secrets.
Appendix A

Effective field theory treatment of gravity

A.1 Summary of the vertex rules

Scalar propagator

The massive scalar propagator is well known:

\[ \frac{\mathcal{P}_\alpha}{q^2 - m^2 + i\epsilon} \]

Photon propagator

The photon propagator is also known from the literature. We have applied Feynman gauge which gives the least complicated propagator:

\[ \frac{-i\eta_{\gamma\delta}}{q^2 + i\epsilon} \]

Graviton propagator

The graviton propagator in harmonic gauge is discussed in the literature refs. [16, 33], but can be derived quite easily explicitly refs. [51]. We shall write it in the form:

\[ \frac{i\mathcal{P}^{\alpha\beta\gamma\delta}}{q^2 + i\epsilon} \]

where:

\[ \mathcal{P}^{\alpha\beta\gamma\delta} = \frac{1}{2} \left[ \eta^{\alpha\gamma} \eta^{\beta\delta} + \eta^{\beta\gamma} \eta^{\alpha\delta} - \eta^{\alpha\beta} \eta^{\gamma\delta} \right] \]  

(A.1.1)
2-scalar-1-photon vertex

The 2-scalar-1-photon vertex is well known in the literature. We will write this vertex as:

\[ \tau_1^\gamma(p, p', e) = \tau_1^\gamma(p, p', e) \]

where:

\[ \tau_1^\gamma(p, p', e) = -ie(p + p')^\gamma. \]  \hspace{1cm} (A.1.2)

2-photon-1-graviton vertex

For the 2-photon-1-graviton vertex we have derived:

\[ \tau_3^{\rho\sigma(\gamma\delta)}(p, p') = \tau_3^{\rho\sigma(\gamma\delta)}(p, p') \]

where:

\[ \tau_3^{\rho\sigma(\gamma\delta)}(p, p') = i\kappa \left[ \mathcal{P}^{\rho\sigma(\gamma\delta)}(p \cdot p') + \frac{1}{2} \left( \eta^{\rho\sigma} p^\delta p^{\gamma \gamma} + \eta^{\gamma\delta} (p^\rho p^{\sigma} + p^\sigma p^{\rho}) - (p^{\gamma \gamma} p^\rho \eta^{\rho\delta} + p^{\rho \rho} p^\delta \eta^{\sigma\gamma} + p^{\sigma \sigma} p^\delta \eta^{\rho\gamma}) \right) \right]. \]  \hspace{1cm} (A.1.3)

2-scalar-2-photon vertex

The 2-scalar-2-photon vertex is also well known from scalar QED. We will write it as:

\[ \tau_4^{\gamma\delta}(p, p', e) = \tau_4^{\gamma\delta}(p, p', e) \]

where:

\[ \tau_4^{\gamma\delta}(p, p', e) = 2ie^2\eta^{\gamma\delta}. \]  \hspace{1cm} (A.1.4)

2-scalar-1-photon-1-graviton vertex

For the 2-scalar-1-photon-1-graviton vertex we have derived:
APPENDIX A. EFFECTIVE FIELD THEORY TREATMENT OF GRAVITY

For all vertices the rules of momentum conservation have been applied. For the external scalar lines we associate a factor of 1. At each loop we will integrate over the undetermined loop momentum.

For a certain diagram we will divide with the appropriate symmetry factor of the Feynman diagram.

2-scalar-1-graviton vertex

The 2-scalar-1-graviton vertex is well known. We will write it in the following way:

\[
\begin{align*}
\gamma & \quad p' \\
\sigma & \quad p
\end{align*}
\]

where:

\[
\tau_5^{\rho\sigma\gamma}(p, p', e) = i\kappa [\mathcal{P}^{\rho\sigma\alpha\gamma}(p + p')_\alpha],
\]

and \(\mathcal{P}^{\rho\sigma\alpha\gamma}\) is defined as above.

For all vertices the rules of momentum conservation have been applied. For the external scalar lines we associate a factor of 1. At each loop we will integrate over the undetermined loop momentum.

2-fermion-1-graviton vertex

The 2-fermion-1-graviton vertex has been derived and is known from the literature. We will write it in the following way:

\[
\begin{align*}
\alpha\beta & \quad p' \\
\gamma & \quad p
\end{align*}
\]

where:

\[
\tau_6^{\mu\nu}(p, p', m) = -\frac{i\kappa}{2} \left[ p^\mu p'^\nu + p'^\nu p'^\mu - \eta^{\mu\nu} ((p \cdot p') - m^2) \right].
\]

2-fermion-1-graviton vertex

The 2-fermion-1-graviton vertex has been derived and is known from the literature. We will write it in the following way:

\[
\begin{align*}
\alpha\beta & \quad p' \\
\gamma & \quad p
\end{align*}
\]

where:

\[
\tau_7^{\alpha\beta}(p, p', m) = -\frac{i\kappa}{2} \left[ \frac{1}{4} (\gamma^\alpha (p + p')^\beta + \gamma^\beta (p + p')^\alpha) - \frac{1}{2} \eta^{\alpha\beta} \left( \frac{1}{2} (p' + p') - m \right) \right].
\]
A.1. SUMMARY OF THE VERTEX RULES

2-scalar-2-graviton vertex

A discussion of the 2-scalar-2-graviton vertex can be found in the literature. In the expression below it will be written with the full symmetry of the two gravitons:

\[ \tau_8^{\eta\rho\sigma}(p, p', m) = \frac{\iota k^2}{2} \left\{ \Gamma^{\eta\rho\delta} \Gamma_{\delta}^{\rho\sigma} - \frac{1}{4} \left\{ \eta^{\eta\rho} \eta^{\rho\sigma} + \eta^{\rho\sigma} \eta^{\rho\eta} \right\} (p_\alpha p'_\beta + p'_\alpha p_\beta) \right\} \left[ (p \cdot p') - m^2 \right] \]  
(A.1.8)

2-fermion-2-graviton vertex

The 2-fermion-2-graviton vertex has been derived. We will write it in the following way:

\[ \tau_9^{\alpha\beta, \gamma\delta}(p, p', m) = \frac{\iota k^2}{2} \left\{ \left( \frac{1}{2} (p' + p') - m \right) p^{\alpha\beta, \gamma\delta} - \frac{1}{16} \left[ \Gamma^{\eta\rho\delta}(p + p')^{\alpha} + \Gamma^{\delta}(p + p')^{\alpha} \right] \right\} \gamma^{\gamma}(1 + p_\beta p'_\gamma) \]

3-graviton vertex

The 3-graviton vertex in the background field method has the form refs. [32, 33]
where:
\[
\tau_{\alpha\beta\gamma\delta}^{\mu\nu}(k, q) = -\frac{i\kappa}{2} \times \left( P_{\alpha\beta\gamma\delta} \left[ k^{\mu}k^{\nu} + (k - q)^{\mu}(k - q)^{\nu} + q^{\mu}q^{\nu} - \frac{3}{2}\eta^{\mu\nu}q^2 \right] \right) 
+ 2q_{\lambda}q_{\sigma} \left[ I_{\alpha\beta}^{\lambda\sigma} I_{\gamma\delta}^{\mu\nu} + I_{\gamma\delta}^{\lambda\sigma} I_{\alpha\beta}^{\mu\nu} - I_{\alpha\beta}^{\mu\sigma} I_{\gamma\delta}^{\nu\lambda} - I_{\gamma\delta}^{\mu\sigma} I_{\alpha\beta}^{\nu\lambda} \right] 
+ q_{M}q^{\mu} \left( \eta_{\alpha\lambda} I_{\gamma\delta}^{\mu\nu} + \eta_{\delta\gamma} I_{\alpha\beta}^{\mu\nu} \right) + q_{\lambda}q^{\nu} \left( \eta_{\alpha\beta} I_{\gamma\delta}^{\mu\nu} + \eta_{\gamma\delta} I_{\alpha\beta}^{\mu\nu} \right) 
- q^2 \left( \eta_{\alpha\beta} I_{\gamma\delta}^{\mu\nu} - \eta_{\gamma\delta} I_{\alpha\beta}^{\mu\nu} \right) - \eta^{\mu\nu} q_{\sigma} q_{\lambda} \left( \eta_{\alpha\beta} I_{\gamma\delta}^{\sigma\lambda} + \eta_{\gamma\delta} I_{\alpha\beta}^{\sigma\lambda} \right) 
+ 2q_{\lambda} \left( I_{\alpha\beta}^{\lambda\sigma} I_{\gamma\delta\sigma}^{-\mu\nu} (k - q)^{\mu} + I_{\alpha\beta}^{\lambda\sigma} I_{\gamma\delta\sigma}^{-\mu\nu} (k - q)^{\nu} - I_{\gamma\delta}^{\lambda\sigma} I_{\alpha\beta\sigma}^{-\mu\nu} k^{\mu} - I_{\gamma\delta}^{\lambda\sigma} I_{\alpha\beta\sigma}^{-\mu\nu} k^{\nu} \right) 
+ q^2 \left( I_{\alpha\beta\sigma}^{\mu\nu} I_{\gamma\delta\sigma}^{-\mu\nu} + I_{\alpha\beta\sigma}^{\nu\sigma} I_{\gamma\delta\sigma}^{-\mu\nu} + \eta^{\mu\nu} q_{\sigma} q_{\lambda} \left( I_{\alpha\beta}^{\lambda\rho} I_{\gamma\delta\rho}^{-\mu\nu} + I_{\gamma\delta}^{\lambda\rho} I_{\alpha\beta\rho}^{-\mu\nu} \right) \right) 
+ \left\{ (k^2 + (k - q)^2) \left[ I_{\alpha\beta}^{\mu\sigma} I_{\gamma\delta\sigma}^{-\mu\nu} + I_{\gamma\delta}^{\mu\sigma} I_{\alpha\beta\sigma}^{-\mu\nu} - \frac{1}{2} \eta^{\mu\nu} \mathcal{P}_{\alpha\beta\gamma\delta} \right] 
- \left( I_{\gamma\delta}^{\mu\nu} \eta_{\alpha\beta} k^2 + I_{\alpha\beta}^{\mu\nu} \eta_{\gamma\delta} (k - q)^2 \right) \right\} \right). 
\]

(A.1.9)

### A.2 Needed integrals in the calculation of diagrams

To calculate the diagrams the following integrals are needed

\[
J = \int \frac{d^4l}{(2\pi)^4 l^2(l + q)^2(l + k)^2(l + m)^2} = \frac{i}{32\pi^2} \left[ -2L \right] + \ldots, \quad (A.2.10)
\]

\[
J_{\mu} = \int \frac{d^4l}{(2\pi)^4 l^2(l + q)^2} = \frac{i}{32\pi^2} \left[ q_{\mu}L \right] + \ldots, \quad (A.2.11)
\]

\[
J_{\mu\nu} = \int \frac{d^4l}{(2\pi)^4 l^2(l + q)^2} = \frac{i}{32\pi^2} \left[ q_{\mu}q_{\nu} \left( -\frac{2}{3}L \right) - q^2 \eta_{\mu\nu} \left( -\frac{1}{6}L \right) \right] + \ldots \quad (A.2.12)
\]

together with

\[
I = \int \frac{d^4l}{(2\pi)^4 l^2(l + q)^2(l + k)^2(l + m)^2} = \frac{i}{32\pi^2 m^2} \left[ -L - S \right] + \ldots, \quad (A.2.13)
\]

\[
I_{\mu} = \int \frac{d^4l}{(2\pi)^4 l^2(l + q)^2(l + k)^2(l + m)^2} = \frac{i}{32\pi^2 m^2} \left[ k_{\mu} \left( -1 - \frac{q^2}{2m^2} \right) \left( -\frac{1}{4}L + \frac{1}{2}S \right) \right] + \ldots, \quad (A.2.14)
\]

\[
I_{\mu\nu} = \int \frac{d^4l}{(2\pi)^4 l^2(l + q)^2(l + k)^2(l + m)^2} = \frac{i}{32\pi^2 m^2} \left[ q_{\mu}q_{\nu} \left( -L - \frac{3}{8}S \right) + k_{\mu}k_{\nu} \left( -\frac{1}{2}L + \frac{1}{8}S \right) \right] + \ldots, \quad (A.2.15)
\]

\[
(q_{\mu}k_{\nu} + q_{\nu}k_{\mu}) \left( \frac{1}{2} + \frac{q^2}{2m^2} \right) L + \frac{3}{16} \frac{q^2}{m^2} S + q^2 \eta_{\mu\nu} \left( \frac{1}{4}L + \frac{1}{8}S \right) \right] + \ldots.
\]
\[ I_{\mu\nu\alpha} = \int \frac{d^4l}{(2\pi)^4 l^2(l+q)^2((l+k)^2 - m^2)} \frac{l_{\mu} l_{\nu} l_{\alpha}}{l^2(l+q)^2((l+k)^2 - m^2)} \]

\[
= \frac{i}{32\pi^2 m^2} \left[ q_{\mu} q_{\nu} q_{\alpha} \left( L + \frac{5}{16} S \right) + k_{\mu} k_{\nu} k_{\alpha} \left( -\frac{1}{6} \frac{q^2}{m^2} \right) \right. \\
+ \left( q_{\mu} k_{\nu} k_{\alpha} + q_{\nu} k_{\mu} k_{\alpha} + q_{\alpha} k_{\mu} k_{\nu} \right) \left( \frac{1}{3} \frac{q^2}{m^2} L + \frac{1}{16} \frac{q^2}{m^2} S \right) \\
+ \left( q_{\mu} q_{\nu} k_{\alpha} + q_{\nu} q_{\alpha} k_{\nu} + q_{\alpha} q_{\nu} k_{\mu} \right) \left( -\frac{1}{3} \frac{1}{2} \frac{q^2}{m^2} L - \frac{5}{32} \frac{q^2}{m^2} S \right) \\
+ \left( \eta_{\mu\alpha} k_{\nu} + \eta_{\mu\nu} k_{\alpha} + \eta_{\nu\alpha} k_{\mu} \right) \left( \frac{1}{12} q^2 L \right) \\
+ \left( \eta_{\mu\nu} q_{\alpha} + \eta_{\nu\alpha} q_{\nu} + \eta_{\alpha\nu} q_{\mu} \right) \left( -\frac{1}{6} q^2 L - \frac{1}{16} q^2 S \right) \bigg] + \ldots ,
\]

where \( L = \ln(-q^2) \) and \( S = \frac{s^2 m}{\sqrt{-q^2}} \). In the above integrals only the lowest order non-analytical terms are presented. The ellipses denote higher order non-analytical contributions as well as the neglected analytical terms. Furthermore the following identities hold true for the on shell momenta, \( (k \cdot q = \frac{q^2}{2}) \), where \( (k - k') = q \) and \( (k^2 = m^2 = k'^2) \).

In some cases the integrals are needed with \( (k) \) replaced by \( (-k') \), where \( (k' \cdot q = -\frac{q^2}{2}) \), these results, are obtained by replacing everywhere \( (k) \) with \( (-k') \). This can be verified explicitly. The above integrals check with the results of ref. [33].

The following integrals are needed to do the box diagrams. The ellipses denote higher order contributions of non-analytical terms as well as neglected analytical terms:

\[
K = \int \frac{d^4l}{(2\pi)^4 l^2(l+q)^2((l+k_1)^2 - m_1^2)((l-k_3)^2 - m_2^2)} \frac{1}{l^2(l+q)^2((l+k_1)^2 - m_1^2)((l-k_3)^2 - m_2^2)} \]

\[
= \frac{i}{16\pi^2 m_1 m_2 q^2} \left[ \left( 1 - \frac{w}{3m_1 m_2} \right) L - \frac{i\pi m_1 m_2}{(m_1 + m_2)p} \right] + \ldots ,
\]

\[
K' = \int \frac{d^4l}{(2\pi)^4 l^2(l+q)^2((l+k_1)^2 - m_1^2)((l+k_4)^2 - m_2^2)} \frac{1}{l^2(l+q)^2((l+k_1)^2 - m_1^2)((l+k_4)^2 - m_2^2)} \]

\[
= \frac{i}{16\pi^2 m_1 m_2 q^2} \left[ \left( -1 + \frac{W}{3m_1 m_2} \right) L \right] + \ldots ,
\]

where

\[
p = \left[ \frac{(s - (m_1 + m_2)^2)(s - (m_1 - m_2)^2)}{4s} \right]^{\frac{1}{2}}
\]

\[
\text{equals the center of mass momentum. Here } \left( k_1 \cdot q = \frac{q^2}{2} \right), \left( k_2 \cdot q = -\frac{q^2}{2} \right), \left( k_3 \cdot q = -\frac{q^2}{2} \right)
\]

\[
\text{and } \left( k_4 \cdot q = \frac{q^2}{2} \right), \text{where } (k_1 - k_2 = k_4 - k_3 = q) \text{ and } (k_1^2 = m_1^2 = k_2^2) \text{ together with } (k_3^2 = m_2^2 = k_4^2).
\]

Furthermore we have defined \( (w = (k_1 \cdot k_3) - m_1 m_2) \) and \( (W = (k_1 \cdot k_4) - m_1 m_2) \). The above results for the integrals check with ref. [99].

The algebraic structure of the calculated diagrams is rather involved and complicated, but yields no complications using our algebraic program. The integrals are rather complicated to do, but one can make use of various contraction rules for the integrals which hold true on the mass-shell.
From the choice \((q = k_1 - k_2 = k_4 - k_3)\) one can easily derive:

\[
\begin{align*}
    k_1 \cdot q &= k_4 \cdot q = -k_2 \cdot q = -k_3 \cdot q = \frac{q^2}{2}, \\
    k_1 \cdot k_2 &= m_1^2 - \frac{q^2}{2}, \\
    k_3 \cdot k_4 &= m_2^2 - \frac{q^2}{2},
\end{align*}
\]  
(A.2.19)

where \((k_1^2 = k_2^2 = m_1^2)\) and \((k_3^2 = k_4^2 = m_2^2)\) on the mass shell.

On the mass shell we have identities like:

\[
\begin{align*}
    l \cdot q &= \frac{(l + q)^2 - q^2 - l^2}{2}, \\
    l \cdot k_1 &= \frac{(l + k_1)^2 - m_1^2 - l^2}{2}, \\
    l \cdot k_3 &= \frac{(l - k_3)^2 - m_2^2 - l^2}{2}.
\end{align*}
\]  
(A.2.20)

Now clearly, \(e.g.\):

\[
\begin{align*}
    \int \frac{d^4 l}{(2\pi)^4 l^2(l + q)^2((l + k_1)^2 - m_1^2)((l - k_3)^2 - m_2^2)} l \cdot q &= \frac{1}{2} \int \frac{d^4 l}{(2\pi)^4 l^2(l + q)^2((l + k_1)^2 - m_1^2)((l - k_3)^2 - m_2^2)} (l + q)^2 - q^2 - l^2 \quad \text{(A.2.21)}
\end{align*}
\]

as only the integral with \((q^2)\) yield the non-analytical terms we let:

\[
\begin{align*}
    \int \frac{d^4 l}{(2\pi)^4 l^2(l + q)^2((l + k_1)^2 - m_1^2)((l - k_3)^2 - m_2^2)} l \cdot q &= -\frac{q^2}{2} \int \frac{d^4 l}{(2\pi)^4 l^2(l + q)^2((l + k_1)^2 - m_1^2)((l - k_3)^2 - m_2^2)} \quad \text{(A.2.22)}
\end{align*}
\]

A perhaps more significant reduction of the integrals is with the contraction of the source’s momenta, \(e.g.\):

\[
\begin{align*}
    \int \frac{d^4 l}{(2\pi)^4 l^2(l + q)^2((l + k_1)^2 - m_1^2)((l - k_3)^2 - m_2^2)} l \cdot k_1 &= \frac{1}{2} \int \frac{d^4 l}{(2\pi)^4 l^2(l + q)^2((l + k_1)^2 - m_1^2)((l - k_3)^2 - m_2^2)} (l + k_1)^2 - m_1^2 - l^2 \quad \text{(A.2.23)}
\end{align*}
\]

or

\[
\begin{align*}
    \int \frac{d^4 l}{(2\pi)^4 l^2(l + q)^2((l + k_1)^2 - m_1^2)((l - k_3)^2 - m_2^2)} l \cdot k_3 &= \frac{1}{2} \int \frac{d^4 l}{(2\pi)^4 l^2(l + q)^2((l + k_1)^2 - m_1^2)((l - k_3)^2 - m_2^2)} -(l - k_3)^2 + m_2^2 + l^2 \quad \text{(A.2.24)}
\end{align*}
\]

or
as seen the contraction of a loop momentum factor with a source’s momentum factor removes one of the propagators leaving a much simpler loop integral. Such reductions in the box diagram integrals help to do the calculations.

For the above integrals the following constraints for the non-analytical terms can be verified directly on the mass-shell:

\[ I_{\mu \nu} \alpha^\beta = I_{\mu \nu} \eta^{\mu \nu} = J_{\mu \nu} \eta^{\mu \nu} = 0, \]  
(A.2.25)

\[ I_{\mu \nu} q^\alpha = -\frac{q^2}{2} I_{\mu \nu}, \quad I_{\mu \nu} q^\nu = -\frac{q^2}{2} I_{\mu \nu}, \quad I_{\mu} q^\mu = -\frac{q^2}{2} I, \]  
(A.2.26)

\[ I_{\mu \nu} k^\alpha = \frac{1}{2} J_{\mu \nu}, \quad I_{\mu \nu} k^\nu = \frac{1}{2} J_{\mu \nu}, \quad I_{\mu} k^\mu = \frac{1}{2} J, \]  
(A.2.27)

These mass-shell constraints can be used to derive the algebraic expressions for the non-analytic parts of the integrals.

### A.3 Fourier transforms

In these section we present the Fourier transforms used to calculate the long-range corrections to the energy-momentum tensor in ref. [37]. The integrals needed in the calculations of the classical corrections are:

\[
\int \frac{d^3 q}{(2\pi)^3} e^{i \vec{q} \cdot \vec{r}} |\vec{q}| = -\frac{1}{\pi^2 r^4},
\]

\[
\int \frac{d^3 q}{(2\pi)^3} e^{i \vec{q} \cdot \vec{r}} q_j |\vec{q}| = -i r_j \frac{\pi^2 r^6}{r^6},
\]

\[
\int \frac{d^3 q}{(2\pi)^3} e^{i \vec{q} \cdot \vec{r}} q_i q_j |\vec{q}| = \frac{1}{\pi^2 r^4} \left( \delta_{ij} - 4 \frac{r_i r_j}{r^2} \right),
\]  
(A.3.28)

while we have for the quantum effects:

\[
\int \frac{d^3 q}{(2\pi)^3} e^{i \vec{q} \cdot \vec{r}} q^2 \log q^2 = \frac{3}{\pi r^5},
\]

\[
\int \frac{d^3 q}{(2\pi)^3} e^{i \vec{q} \cdot \vec{r}} q_i q_j \log q^2 = \frac{i 15 r_j}{\pi r^7},
\]

\[
\int \frac{d^3 q}{(2\pi)^3} e^{i \vec{q} \cdot \vec{r}} q_i q_j \log q^2 = \frac{-3}{2 \pi r^5} \left( \delta_{ij} - 5 \frac{r_i r_j}{r^2} \right).
\]  
(A.3.29)

We also need the integrals:

\[
\int \frac{d^3 q}{(2\pi)^3} e^{i \vec{q} \cdot \vec{r}} \frac{1}{|\vec{q}|} = \frac{1}{2\pi^2 r^2},
\]

\[
\int \frac{d^3 q}{(2\pi)^3} e^{i \vec{q} \cdot \vec{r}} \frac{q_j}{|\vec{q}|} = \frac{ir_j}{\pi^2 r^4},
\]

\[
\int \frac{d^3 q}{(2\pi)^3} e^{i \vec{q} \cdot \vec{r}} q_i q_j |\vec{q}| = \frac{1}{2 \pi^2 r^2} \left( \delta_{ij} - 2 \frac{r_i r_j}{r^2} \right).
\]  
(A.3.30)
and

\[
\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} \log q^2 = -\frac{1}{2\pi r^3}, \\
\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} q_j \log q^2 = \frac{-i3r_j}{2\pi r^5}, \\
\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} \left( \frac{q_i q_j}{q^2} \right) \log q^2 = \frac{1}{2\pi r^3} \left( \delta_{ij}(1 - \log r) - \frac{r_i r_j}{r^2}(4 - 3 \log r) \right) \tag{A.3.31}
\]
Appendix B

Quantum gravity at large numbers of dimensions

B.1 Effective 3- and 4-point vertices

The effective Lagrangian takes the form:

$$\mathcal{L} = \int d^D x \sqrt{-g} \left( \frac{2R}{\kappa^2} + c_1 R^2 + c_2 R_{\mu
u} R^{\mu\nu} + \ldots \right).$$

(B.1.1)

In order to find the effective vertex factors we need to expand: $\sqrt{-g} R^2$ and $\sqrt{-g} R_{\mu\nu} R^{\mu\nu}$.

We are working in the conventional expansion of the field, so we define:

$$g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu}. \quad \text{(B.1.2)}$$

To second order we find the following expansion for $\sqrt{-g}$:

$$\sqrt{-g} = \exp \left( \frac{1}{2} \log (\eta_{\mu\nu} + \kappa h_{\mu\nu}) \right) = \left( 1 + \kappa h_{\alpha\alpha} + \frac{\kappa^2}{4} h_{\beta\alpha} h^{\beta\alpha} + \frac{\kappa^2}{8} (h_{\alpha\alpha})^2 + \ldots \right). \quad \text{(B.1.3)}$$

To first order in $\kappa$ we have the following expansion for $R^{(1)}$:

$$R^{(1)} = \kappa \left[ \partial_\alpha \partial^\beta h_\beta - \partial^\alpha \partial^\beta h_{\alpha\beta} \right], \quad \text{(B.1.4)}$$

and to second order in $\kappa^2$ we find:

$$R^{(2)} = \kappa^2 \left[ -\frac{1}{2} \partial_\alpha \left[ h_\beta^\beta \partial^\alpha h_\beta^\mu \right] + \frac{1}{2} \partial_\beta \left[ h_\nu^\nu (2 \partial_\alpha h^{\nu\alpha} - \partial_\nu h^{\alpha\alpha}) \right] \right. \quad \text{(B.1.5)}$$

$$+ \frac{1}{4} \left[ \partial_\alpha h_\beta^\nu + \partial_\beta h_\alpha^\nu - \partial^\nu h_{\beta\alpha} \right] \left[ \partial^{\alpha} h_\nu^\beta + \partial^{\beta} h^{\alpha\alpha} - \partial^{\beta} h_\nu^\alpha \right]$$

$$- \frac{1}{4} \left[ 2 \partial_\alpha h^{\nu\alpha} - \partial^\nu h^{\alpha\alpha} \right] \partial_\nu h_\beta^\beta - \frac{1}{2} h_\nu^\nu \partial_\nu \partial_\alpha h_\beta^\beta + \frac{1}{2} h_\nu^\alpha \partial_\beta \partial_\alpha h_\beta^\beta + \frac{1}{2} h_\nu^\nu \partial_\alpha \partial_\beta h_\beta^\beta + \frac{1}{2} h_\nu^\nu \partial_\beta \partial_\alpha h_\beta^\beta - \partial^2 h^{\nu\alpha} \right]$$

In the same way we find for $R_{\nu\alpha}$:

$$R^{(1)}_{\nu\alpha} = \kappa \left[ \partial_\nu \partial_\alpha h_\beta^\beta - \partial_\beta \partial_\alpha h_\nu^\nu - \partial_\beta \partial_\nu h_\alpha^\beta + \partial^2 h^{\nu\alpha} \right], \quad \text{(B.1.6)}$$
From these equations we can expand to find \((R^2)\) and \((R^2_{\mu\nu})\).

Formally we can write:

\[
\sqrt{-g}R^2 = \frac{1}{2} h^\alpha_\gamma R^{(1)}_{\mu\alpha} R^{(1)}_{\mu\beta} + 2 R^{(1)}_{\mu\alpha} R^{(2)}_{\mu\beta} + 2 h_\alpha^\mu R^{(1)}_{\mu\rho} R^{(1)}_{\beta\rho} + 2 R^{(1)}_{\mu\rho} R^{(2)}_{\rho\beta} + h_\alpha^\mu h_\beta^\nu R^{(2)}_{\mu\nu},
\]

and

\[
\sqrt{-g}R^2_{\mu\nu} = \frac{1}{2} h^\gamma_{\mu\nu} R^{(1)}_{\mu\rho} R^{(1)}_{\rho\beta} - 2 h^\alpha_\rho R^{(1)}_{\mu\rho} R^{(1)}_{\beta\rho} + 2 R^{(1)}_{\mu\rho} R^{(2)}_{\rho\beta} + h_\alpha^\mu h_\beta^\nu R^{(2)}_{\mu\nu},
\]

Concentrating only on terms which go into the \((3B)^{\text{eff}}\) and \((3C)^{\text{eff}}\) index structures, and putting \((\kappa = 1)\) for simplicity, we have found for the \((R^2)\) contribution:

\[
(R^2)_{(3B)^{\text{eff}}} = \partial^2 h^\rho_\alpha - \frac{3}{2} \partial_\alpha h^\beta_\rho \partial^\alpha h^\beta_\rho - 2 h^\mu_\beta \partial^2 h^\beta_\mu,
\]

and

\[
(R^2)_{(3C)^{\text{eff}}} = -\partial^\rho \partial^\alpha h_\sigma^\rho - \frac{3}{2} \partial_\alpha h^\beta_\rho \partial^\alpha h^\beta_\rho - 2 h^\mu_\beta \partial^2 h^\beta_\mu.
\]
For the \((R^2_{\mu \nu})\) contribution we have found:

\[
(R^2_{\mu \nu})^{(3B)\text{eff}} = \frac{1}{8} h_c^2 \partial^2 h_{\nu \alpha} \partial^2 h^{\nu \alpha} - \frac{1}{4} \partial \partial h_{\nu \alpha} h^\nu h^\alpha - \frac{1}{2} \partial \partial h_{\nu \alpha} h^\nu h^\alpha, \quad (B.1.12)
\]

and

\[
(R^2_{\mu \nu})^{(3C)\text{eff}} = \frac{1}{2} \partial \partial h_{\nu \alpha} h^\nu h^\alpha + \partial \partial h_{\nu \alpha} h^\nu h^\alpha - \frac{1}{4} \partial^2 h_{\nu \alpha} h^\nu h^\alpha - \frac{1}{2} \partial^2 h_{\nu \alpha} h^\nu h^\alpha. \quad (B.1.13)
\]

This leads to the following results for the (3B) and (3C) terms:

\[
\begin{align*}
\left(\begin{array}{c}
\end{array}\right)^{\text{eff}}_{3B} & \sim \text{sym} \left[-P_3 \left(\eta_{\mu \alpha} \eta_{\nu \beta} \eta_{\gamma \delta} \left[3c_1 k_1^2 (k_2 \cdot k_3) + c_2 \left(\frac{1}{2} (k_1 \cdot k_2) (k_1 \cdot k_3) - \frac{1}{4} k_2^2 k_3^2\right)\right]\right] \\
& - P_6 \left(\eta_{\mu \alpha} \eta_{\nu \beta} \eta_{\gamma \delta} \left[2c_1 k_1^2 k_2^2 + c_2 \left(\frac{1}{2} (k_1 \cdot k_3)^2 - \frac{1}{4} k_2^2 (k_1 \cdot k_3)\right)\right]\right], \\
\left(\begin{array}{c}
\end{array}\right)^{\text{eff}}_{3C} & \sim \text{sym} \left[-P_3 \left(k_{\mu \alpha} k_{\nu \beta} \eta_{\gamma \delta} \left[3c_1 k_2 \cdot k_3\right]\right) - 2 P_6 \left(k_{\mu \alpha} k_{\nu \beta} \eta_{\gamma \delta} \left[k_1^2 \right]\right) \\
& + \frac{1}{2} P_6 \left(k_{\mu \alpha} k_{\nu \beta} \eta_{\gamma \delta} \left[c_2 k_1 \cdot k_3\right]\right) + P_6 \left(k_{\mu \alpha} k_{\nu \beta} \eta_{\gamma \delta} \left[c_2 k_2 \cdot k_3\right]\right) \\
& - \frac{1}{2} P_3 \left(k_{\mu \alpha} k_{\nu \beta} \eta_{\gamma \delta} \left[c_2 k_2^2\right]\right) \\
& - \frac{1}{2} P_6 \left(k_{\mu \alpha} k_{\nu \beta} \eta_{\gamma \delta} \left[c_2 k_2^2\right]\right) \\
& - \frac{1}{2} P_6 \left(k_{\mu \alpha} k_{\nu \beta} \eta_{\gamma \delta} \left[c_2 k_2^2\right]\right). \\
\end{align*}
\]

Using the same procedure, we can find expression for non-leading effective contributions, however this will have no implications for the loop contributions. We have only calculated 3-point effective vertex index structures in this appendix, 4-point index structures are tractable too by the same methods, but the algebra gets much more complicated in this case.

**B.2 Comments on general \(n\)-point functions**

In the large-\(D\) limit only certain graphs will give leading contributions to the \(n\)-point functions. The diagrams favored in the large-\(D\) will have a simpler structure than the full \(n\)-point graphs. Calculations which would be too complicated to do in a full graph limit, might be tractable in the large-\(D\) regime.

The \(n\)-point contributions in the large-\(D\) limit are possible to employ in approximations of \(n\)-point functions at any finite dimension, exactly as the \(n\)-point planar diagrams in gauge theories can be used, \(e.g.,\) at \((N = 3)\), to approximate \(n\)-point functions. Knowing the full \(n\)-point limit is hence very useful in the course of practical approximations of generic \(n\)-point functions at finite \((D)\). This appendix will be devoted to the study of what is needed in order to calculate \(n\)-point functions in the large-\(D\) limit.

In the large-\(D\) limit the leading 1-loop \(n\)-point diagrams will be of the type, as displayed in figure B.2.1.

The completely generic \(n\)-point 1-loop correction will have the diagrammatic expression, as shown in figure B.2.2.
Figure B.2.1: The leading 1-loop $n$-point diagram, the index structure for the external lines is not decided, they can have the momentum structure, \textit{i.e.}, as in (3C), or they can be of the contracted type, \textit{i.e.}, as in (3B)). External lines can of course also originate from 4-point vertex or a higher vertex index structure such as, \textit{e.g.}, the index structures (4D) or (4E). This possibility will however not affect the arguments in this appendix, so we will leave this as a technical issue to be dealt with in explicit calculations of $n$-point functions.

Figure B.2.2: A diagrammatic expression for the generic $n$-point 1-loop correction.

More generically the large-$D$, $N$-loop corrections to the $n$-point function will consist of combinations of leading bubble and vertex-loop contributions, such as shown in figure B.2.3.

Figure B.2.3: Some generic examples of graphs with leading contributions in the gravity large-$D$ limit. The external lines originating from a loop can only have two leading index structures, \textit{i.e.}, they can be of the same type as the external lines originating from a (3B) or a (3C)) loop contribution. Tree external lines are not restricted to any particular structure, here the full vertex factor will contribute.

For the bubble contributions it is seen that all loops are completely separated. Therefore everything is known once the 1-loop contributions are calculated. That is, in order to derive the full $n$-point sole bubble contribution it is seen that if we know all $j$-point tree graphs, as well as all $i$-point 1-loop graphs (where, $i, j \leq n$) are known, the derivation of the full bubble contribution is a matter of contractions of diagrams and combinatorics.

For the vertex-loop combinations the structure of the contributions are a bit more complicated to analyze. The loops will in this case be connected through the vertices, not via propagators, and the integrals can therefore be joined together by contractions of their loop momenta. However, this does in fact not matter; we can still do the diagrams if we know all integrals which occur in the generic 1-loop $n$-point function. The integrals in the vertex-loop diagrams will namely be of completely the same type as the bubble
diagram integrals. There will be no shared propagator lines which will combine the denominators of the integrals. The algebraic contractions of indices will however be more complicated to do for vertex-loop diagrams than for sole bubble loop diagrams. Multiple index contractions from products of various integrals will have to be carried out in order to do these types of diagrams.

Computer algebraic manipulations can be employed to do the algebra in the diagrams, so we will not focus on that here. The real problem lays in the mathematical problem calculating generic $n$-point integrals.

In the Einstein-Hilbert case, where each vertex can add only two powers of momentum, we see that problem of this diagram is about doing integrals such as:

$$I_n^{(\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n)} = \int d^Dl \frac{l^{\mu_1}l^{\nu_1}l^{\mu_2}l^{\nu_2}\cdots l^{\mu_n}l^{\nu_n}}{l^2(l + p_1)^2(l + p_1 + p_2)^2\cdots(l + p_1 + p_2 + \cdots p_n)^2}, \quad (B.2.14)$$

$$I_n^{(\nu_1\mu_2\nu_2\cdots\mu_n\nu_n)} = \int d^Dl \frac{l^{\nu_1}l^{\mu_2}l^{\nu_2}\cdots l^{\mu_n}l^{\nu_n}}{l^2(l + p_1)^2(l + p_1 + p_2)^2\cdots(l + p_1 + p_2 + \cdots p_n)^2}, \quad (B.2.15)$$

$$I_n = \int d^Dl \frac{1}{l^2(l + p_1)^2(l + p_1 + p_2)^2\cdots(l + p_1 + p_2 + \cdots p_n)^2}, \quad (B.2.16)$$

Because the graviton is a massless particle such integrals will be rather badly defined. The denominators in the integrals are close to zero, and this makes the integrals divergent. One way to deal with these integrals is to use the following parametrization of the propagator:

$$1/q^2 = \int_0^\infty dx \exp(-xq^2), \quad q^2 > 0, \quad (B.2.17)$$

and define the momentum integral over a complex $(2w)$-dimensional Euclidian space-time. Thus, what is left to do are gaussian integrals in a $(2w)$-dimensional complex space-time. Methods to do such types of integrals are investigated in refs. [20, 100, 19, 21]. The results for $(I_2)$, $(I_2^{(\mu)})$, $(I_2^{(\mu\nu)})$ and $(I_3)$ are in fact explicitly stated there. It is outside our scope to actually delve into technicalities about explicit mathematical manipulations of integrals, but it should be clear that mathematical methods to deal with the 1-loop $n$-point integral types exist and that this could be a working area for further research. Effective field theory will add more derivatives to the loops, thus the nominator will carry additional momentum contributions. Additional integrals will hence have to be carried out in order to do explicit diagrams in an effective field theory framework.
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